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OF THE 37<sup>TH</sup> CONFERENCE OF THE  
INTERNATIONAL GROUP FOR THE PSYCHOLOGY  
OF MATHEMATICS EDUCATION

*» Mathematics learning across the life span «*

**Volume 3**

PME 37 / KIEL / GERMANY  
July 28 – August 02, 2013

**Editors**

Anke M. Lindmeier  
Aiso Heinze



**IPN**

Leibniz Institute for Science  
and Mathematics Education



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# **RESEARCH REPORTS**

**Har - Pal**



# DEFINING THE NEED FOR JUSTIFICATION IN PROCESSES OF CONSTRUCTING JUSTIFICATIONS

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*In the present research, we focus on processes of constructing justifications. According to Abstraction in Context, no constructing process will be initiated without a need (of the learner) for a new construct. In the current research we take a first step toward investigating students' need for justification by giving a definition for one aspect of this need and show the efficiency of that definition by analysing a case study of the role of this need when students construct a justification.*

## INTRODUCTION

### Justification and Proof

Justification and proof are major components of mathematical reasoning and learning. Harel and Sowder (1998, 2007) defined proving as: "the process employed by an individual (or community) to remove doubts about the truth of an assertion" (Harel and Sowder, 2007, p. 6). This process contains two sub-processes: Ascertaining (in which individuals remove their own doubts) and persuading (in which individuals remove others' doubts). Harel and Sowder pointed out that they used 'proving' with the wider meaning of justification.

### Abstraction in context (AiC)

Hershkowitz, Schwarz and Dreyfus (Hershkowitz, Schwarz, and Dreyfus, 2001; Dreyfus, Hershkowitz, and Schwarz, 2001; Schwarz, Dreyfus, and Hershkowitz, 2009) proposed the theoretical framework of Abstraction in Context (AiC). According to them, abstraction is an activity of vertically reorganising previously constructed mathematical knowledge into a new structure (Schwarz et al., 2009, p. 24).

According to AiC, in a process of abstraction the learner passes through three stages: the arising of the need for a new construct, the emergence of the new construct, and its consolidation. A central component of the framework is a model, which provides micro-analytic tools for exploring constructing processes. The model is based on three epistemic actions (Hershkowitz et al., 2001): Recognizing - in which the learner recognizes a previous construct and realizes that this construct is relevant to the problem presently at hand; Building-with - in which the learner acts with recognized constructs in order to achieve a goal such as solving a problem; and Constructing, the central epistemic action of the model, which consists of assembling and integrating previous constructs to produce a new construct. Constructing refers to the first time a new construct is expressed or used by the learner. One reason for using a model of epistemic actions is that these actions are observable. A reason for choosing these specific actions is that they have been found suitable and useful for investigating processes of abstraction.



## Constructing Justifications

Kidron and Dreyfus (2006, 2010) analysed a process of constructing justifications. They found that in a process of justification, several constructing processes may occur in parallel and interact.

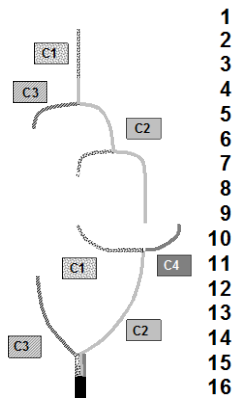


Figure 1: The interacting parallel constructions diagram

Figure 1 presents such a process of justification schematically. It shows four constructing processes marked as C1 - C4. In episode 5 construction C3 branches off from construction C2. In episode 11 constructions C1 and C4 combine. In the current study we will exhibit another case of a process of justification in which constructing actions combine.

### Need for new construct and need for justification

Although AiC has been quite extensively used during the last two decades, research studies have focused far more on the emergence and consolidation stages than on the first stage: The need for a new construct, as well as the need for justification, has hardly been investigated. Schwarz et al. (2009) claimed that without a need for a new construct, there no construction is expected. The term "need" refers to the need of learners that occurs when their constructs are not sufficient for them in order to deal with a given mathematically situation. In the case of constructing justifications several kinds of need may occur. Kidron and Dreyfus (2006, 2010) observed a need for justification that refers to the need for achieving a deeper understanding of the sense of a statement. Kidron, Bikner-Ahsbahs, Cramer, Dreyfus and Gilboa (2010) proposed the notion of General Epistemic Need (GEN), namely a need to make progress in the constructing process. In the current study, we are using Harel and Sowder's (1998, 2007) definition for justification in order to explore still another aspect of the need for justification. Since they defined justification as a process employed by an individual to remove doubts about the truth of an assertion, we consider the need for justification as the need to remove doubts about the truth of an assertion. In the present research we demonstrate and explain this kind of need by means of the analysis of a case study.

## METHODOLOGICAL CONSIDERATIONS

The current research is part of a wider study in which we examine the role of the need for justification in processes of constructing justifications. For this purpose we designed ten activities. Each activity is being carried out by at least three pairs of

students. The activities are carried out as task-based interviews by pairs of students. For each activity, we conducted an a priori analysis, in which we attempted to determine the elements of knowledge that are expected to be necessary or useful to complete the activity, as well as the connections between these elements of knowledge.

### Population and procedures

Here, we present a case study of one pair of students, Hadar and Shaked. The participants were grade 12 students from a high school in Israel. They were chosen from the advanced mathematics stream since they needed to deal with a mathematical situation involving integrals.

### The task

The task given to the students presents two dilemmas. In the first dilemma the students have to decide whether the following claim is true:

In domains in which the function  $f(x)$  takes on positive / negative values, the graph of its anti-derivative increases / decreases.

In the second dilemma the students have to decide whether the following claim is true:

In domains in which the function  $f(x)$  takes on positive / negative values, the computation of the integral:  $\int_a^b f(x)dx = F(b) - F(a)$  for  $a < b$  yields a positive / negative result.

The dilemmas were situated in a story about a teacher who drew two graphs (see Figure 2) and told her class that the drawing contains the graphs of a function and of its anti-derivative.

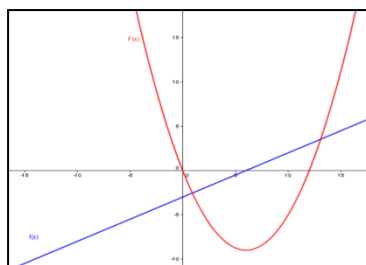


Figure 2: The graphs used in the activity

The story describes two students' controversy about the validity of the claims for all (continuous) functions. This is expected to raise doubts and hence to evoke the participants' uncertainty and need for justification.

### A priori analysis

The analysis below will focus on the second dilemma. In the a priori analysis for the justification process relating to the second dilemma, we predicted the following elements of knowledge as relevant:

E1: In domains in which a function is positive / negative, its anti-derivative function increases / decreases.

E2: In domains in which a function increases / decreases, for  $a > b$  the point  $(a, f(a))$  is higher / lower than the point  $(b, f(b))$ .

E3: The value of the integral over  $f$  from  $a$  to  $b$  is given by the expression:

$$\int_a^b f(x)dx = F(b) - F(a).$$

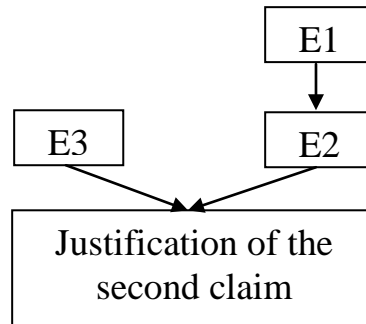


Figure 3: The connections between elements of knowledge

Our working hypothesis that the participants had constructed the three elements before attempting the second dilemma; for example, they had constructed E1 when dealing with the first dilemma. Figure 3 presents our a priori analysis of the connections to be established between these elements of knowledge during the justification process.

### ANALYSIS OF THE CONSTRUCTING PROCESS

Hadar and Shaked had no difficulties figuring out that the first claim is true. For example, Hadar (H) explained as follows:

- 30 H: The graph of  $f(x)$  represents the graph of the derivate of the anti-derivate function... and domains in which the values of the derivate are positive, the function increases.

We will focus on Hadar's process of constructing the justification of the second claim (lines 67 – 214) since she expressed herself clearly. We divided the relevant excerpt into segments, and analysed it as follows (H denotes Hadar, S Shaked, and I the interviewer):

Segment 67-73 contains the first reactions of the students to the second dilemma.

- 67 S & H: Silence (20 seconds)
- 68 I: Hadar, did you understand what she claimed?
- 69 H: I did. But I want to figure out whether it is true or not.
- 70 I: What did she say?
- 71 H: She said that if the values of the function are negative, the anti-derivative of that specific domain will also be negative. Like if I think of the integral as representing the area, so the function... Like the area underneath the function in that specific domain. So if the function is negative the area...
- 72 S: Supposed to be also negative.
- 73 H: Yes... but no. I am trying to find a contradictory example. I am not sure if it is true.

In lines 69 and 73, Hadar expressed doubts about the claim: "I am not sure if it is true". Moreover, she is actively trying to remove the doubts: "I am trying to find a contradictory example". According to our definition, Hadar exhibits a need for justification that the claim is correct. This is the first expression of Hadar's doubts; it exposed her need for justification and led to the first stage of constructing the justification.

In segment 76 – 84 Hadar makes her first steps in constructing the justification.

- 76 H: O.K if the calculation leads to a positive result then  $f(b)$  is supposed to be greater than  $f(a)$  (points to  $\int_a^b f(x)dx = F(b) - F(a)$ ). Like, according to simple mathematics.

In line 76 Hadar recognized (R) the relevance of the expression  $\int_a^b f(x)dx = F(b) - F(a)$ .

In segment 85 – 87 Hadar continues trying

- 85 H: And if this is negative (points to  $\int_a^b f(x)dx$ ) so that (points to  $F(b)$ ) less than this (points to  $F(a)$ ). Now... If the function gets negative values... may I draw here?
- 86 I: You may draw whatever you want. You got there some paper.
- 87a H: Let's say I will do... like in the beginning (she sketches a graph).
- 87b H: O.K it is declining which means that this (points to some point of the anti-derivative function in Figure 2) will be higher than this (points to a lower point on the anti derivative) did you understand what I am talking about?

In line 85 Hadar tried to figure out the connection between the value of the expression:  $\int_a^b f(x)dx = F(b) - F(a)$  and the domains of negativity of the function  $f$ : She is trying to build with (B) the element of knowledge E3 a small step toward the justification.

In line 87a Hadar pointed out the connection between the negativity of the graph of the function and the decline of the anti-derivative function. By that she recognized (R) the relevance of the element of knowledge E1 for the process of the justification. If we relate it directly to the constructing process that began in line 85, we come to the conclusion that Hadar is trying to construct the connection between E1 to E3. In line 87b Hadar examined the meaning of the decrease of the anti-derivative function. When doing that, she mentioned the element of knowledge E2. We point out that in the current segment Hadar mentioned all three relevant elements of knowledge. Furthermore, while building with (B) those elements, she constructed the connection between E1 and E2.

In segment 88-94 Hadar made another step. We focus on line 89:

- 89 H: If this is negative (pointing to the function  $f(x)$  in Figure 2) so that function (pointing to the anti-derivative function) decreases. Which means that  $b...$  but  $b$  comes before  $a$ . No? ...

In line 89 Hadar expressed the need to remove the uncertainty about the question: where should she place the parameter  $b$  (of the expression:  $\int_a^b f(x)dx = F(b) - F(a)$ ).

Therefore, she expressed a need for justification. This is the second expression of Hadar's need for justification. While the need for justification in segment 67-73 referred to the main claim, in the current segment the need for justification refers to a sub-assertion that arises while trying to construct the justification for the main claim. In the current segment, Hadar is trying to combine between the elements of knowledge E2 and E3. As can be seen from the a priori analysis (Figure 2), in order to construct the justification, Hadar needs to combine those elements of knowledge. Therefore, the expression of Hadar's doubts indicated her need for constructing and combining elements that were required for the process of constructing.

In segment 95-100 Hadar asked the interviewer for confirmation that she is on the right path:

- 95 H: Are we doing O.K?  
96 I: I really don't know.  
97 H: seriously?  
98 I: No.  
99 H: So, yes or no?

We interpret this segment as expression of Hadar's doubts and her need to remove them. Therefore we consider it as expressing a need for justification. In this segment Hadar's doubts reflect her need to find a way to justify the claim. This is her third expression of need and it shows Hadar's awareness that she has not yet finished the process of constructing the justification, and this in spite of the fact that she had already pointed out all elements of knowledge that needed for the justification. In other words, she was aware that she had not yet established all relevant links between these elements.

In segment 204-214 Hadar finished constructing the justification:

- 208 H: O.K. domains in which the function is negative, it is negative, therefore,  $F(a)$  will always be higher than  $F(b)$  on the function.  
210 H: Because the function is decreasing.  
210 H:  $F(b) - F(a)$  will be always negative. Because  $F(a)$  will be always higher.

Here Hadar pointed out not only the relevant constructs but also made the appropriate connections between them. After this segment we haven't found any expressions of Hadar's need to remove doubts. Hadar's process of constructing the justification can be summarized as follow: First, she constructed the connection between E3 and E1 (segment 85-87). Then she constructed the connection between E1 and E2 (segment 85-87). After that, she tried to combine between E2 and E3 (segment 88-94). At the end she connected between all the three elements. Figure 4 presents schematically Hadar's process of constructing the justification.

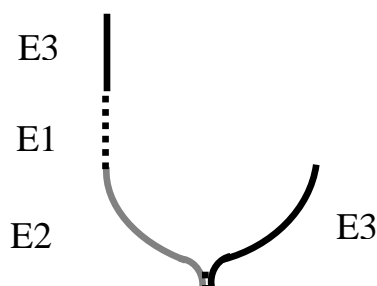


Figure 4: Hadar's construction process

## CONCLUSIONS

In the current study we defined one aspect of the need for justification as the need to remove doubts. The analysis shows that this need can be detected from a student's utterances. Each of Hadar's expressions of her need to remove doubts revealed a different stage of the need for the process of constructing the justification. The first indicated the need for constructing the justification, the second referred to the need for constructing and combining elements that were required for the process of constructing, and the third showed that the process had not yet come to its end. Hence we come to the conclusion that the need to remove doubts can be used for analysing the need for justification in processes of constructing justifications. As a result, it will be the basis of further research studies we plan.

## ACKNOWLEDGEMENT

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# WHEN THE FICTION OF LEARNING IS KEPT: A CASE OF NETWORKING TWO THEORETICAL VIEWS

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*A case of networking two didactic phenomena - the Topaze effect and the Funnel communications pattern - through analyses of the same episode from two theoretical perspectives is presented. It shows how this networked analysis deepens insight into the kind of learning of the same episode through complementary views, how it strengthens the theoretical understanding the two phenomena as parts of the theories and how it uncovers blind spots of the theories themselves.*

## INTRODUCTION

Networking of theories is a new approach of connecting different theories in the same empirical study (Bikner-Ahsbahr et al., 2010). Recent research has shown that the networking of theories may capitalize on the theories' strengths (Gellert, Barbé, & Espinoza, 2012). It usually deepens insight into the theories involved and their concepts (Kidron, 2008), identifies blind spots and boundaries (Font, Trigueros, Badillo, & Rubio, 2012), leads to new methodological considerations (Bikner-Ahsbahr et al., 2010), and possibly to locally integrated theoretical parts (Arzarello, Bikner-Ahsbahr, & Sabena, 2009; Gellert et al., 2012). Networking also may yield enriched research outcomes and deepened insights (Artigue 2009; Drijvers, Dodino, Font, & Trouch, 2013). Different methodologies have been developed for supporting networking activities such as the comparison of research praxeologies (Artigue, Bosch, & Gascon, 2011), cross-experiments (Artigue, 2009) and cross-analyses of the same data. In this paper, we report about a case of networking two theories based on cross-analysis (Bikner-Ahsbahr et al., 2010). On the methodological level, we will show how the cross-analysis of one episode to which researchers associate two different phenomena, the Topaze effect and the Funnel effect, leads to mutually inform the two theories by a deeper understanding of both phenomena, and reveals strengths and weaknesses of the two theories. The empirical part of the paper attempts to answer the questions: Do the two phenomena have a common ground that is deeply rooted in the practice of teaching and learning? How can this ground be described?

## METHODOLOGICAL CONSIDERATIONS

There are two theories involved in this case study: the Theory of Didactic Situations (Brousseau, 1997) and the social constructivist view focussing on social interactions developed by Bauersfeld (1978) and further developed by others (e.g. Voigt, 1983). The method of cross-analyses is used to analyze the same episode respectively from the two theoretical perspectives, followed by an exchange of the results mutually enriching understanding the episode. In order to deepen theoretical insight, a process of



networking the two analyses and their results in terms of the two theories is conducted using the networking strategies comparing, contrasting and coordinating the theoretical understanding of the episode (Bikner-Ahsbahs & Prediger, 2010).

### The episode

Two grade 10 Italian students G and C and their teacher (T) are discussing on what happens to the exponential function for very big  $x$ . Before, the students explored the behavior of exponential functions and its slope by building a secant for small differences of  $x$ -values with the computer (Fig. 1): the computer screen shows a secant built by two points very close to each other leading to a quasi-tangent line. The transcript presents the discourse and some gestures.

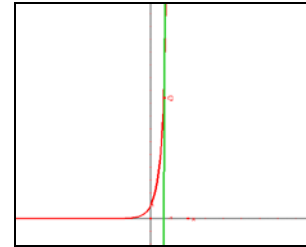


Figure 1. The graph on the computer screen.

- 1 G: but always for a very big this straight line (00:02), when they meet each other, there it is again...that is it approximates the, the function very well, because...
- 2 T: what straight line, sorry?
- 3 G: this here (pointing at the screen), for  $x$  very, very (00:14) big



00:02 G is pointing at the line in the screen



00:14 G's hand goes upwards



00:19 G: two two forefingers touching each other



00:28 T: right hand vertically raised

- 4 T: will they meet each other ? [challenging connotation]
- 5 G: that is [cioè], yes, yes they meet each other (00:19)
- 6 T: but after their meeting, what happens? (continuing to keep the hands in the same configuration as in 00:19)
- 7 G: eh...eh, eh no..., it makes so (00:24: G crossed the left hand over the right one. T keeps the previous gesture)
- 8 T: ah, ok, this then continues, this, the vertical straight line (00:28), has a well fixed  $x$ , hasn't it? The exponential function later goes on increasing the  $x$ , doesn't it ? Do you agree? Or not?
- 9 G: yes [...]
- 10 T (addressing C): He [G] was saying that this vertical straight line (pointing at the line in the screen) approximates very well the exponential function
- 11 G: that is, but for  $x$  that are very...very big
- 12 T: and for how big  $x$ ? 100 billions? (00:51)  $x = 100$  billions?

- 13 G: because at a certain point..., that is, if the function (00:57) increases more and more, more and more (00:59), then it also becomes almost a vertical straight line (1:03)



00:57 G raises his left hand



00:59 G moves his hand upwards



01:03 final position of G's hand after moving upwards.



01:13 T raises his right hand

- 14 T: eh, this is what it seems to you by looking at; but imagine that if you have  $x = 100$  billions, there is this barrier...is it overcome sooner or later, or not? [connotation: suggesting the answer yes]
- 15 G: yes
- 16 T: and so when it is overcome), this  $x = 100$  billions (01:13), how many  $x$  do you still have at disposal, after 100 billions? (01:14 like 01:13)
- 17 G: infinite
- 18 T: infinite... and how much can you go ahead after 100 billion (repeating the gesture as in 01:14)?
- 19 G: infinite points
- 20 T: then the exponential function goes ahead on its own, doesn't it?

## TWO VIEWS ON THE SAME EPISODE

Looking at this episode, two of the co-authors familiar with the Theory of Didactic Situations (TDS) immediately identified a Topaze effect, one of the paradoxes of the didactic contract. The third co-author familiar with Bauersfeld's interactionism saw a Funnel pattern, while Sabena and Arzarello who have proposed the data presented an interpretation in terms of a semiotic game (see Arzarello et al. 2009). This created an evident need for mutual clarification. In what follows, we focus on the first two interpretations resulting in a networking process.

### Topaze Effect

For making clear the reasons for an interpretation in terms of Topaze effect, we associated with the usual discursive description of Topaze effect (Brousseau, 1997) four criteria characterizing a Topaze effect:

- The teacher has a precise expectation in terms of students' answers.
- There is a substantial distance between the students' initial productions and utterances and these expectations.

- One can observe a succession of questions or dialogue piloted by the teacher for obtaining the expected answer drastically reducing the mathematical meaning of it.
- When the expected answer is produced, the teacher tries to maintain the fiction that the answer is really significant and that the didactical contract has not been broken.

Up to what point does the teacher-students interaction in this episode fulfil these criteria? Regarding the first criteria, complementary data collected show that the teacher has precise expectations: students should express that when  $x$  increases, the slope of the exponential function also increases and, with some help, that its evolution is also exponential. However, the Topaze effect if any is not linked to this expectation but to G's first utterance. It is understood by the teacher as expressing that an exponential curve can be approximated by a vertical line and have a vertical asymptote. Regarding this point, the teacher has a precise expectation: he wants the students to reject this claim. Concerning the second criteria, G's utterance is certainly distant from the teacher's expectation, but its exact meaning is not clear. G speaks about *secant* and *almost vertical line* whereas the teacher immediately interprets G's idea as an approximation of the vertical straight line and thus turns the discussion towards the rejection of an asymptotic behavior of the exponential. Regarding the third criteria, we indeed observe a succession of questions piloted by the teacher who develops an argumentation to which students are more asked to adhere than to contribute. At the end of the episode, G's contribution consists merely of words: "yes", "infinite", "infinite points" directly induced by the teacher's questions, which looks like a Topaze effect. The teacher closes the episode. He goes beyond what has been already said, expressing the fact that the line and the curve must separate, and once again looks for the students' agreement. In this episode, we thus find some characteristics of a Topaze effect, however the teacher does neither really hide his expectations nor his arguments to the students. Moreover, in the last exchanges (lines 15-19), the drastic reduction of G's contribution seems to result more from the fact that he has given up and does not want to break the didactic relation than because he cannot contribute anymore to the mathematical exchange. For all these reasons, the final conclusion was that, even if some criteria of a Topaze effect are fulfilled here, interpreting this episode just as a Topaze effect, one would miss essential characteristics of it.

This analysis conducted the co-authors towards deepening the inquiry about the Topaze effect. This inquiry showed that, in the research literature, discussions about the Topaze effect hardly exist; detailed examples are scarce and only partially fulfil the criteria listed above. They more often show teachers who reduce the students' mathematical work to solve simple and isolated tasks than teachers who face the paradoxical characteristics of the didactic contract by the absence of or unexpected answer (see for instance Novotna & Hospesova, 2007). Such an extension of the idea of Topaze effect is questionable. It could make us forget the didactic joint action between students and teachers inherent in classroom functioning (Sensevy, 2012), which imposes some "didactic reticence" to the teacher, but also requires from her to

regularly relax this didactic reticence for making the interaction of the students with the *milieu* cognitively productive.

### The Funnel pattern

In 1978, Bauersfeld identified a Funnel pattern being built in the process of communication between teacher and students in mathematics classrooms. The Funnel pattern starts with an open question followed by four steps of *actions narrowing by answer expectations* (Bauersfeld, 1978, 162, own translation):

- The student does not recognize the mathematical operation or is not able to draw an adequate conclusion. The teacher asks an additional question but gets a false answer or does not get any answer.
- The teacher continues his effort to get at least part of the expected answer. Understanding is not anymore approached basically.
- Missing the expected answer the teacher tends to narrow his efforts aiming to just saying what is expected, no matter who says that. Self-determined behavior of the student decreases and at the same time the situation becomes more and more emotionalized.
- The process is finished as soon as the answer occurs no matter whether the student or the teacher has produced it.

Up to what point does the episode show a Funnel pattern? The analysis method consists of turn-by-turn analyses in which both the teacher and the student are driven by *zugzwangs* to react and reach the aim (Voigt 1983). G begins to answer the question how the exponential function grows for very big  $x$  (1) but the teacher interrupts him. The teacher takes the terms *straight line* and *approximates* as key words indicating the mistake that a vertical straight line approximates the graph for big  $x$  (10). But G does not say vertical (1), he later talks about almost vertical (13) and disagrees (11, 13) expanding what he observes at the computer screen as a result of a DGS-construction for big  $x$ . In line 10, the teacher phrases G's utterance as a claim to be falsified. G's resistance dries down when the teacher evaluates the construction on the screen as misleading (14). In line 10 the teacher begins to socially construct a proof by contradiction. This is not done by narrowing actions to produce the expected mathematical answer but by stage-managing the argumentation process demanding agreement for clear facts. G's reduced answers (15, 17, 19) only partly fulfil the third criteria of a Funnel pattern, since the teacher does not narrow his expectations towards producing the proof. He produces the proof himself to convince the students and seems to approach this goal by approved routines, such as depowering the student to act by depriving him from his argumentation base and demanding confirmation to undeniable facts. This way, another interaction pattern (e.g. Voigt 1983) that reduces the students' contributions is constituted accepting that G maintains his view on a backstage-level.

### Networking the two approaches

Connecting the two different analyses revealed interesting similarities but also differences. The analyses led to the conclusion that the episode finally shows neither a

Topaze effect nor a Funnel pattern; and brought to the fore that a *fiction* of having learnt mathematics is maintained: the students produced answers but not necessarily with insight into mathematics. The differences in the way these negative conclusions have been obtained also clarify the complementary nature of the two theories. Thus the networking process progressed in at least three directions: (1) by having a more mature and dense understanding of each theory, and a clearer awareness of its limitations, (2) by enlarging the units of analysis taking into account new views on the data; (3) by comparing phenomena that turned out to be close and complementary. As an example for the first direction, networking led us to deconstruct, then reconstruct the Topaze concept, and to replace dichotomic considerations by an idea of a degree of proximity with this theoretical object. In the same manner, networking showed that looking only at social interaction as it is methodically done in the interactionist approach may be foreshortened because there can be different levels of acting. Routine actions are actions at the surface level whereas additional views of insight supported for example by individual interest can be underneath but not shown within the social interactions. Limitations were experienced because data constructed from another perspective withstood our analyses demanding to deepen the argumentation and its theoretical foundation. G's resistance, and what it reveals about argumentation in this classroom culture, are only partially captured by TDS constructs. Conversely, TDS's epistemic perspective attests the limitation of the milieu for supporting the expected proof by contradiction. This limitation cannot be identified in a pure interactionist view. Regarding the second way of progressing, TDS cannot totally explain why the situation does not degenerate into a complete Topaze effect. The debate between the researchers assisted the TDS researchers to tackle the data in a way, more sensitive to the classroom culture and to the "emotive dimension" that might explain why finally G gave up. Regarding the last direction some fundamental differences in the principles between the two theories were identified. The analyses neither address the same focuses, nor do they have the same granularity. Bauersfeld's theory is an interactionist approach, reasoning primarily in terms of routines, rooted in patterns of interaction which partially escape consciousness; epistemological concerns are thus not easily captured. Through contrasting, the epistemic strength of TDS became apparent but also up to what point TDS was less sensitive to non-epistemic characteristics of the social interaction. Conversely, the idea of the didactic contract that produces mutual obligations for social interaction and the insight that an insufficient milieu forces the teacher to act deepened the understanding of the episode from the interactionist view: here, the teacher moves the didactic contract and tries to make the students enter a new game - a proof by contradiction that ends with the final alignment of the two actors. Stage-managing a proof by contradiction that itself is part of an interaction pattern "socially correcting a mistake" reduces the student's contributions to one word sentences which gave the impression of a Funnel pattern. At a more general level, the importance of maintaining the didactic contract in the TDS offers an insight into interaction patterns: teacher and students trying to keep their roles may at the level of social interactions orient themselves towards the supposed expectation of the teacher, but the student may at the backstage level have kept his view alive. Referring to the

Funnel pattern, Bauersfeld proposed that the teacher should change the didactical model. This way, he goes into the direction of changing the milieu. However, the interactionist view does not offer a frame of constructing such a milieu as opposed to the TDS-perspective, which might recommend zooming in another screen to evidence that the tangent line never can be vertical.

## CONCLUSION

The networking process reached a joint improved understanding of the episode by the strategy of coordinating the analyses of the two theoretical frames. Cross-analyses showed up to what point the two theories could complement and consolidate each other. It made both research groups progress in their analyses in a way that demanded deepening the understanding of the respective theories by, for instance, re-questioning their theoretical constructs. The two analyses in terms of Topaze effect and Funnel pattern also led to become aware of the resonances between the two phenomena, their respective strengths but also limits. The characteristics of the episode contributed in a large part in making this methodology productive, as it addresses a problem that may often occur in teaching practice. The networking process produced a distinction between the surface level of social interaction and the depth level of epistemic insight. It grasped the same situation with different sensibilities and finally converged in a coherent picture being condensed in a common ground: *the fiction that learning took place*. In our opinion, such a fiction can exist to some degree each time the milieu and forms of joint action (Sensevy, 2012) are not sufficient to produce the new knowledge and to lead the teacher to give substantial responsibility to the students.

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# GENERALIZATION, MENTAL OBJECT, AND STABILITY OF IMPORT IN MATHEMATICS LEARNING

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*The purpose of this study is to clarify how students' generalization arises, and how students know its generality. In this paper generalization is separated from extension. Focusing on generalization, based on Semadeni's (2008) "stability of meaning", we propose the notion of "stability of import" as a new theoretical framework because the semiotic perspective is not enough for studying on generalization. The "import" means more ambiguous, personal, whole, and humanized meaning. It is provided by mental object, and grows in short-term experience like solving a problem. We analyze a classroom episode by using this framework. Consequently, we suggest that stability of import can be driving force of generalization and source of knowing its generality.*

## INTRODUCTION

Generalization is one of most important and interesting research issues in mathematics education because it is an epistemological nature of mathematics and important human ability that should be cultivated in mathematics education. In fact, there are many previous studies related with generalization. Although they do not always focus on generalization itself, Dölfler (1991) clarified what is the process of generalization in mathematics education, analyzed the generalization process, and presented its results as a model of generalization. We want to focus on the feature of his approach; that is, he seemed to implicitly presume an ideal generalization by students in mathematics. In other words, his model is a model of the process of ideal students' generalization. Hence, when we observe a student's generalization in terms of his model, we may distort his/her actual process because we have to judge his/her knowing process. For example, according to his model, two processes of abstraction and generalization must be connected with the phase of "symbols as objects". When we see a student cannot do generalization very well, we just judge that s/he *cannot* attain the phase of "symbols as objects". Therefore, it is important for us to pay more attention to what students are seeing, what they are thinking, how they can attain the phase of "symbols as objects". Thus, the purpose of this study is to answer the following two research questions;

RQ1: How does student's generalization arise?

RQ2: How does student know generality in a process of generalization?

In the previous studies, there are two different types of approach adoptable to answer these questions. The first type is the approach based on semiotic perspective like as Dölfler (1991), and Rina & Peter (2002) (e.g., "symbols as objects"). Some studies in PME also use this approach (cf., Ursini, 1990; Iwasaki and Yamaguchi, 1997; Radford, 2001). In the semiotic perspective, "The learner's behavior is interesting only insofar as it is interpreted by other members of the group as a sign, i.e. as having some



meaning for them” (Sierpinska, 2005). Thus, the semiotic approach cannot satisfactorily answer the above research questions. The second type is “pattern generalizing” approach like as Radford (1996, 2001), Presmeg (1997), and Rina & Peter (2002). Although the term of pattern is used in a broad sense, the term usually means “visual” pattern, and visual patterns are widely accepted as driving force of generalization in this approach. Thus, this approach focuses on the relation between visual patterns and one’s recognition. The focus is close to our interesting in this paper. However, visual patterns are finite. One often sees visual patterns as whatever the individual wants. Thus this second approach is not enough for our purpose to answer the above research questions. Therefore, in this paper we adopt the third approach focusing on the individual’s mental objects that are related to the visuals, but cannot be reduced to them.

## THEORETICAL BACKGROUND

### Distinction between generalization and extension

In this paper, we make a distinction between generalization and extension to avoid confusion. The term of *generalization* means the one’s recognition process that has an epistemological direction from particular to general. On the other hand, the term of *extension* means another recognition process that does not have such direction from particular to general. In previous studies, both of them are implicitly or explicitly integrated into generalization (cf., Dölfler, 1991; Iwasaki & Yamaguchi, 1997; Radford, 2001) without paying attention to this difference.

We formalize generalization and extension with models of Figure 1 and 2 respectively.

$D$  is a field.  $D'$  is a wider field than  $D$ .  $M$  is a known and established meaning in a field  $D$ .  $M'$  is an established meaning in a field  $D'$ .

Generalization: Recognition extending  $D$  to  $D'$  without changing  $M$

Extension: Recognition incorporating  $D$  into  $D'$  such that if  $D'$  is limited to  $D$ ,  $M'$  is equivalent to  $M$ .

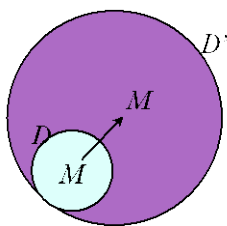


Figure 1: Model of generalization

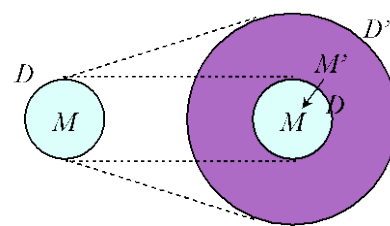


Figure 2: Model of extension (Tomosada, Himeda, & Mizoguchi, 2006, p. 9)

The definition of extension is Iwasaki’s (2003). On the other hand, we defined the term of generalization based on his study. For example, when students noticed that the sum of interior angles is straight angle ( $M$ ) in concrete triangles ( $D$ ), thereby they suppose that it is case of all triangles ( $D'$ ). This recognition is generalization because  $M$  is not changed. On the other hand, when students work on multiplication of decimal numbers

(D') for first time, they cannot solve the multiplication by using the meaning of multiplication as repeated addition (M) in natural numbers (D). The decimal number multiplication can be solved with the meaning of proportion (M'), and this meaning is equivalent to repeated addition in natural numbers. So this recognition is extension.

There are three reasons why we need this distinction:

Epistemological reason: If we ignore the difference between generalization and extension, and include extension into generalization, then we may fail to notice their critical aspect related their direction of recognition.

Methodological reason: Usually extension is more difficult than generalization because extension needs a conceptual change (in the above, M to M'). Hence, previous studies categorized both into "generalization" tend to focus on extension, and pay a little attention to generalization (cf., Dölfler, 1991; Sierpiska, 1994).

Practical reason: The difference between generalization and extension lead to the different structures of practice in mathematics education. The practice aimed at generalization must evaluate an idea and broaden its range of application. The practice for extension must compare some ideas and reinterpret ideas (Tomosada, Himeda, & Mizoguchi, 2006).

For these reasons, in this paper we exclude extension and focus on generalization.

### **Relation between generalization and creating new mathematical objects**

In the paradigm of constructivism, mathematical objects (that is, theorem, proposition, figure, number, and so on) are created by students themselves. There is very famous philosophical problem: Are mathematical objects discovered or invented? According to Giusti (1999), a new mathematical object is invented to solve a problem implicitly at first and subsequently it is going to be discovered as a new mathematical object (I-D process). For example, the notion of limit was invented to solve a physical problem implicitly at first; soon, mathematicians discovered that notion of limit as an important mathematical object.

However, Giusti (1999) did not show the detail of this I-D process. So, we assume that the I-D process is essentially seen as the process of generalization; because a way to solve a problem is particular but a new mathematical object must have generality. In addition, a new mathematical object invented to solve a problem is associated with a certain situation, but one may disassociate it from a situation through generalization; the nature of generality is not limited to a situation. For this reason, in this paper we think about generalization in the context of solving a mathematical problem.

### **METHODOLOGY**

In this study, we basically adopt Semadeni's (2008) the notion of *stability of meaning by deep intuition* and modify the notion for our purpose; we propose the notion of *stability of import* as a new theoretical framework. Semadeni's notion is a part of concept image (Tall & Vinner, 1981), has fixed meaning for a subject, and is interpreted as mixture of mental objects and (visual) images. It is hard to observe

mental objects directly, thus we cannot use empirical method. For this reason, we try to clarify that the mental objects can be driving force of generalization and source of knowing its generality by means of philosophical consideration and analyzing a classroom episode.

## STABILITY OF IMPORT BY MENTAL OBJECT

Semadeni (2008) points out the importance of *stability of meaning* (it's commonly accepted perspective). Although people usually think the stability of meaning can be achieved through defining activity, Semadeni insists that it is not always true.

He focuses on *mental object like image* that is very hard to define by language. It is a concept image (Tall & Vinner, 1981) or a part of concept image which cannot be observed directly. This notion is based on the Davis & Hersh's (1981) suggestion about importance of mental object and its representation because both of them provide us intuition. When one has an object X with a mental object and discusses about X, the basic meaning of X may be stabilized by the mental object which does not depend on a certain context or situation. Semadeni (2008) calls it as a level of deep intuition. The stability is closely related to not only our vision but also daily experience and embodied knowledge.

General verbal rules and formal systems of predicate calculus are of little help here if they are not supported by deep intuition. Language, although crucial for concept formation and communication, is not a sufficient tool to deal with such ambiguities (Semadeni, 2008, p.15).

Semadeni (2008) points out three roles of the stability of meaning by deep intuition; the role of valid mathematical reasoning without reference to precise definitions and known theorems, the role to affect formal and logical interpretation, and the role to understand and appreciate an instance of reasoning with due sense of its necessity. These roles are very suggestive for our purpose in this paper by following three reasons. First, in school mathematics, there are mathematical knowledge should be constructed by students and at the same time have to be taught by teachers. This is a contradictory situation. The notion of stability of meaning by deep intuition is expected to provide us a solution. Second, in generalization in learning school mathematics, previous studies pointed out that even if students represent a mathematical statement by using algebraic characters or verbal words like as "all", "always", the students do not understand its generality (Rina & Peter, 2002). In such a situation, it is worthy for us to consider some influence on students' formal and logical interpretation of algebraic characters by the stability of meaning by deep intuition. Third, we cannot consider generalization without intuition. Mathematical reasoning, its main component is generalization, cannot be analyzed by traditional syllogistics, and one must trust on the power of intuition for generalization except when s/he has known the generalization (Beth & Piaget, 1966). Thus, we have to consider mental object that provides us intuition.

Semadeni (2008) focused on the notion of "stability of meaning by deep intuition" that grows through students' long-term experience and its variation. However, he was less concerned about a similar notion that may grow and change through students' very

short-term experience in solving a mathematical problem. We think that students' mental object grows/changes through their short-term experience, and that it provides a stability of meaning. We should pay attention the fact that such stability of meaning strongly affects students' generalization as demonstrated with an episode in the next section. This kind of "stability of meaning" does not depend on contexts or situations in long-term. It is a different kind of stability. Therefore, we extend Semadeni's (2008) notion wider, and consider mental object which grows/changes through short-term experience in solving a mathematical problem and provides a stability of meaning. We called it "stability of import", because "meaning" in short-term experience may be more ambiguous, personal, whole, and humanized. So we replace "meaning" by "import". It should be precisely called "stability of import by mental object". However it is too long. So, we shorten it as "stability of import" in the analysis of an episode.

## AN EPISODE AND ITS ANALYSIS BY STABILITY OF IMPORT

### An episode in secondary school classroom

The following is an episode observed in a 1st Grade (12 and 13 years old students) classroom of secondary school in Japan. The students had learned basic algebraic characters and equation. Most students well acquired them. The teacher gave to their students the day's problem; "Devise a calculation method,  $1+2+3+4+5+6+7+8+9+10$ ". Teacher intended to generalize students' solution to a formula for sum of arithmetical sequence with common difference 1. The students individually tried to solve this problem, and almost all of them solved it by the method shown in Figure 3.

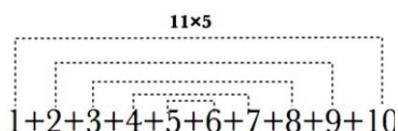


Figure 3: A typical solution of the given problem

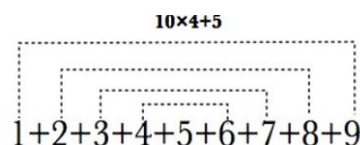


Figure 4: A case of  $n = \text{odd number}$

The teacher asked the students who solved as Figure 3; "Think about other cases". Then, the students generalized their solution. They found that the sum is calculated by  $(1+n) \times \frac{n}{2}$  where the last term is  $n$ . At the same time, the students noticed that this is not applicable in the case where the last term ( $n$ ) is odd number. Thus, they tried to solve some cases like Figure 4, used the same strategy, and noticed that the center number (for example, the center number is 5 in Figure 4) is very important.

Soon, they generalized a new calculation  $(1+n) \times \frac{(n-1)}{2} + \frac{1+n}{2}$  for the case of  $n = \text{odd number}$ . Here, the students understood that both calculations give the same answer. Nevertheless, some students claimed that the correspondence between the two answers is accidental. Other students claimed that they should distinguish two formulas. Thus, the students concluded; "There is no single formula for the sum of arithmetical sequence with common difference 1 and the first term 1", and rejected integrating two formulas by generalization. In other words, they rejected the generality of algebraic

expression. This continued until the teacher showed students the Gauss's strategy. Finally, the students accepted the Gauss's strategy, and agreed to integrate two formulas into one general formula.

### Episode analysis by stability of import

This is a suggestive episode because we can see two different types of generalization in this episode. One is successfully accomplished by the students; the generalization leads to different formulas for the sum of arithmetical sequence with common difference depending on the last term is even number or odd number. Another is not spontaneously accomplished by the students; the generalization leads to the single formula for the sum of arithmetical sequence with common difference 1 and the first term 1. Of course, both types of generalization are valuable for the students. Why were there two types of generalization in the episode? Why could the students not spontaneously accomplish the generalization intended by the teacher?

If the students used the Gauss's strategy to solve the given problem, they might not reject the latter type of generalization. For this reason, we can argue that the first strategy for solving a mathematical problem strongly affects students' success or failure of their generalization and interpretation of its generality. In the above episode, students' generalization derived from the possibility to repeat same strategy. We argue that there was the stability of import that grew through solving a mathematical problem as a background of generalization derived from the possibility to repeat same strategy, of knowing its generality, and of rejecting another generalization. In fact, the generalization in this episode is based on the structure that sum of  $a$ th term and  $(n-a+1)$ th term is always same in an arithmetical sequence with common difference. However, this structure seemed to be intuitively recognized by the students. Therefore, we argue that the stability of import that grew through solving problems of some cases by drawing lines and connecting numbers functioned as driving force for the students' generalization. The stability of import was also source of knowing its generality. On the other hand, this stability of import did not reflect the structure for generalizing the formula intended by the teacher. We argue that the generalization intended by the teacher was rejected by the students because the generalization was inconsistent or incompatible with students' stability of import at that time. In other words, if students' stability of import was consistent with generalization intended by teacher, the students could generalize and know its generality. Therefore, the stability of import should be consistent with the nature of mathematical knowledge created by the generalization. In this episode, for example, it might be effective that the teacher asked students to explain a figurative solution shown in Figure 5.

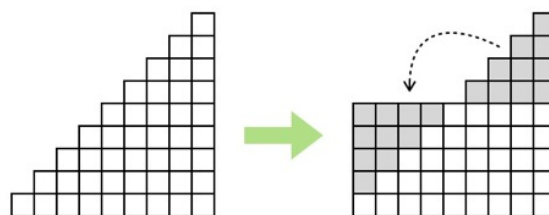


Figure 5: Figurative solution of the sum of sequence

Essentially, operation in Figure 5 is essentially same as operation in Figure 4. Nevertheless, Figure 5 includes the nature of mathematical knowledge of the sum of the sequence; [arithmetical average  $\times$  number of terms]. In addition, the figurative solution in Figure 5 has a kind of dynamics that the average can be always calculated. This kind of dynamics is essential for the stability of import because the nature of generalization relates to infinity and we must see some symbols as general; “Generalization means constructing variables (Dölfler, 1991, p. 84)”.

If students’ stability of import is not consistent with the generalization expected by teacher, the stability of import must be broken or reconstructed by students themselves. Growing stability of import consistent with the nature of a mathematical knowledge seems to be required for generalization.

## CLOSING REMARKS

As a result of this study, about RQ1, we conclude that stability of import can be driving force of generalization and source of knowing its generality. If stability of import is not consistent with generalization, it must be broken or reconstructed by students themselves. And about RQ2, we conclude that if students’ stability of import reflects some features of the new generalized mathematical knowledge, students can know its generality. As mentioned above, our intention was to see student’s actual knowing. The stability of import is probably at level of one’s involuntary thinking, and so it may be much more inaccessible than semiotic thinking. Nevertheless, as we argued in this paper, describing students’ stability of import in solving a mathematical problem is useful to identify the reason why students can/cannot generalize something and know its generality in learning mathematics.

There are two future tasks. First, we have to clarify relation between stability of import by mental object and visual patterns because main component of stability of import is visual image. Second, we analyzed only one episode, so we need to analyze more episodes in mathematics learning so as to more clarify and elaborate our theoretical framework.

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# LEARNING FRACTIONS FROM ERRORS

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*There is empirical evidence that reflections on errors can enhance knowledge and skills. However, there is a lack of studies within school settings. Thus, we conducted an intervention study in order to investigate the question whether 6<sup>th</sup>-grade students (N = 200) can learn fractions by reflecting on errors and if their improvement can be explained by an enhancement of their so-called negative knowledge. We compared two different learning environments in which either incorrect or correct vicarious solutions were reflected. Our findings indicate that students can develop and use a specific negative knowledge via error reflections. With respect to general knowledge of fractions, error reflections were beneficial for advanced school students only. The implications of these results with respect to school instructions are discussed.*

## INTRODUCTION

In educational research, reflecting on errors during the learning process is considered as a potential source to acquire knowledge and cognitive skills in various domains (e.g. Siegler, 2002; Joung, Hesketh, & Neal 2006). Even though research studies have shown that reflections on errors have effects on individual knowledge and skill acquisition, the desideratum stated by Borasi (1996) that instructional recommendations with respect to errors are extremely rare is still of importance: There are open questions about how and under which conditions errors are beneficial for learning. The major objective of the study presented in this contribution is to investigate the question of whether 6<sup>th</sup> grade students can learn fractions by reflecting on incorrect vicarious solutions and whether their improvement can be explained by an enhancement of negative knowledge.

## Learning from Errors

Intervention studies investigating learning processes with an emphasis on errors mainly concentrate on one of two different types of learning environments: error management training or guided error training (cf. Ivancic & Hesketh, 2000). Error management training is characterized by a learning environment in which learners work on problems which go beyond their individual knowledge and, thus, errors are highly probable. In addition, the learners are actively encouraged to accept errors as a positive and natural part of their learning process (Keith & Frese, 2005). In various studies (Chillarege, Nordstrom, & Williams, 2003; Keith & Frese, 2005), participants who received error encouragement instructions significantly improved their task performance – more than participants who got error avoidance instructions or no instructions at all. Guided error training is an alternative to error management training. The main idea is to present learners with selective vicarious errors with a high learning potential. Various studies showed benefits for both professional skills (e.g. Joung et al., 2006) as well as knowledge and skills with respect to mathematical contexts (e.g.



Siegler, 2002). However, findings support the assumption that learners benefit from erroneous examples if, in addition, some correct examples are presented. In particular, this mix seems to be important at the beginning of a learning process (Große & Renkl, 2007). Additionally, following Große and Renkl (2007), the effects of guided error training are influenced by learners' prerequisites. In their study, university students reflected on statistical worked examples and benefits could be found for learners with strong prior knowledge with respect to far transfer, only. All of these findings were derived from strictly controlled experiments with restricted ecological validity. With respect to learning from errors, there is a lack of findings related to ecologically valid school environments and relevant mathematics curriculum topics.

### **The role of negative knowledge in learning processes including errors**

Regarding the question of why error reflection supports knowledge and skill acquisition, there is broad agreement on a cognitive explanation (Joung et al., 2006): Errors encourage learners to test the principles and assumptions of the learners' individual cognitive models. By doing this, they change and extend their cognitive models and incorporate aspects they recognized as incorrect into the error situation. Thus, they build up a deeper understanding of the respective concept (Ivancic & Hesketh, 2000). These extended models can be memorized and recalled well in order to avoid errors in similar situations (Jones & Endsley, 2000). Hence, through error reflection, a more comprehensive cognitive model is developed, including both correct and incorrect facts as well as effective and non-effective strategies (Chillarege et al., 2003). This explanation is in line with the theory of negative knowledge (Oser, Nöpflin, Hofer, Aerni & Spychinger, 2012). According to Oser et al., negative knowledge is defined as the additional knowledge previously mentioned which includes incorrect facts and non-effective strategies. Oser et al. assume that reflecting on errors leads to negative knowledge which prevents the learner from committing errors. Even though they do not use the term negative knowledge explicitly, the idea of negative knowledge also plays a role in different approaches regarding learning (e.g. Siegler, 2002; VanLehn, 1999). In conclusion, negative knowledge seems to play a significant role for learning from errors. However, our thorough search for empirical investigations concerning this mechanism did not reveal any results.

### **The present study**

We investigate the question whether children improve their knowledge of fractions better in an "erroneous learning environment" in which students are encouraged to reflect on erroneous vicarious solutions or in a "correct learning environment" in which only correct vicarious solutions are presented. The use of two different measures, measuring "knowledge of fractions" and "negative knowledge" regarding to fractions allows us to examine the mechanism of learning from errors more precisely. Fractions present a domain of the mathematics curriculum which has an essential impact on future mathematical achievement (Siegler et al., 2012). For the intervention, we followed the idea of guided error training: Vicarious errors were chosen to be presented to the students. Our research was guided by the following hypotheses:

Hypothesis 1: The students in the “erroneous learning environment” build up more negative knowledge than the students in the “correct learning environment”.

Hypothesis 2: The “erroneous learning environment” enhances knowledge of fractions more than the “correct learning environment”. This is because we expect that reflecting on erroneous vicarious solutions supports the construction of more comprehensive cognitive models about fractions than reflecting on correct vicarious solutions.

Hypothesis 3: Following the theoretical approaches sketched above, we assume that negative knowledge is a partial mediator for the acquisition of knowledge, i.e., negative knowledge partially mediates the effect of prior knowledge of fractions on knowledge after the intervention (3a). Moreover, combining Hypothesis 1 with 2, we assume that the increase of negative knowledge partially mediates the effect of the learning environment on the acquisition of the knowledge of fractions (Hypothesis 3b).

Since prior knowledge might have an influence on learning successes (Große & Renkl, 2007) our analysis will account for prior knowledge.

## METHOD

### Design

All students participated in a pre-test, a teaching unit of twelve 45-minute lessons which were part of their regular mathematics classes, and a post-test. Both the pre- and the post-tests contained items designed to measure the students’ “knowledge of fractions” and their “negative knowledge”. Each teaching unit was taught by a team of two trained members of the research group. The allocation of the students to the learning environments was based on the pre-test results so that the treatment groups within each class did not differ significantly from each other.

### Participants

The sample consisted of 200 6<sup>th</sup> grade students (10 to 13 years of age) who belonged to nine classes from German secondary schools. In all classes, the basic conceptions of fractions had already been taught and students had already learned to add and subtract fractions. However, they had not yet learned multiplications and divisions with fractions. Exactly 100 students participated in each of the learning environments. Both groups had a similar distribution of boys and girls (erroneous learning environment 43 % girls, correct learning environment 48 % girls,  $\chi^2 = .50$ ,  $p = .482$ ).

### Pre- and Post-tests

The pre-tests predominantly covered knowledge the students were taught before the intervention: Basic conceptions of fractions, addition and subtraction. The post-test items covered the content of the teaching unit: Multiplication and division of fractions.

The items for knowledge of fractions were developed based on the German PALMA study (Vom Hofe, Kleine, Blum, & Pekrun, 2005). Correct solutions were coded with “1”, incorrect solutions with “0”. Performance scores were represented by the mean of all items. The 14 pre-test items showed satisfactory item parameters with a reliability of  $\alpha = .85$ . Five from the 14 post-test items were excluded from the analysis because of

low corrected item-total correlations. The reliability for the remaining nine items was  $\alpha = .78$ . In order to analyze some results in more detail, we defined extreme groups by the content criteria of the pre-test. There were three easier items in which no arithmetic computation was requested but basic conceptions of fractions were. Three other items demanded relatively complex knowledge of conceptions of fractions. All other items had a medium difficulty and students needed to use familiar arithmetic operations. We defined students with “low prior knowledge of fractions” if they solved three or less items correctly (less than 25% correct items). These students ( $n = 43$ ) only solved some of the easiest three items and, thus, were only capable of applying basic conceptions of fractions. We further labeled students as having a “high prior knowledge of fractions” if they solved eleven or more items correctly (more than 75% correct items). They ( $n = 44$ ) basically only failed to solve all of the three complex items. Thus, they were relatively capable of using arithmetic operations they had already learned.

In order to measure negative knowledge, we developed specific items. There were eight items in both the pre- and post-test that had the following layout: An incorrectly solved fraction problem together with a solution (four to six steps) was presented as the stimulus. The item had to be answered on two multiple choice rubrics: First, the students were asked to identify the wrong solution step and, second, to choose from four given alternatives the best explanation of why the person made this error. The multiple choice rubrics were treated as dependent indicators for two levels of proficiency of negative knowledge: The answers were coded with “1” (full score) if both parts were identified correctly. If only the first was correct, the answer was coded with “0.5” (partial score). In all other cases the answer was coded with “0”.

A principal component analysis with orthogonal rotation (varimax) including items from both pre-tests, indicated that the items measuring negative knowledge load on a different component than the items measuring knowledge of fractions. The same results were obtained for the post-tests. Thus, we can assume that the tests of negative knowledge measure a different construct than the tests of knowledge of fractions.

We further used approved scales to measure different control variables: Students’ interest in mathematics as a subject before the intervention and after the intervention motivation and the perceived cognitive load of the tasks used in the practice phases.

### **Introductory lessons and practice materials to reflect on vicarious Solutions**

During four lessons, new operations with fractions were introduced:  $a \cdot \frac{b}{c}$  in lesson 3,  $\frac{a}{b} \cdot \frac{c}{d}$  (lesson 5),  $\frac{a}{b} : c$  (lesson 7) and  $\frac{a}{b} : \frac{c}{d}$  (lesson 9). The instructions and exercises in the introductory lessons were the same for the students in both conditions. During the eight practice lessons, students had to reflect on vicarious solutions within the two conditions. Thereby, they were practicing the new concepts that they had been taught in the intervention. The exercises were posed as written worked examples (Große & Renkl, 2007) in three steps as follows: (a) An initial problem was presented. (b) Directly below, a solution from the student “Anna” or “Tom” was shown in detail

(correct or incorrect, according to the intervention group). (c) Finally, students were presented with three prompts to initialize their reflections (see Table 1).

Erroneous learning environment	Correct learning environment
Which solution step is not correct and why?	Term the student's solution.
Why do you think the student made exactly this error?	Why is the solution correct?
Solve the problem correctly.	Solve the following problem: [analogous problem].

Table 1: Prompts in the two learning environments.

Overall, 120 minutes of the intervention time was used to introduce the new operations and 240 minutes were used to reflect on vicarious solutions in the two conditions.

## RESULTS

There were no significant pre-test differences between the two intervention groups with respect to “knowledge of fractions” and “negative knowledge”. Thus, the students in both conditions had about the same prior knowledge. Further, there were no significant differences according to the students’ “interest in mathematics as a subject”, the perceived cognitive load nor to motivation. Thus, we can assume that both conditions required similar cognitive demands and motivated the students in a similar way.

### HYPOTHESIS 1: EFFECTS ON NEGATIVE KNOWLEDGE

There was a significant effect of the “learning environment” on “negative knowledge” after controlling for the effect of pre-test scores on “negative knowledge” and “fraction knowledge” ( $F(1, 194) = 5.22, p = .023, \eta^2 = .027$ ). Students in the “erroneous learning environment” showed a higher performance ( $M = .43, SD = .27, n = 96$ ) than students in the “correct learning environment” ( $M = .37, SD = .26, n = 99$ ). Thus, we found that students in the “erroneous learning environment” enhanced their negative knowledge more than students in the “correct learning environment”. Further analysis showed that there was no interaction effect between “learning environment” and “knowledge of fractions” (pre-test) on “negative knowledge” ( $F(1, 192) < 1$ ).

### HYPOTHESIS 2: EFFECTS ON KNOWLEDGE OF FRACTIONS

We did not find a significant effect of the “learning environment” on “knowledge of fractions” after controlling for the pre-test score of “knowledge of fractions” ( $F(199, 1) < 1$ ). Thus, students in the erroneous learning environment did not enhance their fraction knowledge significantly more than students in the correct learning environment. In order to further analyze the prior knowledge of fractions, we computed ANCOVAs separately for both extreme groups with “knowledge of fractions” (pre-test) as the covariate. These analyses indicate for the group with a high prior knowledge of fractions that students in the “erroneous learning environment” ( $M = .86, SD = .13, n = 22$ ) learned significantly more than students in the “correct learning environment” ( $M = .78, SD = .16, n = 22$ ),  $F(1, 43) = 4.88, p = .033, \eta^2 = .109$ ). For the group with low prior knowledge, the “erroneous learning

environment” ( $M = .24$ ,  $SD = .17$ ,  $n = 23$ ) was not as beneficial as the “correct learning environment” ( $M = .37$ ,  $SD = .18$ ,  $n = 20$ ),  $F(1, 42) = 4.50$ ,  $p = .040$ ,  $\eta^2 = .099$ , see Figure 1. Thus, we found that for students with a high prior knowledge of fractions, the erroneous learning environment was more beneficial, whereas for students with a low prior knowledge of fractions, the correct learning environment promoted them more.

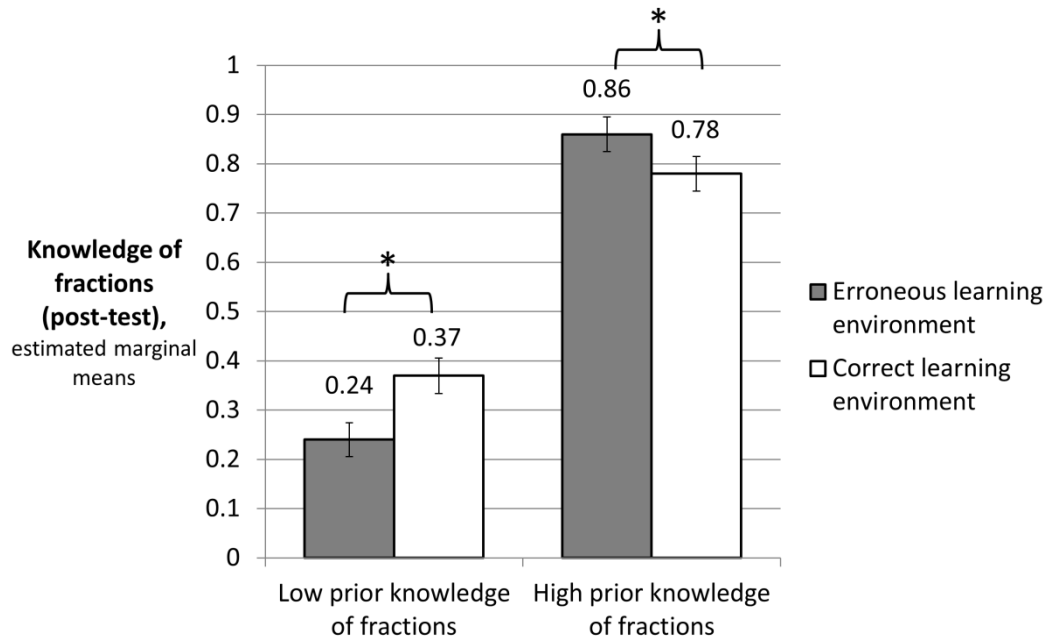


Figure 1: Group differences between knowledge of fractions for students with low ( $n = 43$ ) and high ( $n = 44$ ) prior knowledge of fractions.

### HYPOTHESIS 3: MEDIATOR EFFECT OF NEGATIVE KNOWLEDGE

First, we tested whether negative knowledge is a mediator for the acquisition of fraction knowledge. We controlled for “negative knowledge” (pre-test) and found a partial correlation  $r(192) = .38$  ( $p < .001$ ) between “knowledge of fractions” (pre-test) and “negative knowledge” (post-test) as well as a partial correlation  $r(192) = .45$  ( $p < .001$ ) between “negative knowledge” (post-test) and “knowledge of fractions” (post-test). A hierarchical linear regression showed that the original effect ( $\beta = .66^{**}$ ) of the “knowledge of fractions” (pre-test) on the “knowledge of fractions” (post-test) decreased ( $\beta = .46^{**}$ ) when the “negative knowledge” (post-test) was entered as a further control variable (in addition to the pre-test “negative knowledge”). Sobel’s test confirms that the decrease of the effect is significant ( $t = 4.41$ ,  $p < .001$ ). In order to supplement the linear regression analyses, a structural equation model was conducted so that parameters could be estimated at the same time. The model has an excellent fit ( $\chi^2(1, N = 195) = .01$ ,  $p = .900$ ,  $RMSEA = .00$ ,  $CFI = 1.00$ ). Standardized parameter estimates of the model are presented in the left part of Figure 2. This result supports the assumption that the gain in negative knowledge partially mediates the acquisition of knowledge of fractions.

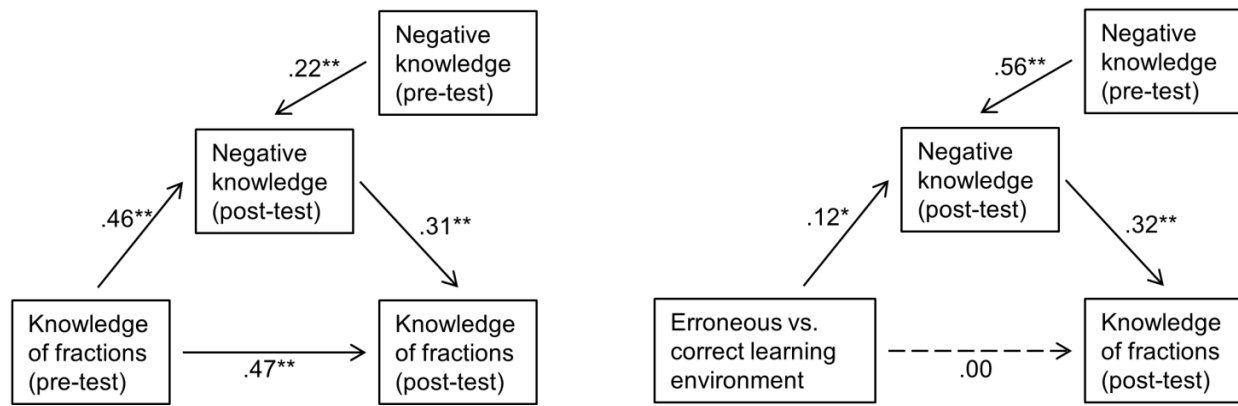


Figure 2: Left: Negative knowledge mediating the effect of the acquisition of knowledge of fractions. Right: No mediation effect of the acquisition of negative knowledge on the effect of the learning environment on the acquisition of knowledge of fractions.

In Hypothesis 3b, we assumed that the effect of reflecting on errors on the acquisition of knowledge of fractions is partially mediated by negative knowledge. However, our findings presented above indicate that the conditions did not differ significantly with respect to “knowledge of fractions”. Thus, we cannot expect the hypothesized mediation effect of negative knowledge in this case (Figure 2, right; we coded the “learning environment” as dummy variable with “erroneous learning environment” represented with “1” and “correct learning environment” with “0”).

## DISCUSSION

Our findings indicate that school students can develop and use a specific negative knowledge as it is described by Oser et al. (2012) via error reflections. Further, error reflections seem to be a promising tool to improve knowledge for advanced middle school students when learning multiplication and division of fraction. For students with low prior knowledge, reflections on errors are not as beneficial as correct examples. Therefore, we would suggest integrating errors in more advanced phases of the learning process. Error reflection supports the construction of negative knowledge and negative knowledge shows a partial mediation effect for knowledge acquisition in general. However, we could not confirm the missing step that negative knowledge mediates the effect of error reflections on knowledge of fractions. If we, again, restricted our analysis to the sub-sample of students with high prior knowledge, we found tentative indications that there might be this partial mediation. However, power analyses suggest that this hypothesis should be tested with samples of approximately 150 students (we assumed a two-tailed correlation of  $r = .23$ ,  $\alpha$ -error probability of .05 and a power of  $1 - \beta = .80$ ). Next to these further analyses, more adaptive approaches focusing on individual errors of students instead of on vicarious errors seem to offer promising instructions that need to be investigated.

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# INVESTIGATING HOW YOUNG CHILDREN MAKE SENSE OF MATHEMATICAL OBJECTS IN A MULTIMODAL ENVIRONMENT: A PHENOMENOLOGICAL APPROACH

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*We describe how certain technologies can be combined to create a multimodal environment for young children to touch as well as see mathematical objects on a computer screen in coordination with a force-feedback device. We describe how a group of children make sense of how certain 3D objects intersect through a phenomenological research perspective.*

## BACKGROUND AND FOCUS OF REPORT

Multimodal interfaces exist as alternative input or combined input/output methods. Common forms of alternative inputs are speech inputs (e.g., voice recognition), touch (e.g., gesture-based interactions) and bodily motion. The latter has rapidly evolved in recent years with the development of multi-touch technologies (e.g., tablet PCs, Interactive Whiteboards, iPads). Combined input/output devices include haptic devices, which integrate visual modes with force feedback loops, offering the user the ability to feel objects or the results of their interactions with the environment. Haptic technology has evolved over the past 10 years. Students can interpret visual, auditory and haptic displays to gather information, while using their proprioceptive system to navigate and control objects in their synthetic environment (Dede, Salzman, Loftin & Sprague, 1999). In this work, multiple sensory representations can offer and mutually reinforce information that a user can collect to develop an understanding of a mathematical or scientific model.

Our research has combined such recent advances in multimodal technologies, in particularly haptic “force-feedback” devices, within a dynamic geometry software environment. We have created a fluid, albeit prototypical, multi-modal interactive environment where young learners can not only click-drag-deform mathematical objects on a screen as in traditional dynamic geometry but also experience force feedback related to mathematical properties through the same device. Our primary research aim is to investigate how young children make sense of mathematical objects and their properties in such a multimodal environment through a phenomenological research perspective.

## THEORETICAL FRAMEWORK

Our theoretical framework incorporates advances in design in mathematics education technology and related fields as well as analytical methods to understand how such design can impact young learners’ investigations of challenging mathematics.

Dynamic interactive mathematics environments offer tools to construct and interact with mathematical objects and configurations. Traditionally, these can be selected and



dragged by mouse movements in which all user-defined mathematical relationships are preserved. In such environments, students are supported in efforts to formulate conjectures and generalizations by clicking and dragging hotspots on an object, which dynamically re-draw and update information on the screen as the user drags the mouse (Drijvers, Kieran, Mariotti, 2009). In doing so, the user can explore and efficiently test an entire parameter space of equivalent mathematical constructions.

Such environments aim to develop spatial sense and mathematical reasoning by allowing conjectures to be tested, offering “intelligent” tools that constrain users to select, construct or manipulate objects that obey mathematical rules (Mariotti, 2003) alongside well-developed curriculum activities. The core features are construction and manipulation allowing constructs to be dynamically reconfigured.

Multimodal interaction has evolved in various research areas and applications including computer vision/visualization, psychology and artificial intelligence with increasing use in education particularly in early learning and developmental psychology. Multi-touch environments are also evolving and Thompson, Avant and Heller (2011) examined the effectiveness of using TouchMath—a multisensory program that uses key signature points on mathematical objects—with students with physical learning disabilities. Using a multiprobe, multiple baseline design, they discovered all students were successful in reaching the criterion with percentages of correct responses to addition problems.

## **DEVELOPMENT & RESEARCH DESIGN**

We used a combination of the H3D programming environment with Sensable’s PHANTOM Omni<sup>®</sup> (hereon referred to as Omni). The Omni is a desktop haptic device with six degrees of freedom for input (x, y, z, pitch, roll, yaw), and three degrees of output (x, y, z). The Omni’s most typical operation is via a stylus-like attachment that includes two buttons (see Figure 1). The environment allows for two levels of programmatic control: precise programmer created feedback—such as vibration—and pre-programmed feedback—such as springs and dynamic/static friction. The Omni provides up to 3 forces of feedback for x, y, and z. It is primarily used in research, with a significant presence in dentistry and medicine.

We developed several activities focusing on transformational geometry in two dimensions and explorations of shapes in three dimensions. In the latter category, there were two main activities that were open-ended in nature. The first was a set of solids and surfaces that our participating children could drag and rotate around the screen and feel characteristics through feedback from the Omni. We asked the children to classify the shapes into their own self-defined categories. The second activity, which we focus on in this paper, allowed children to manipulate the planar intersection of several shapes and feel the line of intersection. We focus on one group of children exploring the planar intersections of a cylinder (see figure 2).

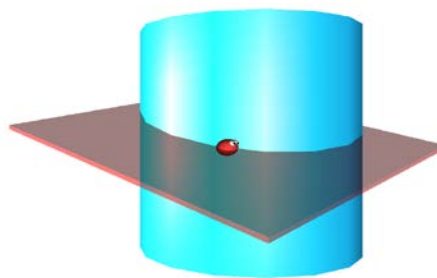
Figure 1: PHANTOM Omni<sup>®</sup>

Figure 2: Intersection of a cylinder

In this type of investigation, the group of children were presented with a cylinder, a movable plane, and a pointer in the form of a bug, which we discovered helped coordinate the spatial location of the Omni with the visual on the screen. The goal of this activity was for the children to explore the various intersections of the cylinder with a plane and describe what they saw and felt. They could construct planar intersections by rotating and translating the plane so that its position overlapped with the cylinder. They could then trace around these intersections by moving the Omni stylus. Since the cylinder was static, the intersection could only be viewed from one orientation. They would gain physical feedback relating to the unseen portions of the shapes by moving the bug around the intersection. For example, each child could feel a magnetic force—which restricted the bug’s movements to the intersection—and a resistant force when they encountered a vertex. Based on prior work (Güçler, Hegedus & Robidoux & Jackiw, 2013), we expected children to hypothesize individually and collectively about the attributes of the intersections they constructed through their visual and physical perceptions and how they made sense of such shapes and attributes. The dynamic, multi-modal features within this investigation provided children with an environment to explore freely.

## METHODOLOGY

Our primary research question focuses on: How do young children make sense of mathematical objects in a multimodal environment? “Making-sense” is operationalized in terms of word use and the conditions under which such words are used (e.g. gestures, bodily motions). What forms or types of word structures (i.e. a speech act or associated motion) are used and in what ways? How mathematical is such discourse from being non-scholastic, though mathematically relevant, to scholastic, though possibly incorrectly used?

Furthermore, we wished to investigate what kinds of words are used for what kinds of purposes including: 1. Linking to prior knowledge, 2. Use of metaphors in a mathematically *relevant* way to build meaning, 3. Words to convince others, 4. Words that are closely linked to the meditational forces of the environment both technological and social.

Given the novelty of the technological environment and the necessity to investigate how children make sense in such environments in rich descriptive ways, we decided to employ a phenomenological research method since our research question requires a qualitative inquiry perspective. The phenomenon in our case is the experience of investigating the cross sections of a cylinder through a multi-modal technology environment with small group collaboration.

Our main study was conducted in a local elementary school in Massachusetts, USA. The school is located in a suburban, middle class neighbourhood of higher socioeconomic status than local cities. We visited all seven separate 4<sup>th</sup>-grade classrooms (10 year olds) over a 2-week period (approximately 150 students). All 4<sup>th</sup>-grade teachers consented for us to visit and conduct their mathematics class for one day. Following this class the teachers randomly selected 3-4 children from their class for us to conduct the Omni activity in the school library. Two cameras were used to capture the screen and the other orthogonal to the screen to capture student discourse, expressions, children's manipulation of the haptic device stylus and any other interactions (e.g. pointing at the screen). A laptop with the H3D software applications and an additional monitor was controlled by one of the research team to initiate the activity so that the students only needed to use the Omni. Given our focus on discourse, high quality microphones were connected to the children. The session lasted 45 minutes. Student utterances were transcribed including physical actions.

### **Analytical Framework**

Phenomenology focuses on the lived experience and what constitutes an experience, in our case, how young children make sense of mathematical objects/ideas/scenarios. It is not just descriptive in its approach but is also an interpretive process in which the researcher mediates making meaning. Husserl's Phenomenology (1931) introduces concepts of noema and noesis. Noema is that which is experienced, the *what* of experience, the object-correlate. Noesis is the way in which the *what* is experienced, the experiencing, the subject-correlate. This links to the *what* and *how*. Noesis refers to the acts of perceiving, feeling, thinking, remembering which is the essence of our research reported here.

Phenomenological methods basically focus on the nature of the phenomenon and its qualities and what appears at different times under different conditions. The challenge of description is to determine the *textural* components (not textual) of experience; the "what" of the appearing phenomenon. From an extensive description of the textures of what appears and is given, one is able to describe how the phenomenon is experienced – the *structural* components – and the context. This means turning one's attention to the conditions that precipitate the textural qualities. In the process of explicating intentional experience one moves from that which is experienced and described in concrete and full terms, the "what" of the experience, "towards its reflexive reference in the 'how' of the experience" (Ihde, 1977). Structures underlie textures and are inherent in them. Texture and structure are in a continual relationship. The relationship

of texture and structure creates fullness in understanding the essence of a phenomenon or experience (Moustakas, 1994).

Some forms of phenomenology wish to “bracket-out” the researcher’s experiences and what influences their experiences Moustakas (1994). It is essentially focused on what participants experience and how they experience it in terms of conditions/context, i.e. an Omni environment activity. We followed this and then used a simplified version of the Stevick-Colaizzi-Keen method as outlined in Creswell (2007, p.159), which includes:

- Develop a list of significant statements that describe how the participants experienced the topic. This is called horizontalization of the data.
- List all nonoverlapping, nonrepetitive statements. These are the invariant horizons or meaning units of the experience
- Cluster the meaning units into themes
- Synthesize the meaning units into description of the textures of the experience – include verbatim examples of the textural components
- Synthesize the meaning units into a description of the structures of the experience – include verbatim examples of the structural components
- Write a composite description (textural-structural) of the phenomenon. This is the essence of the experience and describes “what” the participants experienced with the phenomenon and “how” they experienced it.

We will now present the analysis of the session with four 4<sup>th</sup>-graders as described above using this method with four researchers observing. Due to space limitations we will only be able to offer some verbatim examples of the textural and structural components and not a full listing of the significant statements.

## DATA ANALYSIS & RESULTS

*Personal experience with the Phenomenon.* The researchers had helped design the experience and so were not exposed to experiencing the phenomenon directly during the session (i.e. making sense of cross sections of a cylinder) but as observers probed the children asking them how they were making sense of the activity and what they are doing. Even though we had principalized the design these were no intentional learning goals. We had no script and did not evaluate or correct any statements made by the children during the session.

*Significant Statements.* We reduced 6070 full transcript words and actions to 3443 words. These generally focused on expressive statements on the experience and questions that sought to elaborate such statements made by the children. We did not include questions regarding how to use the technologies such as “can I grab this?” or “how do I click on that?” given this was a first-time use of such an environment unless the response yielded a significant statement.

*Meaning Units and themes.* We discovered that certain meaning units clustered into themes with respect to the researcher’s utterances and others to the participating children. For the researcher, the following themes emerged: 1. Noticing, 2.

Asking-to-Repeat, and 3. Reactions to actions. For the children, the following themes emerged: 1. Story-telling, 2. External references, 3. Aesthetic motivators/regulators, 4. Technological Guiders, 5. Prior knowledge, 6. Perception, 7. Navigation, 8. Motion-based metaphors.

*Structural Description.* We found that certain meaning units highlighted structural components of the researcher's statements (456 words) as well as the children (843 words). The structural components of how a group experienced was mediated by both the technological affordances of the environment and the researcher. Statements by the researcher were limited to asking the children what they noticed after performing such action, or to repeat what they did; either another child repeats the action on the Omni, or the same child repeats for the whole group to observe something on the screen. A few statements asked why some children completed a particular action. Most of these yielded textural statements from the children. We found the structural statements made by the children to be particularly rich and list some below as examples of the themes highlighted above. Children made sense of the shapes through the guiding nature of the force feedback from the Omni and the dynamic visuals on the screen. Color also operated as an aesthetic regulator in the dynamics of the environment. These are examples of structural components from various points in the session:

And when I try – so I'm going to try it. [*Moves bug to right side of cylinder*] I'm going to go in a circular motion for now and then I'm going to try going this way [*Referring to a straight line through the cylinder*] and it won't let me, so I think that it is circular.

I think I know why, Andrew. [*Gestures to demonstrate*] It's because the red is helping the bug guide it along around the cylinder and –

It's guiding it in a circular shape. [*MacKenzie adjusts plane several times and continues tracing intersection*]. And the thing I was trying to say is when we had the red piece going down the middle [*Gestures to demonstrate*] and we were moving it around it would always stop like right now when you're going around it you can tell that it's a cylinder, because you felt what it felt like when you ran into the red and it kind of stops. But now when you go around you don't stop.

[*Drops plane again with an intersection having curved sides*] Yeah it was something like this. See when you go up, it would slant, making like a different thing ...

*Textural Description.* We found that certain meaning units highlighted only textural components of the children's statements (1513 words) which reinforced our approach that we wished to investigate what the children were experiencing and not what the researcher was experiencing with respect to the phenomenon. Children described what they experienced by relating it to worldly shapes (e.g. a bean) again driven by as aesthetic form, prior knowledge with reference to something similarly experienced in mathematics (e.g. a line), perception of what they see and feel (e.g. feeling circular) and motion-based metaphors which fuses the visual-feedback experience (e.g. running up). These are examples of textural components from various points in the session:

[*Whispering*] It looks like a bean

Well it probably is a cylinder, but it could be a half circle and then the flat side could be that way [*Makes flat edge with hand*]

It would slide over there, because it's circular.

I'm feeling the blue. Like when you go to the back it kind of stops the bug.

[*Traces intersection while explaining*] It feels smooth. It feels smooth like a – it doesn't feel like it has any edges. It feels more like a circular motion.

I'm feeling maybe sort of like it's wider at the bottom then it goes like this. [*Gestures trapezoid with hands*]

I felt a slight slant. It feels like a slant a little like it changed direction a little.

...it's pretty much doing the outline of the cylinder but it's not going around in a circle. [*Gestures a circle with hands*] It's going on the flat surface and it's just running up

*Structural-Textural Description.* Many of the textural statements made in this session were followed by structural statements mediated by either the visual-haptic experience or the researcher. Various intersections were explored because of the interactive affordance of the Omni that allowed the plane to be selected and turned by the children often followed by what they saw and felt simultaneously. This often led to a series of statements to be made by the child controlling the Omni or the other observing children. The aesthetic form of the shapes (e.g. color or attributes such as lines or edges) in concert with the force feedback often guided the investigation e.g. “I’m feeling the blue ... it kind of stops the bug”. Many mathematical terms were used to describe what they see although not always accurate (e.g. line circle, cylinder, circular to describe an elliptical intersection) mediated by what they felt, or when the researcher asked them what they noticed vs. what to notice. Many of the shapes were accurately described in terms of the motion of the bug that they felt from the Omni that “guided” them in a magnetized way along the intersection.

## CONCLUSIONS & NEXT STEPS

The environment used in our study has similar affordances to dynamic geometry environments that allow users to navigate and drag objects but in addition to such executable representations the force feedback now appears to aid children in making sense of what they see. The experience yields forms of expression from recalling prior mathematical knowledge of 2D shapes and non-scholastic words and guides their sense-making routines. It appears that such a multi-modal environment guides the structural-textural sequences in contrast to a uni-modal environment where things might be deformed on a screen, or a device vibrates separately without mathematical reasons as to why or how the technologies are doing such things. We will continue to explore the use of such technologies across different students to understand their potential for learning (Güçler, Hegedus & Robidoux & Jackiw, 2013). In understanding how children can come to make-sense in mathematically relevant ways will potentially be useful to teachers in the future who might incorporate such technologies in their mathematics classrooms. There are design tensions in the role of

the researcher in scaffolding the structural elements which will need more attention but which we have begun to parse out here. Our research also assumes a collaborative environment and we have addressed the statements of all children as one but it would be beneficial to parse out different roles that various children might have in collectively vs. individually making sense in the future.

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# PROSPECTIVE TEACHER'S STATISTICAL KNOWLEDGE FOR TEACHING WHEN ANALYSING CLASSROOM EPISODES

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*In this paper, we aim to investigate the statistical knowledge for teaching through investigations of prospective teachers. This knowledge is analysed from their written essays, based on the analysis of classroom episodes and using a framework that links two domains: the statistical knowledge for teaching and the statistical thinking. The study shows the potential of both the conceptual model adopted and the classroom case discussions to provide useful information about prospective teachers' knowledge concerning the teaching of statistical investigations and the development of students' statistical thinking. In particular, it allows us to identify some components as needing further attention in teacher education.*

## INTRODUCTION

In Portugal, curriculum changes have been trying to follow the increasing importance given to statistics and the overall recommendations to emphasize the exploratory data analysis and the progressive involvement of students in investigations in order to develop their statistical thinking (Ben-Zvi & Garfield, 2004; GAISE, 2005). These changes have been quite challenging for teachers since statistics teaching has valued the learning of data representations and the calculation of statistical measures whereas other fundamental aspects inherent to the statistical method, such as the formulation of questions or the planning and data collection have been less discussed (Shaughnessy, 2007). These difficulties call for a need to focus on prospective teacher's education, helping them to develop both a comprehensive understanding of the statistical process itself and a statistical knowledge for teaching (Burgess, 2011).

Several studies reveal teachers' difficulties in developing statistical investigations with students as they tend to focus on procedural aspects, missing out on opportunities to expand the students' statistical thinking. This highlights the importance of developing teacher's knowledge concerning the teaching of statistical investigations in initial teacher education (Burgess, 2009; Heaton & Mickelson, 2002), an area that has also received little attention from research. In this context, it seems relevant to study what are the aspects of statistical knowledge for teaching investigations that middle and secondary prospective mathematics teachers reveal when analysing classroom episodes, in order to clarify which components of teacher knowledge need further attention in teacher education courses.



## **STATISTICAL KNOWLEDGE FOR TEACHING**

Several models have been developed concerning the knowledge required to teach mathematics. However, the knowledge of the teacher should be studied according to the discipline that he teaches, since each knowledge area has its own particularities (Shulman, 1986). Even though statistics is included in the mathematics curricula, several researchers have argued in favour of its specificity, provided the established differences between statistical thinking and mathematical reasoning (delMas, 2004). In this regard, Groth (2007) argues that these differences are usually not considered in research about the teacher's knowledge, even though he recognizes the potential of the models on the knowledge to teach mathematics in defining a model of knowledge to teach statistics.

Accordingly, Burgess (2009) proposed a theoretical framework for examining the teacher's professional knowledge which takes into account the specific needs of the teaching and learning of statistics. The proposed framework for teacher knowledge starts from the dimensions of mathematical knowledge to teach described by Ball, Thames and Phelps (2005), considering two types of content knowledge (CK) – common content knowledge (CCK), specialised knowledge of content (SKC) – and two types of pedagogical content knowledge (PCK) – knowledge of content and students (KCS) and knowledge of content and teaching (KCT). The first one (CCK) refers to a type of knowledge that is not exclusive to teachers but encompasses the statistical knowledge some people use in their professions. The second one (SKC), goes beyond the first, and concerns the knowledge the teacher needs, for instance, to give an explanation, assess the students explanations and errors, in terms of the involved statistical knowledge, and to justify the processes and representations used. The KCS combines the teachers' statistical knowledge with their knowledge of the students, enabling them to anticipate what the students think about a particular aspect of the content, the expected difficulties, erroneous conceptions, and motivations. Finally, the KCT conceals the statistical knowledge with the adequate methodologies to teach each topic in order to promote the students' learning. It includes the capacity to choose appropriate tasks and to sequencing them, as well as to recognize the advantages and disadvantages of the use of certain representations. Burgess combines these four types of knowledge with the components of the statistical thinking model of Wild and Pfankuch (1999), which includes, in what concerns types of thinking, the need for data, transnumeration, consideration of the variation, reasoning with models, and integration of statistics and context. Along with these types of essential thinking, there are others which are more general and that can be regarded as part of problem solving (but not exclusive to solving statistical problems): the investigation cycle (problem, background, data, analysis and conclusions) and the interrogative cycle (produce, seek, interpret, criticize and judge). The integration of these two domains shows the singularity of this analytical framework by Burgess (2009), specifically addressing the statistical work.

## **CONTEXT AND METHODS**

In this paper we are focusing on the statistical knowledge for teaching that prospective mathematics teachers show in the context of a methods course, taught by the authors. The course discusses, from a didactical perspective, the main topics in the mathematics curriculum for middle and secondary levels. In this course, 16 hours are dedicated to the theme of statistics teaching, and, in particular, prospective teachers are provided with opportunities to develop knowledge about how to teach statistical investigations.

Considering that the analysis of classroom case studies has great potential to prepare teachers for making teaching decisions in complex classroom environments (Groth & Xu, 2011), prospective teachers were asked to, autonomously, analyse one classroom case. This concerns an investigation involving the analysis of data from population censuses from three countries (USA, Japan and Kenya). It includes several episodes that provide a view of one teacher's attempts to develop the statistical reasoning of his students in different stages of the investigation, through excerpts of the dialogues between them (taken from Shaughnessy, Chance & Kranendonk, 2009). Students used a variety of data representations from the populations, such as tables, histograms and boxplots, and some statistical measures, such as percentages, medians and quartiles and focus on the discovery and explanation of patterns in the data and in the differences or similarities in the distributions of the population's ages from the three countries.

The ten prospective mathematics teachers were asked to write an essay based on the analysis of the classroom episodes, focusing on their main statistical concepts and on student's statistical reasoning, development of a statistical investigation, and the teacher's role in these processes. Their essays were analysed interpretatively and descriptively, based on the conceptual framework by Burgess (2011) to describe the statistical knowledge for teaching through investigations, using three dimensions of the teacher's knowledge (SCK, CSK, and CKT) in relation to the categories of the statistical thinking. Considering the objectives of the task presented to the prospective teachers, which focused on the teaching and learning of statistical investigations, the CCK is not included in the analysis. From the analysis of the essays, one profile was constructed for each prospective teacher or pair of teachers (in two cases they worked in pairs). These were then assembled to produce a global profile for the class (Table 1) following Burgess (2009). In this paper we illustrate, the statistical knowledge for teaching of one of these prospective teachers, Maria, who we consider to exhibit a typical profile in the group, in each category concerning statistical thinking.

## **PROSPECTIVE TEACHERS' STATISTICAL KNOWLEDGE FOR TEACHING**

### **Prospective teacher's profile**

In table 1 we summarize the aspects of statistical knowledge for teaching in relation to statistical thinking which we identified in prospective teachers' essays (n=8). In each cell, there is often a diversity of relevant knowledge to statistical thinking and thus, the presence of knowledge in relation to a category of statistical thinking does not imply that the prospective teachers have a thorough knowledge of all aspects. Similarly, an

empty cell does not mean that prospective teachers have no knowledge on that category of statistical thinking – it is just that it is not identified in their essays.

		CK	PCK	
		SKC	KCS	KCT
Thinking	Need for data	3	-	4
	Transnumeration	8	6	7
	Variation	-	3	-
	Reasoning with models	8	7	8
	Integration of statistical and contextual	7	2	6
Investigative cycle		4	4	5
Interrogative cycle		5	-	4

Table 1: Profile of prospective teachers' statistical knowledge for teaching

The prospective teachers' profile shows that most of the teacher's knowledge categories in relation to the various components of statistical thinking are present in their essays. SKC and KCT were the ones which stood out the most, showing that the prospective teachers are able to assess students' ideas in terms of the involved statistical knowledge, justify the processes and representations they use and recognize the advantages of different methodological approaches in order to promote students' statistical thinking. These categories did not emerge in relation to variation, maybe because this is a new concept to them and thus they are not yet sufficiently aware of its importance in the students' thinking development or the characteristics of the analysed classroom episodes (the data refers to populations rather than samples) may have limited the mobilization of that type of knowledge. Regarding to KCS, the least present category in the essays, the prospective teachers are able to identify several challenges and to recognize the students' main difficulties in statistical investigations mainly in relation to the thinking components that are more familiar to them (Transnumeration, reasoning with models). The reduced evidence for this category of knowledge may be related to features of the task as well as the much reduced contact with students in the classroom. In the following section we illustrate these results from Maria's profile.

### Maria's statistical knowledge for teaching

**The need for data.** In her essay, Maria reveals that she understands that the available data are not sufficient to answer the questions students started to formulate during the investigative cycle, agreeing that there is a need to lead them to look for data which justifies their statistical reasoning. In this regard, she highlights a passage from the classroom episode in which "the teacher (...) is showing students that sometimes we need more data to allow better conclusions". She adds that "we should discuss with students what information is needed in order to make a valid decision or a justified conclusion". Thus, Maria shows SKC and combines it with the CKT, realizing the need and the importance of the discussions to develop students' understanding of data.

**Transnumeration.** Maria shows SCK in this category, for instance, when she mentions that one may use "other measures, the quartiles, which locate other points of data

distribution, and which define the existing variability among data". She is thus demonstrating the ability to recognize the validity of specific measures to summarize the data. This knowledge was also required to analyse whether students' use of measures, representations and articulations between them were valid and correct, as she refers: "Students make a correct interpretation of the median and can successfully relate it to the histogram and the diagram of extremes and quartiles". Maria also reveals her CKT when she identifies the students' level of education in the classroom episode, based on the representations and measures approached as well as the knowledge she has from the curriculum: "Considering the measures of central tendency and dispersion discussed in this investigation and also the representations, it seems to me that these are middle school students". When she was asked about what she would do differently from the teacher in the classroom episode, she gives suggestions to develop students' transnumeration: "Depending on the students' school level, we could also analyse the interquartile range and standard deviation. Based on the histogram we could address different ways of distribution of the population". Maria also is able to predict difficulties and misconceptions that students may have in the use of several representations, revealing CKS: "For students it is not easy to understand that the histogram represents the data through the areas of bars and not the heights".

**Variation.** The only knowledge identified in this category is KCS, when Maria foresees the students' difficulties in dealing with the data variation. She mentions: "With such a large amount of data, it is possible that students struggle, particularly looking for regularities in the data and the identification of differences in order to describe (...) variability". The fact that in the analysed classroom episodes, the data are referring to populations rather than samples, may limit the emergence of situations where students make use of this important aspect of statistical thinking, particularly deciding based on the data and generalizing.

**Reasoning with models.** Maria understands the importance of using appropriate models (charts, tables, measures and technology) in order to make sense of the data and help students to develop their reasoning. She reveals KCT as she agrees with the approach chosen by the teacher in the episode as he simultaneously presents boxplots for comparing three populations. She finds it adequate (because "it is easier") to motivate students to focus on the main aspects which enable that comparison:

The teacher has presented the boxplots simultaneously, which provided an easier way of comparing these three populations, by accentuating the differences and similarities on how the data are distributed, allowing [students] to compare the location of the median and the quartiles for the different populations, as well as the greater or lesser dispersion of data. In this case, the students have also been able to make a correct articulation between the tables and the diagrams.

Moreover, when the prospective teacher identifies and argues about the appropriate students' use of the models, she indirectly also demonstrates SKC. This knowledge seems to be related with Maria's ability to interpret the students' discourse. Thus, her KCS is required to lead her to anticipate students' difficulties in the reasoning with models: "Regarding the boxplot, a student specifically says that "they provide an

efficient model to see the population variability. Visually, I think that it is the strongest way of representing data”. The KCT in this category is again revealed, when Maria defends and justifies the use of technology to facilitate the construction of graphical representations and the calculation of measures, allowing students to focus on the learning of concepts: “The teacher was right to present the boxplot and histogram already constructed. (...). With the development of technological resources, it makes ever more sense to teach statistics focusing on the concepts and not in calculations”.

***Integrating statistics and context.*** In her essay, Maria is able to identify several situations from the classroom episode in which students formulate conjectures based on both statistical results and context. Thus, her SKC was required to interpret students’ activity. For instance, she refers to an example in which, through the calculation of percentages, students identify a pattern in the data that leads them to formulate conjectures about the context which has originated them:

By adding the percentages in the tables, students (...) establish relationships and comparisons between the data and they also make interpretations leading to conjectures about the past. (...) The analysis and interpretation of data has led these students to develop new relationships and conjectures.

***Investigation cycle.*** Maria mobilizes her SKC to identify and describe the phases of a statistical investigation and to assess the work done in each of them, in particular the correction of the formulated questions and the appropriateness of the methods for collecting data. For instance, the prospective teacher points out the need to formulate statistical questions and an appropriate data planning and collection for later analysis:

A statistical investigation (...), having two initial phases which in this text are not very developed. I am referring to the “problem definition and formulation of questions” and the “Planning and collection of data.” (...) In the first phase we will have to be careful about the questions we will formulate and confirm that they have statistical features. The second phases, involving data collection, require a well defined plan with appropriate data collection techniques.

The prospective teacher claims to be surprised with the work performed by students in the classroom episode, since she is aware of the difficulties they usually face in this kind of task. In regards to data planning and collection, she sees it as a complex phase of the investigation cycle to be worked with the students but also an achievable one, provided that they display some interest in the topic that is going to be investigating: “This second stage seems easier but also very difficult depending on the students’ engagement”. In these examples, Maria reveals CKT, as she identifies the aspects of a statistical investigation which are challenging or problematic for students. Additionally, Maria links the students’ success with a specific and regular work carried out by the teacher in the classroom, using a set of teaching strategies which she describes and seems to know, revealing CKT: “I guess the teacher has worked continuously, specially by conducting investigation activities and tasks that focus on reasoning, statistical thinking, interpretation and critical thinking skills.”

**Questioning cycle.** At a given moment from the classroom episodes, the involvement of the participants is pointed out in the questioning cycle, and Maria commented the teacher's actions when faced with the suggestion from a student to use percentages:

The teacher intervenes [taking advantage of the student's suggestion], and with his questioning, he has led students to move from absolute frequencies to percentages. By adding the percentages in the tables, students (...) establish relationships and comparisons between the data as well as interpretations leading to conjectures about the past.

Thus, even though in an indirect way, the prospective teacher reveals SKC and CKT. She mentions that “the way the teacher intervened seemed correct” as she identifies that the approach suggested by the student to handle the data is more useful in order to allow the search for patterns in the data and the subsequent interpretation of the results.

## CONCLUSION

The conceptual model adopted in this study, which was originally constructed for describing the statistical knowledge for teaching of practicing teachers (Burgess, 2011), has proved to be also useful in the case of prospective teachers. Despite the limitations of the proposed task, the results confirm the potential of classroom case discussion (Groth & Xu, 2011) as a context that inspire prospective teachers to think as teachers, and particularly, on how to teach statistical investigations. In general, the emerging elements provide a good indication of their growing knowledge concerning the development of students' statistical investigations and statistical thinking, which are strongly recommended for statistics teaching nowadays (Ben-Zvi & Garfield, 2004). However, taking into account the complexity of the statistical knowledge for teaching investigations, the evidence emerging from their essays show that they make statements about their anticipated actions as teachers, which are sometimes too general, and, therefore, not very effective for supporting students' thinking. As stated by Groth & Xu (2011), in that context, what is required from the teacher is having viable and defensible arguments, something which refers us to a different level and nature of discussion.

It is, then, important to give prospective teachers the opportunity to make their own investigations, expanding their own CCK and, consequently, their SCK (Burgess, 2009), and that may require more time for statistics, articulated with its didactics, in teacher education curriculum. In addition, their reduced teaching experience greatly limits the development of important categories of their PCK, as KCS, regarding statistical thinking. Therefore, it is their practice, centred in meaningful experiences during initial teacher education, which may broaden and improve their statistical knowledge for teaching.

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# KNOWLEDGE SHIFTS AND KNOWLEDGE AGENTS

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*To better understand the mechanism of knowledge shifts in a learning classroom, we combined two approaches/methodologies that are usually carried out separately: The Abstraction in Context approach with the RBC+C model and the Documenting Collective Activity approach with its methodology. This combination revealed that some students functioned as Knowledge Agents, meaning they were active in shifts of knowledge among individuals in a group, or from one small group to another one, or from their group to the whole class or within the whole class. The teacher as an orchestrator of the learning process is responsible to provide a learning environment that affords argumentation and interaction, in order to enable normative ways of reasoning to be established and to enable students to be active and become knowledge agents.*

## INTRODUCTION

Tracing students' knowledge construction and the shifts of the constructed knowledge in a working classroom are challenges that still need to be achieved (Saxe et al., 2009). This paper is an attempt in that direction, analysing part of a lesson as a paradigmatic example. We use the theoretical framework of Abstraction in Context (AiC) and the RBC+C methodology to analyse construction of knowledge of individuals, mainly while they are cooperating in small groups in a mathematics classroom; and we use the Documenting Collective Activity (DCA) approach to analyse whole class discussions with the aim of identifying practices that become normative in the classroom. The present study combines the theoretical and methodological aspects of these two lines of research that are usually carried out separately. The goal of the study presented in this paper is to learn more about how knowledge shifts between the different social settings in a working mathematics classroom and the role of individuals and the teacher in these learning processes.

## THEORETICAL AND RESEARCH FRAMEWORK

Abstraction in Context (AiC) is a theoretical framework for investigating processes of constructing and consolidating mathematical knowledge (Hershkowitz, Schwarz, & Dreyfus, 2001). Abstraction is defined as an activity of vertically reorganizing (Treffers & Goffree, 1985) previous mathematical constructs within mathematics and by mathematical means, interweaving them into a single process of mathematical thinking so as to lead to a construct that is new to the learner.

The genesis of an abstraction passes through a three-stage process, which includes (i) the need for a new construct, (ii) the emergence of the new construct, and (iii) the consolidation of that construct. A central component of AiC is a theoretical -



methodological model, according to which the emergence of a new construct is described and analyzed by means of three observable epistemic actions: *Recognizing* (R), *Building-with* (B) and *Constructing* (C). Recognizing refers to the learner seeing the relevance of a specific previous knowledge construct to the problem at hand. Building-with comprises the combination of recognized constructs, in order to achieve a localized goal. The model suggests constructing as the central epistemic action of mathematical abstraction. Constructing consists of assembling and integrating previous constructs by vertical mathematization to produce a new construct (see Schwarz, Dreyfus and Hershkowitz, 2009 for a detailed description). Students' consolidation of the new construct is revealed when they use this construct for constructing a later one. Consolidation of a construct by a student is characterized, among others, by their progressively quicker recognition of when and where the construct is relevant (immediacy) and by their progressively more flexible building with the construct in changing contexts (Dreyfus & Tsamir, 2004). AiC researchers usually carry out an a priori analysis of the tasks proposed to the students in terms of the knowledge elements that are necessary or useful for successfully completing the task.

Collective activity is a sociological construct that addresses the constitution of ideas through patterns of interaction and is defined as the normative ways of reasoning that are developed in a classroom community. Such normative ways of reasoning emerge as learners solve problems, explain their thinking, represent their ideas, etc. A mathematical idea or a way of reasoning becomes normative when there is empirical evidence that it functions in the classroom *as if it were shared*. The phrase "function as if shared" is similar to "taken as shared" (Cobb & Bauersfeld, 1995) but is intended to make a stronger connection to the empirical approach which uses Toulmin's (1958) model of argumentation to determine when ideas function in the classroom as if they are mathematical truths (Rasmussen & Stephan, 2008).

The DCA methodology begins by using Toulmin's model to create a sequence of argumentation schemes of every whole class discussion, resulting in an argumentation log. In brief, the core of Toulmin's argumentation model consists of three parts: the data, the claim, and the warrant. If one disagrees with the claim, he or she may present a rebuttal, or counter-argument that shows disagreement. When this type of challenge is made, often a qualifier is provided, which is a way to provide specific conditions in which the claim is true. Finally, the argumentation may also include a backing, which demonstrates why the warrant has authority to support the data-claim pair.

The next step involves taking the argumentation log as data itself and looking across all class sessions to see what mathematical ideas become part of the class' normative ways of reasoning. Rasmussen and Stephan (2008) and Cole et al. (2012) identified three criteria for determining when ideas function as if shared. These three criteria can be thought of as the collective analogue to an individual's process of vertical mathematization. In a DCA analysis it is most often the case that one needs to analyze multiple class sessions to find evidence of the three criteria (e.g., Stephan & Rasmussen, 2002). This is because the three criteria capture functional and structural

changes to elements of an argumentation over time. In this report we analyze only one episode and hence it would be premature to expect much if any evidence for normative ways of reasoning. As a prelude to such a more comprehensive analysis, we carefully examine, with the aid of Toulmin's scheme, the whole class discussion.

### **The concepts of knowledge agent & uploading and downloading of ideas**

A *knowledge agent* is a member in the classroom community who initiates an idea, which subsequently is appropriated by another member of the classroom community. Thus, when a student in the classroom is the first one to express an idea according to the researchers' observations, and others later express this idea, then this student is considered to be a knowledge agent. Shifts of ideas may be observed from a group to the whole class (*uploading*), or within the whole class, or within a group, or from a group to another group, or from the whole class to a group (*downloading*). Shift actions may occur in different time intervals, which can last from seconds to a few lessons. In the present study we aimed at elaborating the role and function of knowledge agents and the role of the teacher in creating an environment that creates opportunities for students to function as knowledge agents.

### **METHODS**

This study was carried out in the framework of a larger project that involved six grade 8 classes who were engaged in learning probability. The data for this study were collected by video recording in one class. The camera was focused either on the whole class discussion, or on a focus group. A unit consisting of a sequence of problem situations embedded in a rich learning environment was designed and implemented. The activities were carefully designed to offer opportunities for constructing and consolidating knowledge and practices in classroom. The unit included about ten lessons.

The present paper focuses on lesson 4 of the unit. The topics considered in lessons 1 to 4 concerned probability of events in one dimensional sample space and included theoretical probability as a ratio of the number of relevant outcomes to the number of all possible outcomes, as well as some experience with the fact that empirical probability values tend to the theoretical value as the number of trials becomes larger. Lesson 4 starts with a Whole Class (WC) discussion followed by small group work, during which we followed the work of a focus group (FG). In the WC, the teacher initiated a discussion on the *chance bar* and its meaning. The WC discussion was focused on *the Dreidel Problem* and the FG discussion was focused on *the Coin Problem* (see below). Both problems dealt with a qualitative appreciation the probability of composite events. Also, in both problems the chance bar had a qualitative role only.

#### **The Dreidel Problem**

A Dreidel is a special kind of top, used as a traditional children's toy. After it is spinning it can fall on one of four sides with equal probability. A Hanukkah dreidel (with four letters N, G, H, and P on its sides) was spun 100 times. Mark approximately

on the chance bar the chances of the following events: (a) The dreidel will fall 100 times on the letter N; (b) The dreidel will never fall on the letter N; (c) The dreidel will fall on N between 80 and 90 times; (d) The dreidel will fall on N between 23 and 30 times.

### **The Coin Problem**

A coin was flipped 1000 times. Mark on the chance bar the chances of the following events: (a) The coin will land 1000 times with heads facing up; (b) The coin will land with heads facing up more than 450 times and less than 550 times; (c) The coin will land with heads facing up more than 850 times and less than 950 times; (d) The coin will fall 1000 times with tails facing up.

### **A priori analysis of the problems**

In this paper we will focus only on part d of the dreidel problem and on part b of the coin problem. The following Knowledge Element (KE) was intended to be developed: The chance of an event to fall into a given range of values, which includes the expected value, is high.

### **FINDINGS**

In the following we bring analysis of excerpts from the whole class discussion followed by an analysis of excerpts from the focus group discussion.

#### **Whole Class (WC) episode on the Dreidel Problem (d)**

In the episode's protocol we code the participants' utterances using Toulmin's Model (1958): Data – [D]; Claim – [C]; Warrant – [W]; Qualifier – [Q] & Rebuttal – [R]. Due to space constraints only a portion of the episode is presented. In the full episode there were 46 turns, 22 of which were the teacher's utterances.

128 Teacher Let's look at d. d says, a top is spun 100 times it will fall on N between 20 to 30 times [D]. What do you think, we will spin the top 100 times, how many times will it fall on N between 20 to 30 times. [To Eliana] Come, you haven't marked yet. (Eliana approaches the board and marks on the chance bar close to the middle [C]).

129 Teacher Adin, what is your opinion, what do you say?

130 Adin I think that it is approximately 30% [C].

131 Teacher that means that you agree with what Eliana is suggesting, explain why!

132 Adin It has more of a chance...

133 Teacher So if it has more of a chance you are marking it on the 30, more chance for what?

134 Adin More of a chance than a, b and c [D]. There is a higher chance that it will happen; it is closer to the middle [C].

135 Teacher So if there is a higher chance you are marking it close to what? Does anyone feel different, want to support or oppose? ... What do you think Guy?

136 Guy I think it is much higher (Teacher asks how high?) 80% [C], because in fact there are 4 sides to the top [D], right? And the chances that it will fall on

one of them is 25% [W] and you said that it will fall between 20 to 30 [D], so...

137 Yael That means that it is 25% not 80% [R]

...

141 Omri What I am trying to see is if I understood Guy: what he is trying to say is that there is a 1 out of 4 chance [D], that means that it is a very high percentage that it will be between 20 to 30 [C]

...

144 Teacher Can you explain again why you are supporting Guy?

145 Omri What he's saying (Guy) is that every time you spin there is a 1 out of 4 chance that it will fall on N [D], meaning, 25% now out of 100 is approximately the number of times it will fall on the N [C], because it is 1/4 out of four [D]

146 Teacher What do you think? You are nodding yes (turns to Rachel), who do you agree with?

147 Rachel With Guy [C]

...

158 Matan I think it is 65% [C], because it can be more or less, there is a chance that it will come out and a chance that it won't [D]

159 Teacher But you think it is more than half but a bit lower

160 Matan Yes

161 Yael I am still not sure, Guy succeeds to convince me. But at first I thought it was half [C], but still...

162 Teacher You are still not convinced

163 Yael I am not sure

164 Teacher Let me ask you this, let's say that you spin the top 100 times and count how many times it will fall on N, what result would you expect?

...

169 Yael 25! [C]

170 Teacher That means that you are expecting an answer between 20 and 30, that is what we are expecting will happen [W]. So if it is what we are expecting that will happen so the chance is close to 1 [C].

*DCA analysis:* As expected, we did not find evidence at this point for any normative ways of reasoning but conjecture that Guy's reasoning in line 136 is a strong contender for a way of thinking that is likely to become normative. Moreover, our coding of the full WC episode reveals nine different arguments that were generated by the students. Only at the end of the episode (line 170) did the teacher add, mathematically speaking, to these arguments. The high number of student arguments revealed by the Toulmin coding prompted us to look more closely at the role of the teacher in fostering an environment that led to so many student generated arguments. In brief, we identified the following three distinct types of contributions from the teacher: Eliciting student reasoning and justification (e.g., line 129); seeking comparisons of student's reasoning (e.g., line 146); clarifying or re-voicing student reasoning (e.g., line 159). In total, we

identified nine turns or portions of a turn in which the teacher elicited student's reasoning, five turns or portions thereof in which the teacher sought comparison of student reasoning, and eight turns or portions thereof in which the teacher clarified or re-voiced student reasoning.

*KA analysis:* In 136 Guy expresses for the first time in this episode a full argument (Data, Claim and Warrant) concerning the intended knowledge element. Omri in 141 and 145 follows him in different words. Additional students are joining (Rachel 147; Yael 161 with some doubts). In fact the teacher herself in 170 closes the discussion related to Guy's argument. Thus we can identify Guy as a knowledge agent in this episode. We argue that Guy's role as a knowledge agent was contingent upon the three different teacher's contributions that elicited, compared, and clarified student reasoning.

### **Focus Group Episode on the Coin Problem (b)**

The focus group includes three girls: Yael, Rachel & Noam.

- 227 Yael (reads event b) "will fall on tails more than 450 times and less than 550". It's logical.
- 228 Noam It's at half!
- 229 Yael No, there are much higher chances (marks b close to 1 on the chance bar.)
- 230 Rachel That's right, she's correct (supports Yael)
- 231 Yael That's what Guy just explained
- 232 Noam Right

The discussion on event b took only 6 turns and resulted in agreement between all three students. Yael reads the question and immediately recognized the claim in the question as "plausible" (227). While Noam claims that the chance is half, Yael objects and marks the chances on the chance bar near one. Rachel backs Yael, with no explanation, and Yael provides support – "that's is what Guy just explained" (231). Noam responds "right" (232). We claim that here we have evidence that Yael (and possibly the other FG's students as well) consolidated the intended knowledge element, after constructing it during the preceding WC discussion: The evidence for constructing it during the WC discussion is strong for Guy and others but only weak for the FG students (Rachel in turn 147 and Yael in turns 137 and 161). However, the present FG discussion shows that at least Yael did construct it then, and that she consolidated it during the present FG discussion. We interpret Yael's turns 229 and 231 as showing consolidation because she uses the construct with immediacy and flexibility, adapting it to a new context, which is characteristic of consolidation. Here we have also clear evidence that Guy was a knowledge agent for Yael, whose ideas were downloaded into the FG discussion. The fact that this group used a way of reasoning from a previous argument as data for a new claim also has strong connections to one of the three criteria for determining when ideas function as if shared (see Rasmussen & Stephan, 2008, for elaboration of the three criteria).

## DISCUSSION

In this study, we offer a way to adapt existing methodological tools in order to coordinate analyses of the individual, the group and the collective in a working mathematical classroom. The proposed combined analytic approach is significant in that it offers a new means by which to document the evolution of mathematical ideas in the classroom, the processes by which ideas move between individuals, small groups, and the whole class, and the role of the teacher in these processes. This way helped us to identify shifts of knowledge in the classroom and the students who are main players in these shifts – the knowledge agents. Identifying the role of the teacher in the knowledge construction process and the function of students who act as knowledge agents helps us to understand the mechanism by which collaborative learning can take place, and be affected by individuals and groups in the collective.

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# FROM IDENTITY TO IDENTIFYING – TOOLS FOR DISCOURSE ANALYSIS OF IDENTITY CONSTRUCTION IN THE MATHEMATICS CLASSROOM

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*This report describes a new method for analysing the ways that students' mathematical identities are constructed in class. Its uniqueness is in tending to the emotional and social aspects of learning, as well as the cognitive ones, using a single, unified set of conceptual tools. These tools are an extension of the tools offered by the communicational (commognitive) theory proposed by Sfard (2008) and are thus tailored specifically for the analysis of mathematical learning. The main conceptual division is made between mathematizing (talking about mathematical objects) and subjectifying (talking about the participants of the discourse). This division forms the basis of an operational set of discursive categorizations for "identifying" activity, enabling the extraction of identity narratives from spontaneous interactions in class.<sup>1</sup>*

## INTRODUCTION

Attention to the relationship between students' identity and mathematical learning has grown considerably in the last two decades (Bishop, 2012; Boaler & Greeno, 2000; Sfard & Prusak, 2005). And yet, the ways in which one can study the actual interaction of identity and learning in a rigorous manner has remained largely obscured. Researchers often use interviews or questionnaires and try to relate them to students' achievements and to their forms of participation in class, mostly according to the interviewees' accounts of their experiences. However, these methods have some important limitations. First, they do not tell us how the students' thoughts and feelings about herself actually interacts with the mathematical activity in the minute-to-minute happenings of the classroom life. Second, these methods are limited mostly to grownups, or to old adolescents, as students of younger ages are usually unable to narrate themselves elaborately enough. Finally, the methods of direct interrogation tend to elicit socially acceptable narratives, thus there may often be a difference between the declared identity and the enacted one.

In order to overcome the above limitations, I found a need to develop a research method that would enable studying emotions, social interactions and mathematical activity concurrently, from transcriptions of classroom videos. This method was developed hand in hand with the research I conducted for my PhD study, which was aimed at understanding how identity interacts with mathematical learning. I shall

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<sup>1</sup> The PhD research reported in this paper was advised by Prof. Anna Sfard and supported by the Mandel Scholars in Education fellowship.



describe this research briefly to give the context of the research method. However, the focus of this paper will be on the tools that I have developed for analysis of the data.

## THEORETICAL BACKGROUND

Many studies have shown that a relationship exists between what a student thinks and feels about herself and about mathematics (conceptualized often as *affective* aspects) and her performance in this subject (Hannula, 2012). From a socio-cultural perspective, researchers have pointed to the relationship between forms of social interaction in class and the effectiveness of mathematical learning (Cobb, Yackel, & Wood, 1992). Research on affect in mathematics education has mostly used different methodological tools than those used by research on cognition and research on social interaction. While the ‘affect’ strand often uses questionnaires and interviews to elicit students’ accounts of their experience, studies of cognitive and social aspects of learning often use tools that look at the learning process itself such as clinical interviewing (diSessa, 2007) and classroom discourse analysis (Atweh, Bleicher, & Cooper, 1998). This difference in methodological tools may be the result of the different lineages of each of these domains of inquiry (social psychology, cognitive psychology and sociology/anthropology) but it also reflects deep theoretical and conceptual differences. Thus in order to devise a method for studying the different aspects of learning it is necessary to rely on a theoretical framework that accounts for all of them based on the same ontological assumptions.

My starting point for developing such a framework is the communicational (commognitive) theory, proposed by Anna Sfard (2008). Its main tenet is that thinking can be thought of as a type of intrapersonal communication, not qualitatively different from interpersonal communication. It thus sees discourse (verbal and nonverbal) as the main phenomena for inquiry in human learning. For my research goals the emphasis on communication is beneficial because spontaneous (including classroom) discourse almost always carries with it emotional, social and cognitive meanings concurrently. Another benefit of this framework is that it has already developed tools for analysing participants’ mathematical discourse from a socio-cultural view. Expanding it to the affective domain would thus enable to look at specific mathematical learning processes as they interact with social interaction and emotional communication.

I take *identity* as the main concept to work with for combining the individual, emotional and social aspects of the student’s experience in class (Wortham, 2005). Within the communicational framework, identity is defined as a *collection of stories* that are *reifying, endorsable, and significant* (Sfard & Prusak, 2005). This definition is operational in that it specifies identity as a type of communication that can be (at least partially) externally observed. It has been beneficial for studies which are mostly based on interviews with students, where long stretches of narratives can be elicited (eg. Lange, 2009). However, for my own research purposes it was not sufficient, as I was interested in the *processes* of learning, not only in identity as an end-product. Therefore, I have moved to analysing *identifying activity*, or the activity of constructing one’s own and others’ identities (Heyd-Metzuyanim & Sfard, 2012).

Before moving on to describe the conceptual tools for capturing and studying this identifying activity, I will give a short description of the research for which these tools were developed.

## THE RESEARCH

The research included 12 7<sup>th</sup> grade students who were divided into three groups according to prior achievements in school mathematics (low, moderate-to-high and advanced). The students received weekly lessons from me in a private learning centre for a period of 5 months. In addition, pre- and post- interviews were held with the students, including a thorough assessment of their mathematical skills. Only the final interview included questions about the students' mathematical identity and feelings about math. All lessons and interviews were recorded using 2-3 stationary video cameras. The students' school teachers were interviewed during mid-course, and their parents were interviewed at the end of the research procedure.

## CONCEPTUAL TOOLS FOR ANALYSING IDENTIFYING ACTIVITY

While most students can easily tell stories about themselves if explicitly asked, the process by which these identity stories are constructed through classroom talk is quite elusive. The first step I take towards capturing this process is by dividing students' discourse (both verbal and non-verbal) into two main categories: mathematizing and subjectifying. *Mathematizing* is defined as talking about mathematical objects, using mathematical words and mathematical signifiers (for instance “two plus two is four” or “an isosceles has two equal angles”). *Subjectifying*, on the other hand, is defined as communicating about *the participants of the discourse* (such as “I don't know how to solve this problem”). In reality, these two activities often overlap, for instance in “I think it's four, but I'm not sure”. However, looking at the rather “pure” forms of subjectifying is an obvious starting point for looking at how participants identify themselves as other. I shall start by describing the ways in which such *direct* subjectifying utterances are analysed.

### Three Levels of Subjectifying

Direct subjectifying utterances, which talk explicitly about a person, can be divided into three levels. The first is the *specific* level. Those are statements which pertain to a participant's actions in a specific context. For instance: “I didn't get this”, or “I think I know the answer”. The second level includes *general* evaluations of one's participation in the discourse. For example “I **never** succeed with these kind of exercises”, or “She's **always** been irritated by fractions”. The highest level is that of attributing stable *properties* or assigning *membership* to a person. Such are statements like “**She's a** straight A math student”, “**I'm** bad with fractions” or “**He has** a learning disability”. This highest level is identifying by definition. However, general participation evaluations and even specific subjectifications can add up to build a clear picture of how a student identifies herself (or is identified by others), provided they are *recurrent* and *consistent*. The next excerpt, taken from the mathematical interview of a student named Idit, exemplifies how mathematizing and identifying can be

interrelated. In it, Idit attempts to compare several different fractions. The first two fractions she's asked to compare are  $\frac{1}{5}$  and  $\frac{1}{7}$ .

**Legend: Specific subjectifying; General participation evaluations; Properties**

186 Idit so um... (looks at the teacher, smiling) **here I'm completely unsure**. I told you **I'm not good with fractions**. (Looks back at the paper), so I think... that this (points to the  $\frac{1}{5}$ ) is bigger... [I'm] not sure.

187 T Why do you think so?

188 Idit 'cause **if I'm not mistaken**, in elementary school **we were told** that a number that is smaller, so it's bigger, that's what **we were told** (shrugs shoulders, smiling)

*Idit hesitates, then decides to change her answer. The teacher asks her "why?"*

193 Idit 'cause it doesn't make sense that seven is smaller than five.

Next, Idit goes on to explain why  $2\frac{3}{4} < 2\frac{2}{9}$ .

223 Idit 'cause.. first of all, the three is bigger than the two (T: um-hmm) but the nine is bigger than the four

227 Idit So.. it's all now like the.. **what you explained** then on the board with the pie and (all) that (T: yes, yes) So I- **I think of it in a different way**

*Idit draws two circles, one with 9 little circles, two of them darkened (for  $\frac{2}{9}$ ), and the other with four little circles, three of them darkened (for  $\frac{3}{4}$ ). When she finishes, she looks at her drawings*

240 Idit So, I guess this one (points to the  $\frac{2}{9}$  circle) is bigger.

245 T OK, why? How is it derived from this drawing?

246 Idit Umm... (giggles slightly) **I don't know. That's how I thought of it**. (T: OK). (Mumbles with a shy smile) **I don't know, there are things I can't do**.

247 T. Alright.

As can be seen in the above excerpt, Idit had considerable difficulties with comparing fractions. Apparently, she was unable to *realize*  $\frac{3}{4}$  both as 3 circles out of 4 and as a number on a number line. This difficulty is far from being unique to Idit (Pantziara & Philippou, 2011). However, it is also clear from this excerpt that Idit has developed a stable identity of herself as "not good with fractions" [186] that could be inhibiting her from what Sfard (2008) terms *explorative* learning of fractions. This can be seen especially in the last general evaluation - "there are things which I can't do"[246]. With this remark, Idit terminated her exploration of the different possible realizations of fractions. Moreover, it seemed there was a strong connection between Idit's identification as "not good with fractions" and her tendency to participate *ritually* in any discussion about fractions (Sfard, 2008, p. 241). As is common in ritual participation, Idit justified her actions solely by relying on externally given rules, ("that's what we were told" [188]), and her discourse about fractions was *syntactic*. That is, she was treating them as separate digits and not objects existing in the world ("the three is bigger than the two... but the nine is bigger than the four" [223]).

Idit's case clarified that there could be, indeed, a strong relationship between the ways a student identifies herself in relation to specific mathematical domains and the ways she mathematizes in them. As shown more elaborately in Heyd-Metzuyanim (2011), her discourse on whole numbers was much more explorative, showing agency and many signs of objectification. However, Idit's case was also rare in that she often identified herself using direct and general participation evaluations. Most of the other participants in the study were not as explicit, so it was necessary to devise ways of extracting identifying messages from *indirect* statements, non-verbal communication and emotional expressions.

### **Indirect Identifying, non-verbal communication and emotional expressions**

I use the term *indirect* identifying, to denote utterances which, though not talking explicitly about a person, still convey an important message about him/her. Such indirect identifying can take different forms. It can be verbal or non-verbal, and it often gains its meaning from the particular emotional expression with which it is uttered. See for instance the following short excerpt from a lesson:

*Algebraic expressions are presented on the board and the three students who are present (Ziv, Edna and Idit) are requested to find a "substitution set" of these expressions. Ziv leads the solution of the first exercise. Then the teacher asks Idit and Edna to solve the next one. After a few attempts, Idit says:*

Idit: (*Smiling, squinting at Edna*) I think **Ziv should solve this**.

Here, Idit is directly identifying Ziv as someone who is capable of solving the algebraic problem. However, she is also *indirectly* stating something about herself (and about Edna), that is, "**We** are incapable of solving this question". Her alignment with Edna is communicated through her squint and smile at Edna, an expression that can also be interpreted as a slight resistance to the teacher's efforts to include them in the discussion. Of course, the context is central here. Such delicate communications can only be interpreted when taking into account as much background information as possible, which has been collected in this research via my own long term acquaintance with the students. Thus indirect identifying can never stand alone as evidence, but collecting these utterances can form converging evidence of a certain identity narrative. The necessity to specify what exactly the message that is indirectly stated (see table 1), forces the researcher to be very explicit about her interpretations and their rational. Of course, interpretations may vary, which is why such analysis demands at least two analysers. Preferably, one of these should be an "insider" to the situation (as I was, being the teacher in the course), and one of them an "outsider" who can see the classroom activity from a less involved stance.

Though the above classification of subjectifying and identifying statements assists operationalizing the emotional and social processes that happen in short episodes, there still remains a challenge of how to gain a wider view of such processes over periods of more than a few minutes. One of the methods I have used to cope with this challenge is collecting direct and indirect identifying statements into "identifying tables" that give a sort of a "snapshot" of the identifying processes that were taking place in class. Table 1

is a very short excerpt from such a table. The utterances in it were collected from the discussion of 4 students during an intensive episode of problem solving, where students' emotional reactions seemed to hinder them from advancing effectively towards a solution. While 3 students (Edna, Idit and Dan) had considerable difficulties in understanding the problem, one student (Ziv) seemed to be coping well with the task. However, when Ziv attempted to explain his solution, the other students stubbornly insisted they do not understand him. Analysis of the mathematizing in this episode revealed that the three resisting students realized the mathematical object that was at the centre of discussion (a 'chocolate bag' containing fractions of a chocolate bar) differently than Ziv did. And yet, it was unclear why these differing realizations were so difficult to negotiate. Therefore, all identifying utterances (direct and indirect) were entered into five different identifying tables (one for each of the participants). Table 2 includes a short segment of Ziv's identifying table, which was the richest in (mostly negative) identifying statements. The five tables that were produced at the end of this analytical process clarified that "identity conflicts" were hindering effective mathematical communication in this episode. Thus, Edna and Idit resisted Ziv's "explanations" because, in their eyes, Ziv was identifying them as "little girls". "Understanding Ziv", would be co-operating with this identity, and therefore better avoided. The benefit of putting all identifying statements into such tables was evident in the fact that these identity conflicts were not unearthed just by reading the transcript, since the episode lasted over 30 minutes, was full of "noise" (chatting, arguments, etc.), and scattered mathematical discussions.

What is said ( <i>Context and non-verbal actions</i> )	Analysis
Dan ( <i>quietly, while Ziv starts explaining again</i> ): Enough, Ziv, <b><u>you won't be a teacher.</u></b>	Dan directly identifies Ziv as incapable of "teaching" or explaining
Edna ( <i>turning to the teacher, in an annoyed voice</i> ): <b><u>He just- he talks to me like I'm his [little] girl!</u></b>	Edna directly identifies Ziv as arrogant and indirectly identifies herself as mature, or not a "little girl".
Ziv: (answering the teacher's request to help the others, in a mocking, "talking to children" voice) <b>so they should make a bit of an effort.</b>	Ziv indirectly identifying himself as more "mature" and "working harder" than his peers.

Table 1 - Ziv's Identifying table - from a problem-solving episode

### Indirect identifying through mathematizing

Up till now, I have talked about extracting data about identity construction from all activities in class but one – that which is most central to mathematical learning – the mathematizing activity. Though at first glance, extracting identity narratives from mathematical talk seems somewhat counter intuitive, on a closer look, one can see that participants use a multitude of communicational tools to identify themselves in a certain way even when they are talking about mathematical objects or performing mathematical routines.

Following is a short excerpt from a lesson with the low-achieving group in which the students were practicing addition of negative numbers. The excerpt focuses on Dana, who was extremely low achieving and failed to advance in any of her mathematical skills during that year (for the complete analysis, see Heyd-Metzuyanin, 2013).

- 77 Dana **What do I do in this situation**, that I have this plus this, like, here (points to  $(+4)+(-1)$ ), **I write** the biggest? (Teacher marks her other exercises) Like, **what do I do**, the closest to zero, for instance plus three, minus three?
- 80 T **You start, look. You do..** uh with the arrows. Plus four is (going) right, right? (*Teacher goes on explaining how to perform the calculation*)

After this explanation, Dana walks back to her place, only to come up a few minutes later, having completed two more exercises.

- 129 Dana **Tell me if I was right, OK? I did (it) like this.** This is the zero, right?  
*Teacher asks which exercise Dana is talking about, Dana points to  $(-5)+(+3)$*
- 133 T Yeah. And **what did you get?**
- 134 Dana Well, I (unclear). So I had here zero-
- 135 T **^You're right^**, **^You're right^** (high tone of voice signalling reassurance)
- 136 Dana (Surprised) Ah!, A-haa! (walks joyfully back to her chair) That's great!

This excerpt exemplifies a typical interaction between Dana and me, her teacher. It highlights the ritual form in which Dana participated in mathematical lessons (often requesting me to evaluate her work, always concentrating on "how to do" the calculations [77], never on the mathematical reasoning). It also shows how rich identifying information can be communicated via tone of voice and specific participation evaluations. For instance, by her surprised reaction to "being right" [136] Dana indirectly identified herself as being "usually wrong". Moreover, this excerpt demonstrates *my own* role in constructing Dana's identity as a "weak" student, who is only capable of following rules blindly. Thus, I, too, concentrated on what she should "do" [80]. Moreover, in [134-135] I even interrupted her in the (relatively rare) occasion where she attempted to explain her mathematizing. By stating she was "right", I was indirectly saying that what is important is only being "right" or "wrong", not the reasoning behind the mathematical calculations. I was thus reinforcing her ritual participation. The fact that Dana seemed to be very content with this reaction [136] demonstrates how we both collaborated in co-constructing Dana's "weak" mathematical identity.

## SUMMARY

The fact that interactions between identity and mathematical performance *exist* is not new (Boaler & Greeno, 2000). Neither is the fact that students' identities are socially constructed (Wortham, 2005). Yet, *how* identity interacts with specific mathematical practices is not yet fully understood. The present method advances us towards gaining this understanding by enabling the study of affective, social and the cognitive processes in mathematical learning as they unfold in actual classroom life. Its main

contribution is in providing a new *lens* – that of mathematizing vs. identifying. This lens has already started uncovering how identities of failure in mathematics may develop (Heyd-Metzuyanin, 2011, 2013). And yet, such findings can only be framed as conjectures due to the limited amount of cases and data that can be examined by this method. Future research and conceptual development is needed to make it applicable for larger sets of data.

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# KNOWLEDGE OF ASSESSMENT AND ITS IMPLICATIONS

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*Assessment is an integral part of the teaching and learning process. Therefore, teachers must master this subject by the time they enter their jobs. This research examines the knowledge of assessment by pre-service and novice math teachers in elementary and secondary schools. The results show a lack of knowledge even in basic concepts of assessment. The research also checks if and to what extent there is a connection between the teachers' declared knowledge and teachers' actual behaviour. It was found that even known concepts are not used in the assessment process. This paper proposes which concepts should be emphasized in order for teachers to succeed in implementing their knowledge of the standards of assessment.*

## THEORETICAL BACKGROUND

Since assessment is an integral part of the teaching and learning process, teachers are responsible for conducting assessment of students' knowledge and achievement on a routine basis and in a reliable manner (Birenbaum et al., 2006; NCTM, 2000). This task is one of the teachers' most important roles, because their assessment results influence how they function. They make numerous decisions about students on a daily basis and the quality of those decisions depends on their ability to evaluate, both formally and informally, in a valid and reliable method (Mertler, 2009). Accordingly, we would expect teachers to have a basic knowledge about tests – how to compose them and how to interpret the results, as well as to be proficient in the concepts of assessment as stated in the standards of assessment (NCTM, 2000). This expectation, however, is only partially met, and their knowledge is sparse and compartmentalized (Valentin, 2005; Maclellan, 2004; Mertler, 2003, 2009). For example, most teachers do not know the difference between validity and reliability, or how to determine if they have applied them in their assessments, especially in tests they make up (Valentin, 2005; Mertler, 2003, 2009).

The most common assessment tool among mathematics teachers is a test or quiz. Most teachers compose their own tests or quizzes to fit the material they have taught, and also use observation as an additional helpful tool for collecting data about the students. This is particularly true among novice teachers (Watt, 2005; Valentin, 2005; Senk et al., 1997). In any case, novice elementary math teachers also use other assessment tools such as projects and performance tasks, as opposed to novice secondary math teachers who rarely use them (Mertler, 2003; Adams & Hsu, 1998).

It has been shown, however, (by e.g. McMorris & Boothroyd, 1993) that the quality of teachers' tests is related to their competence in assessment. Moreover, the most prominent predictor of teachers' test quality was found to be their knowledge of terms related to their training in assessment, with this predictor far outweighing other variables such as teaching experience.



Therefore, it is important to examine the knowledge of new teachers in the area of assessment, but it is just as necessary to examine its implementation in the classroom since knowledge by itself is no guarantee of actualization.

This study is aimed at assessing the knowledge of the new generation of mathematics teachers i.e. pre-service (**PT**) and novice (**NT**) mathematics teachers for elementary school (grades 3-6) and secondary school (grades 7-10), as well as determining to what extent the NT make use of their knowledge in mathematics classrooms. The fact that the research examines both the knowledge and its execution by the same groups, allows us to try and predict which concepts must be focused on in assessment courses so NT will implement their acquired knowledge.

### **THE SYLLABI OF THE ASSESSMENT COURSES IN ISRAEL**

Checking the syllabi of assessment courses in teacher training colleges and interviewing the course lecturers provides trustworthy data about the assessment principles that are given to the PT and to the NT who are trained in those colleges. The main features of those courses are:

- A. The colleges offer the course to PT in their last year of learning (e.g. third year), just before they become NT but after they have some familiarity with classroom dynamics and have seen the way mathematics teachers evaluate students.
- B. All the lecturers define the term 'assessment', talk about its roles and the various types of assessments as related to their use (like formative assessment, summative assessment).
- C. All the lecturers deal with the most common assessment tool – the test, especially the way to construct it, reliable and valid testing and interpreting its results.
- D. All the lecturers emphasize the alternative assessment tools vis-a-vis the traditional ones, and discuss some of them elaborately (such as journals and portfolios).

### **METHODOLOGY**

#### **Participants**

The study focused on two groups: NT and PT in five teacher-training colleges. The NT group was trained to teach mathematics for elementary school or secondary school, and they all have up to three years of experience in teaching mathematics. The PT group is at the conclusion of its studies just prior to becoming NT in elementary or secondary schools. Table 1 shows the distribution of the 139 research participants.

All the participants said they had taken a one semester course in student evaluation at their training institutions. None of the NT took an additional course to the one they had studied.

status \ grade	NT	PT	total
elementary	31	42	73
secondary	33	33	66
total	64	75	139

Table 1: The population's distribution

### Instrument

Three questionnaires were composed:

**Declarative Knowledge questionnaire** contains 18 concepts that represent various aspects of assessment (for example, external assessment, summative assessment, norm reference test, concepts map). All the terms were mentioned by the course lecturers. These concepts were divided into three categories: 1) types of assessment, 2) tests, and 3) alternative evaluation tools. Each participant was asked to rate his knowledge regarding each concept on a scale of 1(not knowing what the concept is) to 4 (proficiency with this concept). The reliability of this questionnaire is  $\alpha$  Cronbach=0.85.

**Actual Knowledge questionnaire** consists of 11 multiple – choice questions in which the subjects were asked to mark the correct answer. The objective was to thoroughly examine the participants' knowledge regarding several of the concepts that appeared in the Declarative Knowledge questionnaire, mainly the various aspects of the concept of testing (see Figure 1 for example). This questionnaire is part of a large one that was composed by McMorris & Boothroyd (1993). The reliability of this questionnaire is  $\alpha$  Cronbach =0.55.

The **Declarative Behaviour questionnaire** attempted to track to what extent the assessment knowledge is expressed in the mathematics classroom. Although PT have some experience in teaching, it is not enough for our needs. Being subordinate to the class teacher, they are not independent in choosing the assessing methods. This is contrary to the NT who have the responsibility for their students' learning and conduct the class as they see fit. Therefore, this questionnaire was given only to NT.

This questionnaire contained the same concepts that appeared in the Declarative Knowledge questionnaire, and were divided into the same three categories. The participants were asked to rate to what extent they made use of these concepts in their work on a scale of 1-4, with 1 representing "not at all" and 4 representing "frequently."

The reliability of this questionnaire is  $\alpha$  Cronbach =0.88.

- Which of the following is characteristic of a test with high reliability?
- A. The test grade is higher for good students than for weak students.
  - B. There are differences in the level of difficulty in the items on the test.
  - C. The test grades are uniformly distributed.
  - D. The two grades of the same test are very close.

Figure 1: An example question from Actual Knowledge questionnaire

## MAIN RESULTS

**Declarative Knowledge questionnaire:** In general, on the average, the participants have a moderate degree of familiarity with assessment-related concepts 2.7 (out of 4).

There were no differences between PT and NT, not in the elementary level nor in the secondary level. However, a significant difference was found between the levels: the average score of the Declarative Knowledge questionnaire of the elementary level was better than the average score of the Declarative Knowledge questionnaire of the secondary level. Significant differences were found in concepts concerning test and alternative assessment tools categories.

**Actual Knowledge questionnaire:** The average score (in percentages) among the respondents was 40%. In other words, on the average, the respondents answered correctly only 4.4 out of 11 questions. The low score exhibits an insufficient level of knowledge of the concepts that were examined. The questionnaire reveals confusion, misunderstanding and obscurity regarding certain terms, for example between the terms validity and reliability or between the terms norm test and criterion test.

No differences were found between the two levels or between NT and PT.

**Declarative Behaviour questionnaire:** In general, on the average, the application of knowledge concepts in the classroom is low 2.18 (out of 4).

For both levels, a significant difference was found between the participants' answers on the Actual Knowledge questionnaire and the participants' answers on the Declarative Knowledge questionnaire. This occurs in all three categories. This means that even the acquired knowledge by the NT is not expressed in their classrooms.

A comparison between NT for elementary school and NT for secondary school reveals that NT in elementary schools use those terms significantly more often than NT in secondary schools. The significant differences between the two groups were found in concepts concerning test and alternative assessment tools categories.

### **Correlations between Terms in the Declarative Knowledge Questionnaire and Terms in the Declarative Behaviour Questionnaire**

The Declarative Knowledge questionnaire and the Declarative Behaviour questionnaire contained the same terms and in the two questionnaires those terms were divided into the same three categories. Therefore, the correlations between the corresponding terms in the two questionnaires were examined. The correlation between all the terms in the Declarative Knowledge questionnaire and in the

Declarative Behaviour questionnaire is extremely high ( $r=.855$ ) and significant. Even the correlations between the corresponding categories in the two questionnaires are very high:

The correlation between assessment type category in the Declarative Knowledge questionnaire and assessment type category in the Declarative Behaviour questionnaire is  $r=.892$  and significant.

The correlation between test category in the Declarative Knowledge questionnaire and test category in the Declarative Behaviour questionnaire is  $r=.833$  and significant.

The correlation between the alternative assessment tools category in the Declarative Knowledge questionnaire and the alternative assessment tools category in the Declarative Behaviour questionnaire is  $r=.691$  and significant.

### Predicting Behaviour According to Knowledge

Due to the high and significant correlations, it was decided to examine which assessment terms from the Declarative Knowledge questionnaire predict knowledge implementation for each of the categories that appear in the Declarative Behaviour questionnaire, and to examine which of the three categories in the Declarative Knowledge questionnaire is important in influencing knowledge application. Table 2 shows the results for predicting categories.

	NT for elementary level			NT for secondary level		
	B	Beta	Sig.	B	Beta	Sig.
(Constant)	-.119			.303		
Type of Assessment	.315*	.503	.028	.225**	.374	.002
Test	.145	.172	.420	.475**	.636	.000
Alternative Assessment Tools	.408	.326	.071	.007	.009	.940
R	.911			.871		
R Square	.829**			.758**		

\*\*significant:  $p<.01$  (2-tailed)

Table 2: Results of Multi-Regression Analysis using Enter Method

For NT in the **elementary** level, the category "type of assessment" is the only predictor. Even for NT in the **secondary** level, this, as well as the "test" category predicts implementation.

An analysis was then performed using stepwise multiple regressions, in which all the terms were used, to predict implementation of the knowledge for each one of the categories.

For NT for **elementary** level: Five terms which predict use of knowledge were found: "internal assessment", "formative assessment", "reliable", "table specification" and "rubric". The first two terms are the dominant terms and belong to the category "type of assessment" and the others – to the "test" category.

For NT for **secondary** level: Seven terms which predict use of knowledge were found: "internal assessment", "formative assessment", "criterion test", "norm test", "rubric", "journal" and "item difficulty rating". Those terms belong to the two categories which were found to be predictors: "type of assessment" and "test" categories.

## **DISCUSSION AND IMPLICATION**

Assessment is a central component of teachers' work, and is evidence of their professionalism. Above all, assessment determines the future of the students (Amit and Fried, 2002). It is thus obligatory that teachers exhibit proficiency in the area of assessment from the first moment they begin their jobs.

The present study examines the knowledge of PT and NT in elementary and secondary schools, with a focus on mathematics teachers. In general, the results of the study point to difficulties in the area of assessment even among the new generation of mathematics teachers. A large portion of the research population lack knowledge of basic concepts of assessment; particularly those connected to tests - the most common assessment tool. Apparently, this is a repercussion of the marginal attention that is given to the topic of assessment during teacher training (Valentin, 2005; Mertler, 2003, 2009).

The research also focused on some special aspects such as differences in knowledge regarding training or grade level (elementary vs. secondary). The results show that PT and NT for elementary school declare they know better terms concerning assessment than PT and NT for secondary school. However, the Actual Knowledge questionnaire reveals that indeed those feelings are not substantiated. One can assume that those feelings come from the differences in focus during their training period. Mathematics teachers training for secondary school focuses on content knowledge, whereas mathematics teachers training for elementary school focuses on pedagogical and methods courses and less on subject matter courses.

The research also checks if and to what extent there is a connection between the teachers' declared knowledge and teachers' practice regarding assessment. It was found that the extent to which teachers use the various assessment terms is significantly less than the extent of their knowledge. This means that even when teachers know a term they don't use it. A positive and significant connection was found between the extent of knowledge and the extent to which it is used. These results clearly illustrate that broad knowledge of the various assessment terms results in more extensive use of these terms during educational practice. If we look at this from the /other side of the spectrum, insufficient knowledge results in failure to use it. Given that no difference was found regarding the knowledge between NT and PT in both knowledge questionnaires (i.e. Declarative Knowledge Questionnaire and Actual Knowledge Questionnaire) and that all the participants come from the same training colleges, we can assume that the PT

will act in the same way as the NT. This means that we can expect no imminent changes regarding this issue in schools.

To change this situation and improve implementation of assessment knowledge, we checked which important concepts should be emphasized so that they could be applied in the classroom. Several concepts that require further discussion were found. Among the two grade groups, knowing the terms "inner assessment" and "formative assessment" of the category "type of assessment" indicates further use. The more knowledge they will have, especially the knowhow of implementing assessment in the classroom, the greater the chance of its actual use.

These terms are general terms of assessments, which all teachers use for evaluation. Therefore, it is clear why they appear as predictors in both grades.

This research, like previous research, shows that NT for secondary school, in comparison to NT for elementary school, use less alternative assessment tools, thus their evaluation is based mainly on tests and quizzes (Adams & Hsu, 1998; Watt, 2005; Senk et al., 1997). Particularly in Israel (place the research was conducted), every year, from the 7<sup>th</sup>-12<sup>th</sup> grades, all students are tested in external mathematics examinations that are carried out by the state. Sometimes international exams like PISA and TIMSS take place instead of those exams. Consequently, NT of secondary schools feel an obligation to prepare their students for success, and therefore they don't allow any disturbing factors such as other tools to interfere with their assessment. All this comes in contrast to NT for elementary school who have to deal only with one external mathematics test – in the 5<sup>th</sup> grade. As a result NT for elementary schools feel free to assess in ways they learnt and know, as well as the traditional tools – tests and quizzes. This fact is reflected by the usage of the test category and its terms as predicting behaviour **only** among NT for secondary school.

The expected connection between knowledge and practice was shown in this article. The results of this correlation should have implications in courses for pre-service and in-service teachers. Since the time dedicated to assessment courses is very limited, it is important to know which terms predict expected behaviour in order to focus on them. This will enhance preservice and inservice mathematics teachers' knowledge, and will ensure that teachers will function, as they are expected to, according to assessment standards. This will help to utilize the full potential of assessment in enhancing students' knowledge and stopping unfair and unjust assessment (Hoch and Amit, 2011).

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# PIVOTAL TEACHING MOMENTS IN A TECHNOLOGY-INTENSIVE SECONDARY GEOMETRY CLASSROOM

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*This study utilized pivotal teaching moments (PTMs) as a framework to characterize cognitive disruptions that occurred over a two year time span in a 1:1 computing, secondary geometry classroom. How the teacher addressed these cognitive disruptions and the likely impact on student learning was also analysed. Results included two new categories of PTMs that emerged in conjunction with the use of technology, technology confusion and incorrect technology use. The likely impact on student learning varied over the course of the study; however, the teacher most often chose to emphasize mathematical meaning when addressing PTM's.*

## INTRODUCTION AND FOCUS

Much research exists on the subjects of mathematical tasks (e.g., Smith, Stein, Boston, and colleagues), technology design and use (e.g., Sinclair, 2003; Pea, 1985; Drijvers et al., 2010; Sherman, 2011; Cayton, 2012), mathematical discussions in technological environments (e.g., Hollebrands, et al., 2011, Cayton, 2012), and teacher actions (e.g., Stockero & Van Zoest, 2012). This study examines the intersection of these areas to address the following research question:

What types of pivotal teaching moments and teacher actions arise in a technology-intensive high school geometry classroom?

## THEORETICAL FRAMEWORK AND RELATED LITERATURE

When planning a mathematics lesson, Smith and Stein (1998) suggest teachers should identify mathematical goals and select cognitively demanding tasks to engage students in accomplishing these goals. Further work (Stein, Engle, Smith, & Hughes, 2008; Smith & Stein, 2011) characterizes five practices to help teachers implement cognitively demanding tasks and facilitate productive mathematical discussions. The first practice, *anticipating*, refers to a teacher's consideration of multiple ways students may approach solving a task, as well as student misunderstanding and difficulties that may arise. The acts of identifying cognitively demanding tasks and anticipation of student solution strategies becomes more complex with the introduction of technology in the classroom because the teacher must not only anticipate mathematical strategies and difficulties, but also technological strategies and difficulties that may arise. The presence of technology may provide opportunities for students to engage in tasks that may be difficult for teachers to anticipate, resulting in a higher incidence of teachers needing to provide an impromptu response to students within a lesson. One way such teacher responses have been framed was described by Stockero and Van Zoest (2012) as a pivotal teaching moment (PTM). They define a PTM as "an instance in a



classroom lesson in which an interruption in the flow of the lesson provides the teacher an opportunity to modify instruction in order to extend or change the nature of students' mathematical understanding" (p. 3). Results from the study identified five categories each for the type of PTM, teacher actions in response to PTMs, and likely impact on student learning. PTMs were characterized as extending, incorrect mathematics, sense-making, contradiction, and confusion. Each PTM was further coded according to high or moderate potential to impact student learning. Teachers' responses to PTMs were coded as extends mathematics and/or makes connections, pursues student thinking, emphasizes the meaning of the mathematics, acknowledges but continues as planned, and ignores or dismisses. In all instances other than acknowledges but continues and ignores or dismisses, a teacher's implementation during a response to a PTM was coded as skillful, moderate, or poor. The PTM was then coded according to the likely impact on student learning as high positive, medium positive, low positive, neutral, or negative. Using PTMs as a framework for analysis, the purpose of our present study was to more closely examine PTMs within the context of a 1:1 laptop computing, high school geometry classroom.

## METHODS

### Context of Study

Mrs. Anderson's selection for this study stemmed from her involvement in a larger, three-year professional development project. She was one of 24 teachers in the geometry cohort of the project. Her participation included two, week-long summer institutes and two years of online professional development with classroom level support as part of her participation. The larger project involved four public, 1:1 computing school districts (each student is issued a laptop) in a southeastern state and used four interrelated interventions: mathematics software programs (e.g., *The Geometer's Sketchpad*, teacher professional development, STEM role models, and cloud computing (VCL). The goal of these interventions was to increase student success in high school geometry so that they would be able to pursue STEM related college majors and careers. During the time of the study, Mrs. Anderson was in her fourth and fifth year of teaching, and all of her teaching experience occurred at a large suburban high school with more than 1500 students. The school operated on a block schedule where students took four, 90 minute classes per semester, and the student population included approximately 20% African American, 5% Hispanic, and 25% economically disadvantaged students at the time of the study.

Mrs. Anderson taught geometry each semester over the two-year period and was observed at least two times per semester. The use of *The Geometer's Sketchpad* was expected for each observation; however, Mrs. Anderson made all decisions regarding the mathematical objectives and nature of dynamic geometry task utilized during instruction (e.g., pre-constructed or student-constructed). She was observed a total of 11 times; however, three observations were eliminated because students were presenting group projects. The remaining eight videos provided longitudinal data for analysis.

**Data Analysis**

The authors analyzed one video recording, collectively to identify PTMs and achieve at least 75% inter-rater reliability for coding the type of PTM, teacher decisions, and likely impact on student learning. Authors independently coded a subset of the final eight videos. PTMs were grouped according to teacher actions and specific episodes of Mrs. Anderson's actions and implementation were transcribed from the video. These episodes were utilized to examine PTMs and teacher actions within Mrs. Anderson's geometry classroom.

**RESULTS**

This section describes overall results for the type of PTM, teacher decisions, likely impact on student learning, and several examples of PTMs that demonstrate Mrs. Anderson's decisions in response to PTMs. Table 1 summarizes the types of PTMs, teacher decisions, and likely impact on student learning for Mrs. Anderson's geometry classroom during the study. Similar to previous research, PTMs included extending, incorrect mathematics, sense-making, contradiction, and confusion. Sense-making was the most prevalent type of PTM observed in Mrs. Anderson's geometry classroom. In addition, two new types of PTM were observed that were unique to the technology-intensive environment, incorrect technology use and technology confusion. Incorrect technology use referred to instances where a student utilized tools within the dynamic geometry environment incorrectly. For example, a student may have drawn what appeared to be an isosceles right triangle, when they should have constructed the triangle using tools within the dynamic geometry environment. Technology confusion occurred when a student did not understand which tools within the dynamic geometry environment to utilize for mathematical exploration. One aspect that makes these PTMs unique lies in the fact that they do not stem from students' mathematical knowledge or understanding. However, their resolution is important to students' mathematical understanding.

As shown in Table 1, Mrs. Anderson's response to the instances of technology confusion led to a neutral likely impact on student learning. In one instance she repeated the technology directions, and in the second case she acknowledged the student's response, but continued with the lesson as another student nearby provided the directions. When addressing two instances of incorrect technology use, Mrs. Anderson acknowledged but continued and ignored or dismissed the students' responses; however, one instance of a high positive impact occurred when Mrs. Anderson pursued student thinking in response to incorrect technology use. The following exemplar details this interaction.

Pivotal Teaching Moment		Teacher Decision		Likely Impact
Type	Potential	Action	Implementation	
Extend (3)	Significant (1) Moderate (2)	Extend (1); Pursue (1); Ignore (1)	Skillfully (2)	Med Pos (2); Neutral (1)
Incorrect Mathematics (7)	Significant (3) Moderate (4)	Pursue (2); Emphasize Meaning (2); Acknowledge/Cont (3)	Skillfully (4)	High Pos (2); Med Pos (2); Low Pos (1); Neutral (2)
Sense-Making (20)	Significant (13) Moderate (7)	Extend (5); Pursue (2); Emphasize Meaning (11); Acknowledge/Cont (2)	Skillfully (14) Moderately (4)	High Pos (7); Med Pos (8); Low Pos (4); Neutral (1)
Contradiction (2)	Significant (2)	Emphasize Meaning (1); Acknowledge/Cont (1)	Skillfully (1)	Med Pos (1); Neutral (1)
Confusion (5)	Significant (5)	Emphasize Meaning (4); Acknowledge/Cont (1)	Skillfully (1) Moderately (3)	Med Pos (4); Low Pos (1)
Incorrect Technology Use (3)	Significant (1) Moderate (2)	Pursue (1); Acknowledge/Cont (1); Ignore (1)	Skillfully (1)	High Pos (1); Neutral (2)
Technology Confusion (2)	Moderate (2)	Repeat Directions (1); Acknowledge/Cont (1)		Neutral (2)

Table 2: Summary of Pivotal Teaching Moments for Mrs. Anderson

### Incorrect Technology Use

The mathematical task used during this episode focused on the Pythagorean spiral. The teacher was working with the entire class on how to construct the first shape in the spiral, an isosceles triangle.

T: Think back to what you did to construct an isosceles triangle. What tool were you able to use? What tool did you use to construct an isosceles triangle out of this tool box over here on the left?

S: The polygon tool.

T: Ok. Kayla thinks you are going to use the polygon tool. So let's try that real quick, ok. (*Teacher uses the polygon tool to draw what appears to be an isosceles triangle.*) Does that kind of look isosceles?

S: Yea.

T: Yea. Alright, so how would I check to see if it is?

- S: Measure the lengths of the sides.
- T: Ok, so I need to measure the lengths of these two sides and see if they are the same. (*Teacher measures the lengths of the two sides.*) They are pretty close aren't they? (*Teacher drags one of the vertices so the two sides are visibly different and so are the measurements.*) How about now?
- Ss: No.
- T: No, so that is not how you construct an isosceles triangle. You could draw an isosceles triangle that way, where it appeared to be isosceles until you manipulate one of the vertices, but how do you construct it so that it is always going to be an isosceles triangle?

The student responded to Mrs. Anderson's question by identifying an incorrect technology tool to use when constructing an isosceles triangle. Rather than dismissing the student's response, Mrs. Anderson pursued the student's thinking by using the polygon tool and measuring the two sides that appear congruent. She then used the drag test to expose the limitation of the polygon tool to construct an isosceles triangle. This episode was coded as having a high positive impact on student learning because the interaction emphasized the difference between a drawing that appears to be isosceles and a construction that will remain isosceles under the drag test.

The next portion of the paper shares several examples that illustrate Mrs. Anderson's decisions when addressing PTMs.

### **Extends Math/Makes Connections**

In the first episode the class is working on a pre-constructed sketch that contains a parallelogram. Mrs. Anderson is facilitating a whole class discussion/exploration about the properties of a parallelogram by asking questions as a student manipulates the sketch displayed for the class. The definition of a parallelogram is written on the board (previous knowledge), and the class measures all angles and sides of the parallelogram. Students noted that opposite angles and opposite sides are congruent. The figure in the pre-constructed sketch is labeled, and the relationships are noted on the board. The teacher then poses the following question:

- T: Anybody know anything else about a parallelogram?
- S1: They add up to 360 degrees.
- T: All the angles in a quadrilateral, any quadrilateral, not just a parallelogram, but they have to add up to 360 degrees. That is going to help us out when we get to talking about some angles.

In this episode the PTM focuses on making sense of properties of a parallelogram. The teacher chooses to extend the mathematics of the student's response to her question to any quadrilateral, not just a parallelogram. She also foreshadows when this information may become useful in future mathematics. The result of the teacher's decision leads to a high positive impact for student learning. The teacher followed this interaction by asking,

- T: What else is true about the angles of a parallelogram?

S2: You can make 'em into triangles.

T: You can make them into triangles because that's how S1 arrived at the 360 degrees.

Again the student response is focused on making sense of the properties of a parallelogram. The teacher chooses to connect the second student's mathematical response to S1's earlier statement that the angles sum to 360 degrees.

### **Pursues Student Thinking**

In this episode the students are working on a task involving the Pythagorean spiral. The class is working together to construct the initial isosceles unit triangle from which the remainder of the spiral will be constructed. During the episode, Mrs. Anderson skillfully pursues student thinking which results in the students' ability to verbalize how to construct an isosceles triangle using a perpendicular bisector of the base. This episode begins with the following question:

T: What's another tool from our toolbox that we could use to construct it to be isosceles? Think of something you could use to where two of the sides have to be the same.

S: A bisector?

T: Ok, so what do I need to do first?

S: A line segment.

T: Ok. Now what?

S: A perpendicular line.

T: Ok. Where would that that perpendicular line need to be in relation to this segment?

S: At the midpoint.

T: At the midpoint. Ok so we could construct a midpoint. And now you want me to do what?

S: Make a segment.

T: That's perpendicular?

S: Yeah.

T: Now what S1?

S: Add a point to that line.

T: Ok. So I'm gonna add a point to this one. Now what?

S: Go ahead and just connect the two endpoints of the line segment to that point.

This PTM has significant potential and is implemented skillfully as it helps students make sense of an important geometric theorem that any point on a perpendicular bisector is equidistant from the endpoints of the segment that is bisected. The likely impact on student learning is high positive because the teacher was skillfully able to

connect the ideas of midpoint, bisector, and perpendicular to constructing an isosceles triangle in a dynamic environment where she checked the construction by dragging.

### **Emphasizes Meaning**

During this episode students are constructing examples of various geometric theorems in the dynamic geometry environment. In this particular example a student is trying to construct an angle bisector. The teacher asks the student the following question:

- T: Which one are you trying to do?
- S: I am doing the definition of the angle bisector.
- T: Okay. So if its an angle bisector what's the big idea?
- S: Half of that is that.
- T: Definition. That's what you are working on.
- S: So angle bisector's (inaudible) I want to do...
- T: So if it's half what do you think about the theorem? What do you know about the pieces?
- S: Congruent.
- T: Nice. So you want to measure those pieces.

During this interaction, Mrs. Anderson emphasized the “big idea” related to the definition of an angle bisector. While the student did seem to understand that an angle formed by the angle bisector would have measure half of the original angle, he was unsure how to use that information in his illustration. Mrs. Anderson was able to focus the student on the “pieces” formed by an angle bisector and the student was able to state that each was “congruent.” In some ways her emphasis on meaning could be described as “funneling.” It was fairly evident from the questions she posed that there was a particular meaning of angle bisector she wanted the student to describe. When he responded with the word she had in mind, her questions ceased.

### **CONCLUSIONS**

One may hypothesize that when students are using technology in mathematics classrooms there will be instances during the lesson when there is a disruption in the sequence of activities planned by the teacher. One may also believe that such disruptions might be related to the technology itself that distracts from instruction and has a negative impact on student learning. While there was variability in how the likely impact the teachers' actions would have on student learning, none of the PTMs that were identified were coded as having a negative impact on student learning. The types of PTMs that occurred often involved questions from students about how to represent a geometric theorem in the technological environment, or how to construct an object so that it maintained its properties. These disruptions required the teacher to make a decision about how to respond. In the case of Mrs. Anderson, we note that the teacher typically responded in a way that “emphasized mathematical meaning.” Her questions and statements focused students on the “big idea,” definition, theorem, or

mathematical relationship. She was also able to respond to responses from students by connecting them to a related mathematical idea or by asking students to think about the meaning of the mathematical object they were discussing. There were also instances when she pursued student thinking by allowing the student to describe his or her strategy and encouraged the student and class to consider whether this strategy resulted in the desired outcome.

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# GRAPHICAL CONSTRUCTION OF A LOCAL PERSPECTIVE ON DERIVED FUNCTIONS

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*A number of epistemological gaps have been identified in the transition from school to university mathematics. One of these that appears crucial is the inclusion of a local or interval perspective of function along with the point-wise and global ones established at school. In this research we constructed a first year university course on calculus based on integrated use of digital technology with the aim of promoting improved versatility of thinking about functions, especially graphical derivative and antiderivative. The results of this approach show the persistence of symbolic world thinking on the part of a significant minority of students as well as the ability to build interval thinking about function by the majority of the students.*

## BACKGROUND

Mathematics is a field that requires students to develop versatile thinking (Thomas, 2008). There are many aspects to this versatility but one that has been identified in the transition from school calculus to university analysis is the ability to move between point-wise, local and global perspectives of function (Artigue, 2009, Vandebrouck, 2011). It has been claimed by Vandebrouck (2011) that "...working at university level on functions implies that students can adopt a local perspective on functions whereas only point-wise and global perspectives are constructed at the secondary school." (p. 2095). This important aspect of supporting students in construction of a local perspective on functions is considered in this paper.

It appears that if schools emphasise an algebraic approach this may work against the construction of this versatility. Hence, a number of researchers (Berry & Nyman, 2003; Vandebrouck, 2011; Yoon, Thomas, Dreyfus, 2009, 2011) have proposed that working on graphical tasks could enrich students' function perspectives. In particular, this may be one way to stimulate the local perspective needed to understand the fundamental concepts of limit, continuity, differentiability, series expansions, and Riemann integration. In order to achieve this more is required than simply adding a graphical approach to an algebraic one. True representational versatility, which is a part of versatile thinking (Thomas, 2008), includes both the ability to address relationships between representations of the same concept, and perform conceptual and procedural interactions with each representation of a construct.

In this study we consider how students construct versatile understanding of graphical derivatives, examining the outcomes in terms of the theoretical framework employed by Thomas and Stewart (Stewart & Thomas, 2007; Thomas & Stewart, 2011), which combines the Three Worlds of Mathematics (TWM; Tall, 2008) and the actions, processes, objects and schemas (APOS) theory (Dubinsky & McDonald, 2001). In this



framework TWM and APOS are viewed as complementary, in the sense that one may consider the nature of actions, processes, and objects in each of the embodied, symbolic and formal worlds of the TWM. Hence, consideration is given here to analysis of embodied actions, embodied processes, embodied objects, symbolic actions, and symbolic processes, etc. involved in any mathematical activity.

## METHOD

The first-named researcher taught a pre-calculus course at a university in Korea, for 15 weeks, one two-hour session per week. There were 143 students in three classes, although seven withdrew after the mid-term test, leaving 136 students in the final term test. This course is a requirement for students who wish to major in a mathematically related subject and hence student entry grades are mixed. It is generally unpopular, and sometimes avoided for as long as possible, since the students have little interest in mathematics for its own sake. None of the students in the study had used any digital technology before in mathematics, other than a scientific calculator. The course covers linear, quadratic, cubic, exponential and logarithmic functions, differentiation, integration, probability and matrices. It was taught using lecturer demonstration with GSP, Autograph and a TI-Nspire CAS calculator. Due to a lack of available technology the students were not able to use the CAS calculator themselves. However, there were some group discussions on the assignments and exercises involving sketching different functions using Autograph where the students had opportunity to use the graphical software.

Prior to the differentiation section the concept of a linear function was emphasised and it was mentioned that it would be useful for differentiation. Using a graphical approach with Autograph and TI-Nspire the effects of a change in the variables  $a$  and  $b$  in  $y = ax + b$  were explored. This was extended to the concept of transformation parallel to the  $y$ -axis. A similar approach, based on changes of variables in general forms was applied to quadratic functions,  $y = ax^2 + bx + c = a(x - m)^2 + n = a(x - \alpha)(x - \beta)$  enabling observation of transformations parallel to the  $x$  and  $y$  axes. Students were then in a position to analyse for themselves the effect (including transformations) of changes of variables on general functions such as  $y = ae^{x-b} + c$  and  $y = \log(x - a) + b$ .

### The differentiation module

For the differentiation module, the teaching was based on the concept of differentiated learning activities into four levels, as suggested by NCTM (1989), but with the addition of a further Level 5. Level 1 defined a rate of change function  $r(h) = \frac{f(2+h)-f(2)}{h}$  for  $f(x) = x^2$  and used the CAS to generate numeric approximations for  $r$ , with  $h$  from 0.1 to 0.000001, to establish the idea of the limit of  $r$  as  $h \rightarrow 0$ . Level 2 established the symbolic relationships:

$$f'(2) = \lim_{h \rightarrow 0} \frac{(2+h)^2 - 2^2}{h} = \lim_{h \rightarrow 0} \frac{4 + 4h + h^2 - 4}{h} = \lim_{h \rightarrow 0} \frac{4h + h^2}{h} = \lim_{h \rightarrow 0} (4 + h) = 4.$$

In addition, using the CAS calculator the tangent line at  $x=2$  on the graph was drawn to visualise and embody the fact that  $f'(2)$  is equal to the slope of tangent at  $x=2$ . The aim of Level 3 was to generalise to the rate of change process. The CAS calculator was used to introduce students to a method of obtaining the derivative at  $x = a$  by defining a function  $slope(h) = avgRC(f(a), a, h)$ ,  $a = \{-1, 0, 1, 2, 3\}$  as the average rate of change over an interval. In this manner students could investigate numerically the change of slope and, by taking the limit, see that it gets close to  $\{-2, 0, 2, 4, 6\}$ , and hence conjecture that the derivative is  $2x$  (see Figure 1).

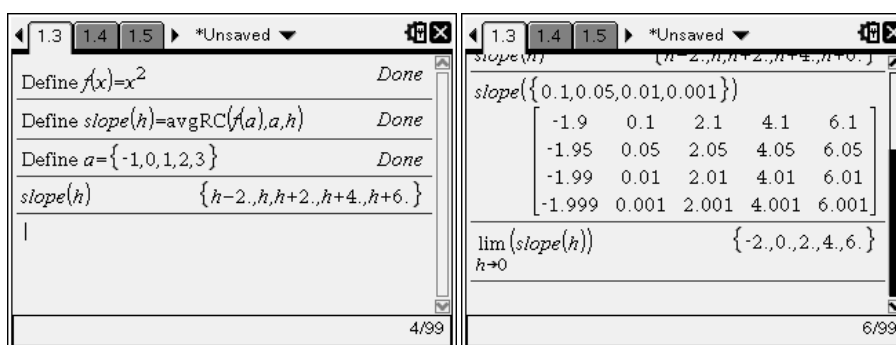


Figure 1: The calculator screens showing the slope calculations.

Linking numeric and symbolic representations to a graph enabled exploration of the relationship between the graph, the slope of the tangent line, and the derivative. When students interacted with the data such as that in Figure 2 they could construct and relate local, interval embodied process ideas, such as if the slope of the tangent line is negative on an interval, the derivative is located under the  $x$ -axis and above if it's positive. Further, they made links to point-wise constructs, such as the turning point is at  $x=0$  since the slope is 0 and the slope is negative on the interval to the left and positive to the right, or vice-versa. Hence, a student could draw the derivative by gaining an embodied process understanding of the variation of the tangent line slope.

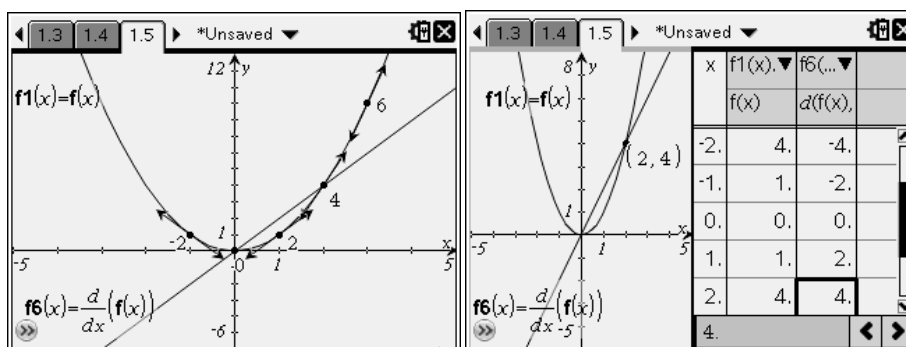


Figure 2: The calculator screens showing the relationship between the slope function and the graphs of  $f$  and  $f'$ .

In this way Level 3 sought to establish links between the graphs and the symbolic process relationships:

$$f'(x) = \frac{d}{dx} f(x) = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} = \lim_{h \rightarrow 0} (2x + h) = 2x.$$

The role of Level 4 was to consider the results above for functions of the form  $f(x) = x^n$  for  $n = \{1, 2, 3, 4, 10\}$ , and hence to generalise the symbolic process and lead students to infer the symbolic object, the derived function  $f'(x) = nx^{n-1}$ . After completing Level 4 students could understand the derivative as the changes in the slope of the tangent line as the  $x$  values are changed, and hence in Level 5 could sketch the derivative without being given an explicit function. For example, with the function shown in Figure 3, students in Level 5 were encouraged to locate the turning points where the gradient of the tangent line is zero, at  $x=0$  and approximately  $x=1.5$ . Based on this they could divide the real line into intervals whose endpoints are the critical numbers 0, 1.5, as shown above, and produce a table of values (see Table 1) of the gradient based on these intervals.

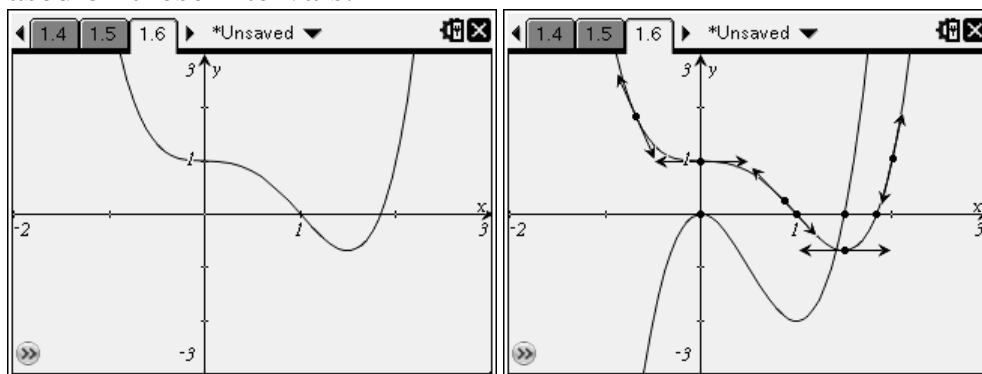


Figure 3: The calculator screens showing construction using interval reasoning of the function  $f'$  for a function  $f$  with no explicit formula.

x	$-\infty$	...	0	...	1	...	1.5	...	$+\infty$
$f(x)$			1		0				
$f'(x)$		-	0	-		-	0	+	
	$-\infty$	$\nearrow$		$\searrow$		$\nearrow$		$\nearrow$	$+\infty$

Table 1: The table using interval reasoning to construct the function  $f'$  for a function  $f$  with no explicit formula.

## RESULTS AND ANALYSIS

The mid-term test required students to sketch the graph of a derived function from a graphical representation, with no explicit algebraic function given. In the final test they were asked to find an antiderivative function in the same way. The derivative question is given in Figure 4. There were a number of different approaches taken to these questions and these form the content of the two cases below.

Sketch the derivative for the given graphs.

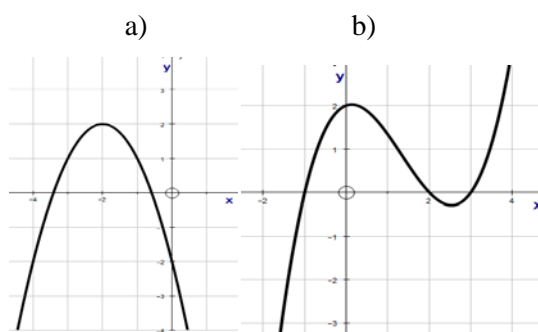


Figure 4: The test questions on graphical derivative.

### Case 1: Symbolic process algebraic thinking

Students whose thinking is dominated by symbolic world thinking initially may find such a question difficult since there is no algebra to work with. The modelling method employed by these symbolic process-oriented students was 1) to assume the graph is a polynomial and determine its order, 2) try to fit it to the general formula for such a polynomial function, using  $y = a(x - b)^2 + c$  or  $y = a(x - b)(x - c)(x - d)$  and information from the given graph to find the parameters and model the function, and 3) differentiate the function obtained and then draw its derived function. In this research 43 students, or 30%, attempted the questions this way. This is typified in the working of students A and B, seen in Figure 5.

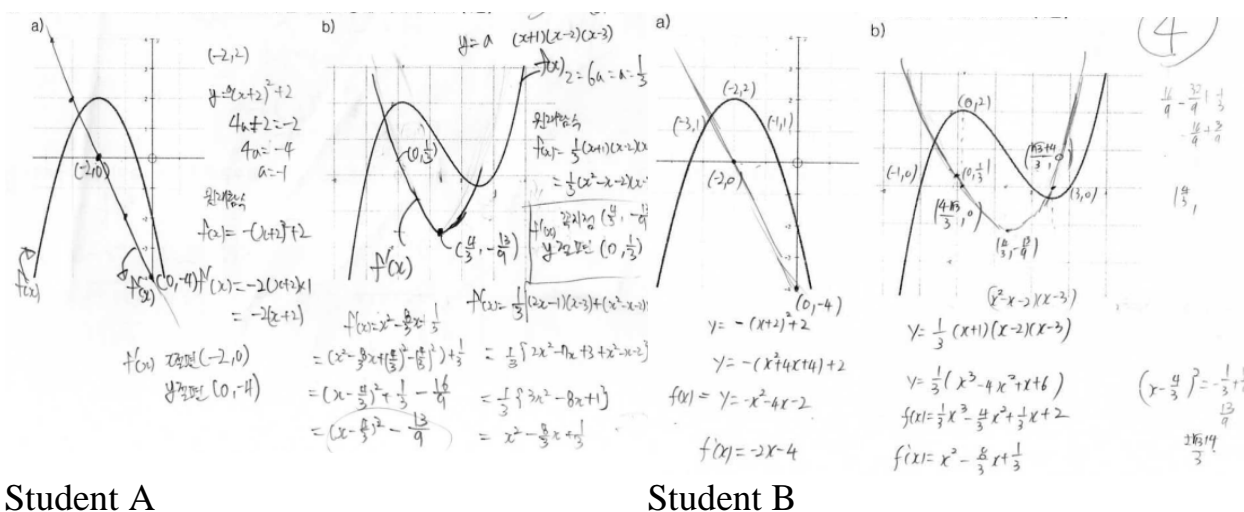


Figure 5: Algebraic solutions of two students to the questions on graphical derivative.

In both cases they have solved problem a) by starting with a function of the form  $y = a(x + 2)^2 + 2$  and using the vertex  $(-2, 2)$  to find the value of  $a$  from the y-intercept  $(0, -2)$ . Once found the function  $y = -(x + 2)^2 + 2$  is then differentiated to give  $f'(x) = -2x - 4$  or  $f'(x) = -2(x + 2)$ . In the second problem b) both defined a function  $y = a(x + 1)(x - 2)(x - 3)$ , then substituted  $(0, 2)$  for  $x$  and  $y$  to find  $a = \frac{1}{3}$ . The function  $y = \frac{1}{3}(x + 1)(x - 2)(x - 3)$  was then differentiated, giving  $y'(x) = x^2 - \frac{8}{3}x + \frac{1}{3} = (x - \frac{4}{3})^2 - \frac{13}{9}$ . Both students gave not only  $f'(x) = x^2 - \frac{8}{3}x + \frac{1}{3}$  but also

completed the square to get  $f'(x) = (x - \frac{4}{3})^2 - \frac{13}{9}$ , enabling them to find the vertex of their parabola.

There is some considerable merit and versatility in the accomplishments of such students; not least a measure of commendable persistence, with all the 43 students successful in their approach. In this method they have to be engaged in a conversion, or translation between representations (or registers), from the graphical to the algebraic in this case, something that Duval (2006) has shown to be difficult for many students. However, we can see that while this symbolic process route requires some point-wise thinking it circumvents the need for local or interval thinking involved in an embodied process when working within the graphical representation.

### Case 2: Embodied process interval thinking

Of the 143 students 80 (55.9%) were able to draw correctly the derived function graphs by a consideration of local or interval thinking (in addition one student used both algebra and interval methods), without requiring an algebraic function. Examples of the working of two students are given in Figure 6. Some of the students made comments such as “if  $f(x)$  is increasing,  $f'(x) > 0$ , if  $f(x)$  is decreasing,  $f'(x) < 0$ , if the gradient of the tangent line is zero on  $f(x)$ ,  $f'(x) = 0$ .”, “If the slope values change from positive to negative, then the values of the derivative change from positive to negative” and “If the slope values change from negative to positive, then the values of the derivative change from negative to positive”.

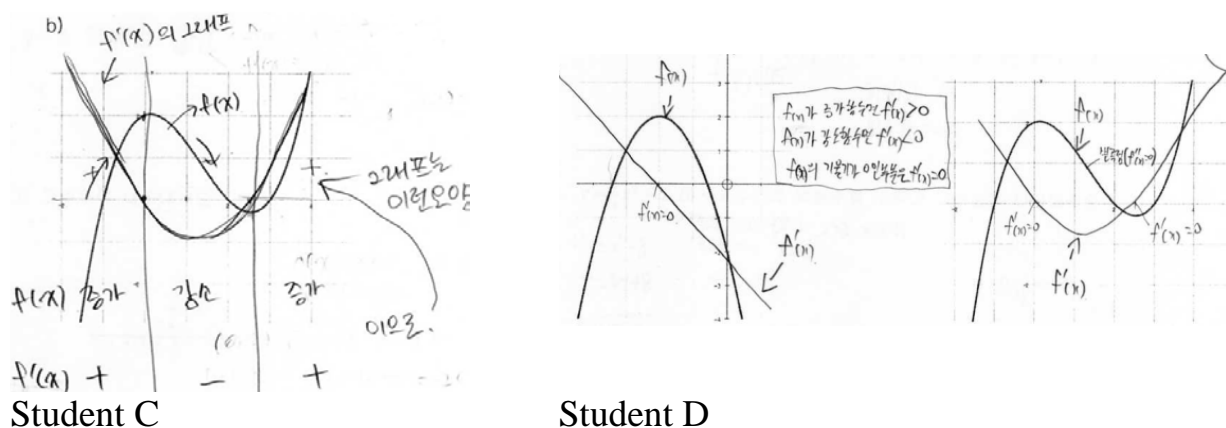


Figure 6: Graphical, interval solutions to the questions on graphical derivative.

These students are thinking in a versatile manner, constructing a method based on principles they have learned. They do not need to work in the world of symbolic algebra thinking but can use embodied process reasoning in their interactions with the graphs of the functions. In addition they do not need to construct the derivative graph in a point-wise fashion, as many students might, but are able to work with intervals. For graph a), having established the point-wise relationship of the stationary point on the parabola, where the gradient is zero, to the point where  $f'(x) = 0$  on the derived function, they can then reason on the intervals  $(-\infty, -k)$ ,  $(-k, \infty)$  to the left and right of this point. Similar reasoning is applied to graph b), where there are two stationary points and hence three intervals to consider. Student D was one of only two students

who also realised that the point of inflection (identified in Figure 6b) corresponded to the greatest negative gradient, and hence the local minimum on the derived function graph.

## CONCLUSIONS

The aim in this study was to design curriculum materials that give an improved cognitive base for a flexible, proceptual understanding of limit, the derivative, integration and other concepts, where the student uses digital technology to develop a more balanced dual view of concepts as process and concept. This accords with Heid, Thomas and Zbiek (2012) who maintain that technology use can help calibrate the balance and interplay of procedural and conceptual knowledge if different concepts are emphasised, concepts studied more deeply, investigations of procedures extended, and increased attention placed on structure. The level method is advantageous in that it engages students in appropriate, staged activities that require them to think through each embodied or symbolic process involved, while the technology allows them to interact actively with what is happening at each level. Thus, rather than a focus on just routine algebraic computations, the learner both knows and understands.

There are some valuable observations arising from this research. First it confirms that an emphasis on symbolic process algebraic thinking in schools produces students with a reliance on this form of working. This thinking strongly persisted for 30% of students, even though the lecture module did not teach this method and the questions did not provide an explicit algebraic function. This is a matter of some concern in the transition from school to university mathematics (Thomas et al., 2012) and didactic situations aimed at avoiding this reliance on symbolic process thinking could be constructed. Secondly, the results imply that a teaching approach, such as that developed here, using digital technology can encourage versatile embodied and inter-representational thinking. Further, developing engagement within the numeric and graphical representations or registers, can support students in the development of local or interval thinking. In this case around 56% of the students were able to apply this kind of strategy to find, correctly, the graphs of the derived functions. Since local or interval thinking is vital in the progression from calculus to the formal world thinking of analysis at university then such a module, supporting a graphical approach to derivative and antiderivative, could usefully be included as part of a multi-faceted strategy to develop it, at school, in bridging courses and in first year university service courses.

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# INVESTIGATING CALCULUS STUDENTS' VERSATILE THINKING: THE CASE OF DEFINITE INTEGRAL

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*This study examined students' versatile thinking regarding the concept of definite integral. Versatile thinking comprised the ability to work within a representation system, transfer seamlessly between the systems of specific concepts, engage in procedural and conceptual interaction with specific representations, the visual images students use to resolve specific problems, and how students handle specific visualizations. Analysis of the interviews and results indicated that coordination between the process concept of the graphic representations and the visual ability of the integral problems is necessary for excellent versatile thinking of definite integral concept.*

## INTRODUCTION

Mathematics education research communities have extensively discussed students' learning and development of mathematical concepts. Topics that have been discussed include the concepts of process and object (Sfard, 1991) and different representations (Goldin, 1998). Students' ability to convert process-objects and representations in mathematical concepts require and involve flexible thinking for mathematical concepts. Flexibility of thinking is essential to the process of mathematical learning. Thomas (2008) proposed versatile thinking, emphasized the effectiveness of cognitive activities, and identified the primary factors that influence problem-solving strategies. Versatile thinking comprises the following three elements: (a) representational versatility; (b) process-object versatility; and (c) visuo-analytic versatility. Although the structure of versatile thinking has recently been applied to mathematical and statistical learning (Graham & Thomas, 2005; Tall & Thomas, 1991; Thomas, 2002), previous studies primarily focused on one or two of the elements. Studies that have investigated the learning and teaching of mathematical concepts by combining the three elements are rare. In this study, we adopted this structure as the theoretical framework to investigate the versatile thinking of calculus students regarding the concept of definite integral.

## THEORETICAL FRAMEWORK

The construction of mathematical concepts is a process that primarily transforms or activates symbols into "do-able" procedures, considering symbols as "think-able" objects. This procedure involves transferring *actions* of known objects and treating these actions as mental objects that can be manipulated. This cycle of mental construction has been described as *action*, *process*, and *object* (Dubinsky, 1991). The chain of events, they suggest, develops as follows. Students acquire the existence of new mathematical concepts through a series of actions before the concepts are abstracted as an object or static structure. Subsequently, students treat mathematical



concepts as an entity when processing the concepts, and no longer consider the particulars. In turn examples of these three link together to form cognitive structures or schemas. Thus, conceptual entities in mathematics often present themselves with two distinct but complementary faces; they may be viewed as dynamic processes or as static objects. To make a mathematical idea readily manipulable and applicable in other contexts, it must be available internally in a concise form and the encapsulation of the process as an object is one way of accomplishing this. Cotrill, Dubinsky, Nichols, Schwingendorf, Thomas, and Vidakovic (1996) emphasized the importance of whether learners could perceive and construct conversions of new objects. They maintained that the process became an object as “the individual becomes aware of the totality of the process, realizes that transformations can act on it, and is able to construct such transformations” (p. 171).

The mathematical objects are symbolized, and this sign may that has become associated with its meaning by usage; it becomes significant simply by virtue of the fact that it will be so interpreted. When signs are thus grouped together, for example in mathematics as integral symbols, or graphs, then we shall call these representations. Thus a representation is a sign associated with a given system of signs. These systems are important since they give a context in which the sign may be interpreted, is an indispensable tool for presenting mathematical concepts, communicating and considering or thinking. Duval (2006) maintained that the process of mathematical thinking required not only the use of representation systems ( registers) but also cognitive integration of representation systems. Based on Duval’s analysis, learning and comprehending mathematics require relatively similar semiotic representations. The way human minds try to understand their world of experience is through the construction of models which lead them toward understanding, or construction of a mental representation of a model. The essence of the concept of definite integral is that the process concept and object concept can be presented by connected but different formats: the graphical representation is typically used in calculations that involve areas under a curve, whereas numerical representations are used for Riemann’s cumulative addition problems. Solving integrals using common integration techniques demonstrates the need for symbolic representations.

Presmeg (1986, p.46) proposes to define a “visual image” as *a mental scheme depicting visual or spatial information, with or without requiring the presence of an object or other external representation*. I consider visual images as the basic element in visualization, and visualization as the kind of reasoning activity based on the use of visual or spatial elements, either mental or physical, performed to solve problem. This mental representation refers to the internal schemata or frames of reference used to interact with the external world.

In this study, I situated my investigation of versatile thinking within the context of definite integral. Specifically, we examined calculus students’ ability to use the relationship between representations to solve integral problems. Based on representations of mathematical objects and process-objects, the versatile thinking structure of the concept of definite integrals is show in Table 1. This structure uses

representations as an intermediary to understand the versatile thinking of students. Students' definite integral concept is divided or placed into two dimensions, understanding and representation, and expressed in matrix format to depict the versatile thinking of students in various representation systems.

Levels	Representations		
	Symbolic	Graphical	Numerical
Action	Can compute integral values using integral formula	Can only calculate areas using symbolic representations	Cannot use numerical approximation to calculate area
Process	Understands the relationship between the integrand and upper and lower limits of integration	Comprehends the relationship between the area above the x-axis and the integral	Can interpret the limiting process of n rectangular area sums
Object	Can interpret and comprehend that a definite integral is an accumulation function	Can interpret and comprehend the relationship between the area and integral	Can interpret and comprehend the limiting process of Riemann sums

Table 1 The versatile thinking structure of the definite integral concept.

## METHODOLOGY

The 18 first-year calculus students who participated in this study were enrolled at a university of technology and had learned the basic rules of integration using primitives, as well as their relationship to the calculation of a number of areas under curves. The instruments used for data collection were a questionnaire containing problems and interviews. The questionnaire comprised seven problems in definite integral (Four of them shown below). These problems enabled the students' performance regarding the coordination of registers and the level of process/object to be analyzed. The results of the questionnaire necessitated further investigation into the versatile mathematics thinking of students.

Task 1. If  $\int_1^3 f(t)dt = 8.6$ , use two strategies to evaluate the value of  $\int_2^4 f(t-1)dt$ .

Task 2. Is it true or false that if  $\int_a^b f(x)dx \geq \int_a^b g(x)dx$ , then  $f(x) \geq g(x)$  for all  $x \in [a, b]$ ?

Justify your answer.

Task 3. If  $\int_1^5 f(x)dx = 10$ , use two strategies to evaluate the value of  $\int_1^5 (f(x) + 2)dx$ .

Task 4. Use two strategies to calculate  $\int_{-3}^3 |x + 2|dx$

All questionnaires and problem-solving drafts were independently analyzed by the study authors and a mathematics scholar after the interviews. To assess and interpret the versatile thinking of the definite integral concept of the students in this study, we constructed the versatile thinking structure of definite integrals, employed this structure to develop clear standards, and used the triad mechanism to describe and then classify the thinking of the students into various levels. The standards were related to the thinking process adopted and presented by the students when problem solving, as

well as their potential to construct relationships among the various representations and properties, and the degree to which they integrated these relationships into their explanation of problem solving. This mechanism divides the versatile thinking of concepts into three levels.

## **RESULTS**

Because students' versatile thinking of the definite integral could be reasonably understood regarding the triad stage, we evaluated the responses to the task interviews, searching for evidence of the non- versatile (NV), local- versatile (LV), and global- versatile (GV) levels.

### **Versatile Thinking of Concept of Definite Integral at the NV Level**

There are nine students categorized into this group. One of the versatile thinking characteristics shared by these students was that they could not recognize the relationship between the area and integral. These students could only process representations within a representation system, and the representations used were influenced by the representation format employed for problems. Finally, they could not reach the process concept stage in the three representation systems. Consider the following excerpt from the interview conducted with Porter, who has a collection of rules that enable him to integrate fundamental functions, such as the integrals in Tasks 3 and 4. However, Porter could not solve these problems using graphical representations. In Task 2, he stated that the proposition was true and provided specific examples of functions  $f(x) = x^2 + 1$  and  $g(x) = x^2$ , and calculated two integrals between 1 and 2 to obtain  $10/3$  for  $f$  and  $7/3$  for  $g$ , without giving graphic representations, failing to provide suitable justifications. Porter was unable to think using graphical representations without algebraic formulae. Therefore, Porter's thinking pattern relied on symbolic but not graphical representation. Students in this group were inclined to rely on analytical thinking instead of visual thinking. They were incapable of visualizing problems. They tend to be cognitively fixed on algorithms and procedures instead of recognizing the advantages of visualizing the tasks; this is a phenomenon that Presmeg(1992) showed that, visualization could be a hindrance for solving a mathematical problem, especially when a mental image of a specific subject controls the student's thinking. In this group students' cases the mental image of standard figure has dominated their thinking when trying to draw a figure to solve problems.

### **Versatile Thinking of Concept of Definite Integral at the LV Level**

I categorized 7 students into this group. These students understood the relationships between representation systems and could change or transfer the representations in some of the representation systems. However, these students had difficulty coordinating these relationships. These students could perform treatments and conversions on the three representations for the procedure concept level. Additionally, these students could also perform treatments and conversions on the three representations for some process concept levels. Helen was one of the students in this group. She could use correct symbolic representations to perform mathematical

thinking and could manipulate the area using graphical representations according to the changes in integral symbols in Tasks 1 and 3. Consider Task 1 for example, Helen assumed that  $F'(t) = f(t)$ , then  $\int_1^3 f(t)dt = F(3) - F(1) = 8.6$ . Consequently,  $\int_2^4 f(t-1)dt = F(3) - F(1) = 8.6$ . Figure 1 shows Helen's graphical representation.

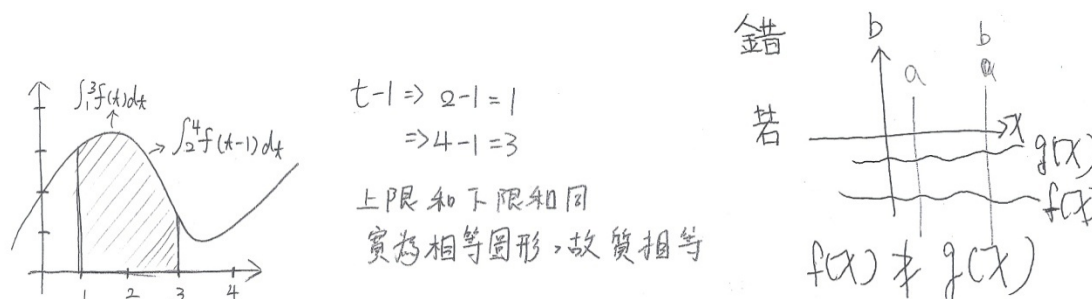


Fig 1. Helen's problem-solving process for Task 1 and Task 2.

However, for Task 2, she says that the proposition is false and gives graphic representations but fails to make suitable justifications.

Interviewer: Can you explain what you think of this task?

Helen: The area enclosed by  $f$ ,  $x = a$ ,  $x = b$ , and the  $x$  axis is greater than the area enclosed by  $g$ , but the function value of  $f$  is smaller than  $g$ .

Interviewer: But the question involves the integral of  $f$  being greater than that of  $g$ .

Helen: The integral value is the area, therefore, a greater integral means a greater area.

Interviewer: Does this have any relevance to the area being above or below the  $x$  axis?

Helen: It is irrelevant to the area being above or below the  $x$  axis.

Similar to Helen, the students in this group could convert symbolic representations and graphical representations, they believed that integral value was the same as area. Although they understood the relationship between the area above the  $x$  axis and the integral, they did not understand the relationship between the area below the  $x$  axis and the integral. The interview results show that students in this group have begun to coordinate the representation systems of definite integrals. These students can perform representation treatments in separate representation systems and generalize, abstract, or interiorize these procedures into processes. Additionally, they can also perform representation conversions in some of the representation systems and interpret the significance of definite integrals in various representation systems. The challenges for these students are that their understanding of the relationship between integral and area has not yet reached the object concept level, and their versatile thinking of the integral concept remains unstable. Students in this group differ from those in the intra stage in that they have developed visual methods to better "see" mathematical concepts and problems. Their visual thinking inclines toward local not global thinking, this restricted visualization actually hinders their solving of the tasks.

## Versatile Thinking of Concept of Definite Integral at the GV Level

Two students were categorized into this group. These students could recognize the relationships among representation systems and convert representations between representation systems. The shared characteristics of these students were that their understanding of numerical, graphical, and symbolic representation systems approached the object concept level, and that they could perform treatments and conversions on the three representations at the procedure and process concept levels. In Tasks 1, 3 and 4, Keven used correct symbolic representations to perform mathematical thinking. He also manipulated the area using graphical representations according to the changes in integral symbols. Consider Task 3 for example, Keven actually employed three methods to solve the problem. The first method was the standard algorithm  $\int_1^5 (f(x) + 2)dx = \int_1^5 f(x)dx + \int_1^5 2dx = 18$ ; the second method was the mean value theorem for integral  $f_{ave} = 1/4 \int_1^5 f(x)dx = 5/2$ ,  $\int_1^5 (f(x) + 2)dx = (f_{ave} + 2) \times (5 - 1) = 18$  and the third method was graphical representation (Fig. 2).

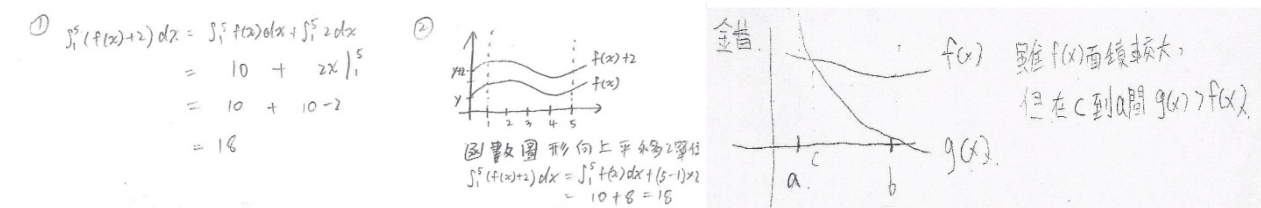


Fig 2. Keven's problem-solving process for Task 3 and Task 2.

For Task 3, Keven stated that the proposition was false and provided graphical representations. Subsequently, the interview progressed as shown below.

Interviewer: If  $f(x)$  is greater than  $g(x)$ , would the integral of  $f(x)$  be greater than the integral of  $g(x)$ ?

Keven: Yes. If  $f(x)$  is greater than  $g(x)$ , the difference of  $f(x)$  minus  $g(x)$  would be greater than 0 and a positive value. The integral of  $f(x)$  minus  $g(x)$  would be a positive value; therefore, the integral of  $f(x)$  would be greater than the integral of  $g(x)$ .

Interviewer: Why do you think Task 3 is incorrect?

Keven: The situation in Task 3 is opposite to that of your question. The integral values of the function in the interval  $[a, b]$  are greater, and the values of the function in the interval  $[a, b]$  are not necessarily greater than that of  $g(x)$ .

Interviewer: Why is that?

Keven: In this graph I drew, the area below  $f$  is greater than the area below  $g$  in the interval  $[a, b]$ ; therefore, the integral of  $f$  in  $[a, b]$  is greater than the integral of  $g$  in  $[a, b]$ . However, the function value of  $f$  in the interval  $[a, c]$  is smaller than the function value of  $g$  in the interval  $[a, c]$ . Therefore, the description in this task is incorrect.

Keven clearly described the process concept of definite integrals, and after encapsulating the process as objects, he also performed seamless representation

treatments and conversions for or based on the objects. The most significant difference between the two students in this group and students in the other groups was that the two students in this group had the ability to perform representation treatments in representation systems, and they could perform representation conversions among various representation systems. This ability may involve or be related to visualization. For Keven, visualization is a powerful tool to explore mathematical problems and to ascribe meaning to the concept of definite integrals and the relationship between them. Additionally, visualization reduces the complexity when considering a significant amount of information.

## CONCLUSION

Versatile thinking involves the capacity to make connections between both mathematical objects and concepts and mathematics and the physical world. The data analysis results show that the main obstacles preventing students from freely shifting within the structure of versatile thinking for the concept of definite integrals were that they had not achieved the process concept for graphical representations of definite integrals, and that this ability involves visualizing the abstracted relationships and non-figural information into visual representations and imagery. Based on the students' thinking performance, we can conclude that visual thinking plays a key role in the development of students' versatile thinking. Visualization is importance for versatile thinking because it promotes versatile thinking and encourages students to consider problems holistically. Aspinwall, Shaw and Presmeg (1997, p.302) quote MacFarlane Smith as saying that "gifted individuals have their own internal 'blackboards' and can visualize complicated structures without being aware that they are doing so". Although not necessarily gifted, the idea of internal blackboard seems to apply to Keven.

From a didactic perspective, the introduction of graphics that illustrate a specific case and a counter-example may focus attention on the key aspects of the representational relationship and render the graphic an integral part of the concept of definite integral. However, adopting a specific method of teaching versatile thinking raises the following questions: What conversion processes are involved in moving versatility among various mathematical representations, including those of a visual nature? How can visual methods be combined in class to improve the versatile thinking of students?

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# DEVELOPING EQUITABLE OPPORTUNITIES FOR PĀSIFIKA STUDENTS TO ENGAGE IN MATHEMATICAL PRACTICES

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*This paper reports on a group of teachers who worked with Pāsifika students to structure an equitable learning environment. The interactional strategies the teachers used to scaffold their students to engage in different mathematical practices are outlined. The use of a Communication and Participation Framework is explained. Explanations are provided of the importance of the teachers drawing on the Pāsifika students' cultural context and using it to engage and connect the students to the mathematics.*

## INTRODUCTION AND BACKGROUND

In recent times a shift in education policies on both a national and international scale has caused changes in the expectations of schooling systems. These shifts increasingly position schooling systems with an expectation that all students within them need to be knowledgeable and able to succeed within a diverse global community (Alton-Lee, 2011). Consistently numeracy is identified as one of the key elements to enable students of the 21<sup>st</sup> century to live within a rapidly changing technological landscape. These expectations require that schooling systems perform better in mathematics education so that all students are supported to engage and achieve higher levels of success; a goal urgently needed for a large percentage of underachieving and disengaged Pāsifika students who have traditionally been marginalised by inequitable schooling practices (Bishop, Berryman, Cavanagh, & Teddy, 2009; Young-Loveridge, 2009). The central focus of this paper is on this group of students and their mathematical development when their participation in the mathematical discourse was addressed. The aim of the paper is to outline the interactional strategies a group of teachers used to equitably reposition their Pāsifika students to access and communicate reasoned mathematical practices.

Notions of equity are a complex and challenging concept within mathematics education. To some, equity in mathematics education is equated as equal opportunities for all to learn through accessing both the mathematics curriculum and qualified teachers; others equate equity with equality of mathematical achievement outcomes across student groups (Foote & Lambert, 2011). However, Gutiérrez (2002) promotes the need to also consider the importance of participation and achievement (as learning). Given the increased emphasis over the past two decades, placed on students communicating their mathematical reasoning equitable participation in the discourse is of prime importance (Hunter, 2007). A body of research (e.g., Civil & Planas, 2004; Hunter & Anthony, 2011; White, 2003) has documented the unequal participation of different groups of students in many discussion-intensive mathematics classrooms. These include reduced participation and contributions, but also often a lack of space



provided for certain students to have a voice in the discussions while others dominate. Considerable evidence is available to illustrate the positive outcomes for students when they are provided with opportunities to participate in reasoned mathematical discourse (Boaler, 2007; Hunter & Anthony, 2011; Lampert, 2001; Wood, Williams, & McNeal, 2006). Hunter & Anthony showed how when teachers attend to the classroom discourse norms more students are empowered to contribute at higher cognitive levels. More specifically, what these researchers highlighted was how when the teachers drew on their Pāsifika students' strengths using pedagogical strategies which built on the core Pāsifika values and provided space which was "culturally, as well as academically and socially responsive (MacFarlane, 2004, p. 61) there was increased participation in mathematical practices and achievement in mathematics.

Mathematical practices evolve through socially constructed interactive discourse. They are specific to and encapsulated within the practice of mathematics (Ball & Bass, 2003) but are not fixed to specific groups of mathematical users, nor are they only invoked in schools. They encompass the mathematical know-how beyond content knowledge which constitutes expertise in learning and using mathematics. Put simply, the term 'practices' refers to the particular things that proficient mathematics learners and users do. Examples of mathematical practices include "justifying claims, using symbolic notation efficiently, defining terms precisely, and making generalizations [or] the way in which skilled mathematics users are able to model a situation to make it easier to understand and to solve problems related to it" (RAND, 2003, p. xviii). In this paper particular focus will be placed on the mathematical practices of explaining, justifying and generalising.

Mathematical practices are constructed and used within situated, social and cultural activity settings. This theoretical position presents a way to understand how teachers, as more knowledgeable members of mathematical communities, socialise students to participate in learning and using the communicative reasoning processes—the mathematical practices—from which deeper mathematical thinking and understanding develops (Wood et al., 2006). The proposal that mathematical practices entail more than what is usually thought of as mathematical knowledge enlarges our view of mathematical learning and requires that we extend our thinking about how to provide more equitable outcomes for students. To develop robust mathematical thinking and reasoning processes, all students need opportunities not only to construct a broad base of conceptual knowledge; they also require ways to build their understanding of mathematical practices; these "ways in which people approach, think about, and work with mathematical tools and ideas" (RAND, p. xviii). However, until recent times the predominant research focus, derived from psychological models, has more generally centred on student construction of cognitive structures. Only recently has research explicitly examined how teachers structure mathematical activity to support student construction and use of mathematical practices. Viewing mathematical learning as embedded within reasoned mathematical discourse offers an alternative way to consider student outcomes. From this position students' opportunities to construct rich mathematical understandings might well be related to the quality or types of classroom

discourse and interactions in which they participate. This provides a different explanation from one that focuses on individual capabilities or the presentation skills of teachers (Lerman, 2001). Theorising that mathematical learning occurs as a result of sustained participation in the reasoned mathematical discourse of mathematical practices is then positioned as a key equity issue.

## **METHOD**

The data presented in the paper were collected in three consecutive studies which spanned six years. In the design research approach (Gravemeijer & Cobb, 2006) used, a Communication and Participation Framework (CPF) (See Hunter & Anthony, 2011) was collaboratively constructed (in the first project) and employed across all the projects. The Framework provided the teachers with a flexible and adaptive tool to map out and reflectively evaluate pathways of pedagogical actions to use, to guide the development of reasoned mathematical practices.

The number of teachers varied in each study; four in Project One and Two and eight in Project Three. They were all experienced teachers. The students were largely of Pāsifika (South Pacific) ethnic groupings, their ages ranged from 8-12 years, and English was spoken as a second language. The study was conducted in New Zealand low income urban primary schools.

Data collection included teacher and student interviews, classroom artefacts, field notes, and a large collection of video recorded lesson observations. A process of analytic induction was used to review the entire corpus of data to identify themes and patterns and generate initial assertions about the interactional strategies the teachers used which successfully scaffolded all their students to engage in mathematical practices equitably.

## **RESULTS AND DISCUSSION**

When the studies commenced, the discourse patterns evident in each classroom were those most often associated with conventional mathematics classrooms, where teacher talk dominates. Although there was variation in the different classroom opportunities provided for different students to contribute, many students offered limited responses or chose to remain silent.

To shift the interaction patterns in the first instance the teachers drew on the Communication and Participation Framework and used it as a tool to map out their immediate and subsequent goals for inducting their students into reasoned mathematical practices. At the same time, consideration was given to how the background of these Pāsifika students could be drawn on in culturally appropriate ways to structure the social norms of the classroom.

### **Constructing mathematical explanations**

A consistent feature of all classrooms was the initial focus on the students constructing mathematical explanations which they could explain in conceptual ways. The students were also required to actively listen and make sense of other's explanations or ask questions if they did not understand. The teachers were aware of some Pāsifika

student's lack of confidence and reluctance to talk and so to strengthen their confidence they provided them with opportunities to practise in pairs or small groups. Likewise, they had them rehearse making mathematical explanations and asking and answering questions in their small groups. Frequently before the students began working in small groups the teachers outlined how they wanted the students to engage in making mathematical explanations. For example, one teacher started with:

Teacher 7: There are different ways to solve problems and you need to use the story in the problem. You can use counters or draw pictures to explain...But you need to get it in your head, practise thinking about each bit for you to explain to the group. So don't be afraid to use pictures, counters to solve the problem because that is our way, we use different ways, that's what we're looking for but make sure your group understands what you're doing. Ask questions, asking a question is very important, don't be shy, and think about being proud for your fono [family] when you speak up. If you're having difficulties and you do not understand, you ask a question like 'Can you explain to me how did we get this? Can you show it to me with the counters or a drawing?' It's very important to keep asking.

Through these means the teacher ensured that the students knew that explainers needed to make explanations as explicit as required by the audience and relevant to the situation. At the same time she ensured that the questions she modelled were framed so that solution strategies were directed toward specific clarification of mathematical explanations. She also indicated that using their voice in the mathematics classroom added to family pride.

Teacher led discussions also ensured that the students could represent explanations in multiple and rich relational ways. Using different materials to explore the mathematical explanations provided ways to explore and re-explore the mathematical reasoning as well as provide them with sufficient evidence to support the explanation. For example, a teacher drew on a Pāsifika context and used it as a metaphor to illustrate the acceptability of different ways of constructing and explaining their reasoning:

Teacher 3: ...it's like when there is a feast and the mamas make puke (a Cook Island food dish) and they all use some of the same things but also different things. Your aunty might use banana but Vavia's mama might use pumpkin but they are still making puke. Your mama might make it quicker too, that's all right some explanation might be quicker than others, when you get your explanation and you might have different bits but when you look at them they still make up your mathematical explanation and how you understand it and others understand it, that is what is important.

Through the on-going discussions expectations were established that mathematical explanations needed to be accessible and understood by the community.

### **Constructing mathematical justification**

Consistently, for all groups of teachers developing the students' confidence and ability to construct and present conceptual mathematical explanations was a process they approached slowly and carefully. When they were sure that they could confidently

explain, the focus shifted from them constructing multiple ways to understand the mathematical reasoning they were using, to them using explanations as a tool for argumentation. The teachers actively positioned the students to take a stance but ensured that they were aware that they needed to be able to justify their stance. For example, one teacher asked:

Teacher 1: Remember our friendly arguing, you need to use it but do it politely. Agree, disagree. Why? Why would you? Anyone disagree? You all agree? OK why do you think that might be right?

Tarai: Cos we're dividing like whole numbers.

Teacher 1: You will need more than that to convince us to agree. What other information can you add either to make us agree or disagree with why or what this group explained? Remember we are all in this together and you and all of us have to be convinced.

As a result explanations became explanatory justification and the explainer learnt that they were required to provide further evidence that supported, refined or refuted the validity of the mathematical concepts in their explanation. In accord with the Communication and Participation Framework the questions modelled by the teachers also shifted from asking 'what did you do' to those which encompassed challenge like 'but why would you'. At the same time, as they scaffolded mathematical justification they continued to ensure that the Pāsifika students were at ease with notions of mathematical argumentation. As illustrated in the example above the teacher promoted politeness norms accepted in Pāsifika communities while also drawing on the core Pāsifika values of reciprocity and collectivism to encourage community responsibility.

### **Structuring classroom communities to work in culturally appropriate ways.**

The Pāsifika students' sense of reciprocity, community, and collectivism were important tools the teachers used. In discussion of how they worked together to construct mathematical explanations and justification they explored notions of whanau (family) and emphasised the strengths inherent in being a collective member and supporting each other. In some instances they drew on the students' family context (for example emphasising that family members take different roles to make a Cook Island haircutting ceremony successful) and used this to illustrate how different mathematical contributions strengthen conceptual understandings. When they observed groups working together they drew attention to their collaborative actions, paralleling these with the actions of a Kapa Haka group (Maori cultural group) who induct, support and challenge members until they achieve similar levels of expertise. One teacher drew attention to similarities in the role of the tuakana (elder brother or sister or cousin in a whanau), linking these to specific individuals she saw taking leadership and actively supporting group members to challenge thinking or to promote development of collective reasoning:

Teacher 2 There's really interesting korero (talk) going on. I really spent most of my time with this group because they were having problems and arguments and Wiremu was really good...you were really good in that position Wiremu,

you were helping your group and you weren't giving out the answers and that's really good but you were pushing them to think. Yes you had everyone talking about and discussing how they were going to sort out the maths ideas. You were challenging and other people were following your lead so the arguing was really kapai (good).

Such explicit discussion promoted the value of mathematical argumentation and made the students recognise how it was used positively in the Pāsifika context. Increasingly, such actions helped the students to see how they could use mathematical argumentation as a tool to develop and deepen their mathematical concepts.

Word problems written within the social and cultural context of the Pāsifika students were also an important tool used by the teachers to promote sense-making and connections. For example, in one classroom the teacher used an open-ended problem which drew on a common Pāsifika scenario (for example, a problem where eis (necklaces) were made from shells and these were divided into groups to make a certain number). This supported the students generating their own numbers at challenging levels and using the context to make sense of the arguments.

### **Constructing mathematical generalisations**

The requirement that the students provide multiple levels of explanatory justification led to increased student recognition and voicing of numerical patterns which the teachers used as position statements to explore and extend numerical connections and patterns. They also introduced the use of a series of problems specifically constructed to support the students making connections from day to day and across problem situations. For example, a teacher has used a more difficult problem which builds on mathematical concepts developed in a previous one. He scaffolds the students to make connections:

Teacher 6: I just want to try and cut in now just to let you know what you can try and do to help you with solving this problem. Now, is this problem similar to a problem you've done before?

Sione: The hula skirt one

Teacher 6: That's right so think about what you did to solve that one.

In accord with the Communication and Participation Framework the teachers asked the students to conjecture solution strategies and then use their 'rethink time' to pattern-see. For example one teacher told the students:

Teacher 3: You are going to pool all your combined collective knowledge to solve these problems. You need to be asking questions. Questions like why did you use that number, how did you do that, is my way more efficient than yours, why is it, why isn't it?

Like this teacher the other teachers both modelled and pressed the students to use questions and prompts to have them compare, evaluate, and make connections between the conjectures. The use of student voiced generalisations also became a consistent part of their practice and often the teachers would structure discussion using specific

questions (would that work if...what happens if...does that work with all numbers...what if...) which they would prompt the students to use. Often the need to provide alternative justification led to validation through use of a generalisation. For example, a group had explained multiplication using a double and halving solution strategy. Their thinking is used to continue the discussion:

Teacher 4: What Ben began with and now Hone is explaining is that he knows that eight times eight equals 64, and then he said minus but he means he halved it. He halved it. He divided it by two, halving it so just take a look at this.

As she spoke she recorded  $8 \times 8 = 64$  then  $4 \times 16 = 64$  on the whiteboard. Then she recorded  $2 \times 4 = 8$  and told the students to explore the relationship by applying the same strategy using different numbers. Similar interventions by different teachers supported the students to explore many generalisations.

## CONCLUSION

At the conclusion of the three studies the teachers had scaffolded their students to engage in a range of mathematical practices. At the beginning of the studies the contention by Civil and Planas (2004) that some students do not have a voice in mathematics classrooms was supported. Through a focus being placed on the student's cultural and social situations, the teachers were able to draw out the Pāsifika student's voice and empower them mathematically in what MacFarlane (2004) termed culturally responsive ways. Similar to the argument by Gutiérrez (2002), evidence from these three iterative studies shows the need to address how students participate in classroom mathematics as a key equity action for Pāsifika students.

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# BEYOND-SCHOOL MATHEMATICAL PROBLEM SOLVING: A CASE OF STUDENTS-WITH-MEDIA

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*This paper addresses mathematical problem solving activity within the context of a web-based beyond-school competition – SUB14. Using a qualitative approach, we aim at finding evidences of the contestants' mathematical competence and technological fluency by analysing four solutions to a particular geometry problem from participants who decided to use GeoGebra. Even though they all make use of the same tool, their approaches to the problem differ in terms of the mathematical and technological fluency they show. We interpret their different ways of dealing with the tool and with mathematical knowledge as instances of students-with-media in problem solving.*

## INTRODUCTION

Several authors have been stressing the need for a deeper understanding of the mathematical activities in which young people engage, particularly in technologically rich environments that can be considered extensions of the school curriculum (Barbeau & Taylor, 2009). The University of Algarve has been promoting a web-based mathematical problem solving competition, addressed to 7<sup>th</sup> and 8<sup>th</sup> graders (12-13 years-old), named SUB14<sup>®</sup>. In this beyond-school and web-based competition there is a mathematical problem published every two weeks that the participants must solve individually or in small teams. Students have to send their solutions electronically, using attachments if they wish so, but those must include a complete and detailed explanation of their reasoning and solving process. Previous results indicate that the SUB14's participants often show sophisticated technological fluency when solving the competition's problems (Jacinto, Carreira, & Amado, 2011), although we know that putting such abilities into practice in the classroom is still rare for most of them.

This study extends the research on understanding mathematical problem solving in a beyond-school technologically rich environment, by characterizing how SUB14's participants reveal their technological fluency and their mathematical competence. Moreover, we aim at understanding how the use of a technological tool, like GeoGebra, supports and shapes four different approaches to a geometry problem.

## THEORETICAL FRAMEWORK

Our conceptual framework is grounded on a sociocultural view of mathematics and draws on the idea that: (i) mathematical competence comprises the ability to use mathematical knowledge, namely, for solving problems; (ii) technology is a powerful mediational means of the mathematical activity, and (iii) technological fluency is expected to be a leverage to face many of the 21<sup>st</sup> century societal challenges.



## **Mathematical knowledge and problem solving**

It is widely accepted that a problem is an intellectually challenging situation for an individual who is willing to solve it, but does not possess an algorithm or a procedure that leads immediately and surely to the answer (Lester, 1983).

In the past years, the Portuguese mathematics curriculum has placed problem solving at the heart of classroom activities, and the current syllabus even puts a renewed and stronger emphasis on this “cross-content skill”, and it acknowledges that improving the ability to solve problems is crucial for the development of other mathematical skills (ME, 2007). Seeing problem solving as the development of a productive way of thinking (Lesh & Zawojewski, 2007) entails a conception of mathematical knowledge that is not reducible to proficiency on facts, rules, techniques, computational skills, theorems, or structures. This conception moves towards broader constructs closer to the notion of mathematical competence (Perrenoud, 1999) and regards problem solving as a source of mathematical knowledge. Considering that mathematical problem solving fosters mathematical thinking (Lesh & Zawojewski, 2007; Schoenfeld, 1992), the solver must adopt a mathematical stance, which impels mathematization, that is, to model, to symbolize, to abstract, to represent and to use mathematical language and tools.

## **Mathematical knowledge under the light of technological fluency**

The impact of digital tools in our society has been a focal point of interest for researchers over the past decades. Changing, reshaping, and affording are some of the keywords that have been recently highlighted to describe and explain such impact. Noss (2001) speaks of the representational transformation as a central feature of post-industrial societies and discusses how computational representations are reshaping the nature of mathematical knowledge. Kaput (1989) had already suggested that the production of mathematical meaning is anchored in the ability to use various representations and stems essentially from making conversions between different representations. Lately, this representational fluency is considered a core competency in the development of mathematical thinking (Lesh & Doerr, 2003), and is acknowledged as a fundamental tool in beyond-school environments, where it mediates decision-making, the interpretation of complex systems, or the use of technologies (Dark, 2003; Lesh, Zawojewski, & Carmona, 2003). While observing that mathematics plays an increasingly significant role in society, Noss (2001) states that some mathematical concepts and processes may be concealed by technological tools. Thus, many authors choose the term *affordance* to define the set of features of a particular technological tool that invite the subject to undertake an action upon it (Artigue, 2007; Noss, 2001).

Researchers have theorised on the representational side of technology-based mathematical activity by looking at the ways students recognise the affordances of the tools to generate mathematical meaning. The semiotic dimension of mathematical knowledge has become more intertwined with the awareness of the mediational role of technological mathematical representations, as semiotic systems are changed by the

introduction of digital technologies. One emergent conclusion is that mathematics and technology cannot be seen as disjoint and the role of technology cannot even be reduced to conversions between representational systems (Artigue & Bardini, 2010).

In the same vein, Borba and Villarreal (2005) argue that the processes mediated by technologies lead to a reorganization of the human mind itself: knowledge is an outcome of a symbiosis of humans and technology – a new entity they named humans-with-media. This concept also discloses a sociocultural perspective of the human mind, in the sense proposed by Wertsch (1991) when assuming that every “action is mediated and (...) cannot be separated from the milieu in which it is carried out” (p. 18). The notion of humans-with-media is supported by two main ideas: (i) cognition has a social and collective nature that (ii) comprises tools which mediate the production of knowledge. The key issue is that media are considered a constitutive part of the subject and cannot be seen as auxiliary or supplementary. The media that are used to communicate, to produce or represent mathematical ideas, influence the kind of mathematics as well as of mathematical thinking that is developed. This means that different collectives of humans-with-media originate different thinking: for instance, the mathematics produced by humans-with-paper-and-pencil is qualitatively different from that produced by humans-with-computers (Borba & Villarreal, 2005).

## RESEARCH METHODS

The broader research project, into which this study is anchored, follows a naturalistic approach, involving qualitative techniques for data collection and analysis (Quivy & Campenhout, 2008). In this particular study, we are looking for evidences of technological fluency and mathematical problem solving fluency of particular, distinctive cases that illustrate variety and do not seek generalization.

Firstly, we gathered all the answers submitted by the 7<sup>th</sup> graders to a particular geometry problem (Figure 1), from the 2011 edition of SUB14. We then selected four productions of participants who have used GeoGebra at some point of their solving process which include their electronic messages and attachments.

**Building a flowerbed**

Rose explained to her gardener that she wanted a triangular area of flowers in her rectangular garden grass. The gardener took a 2-meter stick, held it perpendicularly to one side of the garden, at a random point (E). Then, with a rope, he drew a line through the end of the stick (F) and joining the two opposite sides of the rectangle, thus getting the yellow triangle EGH. On the next day, Rose looked at the triangle and did not like it, moved the same stick to another random point of the garden edge and she got a different triangle EGH.

When the gardener arrived he complained, saying that the area for the new flowerbed was smaller than before. But Rose assured him that it didn't change. Who is right and why?

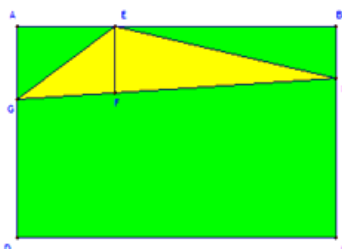


Figure 1 – Problem #6 of the SUB14's 2011 edition

We conducted a descriptive and inductive analysis, considering the theoretical background, specifically aiming at illustrating the features related to the technological fluency and mathematical competence of the participants, namely in terms of the

effective use of a digital tool – GeoGebra – to organize, expand, and sustain mathematical thinking, meaning and knowledge in their problem solving activity.

#### FOUR GEOGEBRA-BASED SOLUTIONS

In this section we analyse four solutions of a geometry problem, all using GeoGebra, to emphasize different mediational aspects that mathematical and technological representations enhance, specifically to: 1) obtain the solution, 2) interpret the solution, 3) confirm the solution, and 4) explore the solution.

##### Using GeoGebra to obtain the solution

Marta and Miguel submitted their solution along with a GeoGebra file (Figure 2). They represented the rectangular lawn as well as the three conditions of the statement: a stick with length 2 (segment FG) is perpendicular to the side AD of the rectangle, and the “rope” (segment JI) passes through the end of the stick, intersecting it at point G. Next, they determined the areas of two triangles, obtained by dividing the triangle FJI through the stick (segment FG). By dragging F they verified that the total area did not change and therefore they concluded that Rose was right.

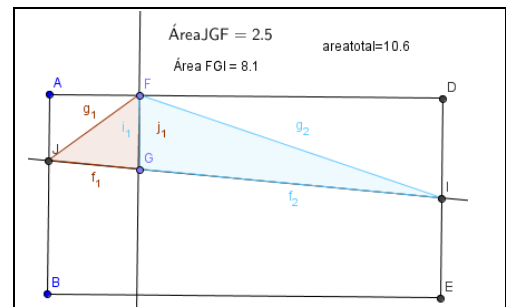


Figure 2 – Marta and Miguel's GeoGebra construction

This solution reveals Marta and Miguel's technological fluency, particularly when handling GeoGebra: they perform constructions that strictly meet the initial conditions and determine areas using the measuring tools. As to their mathematical fluency, and analysing the construction protocol, they seem to be familiar with geometrical concepts such as “perpendicular line” and “parallel line”, “polygon” and “area of a polygon”. Nevertheless, they fail to submit a mathematical reason for the invariance of the areas, which may result from the “certainty” they seem to get from dragging F.

##### Using GeoGebra to interpret the solution

Andreia, Lucas and José also sent a GeoGebra file and a brief text that seeks to validate the conclusion obtained by manipulating their construction (Figure 3). They built a rigorous and robust representation of the garden and the flowerbed, and added two

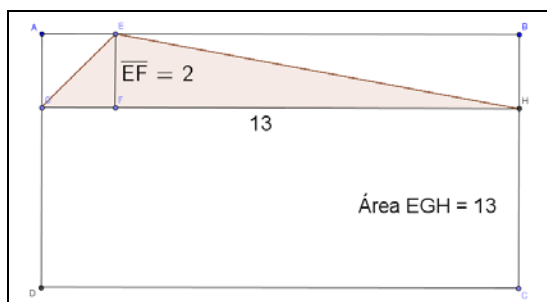


Figure 3 – GeoGebra construction, sent by Andreia, Lucas and José

measures using the tools: the length of the segment GH and the area of the triangle EGH. It seems that the manipulation of the points G and E, the observation of the invariance of the area, and the length of the bottom side of the triangle convince them that the areas do not change, whatever triangle they represent under those conditions.

In written, they try to explain the invariance of the area: “*triangles with the same base and the same height have equal areas*”. This

conclusion arises from the manipulation of vertices “*E and G under the conditions described in the problem*”. However, moving E and G, we observe that the segment GH isn’t always parallel to the side AB of the rectangle but GeoGebra indicates that the length of this segment is always “13”. This may be a consequence of the default rounding – in this case, round to the unit – which, most likely, was overlooked by the participants.

The team revealed its technology fluency in their flexible use of GeoGebra, not only to represent the situation posed, but to obtain the solution and to attempt an interpretation. Although they are somewhat fluent in terms of mathematical knowledge, they seem to be convinced that the length of the segment GH is also invariant, which is not.

### Using GeoGebra to confirm the solution

Sara acknowledged to have felt some difficulty in “*explaining with words*” how she thought about this problem; therefore, she decided to send a screen capture containing her construction in GeoGebra (Figure 4). According to her words, Sara “imagined” that the rectangle had a length of 12 cm and then built a representation of the rectangular garden and the triangular flowerbed, thoroughly following the statement. Therefore she determined the area of the triangle (on the left) and recognized that it matched the length of the rectangle that she initially chose. By making a second construction (on the right) she was already aiming at justifying the earlier result by dividing the flowerbed into two triangles, ONM and OMK. However, Sara explained that the 2m stick corresponded to the base of those smaller triangles, and she represented their heights using two segments,  $a_1$  and  $b_1$ . Finally, she noted that “adding” two segments, i.e., the heights of the smaller triangles, it gives the length of the rectangular garden.

Sara’s technological fluency is quite evident in terms of the effective use of GeoGebra. It is also quite obvious in the diversity of tasks that she was engaged in while solving this problem, as revealed by her desktop’s taskbar: she was also “chatting” online, checking the SUB12’s webpage, and already drafting her answer. Considering mathematical fluency, we highlight the language she used: aside the email limitations regarding symbolic writing, Sara was clearly concerned with making herself clear and she correctly presented formulas and calculations.

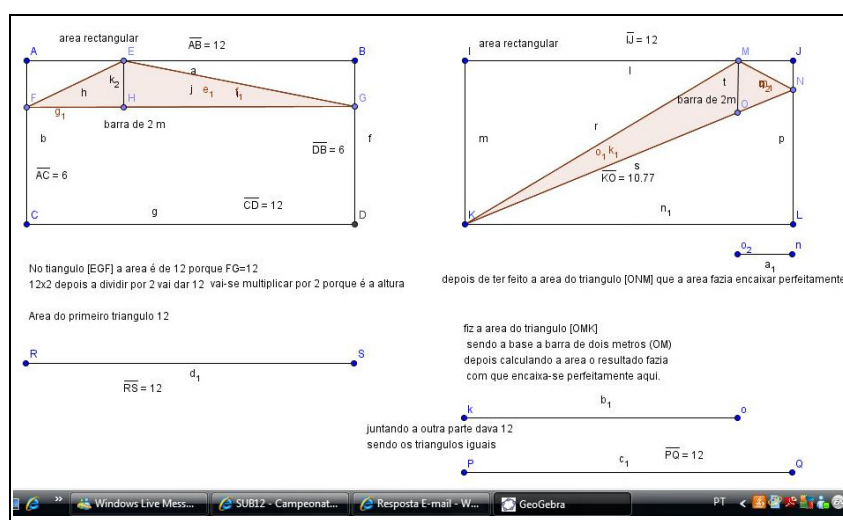


Figure 4 – Screen capture sent by Sara

## The use of GeoGebra to explore the solution

Jessica used GeoGebra to simulate the construction of the rectangular lawn and the triangular flowerbed (Figure 5), but the text that she sent allows a clear understanding of her reasoning. She recognized that the area of the triangular flowerbed equals the value chosen for the length of the rectangle. However, this conclusion arose from the manipulation of the variable "height" of each of the coloured triangles:

The yellow triangle is divided by the 2m stick in two triangles. The base of each triangle measures 2m – the length of the stick. To determine the area of a triangle, we have to calculate:  $\text{height} \times \text{base}/2$ . In order to measure the area of those two triangles, we have:  $\text{height} \times 2/2$ . But it is clear that  $2/2=1$ , so the area of these triangles equals their height.

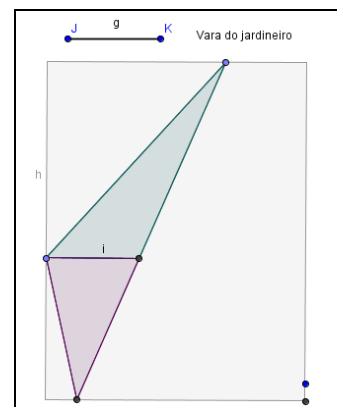


Figure 5 – Jessica's construction

Although Jessica's construction satisfies the three conditions, similarly to the previous solutions, it reveals distinct features in terms of manipulation. Such differences show that Jessica's thinking process is also distinctive: the absence of measurements or calculations stands out, the construction of a slider allows changing the stick's size; and moving the free point on the right side changes the size of the rectangle.

This file reveals Jessica's mathematical and technological fluency in that the GeoGebra construction is built under the perspective of geometrical properties and relations, rather than aiming at measuring or calculating. The quantitative relationship that she explains appears embedded in a geometric representation which is very powerful since it invites at manipulating and therefore generalizing. Adding a slider that controls the length of the stick involves analysing a variable that is not explicit in the statement of the problem; hence Jessica's exploration goes far beyond what was requested to solve the problem.

## DISCUSSION AND CONCLUDING REMARKS

The data presented illustrate the diversity of ways of thinking and modes of action: four groups of solvers, who certainly have very different learning experiences, attend different schools and live in different places, realize and recognize the potential relevance of a single tool, GeoGebra, in solving this problem. These four solutions exemplify the kind of symbiosis described by Borba and Villarreal (2005) since the problem solving strategies and representations they use are revealing of subjects in action with a technological tool; so they can be identified as "students-with-media" or perhaps more accurately as "students-with-GeoGebra".

Still, it is possible to identify common aspects of their problem solving activity: they all represent the rectangular lawn and the triangular flowerbed, they all use "dragging" to check or verify, and they all analyse and conclude. But what each one takes out of that activity is not entirely the same and seems to be closely related to their ability to use, simultaneously, their mathematical competence and their technological fluency.

All participants demonstrated the ability to recognize the affordances of the tool, while their mathematical and technological activity ranged from an elementary and less powerful to an advanced and more sophisticated activity. The data suggest that the differences found are strongly related to the dynamic nature of the mathematical representations afforded by the tool, in depicting the problem conditions. For example, the introduction of additional free elements to the figure led to powerful understandings of the problem, and to generalization. In one production, the invariance of the area is not only numerically recognised but also geometrically explained; in another situation the free elements allow seeing the answer as a particular case of a more general statement; yet another case makes the problem even wider by extending the several conditions stated and allowing the exploration of a more general problem.

The “invisibility” of mathematical ideas is noticeable in the second production. The competitors naively accepted the result given by GeoGebra, and used it for attempting a mathematical justification, without a critical evaluation of such outcome. They lack critical sense in their analysis of the digital representations, which influenced their ability to transform information into knowledge (Noss, 2001).

The link between the solving strategy and the type of GeoGebra usage is clear. In particular, the understanding of the degree of generalisation of the problem and the consciousness of the affordances of the tool to achieve such generalisation are strongly interconnected. These are solid evidences of how the spontaneous use of technology changes and reshapes mathematical problem solving. The spectrum of the problem solutions also highlight the effectiveness of the use of digital tools to structure, support and extend mathematical thinking, meaning and knowledge in students’ problem solving. Further research will focus on studying the mediational role of digital technologies in youngsters’ problem solving activity, in light of what can be called techno-mathematical fluency (Hoyles, Noss, Kent, & Bakker, 2010).

## Acknowledgments

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# MEASURING CONCEPTUAL UNDERSTANDING: THE CASE OF FRACTIONS

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*Developing measures of the quality of understanding of a given mathematical concept has traditionally been a difficult and resource-intensive process. We tested an alternative approach, called Comparative Judgement (CJ), that is based not on psychometric instruments or clinical interviews but collective expertise. Eight mathematics education experts used CJ to assess 25 student responses to a test designed to probe conceptual understanding of fractions. Analysis revealed the CJ assessment process yielded high internal consistency, inter-rater reliability and validity. We discuss the implications of the results for using CJ to measure mathematical understanding in a variety of domains and contexts.*

## CONCEPTUAL UNDERSTANDING OF MATHEMATICS

Many scholars distinguish between conceptual and procedural understanding in mathematics education research (e.g., Hiebert & Lefevre, 1986; Shneider and Stern, 2010; Skemp, 1976). Conceptual understanding is commonly associated with deep, flexible knowledge of underlying abstract principles and procedural understanding is commonly associated with operational knowledge required for stepwise problem solving (Star, 2005). In the research reported here our interest is in the measurement of conceptual understanding.

Traditionally there have been two main approaches to measuring conceptual understanding. The first is to develop and psychometrically validate a bespoke instrument to probe students' understanding of a particular content domain such as calculus (Epstein, 2007) or a particular concept such as equivalence relations (Rittle-Johnson, Matthews, Taylor & McEldoon, 2011). However this has the disadvantage of being a painstaking, resource-intensive process that must be repeated for every concept of interest. The second approach to measuring conceptual understanding, which is sometimes combined with the first, is to record one-to-one clinical interviews and analyse or rate the quality of each participant's understanding (e.g., Knuth, Stephens, McNeil, & Alibali, 2006; Piaget, 1952). However clinical interviews have the disadvantage of requiring skill and consistency on the part of the interviewers and raters, and do not always lead to trustworthy results (Posner & Gertzog, 1982).

The expense, lengthiness and difficulty of measuring conceptual understanding is a barrier to progress in mathematics education. Without valid, reliable and efficient measures it is challenging to evaluate the effectiveness of educational interventions, or to resolve debates in the literature such as whether learning via abstract or concrete representations better aids knowledge transfer (de Bock, Deprez, van Dooren, Roelens & Verschaffel, 2011; Kaminski, Sloutsky & Heckler, 2008). In this paper we report a



study designed to test a novel method to measuring conceptual understanding, called Comparative Judgement, that offers promise for overcoming the drawbacks of traditional methods.

## COMPARATIVE JUDGEMENT (CJ)

The CJ approach to measuring mathematical understanding involves two stages. First participants complete a test question designed to probe their understanding of a particular concept. The test question is likely to be open-ended and allow a wide variety of types of responses from participants. In the study reported here we used a question designed by a teacher for diagnosing teenage students' understanding of fractions, shown in Figure 1.

The second stage of the CJ approach requires mathematics education experts to make pairwise judgements of the quality of the test responses. Each expert is presented with two responses, such as those shown in Figure 2, and asked to decide which is 'better' in terms of a given construct (ties are not allowed), in our case 'conceptual understanding of fractions'. There are no detailed assessment criteria or scoring rubrics and instead each decision is holistic and based solely on the expert's judgement. Many such pairings are presented to several experts and the decisions are statistically modelled (see Methods section) to produce a scaled rank order of test responses from 'worst' to 'best'.

CJ is a well established technique in psychometrics. It derives from the psychological principle that humans are better at comparing two objects against one another than they are at comparing one object against specified criteria (Thurstone, 1927). For example, people are more reliable at stating which of two objects is the heavier than they are at stating how many kilograms a single object weighs. A traditional drawback of CJ is that large numbers of judgements were required to produce a scaled rank order, limiting much past research to the ranking of six or fewer objects. Recent developments in information technology have helped overcome this drawback, enabling the rapid delivery of (virtual) objects for judging by remotely located experts, and making use of algorithms and statistical techniques to reduce the number of judgement decisions required (Pollitt, 2012). Consequently CJ can now be used routinely for educational research (e.g., Kimbell, 2012; Jones & Alcock, 2012) and practice (e.g., Bramley, 2007).

Write down these fractions in order of size from smallest to largest.  
Underneath, describe and explain your method for doing this.

$$\frac{3}{4} \quad \frac{3}{8} \quad \frac{2}{5} \quad \frac{8}{10} \quad \frac{1}{4} \quad \frac{1}{25} \quad \frac{1}{8}$$

Figure 1: Test question for assessing understanding of fractions.

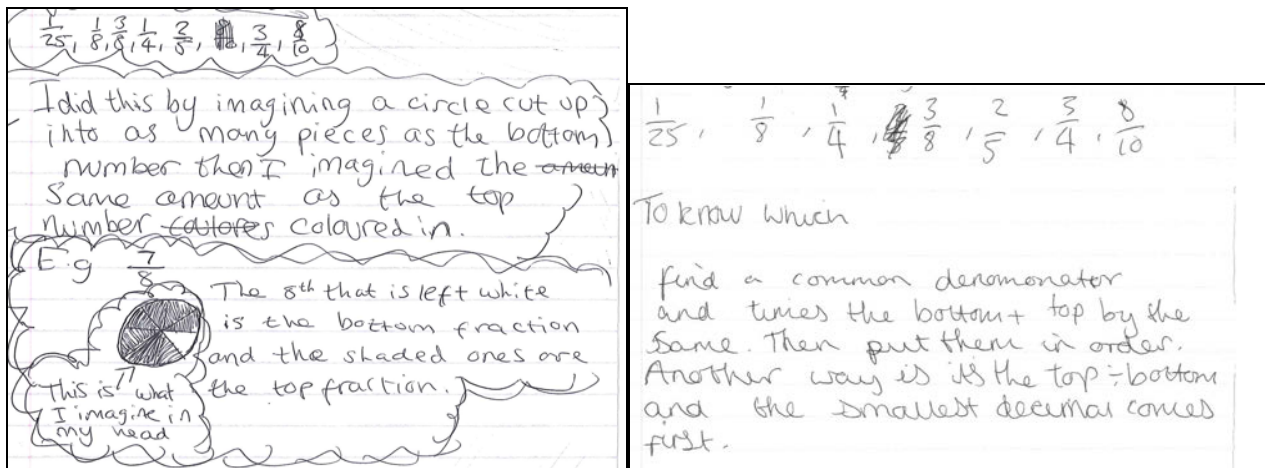


Figure 2: Two example test responses used in the study.

The theoretical strength of the CJ approach is its reliance on collective expertise in the absence of assessment criteria and scoring rubrics. In this sense validity can be thought of as defined in terms of what the experts collectively consider the construct to be in practice. This makes CJ particularly promising for assessing constructs that are recognised and considered important by education experts, such as problem solving and conceptual understanding, but difficult if not impossible to specify comprehensively in rubrics (Laming, 2004). Moreover, constructs such as conceptual understanding lend themselves to open-ended test questions (e.g. Figure 1) that evoke a wide variety of responses (e.g. Figure 2). This variety of responses is difficult to anticipate in rubrics, but is well suited to holistic pairwise judgement by experts. Readers may wish to try judging which of the two responses in Figure 2 they consider to be 'better' in terms of conceptual understanding of fractions.

The practical motivation for studying the use of CJ for assessing conceptual understanding is its potential efficiency. Unlike painstakingly developed psychometric instruments, CJ can be rapidly applied to any target concept with little effort beyond recruiting judges with the requisite expertise. Unlike clinical interviews CJ exploits the long-established psychological principle of pairwise comparisons and yields high validity and reliability with minimal training (e.g., Jones, Swan & Pollitt, submitted).

## THE STUDY

In the remainder of the paper we report a feasibility study into using CJ to measure conceptual understanding of mathematics. Eight mathematics education experts comparatively judged the responses of 25 secondary students to the question shown in Figure 1. The experts' decisions were used to construct a scaled rank order of students' responses from 'weakest' to 'strongest' conceptual understanding. Our research goal was to evaluate the method's internal consistency, inter-rater reliability and validity. We conclude the paper by discussing the implications of the findings for measuring conceptual understanding.

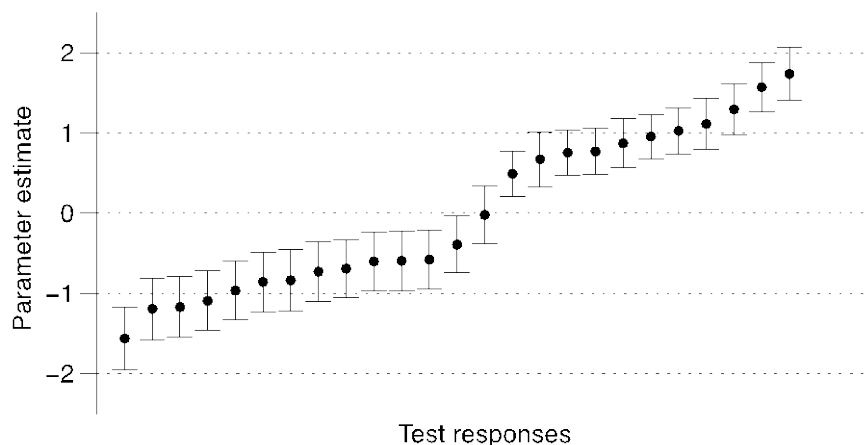


Figure 3: Scaled rank order of test responses from ‘worst’ (left-most) to ‘best’ (right-most), with standard errors for the parameter estimation for each response.

## METHOD

*Materials.* The test question used for the CJ exercise is shown in Figure 1. The question was designed by a mathematics teacher in a school in central England who used it with her classes for diagnostic purposes. For each class the teacher wrote the question on the board and allowed them around ten minutes to complete it. We obtained 25 responses from children aged 12 to 15 years for the study, including the two examples shown in Figure 2.

*Participants.* Eight mathematics education experts (four teachers, two examiners, two research students who were both former teachers) were recruited for the CJ exercise. All had experience of using CJ to assess mathematics from their involvement in previous projects and none required training.

*System.* Pairs of scripts were delivered to the participants online by TAG Development’s *e-scape* system (Derrick, 2012), which uses an adaptive algorithm (Pollitt, 2012) to select scripts in order to minimise the number of judgements required to construct a stable rank order.

*Procedure.* Each participant completed 20 practice judgements then 50 live judgements online within a two-week window. Only the 400 live judgements (8 participants  $\times$  50 judgements) were used in the analysis. For each judgement the *e-scape* system recorded which participant made the judgement, which two scripts were presented, and which script the judge preferred.

## ANALYSIS

*Rank order.* The 400 judgement decisions made by the expert participants were fitted to the Bradley-Terry model using a maximum likelihood estimation procedure (Firth, 2005). This produced a  $z$ -score parameter and standard error for each test response, shown in Figure 3. We then assessed these parameters in terms of internal consistency, inter-rater reliability and validity.

*Internal consistency.* We conducted three checks on the internal consistency of the rank order (Pollitt, 2012). First we calculated the Rasch Separation Reliability

Coefficient, often considered directly analogous to Cronbach's alpha for traditional test items (Bond & Fox, 2008), and found it was acceptably high (.88). Next we calculated and standardised judge 'misfit' figures, which give an estimate of the consistency of a given judge's decisions with the final rank order. A common guideline is to consider those judges whose misfit figures are less than two standard deviations above the mean (i.e.  $z < 2$ ) to be performing consistently. We found that the judges' misfit figures were all well within two standard deviations of the mean suggesting the judges performed consistently. Similarly we calculated test response misfit figures to provide an estimate of how consistently each response was judged relative to the final rank order. The scripts' misfit figures were all well within two standard deviations of the mean, bar one response that was marginally above the threshold ( $z = 2.07$ ). Taken together the Rasch Separation Reliability Coefficient, judge misfit figures and response misfit figures indicate that the final scaled rank order was internally consistent.

*Inter-rater reliability.* Inter-rater reliability measures the extent to which the same rank order would have been produced by a different group of expert judges drawn from the same population. To measure inter-rater reliability we split the eight judges into two groups of four and used their judgements (200 per group) to construct two new separate scaled rank orders. We then calculated Pearson's product-moment correlation coefficient between the two sets of estimated parameters. We repeated this process 36 times, once for every possible unique split of the eight judges into two groups of four, to produce 36 estimates of inter-rater reliability. We found that the correlation coefficients ranged from  $r = .79$  to  $.95$  and the mean was  $r = .87$ , suggesting an acceptably high inter-rater reliability.

*Validity.* We explored the validity of the CJ assessment process in terms of the correlation of the outcomes with measures of students' general mathematical achievement, and their performance on the fractions task measured in purely procedural terms (the extent to which they correctly ordered the fractions).

General mathematical achievement was measured by teacher estimates of ability. For the fifteen oldest (13-15 years old) students in the study predicted grades for the terminal mathematics examination in England (GCSE) were available. These ranged from A\* (highest) to F (lowest). For the ten youngest students (12-13 years old) we obtained a dichotomous (high/low) assessment of their ability from their class teacher.

Procedural performance on the task was assessed by calculating the difference between the correct ordering and that given by each student using the Levenshtein distance metric. This is a calculation of the number of steps required to correct a sequence (Levenshtein, 1966). This produced a score for each student's ordered fractions that ranged from 0 (fractions correctly ordered) to 7 (fractions very out of order).

If our CJ approach was measuring something beyond procedural understanding of the fractions task, then we would expect the teachers' assessments of students' mathematical achievement to be better predictors of the CJ parameters than the procedural accuracy scores. To investigate this we conducted multiple regression

analyses predicting CJ parameters with general mathematical achievement and procedural accuracy scores. In view of the differing measures of general mathematical achievement, this analysis was conducted separately for the older and younger students.

For the 15 oldest students we found that the two predictors explained 53% of the variance in the parameter estimates,  $R^2 = .53$ ,  $F(2, 12) = 6.69$ ,  $p = .011$ . Mathematical achievement (predicted grade A\* to F) significantly predicted parameter estimates,  $\beta = .40$ ,  $t(12) = 2.64$ ,  $p = .022$ , but Levenshtein distance was not a significant predictor,  $\beta = -.07$ ,  $t(12) = -.519$ ,  $p = .613$ . Similarly, for the ten youngest students we found that the two predictors explained 68% of the variance in the parameter estimates,  $R^2 = .68$ ,  $F(2, 7) = 7.33$ ,  $p = .019$ . Mathematical achievement (high or low) significantly predicted parameter estimates,  $\beta = 1.38$ ,  $t(7) = 3.83$ ,  $p = .006$ , but as with the older children Levenshtein distance was not a significant predictor,  $\beta = -.23$ ,  $t(7) = -1.84$ ,  $p = .108$ .

In sum, for both groups of students we found that teachers' assessments of mathematical achievement were better predictors of students' CJ parameters than was a measure of the procedural accuracy on the same task. This provides some evidence to indicate that the CJ method was measuring something other than procedural understanding, and the relationship with general mathematical achievement is consistent with the suggestion that it was measuring conceptual understanding.

## DISCUSSION

We tested an approach to measuring conceptual understanding based on the collated holistic judgement of experts. Traditionally the subjectivity of holistic judgement leads to poor psychometric properties compared to methods based on objective scoring rubrics (Laming, 2004). However the CJ approach reported here yielded high internal consistency (Rasch Separation Reliability Coefficient = .88), high inter-rater reliability ( $r = .87$ ) and high validity in terms of independent student achievement data ( $r = .72$ ). We believe these acceptable psychometrics arising from subjective assessment decisions were due to the underlying principle that people are much more reliable at comparing one object against another than they are at rating an object isolation.

The strong association found between the CJ outcomes and teachers' assessment of students' general mathematical achievement suggests that the experts assessed *mathematics* as opposed to surface features such as neatness or length of response. However, can we be confident that the method measured *conceptual* rather than *procedural* understanding of mathematics? For insight we turn to how accurately the students ordered the fractions in their responses as scored by the Levenshtein distance. The Levenshtein distance can be considered a measure of the procedural knowledge required to complete this task accurately. We found that teachers' assessments of students' mathematical achievement were a predictor of CJ assessment outcomes but the Levenshtein distances were not. Therefore it seems the CJ method produced measures more closely related to general mathematical ability than to specific performance on the fraction ordering test.

Nevertheless, to claim that the CJ process reported here measured conceptual understanding would be subject to two criticisms. First, the distinctiveness of conceptual and procedural knowledge, and the realism of operationalising them independently, has been questioned (e.g., Schneider & Stern, 2010; Star, 2005). For example, the procedures used by students to order the fractions are likely to be strongly influenced by how (and how well) the students conceived the underlying abstract principles of fractions. Second, the student achievement data used to establish the validity of the CJ method provide a general measure of mathematical achievement because they are based on the entire school mathematics curriculum, not just fractions. However teachers commonly use evidence to generate student achievement data that is subject to criticism by the mathematics education community and august bodies for being highly procedural, such as past papers from GCSE exams (e.g. Ofsted, 2008).

Therefore some caution must be exercised in claiming we have demonstrated the measurement of conceptual understanding of fractions. Further work is required to establish the extent to which CJ may offer a method that can be used routinely in mathematics education research and practice. A next step will be to validate CJ for the case of domains and concepts for which psychometrically validated instruments and methods for measuring conceptual understanding already exist.

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# SENSUOUS EXPERIENCE AND MATHEMATICAL CONCEPTIONS

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*Much research based on embodiment theory suggests that mathematical conceptions are embodied, in other words, formed through bodily experiences and exhibit particular physiognomy. The goal of this paper is to propose a basic framework for analyzing the relationship between sensuous cognition and mathematical conceptions, and to imply the importance of multimodal bodily actions for forming mathematical conceptions in space geometry learning through embodiment theory. As part of qualitative data collected from a seventh-grade classroom, an exemplary episode shows that the collaboration of sight and touch is essential for mathematical conceptions and that gestures play roles of orientation and confirmation during a mathematical inquiry.*

## INTRODUCTION

The theory of three worlds of mathematics by Tall (2008) implies that various aspects of human thinking appear during the thinking-development process. For example, in the shift from the embodied-conceptual to the axiomatic-formal world, it might be dramatic for students to change the role of a mathematical definition from expressing to generating objects. This change might be explained as the shift of a domain of mathematical thinking from embodied to collective and cultural. Some research shows that some students reason without referring to definitions. It seems that they participate in communication with particular norms as if they live in the world different from the cultural, formal mathematical world. We might be able to describe those aspects of human thinking as levels of a language system, as does van Hiele's theory, or understand the aspects as something non-linguistic, that is, as a concept image (Tall & Vinner, 1981). From the latter standpoint, this paper aims to describe aspects of students' mathematical conceptions of space geometry, referring to embodiment theory (Lakoff & Núñez, 2000; Radford, et al., 2009; Varera, et al., 1991). The paper focuses especially on the relationship between sensuous cognition (Radford, 2009) of material/mathematical objects and mathematical conceptions.

Embodiment theory breaks away from the dualism of subjectivity and objectivity, deeming that cognition is embodied through sensuous experiences. From this theoretical standpoint, cognitive devices such as point of view, (separately focused on in past research), for example, are interpreted as multi-structured. Multi-structure includes not only the appearance of an object but also the senses of sight and touch, gestures as bodily actions, and a way of understanding of concepts, i.e., "concept-ness", metaphorically termed *physiognomy* later in this paper. Also suggested is that integration of these factors, i.e., added collective and cultural (mathematical) viewing, is essential for mathematical conceptions and meaningful inquiries.



## THEORETICAL BACKGROUND

### Embodiment Theory and Sensuous Cognition

The enactive view argued in Varela et al. (1991), based on the principle that cognition is enacted through embodied actions, suggests that the human mind is embodied within human culture. The basis for mathematical conceptions is found in our embodiment as physical beings and in the manner through which we engage in cultural-social practices. Various bodily experiences and sensuous modalities constitute integral parts of our cognitive process (Radford et al., 2009).

In embodiment theory, meaning and cognition seem to be rooted in physical existence, at least on the phylogenetic, ontogenetic, and microgenetic levels (Edwards, 2011). Therefore, cognition emerges from our bodily actions, in particular actions with the senses as a filter, as well as from the surrounding environment, including others, since individual cognitive subjects exist in historical-cultural environments.

From the anthropological, social-cultural view, Radford (2009) suggests that the nature of human cognition is sensuous. This view holds that the senses, such as sight and touch, form cognition by a kind of plasticity and collaboration with each other. This compensates for the lack of sensory specialization in humans, as other animals have stronger sensory specialization. An implication of considering cognition as sensuous is not only interpreting bodily actions, including gestures, as an incarnation of thinking, but also identifying possibilities that bodily actions create or develop advanced cognitive actions (Yoon et al., 2011).

### Mathematically, How Do Students Know?

Maheux et al. (2009) proposes that for recognizing Fig. 1 (left) as a three-dimensional cube, it is necessary to make a connection between the experience of looking at the figure and the conception of a cube, primarily formed through the bodily experiences of seeing and touching material objects. Needless to say, a figure in mathematics learning is a kind of realization of conceptions with some rules. For example, parallel edges of a solid should be drawn parallel to each other. And thus, selecting information from a conception is implemented intentionally according to an aim of learning. Therefore, cognition of a figure as a cube, and not just as a collection of segments, enhances the understanding of conceptions. Cognition of a figure in terms of conceptions of a cube as “perspective” makes mathematical inquiries possible under the rules of a figure. In this sense, the cognition of a concrete figure and the conception of an abstraction are reciprocally constructive. This state is expressed as “the abstract|concrete dialectic” (ibid, p.73).

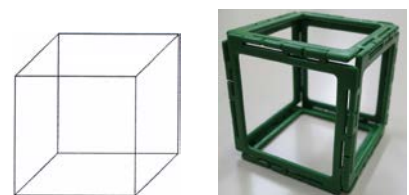


Figure 1: Are these not cubes?

Then, what does make this dialectic possible? And, moreover, develop mathematical knowing? What roles do bodily experiences play as important factors in thinking?

First of all, figures in learning geometry reproduce visual impressions of a concept, but are not strict realizations of what we see. Considering this fact, Fig. 1 (right) is an *appearance* of an object from a particular *viewpoint*, and therefore, the *appearance* of the object changes according to change in *viewpoint*. For example, when a *viewpoint* is exactly perpendicular to a face of a cube, the *appearance* of the cube must be a single square. But, of course, nothing but the position of the *viewpoint* makes the cube appear to be a square. Cognition, as described above, might be possible through a collaboration of the senses such as sight and touch, because material/mathematical knowledge acquired through experiments based on sensuous actions is necessary to change the *viewpoint* and *appearance*.

In contrast, Fig. 1 (left) is a kind of *physiognomy* of a concept from particular *perspective*, that is, a realization of cube-ness. Shape, number of faces, positional relations of edges, and so on are the concrete instances of a *perspective* in understanding polyhedrons and might cultivate polyhedron-ness. Recognition of Fig. 1 (left) as a three-dimensional cube is not just a result of making a connection: Students have formed a conception of a cube to some extent and are asked for ways of knowing that a cube has squares and all opposite faces are parallel to each other. In mathematics, as a famous mathematician H. Poincaré states, it is because of a *perspective* that we can give the same name to different things, and setting a *perspective* is the first step in making categories for forming concepts. In contrast, a *physiognomy* is a sensuous, primary conception, which could consciously or unconsciously control classifying, ordering, and making use of concepts. On one hand, a *physiognomy* of any concept could not exist without bodily/culturally knowing; on the other hand, it is because of the existence of a *physiognomy* that allows dealing with a concept without referring to a definition.

Thus, this paper proposes the ontological coupling <viewpoint, appearance> of material/mathematical objects and the epistemological coupling <perspective, physiognomy> of subjective cognition the basic framework for analyzing the relationship between sensuous cognition and mathematical conceptions.

## METHODOLOGY

We implemented mathematical lessons with 36 seventh-graders in a classroom, with the collaboration of a teacher who had 15 years of teaching experience (Table 1). The lesson plans were proposed by the author and revised by the teacher. These lessons aimed to improve students' spatial sense and reasoning through activities such as manipulating, observing, constructing, drawing, and making mathematical sentences. Each student could manipulate plastic objects such as regular triangles, squares and pentagons, and construct polyhedrons by combining these fundamental figures. The teacher and students could show their manipulations and drawings on the overhead-projector screen at the front of the classroom.

Lesson	Learning Contents	Main Classroom Activities
1	Central projection and parallel projection	Inquiry into the shapes of shadows of various polyhedrons and their changes Manipulation and construction of concrete objects
2	Parallel projection and the properties concerning invariants	Inquiry into the relationship between a direction and position of solids through parallel projection
3-4	A projection chart (1) (2)	Inquiry into shapes of shadows of a solid from various viewpoints Drawing a projection chart of a given solid
5	A projection chart(3)	Inquiry into the relationship between a polyhedron and its projection chart Drawing a projection chart according to given conditions

Table 1: Mathematical Lessons

The entire unit was recorded by the author using one video camera at the back of the classroom. During whole-class activities, the video camera was focused on speakers in order to record all utterances and actions, and the author took field notes as an observer. During group activities, the video camera followed and focused on actions by students in each group. In addition to these data, we collected the students' individual worksheets.

The students' ideas and ways of thinking, either recorded or noted in the classroom, were interpreted with the collaboration of a teacher after each lesson by referring to utterances and written data. We then made transcripts of the video data, analyzed them in terms of the students' activities, and discussed the results with the teacher. Because this study focused on students' mathematical reasoning and conceptions in challenging given tasks, non-participant observation was conducted.

The discussion here refers specifically to the exemplary classroom episode extracted from the collected data because the goal of this paper is to describe and analyze students' actions and utterances in detail through the lens of the two-coupling framework.

### EXEMPLARY CLASSROOM EPISODE

During the first lesson, which was the initial part of the whole-class activity, students found that solids other than a cube—in this lesson, a rectangle and column—could produce the shadow shape of a square through parallel projection. Parallel projection was explained as lighting by rays parallel to each other, directed on a plane by the teacher. This episode describes whole-class and group activities after students were

asked by the teacher whether a triangular pyramid belongs to the group of polyhedrons whose shadow shape is a square.

With four colored plastic frames of the same color shaped in the form of equilateral triangles, each student constructed a tetrahedron by connecting the edges. At first, students adjusted the direction of the tetrahedron so that they could see two edges on a skewed position as crossing, by positioning its faces or vertexes in the front or rotating it based on any edge as an axis. The utterance “I can see only triangles” was predominant during the initial part of the whole-class activity, but students inquired steadily into the possibility of seeing a square. What follows is the transcription of a discussion among four students in a group:

- 1 S1: I can't.
- 2 S2: (*positioning the front edge horizontally and rotating a tetrahedron based on the front edge as an axis*) (Fig. 2)
- 3 S2: (*tracing edges on the front and far side with her fingers*) But these diagonals are equal, aren't they?
- 4 S2: (*placing the square-shaped corner of the ruler on angles of the ends of the solid formed by two faces, including the horizontal edge*) This is now possible.
- 5 S3: (*pointing at some angles of faces of a tetrahedron*) These are less than 90 degrees, since this is 60 degrees.
- 6 S2: (*showing S3 a tetrahedron by keeping an edge horizontal*) But, while we manipulate like this, (*gesturing with a look that parallel rays go on a tetrahedron, and then pointing at the parts she was putting the ruler on*) inquiring into whether this side and another add up to 60 degrees, do they not, do they?



Figure 2: S2's dynamic bodily action



Figure 3: A student's continuous bodily actions

Through exchanging information with the teacher after a lesson, most students who thought that the shape could not be a square reasoned that they could not construct a square from six equal edges of a tetrahedron. As utterances 1 and 5 by S1 imply, some students reason that they cannot find a right angle besides the angles of any two edges of a tetrahedron. S2 continues manipulating continuously as in Fig. 2 until utterance 3 appears, and then pays attention to the positional relation of two edges as “a diagonal” and a shape formed by four other edges.

After S2 stated utterance 3, other students in this group repeatedly traced, with their fingers, two edges in a skewed position, and looked at a tetrahedron closely or distantly. After utterance 6 by S2, touching (just like enfolding) two diagonals and four edges as an outline of a square and changing the direction and position of the tetrahedron by paying attention to *appearance*, they tried to form any positive evidence to show that other edges could form a square (Fig. 3).

During a whole-class activity following the group discussions above, almost all the students seemed to recognize that a direction must exist that would enable them to see a square. A student in another group stated the following as a justification for seeing a square:

7 S4: If the angle is just right.

8 T: How can you be sure that this is exactly a square?

9 S4: (*positioning an edge on the far side so that it is perpendicular to the horizontal edge on the front side and passes through the midpoint of the front edge*) Well, about this, seeing this from the front side, no, not the front. How can I say it?

10 S4: (*attaching his fingers to edges on the front and far sides, as in Fig. 4*) When rotating, the lateral and lengthwise diagonals are, well, because all edges of this solid are equal, that means all diagonals are equal, well, besides, because all edges of a square are equal [...]

11 T: He says that these two edges are equal, but everyone, look at this. (*showing the same polyhedron to the class*) Is it really that the lengths look like equal?

12 T: Someone is saying that these are different.

13 S4: The edges are equal because of parallel projection.

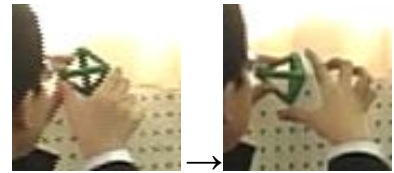


Figure 4: A series of actions for justification

As utterance 11 by the teacher implies, the equality of edges, concentrated on by S2 and S4, was the theme of the whole-class discussion. After utterance 13 by S4, the teacher additionally explained that human sight is rooted in central projection and therefore, perception of depth necessarily appeared. The teacher then verified on the screen that the shape of a shadow sometimes became a square through manipulation of a tetrahedron. During the second to fourth lessons, classroom discussions led to inquiries into invariants (for example, parallel relation of straight lines) and variants (for example, length of edges according to direction). The conclusion was that sometimes two edges, as in this episode, formed a square under the condition of parallel projection because they were diagonals of opposite faces of a cube too.

## DISCUSSION

Sensuous experiences are accumulated through collaboration of the senses, such as sight and touch. As this episode shows, wide experiences, like the process of constructing objects in the physical/mathematical environment, should be included in mathematical lessons. A tetrahedron was familiar to students because they had seen and touched one during their schooling. Therefore, they perceived a tetrahedron as a unique organism formed by only four equilateral triangles connected at each edge; to the students, a tetrahedron had several properties such as equality of all edges and angles of edges and faces. However, as the students' perplexity in the classroom implies, they do not necessarily become conscious of their own tetrahedron-ness (which has positional relations of edges and faces, angles by edges and faces, and so on as *perspective*) just through limited bodily experiences of unconscious sight, touch or

construction. The utterance “I can see only triangles” just after being given a task by the teacher is a typical way of knowing by students. However, the students gradually focused on the relationship between the direction of a solid as a *viewpoint* and a shape and its change in *appearance*. And, they paid attention to a *viewpoint*, i.e., that under certain conditions, that the tetrahedron could produce the *appearance* of a square, as shown by the phrase “the front side” in utterances 7 (S4) or 9 (S4).

As is evident in this task situation, the shift from initial random actions to systematic actions of adjusting the direction of a polyhedron such that it looked like a square (Figs. 2-3) demonstrates that a *viewpoint* and *appearance* underlie cognition as a rigid coupling and that this coupling is a necessary condition to become conscious of mathematical assumptions.

We might then query the following: (1) why the “diagonal” viewing appeared and was used as the justifying term, and (2) why the students were not conscious of the condition of parallel projection until the context led to justification.

In space geometry based on Euclid, a negative aspect of sight, i.e., perception of depth in this episode, often becomes apparent. Sight functions give us sensuous confirmation, but it tends to proceed according to rule or cultural norms. Therefore, in space geometry learning, by separating the effects of sight and the attributes of material objects, we bring relativization of perception to concept, i.e., the ideal properties of mathematical objects. Recognizing a “must be” standpoint under a definition or condition, which might seem contrary to bodily knowing in the material world should be based on another coupling <perspective, physiognomy>.

As mentioned above, the students had already formed naïve conceptions of a cube and tetrahedron, but at the same time, had physiognomies for these shapes, whether individual or collective. Diagonals in this episode are not those of a polyhedron in front of them but those of a cube projected on a fictional plane. Utterances 2 and 3 (S2) implied that an *appearance* of a fictional square resulted in focusing on the positional relation of diagonals as one of the realizations of square-ness. In other words, after achieving the *perspective* that the diagonals were equal and perpendicularly crossing, S2 created the rationale that objects conceptualized by them must be squares.

The important point is that many students, not only S2, were often tracing two edges as diagonals. During such actions, a *physiognomy* helps sensuous cognition become sophisticated, according to the situation. The sense of touch in tracing the diagonals causes it to emerge from the sea of multimodal senses as the critical sense. In this case, the sense of touch and ideal figural properties of a tetrahedron compensates the complexity and ambiguity of visual perceptions with depth; therefore, the sense of touch, including gestures as dynamic actions, functions to orient of particular properties in the situation.

Finally, in this task, to be conscious of a parallel projection means that the collective, cultural (mathematical) aspect takes precedence over the aspect of sight. Through utterances and actions in the context of justification, S4 integrated the senses from

seeing and touching the concrete object of a tetrahedron, physical attributes of concrete objects, and figural properties. Then mathematical conditions of parallel projection, including information of direction, are necessary and sufficient conditions such that two edges of a tetrahedron are diagonals of a square through parallel projection. This consciousness is a condition for forming mathematical conceptions.

## IMPLICATION

This study suggests that sensuous cognition underlies mathematical conceptions and students should become conscious of cultural (mathematical) conditions in an aim-oriented situation for forming mathematical conceptions. The two-coupling framework would focus on students' integrated viewing of various conceptions.

## Acknowledgement

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# THE RELATIONSHIP BETWEEN VISUALIZATION, SPATIAL ROTATION, PERCEPTUAL AND OPERATIVE APPREHENSION

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*This study is a part of a research program designed to study the spatial ability and the geometrical figure apprehension development of students aged 10-13 years. Data were collected from 566 primary school students and 286 secondary school students using a test with two parts: a spatial ability test and a geometrical figure apprehension test. The results of the study give a first insight into the relationship between some major components of spatial ability and geometrical figure apprehension based on the performance of primary school students and secondary school students.<sup>1</sup>*

## INTRODUCTION AND THEORETICAL FRAMEWORK

Spatial ability is generally accepted to be related to skills involving the retrieval, retention and transformation of visual information in a spatial context (Halpern, 2000). As Linn and Petersen (1985) suggested, spatial ability is not a unitary construct, but it is a combination of sub-skills. For example, many researchers (McGee, 1979; Clements & Battista, 1992) support that spatial ability consists of two major components: spatial relations and spatial visualization. *Spatial Relations* are described as comprehension of the arrangement of elements within a visual stimulus pattern (McGee, 1979). They refer to the ability to solve simple rotation problems or to identify reflected versions of the target (D'Oliveira, 2004). *Spatial Visualization* is described as the ability to imagine rotations of objects in 3-D space by folding and unfolding for example (McGee, 1979). The manipulation could be in a holistic, as well as in a piece-by-piece fashion and the movements must be imagined (Clements & Battista, 1992).

In geometry, visualization has a different meaning. A main reason for this is that, geometrical figures are simultaneously concepts and images (spatial representations) (Fischbein & Nachlieli, 1998). They are not mere iconic representations, but semiotic ones. A semiotic representation illustrates the organization of the relations between representational units. In the case of geometrical figures, these representational units can be one-dimensional or two-dimensional shapes. Therefore, in the learning of geometry it is not enough to see and understand what is represented at first glance. Here exists the difference between visual perception and visualization. Although visual perception is one of the most important factors affecting the ability to recognize plane shapes (perceptual apprehension), it only provides a direct access to the shape

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<sup>1</sup> This paper has been written in the context of the research project "Spatial Ability and Geometrical Thinking Development" of the University of Cyprus.



and never gives a complete apprehension of it. On the contrary, visualization is based on the production of a semiotic representation of the concept and gives at once a complete apprehension of any organization of relations (Duval, 1999). Visualization in mathematics requires specific training in order to grasp directly the whole configuration of relations and to handle the figure as a geometrical object.

Duval (1999) proposes a rather comprehensive theoretical model for geometrical reasoning. Within this model a distinction is made between four types of apprehension for a “geometrical figure”: perceptual, sequential, discursive and operative. Each has its specific laws of organization and processing of the visual stimulus array. Particularly, *perceptual apprehension* refers to the recognition of a shape in a plane or in depth. *Sequential apprehension* is required whenever one must construct a figure or describe its construction. *Discursive apprehension* is related with the fact that mathematical properties represented in a drawing cannot be determined through perceptual apprehension. In any geometrical representation the perceptual recognition of geometrical properties must remain under the control of statements (e.g. denomination, definition, primitive commands in a menu). However, it is through *operative apprehension* that we can get an insight to a problem solution when looking at a figure. Operative apprehension depends on the various ways of modifying a given figure: the mereologic, the optic and the place way. The mereologic way refers to the division of the whole given figure into parts of various shapes and the combination of them in another figure or sub-figures (reconfiguration), the optic way is when one makes the figure larger or narrower, or slant, while the place way refers to its position or orientation variation. In a problem of geometry, one or more of these operations can highlight a figural modification that gives an insight to the solution of a problem.

The role of spatial ability in geometry is elusive and complex (Clements, 1997). Battista (1990) found that spatial visualization and logical reasoning were significantly related to both geometry achievement and geometry problem solving. Furthermore, according to English and Warren (1995), the way an individual visualizes a geometrical shape is one of the most important factors affecting the development of spatial ability. In the present paper we will attempt to give a deeper insight into the connection between spatial ability and geometrical understanding. Specifically, our focus is to investigate how two major components of spatial ability, visualization and mental rotation, are related to geometrical figure apprehension (perceptual and operative), as proposed by Duval (1999). Furthermore, we intend to examine how this relationship varies between two grade levels.

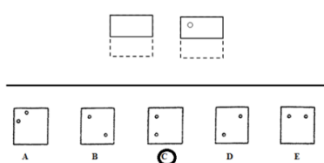
## METHOD

The study was conducted among 566 primary school students and 286 secondary school students, who were randomly selected from schools in Cyprus. It should be mentioned that the mathematics curricula which are applied at primary and secondary schools in Cyprus do not give much emphasis on spatial ability.

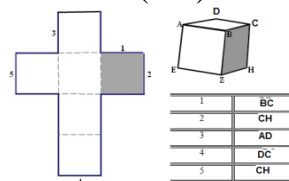
Data were collected through a test which was constructed for the needs of the present study. The test was developed after an extensive review of the relevant literature in

mathematics education and cognitive psychology in relation to spatial ability and geometrical figure apprehension. The test consists of two parts: a spatial ability test and a geometrical figure apprehensions test. The spatial ability components, visualization and mental rotation, were assessed using three marker tests (see Figure 1, Eliot & Smith, 1983). The Paper Folding test (PF) and Surface Development test (SD) were used for the Visualization component, while the Card Rotations test (CR) was used for the Mental Rotation component.

Which of the five figures below the line shows where the holes will be when the paper is completely unfolded (PF)?



Imagine the folding and figure out which of the lettered edges on the object are the same as the numbered edges on the piece of paper at the left. (SD).



Which of the five cards on the right is the same as the card in the box? (CR).

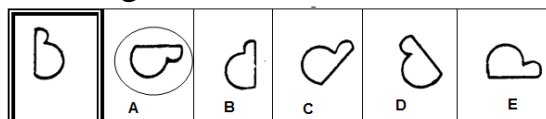
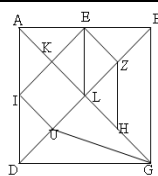


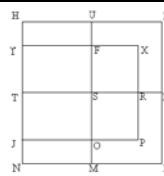
Figure 1: Visualization and Mental Rotation tasks

A total of 6 tasks were included in the test for the following components of geometrical figure apprehension (Figure 2): Perceptual apprehension (PE) and Operative apprehension (OP). The operative apprehension here was assessed on the basis of two ways of modifying a given figure: the mereologic (OPme) and the place way (OPpw). Most of these tasks were used by previous studies which investigated geometrical figure apprehension (Elia, Gagatsis, Deliyianni, Monoyiou, & Michael, 2009).

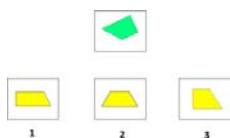
Name the figures KEZL,EZHL, ZEL (PE1).



Name the squares (PE2).



Circle the yellow card that has exactly the same shape with Maria's card (OPpw2).



Compare the area of rectangle 1 and rectangle 2 (OPme1).

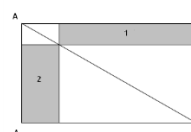


Figure 2: Perceptual and Operative Apprehension tasks



from the geometrical ability components we cannot ignore the similarity connections of the geometrical ability components with some of the paper folding tasks. These connections reveal that students handled the particular paper folding tasks similarly to the geometrical tasks.

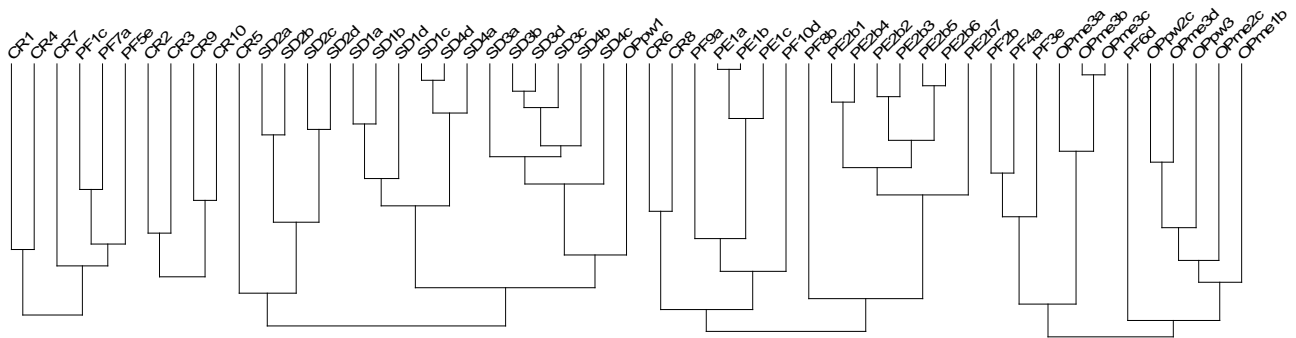


Figure 4: Similarity diagram of primary school students' responses to the tasks.

The results of the similarity analysis on the primary school students' responses (see Figure 4) were not as clear as the results of the whole sample. There are again five clusters which this time involve different types of spatial and/or geometrical tasks. Three clusters can be considered as spatial ability clusters though. The first cluster involves card rotation and paper folding cards tasks, while the second cluster includes four card rotation tasks. The third cluster includes all of the surface development tasks and therefore can be named as the Surface Development cluster. It appears that primary school students apply similar processes when dealing with some card rotation and paper folding tasks, but handle differently the surface development clusters. The last two clusters can be considered as hybrid ones as they include both geometrical and spatial tasks. In particular, the fourth cluster includes all the perceptual apprehension tasks connected to two card rotation tasks and three paper folding tasks. The fifth cluster contains almost all the operative apprehension tasks associated with a group of three paper folding tasks. These results indicate that some paper folding tasks were handled similarly to the geometrical tasks of each type by the primary school students. What is new is that in this grade two card rotation tasks were encountered in a similar way as perceptual apprehension tasks.

The similarity analysis on the responses of the secondary school students resulted in the diagram in Figure 5. The similarity diagram includes five distinct clusters. The former three clusters refer to spatial ability components. The first cluster contains card rotation tasks and surface development tasks. The second cluster includes most of the surface development tasks and can be named as the Surface Development cluster. The third cluster can be named as the Paper Folding cluster as it involves the majority of the paper folding tasks. These results mean that with the exception of one surface development task, the secondary school students dealt with the spatial tasks of each type (card rotation, surface development and paper folding) consistently to each other. The latter two clusters refer to geometrical tasks and can be named as operative apprehension cluster and perceptual apprehension cluster respectively. The fourth cluster includes most of the operative apprehension tasks associated with two paper

folding tasks. The fifth cluster includes all the perceptual apprehension tasks. These findings indicate that the secondary school students dealt consistently also with the geometrical figure apprehension tasks of each type. The operative apprehension tasks were handled though similarly to two paper folding tasks.

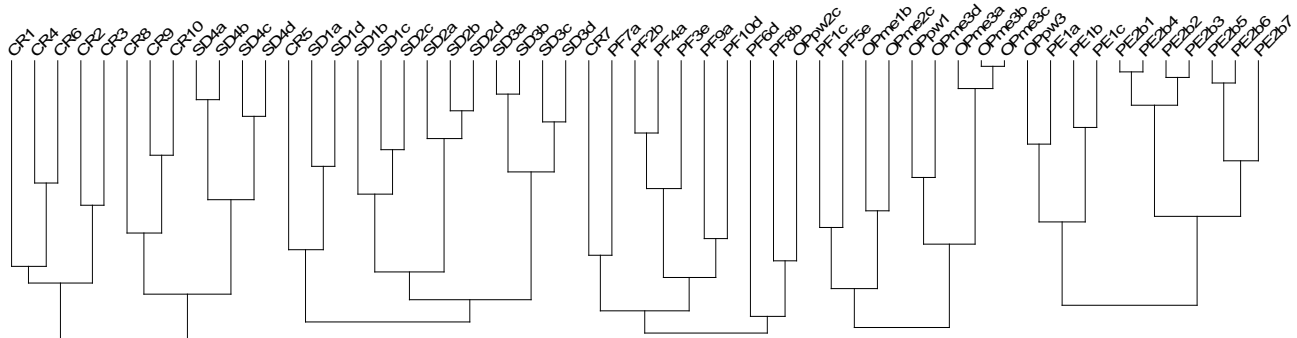


Figure 5: Similarity diagram of secondary students' responses to the tasks.

## DISCUSSION

This study aimed to examine the relationship between some major components of spatial ability and geometrical figure apprehension based on the performance of primary school students and secondary school students. Drawing on the performance of the students, we found a moderate but significant correlation between spatial ability and geometrical figure apprehension. This result concurs with previous findings which pointed out that spatial ability is positively related to geometry achievement and problem solving (Fennema & Tartre, 1985; Battista, 1990). The application of the similarity analysis on the data provided additional and more analytical information about this rather complex relationship (Clements, 1997) and its variation with grade, with a focus on the homogeneity by which spatial ability tasks and geometrical apprehension tasks were handled by the students.

A general conclusion, that is based on the similarity analysis on the whole sample and also on the primary school students and secondary school students separately, is that students dealt with the spatial ability tasks and the geometrical apprehension tasks in a relatively different way. Similarity groups were formed mainly based on the criterion of whether tasks referred to spatial ability or geometrical apprehension. However, a connecting link was found between these two abilities, that is, the paper folding tasks. It could be that some processes applied in the paper folding tasks were common with the processes used in some geometrical figure apprehension tasks, namely, perceptual apprehension tasks and operative apprehension tasks. For example, in folding a piece of paper with a specific geometric form (e.g., a rectangle), one could recognize the new shape that emerges or the one that is folded. This shape recognition process is also found in perceptual apprehension tasks which require a direct access to the shape, to see and understand what is represented at first glance. Moreover, determining whether some shapes within a given figure are congruent, or comparing the area of different shapes, sometimes require the visualization of folding or dividing the whole figure into smaller parts and recombining them into another figure or sub-figures. This includes a

mereologic way of apprehending a geometrical figure, which is a major ability of the operative apprehension of geometrical figures. It should be noted here that for the primary school students, five paper folding tasks were linked both to perceptual and operative apprehension tasks. For secondary school students, though, the paper folding tasks that were linked to geometrical figure apprehension tasks were only two and were connected only to operative apprehension tasks. It is likely that as students get older and receive more advanced teaching in geometry, they tend to use figures not just as spatial representations, like it is the case in the paper folding tasks. They are inclined to use geometrical figures as semiotic representations of geometric objects and grasp the whole configuration of their relations. Thus they activate additional and perhaps distinct and more complex cognitive processes in geometrical figure apprehension tasks relatively to the spatial tasks such as paper folding activities. This conclusion is further supported by the connection between some card rotation tasks and perceptual apprehension tasks, which was found in the primary school students' results and not in the secondary school students' diagram.

A common finding between all the similarity diagrams was that the surface development tasks were distinct from all the geometrical figure tasks, indicating the difference in how students approach these spatial tasks and the geometrical ones. This can be interpreted by the fact that the surface development tasks involve the visualization of the transition from two-dimensional figures to three-dimensional figures. This was not required however in the geometrical figure apprehension tasks examined here, which were based on two-dimensional figures and their two-dimensional figural treatment.

Of interest are also the connections within the spatial ability and geometrical figure apprehension tasks, respectively. Concerning the geometrical figure apprehension, a quite clear distinction was found between the perceptual and the operative apprehension, as in both grades distinct groups of tasks were formed with respect to the geometrical figure apprehension they involved. This distinction is in line with Duval's (1999) cognitive model of geometrical reasoning and the relevant distinction he proposes between these two ways of capturing a geometrical figure.

Regarding the spatial ability tasks, even though the paper folding tasks and the surface development tasks refer to a common spatial ability aspect, that is, the visualization component, they formed distinct similarity clusters, indicating that perhaps students did not use similar cognitive processes to tackle the two types of tasks. On the contrary, some similarity connections were identified between the card rotations tasks which refer to the mental rotation component and the paper folding tasks or the surface development tasks respectively. This indicates that there was not a clear distinction between the mental rotation component and the visualization component based on the responses of the students. This finding needs further investigation to be interpreted and better understood.

To sum up, the findings of this study about the connection between spatial ability and geometrical figure apprehension components, could initiate further investigations of

experimental nature on the relationship between the two constructs. Such investigations may examine the effects of teaching which emphasizes various spatial ability components (e.g., paper folding activities) on geometrical figure apprehension development and how these effects vary with grade.

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# INVESTIGATING THE EFFECT OF GENERAL CREATIVITY, MATHEMATICAL KNOWLEDGE AND INTELLIGENCE ON MATHEMATICAL CREATIVITY

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*This study examines the predictive power of several cognitive factors on mathematical creativity. Data were collected through the administration of four tests - a mathematical creativity test, a general creativity test, a mathematical knowledge test, and an intelligence test - to 476 elementary school students. Data analysis revealed that mathematical creativity is defined across fluency, flexibility and originality, whereas the three abilities are interrelated. Moreover, students' mathematical knowledge, general creative ability and fluid intelligence can significantly predict mathematical creativity. Among these factors holding domain specific abilities as well as domain general abilities are crucial for the appearance of creative behaviour in the domain of mathematics.*

## INTRODUCTION

The concept of mathematical creativity has been firstly reported in 1902 in a publication of the French magazine «L' Enseignement Mathématique». Since then, researchers have been tried to define the concept or to develop an appropriate measurement tool, however, they have not found yet a point of agreement (Leikin, 2007). Beside these epistemological barriers, researchers agreed that mathematical creativity is a necessary condition for achieving high level of mathematical thinking (e.g. Haylock, 1987; Leikin, 2007). Given this fact, the investigation and the enhancement of mathematical creativity is among the priorities of educational systems and research organizations (e.g. NCTM, 2000).

Several attempts have been made to determine the factors that influence individuals' creative potential in mathematics. For instance, intelligence (Sternberg & O'Hara, 1999), prior knowledge (Sheffield, 2009) and general creativity (Hong & Milgram, 2010) are some of the factors that have been investigated in combination with creative ability. The present study aims to examine whether intelligence, mathematical knowledge and general creative ability may predict students' mathematical creativity. In particular, we intend to develop a theoretically driven model, concerning the structure of mathematical creativity and its predictors.

## THEORETICAL FRAMEWORK

### Defining mathematical creativity

There are numerous ways to define mathematical creativity. For instance, mathematical creativity was considered as the ability to solve and pose problems with heuristic methods, to observe patterns with numbers or shapes, to generalize



mathematical concepts, to apply non-algorithmic processes for decision making, and to handle information and processes flexibly (e.g. Sheffield, 2009; Silver, 1997).

Frequently, mathematical creativity is cited by adopting the definition of general creativity as it has been firstly proposed by Torrance (1962). Specifically, creativity was defined across fluency, flexibility, originality and elaboration (Torrance, 1962). However, the scoring was simplified by employing only the concepts of fluency, flexibility and originality, whereas elaboration was not included, due to the fact that there is a difficulty of getting inter-rater reliability (Cramond et al. 2005). Similarly, in the definition applied in mathematics the researchers employed only the concepts of fluency, flexibility and originality, due to the difficulty of determining levels of elaboration in mathematical answers (e.g. Leikin, 2007; Silver, 1997). According to Leikin (2007), fluency in mathematics refers to the ability of producing a number of ideas, flexibility refers to the number of approaches or mathematical ideas that are observed in a solution, and originality refers to the possibility of holding extraordinary, new and unique answers.

### **Intelligence and creativity**

The relationship between creativity and intelligence is not simple and the research results are conflicting (Sternberg & O' Hara, 1999). On the one side, several studies concluded that intelligence and creativity are separate entities and therefore highly creative students are not necessarily highly intelligent, or vice versa. For instance, Getzels and Jackson (1962) concluded that highly intelligent but low creative students had no statistically significant differences in their academic performance with highly creative but low intelligent students. Wallach and Kogan (1965) in a similar study found low correlation between intelligence and creativity. Recently, Silvia (2008) reanalyzed the data by Wallach and Kogan and found that the magnitude of the relationship among creativity and intelligence was higher than the average correlation found in similar studies, but still this loading was low. Kim's (2008) meta-analysis found a negligible relationship between creativity and intelligence scores too, proposing that even students with low intelligence can be creative.

On the contrary, other researchers supported the existence of relationship between creativity and intelligence (e.g. Guilford, 1967; Sternberg & O' Hara, 1999). In particular, in the "Structure of Intellect theory" (Guilford, 1967) intelligence is viewed to be comprised by several components, one of them is creativity. In this view, creativity is an aspect of intelligence, and thus their correlation is high. Furthermore, in the "Threshold theory" (Torrance, 1962) there is a positive correlation between creativity and intelligence throughout most of the IQ distribution, whereas this correlation weakened for IQ above a threshold point.

### **Domain general and domain specific creativity**

For decades, researchers perceived the concept of creativity as a general cognitive ability (Beghetto & Kaufman, 2009). Researchers who supported the universality of the creative ability assumed that a person who has demonstrated a high level of creativity in a specific field is anticipated to show correspondingly high level of

creativity in other fields (Beghetto & Kaufman, 2009). Therefore, measurement tools of creative ability, like the Torrance Tests of Creative Thinking, are expected to predict creative thinking in a range of disciplines (Hong & Milgram, 2010).

The idea of domain-specific creativity appeared in the 90s. Due to the fact that the construct and the applications of creative capacity varied between cognitive domains different definitions and processes were needed (Beghetto & Kaufman, 2009). Low correlations were detected between creative results in different fields, verifying that every cognitive field requires the development of different theoretical and operational definitions for the concept (Plucker & Zabelina, 2009).

Although there is a rich discussion on the discrimination between domain general and domain specific creativity, limited evidences exist concerning the extent of the influence of domain general creativity on domain specific creative ability. Such studies were conducted by Hong and Milgram (2010) and Diakidou and Spanoudis (2002). Regarding the first study, Hong and Milgram (2010) found a statistically significant effect of general creativity on mathematical creativity. Similarly, Diakidou and Spanoudis research (2002) indicated a significant contribution of a domain-independent measure in the content-specific performance in history.

### **Mathematical knowledge and mathematical creativity**

Knowledge is the basis for the production of creative outcomes, since the formation of new knowledge can occur if there is an understanding of what already exists (Shaughnessy, 1998). However, possessing a lot of knowledge can limit creativity. According to De Bono (1968), if an individual knows well what has been preceded in a field he/she focuses on the known frameworks and is unable to propose new ideas.

In the field of mathematics education, knowledge is acknowledged as the variable that contributes more than any other variable to students' creativity (Sak & Maker, 2006). Specifically, the more mathematical knowledge a person holds, the more likely is to be creative in the field. Indeed, Mann (2005) concluded that students' achievement in mathematics contributed to the prediction of their performance on a mathematical creativity test, through a regression analysis.

Seeing that mathematical creativity depends on the ability to link and combine available concepts and information, pre-existing knowledge is the basis on which new information will be organized and developed (Sheffield, 2009). Along the same line, individuals who do not have good numeracy skills are unable to demonstrate mathematical creative thinking, as they lack the necessary knowledge to express their creative thoughts with recognizable forms (Haylock, 1987).

### **PURPOSE OF THE STUDY**

Despite the fact that numerous studies have been conducted to investigate the effect of several cognitive factors on general creativity, there is a lack of corresponding research on domain specific creativity, such as mathematical creativity. The majority of attempts that have been made to this direction examine the effect of isolated factors on creativity and no attempt has been made to ascertain the effect of several cognitive

factors on creativity. Since mathematical creativity is important in fostering high order mathematical thinking (Haylock, 1987; Leikin, 2007), there is a necessity for research studies aiming to reveal the factors that may enhance students' creative potential in mathematics. Moreover, the investigation of the way that a combination of cognitive factors may affect mathematical creativity will reveal significant results that will help teachers to know where to invest in order to strengthen students' creative potential.

Therefore, the present study purports to develop a theoretical model, concerning the structure of mathematical creativity and its predictors and further to assess whether the model fits the data of the research. Specifically, our aim is to investigate whether intelligence, mathematical knowledge and general creativity may predict students' mathematical creativity, as it is defined across fluency, flexibility and originality.

## METHOD

### Sample – Procedure

Four hundred and seventy six students 9–12 years old (Grade 4: N=202, Grade 5: N=165, Grade 6: N=109) participated in the present study. The tests were administered to students in paper-and-pencil form.

### Instruments

To fulfil the objectives of the study four tests were administered to students:

*Mathematical creativity test:* The mathematical creativity test consisted of four multiple-solution mathematical tasks, in which students were required to provide: (a) multiple solutions; (b) solutions that were distinct from each other; and (c) solutions that none of their peers could provide. The mathematical creativity test included problem solving and problem posing tasks. The time allocated to students to solve the test was 40 minutes. Tasks were assessed regarding fluency (number of correct mathematical solutions), flexibility (number of different mathematical ideas) and originality (rarity of solutions).

*General creativity test:* The general creativity test included two tasks taken from the Torrance Tests of Creative Thinking and translated to Greek language, one from the verbal and one from the figural version. The verbal task asked participants to provide unusual uses of a common everyday object whereas the figural task asked participants to draw several pictures from a repeated figure, such as circles. Participants were asked to provide as many answers as they could in 14 minutes (7 minutes for each task). The students' assessment on general creativity test was based on the fluency, flexibility and originality of their answers.

*Mathematical test:* The mathematical test consisted of five tasks: problem solving, numerical operations, patterns-algebra, graph interpretation, area and perimeter of complex figures. Three equivalent forms of this test were developed, one for each grade. Participants had 20 minutes to complete the mathematical test. The items of the mathematical test were marked as correct or incorrect.

*Intelligence test:* Fluid intelligence was measured using the Naglieri Nonverbal Ability Test (NNAT) (Naglieri, 1997). The NNAT is a nonverbal measure of general ability, consisting of seven levels, according to the age of participants. Each of NNAT's levels includes 38 tasks from four categories, namely pattern completion, reasoning by analogy, serial reasoning and spatial visualization. For the purpose of the present study, level D (for fourth graders) and level E (for fifth and sixth graders) were administered to participants for 30 minutes.

### Data analysis

For the analysis of the data we used confirmatory analysis (CFA) using the statistical modeling program MPLUS (Muthén & Muthén 1998). CFA was applied in order to assess the extent to which our data fitted an a-priori theoretical model. In order to evaluate model fit, three fit indices were computed: The chi-square to its degree of freedom ratio ( $\chi^2/df$ ), the comparative fit index (CFI), and the root mean-square error of approximation (RMSEA) (Marcoulides & Schumacker, 1996). According to Marcoulides and Schumacker (1996), for the model to be confirmed, the values for CFI should be higher than 0.90, the observed values for  $\chi^2/df$  should be less than 2 and the RMSEA values should be close to or lower than 0.08.

### RESULTS

The theoretical model consisting of the latent and observed variables are presented in Figure 1. This model consists of four first order factors (Fluency, Flexibility, Originality, General Creativity) and one second-order factor (Mathematical Creativity). The three of the four first-order factors (Fluency, Flexibility, Originality) represent the individuals' performance on mathematical creativity. Furthermore, it is hypothesized that students' performance on Fluency, Flexibility and Originality are interrelated in each mathematical task. The fourth first-order factor (General Creativity) in combination with the observed variables intelligence and mathematical knowledge are hypothesized to lead to the mathematical creativity second-order factor indicating their effect on students' creative ability in mathematics.

The results of the analysis revealed that the model matched the data set of the present study and determined the "goodness of fit" of the factor model (CFI=.980,  $\chi^2=179.199$ ,  $df=101$ ,  $\chi^2/df=1.774$ , RMSEA=.050). The analysis revealed that the statistically significant loadings of the first-order factors, namely fluency ( $r=.95$ ), flexibility ( $r=.94$ ) and originality ( $r=.97$ ), constitute a second-order factor, that of mathematical creativity. It is important to mention that the three first-order factors are composed by students' corresponding performance in the four tasks of the mathematical creativity test. Students' fluency, flexibility and originality are interrelated statistically significant in each task.

The structure of the proposed model also addressed that students' general creativity ( $r=.42$ ,  $p<.05$ ), mathematical knowledge ( $r=.41$ ,  $p<.05$ ) and fluid intelligence ( $r=.22$ ,  $p<.05$ ) can significantly predict mathematical creativity. Out of all cognitive abilities tested, general creative attitude and possession of mathematical knowledge appeared to be the strongest predictive factors. This is indicated by the highest loadings.

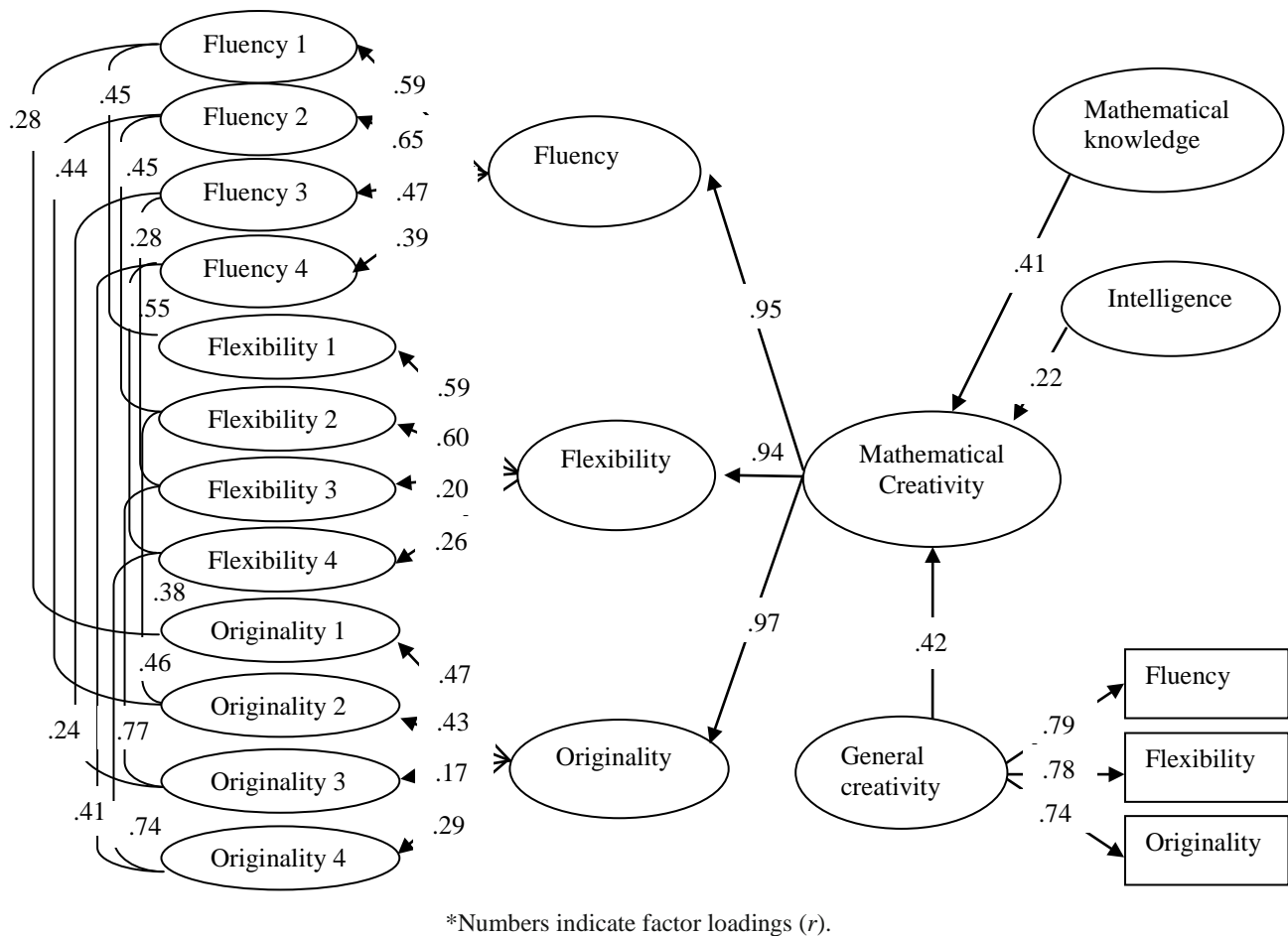


Figure 1: The structure of the model.

## DISCUSSION

Given the importance of mathematical creativity in mathematics education community (e.g. Leikin, 2007; Sheffield, 2009), it is important to clarify in what ways several cognitive factors may affect students' creative potential in the domain of mathematics. Hence, the aim of this study was to articulate and empirically test a theoretical model, indicating the relationship between mathematical creativity with general creativity, mathematical knowledge and intelligence.

Mathematical creativity has been defined across fluency, flexibility and originality. This result is in agreement with the results proposed by Leikin (2007) and Silver (1997), who argued that these three components grasp the concept of creativity in the domain of mathematics. Moreover, fluency, flexibility and originality are correlated within a specific task. This finding suggests that when an individual proposes a number of correct mathematical solutions, there are more possibilities to consider different and original mathematical ideas. According to Diezmann and Watters (2000) this occurs due to the redefinition of the task, as well as the connection or modification of mathematical ideas and representations.

Data analysis also revealed that mathematical knowledge, general creativity and intelligence may predict students' creative potential in mathematics. These results are

in accordance with the findings of other researchers. In particular, Amabile (1996) proposed that domain-relevant skills as well as creativity-relevant skills are essential for creative behavior in a domain. More specifically, the possession of mathematical knowledge in combination with being generally creative are prerequisites for the emergence of an individuals' potential. Given that mathematical creativity depends on the number of mathematical ideas employed in a solution, individuals without strong mathematical background are unable to combine their knowledge and propose unique solutions. Additionally, individuals without flexible thinking skills and original behaviour are not able to consider different ideas, flexibly use mathematical facts and processes and combine knowledge in an effort to propose unique solutions (Hong & Milgram, 2010). Furthermore, intelligence can significantly predict mathematical creativity. However, the low loading indicates that intelligence is a necessary but not sufficient condition for mathematical creativity (Torrance, 1962).

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# CHILDREN'S DYNAMIC THINKING IN ANGLE COMPARISON TASKS

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*This paper discusses findings from a task-based interview with two elementary school children working on angle comparison. This task was designed within the context of a research project investigating the effect of dynamic geometry use on children's geometry learning. The results from the interview show that the children's gestures and motion played an important role in their decision-making on the angle comparison tasks, suggesting a strong presence of dynamic thinking. The intent of this paper is to describe and characterise this dynamic thinking rather than make correlations to children's DGE experiences.*

## INTRODUCTION

Concepts of angle and rotation are central to the development of geometric knowledge (Clements and Battista, 1992). Although angle is a basic concept that is used by humans in analysing their spatial environment, it can pose challenges to learners, even into secondary school due to its multi-faceted nature (Mitchelmore & White 1995). Despite these difficulties, children show sensitivity to the concept of angle from very early years (Spelke, Gilmore and McCarthy, 2011). Angles are normally introduced to children quite late in formal school settings. For example, in British Columbia, they are introduced in grade 6 (12 years old). The strong capacity of young children to attend to and identify angles in various physical contexts motivated us to explore angle learning at the k-1 grade levels. We have chosen a DGE (Dynamic Geometry Environment) approach in order to support the dynamic, angle-as-turn conception (see Kaur & Sinclair, 2012).

In this paper, I report on a one-on-one task-based interview with a split class of kindergarten/grade 1 children (ages 5-6) who were taught the concept of angle using *The Geometer's Sketchpad*. I am interested in finding out whether these angle comparison tasks (Table 1) can elicit dynamic ways of thinking and, if so, how this kind of thinking might be communicated by young children. These questions form the basis of this research study.

## CHILDREN'S UNDERSTANDING OF ANGLE

In the research literature, the concept of angle is shown to have different perspectives, namely: angle as a geometric shape, union of two rays with a common end point (static); angle as movement; angle as rotation (dynamic); angle as measure; and, amount of turning (Close, 1982; Henderson & Taimina, 2005). Much research has been conducted on the development of the concept of angles, focusing at the grades 3, 4 and higher levels. Mitchelmore & White (1995) suggests that angles occur in a wide variety of physical situations that are not easily correlated. Despite the excellent



knowledge of all situations, specific features of each situation strongly hinder recognition of the common features required for defining the angle concept (Mitchelmore, 1998). The situation is more problematic for students where the two arms of angle are not clearly visible.

Research has reported about the young children's difficulties in understanding the turn as an angle as well as connecting static angles to turns (Mitchelmore, 1998; Clements, Battista, Sarama & Swaminathan, 1996). Thus, young children do not spontaneously conceptualize turning in terms of angle and they don't naturally connect static angles to turns. Other popular misconception about angles is related to the relative size of angles. Students think that the length of the arms is related to the size of the angle (Stavy and Tirosh 2000; Clausen-May 2005). Students tend to think that 'the longer the rays, the greater the measure of the angle'. Stavy and Tirosh (2000) reported this misconception could develop among children as a result of intuitive rule 'More A - More B'. Another reasons for such misconception are the introduction of angle as a shape rather than a measure as well as the limited experience of angles as shown in textbook (Clausen-May, 2005). This misconception seems to be very hard to overcome. Lehrer, Jenkins, and Osana (1998) conducted a longitudinal study involving children in grades 1–3 who were followed through grades 3, 4, and 5. Their results show that "the length of the line segments had a substantial influence on children's judgments of similarity [...] the effects of length on children's judgments about angles did not diminish during the three years of the study" (p. 149). We believe that the DGE approach could be helpful in focusing attention on the quantity of turn rather than on the length of the line segments.

## THEORETICAL PERSPECTIVE

In previous research, we have found Sfard's (2008) 'commognition' approach is suitable for analysing the geometric learning of students interacting with DGEs (see Sinclair & Kaur, 2011; Kaur & Sinclair, 2012). For Sfard, thinking is a type of discursive activity. The mathematical discourse has four characteristic features: word use (vocabulary), visual mediators (the visual means with which the communication is mediated), routines (the *meta-discursive rules* that navigate the flow of communication) and narratives (any text that can be accepted as true such as axioms, definitions and theorems in mathematics). Learning geometry can thus be defined as the process through which a learner changes her ways of communicating through these four characteristic features. We are particularly interested in investigating how the students might move between different word uses and to examine the informal language they use to talk about angles.

Additionally, given the importance of gestures in communication of abstract ideas (Cook & Goldin-Meadow, 2006), and their potential to communicate temporal conceptions of mathematics (Núñez, 2003; Sinclair & Gol Tabaghi, 2010), we chose to extend Sfard's approach to incorporate gestural forms of visual mediators. Kita (2000) focuses on the cognitive functions of gestures, which play an important role in communication. He points out that the production of a gesture helps speakers organize

rich spatio-motoric information, where spatio-motoric thinking organizes information differently than analytic thinking (which is used for speech). We thus expect that children will use gestures to convey spatio-motoric information, even though they might not be able to convey the analytic thinking used in speech. Our goal in looking at the gestures will be to see how they communicate different ideas about angles; particularly the mobile ones associate with angle-as-turn.

## **PARTICIPANTS AND TASKS**

We worked with kindergarten/grade 1 children (aged 5-6) from a school in a rural low SES town in the northern part of British Columbia. There are 20 children with diverse ethnic backgrounds and with a wide range of academic abilities. We designed lessons related to angle along with the classroom teacher, who has been developing her practice of using DGEs for a couple of years. The teacher and students worked with angles in different ways, using *Sketchpad* for six lessons in a whole class setting with an IWB (Interactive Whiteboard). Each lesson lasted approximately 30 minutes. After five months, seven students were interviewed on angle comparison tasks. The students were presented with the triads of angle shapes (Table 1) and asked, “Which one is the most different. Give a verbal justification for your choice?” A total of seven triads were printed on different sheets with one triad on each. Four angle triads (1,2,4,5 shown in Table 1) were adapted from Lehrer et al (1998) longitudinal study. Three (3,6,7 shown in Table 1) additional triads were included in this study. Each triad was shown to students one by one. Each interview was roughly ten minutes long and was videotaped with the camera facing the interviewer and the subjects. Some of the interviews are transcribed. This paper analyses the responses of two students Chloe and Mandy, who were chosen because of their contrasting responses.

Two different sketches were used to explore the concept of angle with the children during six angle lessons. The first showed a simple angle, whose vertex could be dragged in order to change the size of the angle. The second was a ‘driving angle model,’ which shows both a static as well as dynamic angle. It includes a car that can move forward as well as turn around a point. The turning is controlled by a little dial (which has two arms and a centre). Four action buttons (Turn, Drive Forward, Erase Traces and Reset) control the movement of the car. The traces of turn offer a visible, geometric record of the amount of turn (details in Kaur & Sinclair, 2012).

## **CHILDREN’S RESPONSES TO ANGLE COMPARISON TASKS**

I begin by describing the results of the interview with Chloe, then Mandy on triad 3, then triad 5 (see Table 1). These triads are chosen to show a sample of detailed responses on triads with different arm length and different orientation respectively.

Interviewer: Have you seen things like that before? (*Showing angle triad 3*)

Chloe: Yeah on the smart board we do that?

Interviewer: Oh yeah...which one out of these three you think is most different?

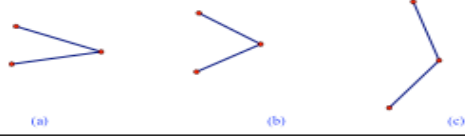
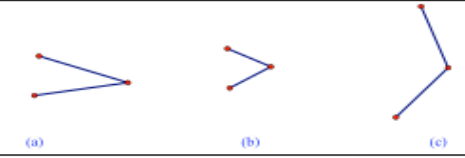
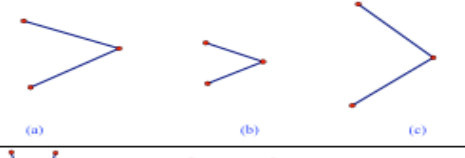
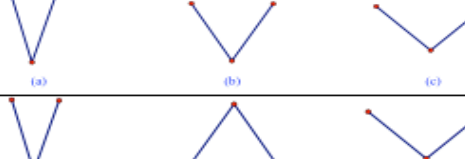
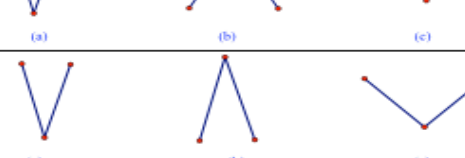
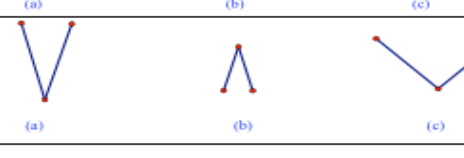
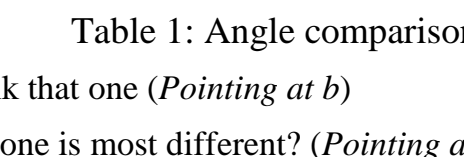
No.	Triads	Description
1		Angles $30^\circ$ , $60^\circ$ and $120^\circ$
2		Angles $30^\circ$ , $60^\circ$ and $120^\circ$ . The length of arms is reduced in 2(b)
3		Angles $50^\circ$ , $50^\circ$ and $80^\circ$ . Angles in 3(a) and 3(b) are same, but arm lengths are different.
4		Angles $30^\circ$ , $60^\circ$ and $90^\circ$
5		Angles $30^\circ$ , $60^\circ$ and $90^\circ$ . The orientation of 5(b) is changed.
6		Angles $30^\circ$ , $30^\circ$ and $90^\circ$ . The orientation of 6(b) is different
7		Angles $30^\circ$ , $30^\circ$ and $90^\circ$ . The orientation as well as length of arms is different in 7(b).

Table 1: Angle comparison triads

Chloe: I think that one (*Pointing at b*)

Interviewer: That one is most different? (*Pointing at b*)

Chloe: Yeah, that one is the smallest

Interviewer: Okay (*Taking out the next sheet of triad*)

Chloe: And those two are pretty big (*Pointing at a and c*)

After looking at triad 3, Chloe instantly said that b was different. The use of the words 'smallest' and 'pretty big' indicates that initially, she was paying attention to the length of the arms of angles. She started the comparison of triads based on the visual appearance of the length of the arms of the angles. As soon as she was presented with the next triad, she gave a new answer and with entirely different explanation.

Chloe: Well in a different way all of those can be different. (*Flipping the next sheet halfway, looking at and talking about the previous task*)

Interviewer: What do you mean in a different way all of these can be different?

Chloe: Well, that one is wider (*Pointing at c tracing, with her right hand index finger (Figure 1a), from the top point to the lowest point and then repeating the gesture from bottom to top*)

Chloe: And that one is just like that one kind of (*gesturing (Figure 1b) with left hand thumb and four fingers to estimate wideness of a and b, then tracing b and a respectively, with left hand index finger, starting from top point to the lowest point*), except that one (*angle b*) is just smaller. That one (*a*) is tiny bit bigger. That one is wider (*tracing c with her right index finger*)



Figure 1: Chloe's gestures for triad 3

Chloe's use of words "in a different way all of these can be different" shows that she started to analyse the triad from more than one perspective. She started to think about the difference in terms of the wideness of the angles. She used the words 'wider', 'that one is just like that one kind of', 'just smaller', 'and tiny bit bigger' for the comparison of this triad. Chloe's statement 'that one is just like that one kind of' while gesturing (Figure 1b) about a and b, shows that she is talking about angles a and b as being the same because they have the same wideness, despite the fact that the length of the arms are different. The tracing of angles with her index finger acted as a visual mediator for Chloe to decide about the wideness of angles. In contrast, while talking about the same triad, Mandy concluded that b was the most different.

Mandy: They go like... This one (*pointing at b*) is going like that one (*pointing at a and then at c, tracing c with right index finger*) and that one... except they are big (*pointing and talking about a and c*)

Mandy's use of pointing gestures and words 'except they are big' reveals that she was focusing on the length of the arms of the angles. Also, her words 'This one is going like that one and that one' hinted at the same orientation of all angles. Her perspective was completely different than Chloe.

Interviewer: Okay...how about here? (*Showing the sheet with triad 5 (Table 1)*)

Chloe: Umm...(thinking)...I think that one (*tracing a with left index finger*)

Chloe: Because that one (*pointing at c and tracing with her right index finger*) all you have to do is to put a tiny bit more like that way (*opening wide hands and then turning them inwards (Figure 2a) to make the angle smaller*) and turn it over (*gesturing with hands to flip the angle (Figure 2b)*). And in that (*referring to a*) you have to put a lot farther (*gesturing to separate both fists apart with index fingers extended out (Figure 2c) to show the wider angle*) and turn it over.

For triad 5, Chloe didn't compare the angles on the basis of their orientation. She compared the angles in terms of the turn and transformations involved and tried to morph angles c and a into angle b. She used her hands as arms of the angles. She her hands wide enough to gesture angle c (Figure 2a) and then said 'all you have to do to



Figure 2: Chloe's gestures for triad 5

put a tiny bit more like that way' and turn it over to get angle b. The use of the phrases 'put a tiny bit more like that way' and 'put a lot farther' along with the gestures (Figure 2a, 2c) shows that she was thinking in terms of angles being in motion through turn. She was using gestures as a visual mediator to communicate (both to herself and the interviewer). This kind of dynamic thinking might be drawing on her DGE-based experiences, where the dial and the car also turn to produce different orientations and sizes of angles.

In contrast, when Mandy was presented with the same triad (triad 5) and asked to find the most different angle, her response was as follows:

Mandy: This one (*pointing at b*)

Interviewer: Okay...why?

Mandy: Because that one is upside down (*pointing at b with right index finger*)...and they are up (*pointing at a and c*)

Mandy's use of phrases like 'upside down' and 'up' shows that she was paying attention to the orientation of the angles, rather than to the relative position of the arms. She does not compare the angles in terms of the concept of angle-as-turn, nor does she talk or gesture in the dynamic terms as Chloe does. Indeed, Mandy's most frequent gestures include pointing at the shape with the finger.

Triad No.	Chloe's response		Mandy's response	
	Most different	Reason	Most different	Reason
1	c ( <i>Tracing c with right index finger</i> )	If you could stretch (a) tiny bit further, it will be just like (b).	c ( <i>Pointing at c</i> )	It is too wide
2	c	This is same as triad 1.	c ( <i>Pointing at c</i> )	It is long and those (a, b) are not longer
4,6	Not presented (N/A)			
7	c ( <i>Gesturing hands wide open and bringing them inwards to make smaller angle as b</i> )	Because you have to put that lot more together and then you have to turn it over.	b ( <i>Pointing at b</i> )	This is upside down

Table 2: Summary of responses of Chloe and Mandy on other triads

It is also interesting to note that Chloe could recognize the triads 1 and 2 being same (see Table 2), whereas Mandy focused on wideness of angles in triad 1 and on length of arms in triad 2. Also, Chloe used tracing with index fingers as visual mediators in case

of triads 1 to 3 and she used gestures of hands as arms of angles in case of triads 5 and 7 of Table 1.

## DISCUSSION AND CONCLUSION

For the comparison of the angles, Chloe developed a routine of assuming her hands as the arms of the angle and then turning them inwards or outwards to compare the amount of turn of angles. It might be in part evoked due to exposure to the car sketch, where the turn of car is traced for a specified angle. Mandy's comparison routines involve either looking at the length of arms of the angles or the orientation of angles.

Chloe's use of words 'wider', 'stretch that one a tiny bit further', 'have to put that lot more together', 'turn it over' shows her propensity to reason in terms of motion and transformation. This gives initial hope that DGE can help in overcoming young children's difficulties in relating turning to angles as reported by Mitchelmore (1998) and Clements, Battista, Sarama & Swaminathan (1996). Chloe used the embodied visual mediators (hands as arms of angles or traces with index fingers) for deciding which angle is different. Her routines for comparing the angles involved use of motion and transformations focusing more on the relative position of the arms as well the amount of turn involved from one arm to another. Thus, it shows that the proposed tasks were able to evoke dynamic thinking to some extent, which may in part was due to DGE based instruction. This shows that the length of arms does not effect her conception of angles and she has understood angle-as-turn.

On the other hand, Mandy did not compare the triads in terms of angle-as-turn. This may be in part due to the fact that all the children's angle work was conducted on the computer, thus the paper-based task was a sufficiently different context. Adding arrows to the triad angles might help evoke children's dynamic imagery, thus supporting their dynamic thinking—this view resonates with ideas of Clausen-May (2005) who suggests the use of arrow for representation of the movement in angle diagram as an attempt to avoid the loss of kinaesthetic concept of angle. Another alternative can be to provide the triad tasks in Sketchpad.

From a methodological point of view, Sfard's (2008) framework can be extended to incorporate the embodied routines as Chloe used the routine (Figures 1a, 2a) to compare the angles again and again, which helped her to navigate the flow of communication. Use of gestures with hands as arms of angle repeatedly enabled Chloe to see the process of turning even in case of static shapes, thus embodied routines could be helpful in looking at dynamic thinking, especially in case of young children.

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# SYMBOLS IN EARLY ALGEBRA: TO BE OR NOT TO BE?

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*As part of a larger project focused on algebraization in pre-school, this paper reports a pre-schooler's engagement in algebraic activities in the absence of algebraic notations. The aim of this paper is to question the necessity of algebraic symbols in early algebraization.*

## INTRODUCTION

Researches in early algebra are different in their approaches to use symbols. Many studies have focused on the use of symbols, while some more recent studies in the realm of early algebra do not necessitate the use of symbols for elementary students.

Despite much research that has been done in the field of early algebra, very few studies have addressed pre-school's algebraization. Since pre-school children in general are not even able to write number symbols, it is not wise to expect them to use written symbols for variables. Thus, the symbolic algebra or any kind of written symbols in algebra is beyond pre-schoolers' ability.

The current study, focussing on the pre-schoolers' algebraization, shows that symbolization is not a necessary condition for algebraization in pre-school, so algebraic thinking can emerge in algebraic activities and in the absence of algebraic symbols for pre-schoolers. This paper reports a pre-schooler's engagement in algebraic activities in the absence of algebraic notations, or any symbolic notations for that matter. The main question that we want to address here is: How do we design an algebraic situation in the absence of symbols? To answer this question, we exploit a newly introduced idea that takes into account some major obstacles to learning algebra in the early grades.

## RELATED LITRATURE

Review of recent research in early algebra shows that there is no common point of view about the use of symbols in early grades among researchers. This is due to differences in both the definition of *algebra* and what the researchers mean by *early*.

Kieren (2004) regards algebraic thinking in the early grades as the development of ways of thinking within activities for which letter-symbolic algebra can be used as a tool but which are not exclusive to algebra and which could be engaged in without using any letter-symbolic algebra at all, such as, analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modelling, justifying, proving, and predicting. In her view of early algebra, algebraization is possible without written symbols. However, Kieren does not give a systematic way for or even any example of algebraization without symbols.



A research by Carraher (2006) has been guided by the idea that multiple problems and representations for handling unknowns and variables, including algebraic notation itself, can and should become part of children's repertoires as early as possible. His classroom studies suggest that children can handle algebraic concepts and use algebraic notation somewhat earlier than commonly supposed. For example his study shows that given the proper experiences, children as young as eight and nine years of age can learn to comfortably use letters to represent unknown values and can operate on representations involving letters and numbers without having to instantiate them. So there may be no need for algebra education to wait a supposed "transition period" after arithmetic. While Carraher's research subjects are at least at second grade, where children are able to read and write, these findings can't be generalized to pre-schoolers.

Kaput (2008) regards generalization and symbolization as the heart of algebraic reasoning. In his view, the only way a person can make a single statement that applies to multiple instances (i.e., a generalization), without making a repetitive statement about each instance, is to refer to multiple instances through some sort of unifying expression that refers to all of them in some unitary way, in a single statement. But the unifying expression requires some kind of symbolic form, some way to unify the multiplicity. Generalizing is the act of creating that symbolic object. This is where symbolization in the service of generalization- and algebra- starts, both within individuals and historically. So as he states, the use of conventional symbol systems is a necessary condition for an activity to be algebraic but it is certainly not sufficient. Indeed Kaput regards a symbolization activity as Algebraic if it involves symbolization in the service of expressing generalizations or in the systematic reasoning with symbolized generalizations using conventional algebraic symbol systems. Moreover, Kaput believes in different levels of algebraic activities. He defines an activity as quasi-algebraic if it satisfies the same conditions except that it may use any symbols, including traditional arithmetic ones, informal ones (including oral speech and physical manipulatives), or idiosyncratic ones. So he qualifies the algebraic use of numbers as quasi-algebraic activity. By algebraic use of numbers he means engaging students in reasoning with numerical statements that are being analyzed not for purposes of computation but for their structure.

A simple comparison between Kieren and Kaput's view of algebraic activities indicates how different understandings of algebraic activities leads to differences in attitudes towards the use of symbols.

While Kaput (2008) regards symbolization as a necessary condition for algebraic activities, Rodford (2011) states that the use of notations is neither a necessary nor a sufficient condition for thinking algebraically. Algebraic thinking is not about using or not using notations but about reasoning in certain ways. What characterizes thinking as algebraic is that it deals with indeterminate quantities conceived of in analytic ways. As he expresses, indeterminacy and analyticity can take several forms. And this is so because algebraic thinking can operate at different layers of generality. Some layers are more concrete, some more general. The most elementary form of algebraic thinking is *factual algebraic thinking*. Indeterminacy appears here in an intuited form: it is

expressed through particular instances of the variable in the form of a concrete rule or formula. This embodied form of algebraic thinking can be accessible to most of Grade 2 students. But, results from his study imply that Grade 2 students can also engage in more sophisticated forms of algebraic thinking which has been termed as *contextual algebraic thinking*. In this level, instead of using special known numbers, they use special but unknown numbers without using alphanumeric symbols. So indeterminacy and analyticity will appear in a more explicit way.

Although both Carraher's and Rodford's Researches are on 2<sup>nd</sup> grade student, they are not unanimous on using symbols. Carraher recommends using symbols in early grades, as early as possible, while Rodford points to other levels of algebraization which are more suitable for early grades.

While Radford introduces an analytical framework in order to distinguish arithmetic and algebra, Asghari (2012) presents the idea of *specularity* that could be used as an operational framemork for algebraization.

## **THEORETICAL FRAMEWORK**

There are major obstacles to overcome to get students to appreciate indeterminacy and analyticity, not least of which is students' reliance on the specific and students' propensity to calculate. The idea of specularity (Asghari, 2012) suggests a systematic way to overcome these obstacles in the generalization situations in which the aim is to get the learner move from "one" specific object to a restricted set containing that object. Indeed, specularity is a direct attempt to use these obstacles as stepping-stones towards the general. Specularity prompts the learner to take *a course of actions on a specific example for the learner* that is begging to be treated as a non-specific, particular example of its kind. Thus, a specular example is just an example whose non-specificity is unique to the individual. Unlike a generic example (in the sense of Mason and Pimm, 1984) that focuses on the structural features of the example at hand, a specular example focuses on the process acted on the example. The process used, supposed to be an already known-for-the-learner general process (like addition) that can be applied to an already known-for-the-learner specific object. However, the specific object is just there to be treated as a non-specific, particular example of its *kind*. In the light of specularity, we may turn a specific-for-the-learner equality into a *specular equality*, that is, an identity. But the variation of the objects (numbers) to which the equality may be applied is determined by the learner's conception of the numbers involved. This would be a major step towards algebra while respecting Rodford (2011)'s indeterminacy and analyticity aspects of algebraic activities at the same time, if skilfully used.

In the light of specularity as a practical framework, our current study tends to show the possibility of early algebraization without using symbols. This paper gives a snapshot of the study.

## **METHOD**

Our current study is a design experiment (Cobb et al., 2003) which is aimed to design algebraic activities for pre-schoolers and investigate how their algebraic thinking and their understanding of variable emerge and improve through those activities. This paper focuses on one pre-schooler to find out the evidences of her algebraic thinking in the absence of written symbols. This case study formed in a pre-school in Iran, in 2012. One of the researchers interviewed Aida—a 6 years old girl—21 times, once a week, while she was doing some tasks designed purposefully in the light of specularity. Aida has a good understanding of numbers, could count objects up to 20 and is able to calculate with numbers less than 5 fluently, sometimes with the aid of her fingers. The whole interviews were video-taped and transcribed for further investigations.

### **Task 1**

In this task there are 4 candies in a bowl on the table and Aida could see the candies. There are also two empty containers on the table. The containers are not transparent, one of them is white and the other one is green. The interviewer shows the inside of the containers to Aida to ensure her that they are empty.

[1] I [Interviewer]: How many candies are there in the bowl?

[2] A [Aida]: four.

[3] I: I want to remove candies from bowl and put them in the containers. Please close your eyes while I do that.

Aida covered her eyes with her hands. The interviewer removes the candies from the bowl and put some of them in the white container and the others in the green one. Aida cannot see how the interviewer distributes the candies among two containers. So she does not know how many candies are there in each container and she cannot see their inside because the researcher has covered them. The interviewer asks her to open her eyes.

[4] I: Do you know how many candies there are in each container?

[5] A: Three in this [pointing to white container] and two, no, one in this [pointing to green one].

[6] I: Three in the white and one in the green? Are you sure?

[7] A: Or if not, one in this [pointing to white container] and three in this [pointing to green one].

[8] I: Ahha... What else may happen?

[9] A: Because they were four, if we take one, this [white] would be 3, this [green] would be one.

[10] I: But I remember none of this happened...

[11] A: If you get one out of three, this [pointing to white] would become two and this [pointing to green one] is two.

**Task 2**

In this task there are two small dolls, a frog and a ladybird, each one attached to an empty container. In this task interviewer puts the same number of candies in each container. Although Aida knows that the frog and the ladybird have the same number of candies, she doesn't know the exact number of candies. In this task interviewer removes three candies, one by one, from the frog's container and asks her which doll has more candies?

[12] A: Ladybird.

[13] I: How many candies does the ladybird have more than the frog?

[14] A: It [the ladybird] has three more.

[15] I: Now I take one candy from the ladybird...

[16] A: this [ladybird's candies] become a bit less, this [frog's candies] is much less.

[17] I: Which one has more?

[18] A: Again the ladybird.

[19] I: How many candies does the ladybird have more than the frog?

[20] A: Two.

**Task 3**

In this task, there is an empty container, a bag of hundred candies and more unpacked candies. Interviewer puts some candies in the container while Aida sees the whole process and count the number of candies in the container. Then the interviewer opens the hundred candies bag (while Aida knows that this bag contains hundred candies) and pours it to the container. Next she removes some candies from the container or adds some. Aida sees the process, and interviewer ask her the number of candies added or removed to be sure that she is informed of this number.

In this experiment, the interviewer puts two candies in the container, add the hundred candies bag to it and then remove three candies.

[21] I: Are there more or less than hundred candies in the container?

[22] A: Less than hundred. Because you added two candies and removed three. So it is less than 100. You should add one.

**FINDINGS**

In this part we are going to discuss the algebraic aspects of Aida's thinking in three preceding tasks. As mentioned before, these three tasks are part of a set of more tasks designed for children to put them in situations where algebraic thinking is possible to emerge. These tasks may help them to make generalizations. Here we explain that how specularity would help them in order to make generalizations, and how this slice of algebraic thinking may happen in the absence of any written symbol.

In task 1 when the interviewer asks Aida to make a partitioning of 4, she says 3 and 1. After Interviewer's insistence on another answer, she announces another answer: 2 and 2. Aida clearly shows that how she makes this new partitioning in line [11]. Removing one from 3 and adding one to 1, gives this new answer. We can translate Aida's saying words to arithmetical writing words as  $4 = 3+1 = (3-1) + (1+1) = 2+2$ . So Aida makes this new partitioning by changing the previous one. But how did Aida make the first one? At the first sight we may think that Aida uses number facts, since she has a good understanding of numbers under 5. But deepening in line [9], shows that she uses the same strategy for her first answer. She makes 3 and 1 from 4 and 0, however she never announces 4 and 0 as an answer while she thinks both containers involve candies. So here is the translation of her words to symbols:  $4 = 4+0 = (4-1) + (0+1) = 3+1 = (3-1) + (1+1) = 2+2$ . Aida skilfully knows when should calculate the answers and when not. She knows that  $3 + 1 = 4$  but she changes the structure of the expression in this form:  $3 + 1 = (3-1) + (1+1)$  and then calculates it in some parts:  $3+1 = (3-1) + (1+1) = 2+2$ . It is possibly the context of the task that leads her to think in this way. So she uses the same restructuring two times in this task with different starting expressions: the first is  $4+0 = (4-1) + (0+1) = 3+1$  and the second one is:  $3+1 = (3-1) + (1+1) = 2+2$ . As it can be seen, she passes from consideration of the specific-for-her starting numbers (4+0 and 3+1), to new set of numbers while keeping the algebraic relation  $X+Y = (X-1) + (Y+1)$  intact. Thus these seemingly arithmetic equalities are more than two specific equalities and they have turned to be *specular equalities*. However, we should be careful about the variation of the numbers to which the equality may be applied. As mentioned before, the variation is determined by the learner's conception of the numbers involved. In the context of this special task, it is possible that for the learner this *general equality* holds just for Xs and Ys with the sum of 4 and not even for all natural sums. However, even with such a small variation in mind, we are intended to think that the task helps Aida to involve in an algebraic activity without using any writing symbols.

Task 2 starts with two equal but unknown quantities: number of frog's candies (say F) and number of ladybird's candies (say L). Aida can identify the doll with more candies after each change in the number of candies. In addition, she can say the difference between these two still unknown numbers. Knowing that  $F=L$ , Aida says that L is three more than F-3 (line [14]). In another language, if  $F = L$  then  $L = (F-3) + 3$  or  $L - (F-3) = 3$ . To put it simpler, while  $L=F$  these equalities could be written as:  $L = (L-3) + 3$  or  $L - (L-3) = 3$ . Moreover she recognized that L-1 is still 2 more than F-3 or  $(L-1) = (F-3) + 2$  or  $(L-1) - (L-3) = 2$  (line [20]). Although these equalities are symbolically hard for the early algebra level, Aida deals with them fluently without using symbols. The starting number of candies, say L, is an unknown number for Aida, but it is also specific for her, while she can disclose the number of candies whenever she likes and check her answer's correctness. Although it is not evident in the preceding episodes, there were many times in the interview with Aida where she wanted to assure that her answer is

right or wrong. The capability of checking the answer is strength of this specular number<sup>1</sup>. The container, involving a special number of candies, plays the role of a specular number for Aida which helps her to pass from a specific-for-her starting number to another number. Again, we should be careful about the variation of numbers to which the above identity may be applied.

The hundred candies bag in task 3 plays the same role as the container for Aida in her algebraizations. Despite the fact that Aida is not capable of doing arithmetic operations with 100, she comprehend 100 as a whole number. When the interviewer asks Aida to compare 100 and  $(2 + 100) - 3$ , she does not calculate  $2 + 100$  and then  $102 - 3$ . She answers the question by restructuring the expression. As she states in line [22], Aida looks at this expression as  $100 + (2 - 3)$  which is one less than a hundred. So algebraically speaking, she uses two rules of commutativity and associativity in order to answer the question in this way:  $(2 + 100) - 3 = (100 + 2) - 3 = 100 + (2 - 3) = 100 - 1$ . It is worth noting that Aida has previous experiences of these two rules, for example in task 1 line [7], Aida make the 1 and 3 partitioning simply from the first 3 and 1 partitioning, which shows her understanding of commutativity. Furthermore, it should be highlighted that Aida's thorough understanding of 100 as a fixed whole helped her to answer the question. Although it is not mentioned here, previous experiments with Aida, when she did not understand 100 as a fixed whole and comprehended it as "very much", shows that she could not give a correct answer to the same question. For example she answered  $2 + 100 - 3$  as 3 less than 100. This answer is due to her incomplete understanding of 100 as "very much" which let her see  $2 + 100$  the same as 100, since 2 more than "very much" is still "very much"!

## FINAL REMARKS

This study focused on the algebraization of preschoolers while they do not have any experience in reading and writing. The experience of Aida, a pre-schooler engaged in some algebraic activities, shows the potential of specularity as a conceptual tool for making generalizations in the absence of writing symbols. As mentioned before, specularity takes its strength from the two weaknesses of students: their reliance on the specific and their propensity to calculate. So specularity could be better used in earlier grades while the tendency to specific is much more than upper grades. Moreover, the preschoolers' propensity to calculate along with their disabilities in calculations (as the disability of Aida in calculation with 100 in spite of her thorough understanding of 100 as a whole), might be used in favour of specularity.

It is worth repeating that the idea put forward in this paper best works when we try to get the learner to see through the specific-for-the-learner numbers while keeping certain algebraic relations intact. Indeed, a specular number is just a specific number for the learner through which non-specificity could pass if skilfully used. Specularity prompts the learner to take a course of actions on a specular number, a specific number

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<sup>1</sup> A computerized version of this task was not as successful as this version, while there was no chance for children to check their answers in reality.

for the learner that is begging to be treated as a non-specific, particular example of its kind. For us, a specific-for-the-learner number is a number that is conceived as a whole. And we are inclined to think that algebra (in the sense of this paper) would be possible with such sense of number. That is why Aida was a good case, and that is why we believe that algebra is possible even within a very small range of numbers, say 1 to 5. It to say that early early-algebra (sic) is possible. This is the possibility that we pursue in our current research.

As a final note we would like to add that the absence of writing makes some difficulties for children. For example, the absence of documentation places a large memory burden on children to keep track of the number involved. So in our main study, this memory burden is to be reduced by the design of activities.

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# THE PEDAGOGICAL CONTRIBUTION OF THE TEACHER AIDE IN INDIGENOUS CLASSROOMS

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*This study examines the pedagogical contributions made by teacher aides in underperforming Indigenous mathematics secondary classrooms in Australia. In addition to interviews and questionnaires over the period of one year, mathematics classrooms were observed for details of pedagogical contributions to the mathematics lessons by both the teachers and the teacher aides. It was found that the pedagogical contributions of the teacher aide included various forms of co-teaching, the provision of menial support and behaviour management. The techniques used by the teacher aide to provide student feedback, to support behaviour management and to undertake questioning vary greatly. The implications of teacher aides who are in a sense “teaching” are discussed.*

## INTRODUCTION

The pedagogical contributions of teacher aides in underperforming Indigenous secondary mathematics classrooms in Queensland, Australia are explored in this paper. Gerber, Finn, Achilles, & Boyd-Zacharias (2001) provided an analysis of how teacher aides have been utilised over time. The studies cited indicate that since the 1950s, the main duties have been in relation to administration, menial tasks, and the helping of individual students. More recently, there has been a trend for the teacher aide to be involved in direct instruction of students in small groups, and in some cases, the entire class. The teacher aide role now is more directed to support the teaching and learning process. Groom (2006) indicated that the role of the teacher aide was “to have a particular focus on supporting learning, including key aspects of the pupil’s personal and social development ... establishing a ‘positive relationship for learning’” (p. 199). A major component of the role of the teacher aide is to support students with their mathematics learning, and behaviour problems (Walther-Thomas, Bryant, & Land, 1996) through interacting with the students on an individual basis or in a small group (Rubie-Davies, Blatchford, Webster, Koutsoubou, & Bassett, 2010). Teacher aides are generally utilised for managing classroom behaviour and assisting teachers during group work in the lower achieving classrooms (Baxter, Woodward, & Olson, 2001).

We adopt the definition that the teacher aide is one who provides direct and indirect *support to the student*, under the supervision of the teacher and the administration staff (Howard & Ford, 2007). This “support to the student” may not be the case for Australian Indigenous classrooms. Warren, Cooper, and Baturo (2004) described a large study in the Cape York Peninsula by Valadian and Randell which told of the Indigenous teacher aide’s role in the classroom to be that of a classroom helper with limited involvement with the students, despite the teacher aide often being in a very knowledgeable position in terms of the cultural and family backgrounds – variables of



high significance in the Indigenous classroom. Westernised styles of teaching mathematics by beginning teachers, many of whom are oblivious to the culture and practices of Indigenous students, result in a gap in the teaching and learning process (Cooper, Baturo, & Warren, 2005). According to Cooper, Baturo and Warren, the lack of experienced teachers in rural and remote schools with Indigenous students makes the role teacher aides play in classrooms vital to the educational success of the students. Teacher aides can be positively utilised to work collaboratively with teachers to bridge the cultural-content gap resulting in better teaching and learning (Warren et al., 2004). Indigenous teacher aides have the potential to bridge the gap between culture and Western schooling, particularly in contextualising (Matthews, 2003) mathematics learning so that mathematics concepts can have relevance and meaning for Indigenous students. In many instances, the Indigenous teacher aide is not trained in their role, they receive no information on how to assist their teacher in the classroom, and may not have a mathematical background beyond that of the students. Warren et al. (2004) found a paucity in the literature with regard to how the teacher and teacher aide work together to support Indigenous students. There is still paucity in such research, especially in the area of the pedagogical contributions being made by the teacher aide in the mathematics classroom.

The current study sought to determine the pedagogical contributions of the teacher aide. Our specific questions are: (a) what are the pedagogical contributions of the teacher and teacher aide in these underachieving Indigenous classrooms? and (b) what are the implications of these contributions for student learning?

## **DESCRIPTION OF STUDY**

### **Participants**

This study is drawn from the larger Accelerated Indigenous Mathematics (AIM) project which investigates the acceleration of mathematics learning for underachieving Indigenous secondary school students. In the larger project there are 9 secondary schools, each with significant Indigenous populations of students studying mathematics in Years 8, 9 or 10. The vast majority of the students have a Year 3 mathematics level – hence they are underperforming. The majority of classrooms in the project consist of a teaching team of a teacher and a teacher aide (neither of whom are trained to teach mathematics). The teacher aide involvement in the classrooms ranges from the teacher aide permanently assigned to the classroom teacher for all lessons, to the teacher aide assisting the teacher for one lesson a week.

### **Data sources**

Three sources of data were available concerning the teacher aides' classroom pedagogical contributions:

General classroom observations had been made by a variety of researchers during scheduled school visits over an eight month period. These observations were recorded on a template document, and included comments about the teacher, teacher aide,

mathematical topic and impressions of student learning. There were approximately six of these subjective observations available for each classroom.

Specific classroom observations of teacher and teacher aide were made during two consecutive mathematics lessons by the first Author in three of the schools. At intervals of five minutes, the classroom actions of both the teacher and the teacher aide were noted on a checklist, as were major disturbances (e.g., entrance of a visitor). Additional pedagogical occurrences were added to the checklist where necessary. Interviews were conducted with each teacher and teacher aide.

Semi-structured interviews followed each lesson and involved a joint interview (teacher and teacher aide) seeking information concerning the lesson just conducted. Individual semi-structured interviews were conducted with each teacher and teacher aide to seek information about the role of the teacher aide in the classroom. All interviews lasted approximately 30 minutes.

### Data analysis

The general classroom observation sheets were analysed for references to pedagogical contributions. A simple numerical count of the pedagogical contribution was made and the semi-structured interviews were transcribed, coded and categorised into frequencies.

## RESULTS AND DISCUSSION

A summary of the results for the pedagogical contributions is provided in Figure 1.

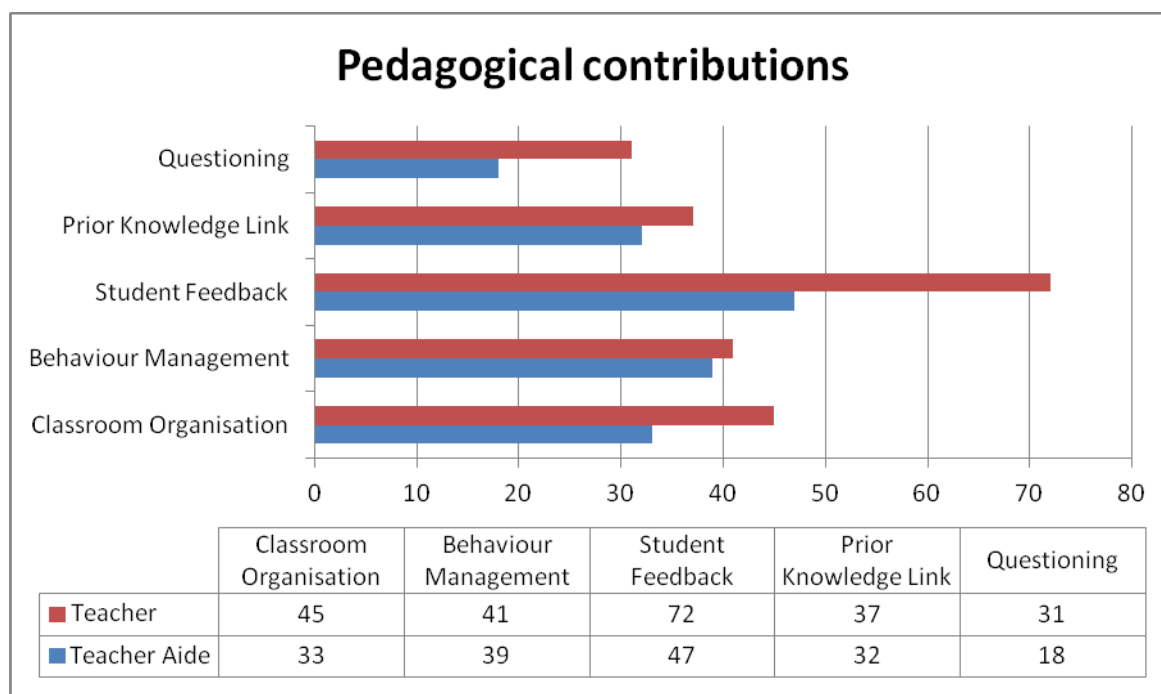


Figure 1: Pedagogical contributions of teachers and teacher aides.

The five pedagogical contribution categories presented in Figure 1 have been expanded in Table 1 to present a deeper analysis. Classroom organisation contributions related to the organising of either the students or materials. Behaviour management

tended to be either proactive or reactive. Student feedback was either task oriented or in the form of praise or criticisms. The links to prior knowledge related to real-world knowledge (a desired component of the AIM project), or to previous lessons. Finally, the types of questions used in the classrooms were found to be either higher order or lower order questioning. This paper will focus on the latter three categories only (feedback, prior knowledge and questioning) as these categories were found to have a direct impact on the development and learning of mathematical concepts and knowledge. The former two categories (organisation and management) had little impact on the development and learning of the mathematics.

	Teacher 1	Teacher Aide 1	Teacher 2	Teacher Aide 2	Teacher 3	Teacher Aide 3	%*
1. Classroom Organisation							
Students	7	6	2	6	12	1	41
Materials	4	6	12	15	4	7	59
2. Behaviour Management							
Proactive	8	6	7	8	6	1	56
Reactive	1	1	1	2	15	8	54
3. Student feedback							
Affect	12	11	6	3	15	3	58
Task	7	10	1	2	8	8	42
4. Prior knowledge							
Reality	14	12	7	9	3	3	62
Classroom	6	7	3	1	12	1	38
5. Questioning							
Higher order	2	4	1	0	4	0	22
Lower order	12	10	8	3	7	0	78

Note: \* denotes percent within a category and not as a total

Table 1: Comparison of teacher and teacher aide pedagogical contributions.

### Student feedback

Fifty-eight percent of the student feedback was in relation to the affect. Feedback from both the teacher and teacher aide was often jovial and, as a result, the students responded well as they were relaxed and seemed to mostly enjoy their lessons. There was generally a very genuine feel to the praise and the students responded well. An example of such feedback comes from Teacher Aide 1: “Yes I know it is tricky, but you can do this, I know you can. Work with me and you will see that you have the ability to get this done well”. Essentially the teacher aide is trying to boost the confidence of the student so that the student does not give up. The comment is about the student achieving, and not about the nature of the task. Unfortunately, there were also considerable levels of critical feedback of a student or his/her work. Rarely did this criticism come from the teacher aide. An example of critical feedback in a situation

to the one just described is from Teacher 3: “No, you won’t be able to do this, because you don’t come to class often enough. Why do you bother to come at all?” The remaining student feedback related to the task being undertaken. Comments regularly heard from Teacher 1’s classroom were similar to this one: “This task is hard. It’s probably going to take a while to get it right, but it is possible for all of us to do it well. We just have to give the task a go”. In this classroom the teacher and teacher aide regularly located the difficulty of content to the task, rather than in relation to the student. The student was not told something was too easy or difficult for *them*; rather it was the *task* that was easy or difficult. The implications of this are that students do not tend to see themselves as dumb and unable to do mathematics. Instead, they see mathematics itself has difficult components that they are not solving at that moment. This approach appears to develop a level mathematical self-esteem in the student, unseen in many Indigenous students. In fact, students in this particular classroom have been known to demand (!) more maths in their lunch hour.

### **Prior knowledge links**

One of the underlying philosophies of the AIM project is for the mathematics to be presented to the students so that it relates to the real world of the student – that is, it has personal relevance. Some teachers and teacher aides in the project are beginning to emphasise the reality of mathematics to the students’ world, rather than linking the present lesson to a previous classroom lesson. Mathematics is not culture-free; culture is closely connected to the way in which we learn and understand mathematics (Ladson-Billings, 1997). The Westernised style of teaching mathematics by Teacher 1 is being complemented by Teacher Aide 1 who brings to light aspects of Indigenous culture and practices of the Indigenous students, relevant to the mathematics teaching and learning process. Linking cultural knowledge to the mathematics lesson is essential with Indigenous learners so that the mathematics becomes meaningful. The lack of experience that Teacher 1 has in the community makes the role of Teacher Aide 1 in the classrooms vital to the educational success of their students. Teacher Aide 1 is working collaboratively with Teacher 1 to bridge the cultural-content gap, possibly resulting in better teaching and learning. A logical flow between the two consecutive lessons is needed to build on conceptual understanding, but this flow may not be apparent to all students. Teacher Aide 1 explained in the joint interview with Teacher 1 that:

Sometimes kids need the links mentioned to what we did in a different lesson, but mostly they respond if we make it match something outside school. School maths is not important to them, but now we show it is in their community, they participate more.

Although Teacher 2 and Teacher Aide 2 relate the mathematics primarily to the real world of the student, these links are not as prevalent as those made by Teacher 1 and Teacher Aide 1. Prior knowledge can simply be related to previous lessons, and not to the real world. When this occurs, very little purpose is given to the learning of mathematics other than what has been presented in previous lessons. If a previous lesson is to be referred to, then it is important that the link is also made to the real world. The implications of this are that personal relevance and continuity of lessons are

both emphasised. Where the teacher aide is involved in creating the personal relevance, the students can see that mathematics does not just come from a book or teacher, that their community members do in fact know maths. This is important for personal identity development.

### **Questioning**

Questioning is not prevalent in many of the project classrooms. Teacher Aide 2 indicated in her individual interview that:

Indigenous students don't really respond to questions. There is this shame issue – they do not want to be “shamed” by giving the wrong answer. But the opposite is also true – they do not want the attention if they give a correct answer. The easiest thing is not to use a lot of questions.

When questioning is used in the classroom, it is generally in the form of lower order questioning, but some higher order questions are asked. Where higher order questions are posed to the students they generally respond with some correctness. In one classroom, higher order questions are asked, but very few students, if any at all, respond with an answer. Classroom observations of the teacher in this classroom were that there was a lack of trust between the members of the class. Where there is no trust, the students do not feel able to respond to questions that require more than a simple yes, no, or rote learned answer. In general, the students do not seem to respond well to higher order questioning, but Teacher 2 is persevering: “I don't have their confidence yet. But we are making progress as they will now respond to simpler questions”. Teacher Aide 3 was not observed to ask questions of the students. In her individual interview concerning her roles in the maths classroom she indicated:

[Teacher 3] is the teacher and I just help. I don't do anything without being asked unless it is obvious. I guess I do the running around while she teaches. But when she is away, I teach ... I put notes on the board for the kids to copy; I might ask a question then.

The implications of this is that whilst there may be a lack of higher order questions being asked, it may not be that the students do not have the ability to respond, it may simply be outside their comfort zone to do so. It is also necessary for the teacher and teacher aide to develop a safe classroom culture where it is acceptable for questions to be asked, and it is equally acceptable for the students to respond without fear of shame. The teacher aide also needs to feel comfortable in the class to be asking questions. Self-confidence, trust and safety need to be established before learning and the sharing of ideas can occur.

### **Interrelationships between pedagogical contributions**

There appears to be a link between student feedback and questioning. From the observations and interview data, Classroom 1 is a happy classroom, the students are at ease, they are regular attendees to class, and both the teaching team and the students appear to engage in a jovial exchange of knowledge. The students respond to some level of higher order questioning. It appears that in this classroom, Teacher 1 and Teacher Aide 1 work together as equal partners in a dynamic and interactive

relationship – a form of co-teaching. Teacher 1 commented in the joint interview that he values the contributions of Teacher Aide 1. Further research is required to explore the potential benefits of this approach to teaching mathematics to underachieving Indigenous students. Classroom 1 is in stark contrast to Classroom 3 where there is little communication between the students and the teaching team. Communication between the teaching team and students is critical in nature and often reactive, and student attendance is irregular for the majority of students. The quantity of student feedback is not as vital as the quality. There seems to be an alignment with high levels of positive affective student feedback, high levels of linking prior knowledge to the reality of the student, and to the emergence of higher order questioning that students feel safe to respond to.

## CONCLUDING POINTS AND IMPLICATIONS

It is beyond the scope of this paper to attribute the pedagogical contributions of the teacher aides to learning outcomes of the students. Whilst the student learning data exists, that story remains untold at this point in time. It is well documented in the literature (see for example the work of Ingersoll) that there exists a high correlation between the quality of the teacher, the teacher's qualifications, coupled with the occurrence of the teacher teaching out-of-field, and student performance on standardised tests. In the classrooms of the AIM study, 89% of the teachers are teaching out-of-field; they are also inexperienced in the Indigenous classroom and most are new to teaching. It is interesting to postulate the confounding factor of co-teaching with a teacher aide who has no qualifications and in many cases less education than the students. In some classrooms, the teacher aide does the "teaching" when the classroom teacher is away from the school – for several days at a time in some cases.

The pedagogical contributions of the teacher aides vary within each classroom. If the teacher wants to control the class and teach in a traditional manner, then the contribution by the teacher aide will be minimal – as in the case of Classroom 3. Conversely, if the teacher permits the teacher aide to use their initiative, then co-teaching can occur, with the teacher aide making a strong pedagogical contribution to each lesson. If the latter is the case, it raises an interesting question for further research: What is the impact of "unqualified teachers" on student learning? If teacher aides are to be working in co-teaching or teacher replacement situations in Indigenous classrooms, then they also need to be involved in the production of curricula so considerations of learning and appropriate teaching strategies are shared.

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# **PEDAGOGICAL CHANGES IN MOVING FROM TRADITIONAL WORKSHEET TO ACTIVE STRUCTURAL CLASSROOM PEDAGOGIES INSPIRED BY AUSTRALIAN INDIGENOUS LEARNING APPROACHES**

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*This study examines the pedagogical changes made by teachers as they modified their mathematical classroom pedagogies from traditional to an activity-based structural form titled YuMi Deadly Maths and inspired by Australian Indigenous learning approaches. It is based on public presentations by teachers in which they described their own and their student changes in roles over a longitudinal study. It was found that the pedagogical changes benefited students in terms of attendance and engagement, mathematical literacy, discussion and argument, mathematical performance, and conceptual understanding and critical thinking. However, the study also found that there were barriers to implementing this new pedagogy.*

Being a teacher is unpredictable. The moment-to-moment interaction between the teacher and the student, the teacher's knowledge of their curriculum, and the teacher's pedagogical prowess all influence and shape the classroom experience for the student. Teaching practice is therefore a fluid system which constantly shifts and changes (Apter, 2001). The teacher is influenced by alternative innovative approaches occurring in other classrooms, in curriculum documents and in universities, but also by her experience of what is successful and unsuccessful. How these influences interact to enable and prevent innovation is the focus of this paper.

The data for the paper comes from teachers' articulations of their classroom experiences with a new pedagogy. This was seen as valuable for the teachers and in terms of data because as Hall (2009, p. 678) stated:

Teachers gain in confidence in articulating their embodied practical knowledge and in translating the contextual understandings of their own classrooms to a wider audience. Moreover, this participation ... fosters the critical engagement with ideas and approaches, which underpins teachers' future decision-making about innovation and change in their practice.

## **INDIGENOUS LEARNING PERSPECTIVES**

The YuMi Deadly Maths program was built on the work of Uncle Ernie Grant (1998), an Indigenous elder and educator, and Matthews (2006), an Indigenous mathematics researcher. Grant's Aboriginal holistic planning and teaching framework focuses on six perspectives in relation to a mathematical idea: Land, Language, Culture, Time, Place, and Relationships (see Figure 1). Grant argued that Place means to focus on interactions that come from a holistic view. In the Torres Strait Islander adaption of this framework, place and relationships were combined to mean Deep Learning.



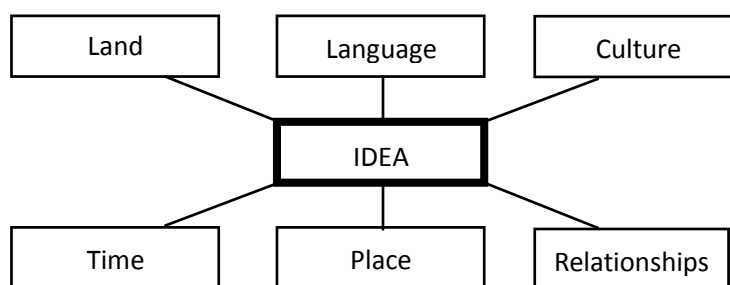


Figure 1: Aboriginal holistic planning and teaching framework (Grant, 1998).

Matthews also argued for deep learning and, therefore, for mathematics learning to be based on the algebraic structure of mathematics. He explored the relationship between the students' perceived reality and the invented mathematics presented in the mathematics textbook (see Figure 2). He argued that abstraction and reflection are central to this relationship from reality to mathematics and return, involve creative and problem-solving acts, symbolic language and structure, as well as cultural bias, and that mathematics knowledge is created, developed and refined through this four-step cycle.

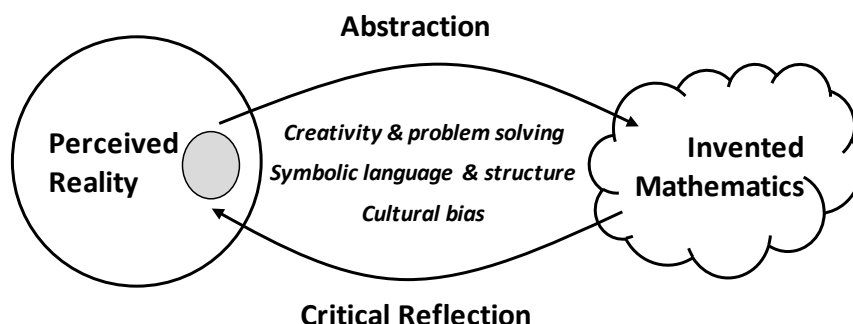


Figure 2: Reality to Mathematics relationship (Matthews, 2006).

## YUMI DEADLY MATHS

The researchers of the YuMi Deadly Centre have been working with Indigenous and low SES schools for many years and the positions of Grant and Matthews represented what they had seen and found to be successful in classrooms. Thus, they developed a mathematics program called YuMi Deadly Maths (YDM) which focused on how to teach and provided ideas for teaching, not lesson plans and worksheets. It was based on two major imperatives.

The first was to ensure that the focus of instruction was on big ideas in terms of concepts, strategies and relationships (or principles) and on connections and sequencing in terms of mathematics structure. The imperative was that Indigenous and low SES students should receive the highest quality mathematics instruction, one that focused on algebraic as well as arithmetical thinking. For example, the teaching of basic addition facts would involve a focus on strategies before automaticity. The strategy that  $8+5$  is found by building 8 to 10 and realising that  $8+5=10+3$  would be used as a starting point for teaching the role of identity and inverse and the compensation and equivalence principles for expressions.

The second was to combine Matthews' cycle in Figure 2 with the pedagogical models of Payne and Rathmell (1977) and Wilson (Ashlock et al., 1983), the generic strategies and levels adapted from Baturo et al. (2007) and the creativity in Matthews et al. (2007), to produce a cyclical pedagogical framework called RAMR (reality, abstraction, mathematics, reflection). The particular attributes of the model were: *R* – always start from the knowledge and culture of the student; *A* – move activities through body (acting out) to hand (physical, virtual and pictorial models) to mind (patterns and mental models), and construct own symbols before appropriating the standard symbols; *M* – build connections as well as practice; and *R* – students validate and apply knowledge back into their own lives and seek to extend knowledge through flexibility, reversing, generalising and changing parameters.

In this way the RAMR model focuses instruction on the underlying structure of mathematics, providing students with a holistic view, and builds mathematics as a tool for problem solving and a life-describing language. The goal of mathematical learning is to empower students to use the most powerful forms of mathematics to solve problems in their lives. Engaging mathematics students in active exploration of real-life scenarios and supporting inquiry processes are major challenges for teachers, requiring a shift in the teacher's role from lecturing and telling to listening, observing, facilitating, and guiding (Yerushalmy & Botzer, 2011). This paper describes this shift.

## DESCRIPTION OF STUDY

The findings presented in this paper are those pertaining to a project, called Teaching Indigenous Mathematics Education (TIME), to train teachers as YDM trainers for Indigenous and/or low SES schools. The project ran for three years, focusing on Years Pre-Prep to 3 in 2010, Years 4 to 7 in 2011 and Years 8 to 9 in 2012, and covering all mathematics strands: Number, Operations, Algebra, Geometry, Measurement, and Probability and Statistics. The purpose of TIME was to improve teachers' mathematics knowledge and classroom practices.

### Aim and research question

The aim of this paper is to identify the impact on teacher and student roles as classroom pedagogy moved from a traditional to a YDM philosophy. Specifically, the research question is: *Which aspects of teachers' and students' classroom practices change as they gain experience in teaching and learning the YDM way?*

### Participants

The participants in TIME were Queensland state schools from low SES areas with high enrolments of Indigenous and/or low SES students. The participants/trainers in the TIME professional development workshops were: (a) principals, deputy principals and heads of curriculum; (b) numeracy coaches, teachers and teacher aides selected by those schools to attend the workshops; and (c) non-school based educational support staff employed by the Queensland Education Department. Limits were placed on numbers that could attend workshops so that no more than 60 schools could be involved in each year of the project. The number of schools was below this in 2010 (39 primary schools),

but approached this number in 2011 (55 primary and secondary schools) and 2012 (59 primary and secondary schools). The participants for this study were predominantly teachers who presented at the 2011 and 2012 TIME Sharing Summit.

### **Data collection and analysis**

Following the TIME professional development workshops, workshop participants documented the implementation of the YDM program in their schools in reflective portfolios. Reflective portfolios included a pre-test, planned activities, post-test and teacher reflection on students' mathematical understanding. At the end of the 2011 and 2012 academic year, many schools chose to attend a sharing summit and present experiences using their reflective portfolios as the basis of presentations to their peers about the implementation of YDM in their classrooms. These presentations were video recorded and the PowerPoint presentations collated for analysis. PowerPoint presentations have been simply coded and analysed for recurring themes within teachers' experiences of implementing YDM. Hall's (2009) key values of teacher autonomy and making public their work were also a basis of the analysis of presentations.

## **RESULTS AND DISCUSSION**

A longitudinal evaluation of the effectiveness of YDM in terms of teacher and student outcomes has begun. It is showing that students' attendance, engagement and class discussion have improved with some evidence, strong anecdotally, of improvement in mathematics performance. Teachers have also found it a much more positive teaching environment and teacher confidence and willingness to try new ideas has improved. However, this paper focuses on the classroom changes as YDM is implemented.

### **Learning and teaching**

Many teachers at the TIME 2012 Sharing Summit described the learning and teaching in their schools and classrooms as a form of "inquiry". These teachers felt that they were undergoing an inquiry process themselves as they taught using the inquiry approach of YDM. The presenting teachers often discussed the inquiry teaching and learning in language and literacy terms and felt that YDM was improving language and literacy skills as well as mathematics knowledge. Many presenters described their classrooms as involving a range of activities with multiple stages that blended oral and written discourses.

The YDM approach was designed to develop deep foundational mathematics knowledge; a depth of understanding often not possible through directive, controlling teaching – or traditional mathematics teaching. It was also designed to relate mathematics to the world of the students. The presenting teachers felt that this was successful; that implementing YDM allowed students to build on and expand their own natural problem-solving abilities allowing parallels between classroom mathematics and the students' real world. They believed that YDM teaching often provided challenging learning opportunities while, at the same time, providing support that allowed such challenging learning to be realized. Although there were instances where some teachers described the learning potential of students as low, the general

consensus was that student learning from YDM usually exceeded the teachers' expectations. They described the students as arriving at YDM classes enthusiastic to learn. The presenting teachers recognized the importance of student ownership of their own knowledge (Matthews et al., 2007), as well as the value of effective guidance and modelling in the development of foundational understandings in mathematics.

The presenting teachers generally saw hands-on teaching and learning as central to mathematics learning in YDM classrooms. All teachers participated in some form of abstraction body-hand-mind teaching with results initially dependent upon the student cohort. This importance, however, does not imply that all the teachers pursued a single approach to the body-hand-mind teaching strategy. In the teaching context of YDM mathematics, this strategy was used in a variety of ways. It was seen both as a characteristic of a desired form of teaching (especially in the primary schools) and as a certain kind of activity (especially in the secondary schools). Overall, in terms of the outcomes of the YDM teaching, the teachers all concluded that the body-hand-mind approach to teaching produces positive results for both teaching and learning. Data on improved performance does exist based on the pre-post testing, but it is beyond the scope of this paper to explore this.

Traditional classroom directions		YDM classroom directions	
Teacher roles			
Dispenser of knowledge	→	Facilitator of knowledge	
Tells students information		Assists with knowledge processing	
Communicates with individuals		Communicates with groups	
Directs students actions		Coaches activities	
Explains links and relationships		Facilitates student thinking	
Knowledge is static		Participates in and models learning	
Knowledge contained in textbook		Readily improvises	
Student roles			
Passive learner	→	Independent learner	
Takes notes of information		Processes and thinks through information	
Memorises information		Interprets, explains and hypothesises	
Follows teacher directions		Designs own investigations	
Goes to teacher for knowledge		Takes ownership of work	
May not engage			
Student work			
Teacher-led activities	→	Student-directed learning	
Worksheets may be attempted		Tasks may vary among students	
Common tasks		Emphasis on literacy skills and communication	
Teacher directed and controlled		Emphasis on problem solving	
		Emphasis on interpretative discussion	

Table 1: Change in key roles in the classroom.

## Changes in roles

The presentations of teachers were analysed for information on changes in teaching prior to and after YDM training. Table 1 provides a summary of the new key roles that presenting teachers felt were appearing in mathematics teaching and learning in YDM schools compared to the traditional mathematics teaching and learning before YDM.

The right-hand column of the table highlights the shift in teacher and student roles through YDM. A brief look at the table highlights that the outcomes include literacy benefits, mathematical thinking process abilities, vocabulary knowledge, conceptual understanding, and critical thinking. The majority of the teachers seemed to have abandoned didactic, teacher-led instruction in favour of more participatory models in which their students were collaboratively engaged in the process of mathematical learning.

These changes identified by the teachers themselves are major and require effort from the teachers. Thus, regardless as to the question of its academic effectiveness, there is the question of whether or not it is possible to place whole-school YDM teaching into practice on a widespread basis. The answer from the presenting teachers was that, in general, it is possible; the teachers even felt that YDM teaching is possible for teachers to initiate without specific training, although they did express that this was difficult – for experienced teachers as well as newly qualified teachers. The length of time was a major concern for all the teachers as it can lead to piecemeal development. Some teachers reported that although other teachers in their school were beginning to take notice of YDM activities in their school, they had yet to attempt to replicate activities.

Importantly, by examining the presentations at the Summits, it appears that the teachers and schools had greater success in implementing YDM if they co-planned with other teachers, and if the development of the YDM philosophy and pedagogy was embedded into lesson and unit plans rather than being used as an introduction to a unit as a form of motivation.

## Barriers to YDM

In designing YDM, researchers were aware that as a program it required much of the teacher. It required teachers to understand the structure of mathematics and the major pedagogies for teaching, and to be able to prepare their own lessons and worksheets, based on the background and culture of their students. This was evident in the presentations; most teachers discussed barriers that must be overcome for teachers to acquire the YDM approach to teaching. For the presenters, much of the difficulty appeared to be internal, experience based and related to the teachers' beliefs and values (Cooper et al., 2004; Lamb, 2010) concerning mathematics, mathematics teaching and learning, the students themselves, and the purposes of mathematics education.

Using YDM does require teachers to learn new knowledge and skills, but to the presenting teachers, it was much more than this. They saw two dimensions of barriers: (a) the *technical dimension* which included the varying levels of confidence to teach constructively, prior commitments (e.g., to a textbook), the challenges of assessment,

the difficulties of group work when they believed that their students were not able to work as cohesive teams, and the challenges of unfamiliar teacher roles and of new student roles (described in Table 1); and (b) the *cultural dimension* which included an established culture of dependence on the teacher or the textbook and/or an established fear of mathematics.

It is evident from the teachers' presentations that the task of preparing teachers for YDM teaching involved the cultural dimension as much as the technical. The teachers agreed that they needed to acquire new assessment competencies, learn new teaching roles, learn how to put students in new roles and foster new forms of student work. However, they also argued that the cultural changes were as important. They felt that they had to change the culture of their classroom to cater for the new understandings and classroom practices of YDM. Some of the changes were from teacher dependence to student-led learning, from individual to group work and from European culture to integrating Indigenous and other cultural perspectives. They argued that it was important to have school staff who supported change in the classroom through a well-established rapport and willingness to share and collaborate. They saw collaborative working relationships among teachers as a very important context for the re-assessment of educational values and beliefs. They felt that teachers should not work in isolation for effective teaching reform. This supports the view that collaboration is a powerful stimulus for the reflection which is fundamental to changing beliefs, values and understandings (Lamb, 2010; Mulford, 2007).

## CONCLUSION

Teachers seeking a YDM orientation to their teaching essentially focus on the nature of student work, the students' role and their own role. To maximise change, teachers and others in positions of leadership should focus on creating a climate of collaboration among teachers and providing a context within which teachers can reflect on their values and beliefs. The core challenge facing each school is not to design some "correct" version of curricula or assessment that will be implemented "as is" by willing teachers, but to develop flexible support structures that facilitate local adaptation and ownership of each curriculum.

Interestingly, the teachers' presentations show that YDM, which emerged in part from a non-European cultural context, is effective when the classroom culture reflects what is powerful in Indigenous learning: holistic deep learning, collaboration and activity.

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# AN ANALYSIS ON THIRD GRADERS' MULTIPLICATIVE THINKING AND PROPORTIONAL REASONING ABILITY

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*The purpose of this study was to examine third graders' multiplicative thinking levels and to analyse how they applied such thinking in solving proportional problems. As a result, many students revealed transitional thinking among the four kinds of thinking levels. This means that they had a difficulty in distinguishing multiplicative situations from additive ones and that they tended to think both additively and multiplicatively. This study is expected to pay more attention to the multiplicative thinking in early grades and to provide instructional implications of teaching multiplication.*

## INTRODUCTION

Multiplication plays a critical role in students' mathematical development. It is widely used in many content areas such as number and operations, measurement, and pattern and function. Also, good understanding of this operation positively influences on students' meaningful use of algorithm and serves a foundation for proportional reasoning (Reys, Lindquist, Lambdin, & Smith, 2009).

To understand multiplication and use it proficiently, students need to think multiplicatively. Multiplicative thinking is different from additive thinking, which involves higher levels of abstraction and more inclusion relationships (Clark & Kamii, 1996). Given that multiplicative thinking plays an entry point to the world of ratio and proportion (Singh, 2000), the development of students' multiplicative thinking ability from the early grades is desirable.

However, in spite of the importance of multiplicative thinking, several studies have reported students' difficulties with multiplication learning (Siemon & Virgona, 2001). For instance, Clark and Kamii (1996) found that even more than half of the 5<sup>th</sup> graders who learned multiplication more than 3 years could not demonstrate solid multiplicative thinking. Other studies also reported that one of the most frequent errors in solving multiplication problems was students' incorrect use of addition. From these results, transition from additive to multiplicative thinking is likely to be a hard work for students.

Given this background, most studies focused on multiplicative thinking and proportional reasoning but they tended to deal with it respectively. Little is known about how one kind of thinking is related or influenced to another kind of thinking. Given that multiplicative thinking is important and is a foundation for later learning of proportion, it is informative to look into multiplicative thinking in relation to proportional reasoning. Specifically, this paper explores how multiplicative thinking works in solving proportional problems. It first explores students' multiplicative thinking levels and their characteristics, and then scrutinizes how they may function in



solving proportion problems. As such, this paper gives some implications of the importance of multiplicative thinking and instructional directions.

## **REVIEW OF LITERATURE**

### **Multiplicative Thinking**

While multiplicative thinking has been defined from various perspectives (Clark & Kamii, Piaget, 1987; 1996; Siemon & Bread, 2006; Steffe, 1994), it is usually defined by distinguishing from additive thinking. For instance, Piaget (1987) described the differences between them in terms of the number of levels of abstraction and inclusion relationship the child has to make simultaneously. In extending Piaget's work, Steffe (1994) regarded children's number sequences as the starting point of operations because operations would be constructed by modifying such sequences. In his teaching experiment, three different number sequences were identified as the process of constructing multiplicative thinking: the Initial Number Sequences, the Tacitly Number Sequences, and the Explicitly Number Sequences. It was reported that children's formation and use of units are progressively elaborated from easy counting up to multiplication. This study implies that multiplicative thinking is not developed at a specific time, but over time with several transitional levels.

With regard to the levels of multiplicative thinking, Clark and Kamii (1996) interviewed 336 children in grades 1-5 using a multiplication task. As a result, four developmental levels were identified in children's progression from additive to multiplicative thinking. For example, the children in Level 1 thought only qualitatively without referring to any numerical reference. The children in Level 2 showed additive thinking with a numerical sequence, whereas their counterparts in Level 4 reasoned multiplicatively and used the term "times" properly. The children in Level 3 demonstrated a mixture of additive and multiplicative thinking, so-called transitional thinking. Siemon and his colleagues (2006) proposed a learning assessment framework for multiplicative thinking. They developed the framework for multiplicative thinking comprised of 8 relatively discrete levels, in which students from Level 1 to 4 usually rely on additive reasoning while their counterparts from Level 5 to 8 can think multiplicatively using a broader range of numbers.

To summarize, multiplicative thinking is different from additive thinking and is sequentially developed. Knowing students' levels is important because it identifies where they are and can serve as a starting point for subsequent learning.

### **Proportional Reasoning**

Proportional reasoning has been emphasized. NCTM (2000) suggested that proportionality is an "integrative thread that connects many of the mathematics topics." (p. 217). Inhelder and Piaget (1958) identified proportionality within Piaget's stage of formal operational reasoning and illustrated it as ability to aware a secondary relationship between two pairs of quantities. Lamon (2007) proposed that proportional reasoning occurs when students recognize co-variation of quantities and in-variation of ratio simultaneously.

It was reported that students use different strategies according to the types of given proportion problems. For example, Cramer, Post and Currier (1993) found that students made use of several strategies such as unit-rate, factor-of-change, fraction, and cross-product algorithm in solving proportional problems categorized as a missing value, numerical comparison, and two types of qualitative situations. This implies that the use of diverse proportional situations is helpful to analyse students' proportional reasoning ability.

However, it is not easy for students to figure out how to solve various proportional problems. Students often use additive reasoning in solving tasks where proportional reasoning is required (Singh, 2000), or non-constructive strategies without reasoning such as avoiding, visual or additive approaches and pattern building (Lamon, 2007). These difficulties result from the lack of profound understanding of multiplicative relationship between quantities, which is the foundation of proportional reasoning. Given this, it seems important to explore the relationship between multiplicative thinking and proportional reasoning.

## METHODOLOGY

### Data Collection

Two kinds of questionnaire were developed for this study. Questionnaire I (see Table 1) was intended to explore students' various levels of multiplicative thinking. It was made by modifying the tasks in Clark and Kamii (1996), and Steffe (1994). Questionnaire II (see Table 2) was to examine how students' different multiplicative thinking levels might work in solving proportional tasks. It was developed by modifying the tasks in Cramer et al. (1993) and Van Dooren et al. (2003).

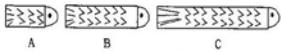
Type	Description	Sample item	Item
M1 (Times)	Recognize 'times' relationships among quantities and find a missing value in the problem context	Three fish shown below eat food in proportion to its length. If fish A eats 1 chip, how many chips will fish B eat? 	5
M2 (Part-Whole)	Understand the multiple inclusion relationships of part-whole and infer quantities	Yujin receives 7 boxes of cookies. Each box has 4 cookies. How many cookies does Yujin have?	5

Table 1: Questionnaire I of multiplicative thinking

	Type	Sample item	Item
Proportional	P1: Missing value (direct proportion)	In a toy shop, if you collect 10 coupons, you get 12 candies. How many candies will you get for 15 coupons?	8
	P2: Missing value (inverse proportion)		
	P3: Numerical comparison		
	P4: Qualitative prediction/comparison		
Non-proportional	NP1: Additive relationship ( $y=x+a$ )	Hyeeun is 12 and her mom is 36 years old this year. When Hyeeun becomes 24 years old, how old will her mom be?	2
	NP2: Constant sum ( $x+y=a$ )		

Table 2: Questionnaire II of proportional reasoning

Questionnaire I was given to 170 3<sup>rd</sup> graders from 5 different elementary schools in South Korea. As multiplication is introduced at the second grade, most of the participants already knew basic multiplication facts and could solve simple problems using them. Six students were chosen according to their multiplicative thinking levels and were asked to complete Questionnaire II. Semi-structured interviews were conducted with these students to probe their thinking processes.

### Data Analysis

The data collected from Questionnaire I were analysed by an analytic framework developed on the basis of the literature review (see Table 3). Four main levels were identified and sub-levels were developed according to the types of multiplication problems, but in this paper we focused only on the four levels.

Level		Type of multiplication problems	
		M1	M2
1	Qualitative thinking	Think only qualitatively	Add just numbers presented in the problem.
2	Additive thinking	In case of $A < B < C$ , regard $B=A+1$ or $A+2$ , and $C=B+1$ or $C=B+2$	Represent the problem situation in addition and solve it so
3	Transitional thinking	Use the term “times”, but add 2 or 3 in place of 2 or 3 times	Confuse relationship between part(cookies) and whole(boxes)  Represent the problem situation in multiplication, but solve it additively
4	Multiplicative thinking	Solve each problem by multiplicative thinking	Represent the problem situation in multiplication, and solve it so

Table 3: Levels of multiplicative thinking

An analysis of the data collected from Questionnaire II and interviews focused on which strategies and errors were used. Such strategies and errors were categorized on the basis of the literature review (see Table 4).

Category	
Strategy	Qualitative Reasoning (QR), Additive Reasoning (AR), Unit Rate (UR), Factor of Change (FC),
Error	Incorrect Qualitative Reasoning (IQR), Incorrect Additive Reasoning (IAR), Incorrect Proportional reasoning (IPR), Only one Value Focused (OVF)

Table 4: Categories for analysis of students' strategies and errors

## RESULTS

### Analysis of Multiplicative Thinking

Table 5 was the results from Questionnaire I. Given that students in Level 3 and 4 can recognize the problem contexts multiplicatively, more than 75% of the participants showed multiplicative thinking. However, the students in Level 3 were more than their counterparts in Level 4 and about 20% of the participants thought only additively.

Type	Level 1	Level 2	Level 3	Level 4	Total
	N (%)	N (%)	N (%)	N (%)	N (%)
M1	7 (4.1)	50 (29.4)	50 (29.4)	63 (37.1)	170 (100)
M2	1 (0.6)	21 (12.4)	108 (63.5)	40 (23.5)	170 (100)
Total	8 (2.3)	71 (20.9)	158 (46.5)	103 (30.3)	340 (100)

Table 5: Frequencies and percentages of multiplicative thinking levels

A close look at the results reveals that the Level 4 was shown most frequently with regard to M1 type, while the Level 3 was with regard to M2 type. This result means that recognizing the 'times' relationship and finding a missing value may be easier for 3<sup>rd</sup> students than figuring out the multiple inclusion relationship of part-whole.

It is noticeable that the students in Level 4 as for M1 type used various kinds of representations such as pictures, graphs, and expressions (see Figure 1). On one hand, given that these representations were rarely found at lower levels, these representations might help students develop solid multiplicative thinking. On the other hand, as for M2 type, higher level students tended to use more abstract representation such as equation (see Figure 2).

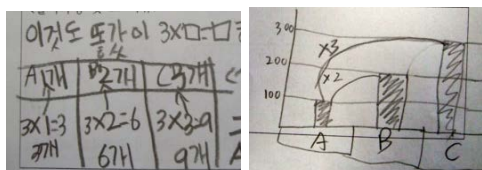


Figure 1: Representation in Level 4 (M1)

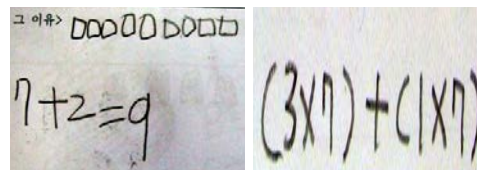


Figure 2: Representation in Level 4 (M2)

## Analysis of Strategies on Proportional Problems According to the Levels of Multiplicative Thinking

Table 6 summarizes how the six students, who were chosen to probe their proportional reasoning ability according to their multiplicative thinking levels, solved the given 10 tasks including their strategies and errors.

Level	Student	Frequency			Type of strategies and errors	
		Correct answer		Incorrect answer	Strategies	Errors
		Correct reasoning	Incorrect reasoning			
2	A	0	1	9	.	IQR, IAR, OVF
	B	4	0	6	QR, AR	IQR, IAR, OVF
3	A	4	1	5	QR, FC	IQR, IAR
	B	5	1	4	QR, AR	IQR, IAR, OVF
4	A	8	1	1	QR, FC	IAR
	B	8	0	2	QR, AR, FC	.

Table 6: Frequencies and types of strategies and errors

The students in Level 2 showed low correct answer rates and some errors. They focused only on the numerical values displayed in the problems without considering the problem situations. They also tended to add these numbers rather than to use multiplication, which led to incorrect additive reasoning (see Figure 3 for students' responses with regard to the toy shop task).

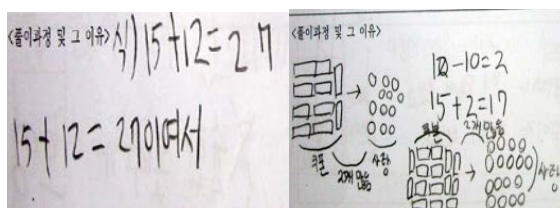


Figure 3: Students' responses in Level 2

For the students in Level 3, both addition and multiplication were used in solving the proportional problems. For instance, student 3A was able to write multiplicative expressions but calculated them using addition, which was perceived as easy and safe leading to the correct answer. It is noteworthy that the students in this level understood the inverse proportional situation by saying that if one quantity was increased, then the other was decreased. However, they were not successful in solving the problems.

The students in Level 4 completed successfully almost all the proportional problems in the questionnaire. They were the only students who could recognize the invariable quantities and found the missing values in the inverse proportional context. The following transcript illustrates a solution of the student 4B.

Interviewer: Can you explain how you solved this problem?

S-4B: To complete the task, 4 people need to work for 6 days. Multiplying 4 and 6 is 24 and dividing 24 by 8 is 3. So 3 days are needed if 8 people can work.

Interviewer: What does the number 24 mean?

S-4B: Completing the given task.

However, even the students in Level 4 often relied on drawing pictures or reasoning additively. These strategies were sometimes helpful to understand or solve the problems, but in other times they were a barrier to reason proportionally.

## CONCLUSION AND IMPLICATION

In this study, transitional thinking was most frequent among the four kinds of multiplicative thinking levels. This means that many students still have a difficulty in distinguishing multiplicative situations from additive situations. Given that the transition from additive to multiplicative thinking is not as smooth or straightforward as most curriculum documents seem to imply (Siemon & Breed, 2006), this kind of thinking is likely to be an obstacle for students to learn subsequent topics such as ratio and proportion. In this respect, multiplicative thinking is needed to be developed by dealing with lots of multiplicative situations.

Another noteworthy aspect is that students' thinking levels were distributed differently according to the types of multiplication problems. Level 4 appeared mostly in solving M1 type, whereas Level 3 did as for M2 type. This result implies that more students have difficulties in understanding various inclusion relationships of part-whole and inferring quantities. Note that constructing composite units and engaging in part-whole reasoning iteratively are regarded as the roots of multiplication (Steffe, 1994). Moreover, M2 type of problems is less frequent than M1 type in mathematics textbooks including Korean ones. Therefore, students need to have an opportunity to deal with many part-whole situations.

The performance of solving proportional problems was different by students' levels of multiplicative thinking. The students in Level 4 were able to solve most of the not-yet-learned proportional problems such as finding a missing value in inverse-proportion tasks or comparing three ratios. However, their counterparts in Level 2 were not able to solve such problems. These results are line with the claim that multiplicative thinking is a precursor for proportional reasoning (Lesh, Post & Behr, 1988; Singh, 2000). Therefore, multiplicative thinking needs to be developed since when multiplication is introduced.

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# WHAT DO HIGH SCHOOL MATHEMATICS TEACHERS MEAN BY SAYING “I POSE MY OWN PROBLEMS”?

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*The study is aimed at identifying mathematics teachers' conceptions of the notion "problem posing" and reasons for which the teachers pose problems for their teaching. The data are collected from a web-based survey with about 150 high school mathematics teachers and eight semi-structured interviews. We found that more than 50% of the participants in the survey see themselves as problem posers for their teaching. We identified five types of the teachers' conceptions of the notion "problem posing" and found that the teachers tend to explain what problem posing means in ways that would embrace their own practices. We also found that some of the teachers' conceptions are aligned with the institutionalized definitions of problem posing.*

## INTRODUCTION

Mathematical problem posing is widely recognized as one of the central activities in mathematics as a science and as a useful tool in teaching/learning mathematics (e.g., NCTM, 2000). However, a glimpse at the professional literature reveals that the notion of "problem posing" is used in variety, not always compatible, meanings and is applied to a variety of, not always comparable, teaching/learning situations. In addition, existing conceptualizations of problem posing, as diverse as they are, reflect the researchers' and mathematics educators' points of view on what counts or not as a worthwhile result of problem posing. The idea to inquire what problem posing means for mathematics teachers is still under explored. The study presented in this paper aims at partial closing this gap by exploring what the notion "problem posing" and the associated notions "a new problem" and "my own problem" mean for in-service mathematics teachers. This study is part of a Ph.D. dissertation, in progress, by the first-named author under the supervision of the two other authors.

## THEORETICAL BACKGROUND

Kilpatrick (1987) conceptualizes problem posing as reformulating an existing problem in order to make it your own. This conceptualization is deliberately poser-centred and depends on one's decisions about whether an existing problem is modified enough to be perceived by the poser's as his or her "own". From this perspective, one may decide that the problem is his or her own after making only a cosmetic change, whereas another person may feel that even the changes that look essential to the readers or solvers of the modified problem, are not enough in order to claim that a "new" problem is born.

Stoyanova and Ellerton (1996) define problem posing in teaching/learning situations as “the process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful



mathematical problems” (p. 518). The subjective nature of this definition – one should decide in which meaning the problem is meaningful and for whom – is apparent (see Koichu & Kontorovich, 2013, for the elaborated discussion of this issue).

Silver (1994) refers to problem posing either as generating new problems and questions for exploring a given situation or reformulating a given problem during the process of solving it. This conceptualization leaves room to inquire in which sense the processes involved in generating new problems and reformulating the given ones could be seen as instantiations of the same process tagged "problem posing". The same question can be asked in relation to the highly inclusive definition of problem posing by Crespo (2003), who refers to *selecting* worthwhile problems and *designing* challenging tasks for teaching as particular cases of problem posing.

Crespo's conceptualization brings us to the question about whether there is room (and need) for using "self-made problems" in teaching given that rich collections of expert-made problems are readily available, for instance, in mathematics textbooks. Extensive research on problem posing by mathematics teachers has not yet provided an unequivocal answer to this question. Indeed, in the majority of studies, mathematics teachers pose problems "on request" in laboratory conditions (e.g., Silver et al., 1996; Koichu, Harel & Manaster, 2013) or pose problems in the framework of professional developmental workshops aimed at enhancing their problem-posing skills (e.g., Crespo, 2003; Lavy & Shriki, 2007). Moreover, the probably most frequently repeated finding on problem posing by mathematics teachers is that not many teachers have skills to pose worthwhile problems (e.g. Singer & Voica, 2013).

Promising results on mathematics teachers' willingness to modify textbook problems and create their own problems is reported in Nicol and Crespo (2006). These scholars found that two out of four participants in their study attempted to extend the mathematical content of the chosen textbook problems in order to make them more complex. When the teachers were asked to prepare collections of problems for teaching in 4<sup>th</sup> grade based on the available textbooks, they preferred to create some of them. Based on these results, Nicol and Crespo distinguished between three ways of using textbooks by the teachers: “adhering, ” (i.e., do not seeing self as a resource) “elaborating,” (i.e., seeing self as a resource) and “creating” (i.e., seeing self as a knowledgeable resource for designing problems). The latter way of using the textbooks "brought forth opportunities to consider connections within and beyond mathematical topics" (p. 347). Note that this study was conducted with pre-service elementary school teachers.

To our knowledge, evidence about whether and how in-service high school mathematics teachers voluntarily pose problems for real use in their classrooms does not yet exist. Accordingly, and in light of the reviewed literature, our study pursues the following interrelated research questions:

1. To which extent high school mathematics teachers see themselves as posers of problems for their teaching? For which purposes they pose problems?
2. How do the teachers perceive the notions "problem posing" and "my own problem"?

## **THE STUDY**

The large-scale data were collected by means of a web-based survey. The survey consisted of an introduction, 6 background questions and 4 questions about teaching practices. The formulations of the questions were validated by 5 experts in mathematics education. The goal of the survey, as indicated in the introduction, was to understand how the teachers select mathematics problems for their teaching. Individually named e-mails with the invitation to respond to the survey were sent to about 500 secondary school mathematics teachers; 151 responded and fully filled in the survey.

From the responses to the background questions, we know that: more than 80% of the respondents teach in high school (grades 10-12); 76% teach the advanced versions of Israeli mathematics curriculum; 82% of the teachers had teaching experience of 10 or more years.

The central question of the survey was: "To which extent do you use the following resources for selecting mathematics problems for your teaching?" The teachers were offered 9 options, one of which was "Pose my own problems". All the options appear in Table 1. For each potential resource, the participants were asked to indicate how frequently they use it. They were given five options, from "Almost never" to "Almost Always" (see Table 1).

The survey also included a question "For what purposes do you pose your own problems?" This question was open-ended. The responses to this question were used in individual in-depth semi-structured interviews with eight teachers. The interviewees represented the groups of teachers who indicated that they pose their own problems "Rarely", "Sometimes", "Often" and "Almost always" (two teachers per group). At the interviews, the teachers were asked to describe how they plan their lessons and select mathematics problems for teaching. They were also asked to provide examples of their "own problems" and explain what "problem posing" means for them.

The survey results were analysed using descriptive statistics; the interviews were analysed inductively, in the meaning specified, for instance, in Thomas (2006).

## **RESULTS**

### **First research question: Teachers as problem posers**

The percentages of responses to the central question, in the order by which the potential resources appeared in the survey, are presented in Table 1.

Problem resource	Almost never	Rarely	Sometimes	Often	Almost always	Number of responses
Textbooks	1.3%	0%	2.6%	23.8%	72.2%	151
Other books	10.1%	9.4%	26.8%	39.6%	14.1%	149
Internet resources	14.4%	17.8%	37.0%	19.9%	11.0%	146
Teacher PDW <sup>1</sup>	18.1%	24.2%	38.9%	11.4%	7.4%	149
Fellow teachers	12.2%	18.9%	36.5%	25.7%	6.8%	148
My academic education	30.6%	26.4%	27.1%	10.4%	5.6%	144
I pose my own problems	19.9%	22.6%	29.5%	21.2%	6.8%	146
Problem posed by my students	51.7%	27.9%	16.3%	3.4%	0.7%	147
Other	56.3%	6.3%	14.6%	14.6%	8.3%	48

Table 3 - Percentage of responses to problems resources

Figure 1 presents the summary of the responses, by the frequencies of using each resource at least "sometimes."

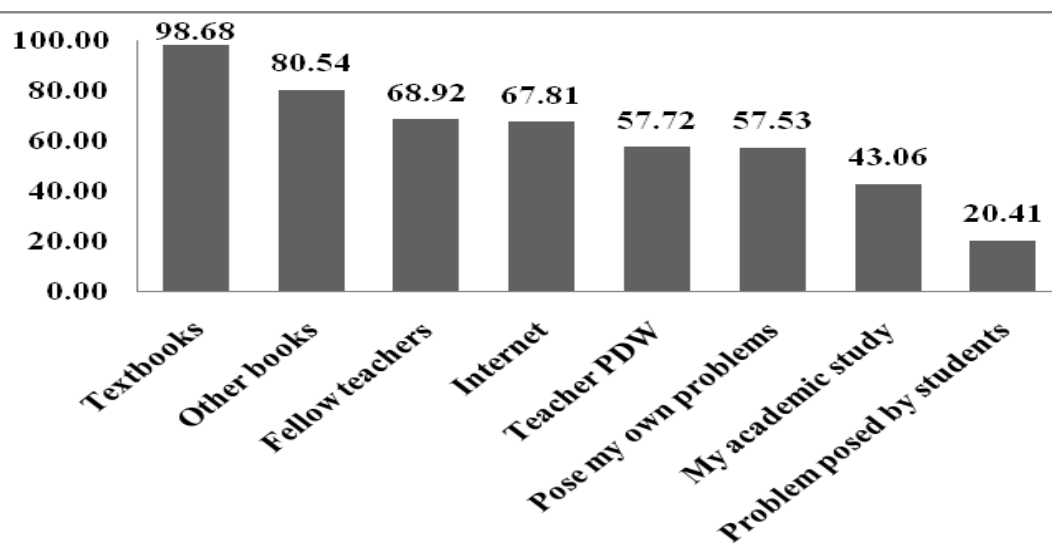


Figure 1: Frequencies of the use of different problem resources at least sometimes

The most frequently indicated reason for problem posing was the teachers' wish to adapt the difficulty level of the textbook problems to the needs of their students and different pedagogical uses. Additional reasons were: "For tests", "For connecting several topics", "When I want to connect a material to real life", "For explaining a new topic", "For encouraging the discussion" and "When I can't find appropriate problems

<sup>1</sup> : PDW stands for Professional Development Workshops

elsewhere". Noticeably, the latter reason underlies the former ones. The same reasons were indicated also in the interviews.

### Second research question: What the teachers mean by “problem posing”?

Five categories are inductively distilled from the eight interviewees' explanations of what “posing a mathematics problem” means to them (see Table 2). The names of the categories are given by us, the authors of the paper, based on the teachers' examples and explanations. Note that in some cases, the teachers could not provide concrete examples of the problems posed by them during the interview, but sent them by e-mail after the interview.

Category	Description
Cosmetic changes	Changing an existing problem by replacing some of its numerical parameters or its story without changing the idea of the solution.
Combining ideas	Creating a new problem that, as a rule, requires for its solving the use of ideas or techniques that have been previously studied and used separately.
Different contexts	Using the same, probably known, problem when teaching different topics in order to encourage the students to solve it by using different tools.
Connecting topics	Using a problem, known to the students, as a starting point for explaining a new concept.
In-the-moment questions	Unplanned using a known mathematical problem or spontaneous generating questions and examples during the lesson.

Table 4: Five categories of the meaning of the notion "problem posing" for the teachers  
Here are examples and clarifications.

**Cosmetic change:** Teacher B posed for a test the following problem: "Sketch a graph of the function  $y = \frac{(x-2)(x+3)}{(x+4)(x-5)}$  without formally exploring it". At the lessons preceding the test, the students of B discussed problems of the same formulation, but with other functions. This above function was invented by Teacher B.

**Combining ideas:** Teacher R and her colleagues combined the previously taught ideas related to arithmetic and geometric sequences in the following problem for the test:

Given an arithmetic sequence containing  $2n+1$  elements. The first element of this sequence is equal to  $k$ , and the sequence difference is equal to  $d$ . From the given sequence, a new sequence was built as follows: The even elements were doubled, and to the odd elements were increased by 4.

- a) Using  $k$  and  $d$ , write the five first elements of the new sequence.

- b) Prove that the odd elements of the new sequence form an arithmetic sequence.
- c) Prove that the sum of the new sequence is  $3n^2d + 3kn + nd + k + 4n + 4$
- d) The second element in the new sequence is 7 times bigger than the first element in the original sequence. The elements in places 1, 6, and 97 in the new sequence form a geometric sequence. Find  $k$  and  $d$  if given that the elements in the sequence are integers.

**Different contexts:** Teacher M uses the problem of finding the area of a triangle given the coordinates of its vertices when teaching plane geometry, vectors and complex numbers.

**Connecting topics:** At the beginning of the topic "Functions", Teacher N reminds the students one of the problems on sequences, and changes the notation from  $a_n$  to  $f(n)$ .

**In-the-moment questions:** All the interviewees indicated that during the lessons they deal with students' misunderstandings and obstacles by means of generating or recalling mathematical questions, problems or examples on the spot, without planning to do so prior to the lesson. Some teachers referred to this practice and problem posing, and the others were unsure whether this practice can be called so. The teachers refused to provide examples related to this category during the interview. Some of them explained that this is because the questions looked too uninteresting out of the context of the lessons. It is in place to note here that various examples of in-the-moment questions were collected when the first-named author observed lessons of Teacher N. However, presenting these data is beyond the scope of this paper.

## DISCUSSION

Either past or recent studies (e.g., Silver et al., 1996; Singer & Voica, 2013; Koichu et al., 2013) give the impression that not many mathematics teachers are active problem posers. In light of this, it is quite surprising that more than half of the participants in our survey indicated that they pose problems at least "Sometimes." This supports and substantiates the Nicol and Crespo's (2006) finding that the teachers can choose problems for their teaching either by elaborating or by creating.

Furthermore, our findings suggest that, in practice, many teachers most readily use the most available resources, and when they cannot find in them problems that fit their teaching needs, they turn to less readily available resources or to posing their own problems. Thus, they seem to choose problems for their teaching by some parsimonious, with respect to the effort needed, strategies (cf. Koichu, 2010, for the principle of parsimony in problem solving). Indeed, the most popular problem resources appeared to be "Textbooks" and "Other books", which are most readily available. One may comment that the Internet is the most available resource, given that in Israel all the teachers, as a rule, are proficient users of the Internet resources. However, it should be taken into account that most of mathematics classes in Israel are currently not equipped with computers, so additional effort – preparing and printing a working sheet – mediates the use of the problems from the Internet in a classroom. This

may explain why Internet is mentioned only as the fourth by popularity resource, after "Textbooks," "Other books" and "Fellow teachers."

Five categories of the teachers' perceptions regarding the notion of problem posing – "*cosmetic changes*", "*combining ideas*", "*different contexts*", "*connecting topics*", and "*in-the-moment questions*" – were identified in our data. The emergence of "cosmetic changes" and "in-the-moment questions" categories in our data supports Silver et al.'s (1996) observation that teachers tend to pose textbook-like problems and to do so also spontaneously during the lessons. In addition, "cosmetic changes" category of problem posing is reminiscent of what Singer and Voica (2013) called "not interesting, being just scholastic" problems, and Crespo (2003) called "non-problematic" or "avoiding pupils' errors" problems. However, our data suggest that problems created by "cosmetic changes" do not bear negative connotation from the teachers' perspective. This is because this type of problems is instrumental in everyday teaching, especially for preparing tests and exams.

The "*combining ideas*", "*different contexts*" and "*connecting topics*" perceptions are reminiscent of the conceptualizations of problem posing by Kilpatrick (1987) and Silver (1994) (see Introduction). Indeed, these conceptions of problem posing presume that teachers pose problems so that their students would not have readily available algorithms for solving them.

Our findings imply that most of the mathematics teachers are result-oriented – as opposite to being process-oriented – when they talk about problem posing. That is, they more value problem posing when it leads to creating worthwhile problems for real use, and less as an activity with the potential to develop their or their students' mathematical skills (cf. Silver, 1997, for the analysis of how problem posing can foster mathematical creativity). This suggestion is indirectly supported by that most of the participants in our study indicated that they rarely use problem posed by their students in teaching, and, as a rule, do not ask their students to pose problems. (In words of one of the interviewees, "Problem posing by students is a waste of time"). Note, however, that this result may also be an artefact of the context, in which we explored the teachers' perceptions.

In any case, the identified conceptions suggest that there can be some discrepancy between how mathematics teachers treat the role of problem posing in their practice and how problem posing is treated in the research literature, especially when referring to situations when the teachers create problems "on request". This suggestion needs further research attention, and, probably, revisiting some of the ways by which quality of the problems posed by teachers is evaluated in the frameworks of various studies on problem posing conducted in the laboratory conditions. Specifically, we believe that the role of relevance of the posed problems to the teachers' needs should be given attention and merit in the future studies.

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# HAVE YOU GOT THE RULE UNDERNEATH? INVISIBLE PEDAGOGIC PRACTICE AND STRATIFICATION

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*This paper describes visible and invisible pedagogical practices, using examples from a mathematics classroom. In the examples the instructional discourse specific to learning mathematics involves implicit rules while the regulative rules about general classroom behaviour are explicit. We reflect on how different social backgrounds may lead to some learners having an advantage in learning in such a context over others, leading to stratification of mathematics achievement in the class.*

## INTRODUCTION

The question “Have you got the rule underneath?” was asked often by a Grade 6 teacher we observed, whom we call Mr. White. Mr. White was referring to an equation relating the two sides of a T-table (Figure 2 shows a T-table), which he required his students to write directly underneath the table itself. We have chosen this question for our title because “the rule” can also refer to the function itself, which figuratively lies underneath the pattern of number in the T-table. In order to write the equation underneath the T-table, the students had to first “get the rule” in the sense of thinking of a function that would produce the numbers given. So the question “Have you got the rule underneath?” can be read as being both a reference to an explicit regulative requirement for classroom behaviour (that the equation should be written underneath and not elsewhere) and as a mathematical action. In Mr. White’s classroom the regulative rules were explicit, but those specific to learning mathematics were implicit, so that Mr. White’s pedagogic practice with respect to mathematics was invisible. We believe this results in stratification of achievement as some learners are better able to learn when a pedagogic practice is invisible than are others, and that a part of their advantage is related to their socioeconomic background.

From the results of the PISA studies (e.g., Lemke, et al., 2004) it is clear *that* socioeconomic background is related to school mathematics achievement. However, it is not clear *how* differences in social background contribute to stratification of achievement in school mathematics, especially in inclusive, non-streamed schools. The research reported here is from a larger project that seeks to identify discursive and interactional mechanisms that provoke stratification of achievement within the mathematics classroom, in order to establish a basis for further research on the role of socioeconomic background in these mechanisms. Mechanisms that we have discussed elsewhere include a focus on procedural activity (Knipping, Straehler-Pohl, & Reid, 2011), lowered expectations, changing pace, and delegation of initiative (Knipping, Straehler-Pohl, Reid, Jablonka, & Gellert, 2011). In this paper we focus specifically on the different types of rules that lie underneath successful participation in school mathematics.



This research complements research from social constructivist (e.g., Yackel & Cobb, 1996) and enactivist (e.g., Reid, Brown, & Coles, 2001) perspectives that also see students' and teachers' mathematical activity as embedded in social contexts. However, it begins from a sociological framework in order to build on prior results in the sociology of education and to keep socioeconomic background in view.

## **THEORETICAL FRAMEWORK**

We have chosen Bernstein's (2000, 2003) theoretical framework as a promising starting point, as it explicitly considers the relationship between socioeconomic background, classroom activity and school achievement. However, empirical research using Bernstein's theories is still fairly rare, especially in mathematics education, so research is needed to determine what aspects of Bernstein's work are most applicable to studying stratification of achievement in mathematics. In this paper we make use of Bernstein's concept of *framing*, and the related concepts of regulative and instructional discourse, and visible and invisible pedagogic practice.

### **Framing**

Framing refers to *how* one can act. The social rules that govern the selection of communication, its sequencing, and its pacing are a part of framing.

As an approximate definition, framing refers to the controls on communications in local, interactional pedagogic relations: between parents/children, teacher/pupil, social worker/client, etc. (Bernstein, 2000, p. 12)

### **Weak and strong framing**

Framing can be weak or strong. Strong framing is linked to explicitness of the social rules. Weak framing is indicated by implicitness of the rules.

Strong framing could, for example, mean that students' behaviours are governed by a number of explicit rules around how they should be involved in classroom activities. Weak framing could mean that students are offered a wide range of strategies of response and involvement during classroom activities, and are given, for example, "the choice of how often to seek out the teacher for interaction" (Bourne, 1992, p. 475). Because weak framing is characterised by the implicitness of social rules it can become a disadvantage for students who do not follow the rules because they are not aware of them (Bernstein, 2003).

### **Regulative and Instructional discourse**

Bernstein differentiates between "two systems of rules regulated by framing" (2000, p. 13): regulative and instructional discourse. The rules in regulative discourse are rules of social order. "The rules of social order refer to ... expectations about conduct, character and manner" (p. 13). Regulative discourse concerns what is right or wrong action in the social realm, specifically in our case in a school classroom. Principles of instructional discourse, on the other hand are rules of discursive order, i.e. rules specific to the instruction of a subject area. Rules of this type "refer to selection, sequence, pacing and criteria of the knowledge" (p. 13). In our context instructional

discourse refers to the school mathematical part of the discourse, e.g. what is right or wrong *mathematically*. While the distinction between regulative and instructional discourse comes from an entirely different theoretical position it is similar in many ways to Yackel and Cobb's (1996) distinction between social norms and sociomathematical norms.

### **Visible and invisible pedagogic practice**

Visible and invisible pedagogic practice can be distinguished according to whether the instructional and regulative discourse are explicit or implicit.

In general, where framing is strong, we shall have a visible pedagogic practice. Here the rules of instructional and regulative discourse are explicit. Where framing is weak, we are likely to have an invisible pedagogic practice. Here the rules of regulative and instructional discourse are implicit, and largely unknown to the acquirer. (Bernstein, 2000, p. 14)

In the following we will consider a case where the regulative discourse is usually explicit but the instructional discourse is usually implicit.

Bernstein points out that invisible pedagogic practice may contribute to stratification because of students' differential knowledge of the fundamental pedagogic principles. Students that have been socialised into similar principles at home may profit more from an invisible pedagogic practice than those whose home experience has been different. Bernstein argues that implicit rules "are less likely to be met in class or ethnically disadvantaged groups, and as a consequence the child here is likely to misread the cultural and cognitive significance" (2003, p. 211) of classroom practices where the rules are implicit.

### **METHODOLOGY**

The data analysed below comes from a research program that examines the emergence of disparity in ways that capture both internal (classroom) dynamics and external (social) factors. We study classrooms where students are beginning a new phase in their schooling, in schools that differ in locale (urban/rural) and selectivity (streamed/inclusive), and in three regional/national contexts: Berlin and Hamburg (Germany); Nova Scotia (Canada) and Norrbotten (Sweden). This comparative approach allows us to investigate in distinct contexts the mechanisms that lead to the emergence of disparity in mathematics classrooms and how systemic differences influence them.

The raw data are video recordings of classroom interactions, and interviews with students and teachers. These are analysed by closely examining key incidents where the level of interaction is high and differences in participation in the activity are evident. This analysis contributes to the development of an "external language of description" (Morais, 2006) that offers mechanisms to account for phenomena observed empirically, based in theoretical concepts derived from Bernstein's work.

Mr. White teaches Grade 6 in Nova Scotia. The students come from different elementary schools in the area. Some of them have been in the same elementary class, but not many. Others might know each other as they have gone to the same elementary

school, but for many the new peers are not familiar. Mr. White is the home room teacher for this class. This means he is teaching them not only math but most other subjects, for example, language arts, science and art. Mr. White is with his home class for more than seventy percent of his weekly teaching.

Mr. White's first day of mathematics instruction is not the first day of school as he spent about a week working with the students to organise their notebooks and giving them pre-tests of basic arithmetic. Mr. White starts the year with a unit on T-tables. On the previous day, when textbooks were handed out to students, he showed a T-table in the textbook to the students (see Figure 1). He asked them to copy this table into their notebooks and to try to "fill in the blanks" for homework.

Kevin's Grade	Alice's Age
6	4
7	5
8	6
9	7
?	?
?	?
?	?

Kevin uses a pattern.  
He predicts how old  
his sister will be  
during each of his  
school grades.

Kevin	Alice
6	4
7	5
8	6
9	7
10	8
11	9
12	10

Figure 1: Textbook MathQuest 2000, p. 8

Figure 2: The T-table as filled in on the board

## RULES IN REGULATIVE DISCOURSE

Mr. White began by organising the class into groups, and giving them some explicit rules for working in groups.

- 1 White This is the first time among probably a number of times that I am going to ask you to work in groups. ... So here is what I would like you to do. If you have just two people in the group usually it is not too hard to keep on task. If you have four people in a group it's a little bit harder. ... I am just going to ask you to try to work together and there is only going to be one person and that person is saying 'we have to get this done, we have to finish off here.' In other words there is a person who you have to nominate as a task master and everybody else is just going to try to help out. So we'll just leave it that way for now. (2007-09-12, 2:20-3:01)

Here the framing of the regulative discourse is strong. Mr. White's students do not have to guess the rules for working in groups. They know that every group will have from two to four members, that they are supposed to stay on task, and that every group will have a "task master". As he goes on, Mr. White states another rule, which is specially marked by being called a "rule".

- 10 White Oh, almost forgot. There is one rule we are going to start out with. ... Listen up ladies, pencils down; eye contact please. One rule: You cannot ask me a question until you have asked that question of all of your teammates. So in other words, if I say oh no I don't know this I'm going to ask Mr. White. Have you asked all of your teammates first? So it comes around then that if all four of you or three of you or two cannot answer that one question you then may ask your group leader or your taskmaster to ask me the question. ... Here is your first task. I'll give it to you orally as opposed to putting it up on the board. ... So you need to be very still and you need to listen to make sure you understand what I would like you to do. (2007-09-12, 5:04-6:23)

In addition to the rule Mr. White calls a “rule” (that they must ask each other a question before asking him), he also reminds them of already established rules like making eye contact and listing. All this is very explicit.

When the homework was assigned the task was to fill in the table, but in the groups there is an additional task, to report how they had arrived at their answers.

- 16 White Excellent, here is your first task, or your first job. I would like you to look at each of your answers, in other words, share your answers to see if you came up with the same results. ... Did you fill them in and what were your answers and how did you get them. Here are the three questions again. Listen carefully. What are your three answers? How did you get them? And, of course, are your answers the same as everybody in your group? The two most important ones: what are your three answers and how did you get them? (2007-09-12, 7:00-8:12)

This shift of the task, from *doing* to *reflecting* on doing, is the first indication that there is more to it than the students were initially told. The nature of the task itself belongs to the instructional discourse, to which we now turn.

## RULES IN INSTRUCTIONAL DISCOURSE

Recall that instructional discourse refers to the knowledge area being studied. In the mathematics class, regulative discourse governs what students can and can not do in general, while the instructional discourse governs what mathematical actions are allowed.

After Mr. White has a student put the answers on the board (see Figure 2) he asked “Is there anybody ... who can tell us how those numbers fit in the way they do?” Max was the only volunteer, and he answered that in each column “you add one on both”.

- 35 Max Because I knew she was two years younger than the grade he was in. So then I just added one on [the numbers?] from there.
- 36 White I have a question. This can come, the answer may come from any group. You may look at the T-Table here or you may look at the one you've created in your notebook. Can anybody figure out or tell me the relationship between the left side of this T-Table and the right side of the T-Table.  
[Max is the only student who raises his hand.]
- 37 White OK.
- 38 Max The difference between the numbers, there's a difference of two on each number.
- 39 White A difference of two. How do you mean difference?

- 40 Max There is, one is two higher.  
41 White So in other words, this one is two higher.  
42 Max Yes. (2007-09-12, 13:40-14:28)

In line 35 Max describes the relationship between the two columns “she was two years younger than the grade he was in” but then in line 36 Mr. White asks the class “Can anybody figure out or tell me the relationship?” Max answers again, but differently, “there’s a difference of two”. We believe Max has recognised an implicit evaluation of his first answer in Mr. White’s ignoring it. In his second answer there is not longer any mention of the context and the word “difference” is used, in its mathematical sense of the result of subtraction. Mr. White also questions the word “difference” however, and so Max reformulates his answer a third time.

It is not clear to the students what kind of answer Mr. White wants to his question. Max finds out the underlying rules in the instructional discourse by trial and error, although he sees to have some ideas of what Mr. White is looking for. Mr. White then changes the question:

- 43 White I have a question. How do you go from this number to this one? Remember you said that we added down or you folks added down. How do we get from this side if you were looking at these numbers and if you say they sort of, they sort of seem to match up in a way? How do we get from this side to this side? ...  
47 Eric You subtracted two each time.  
48 White Ok. Ooh. Would you say that again because I don’t know if Wayne and the boys down there heard it, and the girls.  
49 Eric You subtracted two each time.  
50 White We have a response here that says you subtract two each time. Let’s assume that we are going to give a letter to each of these numbers. Let’s assume we say each of these numbers represents X so anytime we want to put a number in we replace X with the number. And if we say equals a number, (*bell rings*) whoops. Can anybody else, can anybody figure what we would put in here for a little tiny equation? To help fill this out? (*points to Eric*) What did he just say, what did he say we do? (2007-09-12, 14:28-17:10)

Eric’s answer seems to satisfy Mr. White. In contrast to Max’s first answer, it makes no reference to the everyday context of ages and grades, and in contrast to Max’s second answer it is described as an action (subtract two) instead of a situation (the difference is two). Because Mr. White’s goal is to express the relationship as a function expressed by an equation he implicitly requires answers stripped of their context and describing a process.

Knipping (2009) notes that some students may not even recognise that this exchange is important, as in the instructional discourse the pedagogic practice is invisible and the discourse seems far removed from the initial task of filling in the table.

In line 50 Mr. White again changes the question, asking them now to express Eric’s rule in an equation. Now the goal has become clearer, to express the process Eric described using an equation, though the way to achieve that goal is still left implicit.

The examples above illustrate how the instructional discourse in Mr. White's pedagogic practice includes implicit rules concerning what kinds of answers are acceptable, while the rules in the regulative discourse are explicit. This situation should not be confused with constructivist teaching, in which the content to be learned is left to the students to discover but the instructional discourse that provides the framework for their activity need not be. A confusion of the two levels of content and instructional discourse may account for Mr. White's teaching, though it is also possible that he is not explicitly aware of his own rules for instructional discourse, and so they are necessarily implicit.

## CONCLUSION

Learning occurs in a social context, and learners come from a social context. The language of invisible and visible pedagogic practice allows us to describe and important feature of the social context of learning, and to reflect on how the differences in success in learning when the pedagogic practice is invisible might be related to differences in the social contexts learners come from.

Why do Max and Eric look for the rules underneath Mr. White's instructional discourse? Their persistence would be accounted for if they came to Mr. White's class already acquainted with such rules, either from their prior school experience or from their homes. And that fact that they persist means that in the end they learn what the rules are. It is not at all clear that the other students in the class do, and this means that disparities from outside of Mr. White's classroom can manifest themselves in mathematics achievement in his classroom, through the mechanism of invisible pedagogic practice in instructional discourse.

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# CONSTRUCTING THE FUNDAMENTAL THEOREM OF CALCULUS

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*This research report is the fourth one in a series about learning integral calculus in high school. In designing a curriculum that supports an improved cognitive base for a flexible proceptual understanding of integration, we have chosen to use accumulation as core idea. We defined three conceptual milestones: Approximation, Accumulation Function and the Fundamental Theorem of Calculus (FTC). In this paper, we focus on the following questions: What is the structure of the FTC in terms of elements of knowledge? What operational definitions for these elements allow the researcher to follow students' construction of the FTC? How does the process of knowledge construction occur during an instructional intervention?*

## INTRODUCTION

Students have serious problems with understanding the concept of integral (e.g., Thompson, Byerley & Hatfield, in press). Many students have a tendency to see integral calculus as a series of procedures with associated algorithms and do not develop a conceptual grasp that could give them the desirable versatility of thought (e.g., Thompson, 1994). Thus, instead of a proceptual view, they have, at best, a process-oriented view. This may be due in part to a lack of opportunity to experience these processes directly. In order to develop an improved cognitive base for a flexible proceptual understanding (Gray & Tall, 1994) of the integral and integration, it has been proposed to use accumulation as core idea (Thompson, 1994; Kouropatov & Dreyfus, 2009). With respect to undergraduates, Thompson and Silverman (2008) "believe that understanding accumulation ... can be part of a coherent calculus that focuses on having students see connections among rate of change of quantities, accumulation of quantities, ..." (p. 13). Our claim is that with minor changes this is relevant for high school students as well. We designed a curriculum (Kouropatov & Dreyfus, 2012a) supporting students in constructing integration as a conceptual aggregate of knowledge elements from approximation via accumulation to the FTC. The adopted research methodology allowed us to closely observe students' knowledge-constructing processes of approximation (Kouropatov & Dreyfus, 2011) and accumulation function (Kouropatov & Dreyfus, 2012b). Here, we report on teaching episode where students deal, for the first time and in an intuitive manner, with the FTC. Hence, this paper focuses on the questions formulated in the abstract.

## FTC: PRELIMINARY CONSIDERATIONS

Following Newton and Leibniz, the idea of the integral developed in two directions: integral as a limit of a certain sum (definite) and integral as antiderivative (indefinite). The notion of definite integral became an important tool by shedding light on many problems in science. The notion of indefinite integral led to a development in analysis



that is (or at least was until technology put numerical methods at the forefront of calculus) the core of differential equations. The understanding of the connection between these two notions should in our opinion be a main aim of teaching integral calculus. A common approach presents the indefinite integral as a formal "undoing" in the "table of derivatives", and then uses it to compute areas (definite integral). Then it stops. Students never learn why they use antiderivatives to compute areas. The connection is never established. So the question arises: Can integration be taught at the high school level in such a way that the connection is established? Our proposed answer to this is accumulation, which leads simultaneously to the indefinite as well as the definite integral. In our curriculum, students study the FTC after constructing the concept of the accumulation function of a given function (the definite integral with a variable upper bound). In this approach, the FTC says that the rate of change of the accumulation function of a given function is the given function itself.

### KNOWLEDGE CONSTRUCTING AND ABSTRACTION IN CONTEXT

The main aim of this paper is to analyze how students construct the FTC as a result of our instructional intervention. We have adopted Abstraction in Context (AiC) (Schwarz, Dreyfus and Hershkowitz, 2009) as theoretical framework. AiC takes abstraction to be an activity of vertical reorganization of previous mathematical constructs in order to arrive at a new (to the learner) construct. The activity is interpreted in terms of epistemic actions performed by the learners for a specific purpose, in a particular context. The context includes the social setting as well as the learner's personal background, including previous mathematical constructs resulting from previous abstractions. Reorganization includes establishing new connections between such constructs, making generalizations, and discovering new strategies for solving problems. "Vertical" implies building a new level of abstraction over a previous level. An essential component of AiC is a model of three epistemic actions for describing and analyzing at the micro-level the knowledge constructing process:

- R The learner *recognizes* a previous mathematical construct as relevant in the present situation.
- B The learner *builds-with* the recognized constructs to achieve a local goal such as solving a problem or justifying a claim.
- C The learner uses B-actions to assemble and integrate previous constructs so that a new (to the learner) *construct emerges* by vertical mathematization.

In processes of abstraction, R-actions are nested in B-actions, and R- and B-actions are nested in C-actions. These particular epistemic actions have been chosen because they seemed to be relevant for processes of abstraction as well as observable. This working hypothesis has been effective in studies taking place in a large variety of contexts (Schwarz et al., 2009, and references therein).

### DESCRIPTION OF THE RESEARCH

We designed a ten-session unit and implemented it with five small groups of advanced-level mathematics high school student volunteers. The unit was independent of what the students were learning at school. Here, we report on an activity from the

seventh session of a group of two female students (A and B). We worked with groups rather than single students in order to make the knowledge constructing process more observable through their discussions. We decided to analyze the knowledge construction of the pair rather than that of each of the students separately. Our data include audio recordings, transcripts and session protocols.

## THE ELEMENTS OF KNOWLEDGE AND OPERATIONAL DEFINITIONS

AiC requires an a priori analysis of the tasks proposed to the students in terms of the intended knowledge elements, their constituents, and links between the constituents. Based on theoretical considerations, including that the FTC is a hierarchical, nested concept with a proceptual nature, and didactical considerations, including rate of change of the accumulation function of a given function as a complex co-variational process of change (e.g., the rate of change of the area accumulating beneath the graph of a positive function while the "right border is moving"), we focus on the following knowledge elements:

- CAF "Change of Accumulation Function": Changing the independent variable causes changes in the dependent value (of the given function) and in the value of its accumulation function simultaneously.
- iCAF "Infinitesimal Change of Accumulation Function": for any continuous given function, the change of its accumulation function, when the change of the independent variable is very small, can be approximated as an appropriate term of accumulation.
- RCAF "Rate of Change of Accumulation Function".
- iRCAF "Infinitesimal Rate of Change of Accumulation Function": for any continuous given function, the rate of change of its accumulation function at a certain point can be approximated as the ratio between the iCAF and the change of the independent variable.
- FTC For any continuous function defined on some closed interval, the derivative of its accumulation function is the given function itself.

Operational definitions of these knowledge elements have been used to assess whether students have constructed the knowledge element. For example,

- RCAF We will say that students have constructed RCAF if they explicitly (verbally and/or graphically) calculate/represent the rate of change of the accumulation function as a ratio between the difference of the values of the accumulation function and the corresponding difference of values of the independent variable.
- FTC We will say that students have constructed the FTC if they express (verbally and/or graphically) that the derivative of the accumulation function of a given function is the given function itself for any value in the domain of the given function, and explain it using iRCAF.

Students' previous constructs are likely to be relevant during the activity; they include constructs for 'function' or 'variable', as well as others, which we assume to have been constructed in preceding sessions of the unit, for example:

- RM "Rectangles Method": In a rectilinear coordinate system, the accumulating value of any continuous positive function can be approximated as a sum of terms of accumulation that are areas of rectangles.
- CC "Complex Co-variation": When considering accumulation, the value of the accumulation depends on the function that accumulates as well as on where we stop accumulating (upper limit).
- AF<sub>M</sub> "Accumulation Function Meaning": For every upper limit there is a unique value of the accumulation that depends only on the function that accumulates and on the value of the upper limit. The Accumulation Function associates the value of the accumulation to the upper limit.
- AF<sub>P</sub> "Accumulation Function Property": The properties of the accumulation function (e.g. increase) follow from properties of the given function  $f$ .
- RoC "Rate of Change" of a function is the rate at which the dependent variable changes with the independent variable.
- DRoC "Derivative as Rate of Change": The derivative of a function at a certain point can be defined as the limit of the RoC of the function as the difference of the independent variable values approaches zero. The limit can be understood intuitively as a number that the RoC approaches when the change in the independent value approaches zero at that point.

## FINDINGS

During six previous sessions the students had learned about and succeeded to explain the meaning of the CC, AF<sub>M</sub>, and AF<sub>P</sub>. As an introduction to the seventh session, they were asked to consider the accumulation functions of constant functions like  $f(x)=2$ ,  $g(x)=-0.36$  and linear functions like  $f(x)=2x$ ,  $g(x)=-0.5x$  on an interval like  $[0,b]$ . They succeeded to explain the properties of and to produce the algebraic representation of the accumulation functions (A and B denote the students, R the researcher):

- 129 R: Look at your answer for task b. What do you think is the connection between  $Af$  [the accumulation function] of  $x$  and  $f$  of  $x$ ?
- 131 B: According to the rate of change of the function the accumulation grew, like, the accumulation was moderate...
- 132 R: Okay, then what... visually, what is the connection with this to this? Between this function and that function? [R points to the algebraic expressions, which were written by students.] What's the connection between the function...
- 133 A:  $f x$  is used as the derivative of that function
- 136 B: What? What's the relationship? That the accumulation equals the derivative... OK, I got it...

- 138 A: As we saw in the previous tasks  $Af x$  prime equals  $f x$ , when the  $Af$  of  $x$  is the accumulation function of  $f x$ . In these tasks we dealt with certain functions. Now we should check if this relationship exists in each function.

The students recognized CC,  $AF_M$ ,  $AF_P$  as relevant in the present context. When they were asked to connect between  $f$  and its accumulation function, it seems that they established the connection (133) based on the specific algebraic expressions of the example. They conjectured a generalization and expressed a need to justify it (138). The next task asked them to analyze (using guiding questions) the illustration in Figure 1 (the horizontal segment starting at C and the more or less vertical scribbles in the light grey area were not present at this stage but were added later).

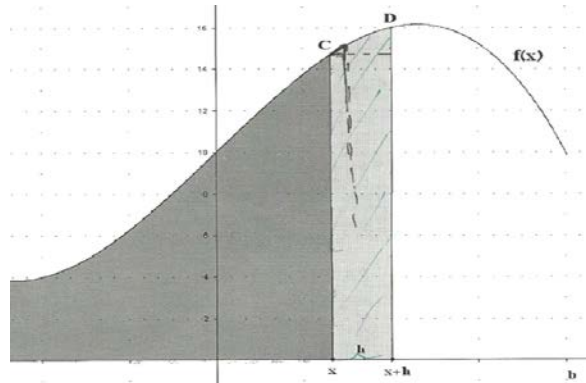


Figure 1

- 190 R: What is the change of the function of accumulation  $Af$  of  $x$  between  $x$  and  $x$  plus  $h$ ?
- 191 B: It adds up like, what does it mean, what's the change?
- 192 R: How, you see the change of the function, what is it?
- 193 B: Yes, one more area
- 194 A: One more area
- 196 B: Which area?
- 197 A+B: This one [point on the light grey area in Figure 1]

From this excerpt we infer that the students are able to refer in parallel to three different entities: the independent variable,  $f$  as represented by the graph, and  $Af$ , which is not represented at all. This short episode allows us to point to the following nested epistemic actions: recognizing their previous constructs for CC,  $AF_M$  and building-with them to reach a new one construct – CAF. In the next task the students were asked to refer to the rate of change of  $Af$ :

- 200 R: Okay. What is rate of change, for now? Rate of change?
- 201 B: It's the size... another segment that you're adding to it, if it's big or small then like...
- 203 B: Slope, no?
- 205 A: Delta  $y$  over...

In 200-205, we can see some signs that fit with recognizing RoC but there is no connection to the task context. We can't identify any building-with here. It seems that

including RoC as additional player in the plot is beyond the students' current grasp. But an explicit hint by the researcher brings immediate results:

- 211 R: No, I want the accumulation function. Accumulation  
 212 B: Aaah...  $Af$  of  $x$  plus  $h$   
 214 A: Minus  $Af$  of  $x$  divided by  $x$  plus  $h$  minus  $x$ ...

$$\frac{A_f(x+h) - A_f(x)}{x+h-x}$$

The students' observable behavior precisely fits the operational definition of RCAF. It appears that their experience from preceding lessons allowed them (with a little help) to construct RCAF in the context of the given task. In the next task the students were asked to refer (in the same context) to a very small change of the independent variable. They succeeded to connect the task with their previous knowledge:

- 219 A: As long as  $h$  is smaller  
 220 B: We can use a rectangle... that almost this area [draws the horizontal segment starting at C in the illustration, see Figure 1]

Here the students recognize as relevant their previous constructs for RM, CC, and CAF. They then start building-with these constructs:

- 229 R: If  $h$  is very small, what can be said about the difference between the area of the rectangle and ...  
 230 B: Which rectangle are we talking about here?  
 231 A: Here  
 233 B: And the area that represents the change of the accumulation function  
 235 B: All of this, all up the top  
 244 B: They're asking what's the difference between the rectangle that was created and all this [points to the light grey area]  
 245 A: Ah, as long as the rectangle is small the difference between the areas is small  
 246 B: Close. We're saying: The change of the accumulation function with a smaller and smaller  $h$  is almost not different from the rectangle

The students combine (build-with) constructs including CAF (233) and RM (lines 230-231) and get (construct) the new one, iCAF. It is also interesting to mention the informal, intuitive and, in our eyes, appropriate use of the idea of limit (245-246). The students were then asked to decide how to approximate the rate of change of  $Af$  when  $h$  is very small. They tried algebraically using RCAF but were only able to simplify  $x+h-x$  to  $h$ . They also tried visually, scribbling vertical dashes in the light grey area in Figure 1, but got stuck with this too. Only a strong hint of the researcher led to the student behavior we were looking for:

- 260 R: How do we calculate the area of this rectangle?  
 261 A: It's  $h$  times  $f$  of  $x$   
 262 R: Does this help you with rate of change?

263 B: Like this? [writes  $f(x)$  times  $h$ ]

After this, a rather long and somewhat confused episode with back-forward considerations followed, during which the students were trying to recognize the elements relevant to the situation, and to build with them what the task required, namely to approximate the rate of change of  $Af$  when  $h$  is very small. Our interpretation is that the origin of their difficulties is rooted in their limited RoC construct. Anyway, in the long run it seems that they recognized relevant constructs including iRCAF and DRoC and combined them to get a rather sophisticated conclusion:

320 A: ... rectangle,  $f x$  times  $h$  because it's almost a change, partly  $h$ , right? What did we get?... So that's  $f$  of  $x$

324 B: The function. The derivative of the accumulation equals the function

336 A: That as long as you take a smaller  $h$  the area actually approaches to being the rectangle and this rectangle is actually equal to the function

$$\frac{A(x+h) - A(x)}{h} \approx \frac{h \cdot f(x)}{h} = f(x)$$

$$A'(x) = f(x)$$

While this does not fit the operational definition of the FTC construct yet, it does point to a beginning of its emergence.

## DISCUSSION

In the study presented here, we tried to find an answer to the question how processes of knowledge construction occur during an instructional intervention in the form of a teaching interview on the FTC. We reported findings about one pair of students. We choose this pair because it represents the sample. The findings about other groups are quite similar. The differences (rate of progress, level of discussion, quality of previous knowledge, completeness of new constructs and so on) depend on the personal, cognitive and social context. The data gathered and their analysis using the RBC methodology allow us to give a tentative answer to the above question. From the excerpts we can see that, at the current stage of learning, the students succeeded to construct several elements of knowledge that were totally new to them. The analysis of the excerpts shows that the students recognize relevant previous constructs, build new constructs with them, and use these new constructs for progressing toward the FTC. The main conclusion we draw from this study is that the notion of accumulation has allowed us to design a didactical tool to support students' knowledge constructing processes on integration, and that the adopted research methodology has allowed us to observe these processes. An additional conclusion is that the understanding of the FTC using this approach depends on students' understanding of accumulation as well as on their understanding of rate of change. Despite the rather impressive general performance of the students, we feel that their construct of rate of change was inappropriate or too fragile. The RBC methodology allowed us to note this weak link in the chain. The next question we hope to research is whether, in the high school context, advanced concepts like accumulation and rate of change might be introduced

separately, one after another, or, perhaps, combined in some dialectical didactical entity.

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# PROBLEMS WITH AND WITHOUT CONNECTION TO REALITY AND STUDENTS' TASK-SPECIFIC INTEREST

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*In this study 192 ninth and tenth graders from 8 German classes were asked about their interest concerning tasks with and without connection to reality. The students were randomly assigned to two experimental groups. The first group was asked about their task-specific interest after and the second group before task processing. The study aimed to answer the following questions: (1) Does students' task-specific interest differ according to the type of problem (intra-mathematical, "dressed up" word problems and modelling problems)? (2) Does task processing influence students' task-specific interest? The analysis showed that there are differences in students' interest regarding tasks with and without connection to reality and that task-specific interest across all types of problems decreases after task processing.*

## INTRODUCTION

The discussion about the types of mathematical tasks that should be treated in the classroom has a long tradition in mathematics education. Over the last decades there has been a strong plea for treating real-world problems in mathematics classroom (Blum & Niss, 1991). From treating reality-based tasks, an improvement in students' interest can be expected. However, recent empirical research studies do not always confirm the assumption that students prefer to solve real-world problems (Schukajlow et al., 2012). A further question remains: how does task processing influence task-specific interest? The present study refers to these points and examines the impact of different types of mathematical problems and task processing on task-specific interest.

## THEORETICAL BACKGROUND

### Interest

Interest is a motivational construct with great importance for learning. An interested learner, for example, engages more in solving problems than an uninterested one. "An interest represents or describes a specific relationship between a person and an object in his or her "life-space"" (Krapp, 2000), such as the relationship between a person and a mathematical task. A special feature of interest is its content-specificity. Content-specificity means that interest is closely related to specific topics, tasks and activities. Educational researchers usually differentiate interest as being either situational or individual. Individual interest is a relatively stable evaluative trait towards certain domains. Situational interest is an emotional, highly variable state aroused by specific features of an activity or a task with physiological, subjective, goal-orientated, and behavioural components. Mitchell (1993) distinguished between two levels of situational interest. First, a possible activity (e.g. an opportunity to solve a



mathematical problem) catches or initiates a person's interest. Second, this activity holds the person's interest with the likely result that deeper individual interest may emerge. The catch-level can be stimulated by content specific activity (Schraw & Lehman, 2001). For increasing situational interest activity, novelty, challenge, exploration intention, attention demand and interactive experience are crucial (Deci, 1992). Engaging in task processing can influence these factors, thus also influencing situational interest.

### **Mathematical problems**

According to Niss, Blum, & Galbraith (2007) there are three types of mathematic problems: modelling problems, ("dressed up") word problems and intra-mathematical problems. The main difference between these types is their strength of connection to the real world.

*Modelling problems.* The core of modelling activities is the transfer process between the real and the mathematical world. An idealized process of solution for a modelling problem can be characterized as followed: (1) understanding the problem and constructing an individual "situation model"; (2) simplifying and structuring the situation model and thus constructing a "real model"; (3) mathematizing, i.e. translating the real model into a mathematical model; (4) applying mathematical procedures in order to derive a result; (5) interpreting this mathematical result with regard to reality and thus attaining a real result; (6) validating this result with reference to the original situation; if the result is unsatisfactory, the process may start again with step 2; (7) exposing the whole solution process.

*Word problems.* Another type of mathematical problem is the "dressed up" word problem. Although also related to reality, the mental activity for the solution of word problems is more simplified than that which is required for solving modelling problems.

- In a word problem the real model is already given in the task.
- The data for finding the solution are given in the text and no other data are needed for development of the solution.
- "Modelling loops" for validation of the real result are unnecessary.

*Intra-mathematical problems.* Mathematical problems without any connection to reality are termed intra-mathematical problems. The solution of intra-mathematical problems begins with the analysis of the situation model. The situation model in these types of tasks is equal to a mathematical model. The problem can be solved using appropriate mathematical procedures. Validation is limited to checking the mathematical activity.

### **Task-specific affect**

In the last decades there have been strong pleas for the development of new measurement devices for the task-focused and subject-specific investigation of affect (Zan, Brown, Evans, & Hannula, 2006). Recently, two studies were carried out where students' affect towards mathematical problems with and without reference to

reality using task-specific questionnaires was investigated. “Dressed up” word problems more enjoyable and caused less anxiety for students than intra-mathematical ones (Pekrun et al., 2007). This result was not confirmed by the other study on this issue. Schukajlow et al. (2012) showed that there are no differences in students’ enjoyment, value, interest and self-efficacy among tasks with and without connection to reality. As the results of both studies differ, there is still an open question as to whether students’ task-specific affect varies according to the task’s connection to reality. In this paper we focus on task-specific interest.

An essential limitation of both studies is the way in which the questionnaire was applied. The studies have in common that they inquired about task-specific affect before task processing. Students were asked to appraise their own enjoyment, interest, self-efficacy etc. without actually solving the problems. Thus their perception of affect was based only on their first impression of the problems. If students were to solve the task before answering the questions they may be able to appraise differences between types of problems more accurately. To prove this assumption we asked students to evaluate their task-specific interest before and after solving problems with and without connection to reality.

## RESEARCH QUESTIONS

This study was designed to answer the following research questions:

- Does students’ task-specific interest differ according to the type of problem (intra-mathematical, “dressed up” word problems and modelling problems)?
- Does task processing influence students’ task-specific interest?
- Does the influence of types of problems on task-specific interest depend on task processing?

## METHOD

### Design und sample

192 German ninth and tenth graders from 4 middle-track and 4 grammar school classes (53.6% females; mean age=16.1 years, SD=0.86) were asked about their interest regarding various types of problems. The students were randomly assigned to two experimental groups. Students of group 1 solved problems first and then reported on their task-specific interest regarding these problems. In group 2, students reported on their task-specific interest first and then solved tasks that were used in the questionnaires (see Fig. 1). Students of both groups worked on the same tasks and had the same amount of time to answer the questions about task-specific interest.

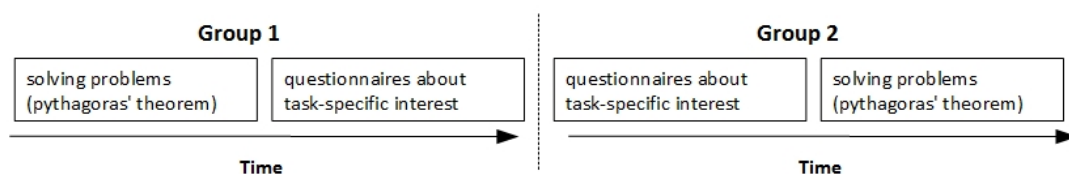


Fig.1: An overview to the study

## Sample problems

Twelve problems on the topic Pythagoras' theorem - four modelling, four word and four intra-mathematical ones - were selected for this study. Sample tasks are presented below.


<p><b>Maypole</b></p> 	<p>Every year on Mayday in Bad Dinkelsdorf there is a traditional dance around the maypole (a tree trunk approx. 8 m high). During the dance the participants hold ribbons in their hands and each ribbon is fixed to the top of the maypole. With these 15 m long ribbons the participants dance around the maypole, and as the dance progresses a beautiful pattern on the stem is produced (in the picture such a pattern can already be seen at the top of the maypole stem). At what distance from the maypole do the dancers stand at the beginning of the dance (the ribbons are tightly stretched)?</p>
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Fig. 2: The “maypole” problem

The maypole problem can be classified as a *modelling problem*. An individual mental model of the given situation has to be constructed when the problem is read and the picture is viewed. In the situation model important data like the distance from one end of the ribbon to the ground are missing and have to be assumed for constructing the real model. The problem solver can assume that the dancers hold the ribbons at 1 m high. An idealised ribbon is 15 m long and the stem is 8 m high. The real model has to be mathematized using a right-angled triangle as a mathematical model. The distance from the dancers to the maypole corresponds here to one leg of the triangle and has to be calculated using Pythagoras' theorem. The calculated distance can be validated using the information from the picture of the dance around the maypole.

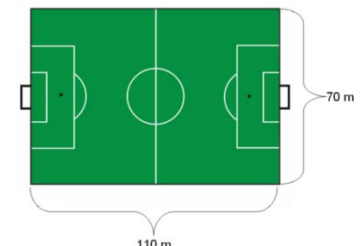
<p><b>Football Pitch</b></p> 	<p>Trainer Manfred would like to carry out a diagonal run with his team. To do so he would like to know how long the diagonal of the football pitch is. Can you help him?  Calculate the diagonal length of the football pitch.</p>
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Fig. 3: The “Football Pitch” problem

The task “Football Pitch” is a “dressed up” *word problem*. The situation is pre-structured by the data presented in the picture and the formulation of the task. Thus, simplifying and structuring the situation model is not essential for solving this problem. As a right-angled triangle in the task “Football Pitch” can be recognised, a direct translation into a mathematical model is possible. Calculating the diagonal using Pythagoras' theorem and the interpretation of mathematical results are activities that are necessary for the solution of this task.

The solution of the *intra-mathematical task* “Side c” can be developed using the same mathematical activities as the problem “Football pitch”.

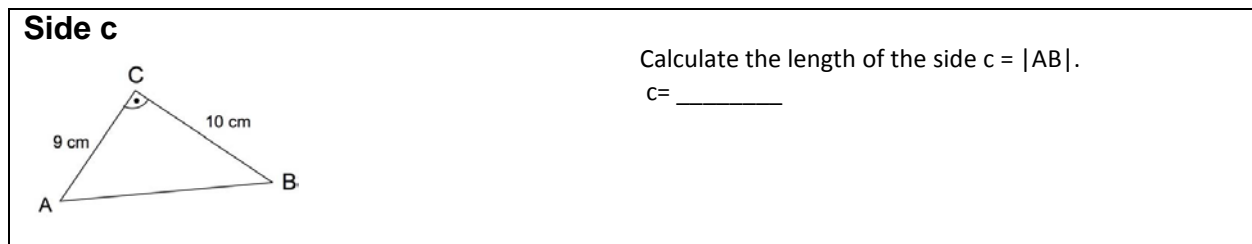


Fig. 4: The “Side c” problem

### Interest scales

In the questionnaire each of the twelve problems was followed by a statement about students’ task-specific interest. The instruction for both groups was: “Read each problem carefully and then answer some questions. **You do not have to solve the problems!**” In group 1 students were asked after task processing to what extent they agreed or disagreed with a statement (“It was interesting to work on this problem”). In group 2 students were asked before task processing with a statement (“It would be interesting to work on this problem”). For recording their answers a 5-point Likert scale was used (1=not at all true, 5=completely true). The statement we used represents a main feature of the construct “interest”. Each of 3 scales for the measurement of task-specific interest was formed across four problems. The reliabilities (Cronbach’s alpha) for the 3 scales were all higher than .81.

### Treatment fidelity

To control the treatment we used a five-point Likert item: “Before I agreed or disagreed with statements (to task-specific interest), I have solved the problems” (1=not at all true, 5=completely true). Means and standard deviations were for group one and two 4.3(1.17) and 2.19(1.01) respectively. An unpaired t test showed that there were significant mean differences between both groups ( $T(179)=13.07$ ,  $p<.0001$ , *Cohen’s d*=1.93). As intended in the study, students of group 1 solved the tasks significantly more often than students of group 2 *before* they reported on their task-specific interest.

## RESULTS AND DISCUSSION

In Table 1 the interest mean scores (Ms) and standard deviations (SDs) regarding the three types of problems and the two groups are presented. A one-factorial repeated-measures ANOVA with the type of the problem as the within-subject factor was used to compare the task-specific interest of the two groups. The crucial assumption while using repeated measures ANOVA is the sphericity. Mauchly’s test of sphericity indicated that the assumption of sphericity by the factor type had been violated ( $\chi^2(2) = 22.13$ ,  $p < .001$ ). Thus we used the Geisser/ Greenhouse correction to adjust the degree of freedom.

	M <sub>1</sub> (SD <sub>1</sub> )	M <sub>2</sub> (SD <sub>2</sub> )	Cohen's d	T(df=190)
modelling	2,74 (1,05)	2,97 (0,88)	0.14	1.638
dressed up	2,89 (1,06)	3,11 (0,82)	0.06	1.622
intra-mathematial	2,83 (1,02)	3,19 (0,86)	0.19*	2.568
* $p < 0.05$ , M <sub>1</sub> (SD <sub>1</sub> ): group 1, M <sub>2</sub> (SD <sub>2</sub> ): group 2				

Table 1: Students' task-specific interest

### Students' task-specific interest, types of problems and task processing

The ANOVA shows that the factor "types of problems" has a significant influence on students' task-specific interest ( $F(1.8)=7.681$ ,  $p < 0.001$ ,  $\eta^2=.04$ ). Thus it can be concluded that students' interest differs according to the three types of problems. To avoid the alpha error accumulation we have used the Bonferroni correction in the post-hoc test. The post-hoc test reveals that the students' interest regarding modelling problems is lower than their interest regarding word and intra-mathematical problems (c.f. Table 2). No differences between students' interest regarding "dressed up" word problems and intra-mathematical problems were found.

(I) type	(J) type	Mean Difference (I-J)	Std. Error (SE)	<i>p</i>
Modelling problems	"Dressed up" word problems	-.15	.04	<.01
Modelling problems	Intra-mathematical problems	-.15	.05	<.01
"Dressed up" word problems	Intra-mathematical problems	-.01	.04	1,00

Table 2: Values for post-hoc analysis of differences in task-specific interest

The ANOVA with task processing as a between factor indicates that this factor has a statistically significant effect on task-specific interest ( $F(1)=4.358$ ,  $p=.038$ ,  $\eta^2=.02$ ). Hence it follows that students' task-specific interest decreases after task processing from 3.1 (SE=.09) to 2.82 (SE=.09).

To answer the third question we analyzed the interaction effect of types of problems and task processing on task-specific interest. No interaction effect could be observed in the data ( $F(2)=1.316$ ,  $p=.27$ ). This result implicates that the task processing has no influence on lower interest regarding modelling problems compared to task-specific interest regarding other types of problems. Modelling problems are less interesting than "dressed up" word problems and intra-mathematical problems before as well as after task processing. The task-specific interest in "dressed up" problems and intra-mathematical problems does not differ before and after task processing significantly. However, interest regarding intra-mathematical problems decreases slightly more strongly than interest regarding other types of problems.

## DISCUSSION

As we found no differences between task-specific interest regarding “dressed up” word and intra-mathematical problems, the results of the study by Schukajlow et al. (2012) can be confirmed. However, unlike previous results students have lower interest to modelling problems than to the other types of problems. One possible reason for this inconsistency is the usage of different topics for the measurement of task-specific interest (Pythagoras’ theorem and linear functions vs. Pythagoras’ theorem only). Another explanation for lower interest regarding modelling problems is that students don’t solve these problems in regular mathematics classes and may be unsure of their ability to solve this type of problem (for similar results in physics see (Hoffmann, Häussler, & Lehrke, 1998)).

Other important results are (1) no interaction effect between task processing and types of problems and (2) the decrease of task-specific interest after task processing. This decrease of interest can be explained by the novelty of the problems for one of the two groups. Task processing can negatively influence the novelty of the tasks and thus also the situational interest. However, the same problems were presented in the performance test as well as in the questionnaires. It is possible that task-specific interest would not change or even increase after task processing if other problems were to be used for the measurement of interest.

The main limitations of this study are that only one mathematical content area was incorporated and only one statement summarized across four problems was used for the measurement of task-specific interest.

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# **PEDAGOGICAL CONTENT KNOWLEDGE AND VIEWS OF IN-SERVICE AND PRE-SERVICE TEACHERS RELATED TO COMPUTER USE IN THE MATHEMATICS CLASSROOM**

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*Computer use in the mathematics classroom can support the development of students' mathematical competency through specific insightful approaches. For a best-practice framing of learning opportunities supported by computer use, mathematics teachers need specific pedagogical content knowledge (PCK), including views about computer use which are likely to filter the further development of such PCK. However, the empirical evidence base in this research domain still needs to be broadened. This paper thus reports findings from two studies about such PCK of 39 pre-service and 65 in-service teachers. The evidence suggests specific professional development needs, but also that it is possible to develop PCK related to computer use.*

## **INTRODUCTION**

The development of practice-relevant theory in mathematics education related to forms of computer use in the mathematics classroom has progressed substantially in the last decades. However, for designing effective learning opportunities for their students, teachers need specific PCK which reflects an awareness of such advances in mathematics education. This PCK also encompasses corresponding views about possibilities of computer use in the mathematics classroom, which merit intensified attention, given the relatively narrow base of existing evidence in that area.

Consequently, this paper addresses this research need, reporting results from two studies with pre-service and in-service teachers. The findings indicate that views and PCK related to computer use should be developed and indeed can be supported through focused university courses for pre-service teachers. The in-service teachers appear to hold views which suggest an even more substantial need of professional development and they report a non-frequent technology use in their classrooms.

The following second section provides an overview of the theoretical background and leads to the research questions of the two studies which are presented in the third section. The fourth section informs about designs and samples of the studies. The fifth section presents results, which are discussed in the concluding sixth section.

## **THEORETICAL BACKGROUND**

Technology use changes the mathematics classroom: If teachers really make use of the possibilities provided by software like e.g. CAS, spreadsheet, and DGS, the activities of students tend to shift from processing algorithms and calculations to constructing models, reflecting, or evaluating results (e.g. Martin, 2012; Hoyles & Lagrange, 2010; Peschek, 1999) and from using static representations to experimenting with dynamic and interactive modes of visualisation and exploration (e.g. Hoyles & Lagrange, 2010;



Jonassen & Reeves, 1996). As described by the distributed cognition theory (e.g. Salomon, 1996) computer tools afford a so-called ‘off-loading’ in thinking processes as learners can profit from the (mental or direct) use of tools in problem solving processes. This implies also changes in the curriculum and in the ways mathematical subject matter is dealt with when using computer tools (cf. Martin, 2012; Hoyles & Lagrange, 2010) – and the goals of competency development change accordingly.

Mathematics teachers should know about the spectrum of possibilities they have when creating computer-based learning environments for the classroom (e.g. Niess, 2005). These professional knowledge requirements are framed by corresponding views, e.g. about advantages with respect to visualisation, interactivity, exploration, ‘off-loading’ when using computer tools (e.g. Martin, 2012), and student-centred teaching methods in the classroom. Perceptions of potential obstacles of computer use or the requirement of curricular change, as well as an individual positive view of own positive learning experience with technology use are likely to influence instructional practice and can be additional indicators of specific PCK. A set of constructs that reflect such specific views has been described in more detail in Kuntze (2011), going beyond rather general teacher knowledge and views about computers (e.g. Hardy, 1998; Cuckle, Clarke & Jenkins, 2000), which is less specific for mathematics instruction.

Teacher knowledge about using technology is hence multi-faceted: Mishra & Koehler (2006) distinguish professional knowledge components around “Technological PCK” inspired by Shulman’s (1986) model (s. also Akkoç et al., 2008). However, even though this model emphasises several knowledge components related to technology and to its use in the classroom, views and convictions of teachers are somewhat hidden in this model despite their potential significance for PCK. Moreover, the spectrum between relatively general knowledge and very content-specific knowledge or knowledge specific for particular classroom situations is hardly reflected in this model – however being able to refer to examples of specific learning situations rooted in exemplary contents appears crucial for PCK related to technology use.

Consequently, as theoretical framework of professional knowledge of mathematics teachers, we refer to a model (Kuntze, 2012) which includes the spectrum between convictions/beliefs and declarative/procedural knowledge under the notion of PCK (cf. Pajares, 1992), as well as levels of situatedness resp. globality (cf. Törner, 2002), which affords focusing on PCK related to specific examples of technology use in the classroom. Views and PCK related to computer use specific for situations in which computer use merits being considered appears as a key influencing factor on classroom practice, hence the spectrum from global views down to content domain-specific or situation-specific PCK should be reflected in corresponding research designs. Accordingly, this paper aims to collect evidence about PCK in such a multi-component approach.

## **RESEARCH QUESTIONS**

This paper presents results from two studies. Study A focuses on PCK of pre-service teachers, the evaluation of a newly developed questionnaire and test instrument and on

PCK development in a corresponding university course. Responding to the research need described in the previous section, both relatively global views about computer use in the mathematics classroom and knowledge related to content-specific and situation-specific examples are in the scope when seeking for answers to the following research questions:

- What PCK and in particular views related to computer use in the classroom do pre-service teachers have?
- Can such PCK be developed through focused professional development activities in a corresponding university course?

Study A calls for collecting first evidence related to technology use also from in-service teachers. Study B hence concentrates on views of in-service teachers and their reported use of technology according to the following research questions:

- What views related to computer use do in-service teachers have and which needs of professional development can be identified?
- How and in particular how frequently do the teachers use technology in their classrooms according to their reported instructional practice?

## DESIGN AND METHODS

Both studies used quantitative methods, in order to gain first overviews. For study A, a new questionnaire and test instrument was developed, comprising of both multiple-choice and open items. The rather global views about computer use in the mathematics classroom were examined through the multiple-choice scales shown in Figure 2 in the following section and presented in detail in Kuntze (2011). An additional indicator-like test of PCK consisted of five items and focused on the pre-service teachers' ability to identify and present examples of contents and potential classroom situations that were linked to specific aspects of PCK related to technology use. A sample question is given in Figure 1. For an indicator-like overview, the answers to these items were coded by two raters using the categories “‘There is no such situation’ / analogous answer marked”, “no answer”, “second alternative marked but without example/explanation”, “second alternative marked but non-adequate answer/explanation”, “idea visible in imprecise explanation”, and “adequate explanation/answer”.

Please describe (if appropriate) a learning situation and the corresponding task, in which the students can explore a content through computer-based experiments.

☐ In my view there is no such situation.

☐ Such a situation could look like this:

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Fig. 1: Sample question from PCK test

The pre-service teachers were asked to complete the questionnaire at the beginning and at the end of a university course, respectively. The sample of study A consisted of 39 pre-service teachers (31 female, 8 male). 25 out of these pre-service teachers (18 female, 7 male) took part in a university course on computer use in the mathematics classroom during one semester, whereas the other 14 pre-service teachers (13 female, 1

male) took part in another course without reference to computer use and hence could be considered as a non-treatment control group.

In study B, 65 German in-service teachers working at academic-track secondary schools (30 female, 31 male, 4 without data; age: 23 up to 35 years, 16 from 36 to 45 years, 14 from 46 to 55 years, 8 more than 55 years, 4 without data) were asked to complete an extended multiple-choice questionnaire and to report on the frequency of different modes of technology use in their classrooms. The teachers had been teaching mathematics on average for 11.7 years (SD=10.7 years).

## RESULTS

In both studies, the multiple choice scales were reliable (Study A: reliability values reported in Kuntze, 2011; Study B: Cronbach's alphas displayed in Fig. 2). The results from the multiple choice questionnaire reported here concentrate on the scales which were included in both studies.

Scale	Sample item	Number of items	Alpha (Cronbach)
Computer use motivating regardless of content	In comparison to the usual classroom and regardless of the subject matter is computer use always motivating for students.	4	.92
Computer use as additional obstacle	Dealing with a computer-based learning environment consumes so much attention of the students, that mathematical learning is constrained.	4	.76
Computer use and demand of curricular change	Computer use in the mathematics classroom requires changed contents, changed instructional goals and a changed task design.	3	.82
Positive view of computer use based on own positive experience	As the work with computer software for the mathematics classroom is fun for me, I would like to provide my students with learning experiences at the computer.	3	.87
Added value through off-loading when using computer-based tools	Computer use can help students to concentrate on the key ideas of a solution when being confronted with a task requiring difficult calculations, because the technology does the calculation job.	4	.87
Added value through experimentation	One of the strong points of computer use is that students can experiment with mathematical contents and hence explore subject matter on their own.	5	.88
Added value through visualisation/interactivity	Computer use in the mathematics classroom brings possibilities of visualising mathematical content, which result in a better understanding of the students.	3	.83
Advantages for methods in learning environments	Students can learn better as they are supported by help and navigation features which are often available in computer-based learning environments.	4	.79

Fig. 2: Scales, sample items and reliability values (Study B)

In the following, the results of studies A and B are reported partly in integrated diagrams, in order to meet the length limit of this paper. However this section will keep to the order of the research questions.

The first research question (study A) concentrates on PCK and views of pre-service teachers prior to the university course. The corresponding data can be found in Figures 4 and 5, indicating that the pre-service teachers were moderately optimistic about the unspecific motivation potential of computer use and about some aspects of its added value for the classroom, such as exploration and visualisation/interactivity (see Fig. 5). However on the level of declarative and procedural PCK, when asked to describe or

explain specific contents and situations for these aspects and others, adequate answers are rather an exception (on average less than 10 per cent, see pale blue bars in Fig. 4).

The second research question (study A) focuses on changes in PCK as a consequence of the university course. For this research interest, the comparisons between treatment group and control group in Figures 3 and 4 are relevant. As far as the views displayed in Figure 3 are concerned, the participants of the computer course show a highly significant increase of the positive view of computer use based on own experiences and of the perceived added value related to off-loading, corresponding to strong effect sizes, whereas these variables show no effect in the control group. In Figure 4, there is a moderate shift of the code frequencies in the direction of more elaborated and more adequate answers for the intervention group, whereas there is no such shift or even a slight opposite shift in the control group.

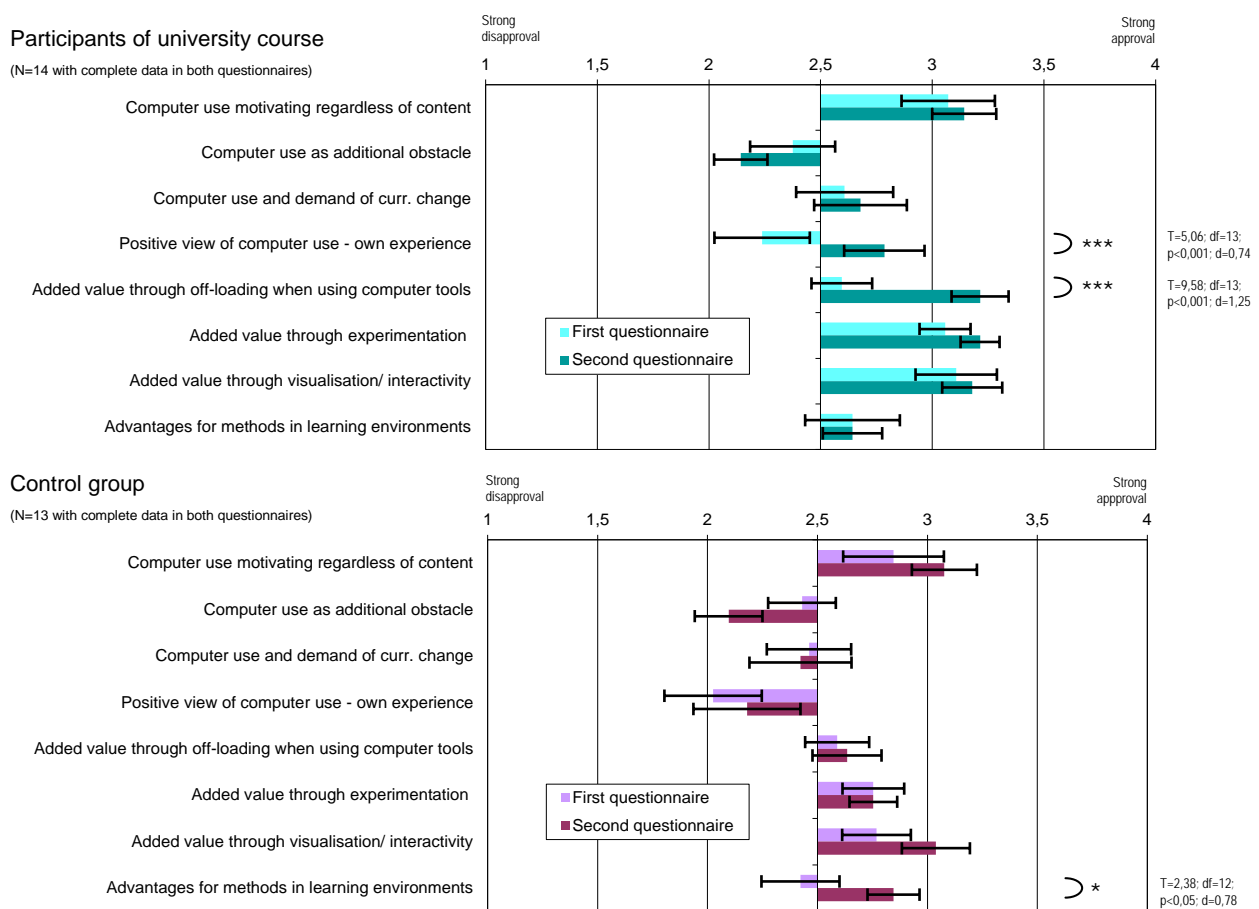


Fig. 3: Views related to computer use in the mathematics classroom at the beginning and at the end of university courses (Study A)

The third research question (study B) is connected to in-service teachers' views about computer use. The data in Figure 5 shows that the in-service teachers held views which differed from the views of the pre-service teachers: They saw computer use as less motivating (regardless of content,  $T=4.35$ ;  $df=67.65$ ;  $p<0.001$ ;  $d=.86$ ) and as less demanding for learners in addition to the contents being taught ( $T=2.61$ ;  $df=102$ ;  $p<0.01$ ;  $d=.53$ ). Moreover, they had a lower perception of the added value of computer

use with respect to visualisation/interactivity ( $T=2.28$ ;  $df=102$ ;  $p<0.05$ ;  $d=.47$ ) and to specific methods aspects in learning environments ( $T=2.27$ ;  $df=102$ ;  $p<0.05$ ;  $d=.45$ ).

The fourth research question was about the instructional practice reported by the in-service teachers. Figure 6 displays the corresponding results, which show that computer use is relatively rare in the everyday mathematics classrooms of the teachers according to their answers.

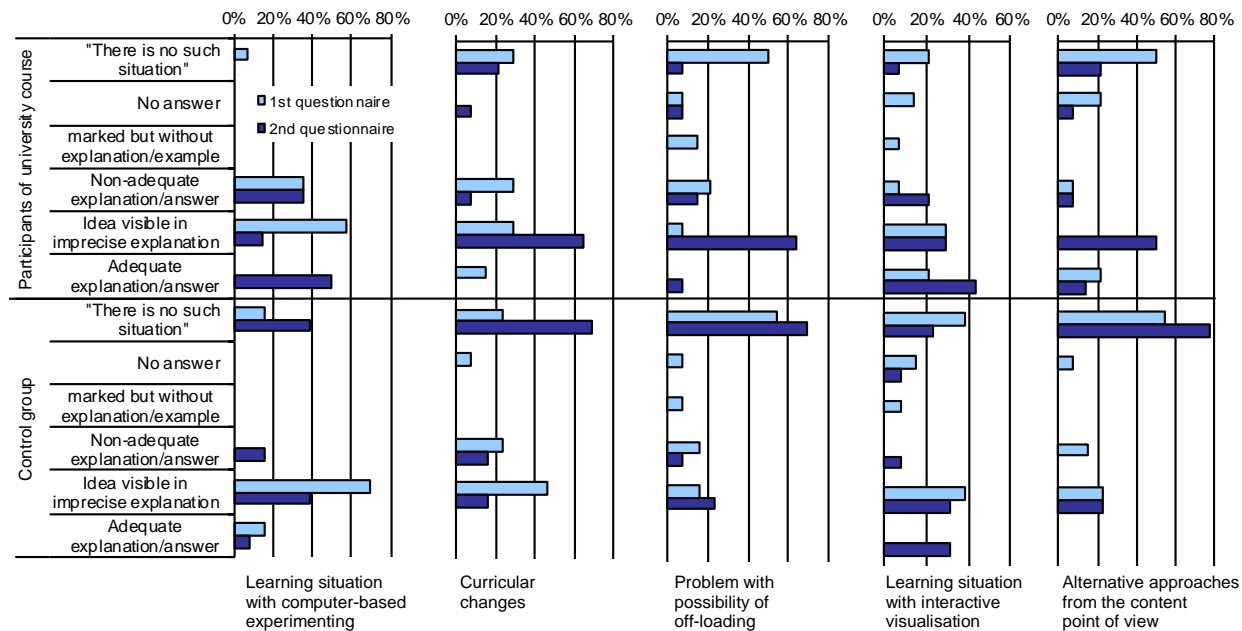


Figure 4: Indicators for declarative PCK related to computer use in the mathematics classroom (questionnaires at the beginning and at the end of the university course)

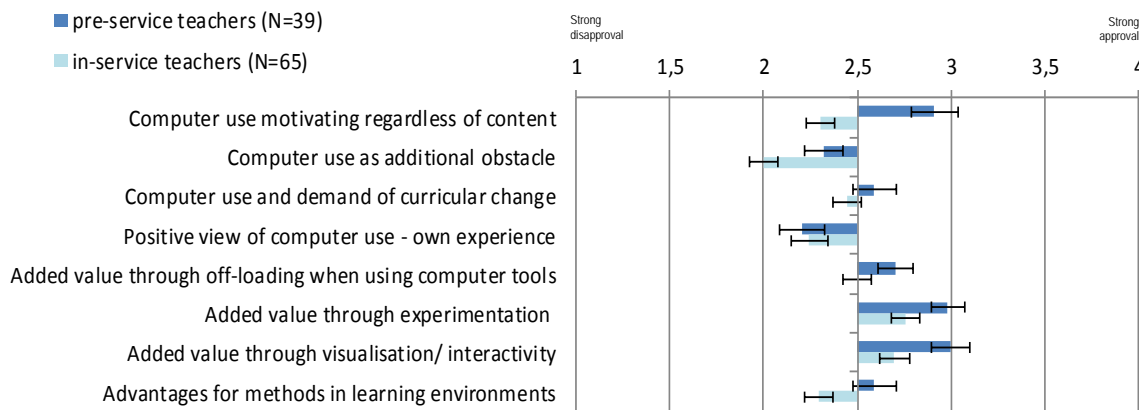


Figure 5: Views related to computer use in the mathematics classroom – comparison between pre-service teachers (study A) and in-service teachers (study B)

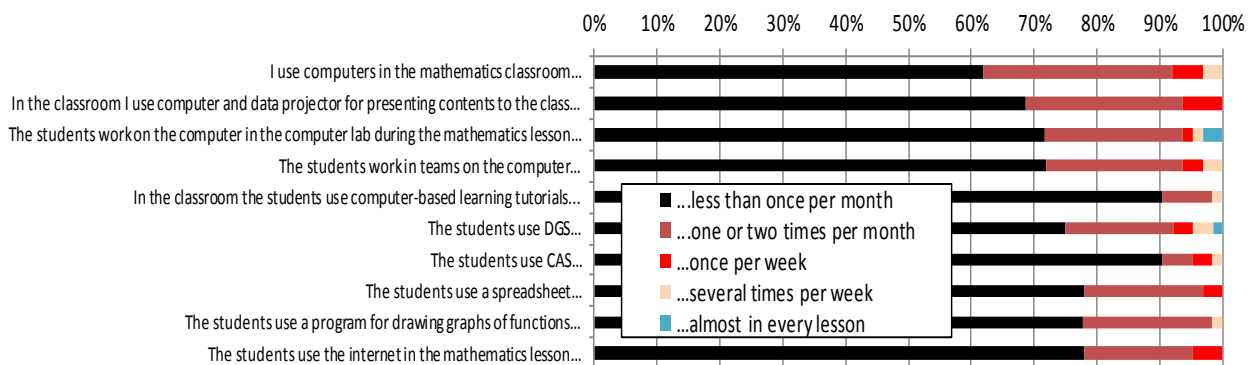


Figure 6: Computer use in the mathematics classroom as reported by the in-service teachers (study B)

## DISCUSSION AND CONCLUSIONS

Given the multi-component approach to PCK related to computer use in the mathematics classroom, the findings give a multi-faceted picture. Whereas the results of study A suggest that pre-service teachers have moderately optimistic views related to computer use, these views need to be complemented by more refined PCK which should be supported by corresponding professional development courses. The findings of study B suggest that in-service teachers might be even more in need of specific professional development. The low reported frequencies of computer use appear to fit to the less optimistic views in Figure 5 and suggest that some of the teachers not only lack PCK but also experience with computer use in the mathematics classroom. Especially the average answers to the “added value” scales indicate that a part of the in-service teachers might be unaware of the possibilities of computer use.

Study A shows that it is possible to improve PCK and specific views in professional development courses. The changes in the pre-service teachers’ views presented in Figure 3 have in the meantime been replicated in a follow-up study with more than 70 pre-service teachers (Kuntze & Bescherer, in press), together with the observation of significant changes in some more of the scales. Furthermore, the findings in Figure 4 indicate a moderate progression in PCK of the pre-service teachers who had taken part in the university course about computer use in the mathematics classroom.

The findings also raise follow-up questions: For instance, in-depth research about reasons for the frequent lack of transfer into classroom practice (cf. Fig. 6) is needed. For implementation, not only PCK, but also specific content knowledge (CK) might play a role: For instance, mathematics teachers’ own work with computer tools might be a key source of development of their content-specific CK and PCK. With respect to the findings about changes in the pre-service teachers’ PCK, the findings call for more evidence about possibilities of supporting such PCK also beyond the detected changes from corresponding intervention studies. Such studies could help to identify best-practice elements of professional development, integrating the research instruments presented here in the evaluation research design.

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# **THE EFFECT OF INSTRUCTION IN COGNITIVE AND METACOGNITIVE STRATEGIES ON TAIWANESE STUDENTS' PROBLEM SOLVING PERFORMANCE IN PROBABILITY**

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*The purpose of this study was to investigate the effect of the use of Metacognitive-Strategy Worksheet (MSW) on Taiwanese ninth-grade students' problem solving performance in probability. MSW was developed based on Montague's (1992, 1995, 1997) cognitive-metacognitive strategies for mathematical problem solving and was used by students to solve probabilistic problems during instruction in cognitive and metacognitive strategies. The results of this study suggested that instruction in cognitive and metacognitive strategies had a statistically significant effect on the ninth-grade students' problem solving performance in probability.*

## **INTRODUCTION**

Probability affects the biggest and smallest decisions of people's lives; however, it is often misinterpreted in the wrong way in everyday life. Studies have suggested students' misconceptions about probability (Kahneman, Slovic, & Tversky, 1982; Jones, 1974; Kapadia, 1986; Fischbein, Nello, & Marino, 1991) Attempts to improve students' problem solving performance in probability through teaching have been made by researchers; however, little was done with metacognitive instructional approaches to improve Taiwanese ninth-grade students' problem solving performance in probability. Lo (2002) found that students with higher metacognitive abilities performed significantly better on probabilistic problem solving than those with lower metacognitive abilities. Instruction in metacognitive skills may be fruitful in improving students' problem solving performance in probability. The investigators designed instructional materials to encourage students' use of metacognitive skills while solving problems in probability. These materials, called Metacognitive-Strategy Worksheets (MSWs), were used in this study.

The purpose of this study was to investigate the effect of the use of Metacognitive-Strategy Worksheet (MSW) on Taiwanese ninth-grade students' problem solving performance in probability. MSW was developed based on Montague's (1992,1995,1997) cognitive-metacognitive strategies for mathematical problem solving and was used by students to solve probabilistic problems during instruction in cognitive and metacognitive strategies.

## **THEORETICAL FRAMEWORK**

Cognitive and metacognitive strategies play an essential role in effective mathematical problem solving (Artzt & Armour-Thomas, 1992; Garofalo & Lester, 1985; Mayer,



1987; Montague, 1992, 1995, 1997; Montague & Bos, 1986; Schoenfeld, 1985, 1987, 1989, 1992). Metacognition is essential to employing efficient cognitive strategies during problem solving processes. Based on the view that metacognitive processes not only focus on self-awareness of cognitive knowledge but also direct and regulate cognitive processes during problem solving, Montague (1992, 1995, 1997) described mathematical problem solving in seven cognitive processes: comprehending problem information, paraphrasing problems in one's own words, visualizing problems through illustrations, hypothesizing solution plans, estimating answers, computing solutions, and checking every step of the solutions, and identified three metacognitive activities associated with each cognitive process: self-instruction, self-questioning, and self-monitoring. In self-instruction, students are involved in identifying and directing problem solving strategies before execution. Self-questioning involves internal dialogue for regulating execution of cognitive strategies. Self-monitoring encompasses appropriate use of strategies and encourages students to monitor their performance. Montague's (1992, 1995, 1997) cognitive-metacognitive model of mathematical problem solving served as the foundation for this study.

Montague and Bos (1986) found that cognitive-metacognitive strategy instruction is effective in improving mathematical problem solving for secondary students with learning disabilities. Montague (1992) also found that coordinated use of both cognitive and metacognitive strategies for mathematical problem solving was more effective for middle school students with learning disabilities than either cognitive or metacognitive strategy alone. Little, however, is known about how to implement instruction in cognitive and metacognitive strategies with average students and the effect of such instruction on average students' problem solving performance in probability. Lo (2002) indicated that students with higher metacognitive abilities performed significantly better on probabilistic problem solving than those with lower metacognitive abilities. In the present study, Metacognitive-Strategy Worksheets (MSWs) were used to engage Taiwanese average students in instruction on probability in cognitive and metacognitive strategies. MSW instruction was developed based on Montague's (1992, 1995, 1997) cognitive-metacognitive model of mathematical problem solving.

## **METHODOLOGY**

This study involved one teacher and forty-nine students. The teacher is one of the investigators and the subjects comprised two intact ninth-grade classes from a suburban junior high school in Taiwan. Data for this study consisted of students' probabilistic problem solving scores on the pretest and posttest. To describe the methods and procedures used in collection of these data, this section includes subsections of Subjects, Materials and instruments, and Instructional procedures.

### **Subjects**

The subjects for this study comprised two intact ninth-grade classes. One class served as the experimental class, and the other class served as the control class. There were twenty-eight students in the experimental class and twenty-one students in the control

class. The experimental class received instruction on probability in cognitive and metacognitive strategies using MSW techniques for fifty minutes, five days a week for a period of two weeks; the control class received traditional instruction on probability for fifty minutes, five days a week for a period of two weeks.

## Materials and Instruments

### Metacognitive-strategy worksheets

Metacognitive-Strategy Worksheet (MSW, see Figure 1) was developed based on Montague's (1992, 1995, 1997) cognitive-metacognitive strategies for mathematical problem solving and was only used in the experimental class to encourage students to use cognitive-metacognitive strategies while solving probabilistic problems. The worksheets consisted of seven sections: read, paraphrase, visualize, hypothesize, estimate, compute, and check. Students were taught to self-instruct, self-question, and self-monitor within each section.

Steps	Descriptions	Please mark $\checkmark$ after checking.
Read	What does this problem mean?	
Paraphrase	What is known? What is unknown?	
Visualize		
Hypothesize		
Estimate		
Compute		
Check	Please mark $\checkmark$ in the box at the right to each step after you have checked the step.	

Figure 1: Sample of Metacognitive-Strategy Worksheet

### The pretest and the posttest instruments

The pre and post tests were developed by one of the investigators and were used to assess each student's problem solving performance in probability in this study. Each student was given the pretest before and the posttest after they received either instruction in cognitive and metacognitive strategies or traditional instruction.

Twenty mathematical problems were adapted from the schools system's instructional materials in probability and made up both pretest and posttest. The problems on these two tests required comparable problem solving knowledge and skills. The value of Cronbach's alpha was 0.906, the average value of item difficulty was 0.65, and the average value of discrimination was 0.48.

## Instructional Procedures

In this study, the experimental class received instruction on probability in cognitive and metacognitive strategies using MSW techniques for fifty minutes, five days a week for a period of two weeks; the control class received traditional instruction on probability for fifty minutes, five days a week for a period of two weeks. Before the study began, the teacher, who was also one of the investigators, introduced students to the purpose of the study, the procedures, and the use of MSWs. The pretest was administered by the teacher to both of the experimental and control classes to assess their problem solving performance in probability before they received either MSW instruction or traditional instruction.

Since MSW was an attempt to integrate cognitive and metacognitive strategies into regular mathematics classes, it was incorporated into the schools system instructional materials which were being used in the class. During each class of the experimental class, the teacher taught the students to use Montague's (1992, 1995, 1997) cognitive-metacognitive strategies to solve probabilistic problems. The students were then asked to do exercise problems on the MSWs. MSW requires each student to use Montague's cognitive-metacognitive strategies to solve mathematical problems in probability. After the students started to work on the worksheets, the teacher moved around the classroom to help them, but her role was to facilitate the students' use of Montague's cognitive-metacognitive strategies.

When most of the students completed their worksheets, the teacher guided the class to check each section of the worksheets. The teacher then continued the class schedule using MSW techniques. After the MSW instructional procedures, the posttest was administered by the teacher to both of the experimental and control classes to assess their problem solving performance in probability after they received either MSW instruction or traditional instruction.

## RESULTS

Descriptive statistics were used to examine score differences between scores on the pre and post tests of both of the experimental and control classes. As evident in Tables 1 and 2, there was a decrease between the means of the problem solving scores on both tests in the control class, and there was an increase between the means of the problem solving scores on both tests in the experimental class. In the control class, the mean for pretest was 55.48 compared to the mean score of 40.48 for posttest. In the experimental class, the pretest score was 41.79 and the posttest score was 60.71.

T-test analysis was conducted on the problem solving scores to detect statistically significant differences between pre and post tests of both of the classes. In the control class, as shown in Table 1, students' problem solving scores significantly decrease from pretest to posttest ( $\alpha=0.05$ ,  $p=0.002$ ). In the experimental class, Table 2 indicated that students made statistically significant improvement from pretest to posttest ( $\alpha=0.05$ ,  $p=0.000$ ).

Problem solving	Mean	Std. Deviation	t	p
Pretest	55.48	31.66	-3.635	0.002
Posttest	40.48	29.45		

Table 1: Quantitative findings of the control class

Problem solving	Mean	Std. Deviation	t	p
Pretest	41.79	34.24	5.542	0.000
Posttest	60.71	31.14		

Table 2: Quantitative findings of the experimental class

Since homogeneity of regression was assumed, as shown in Table 3, in this study, ANCOVA was performed to determine if the posttest scores of both of the classes differed significantly. Table 4 indicated that there was significant difference between the posttest scores of the experimental and the control classes.

	SS	DF	MS	F	p
Group*Pretest	4.897	1	4.897	0.017	0.898

Table 3: Summary of test for homogeneity of regression

	SS	DF	MS	F	p
Group	10861.276	1	10861.276	37.995	0.000

Table 4: Summary of ANCOVA

## CONCLUSIONS

The study provided a glimpse into how instruction in cognitive and metacognitive strategies using MSW techniques could influence students' problem solving performance in probability. The results of this study suggested that instruction in cognitive and metacognitive strategies using MSW techniques had a statistically significant effect on the students' problem solving performance in probability.

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# USING A BOTTOM-UP APPROACH OF TEACHING BOX PLOTS TO OVERCOME THE HEURISTIC MISINTERPRETATION OF BOX PLOTS

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*In this paper we studied an intervention aimed at remediating students' tendency to interpret the area in box plots as representing the proportion or number of observations. This misinterpretation has been shown to be caused by heuristic reasoning. Using a bottom-up approach of teaching box plots, we tried to improve students' interpretation. Most students showed improvement after the intervention, but signs of heuristic reasoning were still found. This means that under some circumstances, such as time pressure or when less attentive, these participants could still be prone to reason heuristically and hence interpret box plots incorrectly.*

## INTRODUCTION

Recent studies have shown that people have great problems interpreting box plots (e.g., Bakker, Biehler, & Konold, 2005; Lem, Onghena, Verschaffel, & Van Dooren, in press a, 2012a). We investigated the effect of an intervention on the occurrence of a specific misinterpretation of box plots in university students.

### The (mis)interpretation of the area of box plots

Bakker et al. (2005) and Lem et al. (in press a, 2012a) have shown that students have great difficulties interpreting box plots, and particularly more difficulties than in interpreting other representations for data distributions, such as histograms and dot plots. These difficulties have been found even among students who were taught about box plots as much as about these other representations.

This study focused on one specific misinterpretation, namely students' tendency to interpret the area in parts of the box plot as representing a proportion or number of observations, while it actually represents density. In Figure 1, this would mean that students would think there are more observed values between 9 and 18 than between 5 and 9, since the former part of the box is larger than the latter. This is incorrect, as the larger area of the box at the right of 9 only shows that the density here is lower (i.e., the data are more spread out in this interval) than in the interval at the left of 9.

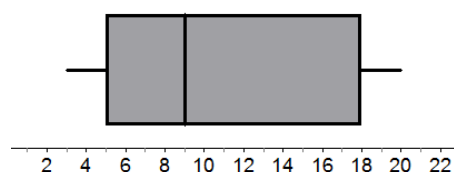


Figure 1: Example of a box plot



## **Heuristic reasoning**

Recent research (Lem et al., in press b), has shown that this specific misinterpretation is caused by heuristic reasoning, triggered by the saliency of the area of the box. Dual process theories (e.g. Kahneman & Tversky, 1972) oppose this heuristic type of reasoning to analytic reasoning. They do so to explain the phenomenon of people failing to give the correct answer to a task for which they do have the required knowledge (e.g., Kahneman & Tversky, 1972). Not only in cognitive psychology, but also in mathematics education research this theoretical framework has been used to study this phenomenon (e.g., Gillard, Van Dooren, Schaeken, & Verschaffel, 2009; Inglis & Simpson, 2004; Lem, Onghena, Verschaffel, & Van Dooren, 2012b). Heuristic reasoning is described as unconscious, automatic, fast, and undemanding of working memory capacity, as opposed to analytic reasoning, which is considered as conscious, deliberative, slow, and effortful (Evans, 2008). Heuristic reasoning often leads to the correct solution of a given problem. However, in some cases it does not, making analytic reasoning necessary. In those latter cases, the heuristic error needs to be detected (i.e., cognitive conflict needs to be experienced) and heuristic reasoning needs to be inhibited in order to obtain a correct answer. For the misinterpretation of box plots that is investigated in this paper, the heuristic reasoning is thought to be triggered by the area in the box plot, which is very salient, but actually does not provide the students with any more relevant information than the distance between two points in the data distribution. This leads students to the incorrect interpretation of the area of box plots (Lem et al., in press b). Moreover, Lem et al. (in press b) showed that this reasoning is very difficult to overcome, even with ample time and extra instructions.

In this study, we used a bottom-up approach of teaching box plots, in order to refresh and activate students' knowledge so that they would more likely be able to overcome their heuristic reasoning. But even when students' knowledge is correct and rapidly available, heuristic errors may occur. When someone with sufficient knowledge is confronted with a box plot, the automatic heuristic processing of salient features of the task in the very first phase of their solution process leads them to an incorrect first impression, which afterwards may or may not be successfully corrected by analytic reasoning. Two possible outcomes are hence possible. First, when a conflict is detected and successful analytic reasoning follows, the correct response is provided, but the reasoning takes longer than for an item where mere heuristic reasoning is sufficient. A second possible outcome is that analytic reasoning does not take over or fails to overrule the heuristic reasoning, leading one to the incorrect response. This means that only when no heuristic errors are found and the reaction time for items in which heuristic reasoning suffices is the same as the reaction time to items in which analytic reasoning is necessary, the incorrect heuristic reasoning is completely overcome.

## **RESEARCH GOAL**

As previous research showed that the misinterpretation of the area in the box of the box plot is caused by heuristic reasoning and that this reasoning is difficult to overcome,

our main goal was to test whether an intervention aimed at improving students' understanding of box plots reduces the occurrence of this heuristic reasoning.

## **METHOD**

### **Participants**

Participants were 44 students of the first year of educational sciences at KU Leuven. They all had followed an introductory statistics course, including box plots, about three months before their participation. Students participated in return for course credit.

### **Intervention**

The intervention was based on "Minitool 2", an applet designed by Cobb, Gravemeijer, Bowers, and Doorman (1997), and revised by Bakker (2004) in order to improve 7th and 8th grade students' interpretation of box plots. Minitool 2 uses a bottom-up approach for teaching students about box plots.

Students first see a dot plot, which is then divided into two and later four equally sized parts. Finally, a box plot is drawn over it. This way students get to see how a box plot is constructed: by dividing a data set into four equally sized groups. According to Bakker et al. (2005), the misinterpretation we were interested in, namely the interpretation of the area as representing frequency of observations instead of density, is prevented when learning about box plots using Minitool 2. They argue that "very few students in Bakker's (2004) study struggled with this density feature, which indicates that these difficulties might be addressed to some extent by using software that allows displaying dot plots and box plots at the same time" (Bakker et al., 2005, p. 171).

The participants were individually presented with a computer animation based on Minitool 2 (see Figure 2). The animation started with a dot plot in which the exam results of 100 students were presented (step 1). The participant was then asked to imagine that he had scored a 13 out of 20 and to find out whether he was in the best half of the class or not. After this, the dot plot was presented again, now with a red line on the minimum, the maximum, and the median (step 2). The percentages of students in both halves were also shown. The participant was then asked to imagine that he had scored 17 out of 20 and to decide if he was with the best 25% of the class. The next slide showed the upper half of the class being divided into two equal parts (by drawing a line on Q3) (step 3). Next, also the lower half was divided into two equal groups (step 4). Placing these additional vertical lines again allowed to respond to the question whether the student was with the best 25%. In a next step, horizontal lines were added in order to construct a box plot image (step 5), and finally the dot plot was removed from the image, leaving only the box plot on the screen (step 6). The different components of the box plot (minimum, Q1, median, Q3, and maximum) and the percentage of observations in each of the four parts of the box plot (25%) were also given in this slide. After this first explanation phase, participants were asked to walk through the different steps of making a box plot three more times, each time adding vertical and horizontal lines on a dot plot by clicking. The whole intervention took about 10-15 minutes.

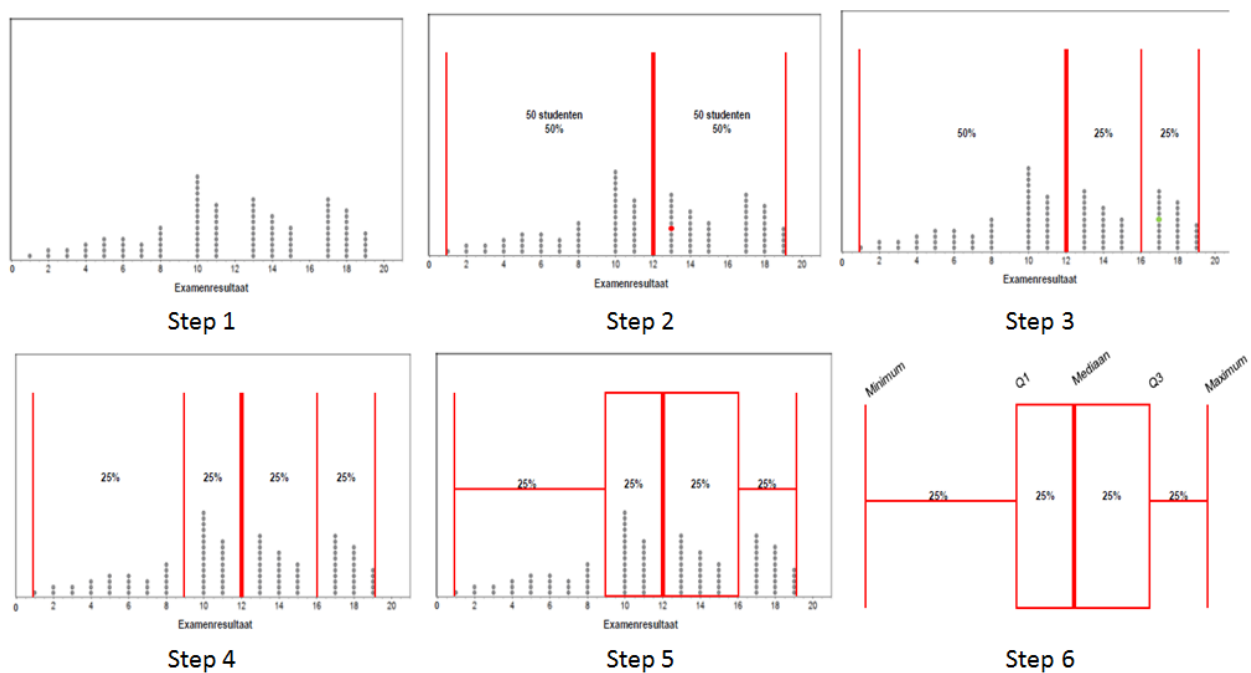


Figure 2: Animation used in the intervention

### Pre- and post-tests

Two measures to assess the effect of the intervention were used: a paper- and-pencil test, administered before and after the intervention, and a computer test afterwards.

Before the participants were exposed to the intervention, a paper-and-pencil pre-test was administered, consisting of five items. The items were selected from the items used by Lem et al. (in press b). In each item, participants were presented with two box plots representing fictional exam results of two classes. Their task was to determine in which of the two classes there were more students with a grade higher than 10 out of 20. A vertical red dotted line was placed in each item to indicate the score 10, in order to assist participants in focusing on the comparison of the box plots. Five different item types were used: two congruent item types and three incongruent item types (see Figure 3). In congruent items the correct response was the same as the heuristic response, whereas these two responses were different from each other in incongruent items. Besides answering each question, students were prompted to explain how they found their answer. After the intervention the paper-and-pencil pre-test (with different items, but the same design) was repeated.

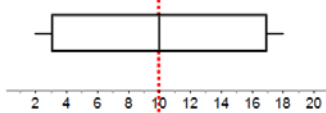
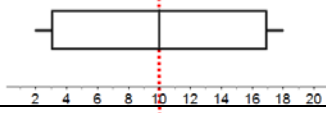
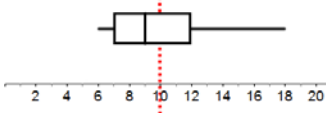
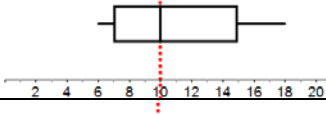

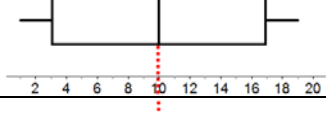
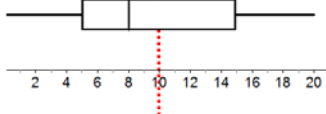
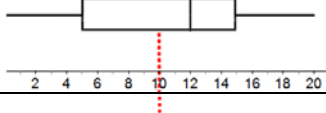

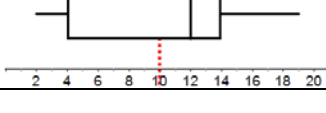
	Correct response	Heuristic response	Example
Congruent equal	The same (the same median)	The same (same area after 10)	
			
Congruent unequal	Box plot with higher median	Box plot with larger area above ten	
			
Incongruent equal	The same (the same median)	Box plot with largest area above ten	
			
Incongruent unequal	Box plot with higher median	Box plot with largest area above ten	
			
Incongruent inverse	Box plot with higher median	Box plot with largest area above ten	
			

Figure 3: The five item types used in the tests.

After the intervention and the paper-and-pencil post-test, students additionally completed a heuristic reasoning test on the computer. This test was the same as the one used in a previous study (Lem et al., in press b). Participants were presented with 40 items in random order. These 40 items were equally divided over the five item types used in the paper-and-pencil tests. Participants did not have a limit on the response time and were told to work at their own pace and to try to always provide the correct response. This test was used for two reasons: First of all to confirm the expectation that

even students who performed optimally on the post-test would still make mistakes on the incongruent items in this computer task, where they are offered much more items and merely have to decide without having to justify their answer, which may elicit more heuristic processing. Second, we wanted to test whether reaction time difference would also be visible, giving even more evidence for the occurrence of heuristic reasoning.

## **Predictions**

First, for the paper-and-pencil test, we expected that the intervention would result in an improvement both with respect to number of correct responses (Prediction 1A) and correct explanations (Prediction 1B), for the incongruent items. Also, we expected to find higher accuracy for congruent than for incongruent items (Prediction 1C), as analytic reasoning is necessary to find the correct response in incongruent items. Second, in the computer test, we expected that even in the students who scored optimally in the paper-and-pencil post-test we would still find evidence for the occurrence of heuristic reasoning based on accuracy rates and reaction times,. Concerning accuracy, we expected less correct responses in the incongruent items as compared to the congruent items (Prediction 2A), like in the paper-and-pencil test. With respect to reaction times, we expected longer reaction times for the correct responses to incongruent items as compared to correct responses to congruent items (Prediction 2B), as more time-consuming analytic reasoning is necessary in the case of incongruent items to find the correct response.

## **RESULTS**

### **Paper-and-pencil tests**

The number of correct responses was higher than the number of correct explanations, both in the pre-test and in the post-test. Paired sample *t* tests showed that for incongruent items, both accuracy and correct explanations significantly improved after the intervention. The number of correct responses for the incongruent items increased from 1.8 to 2.7 out of 3,  $t(44) = -5.15$ ,  $p < .001$ ,  $d = 0.99$ , confirming Prediction 1A. The number of correct explanations for the incongruent items increased from 0.3 to 2.1 out of 3,  $t(44) = -9.81$ ,  $p < .001$ ,  $d = 1.77$ , confirming prediction 1B. Large variation with respect to the improvement was however visible between the participants: While some students improved to answering all items in the post-test correctly ( $n = 26$ ), others did not show any improvement ( $n = 10$ ).

Concerning Prediction 1C, we found, using generalized linear mixed models, statistically significant higher accuracy for the congruent items, than for the incongruent items, both before,  $F(1, 175) = 23.88$ ,  $p < .001$ ,  $OR = 20.96$ , and after,  $F(1, 175) = 5.17$ ,  $p = .024$ ,  $OR = 6.26$ , the intervention.

### **Computer test**

We only looked at the 26 participants who scored optimally in the paper-and-pencil post-test. Even though their accuracy was very high (97.1% for congruent items, 93.6% for incongruent items), we found a statistically significant effect of congruency

on accuracy,  $F(1, 944) = 7.58, p = .006, OR = 3.34$  (generalized linear mixed model). These accuracy differences confirm Prediction 2A. More importantly, the reaction times for correct responses to congruent items (2964 ms) were significantly shorter than to incongruent items (3393 ms),  $F(1, 961) = 4.55, p = .033, d = .14$  (linear mixed model), based on the log transformed reaction times, confirming Prediction 2B. This implies that, as predicted, even for the participants who were able to correctly interpret box plots in the paper-and-pencil part of the post-test heuristic reasoning still occurred and analytic reasoning still had to take over this heuristic reasoning in order to get to the correct response.

## CONCLUSION AND DISCUSSION

Based on both the increase in correct responses and in correct explanations after the intervention, we can conclude that the intervention had a positive effect on participants' understanding of box plots, confirming the results of Bakker et al. (2005). Large differences between participants were, however, found, although the intervention seemed to have improved the reasoning of the majority of the participants. The computer test, however, showed that even the participants who correctly explained all items in the paper-and-pencil post-test were still affected by heuristic reasoning, leading them to longer reaction times and sometimes also to incorrect responses for incongruent items. The intervention was hence not able to alter the participants' heuristic reasoning concerning box plots, although it did help these participants to give correct answers. This means that under some circumstances, such as under time pressure or when being less attentive, these participants could still be prone to reason heuristically and hence interpret box plots incorrectly. We can think of two major reasons for the fact that our intervention was only partially successful. First, the intervention was relatively short and (almost) no interaction with the representations and data was possible during the intervention, both of which are in contrast with the studies of Cobb et al. (1997) and Bakker (2004). Second, the design of box plots seems to not conform to the way people interpret external data representations. This means that even with ample experience, people may continue to have difficulties with the interpretation of box plots, as revealed in reaction times and sometimes even in accuracy rates. For the participants for whom the intervention did have a positive effect, it is not clear to what extent the improvement is long lasting and can be transferred to other contexts and different tasks involving box plots. More research is hence necessary to completely understand the reasoning of students and the possibilities to improve their interpretation of box plots. Possible ways to improve students interpretation of box plots could be found in using instructional interventions (like in Bakker's (2004) and in this study), or by altering the design of box plots (i.e., like proposed by Tufte, 1983).

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# PREPARING STUDENTS FOR DISCOVERY LEARNING – SKILLS FOR EXPLORING MATHEMATICAL PATTERNS

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*To avert the ineffectiveness of ‘learning by discovery’ various models of guided discovery have been suggested. Some of them aim at fostering students’ discovery skills, especially in the area of scientific experiment. In this article we look at mathematical discovery and describe an instructional model for the acquisition of certain ‘discovery skills’ that are useful for students in the investigation of mathematical patterns. Such skills comprise ‘generating examples’ and ‘testing conjectures’ (und weitere?). They can be substantiated theoretically by drawing on epistemological models of ‘mathematical experimentation’ and of scientific discovery. In an experimental study we show that these discovery skills can be effectively and sustainably acquired and furthermore transferred to other content areas.*

## INTRODUCTION

‘Learning by discovery’ (or more briefly ‘discovery learning’) is a rather broad concept that equally permeates educational research and everyday practice. As an instructional model it is a subject of debate, especially with respect to its efficiency (Mayer, 2004). The assumed benefits of discovery learning – the deep elaboration of actively discovered content (Bruner, 1961) – cannot be supposed to apply generally. This is usually explained by the extraneous cognitive load which is created by the discovery process (Sweller, Kirschner & Clarke, 2007). A meta-analysis by Alfieri et al. (2011) clearly shows that effective instructional designs for discovery learning utilize a variety of principles of guiding the learner during discovery (‘enhanced discovery’) whereas pure discovery mostly appears to be ineffective.

However, educational practice remains rather unimpressed by these empirical findings: Reform-oriented programs articulate a need for intensifying discovery learning in the classroom (e.g. NCTM, 2000). Internationally comparative studies (cf. Stigler & Hiebert, 1999) reveal cultural scripts with substantial differences in the prevalence and the quality of discovery processes: The Japanese classroom for example appears to rely substantially on discovery learning and proves empirically successful with respect to student achievement.

This contradiction between research and practice becomes less paradoxical when one acknowledges that the goals of discovery learning are not restricted to the criterion of short-term effectiveness of mere knowledge acquisition. Dean & Kuhn (2006) point to a rather gradual attainment and long-term effects of discovery learning that do not show in short-term evaluation (e.g. Klahr & Nigam, 2004). Mathematical discovery also has to be considered as the context for the acquisition of more general competences such as problem solving, modelling, discovering patterns and proving.



Finally discovery is at the core of problem based concept generation (Freudenthal, 1973) and is assumed to have positive effects on students' epistemic beliefs (Schoenfeld, 1992).

Without doubt, discovery remains a central feature of mathematics instruction – nevertheless it is a challenging activity for students, so the didactical problem to be tackled is how to prepare students for dealing with situations of mathematical discovery. Hence it must be the goal of research not only to investigate basic and fairly general principles for effective discovery learning but to propose and empirically substantiate practically feasible instructional models for mathematical discovery for the everyday classroom.

Although there is plenty of literature on the effectiveness of discovery learning (cf. Alfieri et al, 2011), there are much fewer findings about the skills students need to successfully engage in discovery. Moreover these skills seem to be rather content specific. One of the areas investigated most thoroughly is discovery by experiment and the use of the control of variable strategy (Chen & Klahr, 1999). In this report our intent is to contribute to the issue of conceptualizing and fostering prerequisites for discovery learning in the *mathematics* classroom. We propose an instructional model for fostering certain basic skills for dealing with mathematical discovery, and we evaluate it empirically.

## THEORETICAL FRAMEWORK

To describe the process of “mathematical discovery” theoretically we basically draw on two theories which are broadly used for describing discovery in the domains of mathematics and in science respectively – both of them extend beyond these domains to general epistemology.

**Mathematical discovery:** Although mathematics as a discipline is distinguished among other disciplines by its deductive approach, ‘doing mathematics’ heavily relies on inductive processes. This has been stressed throughout mathematics history as ‘quasi experimentum’ (Euler, 1761), ‘plausible reasoning examples’ Polya (1954) or as ‘experimenting with examples’ (Heintz, 2000). In order to translate this into the context of education, one can rely on general theories for individual processes of knowledge generation, such as the epistemology of Peirce (1960) who identifies three main processes in knowledge generation: (1) generating a hypothesis by examining examples (*abduction*), (2) testing a hypothesis by creating (counter)examples (*induction*) and (3) creating predictions by *deduction*.

**Scientific Discovery:** A model for ‘experimental thinking’ which has become important in the context of science education – but is not restricted to it – was proposed by Klahr & Dunbar (1988). Within their model of *scientific discovery as a dual search* (SDDS) they describe experimentation as a problem-solving process during which the individual moves between two spaces, i.e. a *space of hypotheses* and a *space of experiments*. In the context of mathematics the latter can also be regarded as *space of examples* or as *space of mathematical situations*, since by *experiment* the authors mean testing a hypothesis in a specific situation.

**Synthesis:** Synthesizing these theories (which cannot be discussed in depth in this article) we put forward a model of mathematical discovery as inductive knowledge generation. We call this situation „mathematical experimentation“ and postulate the following central processes: (1) generating examples either intentionally or intuitively, (2) identifying or generating structure among these examples, (3) proposing conjectures, and (4) testing conjectures by generating adequate examples (Leuders, Naccarella & Philipp, 2011). This (idealized) model for mathematical discovery as ‘quasi-empirical experimentation’ can be used to describe and understand discovery processes of students. Furthermore it will be used to identify adequate discovery skills to foster and evaluate in an instructional setting.

## PREVIOUS RESEARCH AND RESEARCH QUESTIONS

Discovery processes with respect to mathematical patterns can be considered quite generally as inductive reasoning (Heit, 2000). The body of research in this area suggests that it is difficult to differentiate between deductive and inductive cognitive processes and that the evidence for transfer effects of training of inductive reasoning abilities is inconclusive (Klauer, 2011).

In previous years experimentation processes in science education have been studied extensively (e.g. Zimmermann, 2000). In mathematics one can find didactical propositions (Polya, 1954; Watson & Mason, 2005) but only few empirical works. Haverty, Koedinger, Klahr & Alibali (2000) investigate undergraduate students while generating algebraic expressions for linear and quadratic numerical data. They identified three types of inductive reasoning activities: gathering data, finding patterns and generating hypotheses. Successful students used all three processes and constantly switched between them building on already constructed knowledge.

Meyer (2008) uses the framework by Peirce for reconstructing single cases of discovery as abduction processes. In a more extensive study Leuders, Naccarella & Philipp (2011) focused on the discovery process of students of different ages (ca. 9yrs to 25yrs) when exploring different arithmetic problems, e.g. finding successive numbers that add up to a given number (cf. Mason, Stacey & Burton, 1982). In an extensive analysis of verbal data and students’ products 23 typical processes could be identified and subsequently be clustered in four types of processes.

This state of research was the basis for devising and evaluating a training concept for discovery skills. In our study we attempt to answer the following research questions: (1) Can discovery skills be fostered within the regular curriculum? (2) In what way does students’ competence growth show up in their solutions? (3) Does transfer on different content occur? (4) Are the acquired skills sustainable?

## DESIGN OF THE STUDY

The training was integrated into a teaching unit dealing with ‘exploring numbers’ (10 lessons, each 45 min). Students were supposed to develop the concepts of *factorization*, *prime numbers* and *divisibility rules* within an enhanced discovery environment described below. The twelve problems used in this teaching unit were

developed in the context of the design-research project KoSiMa ([www.ko-si-ma.de](http://www.ko-si-ma.de)). This teaching unit is part of a regular textbook designed for the German curriculum for non-academic lower secondary education (for an example see fig. 1).

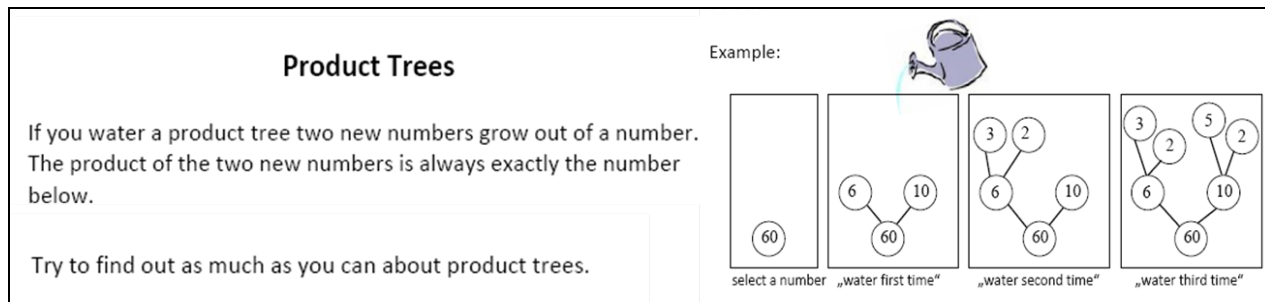


Figure 1: Example problem from the intervention

Students wrote down their ideas and approaches during the discovery phases in a simplified type of journal to provide a basis for verbalizing and reflecting their processes during the lessons. The training of discovery skills was conducted in four steps analogous to a well-established model for teaching problem-solving strategies (e.g. Schoenfeld, 1989). Step 1: Solving a given problem without explicit heuristic support, Step 2: Making strategies explicit with reference to experience, Step 3: Reflecting the use of the strategies, Step 4: Transferring strategies. Accordingly the use of the discovery processes was supported by progressively giving prompts: “Write down some examples; Write down your examples systematically; Find another representation; Write down a conjecture; Test your conjecture by a (counter)example”.

In a controlled quasi-experimental design the intervention was implemented in a sixth grade in lower secondary school. The experimental group (n=126) participated in a teaching unit with the integrated training concept for experimentation while the control group (n=101) was taught the same content using their textbook for 3 weeks (10-12 lessons, each 45 minutes).

In order to measure the discovery skills we constructed a test that empirically differentiated two dimensions (Philipp & Leuders, 2011): (E1) finding structures and writing down conjectures and (E2) testing given conjectures by using examples. The two dimensions comprise a relevant subset of the discovery processes identified previously. In addition to pre- and post-testing a follow-up test was administered after 6 weeks. Further control measures such as classroom observations and teachers’ written feedback were used for monitoring the compliance. Some students were interviewed while working on selected tasks (these data and the written journals will be analysed as part of a subsequent study). To distinguish experimental competence from knowledge acquired in the intervention we used items that asked for example generation and hypothesis testing with respect to different content (*‘number patterns in addition and subtraction’*).

## RESULTS

The non-randomized group assignment requires testing the homogeneity of the two groups. A multivariate variance analysis for the pre-test revealed only a group difference in self-perception with respect to mathematics, this variable was used as covariate in further analysis. To assess the effectiveness of the training (research question 1), in both dimensions (E1) “structuring examples” (6 items, reliability,  $\alpha = 0,755$ ) and (E2) “testing conjectures” (4 items, reliability  $\alpha = 0,740$ ), a covariance analysis with repeated measures was conducted. In both dimensions there are significant differences between the two groups. This effect was found to be remarkably stable between the post-test and the follow-up-test.(research question 4).

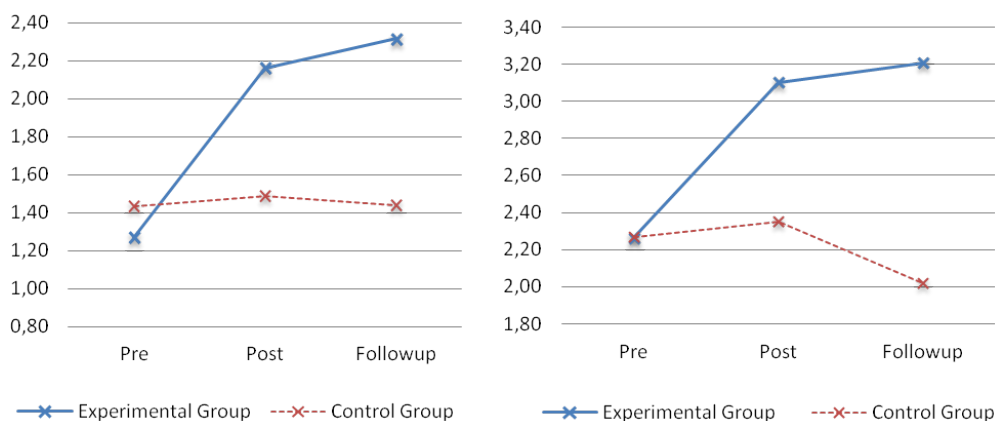


Figure 2: Differences between experimental group and control group in dimensions “structuring examples” (left) and “testing conjectures” (right)

A large effect size in both dimensions (E1:  $\eta^2=0,307$ ; E2:  $\eta^2=0,183$ ) shows that students achieved a higher level of experimentation competence in different areas (2 dimensions), with sustained effect (6 weeks) and the ability of transferring it on moderately new content (dealing with numbers, subtractive and additive structures instead of multiplicative structures) (research question 3). We presume that this is explained by the fact that our theoretical model of experimentation competences is deeply rooted in a previous qualitative study so that the intervention is aligned to actual working processes of students.

The following typical example (fig. 3) of students’ solutions illustrates the growth of discovery skills (research question 2). A similar qualitative development was detectable in the other tasks in both dimensions.

<p>Look at the given examples and continue.          Have a look at the differences.          What do you notice?          Write down one, two or three conjectures.</p>		
<p><i>Pre-Test:</i>          It's increasing.</p>	<p><i>Post-Test:</i>          The difference is the same above.          It's always a hundred less or more.          Always a step of 109.</p>	

Figure 3: Example task in the dimension (E1) “structuring” (translated)

## DISCUSSION

Our results suggest that fostering discovery skills can be highly productive with respect to effect size, sustainability and transfer. We especially emphasize that we conducted our experiment within a group of students from the non-academic strand (“Realschule”) who are distributed approximately within the lower 60% achievers in mathematics in Germany. A previous teaching experiment within the lowest strand (“Hauptschule”, 20% lowest achievers) was not successful.

We interpret this in the following way: The cognitive processes necessary for discovery can be considered as *basic* skills, extending general abilities of inductive reasoning into the domain of mathematics. Thus they can be effectively fostered in students with a certain ‘minimal’ cognitive and metacognitive level and can be applied in areas of mathematics which are accessible to exploration. Further investigation of the relation of general inductive reasoning abilities and domain specific experimentation competences may yield interesting results.

The process of mathematical discovery may be considered as a specific type of problem-solving with a focus on exploration and inductive knowledge generation. The strategies students can rely on for these purposes are rather limited in number and complexity and may be described by: „When you are stuck, try some more examples, or try another guess and test it by yet another example“. These strategies can be fairly generally employed when exploring moderately open mathematical situations without drawing on complex previous knowledge – this is the standard situation of a typical „discovery learning“-lesson. Teachers (even those without exceptional didactical capabilities) can easily integrate the strategy support which we propose. Our teaching experiment was conducted in a regular class and was integrated in regular curriculum (divisibility and prime numbers) within the regular amount of time spent on this subject. Thus we demonstrated not only ecological validity but also showed that strategy acquisition and content learning do not need to be trained separately.

We concur with English et al. (2008) in that it is necessary to develop an understanding of and to measure *basic* strategies before we move on to complex problem solving behaviour. By focusing on discovery skills within ‘mathematical experimentation’ we

may have identified such an area of basic strategies and can contribute to understanding the construct and its constitutive processes.

For further studies we are planning to follow the development of such skills more closely on an individual basis: Are there discernible stages of use of discovery skills? Can we detect different types of students? Which ways of applying the strategies are more successful? Further quantitative analysis may yield an interdependence of discovery skills with other cognitive prerequisites, such as mathematics achievement, mathematical beliefs or inductive reasoning abilities.

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# STUDY ON THE LEARNING OF QUADRATIC EQUATION THROUGH PROPORTION IN A DYNAMIC ENVIRONMENT

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*This study investigated the process of constructing the geometrical solutions to the quadratic equation,  $x^2-ax+b^2=0$  through proportions between lengths of similar triangles in a dynamic environment. To do this, we provided one task to 4 ninth grade students and observed the process of the students' activities. As a result of this study, our research shows the possibility and advantage of geometrical method in learning the quadratic equation.*

## INTRODUCTION

Algebraic manipulation such as using factorization or square root takes up most of the process of teaching and learning quadratic equations in Korean secondary school math classroom. Students also memorize the quadratic formula and learn that the number of solution depends on the value of discriminant of quadratic equation. In this process, students have little opportunity for reflecting on the meaning of the algebraic manipulation in solving a quadratic equation.

In this study, we chose an alternative approach to provide students with geometrical experience. Even those who are comfortable with a symbolic conception of algebra will probably find different points of view to be enriching (Allaire & Bradley, 2001). At all levels of education in mathematics, teaching and learning work better when both geometrical and algebraic ways of thinking are involved and when they complement each other (Banchoff, 2008). Eves (1995) introduced geometric solutions to quadratic equations contained in the Elements, which could be one example of integration of geometric and algebraic approach, i.e. how to construct the solutions of  $x^2-ax+b^2=0$  and  $x^2-ax-b^2=0$  ( $a>0$ ,  $b>0$ ). However, it is not easy for many students to grasp the ideas in the construction process by themselves. So we provided 9th grade students with a guided opportunity by teacher to construct the solution of quadratic equation with dynamic geometry based on the proportion and similarity. In this paper we introduce the result of  $x^2-ax+b^2=0$  ( $a>0$ ,  $b>0$ ) only.

Observing students' construction process, we focused on the students' strategies to explore the geometrical solution of  $x^2-ax+b^2=0$  by the guided discovery learning:

- What are the roles of proportional reasoning and similarity in students' strategies to construct the solution in a paper and pencil environment?
- What are the characteristics of students' strategies to construct the solution in a dynamic environment?



## THEORETICAL FRAMEWORK

This study used proportional reasoning and similarity as an important means of connecting quadratic equation and geometrical construction. Proportional reasoning is one of the big mathematical ideas of the middle school curriculum and a benchmark for a critical foundation of algebra (Christou & Philippou, 2002; Cramer & Post, 1993; Lamon, 1993; Langrall & Swafford, 2000; Lanius & William, 2003; Slovin, 2000; Touriaire & Pulos, 1985). In the sense that students who fail to develop proportional reasoning are likely to encounter obstacles in understanding higher-level mathematics, teachers should diversify the numerical relationships and the context of proportional reasoning problems in both instruction and testing. Particularly, students must be given ample time and a variety of contexts to develop the ability to use proportional reasoning productively in problem solving (Slovin, 2000).

Similarity is a good subject matter for ratio in geometrical context because proportional expressions could be visualized through the lengths, areas and volumes between similar figures. Although similarity-related activities are effective for developing students' proportional reasoning, they have not been considered in mathematics education significantly (Kwon, Park, & Park, 2007). In this study similarity is not only connecting method of algebraic and geometric situation but also problem context which improves proportional reasoning.

Together with the concept of proportions and similarity, this paper introduces the "analysis method" as one meaningful heuristics for constructing the solutions of quadratic equation. The analysis method was developed systematically by Pappus around 300 AD (Heath, 1981; Jones, 1986). The analysis method begins with the supposition that the solution is already given or the theorem is already proven and it tries to find out the previous knowledge or proposition. This process would be done repeatedly until a self-evident assumption is reached. Because the analysis process is not an easy route, students need special help from their peers or teacher through discussions with them. After finishing the analysis phase, students go to the synthesis phase which is the reverse of the analysis phase to prove the solution deductively.

The important point in teaching about construction is to guide students to discover the process on their own. Due to the difficulty of discovering the construction process teachers usually show students how to construct geometric figures and the students uncritically accept and mechanically memorize the process that is given to them. As a result, construction may not be educationally meaningful for students in the classroom. The analysis method might help students overcome this difficulty (Lew & Je, 2010).

If students are familiar with the dynamic geometric environment, it could be possible to start with the construction in a DG (dynamic geometry) environment because DG helps students check their thinking in a visible way. The 'dynamic' characteristic of DG allows a focus on the important geometrical idea of invariance while students observe the entire process the targets undergo. In this visible way, students could check if their construction method is suitable for intention (Jones, 2002; Lew, 2004). Falcade et al. (2007) distinguished two kinds of motions: direct and indirect motion. The direct

motion of a basic element (i.e., a point) represents the variation of this element in the plane and the indirect motion of an element occurs when a construction is accomplished. The use of the dragging tool allows the users experience the combination of two interrelated motions. Nevertheless, the sorts of tasks that students tackle, the form of teacher input and the general classroom atmosphere are all important factors (Jones, 2002)

## METHODOLOGY

**Participants** This study was conducted with 4 ninth graders who had no experience in using GSP. The students were in the upper level of their class but there existed a little difference in self-confidence about mathematical competence. They remembered what they had learned two years ago about the construction by compass and ruler and thought that there was no connection between constructions and quadratic equations. They had also learned quadratic equations in regular school classes.

**Procedure** The students were paired up and the experimental classes were conducted separately. Each experimental class was conducted 2 times and each team was provided with individual worksheet and one computer. The 1st class was about learning how to use GSP and the 2nd class was about quadratic equations. After all of activities, they were asked to reflect and write down the process of solving the problem, including their difficulties and feelings in using GSP. Each class was video-recorded.

**Task** There are two segments which  $\overline{AB} = a$  and  $\overline{CD} = b$  ( $a > 0$ ,  $b > 0$ ). You can make a rectangle which is  $\overline{AQ}$  wide and  $\overline{BQ}$  long and a square  $\overline{CD}$  on a side. Here, Q is a point on  $\overline{AB}$ . Find the position of the point Q when the area of the rectangle and the area of the square are same (Figure 1).

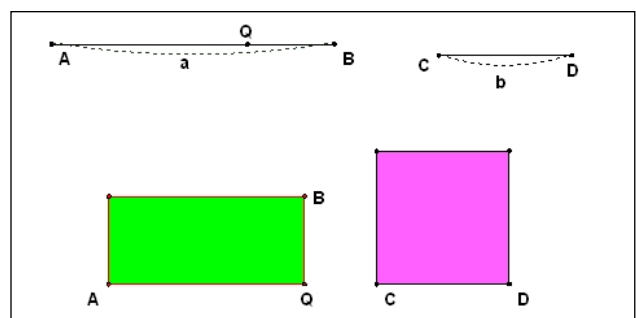


Figure 1: Task

## RESULT AND DISCUSSION

Students assumed that they found the position of the point Q and started finding previous knowledge satisfying the area condition. After these analytical activities in a paper and pencil environment, the students moved to the next stage of construction activities finding the position of Q according to the synthesis method in a DG environment. Students' activities and strategies are as follows.

### Students' strategies in a paper and pencil environment

The teacher explained the task to the students and asked them to make an algebraic expression with the condition. The students inquired if they could substitute  $x$  or  $y$  for  $\overline{AQ}$  or  $\overline{BQ}$  and the teacher said yes. The students substituted  $x$  for  $\overline{AQ}$  and  $a-x$  for  $\overline{BQ}$

without difficulty and transformed area condition into algebraic expression such as  $x(a-x)=b^2$  and expanded it like  $ax-x^2=b^2$  (Figure 2).

Teacher as researcher could know that the students had an ability to transform “condition” into “algebraic expression” and were good at algebraic manipulation.

Teacher asked the students to transform  $x(a-x)=b^2$  into other forms including  $\overline{AQ}$ ,  $\overline{BQ}$ ,  $\overline{CD}$ , etc. Students transformed  $x(a-x)=b^2$  into  $\overline{AQ} \times \overline{BQ} = \overline{CD}^2$  with ease.

Teacher asked questions to connect algebraic expressions with similarity.

Teacher: What comes to mind when you think of similarity?

Student C: We often used proportional expressions.

Teacher: Okay. Then  $\overline{AQ} \times \overline{BQ} = \overline{CD}^2$  is from what kind of proportional expression?

Students: (They think for a while and write down)  $\overline{AQ} : \overline{CD} = \overline{CD} : \overline{BQ}$

Teacher: What kind of figures did you usually deal with while studying similarity?

Students: Triangle!

Teacher: That’s right. Then, can you draw similar triangles satisfying the proportional expression  $\overline{AQ} : \overline{CD} = \overline{CD} : \overline{BQ}$ ?

The students remembered using proportional expression when learning about similarity and recalled that the similarity of triangles was the main theme. They drew two triangles which have one common side and marked “.” and “x” on triangles (Figure 3).

Handwritten algebraic derivation:

$$\overline{AQ} = x$$

$$x(a-x) = b^2$$

$$ax - x^2 = b^2 \quad \therefore x^2 - ax + b^2 = 0$$

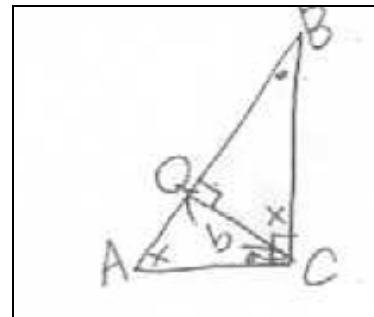


Figure 2: Making an algebraic expression      Figure 3: Drawing similar right triangles

Teacher asked if the triangles are similar one another.

Teacher: Are the triangles similar?

Students: Yes, because two corresponding angles are same.

Teacher: Then on what condition the three triangles are similar one another?

Student A: “.” and “x” meet like this. (He pointed to “.” and “x” of Figure 3). So, three triangles are similar if “.”+x=90°.

Until now, the students had to find similar figures which satisfy  $\overline{AQ}:\overline{CD}=\overline{CD}:\overline{BQ}$  instead of constructing proportion and solving it and in this process they experienced an unusual context for proportional reasoning problems in comparison with ordinary problems. Now, the teacher called the students' attention to a strategy for finding the position of Q.

Teacher: What should we do to find the position of Q?

Student A: The intersection point of the perpendicular line passing through C and  $\overline{AB}$  is Q. (He indicated the point Q of Figure 3.)

Teacher: How can we make the size of the point C  $90^\circ$ ?

Student B: If we draw a circle whose diameter is  $\overline{AB}$  and an inscribed triangle whose hypotenuse is  $\overline{AB}$ , then the angle of C is  $90^\circ$ .

Student A: That is exactly what I was thinking! Then it is enough to find the point C on the circle which the distance from C to  $\overline{AB}$  is b.

Teacher: How can we find the point C which the distance from C to  $\overline{AB}$  is b?

Student A: We can think one rectangle which is  $\overline{AB}$  wide and b long. The intersection point of the circle and the rectangle is the point what we want.

Student B: Right. We can find appropriate position of C while changing the position on the circle.

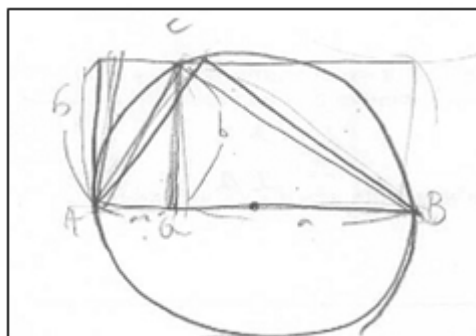


Figure 4: Strategy to find the position of Q

Student A and B's strategy is to use a rectangle which is  $\overline{AB}$  wide and b long to find the intersection point C between the circle and the rectangle (Figure 4). The intersection point between perpendicular line passing through C and  $\overline{AB}$  is exactly the point Q! Student C and D's strategy was also to use a circumscribed circle and a right triangle. All these worksheet activities were completed in a paper and pencil environment and were based on the analysis method. We could observe the roles of proportional reasoning and similarity as connecting method between quadratic equation and geometric construction. The rest of the activities progressed in a DG environment.

### Students' strategies in a dynamic environment

After finishing worksheet activities, the students started construction activity and checked their construction strategy is suitable for their goal. Student C and D constructed a circle whose diameter is  $\overline{AB}$  and used a perpendicular line and parallel

line to find an intersection point (Figure 5). Then they named the intersection point D and constructed an intersection point of a perpendicular line passing through D and  $\overline{AB}$ . Teacher asked them to confirm that it is the point Q and they checked the point Q satisfies  $\overline{AQ} \times \overline{BQ} = b^2$ . They observed the change of  $\overline{AQ}$ ,  $\overline{BQ}$  and the position of Q.

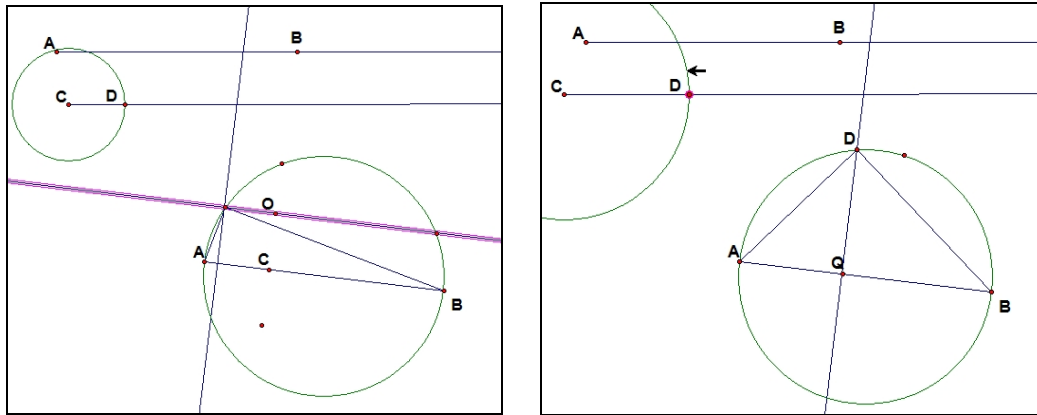


Figure 5: Finding the position of Q

In this process, they realized the difference between the nature of a paper and pencil and dynamic environment. Students can draw or erase arbitrary points and segments in a paper and pencil environment because there is no cause-and-effect relationship between them. But points and segments in a computer screen maintain their logical interrelations so students have to construct more carefully and become to focus on the idea of invariance.

Students: (While dragging D of Figure 6, they discovered the intersection point Q disappeared in a certain situation). What is wrong?

Student A: Right! This is the time when the intersection point is disappearing because the length of  $\overline{CD}$  is longer than radius. There is no intersection point.

Teacher: Then how about the case of boundary? (Student B dragged point D of Figure 6.)

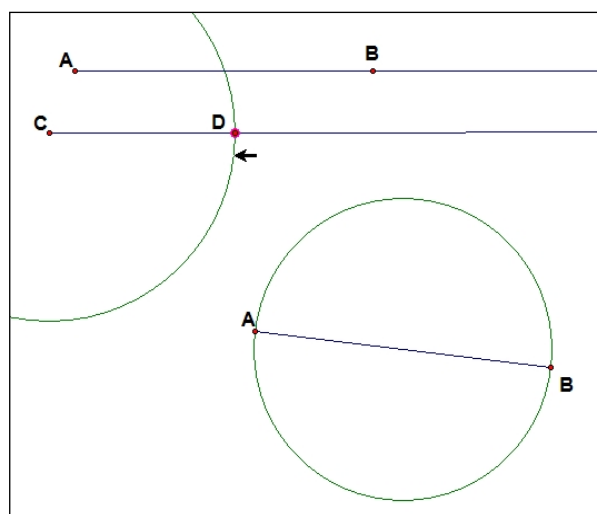


Figure 6: the existence of Q and discriminant

How about this case?

Students:  $\overline{AQ} = \overline{BQ}$ . Equal root!

Teacher: Then what's your opinion about disappearing the intersection point?

Student A: That means there is no root of the quadratic equation.

The students said that the point Q did not exist in case the length of  $\overline{CD}$  is longer than radius (i.e.  $b > a/2$ ) and there were no root of the quadratic equation at this time. It means that the students could connect the existence of the point Q to discriminant of  $x^2 - ax + b^2 = 0$  in a dynamic environment. And they could observe  $\overline{AQ}$  and  $\overline{BQ}$  as the solutions of  $x^2 - ax + b^2 = 0$  when  $\overline{AB} = a$  and  $\overline{AQ} = x$ . The solutions of  $x^2 + ax + b^2 = 0$  are  $-\overline{AQ}$  and  $-\overline{BQ}$ . It means that the solutions of quadratic equation were geometrically and dynamically visualized. This is why dynamic environment is needed (Figure 5 and Figure 6). We will study on this kind of approach to  $x^2 - ax - b^2 = 0$  and possibility of alternative teaching and learning method of quadratic equation. After all activities, teacher as researcher interviewed students. Student A was interested in connecting quadratic equation and construction and said that connecting algebra and geometry is important in solving mathematical problems. Student C and D said that it was refreshing experience but it is not easy to consider the cause-and-effect relationship between the position of Q and the length of  $\overline{AB}$  in a dynamic environment.

## CONCLUSION

This study provided middle school students with an opportunity to construct the solutions of quadratic equation with dynamic geometry based on the proportion and similarity. We observed students' activities and their strategies and our findings are as follows. First, constructing the solutions of quadratic equation offers an alternative approach that gives students an opportunity to connect algebra (quadratic equation) and geometry (construction) because they could observe changing the length of  $\overline{AQ}$  and  $\overline{BQ}$  by dragging tool and recognize  $\overline{AQ}$  and  $\overline{BQ}$  as the roots of  $x^2 - ax + b^2 = 0$  in a visible way. Algebra itself has no visual image although it is a good modelling tool for real world problems and geometry can complement the weakness of algebra. Second, students became to consider discriminant equation of the quadratic formula while they observed the point Q disappearing in a certain condition. They could connect former knowledge about discriminant equation and present construction activities in a dynamic environment. This experience may enrich students' understanding of quadratic equation. Third, in our lesson study, students were able to learn a valuable heuristics called the "analysis method" in the process of exploring the plan to construct, so that their deductive reasoning ability can be improved by way of justifying the choice of construction method. This paper also shows the importance of proportional reasoning and the understanding of similarity which connect quadratic equation with geometric construction. In this research, we investigated new possibilities in learning and teaching quadratic equation by providing students with an opportunity to construct the solution in a dynamic geometry environment. But there still remain restraints in sample selection and size. If we conduct more experiment with

more general students and further quadratic equation-related task, we would be able to get more findings and implications.

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# STUDENTING: THE CASE OF "NOW YOU TRY ONE"

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*Studenting is defined as what students do while in a learning situation. A subset of studenting behaviours, that we call gaming behaviours, subverts the intentions of the teacher. In the research that we present here we confirm a taxonomy of studenting behaviours observed while grade 10 students are independently solving a problem to test their understanding of the days lesson. Results show that 79% of the studenting behaviour observed subvert the intentions of the teacher.*

## INTRODUCTION

In 1986, Gary Fenstermacher introduced the term *studenting* to describe the things that students do to help themselves learn; from paying attention to following instructions, from practicing to studying, from reviewing to seeking help, from trying to understand to ensuring they understand, etc. In 1994 Fenstermacher expanded this definition to also include the other things that students do while in learning situations – things that do not actually help them to learn.

[T]hings that students do such as ‘psyching out’ teachers, figuring out how to get certain grades, ‘beating the system’, dealing with boredom so that it is not obvious to teachers, negotiating the best deals on reading and writing assignments, threading the right line between curricular and extra-curricular activities, and determining what is likely to be on the test and what is not. (p. 1)

Taken together, the understanding of studenting as what students do while in a learning situation expands our ability to talk about student behaviour in classroom settings. More specifically, it gives us a name for the autonomous actions of students that may or may not be in alignment with the goals of the teacher. As such, studenting extends constructs such as the didactic contract (Brousseau, 1997) and classroom norms (Cobb, Wood, & Yackel, 1991; Yackel & Cobb, 1996) to encompass a broader spectrum of classroom behaviours – behaviours that are not predicated on an assumption of intended learning. Consider, for example, the following anecdote:

At the end of a lesson Ms. Teacher assigns some homework from the textbook to be completed by next class. At the same time she provides the students with the answers to the homework questions. Her reason for doing this, she explains to the class, is that she believes that in order for the students to better learn the day's concept they need immediate feedback on their efforts as they try out their new knowledge. One of her students, Stuart, goes home and copies the work from a friend who has already completed the homework assignment. Stuart's reason for doing this is that he wants full credit for having done the homework.

From the teacher's perspective Stuart is meeting all of the benchmarks for learning – he paid attention in class and he did his homework. From Stuart's perspective he is meeting all of the benchmarks for getting a good mark – he is getting full marks for



attendance and homework. There is a rationality to Stuart's actions that is overlooked if we examine it through the oft used lens of learning. Stuart is not learning, at least not in the way that the teacher intended. But he is studenting. Specifically, he is studenting in a way that *beats the system*.

Studenting has appeared infrequently in the literature, and when it has been used it has been limited to only some aspects of studenting, and then only within particular learning situations. Goldin (2011) explores studenting from the teacher's perspective focusing on the historical and sociological aspects of studenting and limiting her study to the nature of student work, the politics of studenting, and what the student brings to the work. Aaron (2010), on the other hand, looks at studenting from the perspective of the student and focuses on the rationality of studenting behaviour within the context of high school geometry instruction. In particular, she looks at those behaviours relating to the work students do in instruction and the tacit knowledge they bring to it. Both of these studies neglect the subversive aspects of studenting that Fenstermacher introduced to the concept in 1994.

It is exactly these aspects of studenting that we are interested in. More specifically, we are interested in the studenting behaviours that are not in alignment with the teacher's goals and expected actions, yet are missed by the teacher during the activities of teaching. We have come to refer to this class of studenting behaviours as *gaming* behaviour, as in the students are *gaming* the system.

## METHODOLOGY

The data for this study comes from an ongoing larger research project in which studenting behaviour is being studied across a large number of mathematics classroom contexts. The data for this research consist of classroom videos, field notes, and post observation interviews with students. Using a grounded theory (Charmaz, 2006) approach these data are continually analysed between observations. From this analysis, over time, a number of interesting studenting behaviours has begun to emerge within a number of contexts. As these behaviours emerge and clarity is gained, coding for these now known studenting behaviours in subsequent observations becomes easier. Over time a form of saturation is reached as new observations of these contexts no longer reveal new studenting behaviours. When this occurs we can say that a taxonomy of studenting behaviour in a certain context has been reached. So it is with the context that is being presented here.

### Context

Having worked in a number of grade 10-12 (ages 15-18) classrooms wherein the teachers use a transmission model of instruction we had reached a saturation point around the context of *now try this one* (as we have come to call them) problems. These are the problems assigned, usually one at a time, by a classroom teacher immediately after s/he has done some direct instruction concluding with some worked examples. We recognize the rather traditional approach in this method of teaching and, although

we would not ourselves approach the teaching of the topics in this fashion, we make no judgement about it here. The purpose of this research is not to try to change teaching but rather to observe studenting behaviour within whatever teaching method we observe. As it is, this method of teaching is the most prevalent method we have encountered at the grade 10-12 levels.

## Data

The data for what we present here comes from a single lesson on completing the square as a way to graph quadratic functions being taught in a grade 11 classroom ( $n = 32$ ). Because saturation had already been achieved our codes were already well established. As such, for this study no video was used. Instead, we simply used our pre-established codes to annotate observed student behaviour on a supplied seating chart of the classroom during the *now try this one* phase of the lesson. Immediately after these observations, while students began to work on their assigned homework, as well as for a few minutes after class, we collected very brief interview data from a number of students selected based on the different behaviours we saw exhibited during our observations. The interviews were short (1-4 minutes) and were audio recorded using a portable digital recorder. For the most part these interviews consisted of a brief declaration of what we had observed them doing and one or two questions regarding their reasons for their behaviour. This was not foreign to the students as the lead author had previously spent several lessons doing similar research in the same class; although not always in the context of *now try this one* problems. In all, data from 15 interviews was collected by the two authors in a time of 25 minutes. Added to this were lengthier interviews with the teacher before and after the lesson in order to ascertain her goals for the lesson in general and the *now try this one* problems in particular. In the post interview we shared with her some of the behaviours we had observed as well as some of the responses the students had given during our brief interviews and asked to her to respond to these vis-à-vis her own goals.

## Analysis

These data were then analysed using a framework of analytic induction (Patton, 2002). “[A]nalytic induction, in contrast to grounded theory, begins with an analyst's deduced propositions or theory-derived hypotheses and is a procedure for verifying theories and propositions based on qualitative data” (Taylor & Bogdan, 1984, p. 127 cited in Patton, 2002, p. 454). In this case, the theory we were attempting to verify was the taxonomy of *now try this one* studenting behaviour that had emerged over time within a variety of classrooms.

## RESULTS AND ANALYSIS

From the analysis of the data our previously established taxonomy of five main studenting behaviours was confirmed. In what follows we present each of these studenting behaviours exemplified with excerpts from the data.

## Amotivation

Of the 32 students observed for this study three (all boys) displayed a general lack of attention towards the lesson. They were generally disengaged and disinterested in the lesson. Visibly they paid little attention, took no notes, and when they were asked to try to solve an example on their own they made no attempt to do so, or to seek help. When asked about their lack of interest they each gave a different explanation.

Frank            "I don't get it." [shrugging his shoulders and looking back down at his cell phone]

Andrew        "My tutor will help me with this tonight."

Jason           "I'm just tired today."

When we shared these comments with the teacher after the class she replied that she was not surprised.

Ms. Duo        "Frank and Andrew are never engaged. They're often absent or late and when they are here they don't do much. Andrew has a tutor and uses that as an excuse to not do anything in here ... but he is still failing the course. Jason is always here but he isn't doing any better."

(Ryan & Deci, 2002) would likely refer to them as amotivated. Amotivation is a deeper problem that goes well beyond the context that we were focused on. As such, we initially considered not including these cases in the taxonomy. However, we decided against this for two reasons – this behaviour was seen in almost every class and its inclusion allows us to account for all of the behaviours seen during the *now try this one* context.

## Stalling

Four students exhibited a behaviour that we came to call stalling. Stalling behaviour are actions that can be seen as legitimate, that are not out of place in a normal classroom or during the course of a lesson. What made these actions interesting to us was their timing. As soon as the students were asked to do a question on their own two students suddenly had to go to the bathroom, one needed to sharpen their pencil, and one couldn't find a calculator (even though the question didn't require one). When we asked the students about these coincidences they had a variety of superficial reasons justifying their actions:

Jessa           "I had to go. That's all."

Barry           "I waiting until there was a break in the lesson."

Jenny           "My pencil broke."

Drew           "Calculators are allowed so I wanted to use one."

When pushed about these reasons, however, two things emerged that were common to each of these four students. First, all of them expressed that the *now you try one* was an unimportant part of the lesson; "like a break". The reason for this, they all revealed, was "because in a few minutes the teacher [was] going to provide the answer". Taken together, these students were seeing a redundancy between their efforts to solve the problem (had they done so) and the teacher presented solutions. This redundancy exists

only within a context where the purpose of the *now try this one* problem is the production of notes.

### Faking

There is one final category of non-trying behaviour – faking. Two students (both girls) exhibited this behaviour. These girls had two things in common – they had impeccable notes and from the front of the classroom they both appeared to be trying to solve the problem. It was only from our vantage point in the back (and side) of the classroom that we were able to detect what was really going on. Physically all of their actions were those of students who were working. Their heads were down and their pencils were moving. In reality, however, neither of them was actually writing anything on their paper, even though one of them even made the pretence of erasing a mistake. When asked about this they both gave the same general answer,

Keesha        "I don't want to mess up my notes".

When pushed on this point they both came back with the same answer that the stallers did – that the teacher will soon provide the solution. However, they added to this a nuance that the stallers did not mention, and perhaps did not care about.

Jennifer        "Not only will she give us the answer, she will give us the best answer. This is the one I want in my notes."

The importance of the best answer, as opposed to just a correct answer, is important when the goal is to produce perfect notes, a goal that both of these girls clearly shared.

### Mimicking

The nine aforementioned students aside, the remaining 23 students all tried, at least in part, to solve the *now you try one* problem. Of these, 17 were mimicking. Visibly these students engaged in the problem and tried to solve it. Some made mistakes, some gave up, but most succeeded in arriving at the correct answer. Successful or not, what these students all had in common was that they referred to their notes, or the notes on the board, OFTEN. Closer observation and our questioning revealed that the students in this category were not so much relying in understanding as much as simply following the solution pattern laid down by the teacher in the example that she had worked through immediately prior to the *now you try this one* problem. The constant referencing to the previously solved problem was symptomatic of the students' attempts to map characteristics of the example problem onto the current task. When asked about this mimicry behaviour these students claimed that they were doing what the teacher wanted them to do.

John            "This is how we do things in this class. The teacher gives us an example and we write it down. Then she gives us one to try and we copy what we did in the example."

When we asked the students who had failed to get an answer about what happened their general response was that the *now try this one* question "must have been" different from the example question.

Samantha "I got lost somehow. I'm not sure where. I thought I was following the rules."

For Samantha, like the rest of the students in this category, the "rules" is a solution pattern to be copied.

### Reasoning

The remaining six students demonstrated a behaviour of reasoning. These students not only attempted the problem but progressed through it in a reasoned and reasonable manner with minimal references to prior examples. This is not to say that the prior examples did not play a role in their solutions, for they did, but as a whole rather than the line by line copying that the mimics performed. Further observation of this group of students, as they tackled additional problems, confirmed that they had a good understanding of the mathematical relationships and skills at play. Given this, we asked these students if the teacher's examples had in any way contributed to their understanding of the *now try this one* problem. For the most part the students indicated that what the teacher's examples gave them was a new combination of things that they already knew.

Kenneth "I don't know. Maybe. ... I mean it all makes sense. If anything maybe the examples just showed me what kinds of questions are possible."

That is, although they seemed to know all of the pieces they had never thought to combine their knowledge in this way.

The one exception to this was Ryan, who on several occasions (during the lesson that was observed for this study as well as others) anticipated the teacher's next example or next question. That is, unlike the others in this category, Ryan was able to combine his knowledge without being shown how to do this.

## DISCUSSION AND CONCLUSION

Having spent time in this particular class before we knew that this teacher made extensive use of *now try this one problems*. As such, prior to our observation we asked the teacher to explain to us what her intentions were with the tasks and what she expected the students to do with them.

Ms. Duo Well, I use them to give the students a chance to check their understanding of what we had just learned. This way, if they don't understand something we can catch it right away.

Researcher And what do the students do with these problems?

Ms. Duo For the most part they do the problems. You'll see when we are in there that there are a couple of boys in the back that don't do them but they don't really do anything. Everyone else, though, does them.

Ms. Duo's expectation is that the students will do these problems as a way to test their understanding and she believes that, for the most part, this is what they do. In the post lesson interview she confirmed her expectation.

Ms. Duo        So, as predicted, those three boys in the back didn't do much. But everyone else was pretty much on task. I mean, they didn't all get the problems right, but they did them. And the ones that made mistakes had a chance to learn from their mistakes when we went over it.

The data does not agree with either Ms. Duo's pre-lesson prediction or her post-lesson reflection. Of the 29 students in the class that the teacher thought were acting in alignment with her goals, only six actually were. The other 23 students were stalling, faking, or mimicking understanding. Their actions were not actually what the teacher thought they were. That is, 23 out of 29 (79%) students were subverting the intentions of the teacher, and doing so in ways that the teacher was not aware of. Now, it could be argued that those students who were mimicking understanding by mapping the solution process from one problem to another were exhibiting expected behaviour, but keep in mind the words of John and Samantha. From the perspective of the students, they were not trying to test their understanding. They were *copying* and *following the rules* – neither of which is what Ms. Duo intended.

These findings are consistent with our research in other contexts as well. Across the board students are finding ways to game the expectations of the teacher in ways that the teacher is not aware of. In many cases these behaviours are centred on proxies for learning and understanding, such as mimicking, that are not actually conducive to learning – but appear to be in alignment with the teacher's goals.

From the perspective of the student, however, there is a certain rationality to their actions that we are trying to understand using theories from behavioural economics, such as minimisation of effort, economy of action, bounded rationality (Simon, 1955), loss aversion, and risk aversion. At the same time we are exploring game theory to try to understand potential performance goals when students 'game the system' (Baker, Roll, Corbett, & Koedinger, 2005), the behaviours and related consequences when students engage in 'playing the game' or 'playing the system' (Dryden, 1995), and students' behaviour in response to incentive grading systems (Newfields, 2007).

Finally, it is worth noting that since we brought to the attention of Ms. Duo the taxonomy of the behaviours we had observed within her lesson she has begun to make changes to her teaching. It seems as though the kind of knowledge generated by research into the gaming aspects of studenting behaviour can be a powerful catalyst for initiating teacher change.

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# ELABORATING STAGES OF TEACHER GROWTH IN DESIGN-BASED PROFESSIONAL DEVELOPMENT

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*The paper aims to present how experienced mathematics teachers learn through participating in design-based professional development. By applying a case study approach focusing on examining teachers' design intention and design practice, we identified three growth stages: self-expression stage, the stage of combining others' ideas into personal design, and the stage of digging into the essences of mathematics learning.*

## INTRODUCTION

A central challenge in mathematics education is the gap between research and practices. The fragmentation of teaching activities often leaves teachers alone to face difficulty in integrating theories and research into the contexts of their teaching. To bridge the gap, one way is to transform the theories into practical materials such as publications to help teachers make sense of the theories so that teachers can properly apply the theories in the teaching contexts. The alternative is rooted in ground-oriented design research which emphasizes the design of innovations in classroom situations to formulate and examine specific teaching and learning theories (Brown, 1992). For teachers, their roles in design projects are just implementers and feedback providers (Swan, 2007).

We used another approach of professional development to facilitate teacher growth by which teachers are designers for creating instructional materials (Hjalmanson & Diefes-Dux, 2008 ). This approach, different from the others, emphasizes teachers' initiatives in engaging in the designs and their role as active learners for their professional growth. We aim to enhance teachers' pedagogical power so that they can scaffold students to experience the essence of mathematics learning in terms of nurturing students' common sense in- and out-of-mathematics, cultivating their mind and leading them to understand how mathematics is developed and formulated from an epistemology perspective (Bishop, 1991). Professional development programs, to this end, can be viewed as supportive environments where teachers and educators work together in "reciprocal relationship of a reflexive nature" (Jaworski, 2001, p. 315). In this paper, we explored how in-service teachers make growth through participating in the design-based professional development.

## ANALYTICAL FRAMEWORK

We analysed teacher growth focusing on two dimensions: the design intention and design practice. We recognize the aim of professional development is to enhance teachers' pedagogical power so that they are able to design tasks which allow students



to experience the essence of mathematics learning. Therefore, it is particularly important to include the two dimensions as the analytical framework because it can probe characteristics specific to the growth with respect to the change in intensions, the design strategies selected based on the intensions, and the evaluation of design products in accordance with the professional development aims.

Design intention is to identify teachers' attempts to initiate tasks and the strategies they use accordingly. The way to investigate the design intention is in line with implementation intention, which delegate the control of goal-directed responses to anticipated critical situations when translating the goal into actions (Gollwitzer, 1999). In this regard, the investigation does not directly focus on the goals set up by the teachers but rather teachers' attempt to secure the goal by strategically calling on the responses and the implementation plans.

For design practice, it aims to classify the growth in terms of the design quality related to professional development aims. As the professional development were arranged to facilitate teachers in designing tasks that can enhance student active thinking, mathematical conjecturing is the approach we used. This is because conjecturing can serve as the backbone for mathematical thinking (Mason, Burton, & Stacey, 1982) as well as enhance students' mathematics literacy (OECD, 2009). Thus, the way to identify the growth stages can be the examination of the design practice in terms of what learning opportunities the designs can offer students to make conjectures. The four principles for designing conjecturing tasks (observation, construction, transformation, and reflection) (Lin, Yang, Lee, Tabach, & Stylianides, 2012) and the five types of conjecturing (Cañadas, Deulofeu, Figueiras, Reid, & Yevdokimov, 2007) are the criteria to evaluate the design practice.

## **METHODOLOGY**

The data for this paper was part of a large project which implemented four professional development workshops, each of which lasted a semester long. The workshops were organized with the aims to enhance students' active thinking by means of designing conjecturing tasks. During the workshops, participating mathematics teachers were required to create tasks and revise the work based on the responses obtained from the discussions with educators and peer teachers and the enactment with students, which together offer them opportunities to enrich their pedagogical power.

Data sources for the analyses included survey, interviews, and self-reflections with respect to how participating teachers initiated the tasks, what sources of information the teachers perceived had influence in the creation of tasks, how they enacted the designs in classrooms and subsequently made refinements. Other data sources were participating teachers' design products, the video corpus of professional development sessions, and the field notes taken by researchers. Among the mathematics teachers participated in the professional development workshops, we particularly selected an experienced mathematics teacher, Adam (pseudonym), who demonstrated significant changes in terms of his design intention and design practice. Adam had about 10 years of teaching experiences and was a co-author for two books which included a variety of

magic activities that can motivate students in learning mathematics. To identify the growth stages, qualitative methods (Merriam, 1998) were applied to analyse and triangulate the data sources in order to clarify the storyline for the growth and to establish the reliability of the analyses.

## FINDINGS

Adam's growth can be generally identified into three stages: self-expression stage, the stage of combining others' ideas into personal designs, and the stage of digging the essence of mathematics learning.

### Stage 1: Self-expression

Self-expression here refers to the situations as teachers intend to show good tasks that they designed previously as they think their design products meet professional development requirements. Self-expression teachers usually focus their teaching and use their prior experiences to assimilate the professional development learning by noticing the similarities instead of the differences. For Adam, he was identified in this stage as he showed his previous designs involving magic activity. Using magic activities as a strategy to motivate student learning of mathematics is common in Adam's instruction as he reflected that *"using magic activities are routine for me to motivate students"*. The "tall and short cats" magic activity is one of the tasks Adam showed in the workshops. For the magic activity activity, its underlying mathematics is the concept related to arc lengths that usually cause students visual illusion and misconceptions. In the magic activity, Adam used congruent arc-length diagrams and named them as Cat A and Cat B. As shown in Figure 1, the representation of the two cat diagrams gives the impression that Cat A looks longer than Cat B. But when Adam exchanged the positions for the two cats, the magic occurs because Cat B looks longer than Cat A at this moment.

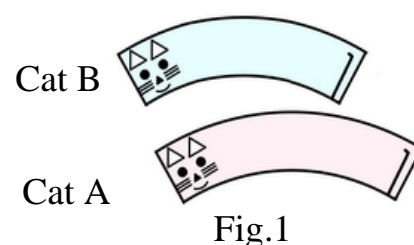
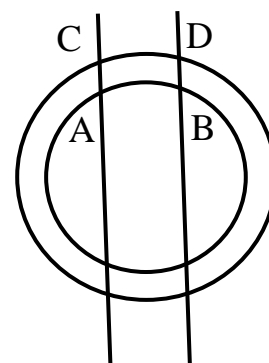


Fig.1

The magic activity led students to notice the inconsistency in visual evaluation and conclude that the visual comparison was not reliable method to assess the arc lengths. Thus, students were motivated to explore the underlying mathematics for the inconsistency. Additionally, Adam also used a series of true-or-false items to clarify students' concepts related to arc lengths after students noticed the inconsistency of the arc lengths.



- ( ) The lengths for arc AB and arc CD are the same.
- ( ) The degrees of central angles for arc AB and arc CD are the same.
- ( ) Arc AB and arc CD can be fully matched.

Recognizing that designing tasks for students to make mathematical conjecture is the aim of professional development, Adam clarified why he presented the task in workshops. He explained true-or-false items can be treated as a type of conjecturing as

the items provide students opportunities to guess an answer in a short time and are proper instructional materials for different levels of students to work on.

The example shows how Adam used his personal understanding of conjecturing to assimilate professional development learning. At this stage, Adam did not or could not distinguish the difference between true-or-false items and the conjecturing types proposed in professional development (Cañadas, et al., 2007). Based on his understanding of conjecturing as discussed in professional development, Adam thought true-or-false can meet professional development requirements, which in turn hindered his growth. As a result, Adam expressed that he did not learn much from participating in professional development in this way.

## Stage 2: Combining other ideas into personal designs

At this stage, teachers create tasks by combining both personal ideas and others in the designs. They make connections of personal design ideas and those presented in professional development, but were not able or possibly put less effort to qualitatively coordinate different ideas into the designs. The example used here to elaborate Adam's design this stage is the simplification of square root. The task includes three parts. The first part is a table of square roots (see Figure 2) in which students have to identify which ones are the simplest square roots and which ones are not. Adam expected students, through working on the table, can become familiarized with the skills in finding the simplest square roots and in turn use the table to answer the second and the third parts of the task.

Fig. 2

$\sqrt{1}$	$\sqrt{11}$	$\sqrt{21}$	$\sqrt{31}$	$\sqrt{41}$	$\sqrt{51}$	$\sqrt{61}$	$\sqrt{71}$	$\sqrt{81}$	$\sqrt{91}$
$\sqrt{2}$	$\sqrt{12}$	$\sqrt{22}$	$\sqrt{32}$	$\sqrt{42}$	$\sqrt{52}$	$\sqrt{62}$	$\sqrt{72}$	$\sqrt{82}$	$\sqrt{92}$
...									

The second part of the task is a series of true-or-false items, each of which involves a mathematical statement about the simplification of square roots. Using true-or-false items is Adam's personal designing approach, which he thought that can offer students opportunities to guess the answers and then promote the learning relevant to the mathematical topics.

- 1.( ) If  $A$  and  $B$  are coprime, then  $\sqrt{AXB}$  must be the simplest square root.
- 2.( )  $\sqrt{\text{multiples of } 3}$  must not be a simplest square root.
- 3.( ) There are six simplest square roots between  $\sqrt{1}$  to  $\sqrt{10}$  and another six simplest square roots between  $\sqrt{11}$  to  $\sqrt{20}$ .
- 4.( ) When is the simplest square root, then  $\sqrt{A+10}$  must also be a simplest square root.

Adam thought that the first parts of the task allowed students to search for a variety of supportive and counter examples that in turn can be used to conjecture and verify statements associated with the second part items. When Adam presented the task in professional development, the educator suggested him to include another problem in which students were required to create mathematical statements themselves. The next is the third part problem that Adam included in the task based on the suggestion from the educator.

*It's your turn to find the troubles. Please use the table from  $\sqrt{1}$  to  $\sqrt{100}$  to create three mathematical statements.*

Combining the first and the third parts of the task can be treated as a mathematical conjecturing—empirical induction by a finite number of cases (Cañadas, et al., 2007), which is one of the strategies introduced in professional development. For conjecturing mathematical statements, students can do so based on the given table for finding both supportive and counter examples. Students' responses to the third part of the task also provoked Adam to see the power of the problem in triggering students' diverse ideas of making conjectures.

The analysis of the task shows how Adam made understanding of professional development strategies and found a way to integrate the strategies into the design without changing the design format of true-or-false items. Although three parts of the task are relevant to the simplification of square roots, both the first and the third parts for mathematical conjecturing do not necessitate the use of second part with the true-or-false items. Adam just combined both his design idea and the idea obtained from professional development in his tasks without making changes of the task qualitatively.

### **Stage 3: Digging into the essence of mathematics learning**

At this stage, teachers intend to go beyond self by constructing tasks which allow students to experience the essence of mathematics learning. To this end, teachers have to transform and coordinate ideas from different learning environments to create novelty for the designs and flexibly use the learned strategies and principles to create this type of tasks. For Adam, he achieved this stage as he changed his design intention and demonstrated his ability to create tasks that aim for students to experience the essence of mathematics. He reflected himself as his prior designs only focused on well-defined mathematics knowledge for students to learn. But he noticed the importance of tasks for students to experience the essence of mathematics.

*Adam: I am thinking whether or not I can use magic activities to explore more essential things instead of just motivating students to learn mathematics...I think teaching for conjecturing "direct and well-defined mathematics knowledge" is not that good as it does not allow students to operate and experience the process of developing mathematics knowledge...In the past, I thought active thinking as the activities of leading students "apply different mathematics knowledge to solve a variety of mathematics problems". But now, I think my teaching goal should focus on helping students develop mathematics knowledge and using different mathematics perspectives to see the problems...I just realize that good mathematics teaching is not how much mathematics knowledge is taught but rather how students can participate in knowing the mathematics. Here, the participation does not mean that students listen to the lectures and effectively imitate teachers' ideas or problem-solving strategies. Participation refers to the discussions between students and teachers aiming for developing mathematics knowledge together...I expect to devolve my learning mathematics experiences to my students.*

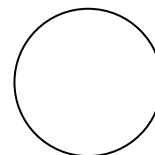
The above transcript reveals several points. First, Adam changed his design intention. He noticed limitations of the magic activities he used previously as those activities

only functioned as a tool to promote students' motivation in learning mathematics. He began to consider alternatives that magic activities can play in facilitating student active thinking. Second, Adam reflected his understanding on mathematical conjecturing. Originally, he thought mathematical conjecturing is for students to learn well-defined mathematics knowledge as the ones being presented in the textbooks. But, he detected the importance of conjecturing is that it allows students to experience and clarify not-well-defined mathematics knowledge. Third, Adam also began to notice what learning opportunities his designs can afford students to experience the development of mathematics and to use different viewpoints in a high position to analyze mathematics problems. Specifically, Adam realized how important classroom discussions can be for students in experiencing the developing process of mathematics knowledge.

Regarding design practice, Adam also demonstrated that he can flexibly use conjecturing strategies provided by professional development and coordinate sources of information experienced from different learning environments into the construction of new tasks. Using Adam's design of "your imagination of centers" as an example, he created this task as he noticed a pedagogical problem in mathematics textbooks. School mathematics textbook arranged circumcenter, incenter, and centroid of triangles in the same unit. However, only circumcenter and incenter are related to circles but not centroid. To challenge the problem, Adam attempted to create a task that allows students to learn the three types of center in triangles naturally. Instead of directly working on each center property in triangles individually, he constructed a task in which students were required to explore centers in a series of geometric diagrams, including circle, square, equilateral triangle, isosceles triangle, and scalene triangles respectively. The following is part of the problems for exploring the centers of a circle.

*Circle*

- Where is the center in the circle?
- How can you construct the center?
- What geometric properties does the center have?



Exploring the centers in a series of geometric diagrams provides students opportunities to try different mathematical ideas in relation to the center and experience the essence of the centers in triangles. For example, students can use equidistant idea to explore the centers in the diagrams. Then they can notice that the way to find the center by constructing distances equally in one geometric diagram may not be proper for another one. In this regard, students have to modify their constructing ideas and methods accordingly as well as conjecture the statements for the centers in different geometric diagrams. For example, formulating the statements of constructing centers in a square and in an equilateral triangle may require of making analogy—a type of conjecturing (Cañadas, et al., 2007). It is also possible that students check with several empirical examples to obtain the mathematical statements which involve the induction by a finite number of examples. After working on the series of diagram shapes, Adam expected students to obtain the meaning for each type of center in triangles.

Additionally, discussions in professional development also gave Adam an idea of including an activity of “exploring the center in a classroom” before working on the series of geometric diagrams. While another teacher, Hui, shared her experiences of leading students to explore the centers in a classroom and the responses that she obtained from her students, Adam thought the activity can further facilitate students in seeing the essence of the centers in triangles. The exploring activity in a classroom can emancipate students thinking as a center in a classroom can be named if it possesses a particular meaning. Similar idea can be applied to develop the types of centers in triangles.

## **DISCUSSION**

This study presents growth stages by analysing an experienced teacher through his participation in the design-based professional development. The evolution of the growth starts from self-expression stage, then the stage of combining others into personal design, and to the stage of digging into the essence of mathematics learning by coordination. Adam’ growth through the three stages is not straightforward, but a cyclic and sophisticated process. Creating a task can vary as it heavily relies on teachers’ understanding of cognitive behaviors of students specific to the mathematics content, the formulation and the history of the knowledge, and the curriculum arrangement.

Additionally, Adam’s reflection on each task also reveals how he gradually clarified his understanding of conjecturing and figured out the ways to create tasks in accordance with his design intention. For example, Adam iteratively asked himself “what am I designing for?” “What should be counted as a good task?” “Why do other teachers design tasks in that way?” Those questions and thought experiments on the questions enabled the critical reflections and lead Adam to dig into the essence of mathematics learning. He also recognized the importance of the experiences of the origins of mathematics and the learning beyond school mathematics. Experiencing the essence of mathematics learning in turn nurtures the common sense of students so that they can use it to solve both in- and out-of-mathematics problems.

Of importance is how to promote teacher growth from stage two to stage three. Teachers at stage three not only need to have the intentions to create tasks that can allow students to experience the essence of mathematics learning but also require the ability to create the tasks that correspond to the intentions. Another peer teacher Sophie (pseudonym) pointed out how hard this can be

Sophie: All of my designs look similar no matter how hard I have tried...I start feeling the constraints of my designs as I could not go further...I would like to analyze Adam’s brain to see how he can create such good tasks.

The transcript shows the difficulty in achieving this stage as having intentions and the ability to create good tasks. How to facilitate mathematics teachers toward stage three like Adam becomes a central issue for professional development research.

Particular attention should be given to the experienced teachers at stage one as they already have clear pictures of self-teaching and use the picture to assimilate professional development learning by finding the similarities in the both. In this regard, they are self-content and do not see the need for growth. How to change teachers' intention so that they are willing to make changes becomes a central issue for professional development.

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# ENHANCING PRE-SERVICE TEACHERS' KNOWLEDGE OF STUDENTS' ERRORS BY USING RESEARCHED-BASED CASES

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*The study examined the effect of research-based cases on pre-service teachers' knowledge of students' errors on fractions. Eight pre-service teachers came from 30 analyzed. The result shows that more pre-service teachers became competent in identifying students' errors and offering possible pedagogical strategies corresponding to specific errors, such as, the strategy of "distinguishing the fractional part consisting of one pieces from more than one piece" for handling the errors of "ignoring participants including experienced teachers participated in cases workshops. Six research-based cases related to students' authentic errors were discussed in heterogeneous small groups. One item including three students' errors on fractions as part of the items conducted in pre- and post-test was the whole" and "no partitioning".*

## INTRODUCTION

Many studies emphasize that knowledge of students' conceptions and misconceptions is essential for teaching (Ding, 2008). The standards emphasize that errors should not be seen as dead ends but rather as springboards for inquiry (NCTM, 2000). Researchers begin to see errors as windows into student thinking (Bray, 2011). Although research has been carried out on pre-service or novice teachers' views of students' errors and responding to errors (Son & Sinclair, 2010), and on in-service teachers' handling student errors in classroom discourse (Ding, 2008), there are limited attention in enhancing pre-service teachers (PTs) knowledge of student errors.

"Failure is the mother of success". The proverb indicates that errors are often considered as catalysts for mathematics learning. Teachers hardly take advantage of mathematical errors in instruction. Many teachers tend to hide and avoid students' errors in classrooms. Handling errors in learning shows to be more effective than avoiding them, if there is clear feedback. However, prior to dealing with errors, teachers need to enhance their knowledge of students' errors, specially, identifying the sources of students' specific errors.

Teachers are learners who need to be taught in innovative ways under the use of reform-based curriculum. It is difficult for teachers to use students' errors as a teaching tool without similar learning experience in their teacher education. Fennema et al. (1996) suggest that improving knowledge of students' misconceptions can probably only acquired in the context of teaching. It implies that PTs can be learned about students' difficulties via authentic teaching or vicarious learning. However, PTs hardly have any authentic teaching experience. The research-based cases (RBC) possessing the nature of practice-based teaching can provide vicious learning, since they can foresee the complexity of the classroom.



The features of RBC used in the study include: (1) The scenarios in each case are excerpted from a real instruction; (2) Students' various correct and incorrect responses to each problem covered in each case are collected from real teaching; and (3) The part of "Discussion Questions" covered in each case are sourced from teachers' concerns being discussed in previous professional programs that we investigated.

## **THEORETICAL BACKGROUND**

### **The Use of Cases**

Case inquiry is an approach adopted for teacher professional development of the study. Cases provide episodes or scenarios how a teacher experienced a problem and strategies she used. Case discussion initiates reflective conversations with peers, and then fosters a collaborative disposition. Through case discussion, teachers learn to be reflective as they learn to think critically about their work and learn to see their work as problematic. As teachers talk about their work, they come to know what they know. Thus, they learn theoretical principles of mathematics and how mathematics should be taught (Merseth, 2008). Thus, the cases have the functions of collaboration, reflection, learning, and knowledge.

To achieve the ends of using cases, heterogeneous small group for discussing cases can be used for this study. Merseth (2008) report that without guidance, it is difficult for in-service teachers to have a coherent focus and promote higher thinking, let alone PTs have weak knowledge and without teaching experience. Thus, experienced teachers are invited to be the facilitators of case discussion in small groups to assist PTs becoming sensitive to students' errors.

### **The Importance of Knowledge of Students' Errors**

Shulman (1986) has addressed the importance of handling student difficulties by putting it as a component of pedagogical content knowledge. Errors can be viewed as a diagnostic tool or as resources for promoting learning. Errors as a diagnostic tool refer to a method for detecting the causes of error. Detecting errors help to revise false knowledge, and then prevent the errors from reoccurring. Otherwise, if one's errors are not corrected immediately, the errors may repeatedly occur or bring about another related error (Brown & VanLehn, 1980).

Errors as sources for promoting learning mean errors as springboards for inquiry for stimulating student inquiry and understanding (NCTM, 2000). Rach et al. (2012) propose two ways of handling errors, outcome-oriented and process-oriented. The outcome-oriented approach proceeds directly from error detection to correction, while the process-oriented approach includes closely analyzing the errors and generating errors prevention strategies. The closer analysis contributes not only to identifying what the error is but also to knowing the sources of error. Consequently, it is more possible for a teacher to generate effective strategies to repair the errors.

Minsky's (1994) theory of negative knowledge refers to two complementary types of knowledge: negative and positive knowledge. Negative knowledge about incorrect facts and procedures is necessary to make a distinction between correct and incorrect

facts and processes. Individuals are usually not taught about incorrect facts or procedures, but individuals are often staying in error situations. However, we do not warrant that all errors situations are good opportunity to learn. Hence, it is essential for individual in acquiring negative knowledge. Thus, theory of negative knowledge supports the process-oriented approach. This study takes process-oriented approach for using research-based cases in workshops. We helped PTs to identify sources of errors and figure out possible strategies for repairing errors.

Fraction requiring complex understanding is a challenge for students to learn because it defies students' intuitions from whole numbers. As a result, students have more errors on fraction than other topics. Moreover, most studies on students' errors focus on whole numbers and geometry. There is limited study on fraction. The research question of the study to be asked was: How were the elementary PTs interpret and respond to students' errors on fractions influenced by the use of research-based cases on?

## RESEARCH METHOD

### Participants

The 8 PTs came from 30 participants who participated in six 3-hour RBC workshops. The PTs were working on practicum in their final year of teacher preparation. The rest of 22 participants were elementary in-service teachers whose years of teaching ranged from 2 to 21. Six of the 22 teachers were master teachers for mathematics teaching.

### The RBC workshops

The RBC workshops aimed at helping the participants to: (1) anticipate students' various solutions to a specific problem; (2) identify students' errors and causes of the errors; (3) discuss possible strategies the teachers could use.

Six RBCs on fractions from a casebook in which the cases we created in previous studies were employed in workshops. Case 1 demonstrates third-graders' difficulties in transforming fraction with iterating units into a part-whole model. Case 2 displays students' difficulties with the problems involving in the number of fractional part more than one. Case 3 displays fourth-graders' difficulties with renaming fraction. Case 4 displays fifth-graders' misconceptions of ordering two fractions with two dislike denominators. Case 5 illustrates fifth-graders' difficulties in equivalent fractions. Case 6 describes fifth-graders' difficulties with the distinction between  $\frac{1}{4}$  box (fraction as part-whole model) and  $\frac{1}{4}$  of a box (fraction as operator).

Prior to each workshop, all participants were required to read an assigned written case in advance. Each workshop began by dividing whole class into small groups with a group of 5. Each master teacher was assigned as a facilitator in each group. 8 PTs were distributed into different groups. It was followed by 1-hour group discussion on the questions listed in the "Discussion Questions". Each workshop was ended by two hours whole class discussion.

One of the authors was the leader and the facilitator of the case discussion. The leader did not provide the teachers extra information. In the whole class discussion, the participants were asked to answer the following questions: (1) What are the sources of

students' errors listed in the case? (2) What evidence is there that students learned the concepts or the difficulties students have in this case? (3) What could be the possible strategies you would like to respond to each error?

### Data Collections

Before taking part in the RBC workshops, we conducted a pre-test with two items including seven sub-items for all teachers. We also conducted a post-test with the same items as the pre-test at the end of the workshops three months apart from the beginning. Due to page limitation, PTs responding to item 1 as shown in Table 1 is reported here.

**Item 1:** Jing gave her fourth-graders to solve the problem:

*A box has 18 pieces of chocolates. Joseph ate  $\frac{4}{9}$  box. How many pieces of chocolates did Joseph eat? Draw a figure to show your thought.*

There were 4 pupils' errors at right column.

- (1) What can be the sources of each student error?
- (2) What could be the possible strategies to respond to each error?

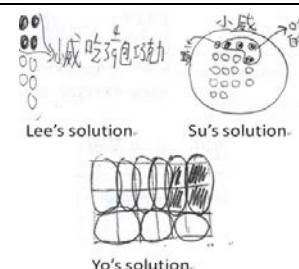


Table 1: Item of pre- and post-test

### Data Analysis

This study conducted cross-case analyses to examine how the PTs identify students' errors and how they would respond to the errors. Cross-case analyses were conducted to identify the similarities across cases and the differences among them, and to search for overall patterns. There were 5 codes with 8 sub-codes emerged for the sources of students' errors, and 4 codes with 9 sub-codes for possible teaching strategies for repairing the errors, the codes can be seen in Table 2 and 3.

## RESULTS

### PTs Enhancing Knowledge of Source of Students' Errors

Table 2 shows the overall results of their identification on the sources of errors. Due to page limitation, we will elaborate in detail in some codes.

#### Source of no understanding the set of objects as the whole

It can be looked into three ways on PTs' responses with the code of not understanding the set of objects as the whole. First, Lee's error was resulted from without considering a set of 18 chocolates as a unit. Second, Lee's error was resulted from the source of denominator always to be the set of the whole. Third, Lee did not consider that the number of each equal set can be more than singular entity. The PTs stated that this is a typical error when students represented a fraction in which the denominator is not equal to the number of the whole. Before the workshops, three PTs (PT5, PT6, PT8) were unable to diagnose Lee's error and no teacher focused on the code of "the number of each equal set is always one".

During the workshops, the teachers were frequently asked to distinguish the difficulties between the number of each equal-sized set is equal to one or more than one. For instance, *P1: There are 8 marbles in a bag. Joe has  $\frac{3}{8}$  box. How many marbles does*

*Joe have? P2: There are 8 marbles in a bag. Joe has  $\frac{3}{4}$  box. How many marbles does Joe have?* The number of each set in P1 is a single entity, while that of each set in P2 consists of 3 marbles. Through case discussion, they were getting recognized the significance of relation between the number of each part and the denominator. On the post-test, the three teachers became able to identify Lee's error by using the code "the number of each equal part is just one".

Sources	Pre- test	Post-test
<b>W:No understand the set of objects as the whole</b>		
w1:ignoring the set of the whole	pt1, pt2,pt3	
w2:no know denominator is always the set of whole	pt4, pt7	pt1, pt2, pt7
w3:misunderstanding the concept of part should be a singular entity	--	pt3,pt4,pt5,pt6,pt8
<b>P:No partition the set of objects into parts</b>		
p1: no attending that 2 is the number of each part	pt1,	
p2: no attending that 9 is not the whole	pt4, pt5, pt3	pt1, pt2
p3: no partition or regrouping	--	pt3, pt4,pt5,pt6,pt7,pt8
<b>D:Confusing the parts to be counted</b>		
d1: misunderstanding 4 pieces as 4 equal parts	pt1, pt3, pt4, pt5,	pt1, pt3, pt4, pt5,pt6, pt8
d2: mixed up 9 equal parts with 18 pieces	pt7	pt 2, pt7
<b>M:Misdiagnosed</b>		
	pt2(2), pt5, pt6(2),	--
<b>O:Other causes</b>		
	pt6,pt7, pt8(3)	--

Table 2: PTs' identification of sources of errors in pre- and post-test.

### Source of no partitioning the set of objects into parts

PTs reasoned that the causes of Sue's error in three ways : (1) Sue did not attend that 2 is the number of each set; (2) Sue was unable to identify that the denominator 9 of  $\frac{4}{9}$  is not the whole; and (3) Sue did not partition the set of chocolates into equal-sized part. The three causes were related to partitioning. As PT5 described, that Sue did not partition 18 pieces into 9 equal parts. Two pre-service teachers (PT2 and PT6) did not have correct identification of Sue's error in pre-test, but they turned out to be able to recognize the source of Sue's error in the post-test. The progress could be resulted from the discussion in Case 5 workshop. The PTs were asked to discuss the problem 3: "A box contains 24 cans of drinks. How many boxes are 18 cans?(1) Joe packs them into small bags. Each bag has 3 cans. How many boxes are 18 cans?(2) Chris divides the cans of drinks into 4 small bags. How many boxes are 18 cans?"

### Source of confusing the parts to be counted

In fraction, the numerator tells how many parts. PTs anticipated the sources of Yo's error in two ways: confusing 9 equal parts with 18 pieces and misunderstanding 4 pieces as 4 equal parts. The latter was the major source identified by the PTs (PT1, PT3, PT4, PT5). PT8 did not accept the use of continuous model for the discrete objects as the source of Yo's error in pre-test. Her pedagogical concept was revised

through case discussion. In Case 1 workshop, we discussed that discrete objects is not necessarily to be represented as discrete model, discrete objects can be represented by a continuous model, such as Yo's. In the post-test, PT8's anticipation of the source of Yo's error was not on the model, rather changed to the source of misunderstanding 4 pieces as 4 equivalent sets.

### PTs Enhancing Knowledge of Possible Pedagogical Strategies Handling Errors

PTs wrote 9 teaching strategies against students' errors on item 1. Table 3 shows the overall results with this matter.

Sources	Pre-	Post-
To understand the set of objects as the whole		
T1: draw out the whole	pt1, pt2,pt3	pt1, pt2,
T2: give more examples	pt4,	pt7
T3: to distinguish the fractional part consisting of one pieces from more than one piece	--	pt3,pt4,pt5,pt6,pt8
To partition equally the set of objects into parts		
T4: start from 1/9 box and move forward to 4/9 box.	pt3	pt3,
T5: to partition or regroup	pt1,pt5	pt2. pt7
T6:to distinguish the fractional part consisting of one pieces from more than one piece	--	pt1, pt4, pt5, pt6, pt8
To understand the parts tell how many to be counted		
T7: start from 1/9 box and move forward to 4/9 box.	pt1, pt3, pt4,	pt1, pt3
T8: distinguish 4/9 box from 4/18 box	pt2,pt5, pt7	pt2,pt5, pt7,pt8
T9: to designate what is one part		pt6
M:Cognitive conflict	pt4,pt7(2)	pt4
O:Others	pt2,pt5,pt6(3),pt8(3)	

Table 3: PTs' possible pedagogical strategies responding to errors in pre- and post-test.

### Strategies for constructing the concept of the set of objects as one whole unit

In pre-test, the possible teaching strategies repairing Lee's error included: (1) by asking to draw out the one whole unit; (2) by providing more examples; and (3) by distinguishing a fractional part in an example with one piece from another example with more than one pieces. Prior to case workshop, most of the PTs mentioned that 18 chocolates as one whole unit must become a conceptual entity, so that their responses to Lee's errors involved "showing/drawing/telling" Lee one whole unit. After cases workshops, the most prominent category PTs responding to Lee's error were by distinguishing a 4/18 of 18 chocolates from 4/9 of 18 chocolates. Making students solve more examples or counterexamples to compare for causing Lee's cognitive conflict was frequently mentioned by T4 and T7. They were cautious that the concept of a whole underlies the concept of a fraction. A whole is treated as a unit.

### Strategies for partitioning equally the set of objects into parts

In pre- and post-test, PTs consistently agreed that partitioning is a fundamental to an understanding of factions. They would use three possible teaching strategies to help

Sue to partition a set of objects into equal parts. First, they would start with  $\frac{1}{9}$  box to partition 18 chocolates into 9 equal sets and continually move forward to  $\frac{4}{9}$  box. Second, the PTs would provide Sue more examples of partitioning. Third, they would help Sue to understand that in discrete model the fractional part consists of two pieces in this instance but may be one piece in others. The third strategy for dealing with Lee's and Sue's errors have never proposed by the PTs in the pre-test. Through cases workshop, as problem 3 described previously, this strategy became most popular when the PTs helped Sue to construct the concept of whole-unit and partition, especially for PT4, PT5, PT6, and PT8.

### **Strategies for understanding the parts tell how many to be counted**

In discrete model of fractions, even though understanding the two fundamental concepts including unit-whole and partitioning, it is possible to have difficulty in determining the fractional parts corresponding to the numerator of symbolism. Regarding this error, PTs would take two pedagogical strategies to handle. First, starting from  $\frac{1}{9}$  box and gradually moving forward to  $\frac{4}{9}$  box. Second, distinguishing  $\frac{4}{9}$  box from  $\frac{4}{18}$  box. As T1 described that "I would ask Yo to figure out the number of chocolates in  $\frac{1}{9}$  box, then stepped forward to find out the number of chocolates in  $\frac{4}{9}$  box. Moreover, the language "part" cannot be ignored". In the pre- and post-test, three teachers (PT2, PT5, PT7) coherently used this strategy for making a distinction between  $\frac{4}{9}$  box and  $\frac{4}{18}$  box by drawing a picture.

## **DISCUSSION**

This study attributed enhancing PTs' knowledge of students' errors on fractions to the use of research-based cases. The following factors were analysed. (1) Students' errors in the written cases were authentic rather than imagined. These cases helped PTs to foresee the errors patterns students made. Such knowledge would be useful for PTs to anticipate what errors students would have in related mathematics topics. (2) Experienced teachers worked with the PTs in case discussion. Each experience teacher in each group played a role of abler in case discussion. Case discussion initiated social interaction. Hence, knowledge of students' errors was improved.

The sources of errors PTs identified were almost related to epistemological causes which refer to the nature of mathematics concepts. This finding was not consistent with previous studies that teachers tend to attribute students' errors to psychological cause (Bingobali et al., 2011). The difference of the study from previous study was in that the task used in the study was involved in students' actual errors instead of by asking superficial questions, such as "what can be the causes of the students' error in learning a mathematical concept?".

Various possible teaching strategies for repairing students' errors were mentioned by the PTs. Such strategies would be more efficient than the strategy of "clear explanation and description" mentioned in previous studies (Son & Sinclair, 2010). Since such strategies do deal with the sources of accurate students' errors, this makes it possible for the PTs to anticipate exactly the sources of students' errors and to use acceptable teaching strategies for repairing errors in the future. The authentic students' errors as

the task used in the study became the feature of the study. This study also provided closer analysis of students' errors. The Rach et al.'s (2012) process-oriented approach suggests that it is more possible to generate effective strategies to repair the errors. Whether the possible pedagogical strategies through closer analysis for handling students' errors on fractions are efficient, it should be further examined in classroom teaching. The issue becomes a research question for future study.

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# WHAT IS IN THE BAG? ELEMENTARY SCHOOL CHILDREN'S UNDERSTANDING OF ACCUMULATING EVIDENCE

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*We report results of a study with elementary school children on their understanding of accumulating evidence. The study used probability experiments asking for inferential reasoning. Children were asked to predict the contents of bags with cubes in different colors after seeing the results to drawing cubes randomly from the bag. The study suggests that abilities of children can be elicited for easy inferential tasks with an age-adequate task design.*

## INTRODUCTION

Data and probability is regarded an important topic in mathematics curricula, however, there is still no consensus whether an early introduction is reasonable. While the common core state standards for mathematics locate the starting point – in accordance to more traditional concepts – in grade 6 or 7 and thus at the transition from primary to secondary education (CCSSI, 2010), German standards suggest an early instruction at primary school age and thus follow the NCTM point of view (KMK, 2004; NCTM, 2000). The German curriculum focuses on basic concepts of probability and their application in easy games of chance.

There is some empirical evidence that even young children might master stochastic concepts. Our own research suggests that grade 2 students have a basic understanding of simple probability concepts though difficulties were found when they had to discriminate between improbable and impossible events (Reiss et al., 2011). Grade 4 and grade 6 students showed a bias when comparing probabilities of events due to the use of representativeness heuristics. However, they were able to use structures of events and their sub-events for facilitation (Lindmeier et al., 2012). This basic understanding of probability and chance is found in familiar situations and can be assumed when full information concerning the elements to be judged is provided. However, stochastic thinking includes more than the mastery of basic concepts. In particular, it encompasses working with the concepts, e.g. in processes of inferential reasoning. The study described here addresses this aspect. We describe first characteristics of inferential reasoning and review the relevant research on related difficulties.

## Inferring from accumulating data on a property of the population

Evidence-based thinking is regarded an important aspect of scientific reasoning and a prerequisite for understanding how knowledge is acquired in science (Kuhn & Pearsall, 2000). As a consequence, there is a strong research tradition in psychology focusing on scientific reasoning and inferential thinking. For example, Koerber et al.



(2005) could show that even young children were able to draw correct conclusions from data. Research on the evaluation of contingency tables revealed an inadequate use of strategies by primary and secondary school students who relied on differences and absolute values instead of proportions (Shaklee et al., 1988). Our own results suggest, that most students in grade 6 and even some students in grade 2 recognized the need for base rate information in such tasks. However, the correct integration of information integration was a demanding problem for all students (Reiss et al., 2011). However, the study could not reveal how inferential thinking develops. E.g., easy covariance evidence has a 2x2-structure that can be modeled in an urn-comparison model: Two different urns of unknown composition are compared in respect to a property based on samples from each urn. So, if students fail to solve covariance tasks, they might either not understand inferential reasoning itself or might be unable to carry out the comparison of inferred properties.

Ben-Zvi (2006) described generalization as a basic task of inferential reasoning. Therefore, a property of a population has to be inferred from evidence. Typically, this evidence is generated sequentially (e.g., data point by data point), whereas the proper inference is based on the accumulated data. Moreover, the strength of the inference depends on sample size.

Tversky and Kahneman (1982) summarized common heuristics in probabilistic reasoning that lead to certain misconceptions and biases. The following phenomena were identified as being relevant for basic inferential reasoning tasks: In stochastic situations, humans tend to compare events and (possible) populations and judge whether one is related to the other by the degree of similarity. For inferential tasks, this heuristics is a powerful one. However, it also bears difficulties. If people rely on representativeness heuristics only, they are probably not sensitive with respect to sample size and the strength of evidence might be perceived as being independent of the sample size. In a strong version, the so-called misconception of a “law of small numbers”, people would expect even a small sample to be highly representative for the population. The well-described effects of position and anchoring as well as adjustment processes can be relevant in inferential tasks if data generation is transparent. Summative data protocols that foster the aggregation of data should lower these effects (Wollring, 2007).

### **Research questions**

On the basis of the literature review we performed a feasibility study. One aim was to design an appropriate task format for young children with no prior instruction on data and chance in general and no instruction on inference in particular. Moreover, the following research questions were addressed.

- Are grade 4 or 6 students able to design an appropriate experiment for solving a basic inference task?
- Can students deduce a property of the population from given data?
- Are students sensitive to the impact of the sample size?
- Which strategies underlie the students' decisions?

## DESIGN OF THE STUDY

### Task design

A major aim of the study was to develop an appropriate task design for young children. Especially, paper-pencil items were not seen as viable as the understanding of the children could hardly be evaluated post-hoc from written statements. Therefore, we developed a task to be presented in a standardized interview. This format proved to be feasible for probability problems. According to the children's age and stochastic knowledge, the problems were presented in everyday language. All problems made use of a bag, which contained 15 red and blue cubes. The children were supposed to determine the distribution of colors in the bag. The drawing mode was set as "drawing with replacement", as the distribution of the colors in the urn stayed constant in this case.

We assessed first whether the children could spontaneously define an experiment, which would enable them to reconstruct the contents of the bag. Accordingly, we asked how they would solve the inference problem if they could not look inside the bag and were only allowed to draw from it with replacement one item once at a time. It was recorded if students could explicitly state an inferential approach and how often they would draw from the bag.

Another series of items used a closed answer format. These answer formats had been found to elicit early abilities of children in probability contexts (Anderson & Schlottmann, 1991). Thus, we presented three different possible distributions of the red and blue cubes inside the 15-item bag. The representations were structured in order to facilitate proportional reasoning (Figure 1). The ratio of blue to red cubes inside the three possible bags was given as 3:12, 5:10, and 14:1. The children had to decide which distributions they would prefer and had to give reasons for their answer. We thus collected information about the strategies leading to a certain decision.

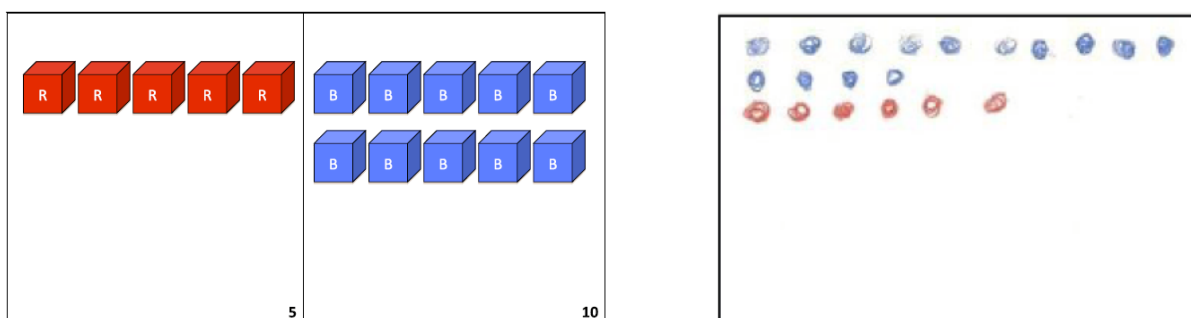


Figure 1: Examples of answer option (left) and data protocol (right)

When abilities to handle accumulating evidence are targeted, this evidence can either be generated by the children or be provided by the interviewer. In the first case, there are two problems: Children – especially if they are not aware of the importance of the sample size – tend to generate only very few data (Wollring, 2007). Moreover, the number of trials and the results may vary significantly as children might generate

miscellaneous items. This would make a comparison of abilities across children quite difficult. If the interviewer would provide evidence, the process of generation might be not comprehensible to the children and thus hinder their understanding of the task. In consequence, we used a video to make the generation of data visible and simultaneously provided the same evidence per item for each child. We used hand puppets as actors, let them draw from the urn and protocolled the results. In piloting studies, we tested the acceptance of the videos and found no evidence that students assumed manipulated trials or other kinds of problems that would intervene with an appropriate understanding of the task.

Finally, we used a summative protocol. The events were recorded by color so that the sequences of events were not visible in the protocol. This kind of protocol may be regarded as facilitating the comparison of population and data, fostering proportional strategies, and diminishing possible effects of position (Figure 1).

The items were varied according to the number of trials: One puppet always drew ten times from the bag, the other puppet twenty times. We matched the items so that the proportion of red and blue results was equal for one experiment with ten and one with twenty trials. Moreover, we made sure that the events lay within the acceptance region of exactly one of the three provided bags with a confidence level of .05, although the acceptance errors are of course (especially for trial length ten) substantial. Table 1 summarizes the characteristics of the eight items we used.

### Design of the study and sample

We conducted a cross-sectional study with children of grade 4 and 6. The items were presented in individual standardized interviews with video-based item presentation. Only age and gender of the children were recorded as individual background data. The interviews were videotaped and the coding of decisions and strategies was done with the help of these videos.

Item	P(Blue)	Number of Trials	P(Event from Target Bag)	Solution Rates		
				4th Grade	6th Grade	All
WBI1	50%	10	13.6%	1.00	0.89	0.93
WBI6		20	5.4%	1.00	1.00	1.00
WBI2	30%	10	30.2%	1.00	1.00	1.00
WBI8		20	21.8%	1.00	1.00	1.00
WBI3	100%	10	50.2%	0.92	0.94	0.93
WBI5		20	25.2%	1.00	1.00	1.00
WBI4	70%	10	26.0%	1.00	1.00	1.00
WBI7		20	18.2%	0.86	0.74	0.78

Table 1: Characteristics of test items with solution rates

In total, 40 students (19 female, 21 male) participated in this study. In detail, 17 students (8 female, 9 male) were from grade 4 and aged  $M = 10:0$  years ( $SD = 0:5$ ) and 23 (11 female, 12 male) from grade 6 aged  $M = 12:7$  years ( $SD = 0:7$ ). Participation in the study was voluntary.

## **RESULTS**

### **Understanding of the task**

Despite of the carefully chosen task and an item design that was likely to be understood even in early years, some children had difficulties in understanding the task. The phrase “draw an item from the urn with replacement” was explicated at the beginning of the interview and the interviewers were trained to make sure that this procedure was understood. However, five students from each grade did not understand the situation when twenty trials were shown in the video sequence. They argued that it was not possible to draw twenty times from a bag with 15 items. This 25% of students were excluded from the sample. Thus, the sample sizes for the following analyses are  $N = 12$  (grade 4) and  $N = 18$  (grade 6).

### **Spontaneously produced strategies to solve the task**

Most of the students could explicitly state the core idea of inferring from data on a property of the population (50% grade 4, 67% grade 6; example statement: “I would draw for example ten times and then see what came most, regarding the color”). However, no child was able to explicitly state a proportional argument.

The grade 4 students stated to draw in mean  $M = 10.6$  ( $SD = 4.5$ ) times from the 15-item bag. In grade 6, the mean was  $M = 13.0$  ( $SD = 5.7$ ) and slightly higher, but the difference was not statistically significant ( $t(24) = -1.09$ ,  $p > .05$ ). It is noticeable that the mode in both grades was exactly 15 trials for the 15-item bag. Only one grade 6 student suggested to draw 30 times and hence suggested in particular more than 15 trials.

### **Abilities to make an inference from accumulating data**

As argued above, we excluded 10 students from the analyses as they had problems with the task design. The remaining students of both grades showed good abilities to draw inferences in our specific setting which lead to a ceiling effect for a scale of length 8 with mean solution rates of  $M = 7.75$  ( $SD = 0.45$ ) for grade 4 and  $M = 7.61$  ( $SD = 0.61$ ) for grade 6 (see also Table 1 for solution rates of items). No differences in the accumulated solution rate regarding the grades were found (Mann-Whitney U-Test  $U = 140.5$ ,  $p > .05$ ).

As described above, we constructed the items as pairs of items with different numbers of trials (10 vs. 20 trials) and asked the students to give reasons for their answers. Thus, a detailed analysis based on these open answers might give further insight, and may especially inform about the different strategies that led to the solutions.

### Sensitivity to sample size

Almost all of the students could explicitly state the difference in chain length when asked to compare the solution approach of both puppets (grade 4: 10 students, grade 6: 17 students). However, only 58.3% of the younger children and 77.8% of the older children were also able to evaluate the more comprehensive approach as leading to a better data basis and thus judge the strength of evidence correctly. Again, this difference was statistically not significant ( $t(38) = -1.13, p > .05$ ).

### Nature of reasons and explicated strategies

For the in-depth analysis of the students' reasons we developed a coding rubric. Therefore, we distinguished reasons that had no reference to the probabilistic nature of the task from such referring to the probabilistic nature. For example, animistic beliefs (Wollring, 1994), physical explanations ("the red one was covered by blue cubes") or re-statements of the decision were classified in the former rubric. Within the probabilistic reasons we distinguished mere references to luck ("if you are lucky, you could get such a result from this bag") from explicitly stated probabilistic decision strategies. Here, decisions could be based on a comparison of (absolute) values ("because here are 5 blue ones and here too"), a comparison of differences ("because there are far more red than blue ones"), or a proportional strategy ("because these are two-times as many blue than red cubes and here are approximately also two-times as many").

		WIB4		WIB7		Cumulated over items and students
		Grade 4	Grade 6	Grade 4	Grade 6	
<i>No reason</i>		-	-	1	-	0.4%
<i>Reason refers not to probabilistic nature of task</i>		2	2	1	1	7.9%
<i>Reason refers to</i>	Good/bad luck	-	-	1	1	5.8%
<i>probabi-</i>	Abs. Values	7	6	7	8	42.5%
<i>listic nature</i>	Differences	3	9	2	8	41.7%
<i>of task</i>	Proportions	-	1	-	-	1.7%

Table 2: Strategies for selected items and overall items and students

The last column of Table 2 summarizes the nature of reasons accumulated over all items and all students. Whereas almost no proportional strategies were explicated, most students could explicate a strategy relying on values or differences (84.2%). The strategies of the 6<sup>th</sup> graders were more often based on differences than absolute numbers, with a statistically significant difference in the reasons provided by the children ( $\chi^2(5) = 54.98, p < .001$ ). Table 2 illustrates this tendency to more sophisticated strategies in grade 6 for the pair of items that proved to be the most difficult.

## DISCUSSION

The task we designed proved to be understandable for most of the students in grade 4 and 6. However, the items were easy and did not allow differentiating between children with higher and lower abilities based on solution rates. This is probably due to the special task design with closed answer formats and results from our effort to create a very basic inference task. Thus, students of this age can infer the property of the population from given data in our setting. However, students were not as confident in explicating an appropriate experiment to solve this inference task. Especially, we found indication for insensitivity to sample size, as most children considered low sample sizes as sufficient and even in grade 6, about a quarter of the children did not realize the difference in strength of evidence for a small and a larger sample.

The analysis of students' explicitly stated decisions strategies showed that proportional strategies were hardly used. Most strategies were based on a comparison of absolute numbers or differences for possible populations and results, a result in line with prior findings for covariance analysis and probability comparison tasks. This indicates, that the students relied on representativeness heuristics and – most likely due to a lack of proportional thinking abilities (Inhelder & Piaget, 1958) – interpreted representativeness in terms of absolute values and differences. The insensitivity to sample size may also be regarded as an indicator of representativeness heuristics as explained above.

We would like to stress the finding that a quarter of the students in this exploratory study showed problems in understanding the task even after explanation and with a clear video-based item presentation that made the generation of the data transparent. A possible explanation is that the students' were unexperienced in inferential reasoning tasks. Children's own experiences and a participation in instructional approaches like the "growing samples heuristics" (Ben-Zvi, 2006) proved to be powerful in developing a better understanding, better strategies, and more sophisticated language for inferential reasoning tasks. It has to remain an open question at this point, whether these findings indicate some developmental restriction at the elementary school level.

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# UNDERSTANDING PROPORTIONALITY: FROM CHILDREN'S QUALITATIVE INTUITIONS TO QUANTIFICATION

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*We report on 20 elementary school children (ages from 8 to 11) who made significant progress in understanding proportionality in the context of quantifying relationships among weight, volume, and density. We compare the pre- and post- performance for the whole sample and we examine transcripts from two students whose responses typify those of many of the participant children. We attempt to show how the particular materials used (cylinders of different materials, an unconventional scale, and an unconventional ruler) and the design of the tasks supported the emergence of proportional reasoning based on ratios among measures.*

## QUALITATIVE AND QUANTITATIVE PROPORTIONAL REASONING

The study aimed to evaluate whether certain kinds of activities might allow students to reconcile and extend their qualitative ideas about proportionality and help them develop a quantitative understanding of the proportional relations between mass and volume of objects made of the same material, that is, objects of same density.

Many studies (e.g. Piaget, 1951, 1958; Inhelder & Piaget, 1958), Karplus et al., 1981; Noelting, 1980; Vergnaud, 1980), have been interpreted as showing that children grasp quantitative proportional reasoning after about age 12, when they reach the so called formal operations stage. However, interview protocols by Inhelder and Piaget (1958) show younger children expressing awareness of proportional relations in a qualitative manner. And, as Piaget et al. (1977) describe, younger children can solve proportionality problems relating, for example, the size of a fish and the amount of food it eats, when the sizes and/or the amounts are quantified by the interviewer.

In everyday life, outside of schools, children develop intuitions about the relationships among mass, volume, and kind of material. These include an understanding of (a) the qualitative co-variation between mass and volume expressed as “this one is lighter, only because it's smaller” and (b) part-whole relationships expressed as “if an object is made of aluminium, then part of the object is made of aluminium too.”

Everyday experience may lead to the understanding of important mathematical concepts. But everyday situations are not sufficiently varied and provocative to capture a broad range of mathematical content (Schliemann, 1995; Carraher & Schliemann, 2002). As such, everyday experience may not lead to a fully quantified awareness of proportional relations for many reasons. As continuous quantities, mass, volume, and density need to be quantified before one can determine the proportional relations between them. However, quantification with standard measuring tools may not be the best approach for understanding density. Ratios among quantities one encounters (e.g. 1.52 g of aluminium may occupy 4.27 cubic cm of space) may require that one resort to



computations (for which one may have a formula), in lieu of thinking about the relations among the stimuli properties.

Providing students with accurate measurements does not ensure, and may even hinder understanding the relations among mass, volume, and density (Hewson, 1986; Leoni & Mullet, 1993; Rowell & Dawson, 1977). We need to develop and evaluate instructional approaches to help children quantify measurements and ratios between measures in ways consistent with their experience with mass, volume, and materials. To this end, it would appear beneficial to first use stimuli such that the relationships between properties of stimulus pairs involve simple ratios (e.g. 1:3, 2:3).

In this paper (see Liu, 2012, for the full study, developed with partial support from the National Science Foundation, grant #0628245, The Inquiry Project), we examine whether quantification of the mass and volume of objects could anchor children's qualitative intuitions and extend these intuitions into considering proportionality in the second degree relationship between (a) the quantitative relationship between the masses of objects ( $M_1/M_2$ ) and (b) the quantitative relationship between their volumes ( $V_1/V_2$ ). For example, if the mass of an object is three times as much as that of another one and the volume of the first object is three times as large as the second one, then those two objects may have been made of the same kind of material. We also analyse whether children could find out that the relationship between mass and volume is the same for objects made of a given kind of material.

During the individual interviews, children employed an unconventional scale and an unconventional ruler as measuring tools, labeled them with numbers, and assigned numbers to the masses and volumes of cylinders having equal bases. In the interviews we used the word weight, instead of mass, because this is the word children use; we used the word "size", instead of "volume", because most children at these ages may not have a formal concept of volume and their intuitive understanding of volume may be expressed by the word "size." The children were not given a definition of density or instruction about the relationships between weight and size. Instead, they were guided to make comparisons between cylinders made of the same or of different kinds of materials and to answer questions that could help them figure out the weight and size relationships by themselves. We examined the impact of this experience on their ability to quantify proportional relations. We hypothesized that, before the formal operations age, the process of assigning numerical values to the weight and size of objects they compare and manipulate helps children coordinate these two dimensions and construct an explicit and quantified proportional relationship between them.

## **METHOD**

### **Participants**

The 20 participants in this study were eight third graders, eight fourth graders, and four fifth graders, all of them attending public schools in Massachusetts. Their ages ranged from 8 to 11. Eight were boys and twelve were girls.

## Materials

During the *intervention part* of the interview, materials consisted of 31 cylinders, a simplified ruler, and a modified scale (Figure 1). Each cylinder was made of a single material, Delrin (a kind of plastic), aluminium, or brass of uniform density. The ratios of the densities of brass, aluminium, and Delrin are roughly 6:2:1. Because they had the same base ( $10\text{cm}^2$ ), height could be used as an index of volume or size. Some cylinders were covered with thin white paper to hide the kind of material they were made of. Small identical copies of cylinders could be stacked up to build cylinders of the same height as the larger ones.

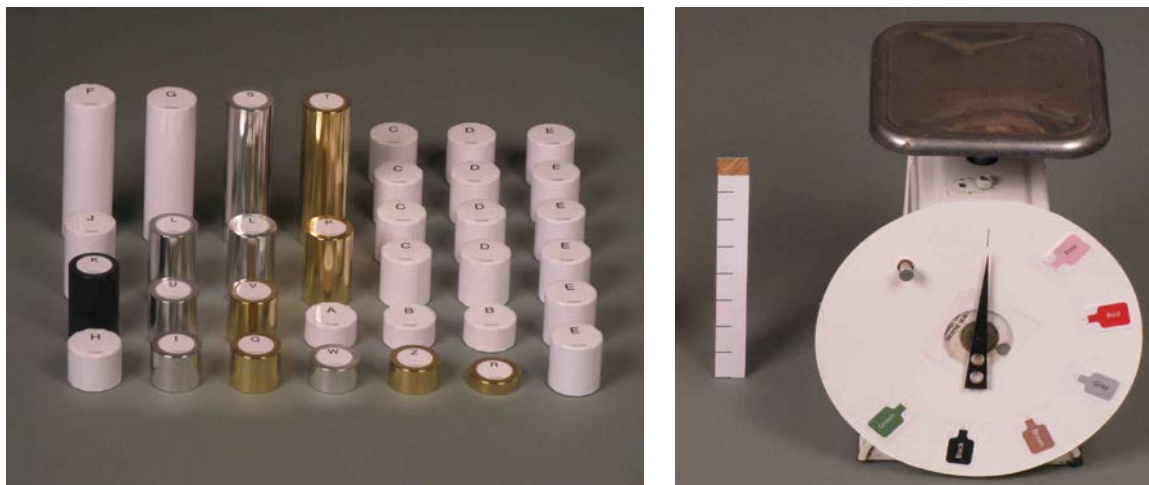


Figure 1: Cylinders, ruler and scale used in the intervention

The scale (modified from a dial scale), labelled with stickers of different colours but no numbers, allowed to describe the weight of each cylinder by a colour. In the interview, all students came to realize that integers could be assigned to each colour, with the first coloured tick assigned the value 1 and serving as a unit measure for the values of the weights associated with the remaining colours. Any of the cylinders would cause the scale pointer to move to one of the colours of the scale. The simplified ruler, with one level of same-space marks (each space was the same as the height of the smallest cylinders), allowed children to quantify volume by counting how many spaces the height of a cylinder matched the marks on the ruler.

In four *subtasks of the pre- and post-tests*, we used 13 cylinders (Figure 2) and a **two-pan** balance scale that only allowed children to know whether two objects were of the same weight or different (and, if different, which one was the heavier). We told the children that E was made of brass, F of aluminium, and G of plastic. Then we asked them whether B, D, A, and C could be made of the same kind of material as the materials of cylinders E, F, or G (brass, aluminium or plastic), or of something else.

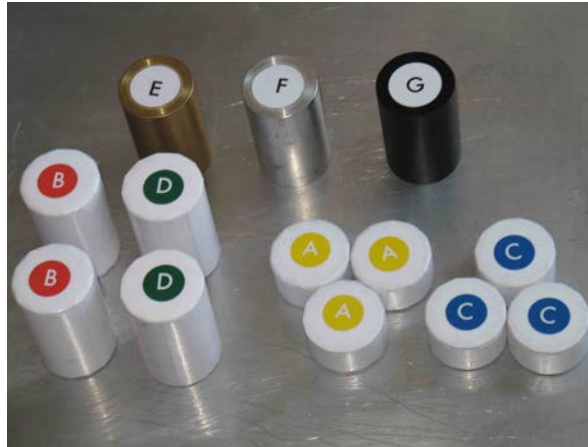


Figure 2: Cylinders used in the pre-test and post-test

## Procedure

Each child participated in a three-part individual interview of about one hour. The first and last parts of the interview were tests to evaluate student's proportional reasoning before and after the intervention part. The middle part constituted an intervention designed to allow children to experience the process of quantification based on their qualitative intuitions, as well as by using the simplified ruler and the modified scale to measure the relative weight and size of the cylinders. The process of quantification included: exploring how the scale worked, labelling the scale with numbers, and answering questions about the relationships among weight, size, and material.

The intervention part of the interview followed the principles of the open-ended clinical interview, as proposed by Piaget (1976). It included pre-prepared open-ended questions and alternative follow-up questions. Different from Piaget's approach, which aimed at determining children's current reasoning, we aimed at helping children develop new understandings, taking into account their previous understandings. Follow-up questions were posed, depending on children's responses. The pre- and post-tests sections were structured interviews with fixed questions.

## RESULTS

The average number of correct answers for the four subtasks was 2.8 in the pre-test and 3.9 in the post-test. This difference was statistically significant (Wilcoxon  $W = -66$ ,  $z = -2.91$ ,  $p = .0018$ ). We found no differences across grade levels.

Due to space limits, we will only show examples of answers for the subtask where children were asked to determine the material cylinder C could be made of. The correct answer is: C may be made of aluminium (it fact, it was). C was  $\frac{1}{3}$  the size of cylinder F, known to be made of aluminium; it was also  $\frac{1}{3}$  the weight of F. In the pre-test, 50% of the children answered correctly. In the post-test, all the children gave a correct answer. Moreover, 69% of them explicitly considered weight and size, against only 39% who did so in the pre-test. Table 1 shows typical responses to this task and illustrates the changes in children's reasoning.

	Aaron (9 years old, 4th grade)	Henry (8 years old, 3rd grade)
Pretest	Aaron: [Holds C and G, C and F] I think it is made of something else.	Henry: [Puts G and C on the balance scale] It may be this material, because it's smaller. [Puts F and C on the scale] Probably not this material. [Puts E and C on the scale] And not this material. So I think it's plastic.
	Chunhua: How do you know that?	Chunhua: Do you mean this plastic [G]?
	Aaron: Because this [C] is heavier than this [G]; this [F] is heavier than this [C], so I think this [C] is between this and this.	Henry: This plastic.
	Chunhua: Between G and F?	Chunhua: Why do you think C is made of this plastic?
	Aaron: Yeah.	Henry: [Puts G and C on the scale] Because even though this [G] weighs more, that's because there are more mass.
Posttest	Aaron: [Stacks up three Cs and puts them on the scale to compare with F] I think C is made of aluminium.	Henry: [Put F and 3Cs on the balance scale] I think it's the amount of F.
	Chunhua: How did you figure that out?	Chunhua: Why do you think that?
	Aaron: They weigh about the same.	Henry: Because it's one third as tall and three of those equal F in weight and height.
	Chunhua: I am still wondering why you use 3 Cs, not just one C, to compare with F.	
	Aaron: Because you kind of need three to get the same height.	

Table 1: Examples of Responses to the Question “What is C made of?”

In the pre-test, when attempting to infer the material C was made of, Aaron only considered the cylinders’ weight. He concluded that C would be made of something other than brass, aluminium or plastic because, using his hands, he felt that C was not the same weight as G or F. He did not compare C with E, perhaps because he already knew that E was heavier than C. Henry, instead, used the balance scale and, although he considered the cylinders’ weight and size, could not find the correct answer, because the balance scale only shows whether or not two objects are equal in weight.

In the post-test, both Aaron and Henry not only answered correctly, but also showed clear proportional reasoning to justify their answers. Aaron used the strategy “same weight and same size ( $W_{3Cs}=W_F$  &  $V_{3Cs}=V_F$ ) means the same material”. Henry used the strategy “two objects are made of the same kind of material if one object has 1/n weight and 1/n size of the other one” (or  $W_C/W_F=V_C/V_F$ ).

### Proportional Reasoning in a Task During the Intervention

Towards the end of the intervention part of the interview, children's responses to one of the tasks showed their emerging use of proportional reasoning, when they considered a second-degree relationship or the linear relationship between weights and volumes, not only with unit ratios (1:2, 1:3 or 1:6), but also with a non-unit ratio (2:3). In one of the subtasks, they were shown a picture of two cylinders and their readings on a ruler and on a scale (Figure 3) and asked whether they could be made of the same material or not. The following are responses from Aaron and Henry.

Aaron: If you have this regular [I] and then this [I] cut in half and stack this [half of I] there [on the top of I], it would be the same height and it will also weight the same [as O].

Chunhua: How do you know this?

Aaron: Because this [half of I] will weigh two. Two plus two is four. If you add that [the half one on the top of I], it will go to green [on the modified scale]. Because this [I] weighs four and this [the half of I] weighs two.

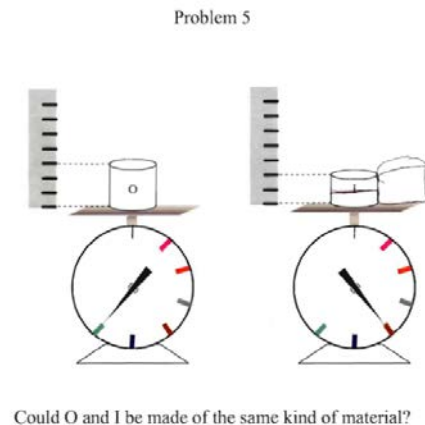


Figure 3 : The picture used and drawings by Aaron

Aaron considered that “same weight and same size means the same material”. In his reasoning, he used a functional approach, considering the linear relationship between weights and volumes (see Vergnaud, 1980, for the distinction between functional and scalar approaches). He figured out the weight of half of I and used this function operator to determine the weight of a cylinder made of same material as I and the same size as O, which was three times as tall and heavy as half of I.

Henry's responses to this task show how his reasoning evolves:

Chunhua: Could O and I be made of the same kind of material or not?

Henry: I don't think so.

Chunhua: Why not?

Henry: Because this [I] is one shorter [in size], and [its weight is] two of these [shorter] (marks on the scale).

- Chunhua: Two of these?
- Henry: It's [I] only two tall. This [O] is three tall. Wait, wait. I think they are the same material actually.
- Chunhua: Okay. Why do you think they are the same material?
- Henry: Because it's [I] two thirds of this [O] tall and two thirds of the weight.
- Chunhua: Two thirds of this tall?
- Henry: Two thirds of the height.

Because the difference in size between O and I was one (as measured by the ruler), and the difference in weight was two, Henry first claimed that they could not be made of the same material. Then he found out that the relationship between O and I sizes ( $2/3$ ) was the same as the relationship between their weights ( $2/3$ ). He then concluded that O and I could be made of the same kind of material, using a scalar approach, that is, considering the second degree relationship between  $W_I/W_O$  and  $V_I/V_O$ .

## FINAL REMARKS

Our results show that quantification does not necessarily lead to suspension of sense making and that it can even emerge from students' observations and experiences. Furthermore, it would seem that, by having students themselves quantify properties, rather than simply accepting standard measures, they can enrich their understanding of proportionality and of scientific concepts, the meaning of which stems in part from the interrelations among related concepts. Having children assign numerical values to the weight and size of objects, as they directly compared them, allowed children to merge quantification of the weight and size dimensions into their intuitive understandings and thus promoted their understanding of proportional relationships. We suggest that educators and researchers begin to invest on making connections between children's qualitative and quantitative understandings of scientific and mathematical concepts. Emphasis on only one of these aspects does not ensure that children will come to understand science or mathematics content. Instead, it is necessary to integrate them. This study has shown a fruitful way for integrating qualitative and quantitative features for mass and volume. More studies are needed for us to better understand how children's qualitative and quantitative understandings interact to promote mathematics and science learning.

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# MEDIATING THE MEANING OF ALGEBRAIC EQUIVALENCE: EXPLOITING THE GRAPH POTENTIAL

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*Within the Theory of Semiotic Mediation, this research study investigates how the Graph representation provided by the Aplusix CAS can become a tool of semiotic mediation to make the meaning of algebraic equivalence emerge. The analysis of the potentialities of the Graph and the design of specific tasks aiming at exploiting them, are presented. More precisely, it is investigated how this representation may contribute to develop the meaning of equivalence as a structural relation which condensates the procedural processes underlying the transformation of algebraic expressions. A teaching experiment is described and some results are discussed.*

## INTRODUCTION

This contribution aims at exploiting the potentialities of a specific component of the Aplusix tool (Nicaud et al., 2004), namely the representation of algebraic equivalence by means of a graph. The 'Graph representation' in Aplusix is a particular type of representation of a set of algebraic expressions which become the nodes of the graph. A relevant body of research which concerns the issue of equivalence between algebraic expressions is focused on the difficulties encountered by students in understanding algebraic equivalence as well as in dealing with manipulation of various forms of algebraic expressions (e.g. Ball et al., 2003; Kieran, 1984). As far as symbolic manipulation is concerned, the issue of addressing the dichotomy between syntax (rules stating how to deal with symbols) and semantic (meaning of equivalence) arises. This distinction can constitute a cognitive obstacle to deal with algebraic manipulation. In fact, as argued by Booth (1989)

“our ability to manipulate algebraic symbols successfully requires that we first understand the structural properties of mathematical operations and relations[.]. These structural properties constitute the semantic aspects of algebra” (pp. 57-58).

A valuable contribution to the research in this area comes from studies involving the use of Computer Algebra Systems (CAS) devoted to foster the equivalence meaning (e.g. Kieran & Drijvers, 2006; Lagrange, 2000; Artigue, 2002). Artigue underlines how the potential for these tools, compared with paper and pencil, consists mainly in giving the opportunity to make students face tasks which, at the same time, can foster both syntactic and semantic aspects.

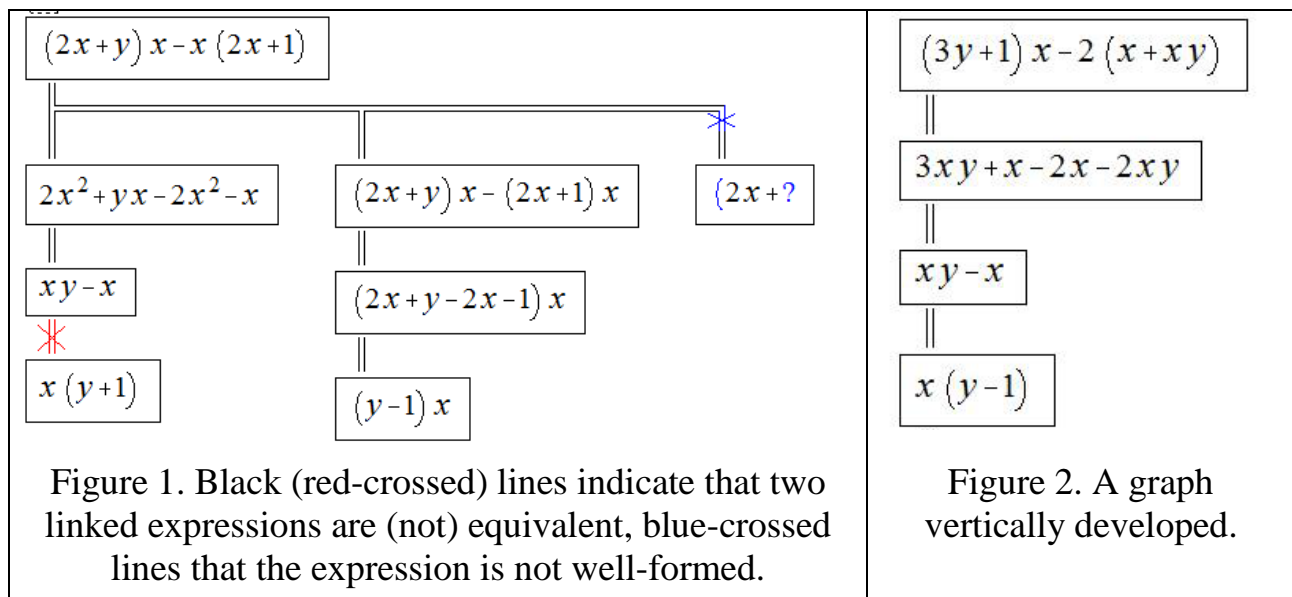
Elaborating on these studies, our research is centred on the idea that symbolic manipulation can be conceived as a condensation of procedural relations – transformation of algebraic expressions according to rules respecting the equivalence – and structural relations – which are established between expressions obtained by procedural manipulation. According to Booth (1989), this makes syntactic and



semantic aspects of algebraic manipulation be combined. We formulated the hypothesis that the Aplusix CAS, and in particular the use of the Graph representation to carry out algebraic manipulation, allows one to refer to the mathematical meaning of equivalence class of algebraic expressions.

## THE GRAPH IN THE APLUSIX CAS

Aplusix is a CAS in which the user can perform both numerical and algebraic calculations. The computational objects of the CAS consist of algebraic expressions; each of these is enclosed in a box which constitutes a node of the Graph (Fig. 1, Fig. 2). The treatment of an expression is carried out within a box; new boxes can be added to the Graph. Two settings are available for creating a new box: the student can choose to create an empty box or to create an identical box to the previous one.



Each box enclosed in the Graph can be linked by three different signs to both the box from which it has been obtained and to another box, in case another one originates from it. In fact, when students carry out a calculation, they can exploit the feedback provided by these signs. Three different ‘feedback-signs’ signal whether the equivalence between the expressions filling two consecutive boxes has been respected or not. Black lines indicate that the equivalence has been respected, blue crossed lines indicate that one of the two expression is not syntactically well-formed (e.g. a parenthesis is missing), red crossed lines indicate that the two linked expressions are not equivalent (Fig. 1). Each box containing an expression can be the root for one, or more than one other boxes, in which other expressions can be represented. In other words, the Graph representation in Aplusix, which is two-dimensional, offers a very peculiar way of expressing equivalence between algebraic expressions (Fig. 1). As a consequence, the Aplusix CAS overcomes the mere top-down flow from the given expression to the required expression which is usually carried out by means of a single chain (starting with the given expression and ending with the required expression).

Actually, the iteration of generating new boxes containing equivalent expressions produces a web of expressions, that is a set of expression one related to the other by a link representing the equivalence between them; in such a web the order in which one

expression is obtained from another disappears, whilst what becomes evident is the equivalence relation linking the different expressions.

## THE THEORY OF SEMIOTIC MEDIATION

The Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008) provides us with theoretical tools to frame the analysis of the semiotic potential of the Graph representation in mediating the meaning of algebraic equivalence and to plan the didactical intervention based on its use. According to this theory, the Graph as it is implemented in the Aplusix CAS, can be considered as an artefact. The use of it, in order to carry out specific tasks, make students construct personal meanings which are deeply related to the actual use of it. Then, the role of the teacher consists in making students' personal meanings gradually evolve to mathematical meanings. An artefact, when intentionally used by the teacher in promoting this process, becomes a tool of semiotic mediation (Bartolini Bussi & Mariotti, 2008, p. 754). The semiotic potential of an artefact emerges through the twofold relation that such an artefact has both with the meanings emerging from its use to accomplish a specific task, and with the mathematical meanings which are evoked in such a use, as they are recognized by an expert (Bartolini Bussi & Mariotti, 2008, p. 754).

Planned didactical activities, which aim at making meanings emerge, constitute the basis on which the teacher orchestrates the evolution from personal meanings to the mathematical meanings. In the Theory of Semiotic Mediation, the organization of a teaching/learning sequence is based on didactic cycles (Bartolini Bussi & Mariotti, 2008, p. 754). Each didactic cycle pursues a specific didactic goal by means of a sequence of different activities aimed at developing personal meanings (solving tasks using the artefact), producing individual signs (writing reports related to the task solution), making students' personal meaning evolving towards mathematical meaning (discussing under the guidance of the teacher).

## THE SEMIOTIC POTENTIAL OF THE GRAPH REPRESENTATION

The leading principle of the research study is that the Graph representation in the Aplusix CAS can be considered an artefact fostering the meaning of equivalence between algebraic expressions. As explained above, any graph can develop in two directions (Fig. 1, Fig. 2): vertically (creating box containing expressions one under the other) and horizontally (creating boxes of expressions which are one next to the other, starting from the same expression). The final product of this representation refers to an equivalence class of expressions; thus, we can refer to it as Equivalence Graph. Aplusix manages potentially infinite webs of expressions. The navigation of very large graphs is facilitated by the use of the vertical/horizontal slide bar. The possibility of representing algebraic expression through a graph permits a new approach to symbolic manipulation and, in particular, to the meaning of equivalence among expressions. Producing an Equivalence Graph means transforming a given expression into equivalent forms which can be transformed themselves into equivalent forms and so on. The iteration of this procedure makes emerge the symbolic manipulation as a condensation of procedural relations into a structural relation. In fact, the Graph can be

considered as a representation of the structural relations between algebraic expressions that, at the same time, remain linked one another through the transformation procedures that have been carried out to generate each of them. The artefact, thanks to both its structure and the ways to act on it, allows the user to break the rigid *direction of the manipulation* towards either an expanded form (canonical form) or a factorized form (if it exists) of an expression. In this way, not only can the Equivalence Graph mediate the meaning of equivalence between expressions, but also the produced web of equivalent expressions makes it possible to refer to a complex system of expressions as differently related through symbolic transformation, and structurally related through the equivalence relation.

## THE TEACHING/LEARNING SCENARIO

A teaching experiment was designed with the aim of exploiting the semiotic potential of the Graph representation in Aplusix for developing the mathematical meaning of algebraic equivalence, as well as the meaning of algebraic equivalence between transformed expressions. The implementation of the teaching/learning experiment involved two 9th grade Italian classes in the second semester of the school year.

The students were familiar with the Graph that had been already used to perform calculation tasks with numerical expressions. They were also familiar with the feedback-signs, and their meanings had already been shared, as well as the meaning of algebraic equivalence. Then, in the introduction to the algebraic calculation, the same artefact, that is the Graph, is exploited to carry out manipulation of literal expression. The teaching sequence is composed by some didactical cycles; we will consider only a few of them. The first task consists in asking students to state if a set of three expressions contains expressions which are equivalent and, in the affirmative case, to prove the stated equivalence. Two different approaches to the solution are possible, each of them refers either to the structural or to the procedural meaning of symbolic calculation. Equivalence can be stated by means of creating a graph containing the whole set of expressions and checking whether or not the lines linking the three expressions are black (indicating the equivalence) or red (indicating the non-equivalence). Otherwise, it is possible to start a transformation process producing a graph per each expression with the aim of transforming one expression into the other.

The analysis of the graphs may lead students to realize that if two of these share at least one expression they can be considered as part of the same larger graph. The comparison of students' solutions in the collective discussion will be aimed at developing the mathematical meaning of equivalence class of algebraic expressions: any graph will become an Equivalence Graph. Building on this, new activities will be proposed with the aim of introducing a classification with respect to the form of the expressions so that, finally, identifying the canonical form to be selected as a specific representative of the equivalence class.

## CONDENSING PROCEDURE AND STRUCTURE INTO THE GRAPH REPRESENTATION

We are going to analyse some excerpts from the classroom discussion after the first activity concerning the comparison of three expressions. Objective of the discussion is fostering the emergence of the mathematical meaning of equivalence class, elaborating on the personal meanings emerging from the use of the Graph and, in particular, on the idea of two-dimensional web of equivalent expressions. The teacher opens the discussion coming back to the task by asking students how they have solved it. The teacher shows to the classroom files in Aplusix containing the solution given by students; she also uses the CAS to transform some expressions given in the task so as to give evidence of specific issues emerging during the discussion.

Marco: We made three schemes and in the B scheme the third expression in the second column (Fig. 1) is identical to the fourth expression in the C scheme (Fig. 2). We also copy this latter expression in the B scheme and everything was OK.

Teacher: Is that enough to prove the equivalence?

Martina: Every expression of the B scheme has to be equivalent to every expression to the C scheme.

Teacher: So, do we have to check any expression of a graph with any expression of another graph to say that the two expressions are equivalent?

Chorus: No.

The teacher tries to move from a local approach, in which the equivalence is respected if two (or more) graphs contain two identical expressions, to a more global one where the sufficient condition to state that two expressions are equivalent consists in proving the equivalence between two specific expressions: one belonging to a graph, the other one to another graph.

Teacher: So if I see the presence of at least two equivalent expressions, one belonging to one graph, the other to the other graph, what can I do?

Matteo: A new graph.

Teacher: A new graph [...] Do you agree with him? How can I do?

Matteo: You can attach the second graph to the first graph by linking by chance one of the expressions of the first graph to one of the expressions of the second graph. In this way, you obtain a single graph containing the two graphs.

Teacher: Well, is there something which characterizes the expressions of the new graph?

Chorus: Are equivalent!

After having expanded a single graph into a new bigger one, by connecting two graphs (within the CAS environment), the teacher tries to move a step forward selecting a single expression which can stand for the whole graph.

Teacher: The procedure of expanding graphs by linking two or more of them, may suggest us the idea that is possible also to reduce a big graph to a smaller one, do you agree?

Chorus: Yes.

Lorenzo: Yes, we could also compact it, reducing it to a smaller one.

Teacher: How small?

[...]

Lorenzo: Since all the expressions which the graph is made of are equivalent, the graph can be compacted into a single expression.

[...]

Teacher: Well, since we all agree that it is possible to reduce a graph to a single expression, is there any idea of how it could be convenient to choose this expression?

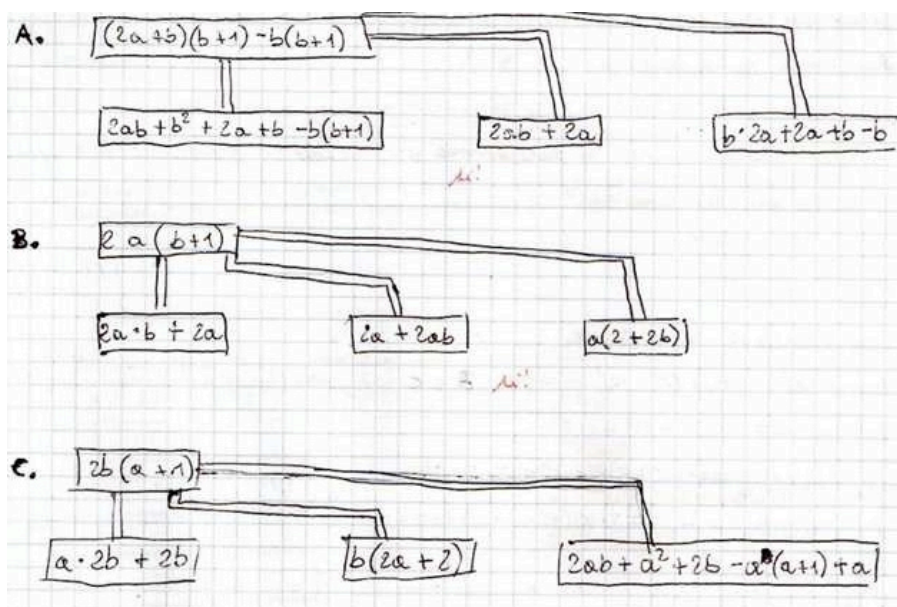
Emma: I would choose the less complicated expression.

The discussion gives clear evidence of how the teacher uses the Graph representation as a tool of semiotic mediation in respect to the equivalence relation, as it is established either by expanding a graph or by compacting two graphs. Focusing on any pair of expressions belonging to the same graph it is possible to state the equivalence between them; at the same time a sequence of connected expressions may represent the procedure used to transform the former into the latter. In other words, the structural meaning of equivalence maintains memory of the procedural meaning.

The teaching sequence goes on with the aim of introducing the idea of representative of an equivalence class. The introduction of specific forms of expressions - canonical (or expanded) and factorized, if it exists - is made through classification tasks in which students are asked to recognize in a graph which are the expressions of a particular form (Maffei & Mariotti, 2011). Then, in the subsequent didactical cycle, students are asked to produce themselves expressions having a required form.

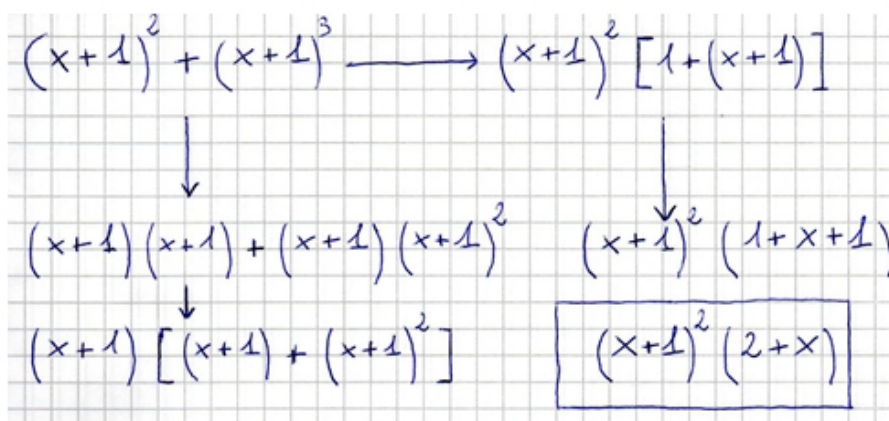
### **THE GRAPH REPRESENTATION AS AN INTERNALIZED TOOL**

Interesting evidences of the role played by Graph representation in the construction of the meaning of equivalence comes from the results of a delayed test passed two months after the end of the teaching sequence. The test is performed in paper and pencil; in the written productions traces of the use of the Aplusix Graph representation clearly appear: students draw a graph (sometimes quite large) representing the web of equivalences where to select the expression(s) they need. Moreover, the solutions reveal how students has appropriated to this tool, transforming it in a personal mode of representation to be exploited in treating algebraic expressions, specifically to obtain particular forms and relate them in the equivalence class. In a vygotskian perspective, the Graph has been internalized (Vygotsky, 1978).



Yes, it is possible to compact graph A with graph B since graph A includes an expression which is identical to one included in graph B.

Figure 3. Internalization of the Graph representation.



I made the graph so as to obtain all the equivalent expression. The expression looked for is that enclosed in the box.

Figure 4. Obtaining the factorized form of an algebraic expression in a graph in paper and pencil.

expression, Giacomo (Fig. 4) carries out different treatments of the given expression and produces an Equivalence Graph representing them, the solution is identified within such a graph. In general, the analysed protocols show how the Graph representation has become a tool suitable to refer to both the procedure (making vertical chains of equivalence possible) and the structure of the equivalence relation (embedding the vertical/horizontal chain in a single object).

## CONCLUSIONS

The last protocols, referring to the delayed test, witness the students' appropriation of a way of representing algebraic expression, which clearly comes from the Aplusix Graph. Different vertical/horizontal chains, representing the transformations of an

Fig. 3 shows the solution given by Francesco to a task asking to state if some of the three given expressions are equivalent. The student reproduces three Equivalence Graphs – one per each given expression – as if he worked in the Aplusix environment. Then he proves the equivalence recognizing that two expressions belonging to two different graphs are identical.

The internalization of the Graph representation as a tool to think about equivalence and equivalent forms of an expression emerges with more emphasis in the task asking to produce specific forms of an expression.

When solving a task asking to produce the factorized form of an

expression into another, are finally embedded in a single object, representing a class of equivalence, within which different forms of expressions can be identified to solve the task. According to the Theory of Semiotic Mediation, the designed teaching sequence has combined activities with the artefact and mathematical discussions to make personal meanings evolve into mathematical meanings: the Graph, thanks to its potential to referring to both the procedural meaning and the structural meaning of the equivalence relation, has become a mathematical tool of representation.

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# PROVIDING PROFESSIONAL DEVELOPMENT AT SCALE: RECOMMENDATIONS FROM RESEARCH TO PRACTICE

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*In this paper we present the process and outcomes of a project aimed at examining what it takes to create, sustain, and assess professional development. We present a set of nine recommendations generated through discussions among a diverse group of research experts regarding the scaling up of professional development in light of the new Common Core State Standards for Mathematics in the United States.*

## COMMON CORE AND PROFESSIONAL DEVELOPMENT

In the United States (US), schools are preparing to transition to a new set of standards for mathematics. The new Common Core State Standards for Mathematics (CCSS-M), as the standards are called, were generated through a process led by National Governors Association for Best Practices and the Council of Chief State School Officers. Starting in the 2014-2015 school year, these new standards will shape what students are expected to know and be able to do at each grade level from Kindergarten to Grade 12.

Education in the US is a responsibility of the state and there is no national curriculum in the country. Up to now, each state in the US developed and implemented its own standards for mathematics teaching and learning. Now, with the new CCSS-M, 46 out of 50 states and two territories have agreed to implement the common set of mathematics standards. Therefore, the shift to this new set of core standards represents a unique, nation-wide effort to implement shared expectations for mathematics. It will require that large numbers of teachers throughout the nation be informed about and prepared to teach according to the CCSS-M. As Daro noted "...at the end of the last mile on the journey from noble intentions of common standards to the reality of students learning, our hopes are in the hands of teachers" (Sztajn, Marrongelle & Smith, 2012, p. 2).

The CCSS-M is organized around sets of content and practices. For grades K-5, the content standards include the categories of counting and cardinality; operations and algebraic thinking; number and operations in base ten; number and operations—fractions; measurement and data; and geometry. In grades 6-8, the content includes ratios and proportional relationships; the number system; expressions and equations; functions; geometry; and statistics and probability. For High School, grades 9-12, the content is organized around the categories of number and quantity; algebra; functions; modelling, geometry; and statistics and probability. Cutting across all grade levels are eight mathematical practices: (1) make sense of problems and persevere in solving them; (2) reason abstractly and quantitatively; (3) construct viable arguments and critique the reasoning of others; (4) model with mathematics; (5) use appropriate



tools strategically; (6) attend to precision; (7) look for and make use of structure; and (8) look for and express of regularity in repeated reasoning. Thus, the CCSS-M emphasizes both what students need to learn and what it means to do mathematics.

In light of the nation-wide effort to promote the new standards, questions about the scaling up of professional development are of utmost importance. The project described in this paper was designed to address the following questions:

How can current knowledge from existing research be best used to make sure we provide high-quality, effective mathematics professional development at scale, addressing both the CCSS content and practices?

How can different fields of knowledge such as mathematics, mathematics education, policy studies, and systems research contribute to addressing the emergent national need of providing high-quality, effective mathematics professional development at scale?

Two assumptions guided the project. First, to scale up quality professional development, current knowledge and research results from diverse fields are important for the CCSS-M implementation process. Second, collaboration across research fields is necessary for researchers to answer the questions related to best ways for bringing mathematics professional development to scale. We argued that research results from diverse fields had to be articulated into a coherent framework, creating paths for successful large-scale, system-level professional development.

Over the course of the project, researchers and expert practitioners worked collaboratively to integrate research-based perspectives into a set of design recommendations for creating, sustaining, and assessing professional development systems for practicing mathematics teachers. A three-phased approach was used to develop the recommendations. This process is described next, followed by a discussion of the resulting recommendations.

## **PROJECT DESIGN AND PROCESS**

The first phase of the project included building on current research results and gathering experts. Findings from research on effective professional development features are robust. Thus, the general agreement across different content domains on these features (e.g., Desimone, 2009; Elmore, 2002; Guskey & Yoon, 2009) provided the starting point for the project. These include designing professional development that is: (1) intensive, ongoing, and connected to practice; (2) focused on student learning and address the teaching of specific content; (3) aligned with school improvement priorities and goals; and (4) built on strong working relationships among teachers (Darling-Hammond, et. al., 2009).

Taking these principles into account, research experts from fields such as mathematics education, mathematics, professional development, assessment, policy, equity, and systems research were gathered to work through a set of activities intended to integrate varied perspectives on the problem of providing professional development at scale. Participants were selected because of the contributions their specific knowledge within

their fields could make to mathematics teacher education. All invited participants prepared a two-page brief in which they: (1) described, based on their expertise and knowledge of research, what it would take to design, implement, or assess the quality of mathematics professional development that would support the implementation of the CCSSM content and practices; (2) described, in some detail, one idea they had for how professional development could be done at scale (i.e., with a whole district or entire state) to support the implementation of the CCSSM; and (3) identified a small number of articles that influenced their thinking.

The collection of briefs the basis for the second phase of the project, a two-day, face-to-face meeting of all experts during which time the ideas were discussed and synthesized. Also participating in the meeting were representatives from five national professional organizations whose missions address aspects of mathematics teachers' professional development: Association of Mathematics Teacher Educators (AMTE), Association of State Supervisors of Mathematics (ASSM), Mathematics Association of America (MAA), National Council of Supervisors of Mathematics (NCSM), and National Council of Teachers of Mathematics (NCTM). Prior to coming together, all project participants were assigned to mixed-expertise small groups and received the two-page briefs written by members of their small groups. They were asked to read these briefs and identify what they saw as common recommendations for professional development as well as conflicting points of view. Participants came to the meeting prepared to engage in conversations among experts with diverse perspectives.

On the first day participants met in their mixed-expertise small groups to brainstorm, elaborate, prioritize and finalize recommendations. The use of groups with mixed expertise allowed researchers to embrace the various perspectives and carefully consider what each field of knowledge could contribute to the generation of recommendations for professional development at scale.

Each group was asked to create a list of recommendations and, ultimately, to comment on the recommendations produced by other groups. Groups were encouraged to look for similarities and differences across recommendations and to identify recommendations that seemed particularly important. Additionally, participants were asked to consider the extent to which their recommendations were applicable to working at scale and the extent to which their recommendations specifically supported the CCSSM.

On the second day, the meeting began with a synthesis of the issues that emerged on day one. The group was challenged to think about how recommendations made the previous day could be made more specific to the new issues raised by the CCSS-M.

Unlike the groupings on Day 1, which, by design, brought participants with different perspectives together, the intent of Day 2 was to give participants with similar expertise an opportunity to review the emerging recommendations from their professional perspectives. Therefore, participants were grouped by areas of expertise and asked to prioritize recommendations and consider how the recommendations

would be useful to their peers. Recommendations that emerged from these similar-expertise groups were then shared and discussed with the entire group.

A panel presentation by the representatives from the professional associations followed. Each representative shared the perspectives about and current activities related to professional development and the CCSS-M as they related to each organization's constituents. The meeting concluded with a large group discussion.

The third and last phase of the project including analysing the data collected from the meeting and generating the recommendations. After the meeting, recorders who had been assigned to small groups during both meeting days prepared a summary of the major discussion points from their groups, as well as a list of issues discussed in the groups, but not included in the final list of recommendations. They also compiled the recommendations that were listed in each of the group posters from the first day, indicating the extent to which each group viewed the recommendations as important. Recommendations from the second day were also compiled in a list.

The project researchers analysed all compiled data and organized them into categories, first individually and then collectively, looking for emerging common themes across the work of the various groups. After a cycle of creation and refinement, the researchers crafted a set of initial recommendations that included a research-based elaboration and a set of specific action steps for different actors in the educational system. For member checking, the initial recommendations were sent to all project participants, who were encouraged to send feedback and suggestions on the emerging recommendations. In addition, four participants with different areas of expertise were invited to provide a more detailed review of the recommendations, answering a set of specific questions.

The final set of nine recommendations discussed below resulted from the analysis of the feedback and detailed commentary the researchers received. This set was then sent to the partner professional associations for their final comments and was reviewed one more time.

## **PROJECT RESULTS: THE SET OF RECOMMENDATIONS**

### **1. Emphasize Substance**

Professional development provides opportunities for practicing mathematics teachers to engage with both the new content and the practices in a focused and integrated way. The substance (Kennedy, 1998) and content-focus (Desimone, 2009) of any professional development is a key component for providing teachers with opportunities to learn. Professional development targets a defined and focused set of content and practices, making it salient how students develop mathematical ideas over time and the ways in which the mathematical practices support the learning of the content.

### **2. Create and Adapt Professional Development Materials**

Professional development materials are needed to explicitly address the mathematics content and practices of the CCSS-M, providing vivid images of teaching and learning

that are consistent with the new standards. Materials target specific content and/or practices and provide opportunities for teacher learning that are grounded in practice, that is, the everyday work of teaching is the object of ongoing investigation and thoughtful inquiry by teachers (Ball & Cohen, 1999; Smith, 2001).

### **3. Design Professional Development to Support Teacher Learning**

Professional development takes into account existing knowledge about effective ways to organize learning experiences for teachers of mathematics. Whereas two of these features were highlighted in recommendations #1 and #2 due to their importance, the design of professional development takes into account all other features that relate to the ways in which professional development promotes teacher learning, such as: offering a substantial number of professional development hours; spreading these hours over time; aligning the professional development goals with school improvement priorities; attending to student learning; and fostering strong working relationships among teachers. Features of effective professional development, however, do not prescribe the means through which professional development is delivered. Using a variety of delivery mechanisms to make professional development available to teachers assures that such initiatives fit a myriad of teacher schedules and working conditions. Further, recent research indicates that although these features are necessary, they alone may not be sufficient to impact instruction (Garet et al., 2010; Garet et al., 2011). This suggests that the scale up of professional development also incorporates practices the field has begun to see as promising, such as attention to discourse, high-leverage practices, student thinking, formative assessment, and cognitively challenging mathematical tasks.

### **4. Build Coherent Programs of Professional Development**

Programs of professional development provide a continuous and coherent set of experiences in which practicing mathematics teachers engage over an extended period of time. Such programs warrant that the collection of professional development experiences provide an overall consistent message about what the new expectations are and how the standards should be incorporated into instruction to improve mathematics teaching and learning. Thus, in addition to ensuring that each professional development initiative emphasizes substance, uses materials that are tightly connected to teaching practices, and incorporates features that support teacher learning, a program of professional development takes into account teachers' experiences across all initiatives, making sure the set of experiences is coherent (Cobb & Smith, 2008).

### **5. Prepare and Use Knowledgeable Professional Development Facilitators**

Professional development uses expert facilitation to ensure teacher learning at scale. Professional development participants are far more likely to achieve the targeted learning goals when skilled and knowledgeable providers facilitate the professional development (Banilower, Boyd, Pasley, & Weiss, 2006; Elliott et al., 2009). The capacity problem for developing expertise in professional development facilitation is large and one that needs to be addressed to support the implementation of professional development at scale. Working to develop facilitation capacity requires the

identification and adequate preparation of a new cadre of professionals within the educational systems.

## **6. Provide Professional Development Tailored To Key Role Groups in Addition To Teachers**

Strong programs of professional development target a variety of role groups within the education system attending to the professional needs of each group as the system builds capacity at all levels. Because the CCSS-M represents changes in both what mathematics is taught and how it is taught, professional development to attend to key role groups in the system such as department chairs, instructional leaders, school administrators, and superintendents to ensure that all professionals in the system understand the new content and practices and share a vision for mathematics instruction. Professional development experiences conducted with a diverse group of professionals foster coherence and align expectations within the system for supporting teaching and learning. Different professionals in the system have different needs, and role-specific professional development opportunities target the needs of the specific groups.

## **7. Educate All Stakeholders**

Members of the general public need to be appraised on how the CCSS-M will impact instruction and learning in US classrooms. Mathematics teaching will be transformed as the new standards are implemented. Those who understand how teaching will look different need to educate parents, politicians, school boards, businesses partners, industry representatives, and other interested parties about what to expect. It is also important to inform stakeholders of the reasons why new content standards progressions and practices are proposed and why teaching approaches being used in classrooms are likely to be more effective in supporting student learning. In particular, parents need to understand what their children are working on in school and be positioned to better support children's learning (Civil & Bernier, 2006).

## **8. Continuously Assess Professional Development**

Professional development programs are regularly assessed to provide formative information for program improvement and revision, and to establish the effectiveness of the programs. Professional developers incorporate assessment and evaluation practices into their work so that regular, cyclic patterns of feedback shape professional development materials and programs. Consistent with current emphasis on evidence-based practices and programs, mathematics professional development needs to provide evidence of effectiveness and the use of data-driven decision making. Assessments of professional development are tied to specific professional development goals. Similar to design experiment cycles that have informed mathematics curriculum development and improvement (e.g., Cobb, Confrey & diSessa, 2003), formative assessment cycles support the improvement of professional development.

## 9. Create Professional Development Consortia

Professional development consortia are needed to oversee and improve the role professional development plays in successful implementation of the CCSS-M. These consortia serve as clearinghouses for mathematics professional development materials, programs, and providers in order to support states and districts in making decisions about mathematics professional development. The creation of professional development consortia addresses the current lack of certification for professional development providers and these consortia can work with professional organizations to develop models of instruction that are aligned with the standards. The professional development consortia help facilitate the sharing of information between assessment developers and professional development providers, contributing to a streamlined system of mathematics education.

## CONCLUSION

The project described herein generated nine recommendations intended to support large-scale, system-level implementation of professional development initiatives aligned with the new mathematics standards in the US. These recommendations emerged from the work done by experts from diverse fields. They build on state-of-the-art research findings from mathematics education, professional development, organizational theory, and policy. These recommendations focus on the important role that professional development plays in ensuring the successful implementation of standards. It is important to emphasize the importance of the diversity in the experts convened to generate the recommendations: without a diversity of viewpoints and experiences, the recommendations would not have the scope needed to provide quality professional development at scale.

## NOTE

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# THE VIEWS OF MATHEMATICS OF MEXICAN HIGH SCHOOL STUDENTS

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*This paper reports a qualitative research that identifies the views of mathematics of Mexican high school students. For this purpose, the social representations of 'mathematics', 'learn mathematics' and 'teach mathematics' were identified in a group of 67 students. In order to obtain information an open-ended questionnaire and focus group interviews were carried out.<sup>1</sup>*

## INTRODUCTION AND THEORETICAL BACKGROUND

In the field of mathematics education for at least three decades there is great interest in the investigation of personal views that students and teachers have about mathematics and its teaching and learning. In international research the personal views of mathematics of teachers and students have been identified through concepts such as: *conceptions* (Crawford & Gordon, 1998; Petocz, Reid, Wood, Smith, Mather, Harding & Engelbrecht, 2007; Reid, Petocz, Smith, Wood & Dortins, 2003), *beliefs* (Schoenfeld, 1992; Kislenko, 2009; Kaldo, 2011), *views of mathematics* (Törner, 1998; Roesken, Hannula, & Pehkonen, 2011) or *epistemological beliefs* (Liu, 2010). Each of these ways of approaching to the personal views matches different theoretical and methodological traditions.

The general finding of that research domain indicates that people experience and give meaning mathematics in different ways. Mathematics, for some, is a passenger inconvenience is necessary to advance beyond their school trajectory, for others are meaningless series of algorithms and techniques, for others is so important that permeate all spheres of human life and other consider them more central part of their personal life, their school life and their professional life. Some others consider mathematics as a static discipline developed in an abstract world and ideal, while others see mathematics as a discipline dynamic, constantly changing as a result of new discoveries of experimentation and application. These same studies have shown that the meanings given to mathematics strongly influence other dimensions of both the affective domain and the cognitive domain.

In Mexico and Latin America, there has been no research of personal views that students and teachers have about mathematics and the teaching and learning of mathematics. This research aims to start filling that gap that void by answering the

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following research question: What are the personal views that Mexican high school students have about mathematics?

There are many ways to carry out the definition of personal views. In this research I have chosen to approach the concept of *views of mathematics* used by Törner (1998) who states that the view of mathematics form a structure composed for a wide spectrum of beliefs (and conceptions) contains, among others, four main components: (1) beliefs about mathematics, (2) beliefs about oneself as an user of mathematics, (3) beliefs about teaching mathematics and (4) beliefs about learning mathematics.

In the definition of Törner in this research I have chosen to replace the concept of belief by the *social representation* (Jodelet, 1986; Moscovici, 1976). This I have done with the intention from the beginning to incorporate the theoretical consideration that the conceptions of students are the product of a social construction. Then, the research question becomes: What are the views of mathematics of Mexican high school students? Which leads to the following questions: (1) what are their social representations about mathematics? (2) What are their social representations about oneself as a user of mathematics? (3) What are their social representations about teaching mathematics? And (4) what are their social representations about learning mathematics? This research report only includes the answers to the questions 1, 3 y 4.

In this research a social representation is conceptualizing as an expression of *common sense knowledge*; it is the knowledge that gives certainty that the phenomena are real and that they possess specific characteristics (Berger & Luckmann, 1966). This choice comes from the consideration that common sense knowledge constitutes the most basic, primary, immediate knowledge that all individuals as a member of a community, group or society, the integration which essentially depends on the existence of this knowledge.

The theory of social representations is developed within social psychology and concerns human beings and their common, everyday knowledge (Moscovici, 1976). According to the theory, groups of people who share daily life develop common ideas, norms, values and 'truths' about phenomena and events. Through various forms of communication, habits and activities such social representations of the shared life and its conditions are developed, confirmed or changed. According to Jodelet (1986) social representations are forms of common-sense knowledge among groups of people, such as citizens in a community, colleagues at a working place, students in a university class or children and staff in a school. They include the set of beliefs, knowledge and opinions produced and shared by individuals in the same group in relation to a particular specific social object (Guimelli, 1999). A social representation allows guiding the people action in front of a specific social object. Therefore, the study of social representations is particularly important since the way in which they are produced and transformed helps to understand human behavior. The representation operates as a system for the interpretation of the reality that governs the relationships of individuals with their physical and social environment, due to the fact that it

establishes their behaviors or their practices. It is a guide for action; they guide actions and social relations.

In other words, social representation is practical knowledge. It gives meaning, within incessant social movement, to events and activities that end up becoming commonplace to us and this knowledge forges evidence of our consensual reality, as it participates in the social construction of our reality (Jodelet, 1986). Consequently, social representations are characterized by their significant, shared character, where their genesis is composed of the interactions and their functions fulfill practical purposes and are thus a form of knowledge created socially and shared, with a practical purpose that takes part in the construction of a shared reality for a social group, the function of which is to create behaviors and communication between individuals.

## **METHODOLOGY**

### **Procedure**

In order to obtain the data an open-ended questionnaire and focus group interviews were carried out. The purpose of these two techniques was to generate written and verbal discourse<sup>2</sup>, allowing us to find out the social representation.

The questionnaire was composed of open-ended questions so as not to limit the answers of the participants and in order to allow them to openly express their opinions, reducing to a minimum the influence of the questionnaire. Three questions were asked in order to discover the conceptions of mathematics: 1) in your opinion, what is MATHEMATICS? 2) in your opinion, what is LEARN MATHEMATICS?; and, 3) in your opinion, what is TEACH MATHEMATICS? In the questionnaire given to the students, the capital letters were used to emphasize the purpose of the object of interest in each question. The questions asked in the focus group were the same as those in the questionnaire and the role of the interviewer was to ask for more specific information in relation to answers regarding the use, meaning of words and phrases used by the students. For this purpose, questions such as, in your opinion, is a dynamic class?, why do you say that the class is boring?, what does it mean that a teacher knows how to explain?, etc.

According with Morgan (1996, p. 130) “the focus groups as a research technique that collects data through group interaction on a topic determined by the researcher”. Focus groups are an appropriate method data collection when one is interested social representations because they are based on communication and it is the heart of the theory of social representations (Kitzinger, Markova, & Kalampalikis, 2004). The primary aim of a focus group is to describe and understand meanings and interpretations of a select group of people to gain an understanding of a specific issue from the perspective of the participants of the group.

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<sup>2</sup> In this paper the word discourse is used in a very broad sense to refer to narratives generated by students through the instruments used in this research: narrative written to answer the open questions in the questionnaire and oral narrative through focus groups interviews.

Both, the questionnaires and the focus groups, were carried out after school in approximately hour and a half in a classroom, this allowed the students to gather at tables. We worked in sets of students with two interviewers, none of which were teachers of the students (two sets of twelve, two sets of fifteen students and a set of thirteen students). The procedure were as follows: 1) Individual application of the questionnaire, 2) Creation of four focus groups of between three and four students as students themselves decided, 3) Collective answering the questionnaire in each focus group, 4) Commenting and providing more specific information in relation to the answers with the interviewers. The second, third and fourth part were videotaped.

### **Participants and context**

The IPN (National Polytechnic Institute for its acronym in Spanish) is a public institution that provides free in Mexico City, or very low cost, in high school, undergraduate and postgraduate level in the area of science and technology. The CECYT (Centre for Science and Technology Studies for its acronym in Spanish) are part of the education offered by high school level of IPN dedicated to the training of technicians. The participants were a non-statistical sample of 67 fifth-semester students, 16 to 18 years. It is important in focus group research that participants have some form of homogeneity. Therefore it was decided that the participants were students enrolled in the same school in the same math class with the same mathematics teacher.

### **Data Analysis**

The general strategy for this analysis was a constant comparative style (Strauss, 1987; Glaser and Strauss, 1967), which permitted the categories to emerge from the data. The analysis involved two stages. The first stage of analysis was run for each questionnaire and focus group transcripts separately, that is, each utterance about views of mathematics in every interview transcript was identified and labeled. The second stage of analysis was to group those utterances across different transcripts that had a similar meaning into a common category. After this, a set of categories was established and further, intensive analysis of each particular category was carried out to identify a distinct meaning for them. Each category was used to identify the different social representations that were expressed through a phrase that represents the global meaning that summarizes and condenses the form in which subjects capture the represented object.

The students were identified with the labels An (with n being from 1 to 67). The En label identified either of the two interviewers in the focus groups (one of the interviewers was the author of this paper). I used a diagonal line between two words to note that two words have the same meaning from the perspective of students. In addition to the above, square brackets were used to establish when the same meaning exists in two phrases and between a phrase and a word. Thus, for example, daily/everyday indicates that for students the adjectives ‘daily’ and ‘everyday’ are equivalent, and apply/[put into practice] indicates the same meaning between the word

‘apply’ and the phrase ‘put into practice’. Such equivalency of meanings was identified in the focus groups.

## RESULTS

The following section titles show, respectively, the students’ social representations of ‘mathematics’, ‘learn mathematics’ and ‘teach mathematics’. I present some examples of what students declared.

### **Math is important because they are all that surrounds us**

A2: Mathematics is a science that allows us to know and understand how everything around us, science is enriched over why it is so important.

### **Mathematics is important in everyday life**

A15: Mathematics is essential in our daily life, which we must always bear well in mind.

### **Mathematics is important for school**

A23: Mathematics is a fundamental subject you use up your daily life.

### **Mathematics is important for the professions and for everyday life**

A58: Mathematics is essential for all, both career to the problems of daily life.

### **Mathematics used to solve everyday problems**

A13: Mathematics is a way to solve problems of everyday life, but with numbers and the like.

### **Mathematics is numbers and operations to solve problems**

A4: Mathematics is performing basic and complex operations.

### **Mathematics is complicated and difficult**

A12: Mathematics is complicated and difficult, which although very useful, are always very complicated.

### **Mathematics is exact**

A10: Mathematics is exact because when you do a problem usually there is only one answer. In equations around there is an answer, not several.

### **Learn mathematics is to possess/acquire/have knowledge to apply/[solve problems]**

A55: Learn mathematics is to know and apply mathematics at any time.

### **Learn mathematics is be able to solve problems of daily living**

A16: Learn mathematics is to know exactly what to do when we get a problem or mathematical operation.

### **Learn mathematics is to do calculations and operations to solve problems of daily living**

A30: Learn mathematics is to know the different techniques or ways to calculate certain volumes, areas, etc.

### **Learn mathematics is reasoning/[think logically] to solve problems**

A14: Learn mathematics is important because it makes to someone thinking logically

**Teaching mathematics is transmit / give / provide knowledge**

A6: Teaching mathematics is to convey the little knowledge I have about the topic to help others.

**To teach mathematics is to know/understand to pass/share/give**

A12: To teach mathematics must fully understand and we must learn to understand others give them.

**Teach mathematics is transmit/give the [reasoning ability]/[the understanding]/logic**

A16: Teach mathematics is someone trying to understand what the subject is.

**Teach mathematics is [help] / [show as] solve problems**

A13: Teach mathematics is to show others the knowledge and the different steps and ways to solve problems

**CONCLUSIONS AND DISCUSSION**

For participating students in this research, mathematics has the function of solve problems of daily life. Everyday life is an existential universe composed of at least three complementary sub-universes: 1) Daily life in school, 2) daily life outside of school, 3) the ideal life of employment associated with the professions and technical specialties. In all these sub-universes mathematics is considered very important. In the world of school the importance is highlighted by the view that mathematics is a subject that is the basis for other subjects (such as physics or chemistry). Outside of school, mathematics is considered necessary for a wide range of social practices related to the number, size and the business transaction. In addition, students say that to solve problems have to use numbers and operations from the "simple" as addition, subtraction, multiplication and division, to more "complex" as the differentiation and integration. Two characteristics are associated with mathematics: 1) mathematics is difficult and complicated, in the sense that it requires more dedication and time compared with other study materials and 2) mathematics is considered accurate, because the answer of an operation or a problem is unique. Learn mathematics is closely linked to the vision of the role given to mathematics to solve problems of daily life. To make sense of the phrase "learn mathematics" students resort to various metaphors<sup>3</sup>, where, in linguistic terms, one can observe that students use transitive verbs (have /acquire/have) where action objects correspond to the functions conferred to mathematics. Thus, learn math is: 1) Possess/gain knowledge to apply/[implement]/[solve problems], 2) power/knowledge to solve problems of daily living, 3) power/knowledge do calculations and operations and 4) Reason/[think logically]/[the ability] to power/knowledge to solve problems. In complement **teach mathematics** is closely linked to the metaphor of the transfer of a good or a possession by the person who teaches through the explanation. In linguistic terms can be observed that the word "teach" is associated with other transitive verbs where the object of the

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<sup>3</sup> For Lakoff & Núñez (2000) metaphors are a way of conceiving one thing in terms of another and give expression to abstract realities in more concrete terms.

action are the knowledge, [reasoning ability]/understanding/logic or [solve problems]. Thus, teach mathematics is: 1) Transmit/giving/sharing knowledge, 2) L To teach mathematics is to know/understand to pass/share/give, 3) Transmit/give the [reasoning ability]/ understanding/logic and 4) [Help]/[show how] solve problems.

As other research (Petocz et al., 2007, Reid et al., 2003) in this investigation were detected conceptions of mathematics conceptions ranging from narrow (mathematics are calculations with numbers) to broader visions of mathematics (the mathematics as a way of thinking). In our participants did not find the conception of mathematics as a system of knowledge or consideration of mathematics as models, which is explained by taking into account the educational level that enrolled participants which does not include the study of mathematical structures or systems or work with explicit mathematical model. The similarities and differences with the results found by Petocz et al. (2007) and Reid et al. (2003) can be explained by the fact that the experience with the mathematics of the participants in our research is more limited than that of college students participating in such research.

The conceptions found here of learning as metaphor is consistent with other research that has delved into the metaphors used by teachers of mathematics and its teaching. By example metaphors categorized as production were most common in the Reeder, Utley & Cassel (2009) data and indicated that students passively receive knowledge from teachers. For example, "teacher is as a sponge full of knowledge, squeezing it out into the empty glass". Also in Martínez, Saulea & Huber (2001) study majority of experienced teachers as well as prospective teachers shared traditional metaphors depicting teaching and learning as transmission of knowledge. Future research could explore with more detail the metaphors used by students and teachers in their views of mathematics, as already reported by Reeder et al. (2009).

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# VIRTUAL SIMULATIONS FOR MATHEMATICS TEACHER TRAINING: PROSPECTS AND CONSIDERATIONS

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*A key aspect of this case study was to exploit the affordances offered by digital simulations for bridging the classic gap between teacher preparation and practice. Using the simulated classroom simSchool as a virtual field experience, a teaching experiment focused on mathematics education and on technology use was conducted. Both undergraduate and graduate education students participated in the study. Findings suggest that participants appreciated simulations as virtual environments that provide the opportunity to practice and experiment on particular teaching approaches, in a safe environment. Nevertheless, they also expressed a number of concerns related to how simulations compared with real classroom experiences.*

## INTRODUCTION

Teacher education programs worldwide have been criticized for failing to equip their graduates with the knowledge and skills required to teach quality mathematics and to produce reform-based classroom change. A chief criticism is that they are disconnected from the school system, overly theoretical and not as relevant as practitioners demand (Darling-Hammond, Hammerness, Grossman, Rust, & Shulman, 2005). Alternative technology-based approaches to teacher preparation have lately received increasing attention in the press and in research studies. One promising approach explored is that of digital games and simulations. Simulations are routinely used in a variety of professions (e.g. healthcare, aviation, military) to better educate and prepare people for the real-life situations they are likely to encounter. Several researchers have recently explored the possibilities of linking simulations with field experiences for novice teachers, with promising results (e.g. Girod & Girod 2008; Hettler, Gibson, Christensen, & Zibit, 2008). Reported benefits of virtual field experiences include exposure to multiple teaching strategies and learning styles in a short period of time, and better understanding of how theoretical knowledge presented in pre-service teachers' college courses relates to actual classroom practices and student behaviors and learning (Christensen, Knezek, Tyler-Wood, & Gibson, 2011).

The article shares some of the experiences gained from a case study that took place within the *simSchool* Modules Project, an international technology project designed to enhance the *simSchool* platform, a "flight simulator" for teachers to become a fully-realized teacher training platform for higher education. The project was sponsored by the Association for the Advancement of Computing in Education (AACE), in collaboration with CurveShift (*simSchool*) and Pragmatic Solutions, Inc. (Leverage). It was selected in the USA from a field of more than 600 pre-proposals and 50 finalists as a 2011 Next Generation Learning Challenges (NGLC) awardees.



*SimSchool* is a classroom simulation developed by CurveShift with funding from the U.S. Department of Education's Preparing Tomorrow's Teachers to Use Technology program. It has been designed to serve as a "virtual practicum" that augments teacher preparation programs (Zibit & Gibson, 2005). Using *simSchool* has demonstrated effectiveness in increasing pre-service teacher perceived instructional self-efficacy, causing a shift in attitudes about the locus of control for bringing about student success, and developing teaching skills on a par with more expensive and time consuming methods (e.g. Christensen et al., 2011; Ellison, Tyler-Wood, & Sayler, 2009).

The *simSchool* Modules Project aimed to develop, pilot test, and disseminate modules for learning to teach via *simSchool*, through a network of 39 colleges of education around the world. Participating institutions undertook both local and project-wide collaborative research on the benefits and challenges of using *simSchool* in teacher training. Within the project, we conducted a pilot case study which sought to exploit the affordances offered by *simSchool* for contextualizing pre-service and novice teachers' learning of mathematics pedagogy. We studied the potential of digital simulations to create reality based learning contexts that foster opportunities for teachers to translate what they had learned in their preparation programs into classroom practice.

## **BACKGROUND TO THE STUDY: SIMSCHOOL FEATURES**

The *simSchool* simulator is driven by an artificial intelligence (AI) engine built based on a complex systems framework for simulating teaching and learning. When launching a *simSchool* session, the system simulates a living classroom populated by students (*simStudents*) sitting in rows of individual desks (see Figure 1a). Flexible classroom demographics (e.g. variation in class size, student gender, race, and academic ability) allow the construction of a nearly countless number of classrooms.

Each *simStudent* can be drawn randomly or be custom-made. He/she has an initial personality profile with settings on five psychological dimensions, two cognitive dimensions, and three physical-perceptual variables. The student's emotional make-up is built on the OCEAN or Big Five model of personality (McCrae & Costa, 1996) which includes the following characteristics: openness to learning, conscientiousness, extroversion or introversion, agreeableness, and neuroticism or emotional stability. The cognitive dimension is represented by two variables: overall academic performance and native language proficiency. The physical-perceptual dimension is represented by three variables: auditory, visual and kinaesthetic awareness.

The players can review student records or start the class session. When the class starts, they can select instructional tasks, monitor how learning is progressing, decide whether to talk or not, whether to change tasks, and when to end the class. If, for example, they choose to assign a new task, they are given options such as recall task (e.g. take a pop quiz), skill/concept task (e.g. apply a formula), strategic thinking task (e.g. compare and contrast), or extended thinking task (e.g. develop a hypothesis). Task environments exert performance requirements (cognitive load) independently on each student, causing some to learn and others to get either frustrated or bored. Each task

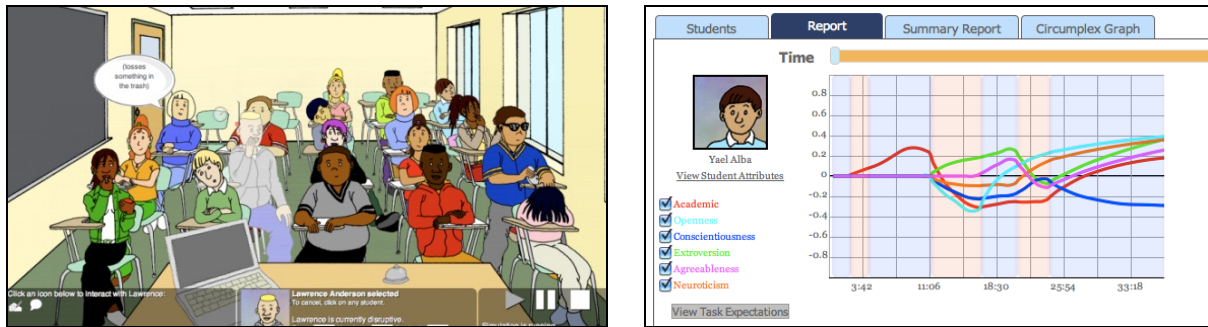


Figure 1: A simulated *simSchool* classroom (Figure 1a) and a post-game report of *simSchool* dynamics over the course of one simulation (Figure 1b)

environment is characterized by settings which interact with each student current profile setting to produce classroom and academic behavior. If a task is within the student's zone of proximal development (Vygotsky, 1978), the student will independently learn without need of further scaffolding or task adaptation. The student will not learn and will become bored if the task is too easy, and the student will not learn and will become frustrated if the task is too difficult. SimSchool, therefore, lets future teachers practice designing tasks that match the individual differences of students. Dispositions in teacher-student interactions are modelled based on the 'interpersonal circumplex theory' (Kiesler, 1983) which proposes that humans negotiate between 'power' and 'affiliation' in their interactions, ranging from dominant to submissive and from friendly to hostile respectively.

Detailed reports about teaching performance are made available at the conclusion of any simulation, and give a moment-by-moment analysis of the user's decisions and their impact (see Figure 1b). Additionally, *simSchool* allows the instructor to embed questions after each simulation for student reflection. Answers to these questions are saved in the online student portfolio section of *simSchool*. They can be accessed, along with the other reports provided by the system, by both the instructor and the students.

## METHODOLOGY

A case study aimed at determining best practices for *simSchool* use at our institution and the major challenges of such an approach took place within the context of two education courses: an undergraduate methods course on the Integration of Contemporary Technologies in the Teaching of Mathematics attended by students in the Primary Education program ( $n=12$ ), and a graduate course on New Technologies in Education attended by students in the M.A in Educational Leadership program ( $n=9$ ). The case study involved the design and pilot-testing, with these two different groups of students, of a series of modules which utilized *simSchool*. For both groups, emphasis was given on exploiting the simulation's potential to foster opportunities for students to experiment with the system in teaching activities. The project was integrated as part of the course requirements, in order to provide incentives for participation.

The teaching experiment spanned approximately one half of the 15-week semester. It was designed in several stages, which included online modules and face-to-face meetings with instructors. The first stage included an introduction to the *simSchool*

interface, and the theoretical basis behind its creation. The second stage involved the completion of two introductory modules. The third stage was a face-to-face meeting in which technical and procedural issues were discussed. At the next stage, participants worked on two new modules in which they designed and improved lesson plans for their simulated classroom. The fifth stage involved two modules where participants had to create *simStudents* and tasks. These modules required students to use the tools available in *simSchool*, in order to specify custom profile characteristics for their *simStudents* as well as custom levels and content of their tasks. In the final stage, students were provided with scenarios of teaching mathematics in mixed ability mathematics classrooms, and were asked to use their previous experience of *simSchool* to design their lesson plans, observe their diverse *simStudents*, manage their classroom, and improve their *simStudents*' learning outcomes. Pre-service teachers enrolled in the undergraduate mathematics methods course were also asked to implement their lesson plans in a real classroom setting during their teaching practice placements, and to compare their experiences with those gained from the simulated classroom.

Two main data collection methods were used to document evidence of participants' perceptions and attitudes towards the use of simulated classrooms: interviews and reflective reports. Interviews were standardised open-ended. Standardised interviews, although criticized for limited flexibility, were chosen as a suitable means of receiving feedback on the teaching experiment, because respondents were coming from different backgrounds and had varying levels of teaching experience. Hence it was important to keep the topics addressed in the interviews constant, for comparable results. Flexibility in data collection was provided by the second method of self-reflection reports, where participants had the opportunity to record any thoughts, considerations and other issues related to different aspects of their experience during the teaching experiment.

In the next section, we provide a synopsis of the main insights gained from the study. For a more detailed analysis see Mavrou and Meletiou-Mavrotheris (in press).

## RESULTS

Based on the analysis of the collected data, we were able to verify SimSchool's overall efficacy and potential to create more rapid expertise in novice teachers of mathematics. Participants, both pre-service and in-service teachers, appreciated the use of simulations as virtual environments that provide the opportunity to learn to teach or practice and experiment on particular teaching approaches, before entering a classroom. During the interviews, they stressed several benefits of simulations. Firstly, they highlighted the benefits of using *simSchool* as a safe environment to practice teaching skills. Undergraduates, in particular, found very useful the opportunity provided by the system to freely practice teaching without the fear of their difficulties as inexperienced teachers: "It was quite helpful in practicing certain approaches and reflecting on our work, even if we faced some difficulties". Secondly, they underlined the way the system guides players through a well-structured lesson plan design.

Participants also appreciated the ability to work with a wide diversity of virtual students, which helped them to focus on particular types of learners that may be

disregarded in large groups of real classrooms settings. They acknowledged that one of the most beneficial aspects of the simulation was the fact that they had to continuously monitor the effect of their actions on students' learning and motivation, and to adjust their instruction by taking into consideration each *simStudent*'s unique learning style, academic performance, and disability status. Although handling the diversity faced in their virtual classroom was particularly challenging, and caused considerable frustration, they viewed this as an important learning opportunity that forced them to continually contextualize decision making by individualizing instruction: "Interesting. I now pay more attention to the indifferent students. It was challenging." In their self-reflection reports participants also expressed their satisfaction about the opportunity they had to control the number of students in a classroom. This option provided the opportunity to more carefully observe different learners' characteristics, and to be able to pay attention to them later on, in larger groups of students.

In sum, participants' appreciation of *simSchool* advantages are in line with prior research findings about the benefits of simulations as teaching practice environments, which for further reference are very well summarized by Gibson and Kruse (2011).

Although participants recognised the innovative perspective of the simulated classroom environment and its beneficial educational effects on their training, analysis of their interviews and reflection reports led to the identification of several issues and concerns regarding the online simulation. These are summarized next.

### **Real Vs Simulated Experiences**

Howard-Brown (1999) argues that "the tradition of field experiences is so firmly entrenched that is often difficult for students to see any value in alternative activities – they are just not 'real' enough" (p. 307), something that was evident in this study findings. Our participants believed that their experience in real classroom situations cannot, in any way, be replaced by virtual activity. Both experienced and less experienced teachers held similar views. Even undergraduates, who had very limited real classroom teaching experience, firmly stated that a simulation cannot reach reality's potential: *"The design and implementation of activities in simSchool has nothing to do with the real classroom. I believe that in reality things are totally different."* Despite the reported benefits of simulations as safe environments for experimenting, participants stated that they had difficulties adapting and changing their teaching: *"There is no opportunity in adapting the tasks during their implementation, thus not leaving space for meeting the students' needs and interests."* It seems that in their urgency to complete the tasks as part of their course requirements, participants disregarded the nature of simulations as play, and handled them as a real classroom experience, overlooking the undo and redo options of virtual worlds. Comparison between the reactions of real and simulated students also affected their unenthusiastic evaluation of simulated classrooms. They expected more intense and dynamic reactions from *simStudents*: *"During my school practice, students responded to questions, asked questions, most of the time collaborated, something that was not obvious in simSchool"; "In reality our students were excited with the use of Sketchpad"*

[...] in the virtual classroom they were not focused, they talked to each other about various things.” Participants seemed more satisfied with their *simStudents*’ learning outcomes than their responsiveness as social practice: “The only thing in common with the lesson taught in a real classroom was students’ academic performance. As far as feelings are concerned, in *simSchool* it was very difficult to discern students’ feelings.”

### **Interpersonal Relationships in *simSchool***

Participants had difficulties in developing interpersonal relationships with their simulated students: “I considered my students virtual and hence it was difficult for me to develop any interpersonal relationships”; “I felt that there was no space for expressing emotion...facial expressions and gestures are of great importance, but I couldn't express them through the simulation”. They believed that their lack of emotional engagement and understanding of their interpersonal involvement with *simStudents* was a consequence of either lack of time, or their own lack of experience in using virtual environments: “I believe that the reason the real lesson was much better than the virtual one, was that we are not quite familiar with *simSchool* and so very likely we did something wrong during the simulation.”

### **Technological constraints of the simulation environment**

*SimSchool* attempts to re-create the complexities of real classrooms through a complex mathematical model of how people learn and what teachers do when teaching. However, all models have inherent limitations due to their simplification of reality, and *simSchool* is no exception. Participants found the system too simplistic for simulating the complexities of real mathematics classrooms. An issue raised by most participants concerned the simulation environment’s graphical design, which they found to be poor and unrealistic, and to lack the sophistication of an immersive experience since it does not include any animated images, video, or sound, but is restricted to photographs and text. Additionally, comments on the design of the simulations indicated issues of inflexibility. Participants required more freedom in working with the avatars and other features in the simulations: “I didn't really like it because there was no opportunity in organising the desks the way I wanted to”. They also noted that the options available for interaction are constrained to basic knowledge-level and behaviour comments. The difficulties mentioned by participants also indicated issues of conveyance of feedback and messages from *simStudents*: “It was difficult to understand students’ state, because there seemed to be a contradiction between what really happened and how they expressed themselves”; “Feedback from students was not clear”.

### **Other technical concerns**

Some technical problems experienced during the study, seem to have discouraged some of the students from fully participating in the teaching experiment. The *simSchool* simulator is Web-based, so it relies on the internet to be accessible and to deliver its functionality to users. Although it is, overall, a very reliable system, there were a couple of occasions when students could not use it due to servers going down, or because of database connectivity failure. Access of technology at home was also an important factor affecting student online participation. Although everyone had an

internet connection at home, it seems that online access was difficult and time consuming for a few of the participants due to issues such as low speed connectivity, outdated browsers, forgetting of passwords, etc.

## DISCUSSION AND FURTHER CONSIDERATIONS

The research literature and our own experiences (e.g. Bell, 2009; Gibson & Baek, 2009; Meletiou-Mavrotheris and Mavrou, in press), have led us to the conclusion that classroom simulations hold a lot of promise as an instructional tool in teacher preparation programs. There is a lot that pre-service and novice teachers could gain from realistic simulations that provide a safe environment for experimenting with teaching techniques. Despite, however, the benefits of virtual environments, there are several issues that need to be resolved to ensure their effective employment in mathematics teacher training. Findings from this case study have further illustrated the generic effectiveness of *SimSchool* to increase pre-service and novice teachers' perceived instructional self-efficacy. At the same time, however, they have indicated that the system is only partially successful in depicting the complexity of interactions occurring in actual classrooms. Study participants stressed the need to make the simulation more realistic and relevant for mathematics instruction. They pointed out several key areas that ought to be improved in the program, including the need for enhancements to the simulation's ability to capture student-teacher and student-student interactions, and the inclusion of rich multimedia to add to the simulation's realism.

Virtual settings have inherent limitations due to the fact that they are just models which cannot fully replicate real experience. Despite their limitations, we are very interested in further exploring the possibility of linking simulations with field experiences. In particular, we are profoundly interested in expanding the *simSchool* capacity to make it subject-specific, focusing on the provision of virtual field-based experiences for pre-service mathematics teachers. To achieve this, we currently aim at securing funding that would allow us to extend the *simSchool*'s academic component, currently represented by a single variable representing overall academic performance, by modifying the system's underlying algorithms and representations to support inquiry-based mathematics teaching and assessment approaches to each localized base layer of the platform, based on appropriate teaching and assessment constructs.

We are strong advocates of an important underlying principle of *simSchool*: “*practice in a variety of settings builds expertise*” (Gibson & Kruse, 2011). Simulations could provide pre-service and novice teachers with the opportunity to practice and refine their instructional skills through exposure to multiple teaching strategies and learning styles in a short period of time. Coupled with actual field practice, they can better prepare them for their transition from college to classroom teaching.

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# THE NEED FOR PROOF IN GEOMETRY: A THEORETICAL INVESTIGATION THROUGH HUSSERL'S PHENOMENOLOGY

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*The students' internal need for proof is at the crux of their learning advanced mathematics. In this essay we adopt a Husserlian perspective in order to read the sociocultural factors that lead to the genesis of geometrical proof in ancient Greece, with the purpose to investigate the contribution of Husserlian phenomenology in pedagogies aiming to foster the students appreciation for proof. We argue that the transcendental character of Husserl's phenomenology may contribute in a coherent framework for addressing the whole spectrum of the students' identified internal needs for proof (notably, structure), thus contributing in more effective pedagogies.*

## INTRODUCTION

The students' learning and thinking about mathematics have been discussed through phenomenological ideas stressing the importance of communication and active argumentation (Gravemeijer, 1994; Radford, 2003). Nevertheless, it appears that little explicit discussion has been made about the contribution of phenomenological ideas about the students' appreciation of and need for mathematical proof, which includes the coordination of logic and the axiomatic structure to investigate the validity of a statement. Such a view of proof does not exclude its various functions (for example, verification, communication, explanation, systemisation; Balacheff, 1991; de Villiers, 1990; Hanna, 2000). Recently, Zaslavsky, Nickerson, Stylianides, Kidron and Winicki-Landman (2012) addressed both mathematical and pedagogical perspectives about the need for proof, while Grabiner (2012) discussed the reasons that lead to proof from a historical perspective. In this theoretical essay, we consider historical, philosophical and empirical evidence to investigate the contribution of Husserlian ideas in this discussion. We argue that Husserl's transcendental phenomenology (Husserl, 2001) may contribute in an interpretative framework for gaining deeper understanding about the sociocultural factors that lead to the genesis of proof in geometry in ancient Greece and for understanding and pedagogically fostering the students' intellectual need for proving in geometry. Hence, we consider the fundamental question: *What is the contribution of Husserlian ideas in fostering the students' need for proof in geometry?*

## HUSSERL'S TRASCEDENTAL PHENOMENOLOGY

*Intentionality* is central to Husserl's phenomenology, referring to "the conscious relationship we have to an object" and, thus, every "intending has its intended object" (Sokolowski, 2000, p. 8). The *natural attitude*, "our straightforward involvement of things and the world", is contrasted with the *phenomenological attitude*, "the reflective point of view from which we carry out philosophical analysis of the intentions exercised in the natural attitude and the objective correlates of these intentions" (Audi,



1999, p. 405). The intersubjective experience of the communicated shared meaning is contrasted with transcendental subjectivity in which there is an awareness of a phenomenon outside our subjective perceptual experience. For Husserl, the “objective and the subjective are correlative, but never reducible to one another” (Audi, 1999, p. 405) and the phenomenologically described given object (*noema*) is not to be confused with the subjective activity (*noesis*). Transcendental subjectivity is viewed “as a possible communicative subjectivity [...] through possible intersubjective acts of consciousness, it encloses together into a possible allness a multiplicity of individual transcendental subjects” (Husserl, 1974, p. 31). His method of phenomenological reduction (*epoché*) includes *bracketing* out the natural attitude and moving towards a phenomenological attitude by investigating the sedimented intentional history of the object to “seek for its “constitutive origins” and to reproduce its “intentional genesis” (Klein, 1940, p. 150). During epoché, *explicit thinking* is activated (not just “passive, thoughtless repetition of words”; Audi, 1999, p. 406). Language (oral and written) is central to Husserl’s phenomenology, constituting the means for the *objectification* of the subjective experiences. Husserl’s idealities differ from platonic ideas in that new knowledge is intentionally subjectively constructed *once* in history and every subsequent knowing requires the reactivation of this objectification (Derrida, 1989).

## **PROOF AND PHENOMENOLOGY IN MATHEMATICS EDUCATION**

### **The need for proof in mathematics education**

Research suggests that the students lack a comprehensive understanding of the various functions of proof. For example, 14-15 years old students appeared to most commonly identify the verification function of proof, followed by explanation and communication (Healy & Hoyles, 2000), while the vast majority of younger students did not express a need for proving or specifically for ‘proving the obvious’ (Kunimune, Fujita & Jones, 2009; Williams; 1980). With respect to older students, first year advanced-level mathematics students appeared to obtain a theoretical understanding of the systematisation function of proof, but they were not convinced of its verification function, which may be linked with their reluctance to employ proof (Coe & Ruthven, 1994). Furthermore, in exam-type situations, second year mathematics undergraduates were found to consider the verification function of proof in their persuasion of others, without necessarily employing proof in ascertaining themselves (Moutsios-Rentzos & Simpson, 2011). Though the students’ appreciation of proof widely varies depending on the identified by the students internal or external requirements (ibid), they need a reason to produce a proof (Balacheff, 1991). The external need for proof may be realised as an ‘external conviction’ proof scheme (Harel & Sowder, 1998) and/or may be embedded within the situation (Moutsios-Rentzos & Simpson, 2011). Considering internal needs, Zaslavsky et al (2012) identify: *certainty* (verification of the truth of a statement), *causality* (explanation of the reasons why a statement is true or not), *computation* (quantification of definitions, properties or relationships through algebraic symbolism), *communication* (formulation and formalisation in conveying ideas) and *structure* (logical re-organisation of knowledge). These needs may be viewed as an extension of and cultivated through everyday activities. For example,

learning environments that promote inquiry appear to promote the students' internal needs for producing a proof (ibid). Nevertheless, we argue that the internal need for 'structure' is more demanding, especially when other internal needs have been met (note, for example, the aforementioned reluctance to prove the obvious).

### **Phenomenology in mathematics education**

Considering phenomenological ideas in mathematics education research, it appears that mathematics educators seem to differ from Husserl's. For example, though Radford (2006) shares Husserl's view that objectification occurs within the semiotic system employed to signify an ideality, he argues that "transcendentalism [...] leads to an irresolvable tension between subject and object" (p. 39). Radford (2003) proposed a semiotic-cultural approach to the means of *desubjectification* (cf. objectification), stressing the subjective nature of the constructed through semiotic activities meaning. He discussed the sociocultural and the psychological aspects of algebraic thinking, delineating the semiotic means employed in the students' construction of meaning in algebraic activities. Though his work sheds light in the development of the students' algebraic thinking, it seems to lack explicit investigation about the students' internal need for proof. Moreover, the geometrical signs are of qualitative different nature, as "it is necessary to combine the use of at least two representation systems, one for verbal expression of properties or for numerical expression of magnitude and the other for visualization" (Duval, 2006, p. 108). We argue that a Husserlian phenomenological framework may help in addressing the students' internal needs for geometrical proof.

Drawing upon Freudenthal's didactical phenomenology, realistic mathematics focus on the students' needs to propose teachings in which the learners are guided by the teacher to actively organise a 'real' situation with mathematical tools, thus coming to the *guided reinvention* of a mathematical idea (Gravemeijer, 1994). By putting the learners in a 'real' for them problematic situation, they experience a 'real' need to successfully survive it. Through a process of *mathematisation* of the real-world situation and the teacher's guidance, the students actively re-invent important mathematical ideas as meaningful resources required for their lived reality. Though mathematisation may allow for the need of mathematically modelling and resolving a real situation and, subsequently, incorporating these mathematical ideas within the mathematical world (Freudenthal, 1983), it does not address the ways of incorporating this new knowledge; for example, we posit there is no 'real' internal need for a proof logically founded on already accepted as (or logically derived) true notions.

Consequently, it is crucial to identify pedagogies that explicitly address the students' internal need for producing a desubjectified, axiomatically and logically derived argument in order to be ascertained and/or to persuade. Should the 'institutionalisation' of the inductively and actively constructed by the students ideas be based solely on the teacher? Is mathematical proof only a linguistic shift to a symbolic, official language? To what extent semiotic changes and mediation may provide internal support to proof? In the following sections, we adopt a Husserlian perspective to read the historical genesis of proof in ancient Greece, attempting to address these questions.

## THE TRANSCENDENTAL ARGUMENT IN ANCIENT GREECE

### The reign of the argued thesis over the arguing subject

In this essay, we employ a Husserlian perspective in discussing the genesis of proof in ancient Greece. Reading history reveals a network of possible necessities, rather than a linear assortment of events, which requires the unfolding of the sedimented historical layers within which the mathematical notions were objectified. Though visual, empirical or measurement arguments existed in pre-Greek mathematics, specific factors worked in the ancient Greek city (*polis*) so that geometrical assertions were proved based on logic and accepted or proven to be true assertions, including the

disagreement between older results, the desire to establish elementary first principles, the logical structure produced when problems are solved by reduction to simpler problems, the role of argument in Greek society, the central importance of philosophical argument in Greek thought, and the major contributions to mathematics resulting from using proof by contradiction (Grabiner, 2012, p. 152).

In the *polis*, the equality of the citizens allows their quantification (and objectification) within a democratic power system that a citizen corresponds to a vote. The ruling power is not a subjective quality, but it is the objectively measured sum of all the favourable votes. The qualitative unity of the subject fades out to an objective countable unit. By objectifying the power relationships, the city ensures the continuity and coherence of its structure regardless of the subjects who hold the various posts. Furthermore, the *polis* is characterised by the importance of the transparency of the most important events of the citizen's life. The private life that carries a significant weight is publicly shared to actually obtain its importance. The private-centeredness was generally frowned upon and the Athenian 'idiot' was the person who lacked the reasoning skills or the will to positively contribute in the public affairs. The citizen-subject is subjected to (*ypo-logos*) the oral communication of shared reasoned ideas (*logos* refers to both reasoning and oral speech in Greek). Vernant (1983) stresses that within this strong sense of belonging and the pursuit of a commonly shared reality, the personal identity was not lost. Heraclitus noted that "although *logos* is common to all, most people live as if they had a wisdom of their own". The *logos* is common in the sense that all private understandings and reasoning are in agreement with (*homo-logia*) with the public *logos*. Moreover, the Greek divine appeared to be strongly linked with human activities; for example, the arguer's power of convincing the many was subsequently attributed to the goddess Peitho (persuasion). Hence, we posit that the divine, the private and the public appeared to be strongly interconnected in the *polis*, thus constituting a transcendental with a strong anthropological character (rather than a non-human transcendental idealism). Within this framework the psychology of the self carries the transcendental references of the *logos* and becomes a multiplicity of higher mental internalised social relationships (cf Vygotsky, 1978). Moreover, Vygotsky (1978) notes that the close relationships of (external) social processes and (internal) psychological processes that in "their own private sphere, human beings retain the functions of social interaction" (p. 164). Stressing a Marxian perspective, Godelier (1977) investigates the circumstances and the reasons under which "a certain factor

assume[s] the functions of relations of production” (p. 36), while Vernant (1975) identified the formation of the *polis* as the decisive event that allows *logos* to gradually become the utmost measure of power, replacing bloodline or even economic status with the power to convince the majority of the citizens. Thus, the matters of the common interest become an issue to be debated and to be ruled by the rhetoric of the speaker; *it is not important who the arguer is, but what the argument is*.

### **The genesis of proof in geometry**

The idea of proof historically arose in ancient Greece (Katz, 2009).

Logic lets us reason about things that are beyond experience and intuition [...] The Greek proofs by contradiction changed the way later mathematicians thought about the subject-matter of mathematics. Mathematics now had come to include objects whose existence cannot be visualised and which cannot be physically realised [...] Logical proof created these new objects [...] (Grabiner, 2012, p. 152)

Szabó (1978) claims that the study of incommensurable magnitudes and the irrational numbers and the notion of deductive proof did not meet any practical needs, but autonomous conceptual needs of a transcendental nature driven by the necessity of finding a shared reasoning. He studies the contemporary language of the market and the everyday life and discusses the ways that these words entered the mathematical language. He posits that proof was evolved as a shift of the word ‘show’: from making something literally visible to making “the truth (or falsity) of a mathematical statement visible in some [not necessarily visual] way” (p. 189). The movement towards showing the non-perceptual constitutes the quest for the commonly accepted *logos*, since the argument was not bounded from the subjective perception, but laid within a conceptual extension of the perceived reality, thus allowing the discussion of, for example, dimensionless objects (points) and unidimensional objects of infinite length (lines).

In addition, the quest for achieving the widest acceptance of an argument may have resulted in constructing geometry from the fewest commonly accepted truths and logic. The term ‘common notion’ may draw our attention to the common understanding of the humans’ bodily experience. Hence, Euclidean Geometry may be viewed as the result of the ways that world perceptually appears to humans (visual, straight-looking lines). Common perceptually derived notions, objectified reasoning and quantified qualitative relationships may have been some of the elements that lead to the objectification of the mathematical argument through the notion of proof. It required many centuries before it reached its Husserlian ideal form, by challenging the a priori perceptually derived necessities within which geometrical ideas were built, thus allowing for the mathematical ideas to be constructed within whichever framework the mind chooses: axiomatic logically reigned worlds of infinite possibilities and choices. This allowed for mathematical counter-intuitive objects (such as the Weierstrass function) to be constructed/defined and their inescapable (within the chosen framework) consequences could be revealed through an anthropologically objective proof that transcends both the human bodily experience and its idealised extensions.

## **FOSTERING THE STUDENTS' NEED FOR GEOMETRICAL PROOF**

In which way such a reading of history may inform a pedagogy aiming to foster the students' internal need for proof? The central phenomenological idea is summarised in going "back to 'the things themselves'" (Husserl, 2001, p. 168). From our perspective, this implies a pedagogical design within which the students' 'natural attitude' (for example, not to prove something obvious) can be suspended, allowing the possibility for the reactivation of the intentions that lead to the genesis geometrical proof.

We posit that at the crux of the students' appreciation of proof lies their acknowledging the existence of commonly accepted notions within which the geometrical objects are built. For example, these common notions may be related to the shared bodily perceptions and may be viewed as mental ideal extensions of the perceived world (cf. Lakoff & Núñez, 2000). The importance of the practical use of geometrical ideas may also strengthen the students' need for addressing a mathematical idea. In this way, the students may build a coherent framework deductively linking commonly accepted 'perceived' notions with the geometrical ideas. The need for proof may still not be evident, but we argue that by acknowledging that the geometrical ideas are corollaries of a priori notions, which, though related with human experience, are clearly beyond the reach of human perception, the students would be in the position to take the next step: to challenge these notions, to allow their mind to choose an alternative a priori framework within which reason can act. Acknowledging the a priori framework may internally justify the metacognitive processes of a 'what if ...' analysis to be applied. Thus, the need for proof derives from the metacognitive ability to identify and question the sedimented within language premises of the geometrical object. Harel's (cited in Zaslavsky et al., 2012) suggestion to teach 'neutral geometry' fits with this perspective, embodying a way of investigating alternative frameworks and their consequences. Within a mind game of choosing different a priori combinations, proof becomes the only way of accessing and evaluating the mathematical truth of the assertions, since the constituting framework is not perceptually derived, nor reachable. Language and symbolism allows the communication of those ideas, while reasoning and the chosen a priori are the bedrock upon which the truth of an assertion is proved. Hence, the students may extend their awareness of the transcendental as a conceptual non-perceptually bounded experience of the mind.

In this essay, we discussed the contributions of the phenomenological method in reading the sociocultural factors that lead to the genesis of geometrical proof, in gaining deeper understanding about the psychology of the proving subject, as well as in appropriately designing pedagogies aiming to foster the students' internal need for proof. Zaslavsky et al (2012) note that in school "proof establishes truth rather than validates assertions based on agreed axioms" (p. 219), masking the fact that "modern mathematicians adopt axioms or hypotheses without perceiving them as evident or absolutely true" (p. 218-219). We argued that the transcendental aspect of Husserl's ideas may contribute in a framework for addressing the internal need for 'structure' of proof. In phenomenological analyses "sedimented thought must be reactivated and its meanings revived" (Audi, 1999, p. 406). 'Going back to the things themselves' allows

the metacognitive processes to challenge not only the solution of a given problem, but the very fabric upon which the problem is posed and solved. We posit that Husserl's phenomenology allows for the identification of the anthropological character of the complex processes that necessitate employing proof in learning mathematics, as well as their meaningful convergence within a coherent framework, incorporating semiotic structures, sociocultural conventions and subjective understandings.

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# **AN EVALUATION OF THE AUSTRALIAN 'RECONCEPTUALISING EARLY MATHEMATICS LEARNING' PROJECT: KEY FINDINGS AND IMPLICATIONS**

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*The Pattern and Structure Mathematics Awareness Project (PASMMap) has investigated the development of patterning and early algebraic reasoning among 4 to 8 year olds over a series of related studies. We assert that an awareness of mathematical pattern and structure (AMPS) enables mathematical thinking and simple forms of generalization from an early age. This paper provides an overview of key findings of the Reconceptualizing Early Mathematics Learning empirical evaluation study involving 316 Kindergarten students from 4 schools. The study found highly significant differences on PASA scores for PASMMap students. Analysis of structural development showed increased levels for the PASMMap students; those categorised as low ability developed improved structural responses over a short period of time.*

In our PME 28, 29 and 30 research reports we describe a broad descriptive study of 103 first graders and 16 longitudinal case studies that found children's perception and representation of structure generalised across a wide range of mathematical domains. Children's strategies showing use of pattern and structure were determined from task-based interviews. A high positive correlation (0.944) was found between children's performance on 39 Pattern and Structure Assessment (PASA) tasks, and four stages of structural development: pre-structural, emergent, partial, and structural. Multiplicative structure, including unitising and partitioning, and 'spatial structuring', were found as critical to development of pattern and structure. It was found not only that each student tended to show a single structural level in all their responses, but also that this level was strongly correlated with the total number of correct responses. Mulligan and Mitchelmore (2009) postulated the existence of a general construct called Awareness of Mathematical Pattern and Structure (AMPS). They therefore argued that AMPS could be measured using the PASA interview, and that AMPS was indeed associated with mathematical understanding.

At PME 32 we introduced a new evaluation study, Reconceptualising Early Mathematics Learning, describing the broad aims, design and instruments and pilot work. The purpose of this paper is to provide a summary of the key findings of the project: the implementation of a structural approach to early mathematics learning through the Pattern and Structure Mathematical Awareness Program (PASMMap) and measured AMPS by the Pattern and Structure Assessment (PASA) interview. The theoretical bases for our study and background studies are highlighted in our recent volume (Mulligan, English, Mitchelmore & Crevensten, in press) and in our related PME37 symposium presentation, Reconceptualizing Early Mathematics Learning.



## THEORETICAL PERSPECTIVE

Virtually all mathematics is based on pattern and structure. By mathematical *pattern*, we mean any predictable regularity involving number, space or measure. Examples are friezes, number sequences, units of measure and geometrical figures. By *structure*, we mean the way in which the various elements are organised and related. Thus, a frieze might be constructed by iterating a single “unit of repeat”; the structure of a number sequence may be expressed in an algebraic formula; and the structure of a geometrical figure is shown by its various properties. Structural thinking can emerge from, or underlie mathematical concepts, procedures and relationships. Mason, Stephens and Watson (2009) view structural thinking as more than simply recognising elements or properties of a relationship but having a deeper awareness of how those properties are used, explicated or connected.

### Early childhood research on pattern and structure

There is an increasing body of research into young children’s structural development of mathematics and early algebraic reasoning. Research in the area of number (Hunting, 2003; Mulligan & Vergnaud, 2006; Thomas, Mulligan & Goldin, 2002; van Nes & de Lange, 2007), patterning and reasoning (Clements, Sarama, Spitler, Lange, & Wolfe, 2011; English, 2004; Papic, Mulligan & Mitchelmore, 2011), spatial measurement (Outhred & Mitchelmore, 2000), early algebra (Blanton & Kaput, 2005; Carraher, Schliemann, Brizuela, & Earnest, 2006; Warren & Cooper, 2008), and data modelling (English, 2012) have all shown how progress in student’s mathematical understanding depends on a grasp of underlying structure.

A suite of studies by Mulligan and her colleagues (Mulligan, 2009) suggested that children who have developed an awareness of structure in one aspect of the early mathematics learning also tend to show a structural awareness in other aspects.

The questions naturally arise, is it possible to improve students’ AMPS by an appropriate intervention, and if so, does their general mathematical achievement also improve? Mulligan and colleagues developed a Pattern and Structure Mathematics Awareness Program (PASMAMP) that focuses explicitly on raising primary school students’ awareness of mathematical pattern and structure via a variety of well-connected pattern-eliciting experiences. Studies have included an extensive, whole-school professional development exercise across Kindergarten to Year 6; two year-long, single teacher studies in Years 1 and 2; and an intensive, 15-week individualised program with a small group of low-ability Kindergarten children (For details, see Mulligan, 2009). Many individual cases have been documented showing marked changes in children’s structural awareness and development of mathematical concepts well beyond that expected for their age level. Some evidence has emerged that PASMAMP also has an effect on their scores on independent mathematics assessments. More importantly the PASMAMP aims to promote simple or ‘emergent generalisation’ in young children’s mathematical thinking across a range of concepts.

The studies cited above lend strong support to the hypothesis that teaching young children about pattern and structure should lead to a general improvement in the

quality of their mathematical understanding. However, none of the studies had a sufficiently large or representative sample, most lacked a comparison group and there was insufficient opportunity to track and describe in depth, the growth of structural development. The current study was therefore designed to evaluate the effects of PASMMap on student mathematical development in the first year of formal schooling.

## **METHOD**

**Participants:** A purposive sample of four large primary schools, two in Sydney and two in Brisbane, representing 316 students from a diverse range of socio-economic and cultural contexts, participated in the evaluation throughout the 2009 school year. Two different mathematics programs were implemented: in each school, two Kindergarten teachers implemented the PASMMap and two implemented their standard program. The PASMMap framework was embedded into the standard Kindergarten mathematics curriculum, enabling schools to meet the required system-based learning outcomes for New South Wales and Queensland, respectively.

**Procedure:** Two different mathematics programs were implemented. In each school, two Kindergarten teachers implemented the PASMMap and two implemented their regular program. A researcher visited each teacher on a weekly basis and equivalent professional development was provided for all teachers. The PASMMap framework was embedded within but almost entirely replaced the regular Kindergarten mathematics curriculum. Features of PASMMap were introduced by the research team incrementally, at approximately the same pace for each teacher, over three school terms (May-December 2009).

### **Assessment Interviews and Classroom Data**

All students were administered the I Can Do Maths (ICDM) standardized test of general mathematics achievement (Doig & de Lemos, 2000) at the beginning and end of the 2009 school year and again in mid-2010. From the pre-test data, two focus groups were selected in each class consisting of five students from the upper and lower quartiles, respectively. These students were interviewed in more detail using the PASA in February 2009, December 2009, and September 2010, the number of students varying from 190 to 170. An additional “extension” version of PASA was also administered in September 2010. The PASA items were parallel on all three occasions, but increased in complexity to take account of students’ development.

Other evaluation data included video for a sample of PASMMap lessons for evidence of AMPS and students’ articulation of emergent generalizations. Analysis focused on the high ability and low ability focus students. Students’ explanations and drawn representations, and photos of their responses to tasks were collected during the implementation of PASMMap and were coded immediately after each lesson.

## RESULTS

### *Quantitative outcome analysis*

Initial analysis of assessment scores confirmed the equivalence of the two program groups. The expected differences between ability levels across groups were confirmed through the analysis of PASA and ICDM scores. Students in the two Brisbane schools scored lower overall than those in the two Sydney schools, showing significant differences between states. No significant interactions were observed.

Source	Type III SumSquares	df	Mean Square	F	Sig.
Corrected Model	1048.432 <sup>a</sup>	17	61.672	10.380	.000
Intercept	53.229	1	53.229	8.959	.003
Covariate: PASA	158.346	1	158.346	26.650	.000
Covariate: ICDM	14.071	1	14.071	2.368	.126
School	117.125	3	39.042	6.571	.000
Ability	15.259	1	15.259	2.568	.111
Treatment	61.653	1	61.653	10.376	.002
School * Ability	11.643	3	3.881	.653	.582
School * Treatment	43.663	3	14.554	2.450	.066
Ability * Treatment	.217	1	.217	.037	.849
School * Ability * Treatment	13.589	3	4.530	.762	.517
Error	802.130	135	5.942		
Total	13412.000	153			
Corrected Total	1850.562	152			

R Squared = .567 (Adjusted R Squared = .512)

Table 1. Analysis of covariance of PASA scores at retention point

The PASA and ICDM were administered post PASMMap intervention (December 2009) and at the retention point (September 2010). Total scores on each test for focus students were analysed using analysis of covariance (ANCOVA). In each case, the covariates were the initial PASA and ICDM scores and the factors were school (one of four), ability (high vs. low) and program (PASMMap vs. non-PASMMap).

It was expected that the ICDM test scores would not provide any relevant measure of pattern and structure; these data indicated no significant interactions or main effects apart from a school effect. Essentially, PASMMap and regular students made very similar gains on ICDM over the period of the study, but Sydney students gained more. The analysis of the PASA scores also showed no significant interactions. However, there were two significant main effects at each point: a difference between schools, with the Sydney classes showing higher adjusted means than the Brisbane classes, and a difference between the program groups on each PASA assessment—modest at the

end of the intervention ( $p < 0.026$ ), highly significant at the retention point ( $p < 0.002$ ), but only borderline ( $p > 0.11$ ) for the extension section of the PASA. The most important finding was that the PASMMap group scored higher than the regular group at each interview point. Table 1 provides a summary of the ANCOVA for the PASA at the retention point, one year later when students were in Grade 1. We inferred that the PASMMap treatment was effective in promoting the structural development of key concepts in early mathematics, as measured by the PASA but not in improving mathematical achievement as measured by ICDM.

### **Rasch scale analysis**

The PASA total scores and the ICDM scores were used to construct a single Rasch scale that incorporated all items along a continuum. The main advantage of using Rasch analysis for constructing the PASA scale was that it could be used to link different versions of the PASA used in this study. The item map indicated that the PASA items and the students were reasonably well matched; in comparison, the ICDM items at the lower end of the scale did not sufficiently challenge the majority of students, although some more difficult ICDM items filled a gap between the PASA items. The scale's order of item difficulty on PASA items provided a measure of the students' overall level of AMPS. Thus a conceptual analysis of the item and its position on the scale reflected the complexity of the task in terms of pattern and structure as well as the reasoning required to complete it successfully. What we aimed to achieve with the scale was a picture of how the PASA measure of AMPS fitted with a standardized measure of general numeracy ability over time.

### **Structural outcomes analysis**

As an example of student learning student responses on four PASA items requiring a drawn response at the three administrations were systematically coded for level of structural development. Coding showed an inter-rater reliability of 0.91. Figure 1 summarizes the results for the Sydney students. It can be seen that the PASMMap students were initially slightly more advanced than the regular program students, with about 5% more students in the partial structure and structural levels than the regular students. However, this difference grew in the subsequent administrations, reaching about 20% at the retention point. Further details are provided in Mulligan et al. (in press).

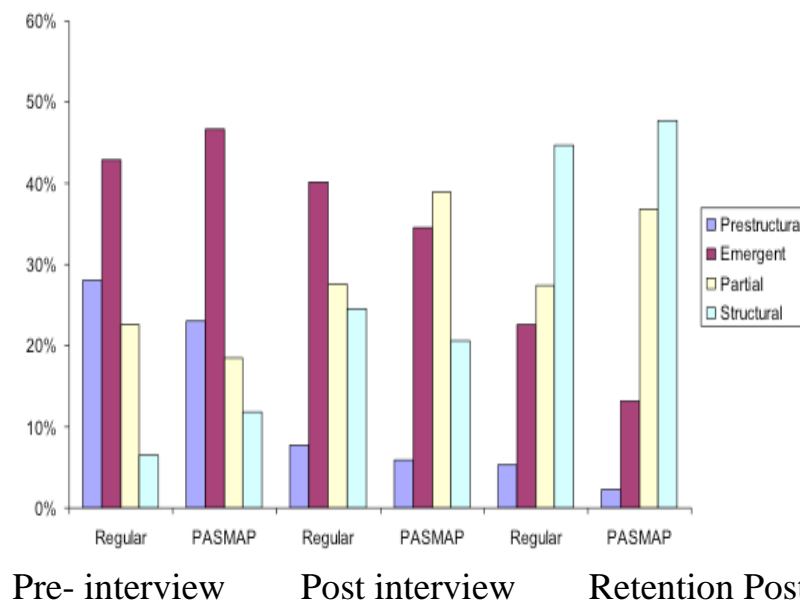


Figure 1. Structural development across selected PASA items at (Pre-intervention), (Post-intervention), (Retention) in two Sydney schools.

## DISCUSSION

The study produced a valid and reliable interview-based measure and a scale of AMPS that revealed new insights into students' mathematical capabilities at school entry. Clearly young students were able to solve a broad range of novel mathematical tasks, including repetitions and growing patterns, and multiplicative problems, not usually asked of students of this age.

PASMAT explicitly focused on the promotion of students' awareness of pattern and structure (AMPS). Particular gains were noted in the related areas of patterning, multiplicative thinking (skip counting and quotition), and rectangular structure (regular covering of circles and rectangles). As expected, a focus on pattern, structure, representation, and emergent generalisation advantaged the PASMAT students. However, students in the regular program were also able to elicit structural responses but had not been given opportunities to describe or explain their emergent generalised thinking that may have been developing. Thus, it was not possible to determine whether more advanced examples of structural development could be directly attributed to the program or innate developmental advances of more able students. One of the most promising findings was that the focus students categorised as low ability were able to develop structural responses over a relatively short period of time. Further analysis of the impact of PASMAT on structural development must consider individual teacher effect and school-based approaches to evaluate the program's scope and depth of achievement.

Our research has established that the development of children's mathematical thinking can be described in terms of a growing awareness of pattern and structure. We have shown that children's levels of structural development can be reliably categorised, and that individuals tend function at same level across different conceptual areas on tasks

that measure pattern and structural development. This finding confirmed the existence of our proposed construct of Awareness of Mathematical Pattern and Structure (AMPS) which is prominent in children who achieve highly in mathematics in school and low in those who do not progress easily or develop learning difficulties. We regard the AMPS construct as a significant contribution to research into early childhood mathematics education. It provides a lens with which to examine children's thinking at a fundamental level and, in particular, to assess the deeper effects of early mathematics teaching.

## FURTHER RESEARCH

We aim to explore further aspects of AMPS: the possibility that low AMPS in early childhood could predict poor performance in mathematics throughout schooling, particularly in relation to algebraic thinking. Extending the AMPS construct to the later years of schooling will involve studies of learning trajectories of students beyond the early years of schooling whose mathematical and scientific reasoning is enhanced by a structural approach. Our interest also lies in the application of the PASMAT approach to assisting those students with special needs, students with low levels of AMPS who may be prone to difficulties in learning mathematics, and students with advanced AMPS who are gifted at mathematics (Mulligan, 2011).

A new phase of the research program is currently in progress, *Transforming Children's Mathematical and Scientific Development*, enabling the extension and application of this study utilising the same research team. This 3-year longitudinal study integrates the PASMAT pedagogical approach through novel experiences in data modelling and problem solving linked to the work of English (2012). An emphasis is placed on developmental features of how students structure data. The study tracks three cohorts of students initially employed in the *Reconceptualizing Early Mathematics Learning* project when in Kindergarten, through to Grades 2, 3 and 4. Two new cohorts of mathematically able students are being tracked from Kindergarten to Grade 2. Other research applications include the *Patterns and Early Algebra* (PEAP) *Professional Development* (PD) *Program* (Papic, in press) which focuses on young children's patterning, early algebraic and mathematical thinking skills with the aim of closing the gap in numeracy achievement for Indigenous children in rural and regional early childhood settings in New South Wales.

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# **MULTIPLE LENSES FOR LOOKING AT TEACHERS' IDENTIFYING TALK IN A MATHEMATICS CLASSROOM**

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*Previous research under the communicational framework has started developing systematic methods that unify the study of cognitive, affective and social aspects of learning under a single set of theoretical assumptions. This study enhances the communicational methodology into the quantifiable domain, thus enabling analysis of large quantities of classroom talk and comparisons between teachers to study identity construction processes in the mathematics classroom. We demonstrate three forms of discourse analysis: qualitative, 'coding and counting' and corpus analysis to analyze the ways in which two instructors at a teachers' college differ in their "identifying talk" or the way they construct their students' identities. Differences between the two instructors were ostensible in all forms of analysis.<sup>1</sup>*

## **INTRODUCTION**

Many studies have shown that the effectiveness of teaching in a mathematics classroom is determined not only by content (or mathematical) actions of the teacher (e.g. Skinner & Belmont, 1993). Rather, issues characterized as 'affective' or 'social' relationships of the teacher with her students are equally and some would say even more important for the effectiveness of her teaching. In the present study, we shall aim at operationalizing these interpersonal actions using a set of communicational tools, extracting them from the teacher's talk. Building on the communicational method for studying identity construction processes (Sfard & Prusak, 2005; Heyd-Metzuyanim & Sfard, 2012), the present study enhances the methodological toolset for studying into the quantifiable domain, thus enabling comparisons between different teachers and classrooms, and analysis of large quantities of classroom talk.

## **THEORETICAL BACKGROUND**

Previous research (Heyd-Metzuyanim & Sfard, 2012; Heyd-Metzuyanim, 2013) has started developing systematic methods for analysing affective processes as they take place in class through classroom discourse analysis. This "communicational" method has been developed with the need to unify the study of cognitive, affective and social aspects of learning mathematics under a single set of theoretical assumptions.

The main tenet of the communicational (commognitive) theory is that thinking can be viewed as a type of intra-personal communication, not qualitatively different from inter-personal communication (Sfard, 2008). Sfard & Prusak (2005) enhanced the communicational theory of learning to the domain of identity, defining it as "[a] collection of reifying, endorsable and significant stories" (p. 16). By defining identity

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in plural (stories), Sfard and Prusak highlighted the social aspect of identity. Their definition is lined with contemporary views of identity as socially constructed within a certain set of culturally defined norms and available narratives (Wortham, 2005).

The collection of stories that form a persons' identity includes several types of narratives. Usually when studying identity, researchers rely on *1<sup>st</sup> person* stories elicited by interviews and questionnaires. And yet, when looking for the ways in which teachers motivate and engage their students, it is more reasonable to look at the *2<sup>nd</sup> P stories* told by the teacher to her students about them ("you're doing great"). In the present study we look at how two instructors at a teachers' college engaged their students in mathematical learning, by analysing *2<sup>nd</sup> person* identity stories.

The definition of identity as a collection of stories has been productive for operationalizing this concept, especially in studies that look at it as a product. However, it is not sufficient for analysing the *processes* by which identity is constructed. Heyd-Metzuyanim and Sfard (2012) have introduced a new set of conceptual tools for studying *identifying* talk, defined as talk about what people *are*, or their stable properties and memberships. The main division they used was between two processes that occur concurrently in almost every learning situation: *mathematizing* (talking about mathematical objects) and *subjectifying* (talking about the participants of the discourse). E.g., "X minus one divided by three" is a mathematizing utterance while "let her (do it) alone, she's smart enough" is a subjectifying utterance. To pick out identifying from subjectifying talk Heyd-Metzuyanim and Sfard suggested three categories of classification: *specific* subjectifying, which refer to specific actions of the participants; *general* participation evaluations which refer to what participants normally do ("you *always* come late to class"), and statements about *properties* or membership categories of the participants ("you are grownups" or "you are prospective teachers"). Heyd-Metzuyanim and Sfard claimed that a student's identity profile can be re-constructed from these different levels of subjectifying utterances made in class, in particular, from the *2<sup>nd</sup>* and *3<sup>rd</sup>* level utterances. In order to deal with the fact that much of the talk about participants is often not stated explicitly because of social norms, they differentiated between *direct* identifying which talks about the participants explicitly and *indirect* identifying, which tells something about a participant without mentioning her explicitly. Though this categorization system provided a step forward towards operationalizing affective and interpersonal communication in the classroom, it has not been suitable for quantitative analysis and analysis of large excerpts of data since it lacked an exhaustive set of criteria for classifying each and every utterance made in the classroom. Thus it was unsuitable for comparing different teachers and classrooms. In this study, we aim to extend the communicational method for studying identifying processes to allow such quantitative analysis. Also, we explore the possibility that corpus-analysis tools can provide an even wider view on a teacher's identifying talk over a period of two whole lessons.

## METHOD

### Data collection and participants

The data for this paper were taken from an ongoing project that focuses on identifying learning and teaching processes in a first-year course about functions, for prospective elementary school teachers studying at a college of education in Israel. The study included two similar courses taught by different instructors in consecutive years. In each course, 11 out of 12 lessons of the course were videotaped and transcribed. During the first lesson in each course the study was described and consent forms were obtained from all participants. During the lessons, students' written work was collected and field notes were taken by two of the authors that were present at all lessons.

In each course, the research population included a group of about 20 prospective elementary school teachers. The course was a part of their first year requirements. All participating students had already learned functions and graphs in high school algebra and had passed the matriculation exam in mathematics after finishing their 12<sup>th</sup> grade. That is, in this course the students studied functions for the second-time. All the students were older than 19, and came from various socio-economic backgrounds. Both course instructors, Ellice and Talli (first author), have a PhD in mathematics education and have been teaching in this college for over 10 years.

The work in the course was organized mostly in small groups, followed by whole class discussions. It focused on connections among various mediators of linear functions: verbal, graphic, numeric and symbolic. All the students scored high grades in the final exam (average of 90) relative to grades earned at similar courses in the college.

### Method of analysis

We used three forms of discourse analysis that yielded a multiple view on the ways in which the identifying talk of two instructors differs: qualitative analysis of specific pedagogical moves, coding and counting identifying talk, and using corpus analysis to count pronoun use by the instructors.

Regarding the coding of identifying talk, Heyd-Metzuyanim and Sfard's (2012) framework provides relatively straightforward criteria for classifying direct subjectifying utterances. However, exhaustive criteria for classifying indirect identifying have not been provided in their work. We dealt with this lack by relying on Searle's (1985) Speech Act theory for classifying direct and indirect speech acts. Indirect speech acts were defined by Searle (1985, p. 13) as those which, in addition to carrying a literal meaning, also have another *illocutionary force*. For instance, when a teacher states, after a student has erred "we're always learning from mistakes in life" she not only describes a general feature of human learning, but also indirectly assures the student that her mistake would not identify her in an unfavourable way. In other words, though her utterance talks about "we", she indirectly talks about "you". Relying on Searle's distinction, we thus attempted to articulate the indirect message in each utterance that seemed to have an identifying message but did not explicitly state

something about “you”. As such explication is interpretative, we relied on coding of the three authors done separately.

One of the main issues that had to be tackled when attempting to quantify identifying talk was that of segmentation. Since we looked only at the instructor's talk, there was not much point in segmenting according to turns. Neither was there a point in merely counting words, since identifying messages is often beyond the word. Inspired by conversation analysis and linguistic studies, we found that the most useful segmentation would be according to *clauses* (see excerpt 1 [421], in which the separation to clauses is signalled with [a], [b], etc.). The advantage of segmenting into clauses lies in its relative simplicity (which relies on grammatical features of the utterance) and the fact we found most clauses to be relatively “pure” in terms of subjectifying vs. mathematizing content. That is, clauses could usually be classified readily as mathematizing or subjectifying.

## FINDINGS

Scanning of the first few lessons of the two instructors, revealed that Ellice's lessons included what we termed “speeches” in which she explicitly identified the students, while such speeches were less frequent and much shorter in Talli's talk. We start out by examining and comparing these relatively short episodes.

### Looking at identifying activity in short excerpts of the instructor's talk

Two examples of identifying discourse, taken from Ellice and Talli's talk in whole classroom discussion in lessons 2 and 3, were chosen for comparison. They were chosen since they enacted similar pedagogical moves – assuring the students that it was OK to err or “feel lost”, and motivating them to hand in their homework assignments. Let us first look at the way in which Ellice reacted to students that declared they “got lost” during her mathematical explanations.

#### Excerpt 1: Ellice assuring the students it's OK to be “lost”

- 418            Ariel     Ellice, I'm a bit lost [more students complain that they didn't understand]
- 421            Ellice     No problem, very good [a]. Who's got lost on me? [b] Alright, **you'll get back to me** [c]. **Don't worry** [d]. **It's excellent that you're getting lost** [e]. *'Cause where we're getting lost, when we find (ourselves), so there it sticks with us the best* [f]. Look [g], I now take a time-out of 30 seconds [h]. *My lessons are all built this way, on tasks* [i], *on some sort of thinking activity of yours and discussion* [j]. **And where you get lost, I promise you that there afterwards it will stick the best** [k]. (To Ariel) You'll (singular) find yourself [l]. If not I'll help you [m]. **Now it's great that you tell me [n]. It's very important for me that you tell me** [o]. *When we finish the discussion on this task completely then if there still be someone who feels that this task is not completely* [p]– **Not everything has to stick with you** [q] *but the things that we talked about them* [r], *so either we come back to it* [s] *or I'll be happy to sit with her alone* [t]. **But it will come back to you in a short while** [u], **don't worry** [v].

**Bold:** direct identifying; *Italics:* indirect identifying; underline: talk about feelings.

Talli arrived at the need to reassure students about making mistakes during lesson 2, following a student's question that showed she had misunderstood Talli's explanations:

- 544 Student He who doesn't ask doesn't know. I've learned that from personal experience
- 545 Talli Good.[a] *We're always learning from mistakes in life.*[b] *It's totally OK to be wrong.*[c] **It's very important that you talk about the stuff** [d] so if there is- [e] *(so) we will be really able to make a change* [f] *and see where the problem is* [g] *so we don't get stuck there.* [h]

Though the two excerpts have a similar message, they are very different in their "linguistic style" (Pennebaker & King, 1999). The first obvious difference is the length (number of words). Where Ellice takes 178 words, organized into 22 clauses to deliver her message, Talli does it in 46 words and 8 clauses. Another difference lies in the directness of subjectifying addressed at the students: where 9 out of 22 clauses (40%) of Ellice's talk directly subjectifies the students (using the pronoun 'you'), only 1 out of 8 clauses (12%) of Talli's subjectifying is direct. Ellice's talk is amplified through several means: her use of relatively extreme evaluative language ("excellent", "very important", "great", "it will stick with you **the best**"); repetitions ("don't worry" [d] & [v]; "it's great that you tell me" [n] & [o]); use of temporal words ("**all** my lessons", "it will come back to you in a **short** while"). Talli's talk is amplified too ("always", "totally", "very" and "really") but without repetitions. Finally, Ellice's talk is rich with emotional references. This can be seen both in her direct talk about emotions and feelings ("don't worry", "I'll be happy") but even more so in her *meta-emotional* talk ("it's great that you tell me") in which she indirectly says something like "it's good that you're not ashamed to talk about feeling lost". Any reference to feelings or meta-emotional talk is absent from Talli's talk.

In the above examples, the major difference between the instructors was in the *style* of their 2<sup>nd</sup> person identifying talk. The next two excerpts are rather different in their message. The similarity between them, which makes them suitable for comparison, is in their overall goal: convincing students to hand in their homework.

### Homework – Ellice

- Ellice Naomi and Idit raised a point that I just wanted to [a] **I'm glad that you've raised it** [b]. First, **the tasks that you hand in are without any grade** [c], (that's how) it is written in my (books) [d], *and I don't remember the names so no problem* [e]. It's written in my (book) 'handed in' [f]. *So let's suppose someone did* [g], *I would have had let's say the grade that someone handed in that is worth 5 points out of 100* [h] *and I marked it as 'handed in'* [i], *I have it in my computer only as 'handed in'* [j]. A 'V' [k]. *Even if someone could have gotten a zero and someone should have gotten a 100, she got a 'V'* [l]. *The same 'V'* [m]. *I don't remember that well the names in any case* [n] **and 5 minutes after you finished it I don't remember who's done well and who not** [o]. **So you don't have a problem with it** [p]. *It's my senility so everything is alright* [q]. (Lesson 3, turn 3)

## Homework – Talli

Talli I'll hand it back, you'll get it back, alright? **Do it** [a], it's for, *now about handing in* [b], *the grade* [c], **you are not graded for its value** [d], **for the quality of the work** [e], **for the correctness of your answers** [f], **rather you're graded for handing in or not handing in** [g]. *I must know where you stand* [h]. Alright? **I want you to work at home** [i] **and if you need to hand it in then you'll work** [j]. *Besides, I wanna see what, what's happening with you in the meantime* [k], alright? (Lesson 3, turn 1187)

Again, we see a clear difference in style. Ellice's talk is interwoven with emotional expressions whereas Talli's has no reference to feelings or "problems". Yet the main difference between the two excerpts is thematic. Both start out with a similar claim: "You are not getting a grade (on the homework assignments) so.." yet they derive from it different conclusions. While Ellice emphasizes that handing in homework assignments would not *identify* the students in her eyes in any way (since she's "senile" and "doesn't remember the names") Talli does not refer to the students' identities at all. She simply states that it is necessary for the teaching-learning process that she knows "where they stand" mathematically. Thus, she makes it clear that the homework will identify the students' *mathematical skills* yet does not attach emotional value to that.

## Looking at identifying activity during two whole lessons

We now zoom out to the whole-lesson timescale attempting to capture the general identifying activity happening throughout lessons 2 and 3 in both courses. Those two lessons were chosen because we believe the first lessons to be indicative of the ways in which a teacher first approaches her students and attempts to engage them in mathematical discussions. The method of analysis here included first extracting all the identifying talk (general participation evaluations and properties/membership attributions) from the two instructors' talk addressed at the whole classroom. We then counted the words included in clauses that were coded as "identifying" and calculated the frequencies of this word-count against the total word count of the instructor's talk. Table 1 summarizes the frequencies of words in identifying clauses in the two lessons.

	Ellice	Talli
Lesson 2	616/4649 (13.2%)	27/5188 (0.5%)
Lesson 3	433/4059 (10.6%)	106/4637 (2.0%)

Table 1: The number of words in identifying clauses uttered by the instructors

As seen in table 1 there is a notable difference between the frequencies of identifying clauses in the two instructors' talk. This finding is coherent with the findings of the event-scale analysis, giving another indication for the stronger emphasis that Ellice put on identifying than Talli. And yet, this analysis only pertains to a very small portion of both instructors' talk. We therefore wanted to get a view of the differences in the ways that the two instructors positioned themselves and the students through their talk.

## Corpus analysis in the service of analysing identifying talk

Though direct identifying talk may be quite rare in teacher's talk, we assume that identifying activity is an ongoing process. However, catching this quality, which is mostly indirect, is not a straightforward task. One of the tools that may assist is looking at pronouns, through which much of the talk about participants is achieved. As pronouns are spread throughout the talk, this examination has the advantage of characterizing a much greater portion of the instructor's talk. We thus counted the frequencies of these pronouns in the instructors' talk: "you" (plural), "we" and "I".

	Lesson 2: Talli		Lesson 2: Ellice		Lesson 3: Talli		Lesson 3: Ellice	
You <sup>2</sup>	26	0.50%	89	1.91%***	63	1.36%	71	1.75%
We	71	1.37%	37	0.80%**	65	1.40%	32	0.79%**
I	46	0.89%	148	3.18%***	43	0.93%	111	2.73%***
Total*	5188		4649		4637		4059	

\*Total number of words uttered by the instructor during the lesson

Significance of difference between Talli and Ellice for the same lesson: \*\*  $p < 0.01$ ; \*\*\*  $p < 0.0001$

Table 2 - Comparison of Lessons 2-3, Ellice and Talli, in pronoun use.

Table 2 shows that the patterns of pronoun use were significantly different between the two instructors in almost all the comparisons, for the same lesson (2 or 3). While Ellice used "I" and "you", Talli mainly used "we". Almost all differences were found to be significant, based on the Log Likelihood test (Rayson & Garside, 2000), a variant of the Chi Square test that does not assume normal distribution of data.

## DISCUSSION

The aim of this study was to develop operational and quantifiable methods for capturing identifying activity in the classroom. Since this activity is very complex, the first step we took was to zoom in and examine it in short episodes of talk that identifies the students. This examination yielded a clear difference between the two instructors. Whereas Ellice's talk included a lot of direct identifying clauses, many of which were interwoven with emotional expressions and amplified talk, Talli's excerpts were relatively unemotional and identified the students only indirectly. In order to gain a wider view of the instructor's identifying talk we then coded and counted all the identifying clauses in two lessons. Again, we found that Ellice's talk had a significantly higher frequency of identifying clauses than Talli's talk. Finally, we looked at the general quality of the instructor's identifying activity through examining their use of pronouns in the two lessons. Though it is not completely clear what part of the differences in "linguistic style" (Pennebaker & King, 1999) is a matter of stable attributes of the instructor (commonly referred to as her "personality") and what is a

<sup>2</sup> This 'you' refers only to the plural you in Hebrew ('Aten' and 'Atem'). Singular 'you' was not included in searches.

matter of differing teaching styles, the present case shows that corpus analysis of pronoun use may, in fact, show differences in patterns of identifying talk. The results showing greater use of “I” and “you” in Ellice’s talk and greater use of “we” in Talli’s talk are congruent with the findings of the two former analyses in which Ellice used more identifying talk than Talli. Whereas “I” and “you” may construct clear differences between the identities of the instructor and those of her students, the use of “we” blurs this distinction. By providing this set of methods for analysing identifying talk, we attempt to overcome the inherent difficulty in analysing “affective” and interpersonal processes in the classroom, which are often difficult to capture and compare using a single method.

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# **PRESERVICE TEACHERS' VIEWS OF THEIR MATHEMATICAL PREPARATION IN ADVANCED MATHEMATICS**

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*This paper reports on a study about the mathematical preparation of high school preservice teachers through advanced mathematics courses. Interviews were conducted with 7 preservice teachers registered in a teacher education program in which the mathematical preparation consisted of advanced mathematics courses. The intention was to better understand, from the preservice teachers' point of view, how they experienced this mathematical preparation and how it prepared them to become mathematics teachers. The analysis reveals a complex interplay of experiences and issues, which are addressed and combined with current literature findings.*

## **RESEARCH PROBLEM**

A number of teacher education programs orient the mathematical preparation of their high school teachers through delivering advanced mathematics (AM) courses. Albeit this model has been amply critiqued (CBMS, 2001; Moreira & David, 2005, 2008; see our review in Proulx & Bednarz, 2008), research is still lacking to understand in fine details what preservice teachers (PT) experience through this mathematical preparation for becoming teachers. Most of what is known about these experiences offers divergent results, ranging from possibilities of diverse reinvestments in teachers' practices to ruptures with that practice. These are discussed below.

### **Reinvestments of AM in preservice teachers' practice**

Even (2011) and Zazkis and Leikin (2010) have taken an interest in better understanding what teachers reinvest from their AM preparation in their classroom practice. They have highlighted two types of reinvestments: at the content and the metamathematical levels. At the content level, teachers explain reinvesting in their practice the content of their AM courses that is closely aligned to the content of the curriculum they teach (e.g. derivatives, limits, asymptotes). Beyond content, at the metamathematical level, teachers explain other sorts of reinvestments of their AM courses for their teaching practice, such as: (1) manners of teaching and doing mathematics of their professors; (2) sensibilities toward their students' mathematical difficulties, gained through experiencing similar difficulties in AM; (3) larger panorama of what mathematics is; (4) establishment of links between mathematics and daily life.

These reinvestments being considered, they can also be contrasted with what other researchers have pointed to, that is, the presence of ruptures between PT mathematical experiences in AM and in their teaching practice.



## Ruptures between AM preparation and teaching practice

A number of researchers have put forth the notion that PT mathematical experiences in AM are ‘disconnected’ from the mathematical experiences of high school mathematics teaching. In a review (Proulx & Bednarz, 2008), we point out three dimensions of this rupture. The first rupture refers to the much bigger and central place that formalism and symbolism occupy in AM in comparison to high school mathematics (e.g. Corriveau & Tanguay, 2007). The second rupture concerns the compressed nature of AM concepts, where it is argued that high school teachers need not to compress but to decompress mathematical concepts in their teaching (e.g. Adler & Davis, 2006; Ball & Bass, 2003; Huillet, 2007). The third rupture concerns the way in which mathematics is taught in AM courses, generally in a lecturing mode (Bauersfeld, 1998; Burton, 2004), an environment that appears at odds with the one wished for in high school mathematics classrooms (NCTM, 2000).

In light of these various outcomes, from reinvestments to ruptures, emerges a need to better understand the lived-experienced of PT. This is the goal of this study, aimed at better understanding PT experiences in AM courses and how they connect this experience to their mathematical preparation for becoming mathematics teachers.

## RESEARCH METHODS

To investigate the experience of PT in AM, semi-structured interviews were designed, composed of 5 questions and 3 tasks aimed at bringing out possible reinvestments and ruptures that PT may (have) experience during their mathematical preparation. The questions were: (1) What did you expect from your university studies? (2) What did you learn in your AM courses that was most significant for you? Were there any striking events? (3) Do you consider that your studies prepare you well to teach mathematics? (4) What are the advantages and disadvantages of studying AM as a preparation to teach high school mathematics? (5) Are you, to this day, satisfied with your mathematical preparation? The tasks represented events that could happen in a mathematics classroom (e.g. classroom vignette, student solution) and were inserted in the interview (at the end or when time was right) to enable PT to interact with mathematical issues emerging in the classroom and comment on them.

Two students solved this logarithmic equation and arrived at two different solutions. What happened? Are both strategies accurate? What would you answer these students?

$2\log_3(2x+10)=6$		$2\log_3(2x+10)=6$
$\frac{2\log_3(2x+10)}{2}=\frac{6}{2}$		$\log_3(2x+10)^2=6$
$\frac{\log_3(2x+10)}{1}=3$		$(2x+10)^2=3^6$
$2x+10=3^3$		$4x^2+40x+100=729$
$2x+10=27$		$4x^2+40x+100-729=729-729$
$2x+10-10=27-10$		$4x^2+40x-629=0$
$2x=17$		$(2x-17)(2x+37)=0$
$\frac{2x}{2}=\frac{17}{2}$		D'où
$x=\frac{17}{2}$		$x=\frac{17}{2} \text{ OU } x=\frac{-37}{2}$

Figure 1: Example of one of the three tasks given in the interview.

Participating PT were part of a 5-year teacher education program in a Canadian institution combining studies in AM (51 credits) offered in the science faculty with pedagogical studies offered in the education faculty (54 credits, with 3 practica). Seven students participated in the study (one in year-2, one in year-3, four in year-4, one in year-5). Interviews were audiotaped. A logbook was kept to write on-the-spot impressions. The data analysis was guided by the above-mentioned reinvestments and ruptures, and other emergent categories were considered in the process.

## FINDINGS

First, a note about the structure of the AM courses in this program. As the PT explained it, their courses were mostly in the form of lectures, where professors hold knowledge and transmit contents to students who take notes and practice exercises. This structure has significance in the understanding of the findings.

Given space constraints, four of the seven cases (one for each year) are presented: Chloe (year-2), Alexa (year-3), Josie (year-4) and Remi (year-5). A brief portrait for each is offered, highlighting how they are personally experiencing their studies in AM. These four cases are afterwards used to establish comparisons with current findings in the research literature in order to nuance, question, and complement current understandings about PT mathematical preparation through AM.

**Chloe** (year-2). For Chloe, AM courses focusing on high school content (e.g. matrix, derivatives, probabilities) are beneficial because they help her understand better these concepts and give her more confidence for teaching them. Other AM courses that work on contents aside of high school and that she will not be asked to teach are challenging for her, but beneficial as they render high school content easier for her. As well, the difficulties and challenges she experiences in her AM courses make her realize what helps her learn better (e.g. when professors offer memorization tricks, examples or drawings to make the mathematics more accessible, or her own memorization tricks). This sensitizes her to what she could do, as a teacher, to help her students when they are experiencing difficulties in mathematics. Regardless of her difficulties, Chloe is satisfied with her preparation through AM, which she sees as essential for a future mathematics teacher.

**Alexa** (year-3). Like Chloe, Alexa finds relevant the courses that allow her to work on high school content. However, she does not feel comfortable in her other AM courses, because she does not see how the contents are linked to the mathematics she will be teaching. For her, AM are not the ‘right’ mathematics, stressing the fact that she does not aim to be a mathematician, but a high school teacher. In her view, the ideal scenario would be to have mathematics courses that deepen her knowledge of high school mathematics (like preparing to answer pupils’ questions and solutions, in similar ways to the tasks offered in the interview). It is not that Alexa dislikes studying AM; she indeed quite enjoys it. However, she does not see how her AM courses are relevant to her preparation as PT, particularly because no links are made between high school mathematics and AM in her courses. After three years, Alexa does not feel mathematically ready to face a high school mathematics class.

**Josie** (year-4). As a PT, Josie shows ambivalence towards AM: she loves studying AM but feels it does not prepare her for the mathematics she will be teaching. Josie wants, in her teaching, to emphasize meaning in mathematics (the ‘why’ and not only the ‘how’) and does not feel that her mathematical preparation through AM prepares her for that. This leads to another aim: she wants to know how to teach mathematics through *other* means than lectures. She wants to teach in the way her education professors tell her (e.g. getting pupils to be active in their learning, favouring group work, differentiated activities). However, she does not know/see how to combine her education and mathematical preparation to do so, as both are worked in isolation.

**Remi** (year-5). For Remi, AM courses are beneficial because they gave him ‘tools’ for his future practice (e.g. anecdotes from his history of mathematics class, practice problems from his calculus class). However, what distinguishes him from the other PT is that beyond the mathematical content, Remi *learned to learn* new mathematics through his study of AM. In other words, through his AM courses, Remi developed mathematical reasoning and ways of doing that enable him to do or learn new mathematics. This leads Remi to feel comfortable learning and teaching any mathematics content in high school, even content for which he did not receive specific training. Thus, he is satisfied with his mathematical preparation, because he feels competent in (doing) mathematics and ready to teach it.

## LINKING WITH CURRENT FINDINGS ABOUT AM PREPARATION

In this section, we describe the study results in relation to current findings about teachers’ mathematical preparation through AM, and organize these around issues of reinvestments and ruptures mentioned above, accounting for other emergent issues.

### Reinvestments (and non-reinvestments) of contents

The PT in this study share similar views with teachers in Even (2011) and Zazkis and Leikin’s (2010) studies; which is that AM courses that go over high school contents are relevant for their mathematical preparation. In sum, these courses are useful because they help them better understand the contents, and they make them deepen their knowledge of the mathematics they will teach. E.g. Chloe explains that it helps her develop a mastery of the procedure, the ‘how’. Remi explains that these AM courses offer him tools to reinvest in his classroom, like a repertoire of problems a little harder than what he usually finds in textbooks or instructive anecdotes from his history of mathematics class to make his teaching more interesting.

Thus, the AM courses that help deepen PT knowledge of high school mathematics are ones that directly cover these contents. None of the seven future teachers mentioned wanting to reinvest, or saw possibilities for, the AM contents (that go beyond high school mathematics) into their classrooms. Studying these AM contents did not really help them understand better the mathematics they will teach. There was thus an absence of reinvestments for their teaching, at the level of contents, of AM.

### Metamathematical reinvestments

*The way mathematics is taught.* It is interesting to point out the differences between

Even's (2011) study and this one in regard to what students learn from the way they are taught mathematics. In Even's study, the AM courses offered were constructed to enable teachers to live rich mathematical activities in which mathematics evolves and is not static. The teachers were then inspired to reproduce this environment with their own students in their classrooms. In this study, as the PT put it, most AM courses were given in the form of lectures. Thus, no PT was inspired by the manners of doing and of teaching mathematics of their professors in their AM courses. In fact, lecturing is for them in contradiction with the way they are learning to teach in their education courses (e.g. getting pupils to be active, group work, differentiated activities). Alexa and Josie are explicit about this when they explain that they want to teach by other means than through lectures: they want to teach like it is suggested in their education courses, yet they do not really know how to do it for teaching mathematics.

*Experiencing mathematical difficulties lead to better understand pupils' difficulties.* AM courses were a means for PT to experience difficulties (sometimes for the first time) in mathematics. Like teachers in Even's (2011) study, PT in this study explain that they learn a great deal through the difficulties they have in their AM courses. In Even's study, teachers became sensitized to their students' frustrations when they do not understand mathematics. The contribution of these experiences for PT in this study are a little different: e.g. for Chloe it made her realize what helps her succeed in mathematics (like tricks, drawings and examples) leading her to see what type of explanations she could provide as a teacher to help her (future) students when they do not understand; for Remi it sensitized him to the conceptual difficulties of some concepts taught at the end of high school. These PT learned from their difficulties in AM from a metamathematical point of view, enabling them to learn about teaching mathematics and how to interact with students that do not understand.

*Learning to learn mathematics.* One metamathematical reinvestment not highlighted in the current literature that emerges from this study is one about learning to learn mathematics. E.g. Remi shares that through his courses he developed the ability to learn new mathematics on his own. Even if they were in the form of lectures, Remi immersed himself in a mathematical activity in his AM courses; by having to get by on his own, Remi learned to *do* mathematics. Of interest is that some authors (e.g. Bauersfeld, 1998) mention the importance of delivering AM in an 'active' manner, like the AM courses in Even's (2011) study, in order to make PT active and plunging them in a mathematical activity (of problem solving, of reflections and debate, etc.). In this case, Remi plunged himself in a mathematical activity on his own, through his work; and not because of the manner of delivery of the course. This led him to develop confidence for learning new mathematical content and teach it, even if he did not take specific courses about it.

This is significant, since teaching through a lecturing mode has often been critiqued for offering a limited experience about mathematics (e.g. constrained to take notes and learn already digested mathematics, see our review in Proulx & Bednarz, 2008). Remi's reinvestment of AM reminds us that beyond the contents and the way by which they are taught, AM courses can successfully get students to reflect upon and question

mathematics, solve problems, learn to reason in mathematics, and learn to learn mathematics. Remi's experiences show that mathematical preparation in AM, even if it is through lectures, can lead PT to learn to learn mathematics.

## Ruptures

*Global rupture.* The PT shared their difficulties in connecting AM contents with high school mathematics contents (an issue often raised in the literature, see e.g. Hache, Proulx & Sagayar, 2009). This lack of connections, for the PT, conveys itself into a global rupture between the AM and the high school content. With the exception of year-1 AM courses covering contents precisely introduced in high school, AM content differs greatly from those taught in high schools for PT. E.g. Alexa explains that she loves her applied AM courses, on differential equations and numerical analysis, but that she sees no links between those and high school contents (like e.g. the notion of line, of system of linear equations, of quadratic or logarithmic functions). For the PT, there is a rupture as AM and high school mathematics are seen as two different and distinct things; and they would like to connect them.

*The compressed nature of AM.* High school mathematics is used in a compressed way in AM, which creates an obstacle for PT who need to decompress it in their teaching (Adler & Davis, 2006; Ball & Bass, 2003; Huillet, 2007). Josie's case questions this rupture about compression/decompression. Her experience with compressed high school mathematics in AM is at first glance positive. For her, being able to re-use school mathematics in a compressed way within a new context (AM context) requires good knowledge of this mathematics and leads to a deepening of her knowledge of it. Thus, the context of AM, which allows her to reinvest high school contents, is an opportunity for her to develop ease with high school mathematics and work with it even more. However, while solving the logarithm task, Josie added that she realizes needing a different way of understanding logarithms (e.g. a way of interpreting the intricacies related to transforming exponents and its impact on the solution), than what her AM courses offer; she mentions needing an understanding more connected to high school content and its intricacies. Consequently, Josie points out a dual-issue regarding the compressed nature of high school mathematics in AM: she needs to understand well high school mathematics to use it in AM, but also needs to know it in a different way for her teaching (a way her AM courses do not provide).

*Rupture between preparations in mathematics and in education.* For PT, the lecturing mode of AM courses is disconnected from the way in which they learn to teach in their education courses. E.g. although Josie and Alexa explain liking both their AM and education courses, they experience them in isolation and do not know how to combine them. PT share that they long to put into practice the theories and principles their generalist teacher educators show them in education courses, but that they are unable to concretely conceive what these ideas and principles could resemble in a *mathematics* classroom. They want to work on adding differentiated or group activities in their teaching, but the basis of their preparation to become mathematics teachers is anchored in an ideology that the teacher, as the bearer of knowledge, transmits the

content to pupils in the form of a lecture; most PT explain not knowing how to teach mathematics, and have their (future) students learn it, by any other means. PT desire is that their courses in education and AM be linked, in order to feel prepared to *teach mathematics*, and not only to be knowledgeable in mathematics on one side and knowledgeable in education on another. This rupture goes beyond the way mathematics is taught in AM courses. For PT, this translates itself in a rupture between their preparation in mathematics and the one in education, which remain separate despite their own efforts to link them.

*The identity gap.* Another rupture, not highlighted in the current literature, emerged concerning PT professional identity. Josie's case is an illustration of this, as she feels that AM courses are not intended for her as a PT, because professors 'speak' to future mathematicians (the ratio of PT in AM courses reaches one for 30 students). Josie explains that in her AM courses she doesn't feel free to ask mathematical questions that are linked to teaching, like asking about the context, the reason for opting for a specific procedure, the underlying meaning of symbols; being scared of wasting other students' time. Thus, the AM courses become less and less relevant for her *as a PT*, as she becomes disinvested. Her remarks underline the issue that attending to AM courses that are mostly meant for future mathematicians is not without incidence on PT identity as mathematics teachers and involvement as learners.

## Confidence

Another finding concerns issues of confidence. Studying AM has had different sorts of impact on PT confidence as mathematics teachers. On one hand, as in Zazkis and Leikin's (2010) study, PT developed confidence in mathematics through their AM courses: e.g. Chloe explains having developed ease in solving high school mathematical problems; Remi explained gaining experience in learning (about) mathematics for himself. On the other hand, some PT explained the opposite: some lost confidence through their preparation in AM. E.g. Alexa and Josie explained having lost confidence in their knowledge of high school mathematics, as they do not consider having developed a deep enough understanding of these contents, and its teaching, not knowing how to teach mathematics in ways their education courses promote. Thus, they do not feel they have the tools to understand the mathematics better *nor* to teach it: their AM mathematical preparation did not strengthen their understanding. Unlike Remi, both Alexa and Josie did not really learn to learn mathematics and did not explore concepts deep enough. For them, studying AM separated them from the mathematics they will be teaching in high school.

## CONCLUDING REMARKS

The current literature highlights a variety of experiences teachers live and learn from in their mathematical preparation through AM: content reinvestments, metamathematical reinvestments and ruptures. The study reported offered additional data to these experiences, through e.g. concrete examples and nuances to current ideas as well as raising new issues not highlighted in the literature (e.g. lack/gain of confidence, development of identity as mathematics teachers). In that sense, this study

complements and adds to current knowledge about PT experiences in AM courses. This data, in combination with the current literature, illustrate that the question of the mathematical preparation of high school PT is a complex one, where a variety of experiences and dimensions are to be considered and play a (often simultaneous) role. From the variety of issues and questions raised by PT in this study, it is now becoming obvious that the issue of PT mathematical preparation is not one of “all good, all bad” phenomena. Hence, research is still needed to continue developing a deeper understanding of issues related to PT mathematical preparation.

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# **GESTURES AND TEMPORALITY: CHILDREN'S USE OF GESTURES ON SPATIAL TRANSFORMATION TASKS**

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*This paper discusses findings from a task-based interview with five elementary school children working on a spatial transformation task. The paper focuses on children's gestural and verbal communication when engaging in the task. Findings suggest that children use gestures as multi-modal resources to communicate temporal relationships about spatial transformations. Although research has shed light on the use of gestures to represent functions deictically, iconically and metaphorically, this work has not addressed aspects of temporality and the dynamic nature of gestures. This paper raises questions for further research in the area of gestures and communication to address the temporal aspects of mathematics.*

## **INTRODUCTION**

The goal of our research has been to extend Presmeg's (1986) work on "dynamic (moving) imagery" for describing the dynamic roots of certain geometrical concepts that are visualized and understood by high school mathematics students. Our work with dynamic geometry technology has prompted us to try to better understand the effects that such technology might have on student thinking, particularly in relation to its dynamic nature. Despite the power of dynamic thinking in mathematics, prior research suggests that formal schooling can compromise this kind of thinking in geometry (Lehrer et al. 1998). We are thus interested in finding out whether the use of DGEs can support dynamic thinking and, if so, how this kind of thinking might be expressed by young children. The latter question forms the basis of this research study.

## **THEORETICAL PERSPECTIVES**

### **Gestures, Utterances, and Mathematical Thinking**

The embodiment of mathematics has led to growing interest and development in the field of gestures in mathematics education research. Recent gesture studies in a mathematical context have focused on different functions of gestures: iconic, indexical and symbolic (Radford, 2003), the synchrony of speech and gestures for forming mathematical meaning on a new level (McNeil, 2005), and in the way students mimic teachers' gestures in mathematical communication (Singer & Goldin-Meadow, 2005). While these studies have provided insights into the multidimensional nature of mathematical thinking, there is a lack of common framework theorizing the relationship between gestures and mathematical thinking. Moreover, much of this work focus on the relationship between speech and gestures and study gestures as independent from cognitive processes. With this dualist approach, gestures are external acts that represent the mathematical thought from within and embodied acts that make cognitive processes explicit.



In contrast, Sfard's (2008) commognitive framework is helpful for examining the relationship between gestures and mathematical thinking. Her non-dualist approach disobjectifies *thinking* as part and parcel of the process of *communicating*. She defines *language* as a system that includes all kinds of symbols in communicational acts, and *gestures* as bodily movements fulfilling communicational function (Sfard, 2009): "Language is a tool for communication, whereas gesture... is an actual communicational action" (p. 194). This communicational act can be interpersonal or intrapersonal. With this view, the actors may be conscious or unconscious of their gestures. Sfard's approach highlights the way in which "talking and gesturing stop being but 'expressions' of thinking and become the process of thinking in itself" (p. 195).

Furthermore, Sfard's (2009) approach suggests that gestures and utterances take on different roles in mathematical thinking. "Each of these modalities contributes to commognitive processes at large and to mathematical commognition in particular" (p. 195). *Recursivity* is a linguistic feature in mathematical discourse offered by utterances. The unlimited possibility to expand linguistically allows human to work in meta-discourse, or thinking about thinking. On the other hand, gestures enable effective communication to ensure all interlocutors "speak about the same mathematical object" (p. 197). Gestures are essential for effective mathematical communication: "Using gestures to make interlocutors' realizing procedures public is an effective way to help all the participants to interpret mathematical signifiers in the same way and thus to play with the same objects" (p. 198). Moreover, gestures can be realized *actually* when the signifier is present, or *virtually* when the signifier is imagined. Sfard illustrates how a student uses 'cutting', 'splitting', and 'slicing' gestures to realize the signifier *fraction*. Since these gestures were performed in the air, where the signifier *fraction* is imagined, it was an instance of virtual realization.

### **Gestures and Temporality**

Although gestures have been widely examined in mathematical discourse in recent years, studies have yet to address the temporal functions of gestures in mathematical discourse. Leading gesture specialist David McNeil's (1992) categorization of gestures into deictic, iconic, metaphoric, and beat, only broadly characterizes the type of functions served by gestures. For example, deictic gestures serve as pointing devices, while metaphoric gestures serve to represent the mathematical objects themselves. These categories have not captured the dynamic nature of gestures, in particular, when gestures are used to convey temporal relationship. Although they are useful for identifying the general functions of gestures, they do not distinguish between the static and dynamic nature of gestures. For example, when a person makes a metaphoric gesture to realize the signifier, *a linear function*, it could be a static one, with the arm or hand representing the function, or a dynamic one, with the hand or finger tracing the motion of the path. In the latter case, the dynamic gestures communicate temporal relationships of the linear function as opposed to the shape of the linear function statically.

Núñez (2006) studied how mathematicians use hand gestures as a way to express dynamic thinking of functions, continuity, and other abstract mathematical ideas.

The gestures (and the linguistic expressions used), however, tell us a very different conceptual story. In both cases, these mathematicians are referring to fundamental dynamic aspects of the mathematical ideas they are talking about. (Núñez, 2006, p. 177)

Furthermore, these mathematicians say “approaching,” “tending to,” “going farther and farther,” to express a sense of motion, while producing metaphoric gestures tracing the trajectory of the point or particle with their fingers. Sinclair and Gol Tabaghi (2010) also examine motion in gestures, in particular, mathematician’s hand gestures depicting movement of vectors, providing evidence of time and motion-based conceptualization of vectors. These two studies point to the dynamic aspect of mathematicians’ thinking, but we are interested in the way that younger learners might express this mathematical and temporality and mobility.

To summarize, we use Sfard’s commognitive framework to study children’s gestures as part of their mathematical discourse (thinking) while they engage in mathematical tasks. In addition, we focus on children’s use of dynamic gestures in spatial transformation tasks. We aim to observe the type of gestures that they communicate and instances that their gestures realize aspects of temporality in mathematics.

## **METHODOLOGY OF RESEARCH**

### **Participants and Task**

We adapted the spatial transformation task used by Goldin-Meadow et al. (2006) and Levine et al. (1999) with minor alteration for our task-based interview. The task contains two halves of a shape that could be spatially transformed to form a vertically symmetric figure. From Levine et al. (1999), we selected two questions from each of the four problem types and ordered them randomly to form a set of eight questions. The four types of spatial transformation are: direct translation, diagonal translation, direct rotation, and rotational translation. A 2x2 choice array accompanies each question; it contains four whole shapes of which one of them is the correct answer. Figure 1 shows the problem types and a sample choice array.

During the interview, subjects were asked, “if you were to put these pieces together, which one of those will they make?” After the children provided their answers, they were asked to justify their choices or show their reasoning.

We conducted a semi-structured interview with five children in the classroom of their elementary school. The subjects are kindergarten (age five to six) students who live in a low socioeconomic, suburban neighbourhood in Northern British Columbia, Canada. Each interview was roughly ten minutes long and was videotaped with the camera facing the interviewer and the subjects. All voice and bodily movement during the interview was recorded. The interviews were transcribed for data analysis.

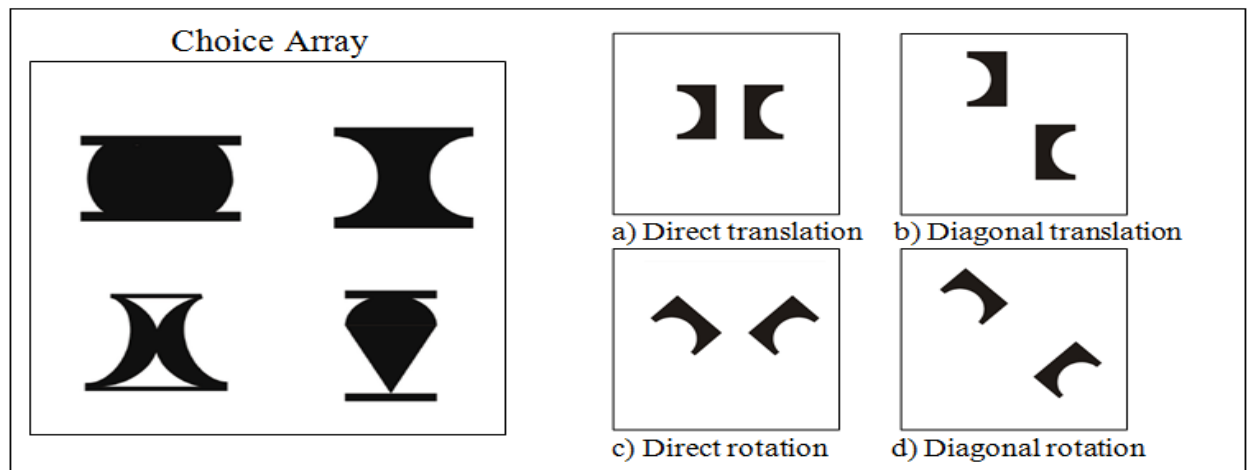


Figure 1: Example of a spatial transformation task and problem types. The choice array is given to the participant along with one of a) to d). The participant is asked to choose which figure in the choice array can be transformed from the two halves.

We adapted the spatial transformation task used by both Goldin-Meadow et al. (2006) and Levine et al. (1999) because the task allows children to express static features of the figures as well as dynamic movement in their reasoning. Using the task as-is, we would like to compare results on children's dynamic thinking with past studies. The task is age and level appropriate; data from their studies shows that children of 5 to 6 years of age used a variety of strategies when completing the task. These strategies include:

- Movement: Any indication of movement of the pieces
- Feature: Any indication that the child is focusing on a particular feature of either the piece(s) or the whole
- Whole: Any indication that the child is seeing the pieces as a whole
- Alignment: Placing the piece(s) on top of the corresponding portion of the whole
- Other: Any strategy other than one listed here

Goldin-Meadow et al. (2006) found that children used the "movement" strategy most frequently, on average five times out of 8 questions. The children frequently used strategies expressed in gestures without accompanying speech. Their study focused on observing sex difference in using the "movement" strategy as well as comparing overall girls' and boys' performance on the tasks. Their methodology was to record the frequency of each type of strategies used; they did not report that children used more than one strategy at a time when completing a given question.

## ANALYSIS OF DATA

### Children's Strategies for Solving Spatial Transformation Tasks

We observed a significant amount and wide variety of gestures used across all five children engaged in the task. They gestured whether or not they answered the questions correctly, and some gestures were expressed with accompanying speech while some were not. In general, the children gestured both deictically to point to the *feature* or the *shape*, and metaphorically to explain *movement*, findings that resonate with

Goldin-Meadow et al. (2006). Table 1 illustrates some examples of children's gestures and utterances in their reasoning.

Another interesting observation was that, overall, the children used more gestures in the beginning of the task than when towards the end. All children, except for one (who did eventually begin to make that gesture in the last four questions), used a "movement" gesture in the beginning of the task. This movement gesture involves moving the fingers or hands to mimic moving the pieces together. Some used this gesture extensively, and in general, all children used less of this gesture as they progressed in the task.

Categories	Utterance examples	Gesture examples
Movement	"Because them shapes, when they stick together, they can make that thing." (Child 3)	Place one index finger on each piece, and then move the fingers so that the tips of the fingers touch. (Child 5)
Features	"Because of the pointy things." (Child 1)	Point to the specific feature (corner) of a shape with the index finger. (Child 3)
Whole	"...like a fortune cookie." (Child 4)	Make a circle with her arms above her head (Child 1)
Alignment	"If you put that... you can see through the paper, and you can measure the sides, and you will know what shape it will make." (Child 2)	Place the choice array sheet on top of one of the original piece. (Child 2)
Other	"Because I know this." (Child 5)	Gestures that are not categorized in the above list.

Table 1: Examples of Strategies

### Dynamic Gestures and Temporality

As mentioned above, we observed that all children made a "movement" gesture in the beginning of their interviews. Some used one hand, connecting their fingers to gesture the moving together, while some used two hands. This sliding gesture is a metaphoric one: it expresses the sliding together of the two pieces to form one whole, where the fingers enact the pieces and the transformation. This gesture provides evidence that the children were thinking dynamically about the movement of the pieces. It expresses temporality by tracing the motion of the pieces and their location in time from the beginning to the end of the movement.

All children in the present study used different strategies when completing the task: by pointing to the specific *features* of the shapes ("because of these corners"), perceiving the shape as a *whole* ("it's a trophy"), and suggesting *movement* of the pieces ("because if you put them together, it makes this shape"). They also made the "movement" gesture most frequently, findings that resonate with Goldin-Meadows et al. (2006). However, when we compare children's use of "movement", "features", and "whole" gestures in our study with Goldin-Meadows et al. (2006), some aspects of temporality emerge. The children in our study were making "movement" gestures first and then followed by another strategy. Some children even made this "movement" gesture twice

in the same question, first when they attempted to answer the question, and second when they were asked to justify their answers. This suggests that they had been *dynamically* thinking about the movement of the pieces before they provided their reasoning. The key here is that they were not just reasoning in one category at a time. The fact that they used the "movement" gesture before justifying with "features" and "whole" suggests that they were *both* thinking dynamically and attending to details of the shape. This was not discussed in Goldin-Meadows et al. (2006).

### Example: Child 1's Gestures

In the following excerpt, we illustrate Child 1's sequence of gestures and utterances in response to her third question in the task (see Table 2). The child was given the same question as seen in Figure 1, with problem type "b)"

		Utterance	Gesture
1 2	Interviewer:	How about this one here, these two, which one will it make?	
3	Child 1:	<2 second pause>	See Figure 2a)
4		It will make this one.	
5	Interviewer:	That one?	
6	Child 1:	Yea.	
7	Interviewer:	How do you know?	
8	Child 1:	'Cause, see these pointy things?	See Figure 2b)
9		Then that goes down,	
10		and this goes down,	
11 12		and that's how it makes a... <3 seconds pause> window!	See Figure 2c)
13	Interviewer:	Oh they are windows.	

Table 2: Excerpt of Child 1's Utterances and Gestures

Child 1 first made a "movement" gesture (line 3) by moving and touching the pieces back and forth with her finger. Then, when asked how she solved the question, she explained that, "'cause see these pointy things? And that goes down, and this goes down, and that's how it makes a window!" Her utterances, in "pointy things" (line 8) and "windows" (line 13) were accompanied by gestures and exemplified using "features" and "whole" strategies respectively. Therefore, Child 1 has used all three strategies, "movement", "features", and "whole" in solving this question. This observation is consistent with the other four children, who occasionally used a combination of strategies both verbally and in their gestures after initially gesturing the

movement of the pieces. Figure 2 shows Child 1's gesture sequence in the above excerpt.

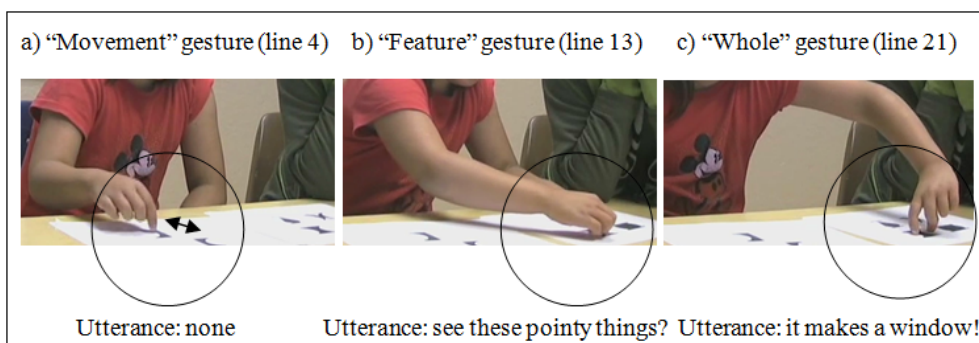


Figure 2: Child 1's Gesture Sequence.

## DISCUSSION

Our data provides strong evidence that children rely on gestures to communicate mathematically: to communicate both with the interviewer and with themselves to internalize their thinking. This claim can be supported by the amount and variety of gestures that the children produced, as well as in the way that the children frequently gestured without accompanying speech. Data also shows that the children used a combination of "movement", "feature", and "whole" strategies in the same question, as illustrated in Child 1's excerpt. Child 1 first produced a "movement" gesture, followed by utterances that characterized "feature" and "whole" strategies while using her fingers to refer to the parts of the figure that she was speaking about (see Figure 2). Sfard (2008) provides a lens for explaining this phenomenon. Using her definition of gestures as a communicational act that serve to "speak about the same mathematical object" (Sfard, 2009, p. 197), the children effectively used a combination of utterances and gestures to communicate their mathematical thinking. Their "movement" gestures enable them to effectively communicate their dynamic thinking by enacting the transformation of the pieces metaphorically. On the other hand, the children also made use of deictic gestures to point to the features and trace the outline of the figure as they spoke about them in speech.

In Sfard's term, the "movement" gestures are *actual* realization since they signified the pieces on the paper actually moving towards each other. These dynamic gestures were used most frequently by the children and exemplified aspects of temporality in children's communication about mathematics. By analyzing the sequence of the children's gesture, we found that the children were both thinking about the dynamic movement of the pieces and attending to the static properties of the shape. This constitutes a very new and important finding in our study. Our study suggests that young children are not only capable of thinking about spatial transformations dynamically, but they can also communicate the dynamic nature over static properties of geometry. This discussion aligns with Núñez (2006), who argues that "motion... is a genuine and constitutive manifestation of the nature of mathematical ideas" (p. 168). Given that temporality is not captured by formalisms and axiomatic systems in the

mathematical discourse, our findings discuss the possibility for more opportunities for young children to interact with temporal mathematical relationships and DGEs.

Finally, more research is needed to explain the finding that children used fewer “movement” gestures during the course of the interview. We speculate that this could be due to the children’s compromise to move towards an adult discourse: the mathematical discourse. In Sfard’s (2008) term, there exists a commognitive conflict between the children’s and the adult’s discourse; the children may well have compromised the way they communicate mathematically as a result of negotiating with the leading discourse of the adult, one that dismisses temporality and dynamism. Further research that examine the change of children’s gestures in expressing temporality over time will be needed to warrant this claim about the commognitive conflict between children’s and adult’s discourse.

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# ADAPTING THE TASK: WHAT PRESERVICE TEACHERS NOTICE WHEN ADAPTING MATHEMATICAL TASKS

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*This paper reports on an in-depth study that explores preservice teachers' pedagogical adaptations to a rich mathematical task. Data were collected from six elementary preservice teachers working in pairs to first solve a mathematics problem and then design adaptations to make the problem more accessible and more challenging for diverse learners. Results indicate that preservice teachers are able to draw upon a range of strategies to vary the mathematical content, the context, and the question asked. However, they also did not notice or attend to how their adaptations changed the mathematical structure of the problem. This study provides insights into what is involved in learning to adapt classroom mathematical tasks as an important pedagogical practice.*

## INTRODUCTION

Good questions, according to Small (2009), involve teachers in adapting, extending and modifying mathematically rich tasks to meet the diverse needs and interests of learners. Textbooks can be a source of such problems. They can provide access to research-based mathematics problems and worthwhile tasks as opportunities for students to explore, practice, and model mathematical concepts. Problems or tasks selected by teachers give students implicit images about what counts as mathematical inquiry or what it means to do mathematics (Schoenfeld, 1992). Problems contextualize, provide possibilities for inquiry, and can pedagogically frame students' attention toward noticing mathematical ideas. However, learning to design and adapt problems for mathematics teaching that meet the diverse needs and interests of students is not a trivial endeavour (Bragg & Nicol, 2008, Crespo, 2003; Nicol & Bragg, 2009; Sinclair and Crespo, 2006). Experienced teachers can draw upon a range of resources to design and adapt tasks including: their own experiences with the problem, their understanding of the mathematics and the curriculum, their knowledge of how students develop mathematical understanding, and their experience in using what they know about how students might respond to inform their instruction. However, creating rich tasks and maintaining the richness while teaching is extremely difficult, even for experienced teachers (Stein, Grover, & Henningsen, 1996). Therefore, it is important that both practicing and prospective teachers have access to opportunities to implement and design worthwhile tasks.

Watson (2008) calls for teachers to learn more about working on "task design rather than only on task implementation" (p. 153). In this paper we examine how preservice elementary teachers adapt and extend a task to meet the diverse needs and interests of their future students. We explore what preservice teachers notice about a task, how



they describe potential adaptations, and how they actually adapt a task for future teaching. By analyzing the ways they adapt a specific task it is hoped that insight can be gained into what preservice teachers find important in order to better support their learning to teach.

## **THEORETICAL CONSIDERATIONS**

Professional development, according to Mason (2002) is about developing the sensitivity to notice. Expert teachers, for instance, notice and see aspects of classroom practice in ways that beginning teachers do not. Mason offers the idea of noticing as an intentional stance—“a collection of practices both for living in, and hence learning from, experience, and for informing future practice” (p. 30). The discipline of noticing is about making the effort to notice particular aspects, for instance, in classroom teaching or research, and to be able to notice these aspects when needed or “when it would be useful to have noticed [them] (and not merely later, in retrospect)” (p. 31). Therefore learning to notice does not just involve noticing aspects of teaching that before went un-noticed but also includes the sensitivity and inclination to be aware.

Noticing as a theoretical construct is receiving increased attention in mathematics education for understanding teaching. In their recent edited book Sherin, Jacobs, and Philipp (2011) present teacher noticing as including two aspects that are interrelated and cyclical: “attending to particular events in an instructional setting” and “making sense of events in an instructional setting” (p. 5). Although Sherin and colleagues focus on in-the-moment teacher noticing that occurs during instruction, Marton and Booth (1997) in their conceptual analysis and systematic research of learning emphasize the relationship more generally between learning and noticing. Marton and Booth refer to learning as the ability to notice critical aspects of a phenomenon while at the same time being focally aware of these aspects. They suggest not analyzing the learning of content, the acts of learning, or the context of learning as separate but rather as being united aspects of the learner’s experience.

As preservice teachers learn to teach they draw upon their vast experience as observers of teaching—and they do so both as students and as prospective teachers. Although new research focuses on preservice teachers’ noticing of classroom events (e.g., Star, Lynch and Perova, 2011) there is still much we do not know in terms of preservice teachers’ noticing in preparing to teach. If designing and adapting mathematical tasks is considered an important aspect of classroom teaching then learning more about what preservice teachers attend to when analyzing a potential classroom task is important.

McDonald and Watson (n.d) state the richness of a task “is created by the level of questioning, diversity of approaches, and exploiting the potential depth and connectedness of mathematics, whatever the starting point” (p. 3). Although many tasks can be made worthwhile and rich with good questioning, learning to adapt a task to meet the needs and interests of students’ is not obvious. Small (2009) refers to the need for teachers to understand student differences and therefore to differentiate instruction in terms of the main lesson goal. This requires practicing and prospective

teachers to adapt and extend mathematical tasks. How then do preservice teachers adapt a task and, in doing so, what do they notice?

## DATA COLLECTION AND ANALYSIS

This report draws upon an in-depth study of six elementary preservice teachers who had completed their mathematics methods course as part of large Canadian university teacher education program, and were ready to begin their extended field experience in an elementary school. At the time of the study the participants had completed eight months of a 12-month teacher education post-baccalaureate program. Data collection involved participants working in pairs to solve a particular mathematics problem called “Three Hungry Monster Problem” (adapted from Watson, 1988) and to discuss how they would adapt and extend the problem to increase its accessibility for a range of learner interests and needs. This problem was chosen because of its richness in eliciting a range of solution strategies and its potential for adaptation through variation in the problem variables (see Watson, 1988).

*Three Hungry Monster Problem:* Three tired and hungry monsters went to sleep with a bag of cookies. One monster woke up, ate  $\frac{1}{3}$  of the cookies, then went back to sleep. Later, the second monster woke up and ate  $\frac{1}{3}$  of the remaining cookies then went back to sleep. Finally, the third monster woke up and ate  $\frac{1}{3}$  of the remaining cookies. When she finished there were 8 cookies left. How many cookies were in the bag originally?

Questions to learn more about what preservice teachers were noticing were asked of each pair following their solution to the Monster Problem as well as before and after their attempts to make adaptations to the problem (e.g., How do you think students would solve this problem? How could you adapt the problem to make it more accessible? More challenging? How would you use the problem with students?). Working in pairs allowed for a free flow of discussion between participants with a more genuine think-aloud protocol.

During each session pairs of participants first worked individually on solving the problem, discussed together how they might adapt the problem, then worked individually on their own adaptations before sharing their work with each other and the researcher. Data collected included audio records of each session, copies of participants’ written work in solving the problem and their written adaptations to the problem, as well as researcher field notes. Analysis involved examination of the written adaptations to the problem developed by each participant. Adaptations were sorted and analyzed according to what participants varied in the problem (mathematics content, context, and questions asked). Each participant’s written adaptations to the problem were first analyzed and this was followed by an analysis of task adaptations across participants. Audio records were transcribed and analyzed for the reasoning, reflections, and pedagogical questions participants asked of themselves and each other.

## RESULTS

We report our results by focusing on the strategies preservice teachers used to make the task more accessible to students and strategies used to make the task more challenging.

### Strategies to make the task more accessible

To make the problem more accessible to a range of students participants suggested adaptations that can be categorized as the following: a) varying the mathematical content (introducing numbers that are more ‘familiar’); b) varying the context (using fewer words and more diagrams; using manipulatives); and c) varying the question asked (working forward; decreasing the number of problem steps; and providing structured support).

Almost all (5 out of 6) participants suggested varying the mathematical content to create a more accessible problem by making the numbers ‘easier’ to work with. All participants suggested replacing the fraction  $\frac{1}{3}$  with  $\frac{1}{2}$  and using a number for the total cookies in the bag to be divisible by 2. For example, one participant stated:

I think  $\frac{1}{3}$  is a harder fraction for students to work on than one half, for younger grades. If we started with numbers like 20, using an even number and then halving it, then I think a logical process they have to go to is they have to be able to count by 10 and by doubling it and something that emphasize[s] younger grades. So if we started from 20 and half it and then half it again as each monster wakes up, we simplify the question. (PT#2).

Participants also suggested adapting the problem by varying the context. In this case preservice teachers (4 out of 6) suggested changing the problem wording or decreasing the number of words used to support students’ access to the problem. All participants suggested visual aids or manipulatives would make the problem easier and some suggested different contexts such as candies eaten by sisters rather than cookies eaten by monsters. Diagrams were introduced to allow students to count off the number of cookies eaten and to lessen the perceived language burden the problem may create. As this preservice teacher stated:

Some kids relate better to pictures rather than words, especially if they do not have the vocabulary. If their reading level is low, you need to look at what you’re marking here because when you look at word problems in math, are you marking math or reading comprehensive? ... So by changing a word problem and making a pictorial question, then you make [reading and math] separate so you can actually look at math skills. (PT#3).

Varying the question asked was a common strategy suggested by participants to make the problem easier. For example most (5 out of 6) preservice teachers suggested revising the question so that it focused not on determining the total number of cookies in the bag when given the number remaining, but instead on determining the remaining cookies given the total number. This strategy is represented by PT#6 below.

Three tired and hungry monsters went to sleep with a bag of 27 cookies. The Monster woke up and ate  $\frac{1}{3}$  of the cookies, then went back to sleep. The second monster woke up and ate  $\frac{1}{3}$  of the remaining cookies, then went back to sleep. The third monster woke up and ate  $\frac{1}{3}$  of the remaining cookies. How many cookies were left? (PT#6)

Participants described this strategy as “flopping it with the end to the beginning” (PT#3) by switching the direction of the sequence of problem steps. Preservice teachers’ reasons for adapting the problem in this way drew upon their own experience in solving the problem. “I flipped the original question around because most students

may read from the start and work through, rather than realize they should work backward like in the original” (PT#4). Most participants made reference to their own difficulty realizing that they could solve the problem by working backwards (i.e. finding  $\frac{2}{3}$  of what number is 8) and thus sought to eliminate that challenge or difficulty from the problem.

Varying the question also included the strategy of reducing the number of steps in the problem. The original problem is a multi-step problem. Preservice teachers recommended adapting the task to include one or two steps or to provide more specific direction to students to guide them through the problem steps. Adapted problems of this type included “going with only one monster, you say one monster woke up and  $\frac{1}{3}$  of the 18 cookies; how many cookies did he eat?” (PT#6). In this case, as with others, participants first adapted the direction of the problem as well as the multi-stepped nature of the problem. Preservice teachers also suggested providing more structured support to students as a way to make the problem more accessible as seen in this example by PT#1 representative of other participants.

Three tired and hungry monsters went to sleep with a bag of 40 cookies. One monster woke up and ate  $\frac{1}{2}$  of the cookies. (How many cookies are left? \_\_\_\_). Monster #2 woke up and ate  $\frac{1}{2}$  of the remaining cookies (How many cookies were left? \_\_\_\_). The Third monster woke up and ate  $\frac{1}{2}$  of the remaining cookies. How many cookies were left in the bag the next morning?

Preservice teachers drew upon their own experience in solving the problem to make it more accessible to potential students. They suggested that structured support would lead to less misinterpretation in finding the correct solution.

### **Strategies to make the task more challenging**

Using a similar categorization scheme for strategies on making the problem more accessible, adapting the problem so that it is more challenging involved preservice teachers in: a) varying the mathematical content (introducing different fractions); b) varying the context (including more or extraneous information); and c) varying the question asked (providing open-ended questions; constructing original questions).

Most preservice teachers (5 out of 6) made the task more challenging by changing the mathematics content and introducing fractions that they considered to be less familiar to and therefore more difficult for students (e.g.,  $\frac{1}{5}$ ,  $\frac{1}{6}$  or  $\frac{1}{8}$  rather than  $\frac{1}{3}$ ), or mixing the fractions (e.g., taking  $\frac{1}{3}$ , then taking  $\frac{2}{3}$  of the remaining cookies), or introducing numbers that were comprised of mixed fractions. Participants suggested such changes increased the challenge for students “because it requires more computation” (PT#1) or because it allowed students who had strong understandings of fractions to draw upon and use this knowledge to solve the problem (e.g., PT #3).

Participants’ strategies to increase the challenge of the problem included varying the context by adding more information. For example, instead of explicitly stating “8 cookies remained,” PT#4 suggested: “...Their parents split the remaining muffins equally. Using only 30 seconds to eat each muffin, their father had eaten all of his share in two minutes...” Preservice teachers did so in order to “require more steps and get

[students] to think not just about the same steps over and over (i.e.  $\frac{1}{3}$  and  $\frac{1}{3}$ ).” Preservice teachers also suggested adding extraneous information (e.g. time, number of chocolate chips in the cookies) so that students would need to determine the necessary data for solving the problem, this they suggested would develop students’ critical thinking.

In order to develop challenging problems participants also focused on varying the question asked. Three preservice teachers developed open-ended tasks that gave students a choice. For example, PT#4 replaced reference to eating  $\frac{1}{3}$  of the remaining cookies with “... ate a fraction of the marshmallows ...” and asked students to determine not only how many remained but also the fraction of marshmallows eaten. Similarly PT#6 adapted the problem so that students could choose their own remaining number of cookies instead of 8. Adapting the problems in this way allowed “students less confident in their math skill to choose a lower number for the remaining cookies, while other can challenge themselves” (PT#6). In addition to open-ended problems, 4 out of 6 participants suggested asking students to design their own tasks. For example, one participant wrote “now design a question for your friends that has a mystery monster eating a fraction of the cookies, where you could create your own question” (PT#5). This adaptation could “require students to use higher-level thinking as they are constructing their own example and not simply applying procedural knowledge” (PT#1).

## CONCLUSIONS

Results of our study suggest that with opportunities to adapt and extend a mathematical problem preservice teachers do have strategies to employ that can make a problem more accessible or challenging for a range of potential students. In making the problem more accessible or challenging preservice teachers noticed and varied the mathematical content, the context, and the question asked. In terms of the mathematical content preservice teachers’ adaptations focused on the fractions—choosing familiar fractions to make the problem more accessible and choosing multiple and less familiar fractions to make it more challenging. Small (2009) offered principles and approaches to differentiating instruction through adapting and extending tasks that include focusing on the big mathematical ideas and providing some aspect of choice for students.

Interestingly none of the preservice teachers in our study noticed or made mention of the big mathematical ideas in the Three Hungry Monster problem. One of the big mathematical ideas in the Monster Problem focuses on the relationship between the fraction and the whole. It is possible to have two fractions that are each halves yet not equal to each other (because their wholes are different sizes). In the case of the Monster Problem one-third is taken multiple times but in each case the whole for each one-third is different. None of the participants in our study noticed this big idea. In fact some preservice teachers interpreted the repeated taking of one-third as procedural and offered to make the problem more challenging by changing the fraction. Certainly varying the fraction of cookies eaten can make the problem more accessible or

challenging but what is important is to make such variations by also attending to how these variations alter the intended big mathematical ideas in the problem. Our data suggest that preservice teachers did not use the salient mathematical ideas in the problem as a framework to guide their thinking in adapting the problem. Nonetheless, the preservice teachers in this study demonstrated the ability to consider and offer ways of adapting mathematical tasks, even without explicit instruction in doing so.

Some preservice teachers in the study also provided adaptations that offered students choice in selecting the fractions or the number of monsters, allowing students to participate in differentiating the instruction to meet their needs. Small (2009) states that “few math teachers are comfortable with the notion of student choice except in the rarest of circumstances. They worry that students will not make the ‘appropriate’ choices” (p. 5). Although some preservice were interested in adapting the problem to include student choice it was done so only as a strategy to make the problem more challenging rather than to also make it more accessible. Concern about whether or not students would make the “appropriate” choices do not appear to be important to preservice teachers at this stage. One preservice teacher’s adaptation stated “Take any number you wish as the remainder of cookies—e.g. instead of 8 cookies, there are 12 cookies left over. How many cookies will there have been originally?” (PT#5). An interesting aspect of this adapted problem is to explore what fractional amounts are needed so that 12 is the remainder. Yet no preservice teachers in this study pursued such explorations.

The results of this study are both encouraging and troubling. The fact that all preservice teachers were able to and were interested in exploring adaptations to the problem that involved the content, context and question asked is encouraging. However, that none of the preservice teachers pursued a solution to the adapted problems or noticed that adaptations have consequences for the intended big mathematical ideas is of concern and points to the need for further opportunities for preservice teachers to explore and practice strategies for making task adaptations.

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# HIERARCHICAL LEVELS OF FRACTION UNDERSTANDING AT THE ELEMENTARY SCHOOL

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*This article examines whether the seven factors found in a previous study carried out by the authors to constitute fraction understanding determine hierarchical levels of fraction understanding. The seven factors were inductive reasoning, definitions and mathematical explanations, argumentation and justification, sense about the magnitude of fractions, representations, connections of fractions with decimals, percentages and division and reflection. Students were clustered into three categories by means of latent class analysis: those of low fraction understanding, those of medium fraction understanding and those of high fraction understanding. It was found that low fraction understanding students were sufficient in inductive reasoning and sense about the magnitude of fractions, those belonging to the medium category in inductive reasoning, sense about the magnitude of fractions as well as in connections, representations and justification, while high fraction understanding students were sufficient in all the seven factors.*

## INTRODUCTION

A few studies examined levels of understanding fractions so far (Pantziara & Philippou, 2012; Pirie & Kieren, 1994). Pirie and Kieren (1994) proposed a model with levels of understanding, and then used the concept of fractions to justify and explain the levels they proposed. Pantziara and Philippou (2012) examined sixth grade students' conceptualization of fractions on the basis of the three stages proposed by Sfard (1991) to acquire the concept of fractions and they found six difficulty levels in the way to conceptualize fractions. In this study, we search for hierarchical levels of fraction understanding on a different perspective than that utilized in previous studies, and more specifically, on the basis of the seven factors found in a previous study carried out by the authors (Nikolaou & Pitta-Pantazi, 2011) to constitute fraction understanding. By the term "factors" we refer to the abilities students must possess in order to understand the concept of fractions. The seven factors found to constitute fraction understanding were inductive reasoning, definitions and mathematical explanations, argumentation and justification, sense about the magnitude of fractions, representations, connections of fractions with decimals, percentages and division and reflection. With the term "hierarchical levels" we refer to levels of fraction understanding on the basis of the seven factors; we examine whether these factors form a hierarchy on the basis of their difficulty.



## **THEORETICAL BACKGROUND**

### **Levels of fraction understanding**

Investigation of levels of fraction understanding is a topic that has not been given much attention in the past. This is evident by the small number of studies dealing with the issue so far. Pirie and Kieren (1994) proposed a dynamic model for the growth of mathematical understanding. The model they proposed was a “general model” for understanding mathematical concepts and encompassed eight levels: primitive knowing, image making, image having, property noticing, formalizing, observing, structuring and inventizing. Pirie and Kieren (1994) then used the concept of fractions to explain and persuade about the proposed levels. In a more recent study, Pantziara and Philippou (2012) examined levels of sixth grade students’ conception of fractions. The examination was based on the three stages proposed by Sfard (1991): interiorization-condensation-reification. Pantziara and Philippou (2012) found six difficulty levels. The first two levels were characterized by procedural understanding and corresponded to the interiorization stage. The third level was a transitional level from the interiorization to the condensation stage for the development of fractions. The fourth level corresponded solely to the condensation stage and reflected students’ ability to think of a process as a whole, combine various processes, make comparisons and alternate between different representations. The fifth level was a transitional stage encompassing characteristics of the condensation and the reification stages, while the sixth level corresponded only to the characteristics of the reification stage.

In the present study we sought for hierarchical levels of fraction understanding on the basis of a different framework than those already examined, focusing on the factors that constitute fraction understanding (Nikolaou & Pitta-Pantazi, 2011). We consider important for teaching and understanding fractions to examine whether such levels exist and what factors differentiate each level from the other. In the space that follows the seven factors are explained in more detail.

### **The factors**

In a previous study (Nikolaou & Pitta-Pantazi, 2011), we proposed a theoretical model with factors that constitute fraction understanding at elementary school. The seven factors encompassed in the model were inductive reasoning, definitions and mathematical explanations, argumentation and justification, sense about the magnitude of fractions, representations, connections with decimals, percentages and division and reflection. The proposed model was confirmed in a sample of sixth grade students in three different measurements and was found to have very good fit to the data. The results indicated that the proposed model provides a comprehensive description of fraction understanding at the elementary school and that the seven factors compose fraction understanding, as it was our initial hypothesis. In the space below brief reference is made to each of the seven factors.

Inductive reasoning is defined as the process that permits the extraction of general conclusions or rules from specific cases (Demetriou, Doise, & van Lieshout, 1988). Inductive reasoning is very important for understanding the concept of fraction as

students are expected to engage in activities of equal partitioning (e.g. a piece of chocolate), of a surface or of a group of objects that gradually can lead to the identification of the concept of fraction.

Definitions and mathematical explanations refer to students' ability to define in their own words what a fraction is and also to explain in various ways (verbally, by using drawings, examples etc.) other issues concerning fractions. According to Niemi (1996) students' ability to explain for fraction topics is essential for understanding fractions as it can reveal students' errors and misconceptions and can thus become a powerful assessment tool.

Argumentation and justification refers to a kind of "informal proof" at the elementary level. Argumentation and justification are also very important for understanding the concept of fractions because they can reveal students' conceptions about fractions, their knowledge of fractions and their errors.

Students' sense about the magnitude of fractions is crucial for fraction understanding, since in the case a student is not able to compare and order fractions and possess a feeling of "how big" is a fraction, then he/she probably does not understand the meaning of fractional numbers. It is very common, for example, for some students to consider the nominator and the denominator of a fraction as two different numbers that do not constitute a unique entity, i.e. the fraction (Clarke & Roche, 2009).

Representations are also very important for understanding the concept of fraction (Lesh, Post & Behr, 1987; Newstead & Murray, 1998). In the context of teaching fractions, children come across a great variety of representations. Further for the recognition and flexible use of various representational systems, a basic goal of teaching and learning fractions should be the development of ability to translate from one form of representation to another (Lesh, et al., 1987).

Students' ability to connect fractions with other forms of rational numbers is also an indicator of understanding rational numbers (Oppenheimer & Hunting, 1999). Moreover, it seems that students' ability to see fractions as division of the numerator by the denominator is an indicator of understanding fractions (Newstead & Murray, 1998). Finally, we feel that in order to understand fractions, students must be able to reflect on the concept of fraction. This means that students should be able to understand that fractions are not two numbers divided by a line, but they are the relationship of the two numbers: they can be seen as part of the whole, as a number that can be placed on the number line, as a ratio etc. This would be necessary while solving problems with fractions where students should be able to treat fractions in various notions for solving the problem. In this case, students should also be able to reflect on the solution, verify the solution and seek other approaches to solve the problem.

### **Aim of the study**

The aim of the present study was to examine whether the abovementioned factors determine hierarchical levels of understanding the fraction concept.

## METHODOLOGY

A test with 56 tasks was developed to measure the seven factors. The test was administered to 349 sixth grade students three times: at the beginning of the school year and more specifically in October, in the middle of the school year, in February and at the end, in May. The conduction of three measurements was required to test the reliability and stability of the proposed model across time. For the purpose of the present study, the three measurements were required to investigate whether the hierarchical levels were stable across time. It must be noted that during the period of the three measurements, students were taught according to the Cyprus National Mathematics Curriculum which gives unequal emphasis in the teaching of the seven factors. More specifically, up to the sixth grade most emphasis is given to the teaching of inductive reasoning, the development of the sense of the magnitude of fractions and connections with other forms of rational numbers. Less emphasis is paid to representations and even less emphasis in developing students' ability in argumentation and justification, reflection and mathematical explanations.

The results of the Confirmatory factor analysis (CFA) applied to the data showed that the proposed model had very good fit to the data in all the three measurements, supporting our hypothesis that the seven factors constitute fraction understanding. For the purpose of the present study, the z-scores that emerged from the CFA were used for each of the seven factors, but they were transformed so that for each factor the maximum score was 1 and the minimum 0. The required score for adequacy in the respective factor was at least 0.5, that is half the maximum. Students scoring less than 0.5 were not considered adequate in the respective factor. To examine for hierarchical levels of fraction understanding, students were clustered into categories; latent class analysis (LCA) was used for this purpose. LCA was expected to reveal categories of students that reflect different types of behavior with respect to fraction understanding. LCA was performed on the basis on students' computed score for fraction understanding in the first measurement. We tested the validity of three different models that hypothesized that students can be clustered in two, three or four categories respectively according to their level of fraction understanding. The results of LCA showed that the best solution with the biggest value for entropy and the smallest value for AIC and BIC indices was the three categories model (Entropy=0.829, AIC=1990.117, BIC=2012.708). Additionally, the average latent class probability on the basis of the three categories' model was satisfactory. There were 145 students in Category 1, 100 students in Category 2 and 74 students in Category 3. Means and standard deviations for fraction understanding were then computed for each of the three categories and it was revealed that Category 1 comprised students of low fraction understanding (L), Category 2 comprised those that had medium fraction understanding (M), while students of the third category had high fraction understanding (H).

## RESULTS

The results for the three measurements are presented in Table 1 that follows.

	First measurement			Second measurement			Third measurement		
	L*	M	H	L	M	H	L	M	H
Factor	<i>X</i>	<i>X</i>	<i>X</i>	<i>X</i>	<i>X</i>	<i>X</i>	<i>X</i>	<i>X</i>	<i>X</i>
Inductive reasoning	0.63	0.79	0.92	0.74	0.88	0.94	0.71	0.87	0.94
Definitions & explanations	0.11	0.30	0.54	0.27	0.33	0.50	0.27	0.45	0.56
Argumentation & justification	0.18	0.40	0.62	0.28	0.51	0.63	0.35	0.54	0.66
Sense about the magnitude	0.42	0.72	0.90	0.60	0.79	0.87	0.61	0.83	0.92
Representations	0.48	0.60	0.77	0.47	0.63	0.70	0.46	0.61	0.77
Connections	0.22	0.59	0.92	0.49	0.72	0.77	0.45	0.68	0.85
Reflection	0.26	0.28	0.58	0.20	0.41	0.58	0.24	0.36	0.59

\*Each category of students, L: Low fraction understanding, M: Medium fraction understanding, H: High fraction understanding

\*\* Values for standard deviation were omitted due to lack of space

Table 1: Mean values of students' ability for each of the seven factors for each of the three measurements

For the first measurement, it is evident that the means of the students belonging to the low category were less than 0.5 for six of the seven factors with the exception of inductive reasoning. The results for the low category suggest that students had adequate abilities only in inductive reasoning, while this was not the case for the other six factors. Concerning the medium category, the results differ since for four out of the seven factors, namely inductive reasoning, sense about the magnitude of fractions, representations and connections students had adequate abilities, while for the other three factors, their means were less than 0.5. In regard to the high category, the means for all factors were greater than 0.5, therefore students of the high group possess adequate abilities in all the seven factors. The results for the first measurement suggest that the answer to the question of the study is affirmative and the seven factors determine hierarchical levels of fraction understanding.

Table 1 shows that in the second measurement, low category students' means in the seven factors were low similar to the trend of the first measurement. Specifically, it was found that in only two factors, inductive reasoning and sense about the magnitude of fractions students of the low category possess adequate abilities, while in the other five factors their means were less than 0.5. Students of the medium category had

adequate abilities in five of the seven factors: inductive reasoning, sense about the magnitude of fractions, connections, representations and argumentation and justification, while in the other two factors, their means were less than 0.5. Concerning the high category, it was found that students had adequate abilities in all the seven factors, but in definitions and mathematical explanations, their score was on the cutting line. Again, the results of the second measurement suggest that the seven factors determine hierarchical levels for fraction understanding.

Concerning the third measurement, Table 1 illustrates that the results were similar to the results of the second measurement. More specifically, low category students were sufficient in only two factors, inductive reasoning and sense about the magnitude of fractions, those of the medium category in five factors: inductive reasoning, sense about the magnitude of fractions, connections, representations and argumentation and justification, while high category students were sufficient in all the seven factors.

The results of the three measurements reveal that the seven factors determine hierarchical levels with respect to understanding the concept of fractions and additionally, these levels were stable over time. It was found that students belonging to the low category had adequate abilities in only two of the factors: inductive reasoning and sense about the magnitude of fractions, those of the medium category in the previous two factors and additionally in connections, representations and argumentation and justification, while high category students had adequate abilities in all the seven factors. The results for the three measurements are presented in Table 2.

L	M	H
Inductive reasoning	Inductive reasoning	Inductive reasoning
Sense about the magnitude of fractions	Sense about the magnitude of fractions	Sense about the magnitude of fractions
	Connections	Connections
	Representations	Representations
	Argumentation and justification	Argumentation and justification
		Reflection
		Definitions and mathematical explanations

Table 2: Hierarchical levels of fraction understanding across the three measurements.

## DISCUSSION

The results of the present study indicate that the seven factors determine hierarchical levels of fraction understanding for sixth grade elementary school students. Moreover, it is noteworthy that these levels were found to be stable over time, thus enhancing the reliability of the results of the present study. We can distinguish three levels: at a first level, students are adequate in inductive reasoning and sense about the magnitude of

fractions, at a second level there is adequacy in the previous two factors and additionally, in connections, representations and argumentation and justification, while in a third level there is adequacy in the previous five factors, as well as in reflection and definitions and mathematical explanations.

Inductive reasoning and sense about the magnitude of fractions were found as the easiest factors that students of all categories were able to capture. Inductive reasoning was found as the easiest factor at the starting of the hierarchy and this could be explained by the fact that students at the elementary school first deal with the specific cases and with concrete objects, gradually ending at a more general conclusion. Moreover, the results indicate that the development of a sense about the magnitude of fractions is necessary at an initial stage for students to acquire fraction understanding. Therefore, students should be presented with activities such as comparison, ordering and other that foster the development of sense about the magnitude of fractions at an early stage (Clarke & Roche, 2009).

At a second level, students were able to connect fractions to other forms of rational numbers and division, to recognize representations and translate from one kind of representation to another and engage in argumentation and justification. These factors were found to be more difficult than the factors of the first level and thus require more advanced abilities. Only the students of medium and high fraction understanding were found sufficient in these factors, while those of low fraction understanding were not.

Finally, the third and higher level was found to comprise the factors of the previous two levels, as well as reflection and definitions and mathematical explanations, which were found as the most difficult factors. Reflection and definitions and mathematical explanations demand higher skills than the other factors and could be captured only by that portion of students showing high fraction understanding. More specifically, reflection refers to students' ability to think about their own thinking, to reflect on their solutions in problems with fractions, verify their solutions and seek for other approaches (Sierpinska, Bobos, & Pruncut, 2011), and all these abilities rest at a higher level. Similarly, according to Niemi (1996), only a small percentage of students were able to accomplish the explanation tasks and those students were considered to show exceptional performance and deep understanding of fractions.

The findings of the present study are important for two reasons. First, they reveal a hierarchy of the factors based on their difficulty. A second reason is that this hierarchy has implications for teaching, since they guide teachers as to which factors demand more advanced abilities, especially for students that are situated at the first levels of the hierarchy. Further research may also reveal teaching approaches related to these seven factors which may accomplish the maximum of fraction understanding. Moreover, the Cyprus National Mathematics Curriculum probably served as a factor that affected the results of the present study, thus future research might examine the impact of other curricula on this hierarchy.

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# **ARE GIRLS OR BOYS BETTER AT MATHEMATICS? A COMMENTARY ON THE GAME OF REPORTING GENDER DIFFERENCES**

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*In Norway, gender differences are traditionally reported for mathematics assessments. These results are used at a policy level to discuss the education system. However, when girls are found to consistently outperform boys on secondary school exams, while boys in previous grades have outperformed girls on national tests, what do these differences indicate about the competence of the students? More importantly, what do we risk if gender differences in test outcomes are used to make judgements about the effectiveness of the education system regarding equal opportunities to learn mathematics? An analysis of test frameworks and reported outcomes for three tests is presented before a discussion of the possible consequences of using such results in decision making.*

## **INTRODUCTION**

This paper presents a discussion about the reporting and use of gender differences on the many mathematics assessments Norwegian students take during their schooling. As in other countries, national governments administer a range of tests to help teachers monitor the learning of their students or to test the level of students' mathematical knowledge. Some tests are, in addition, used to monitor the education system. Whenever results are reported to the public, much emphasis is put on any identified gender difference. However, for some assessments, boys consistently outperform girls, while for other tests girls outperform boys. Can we then state that girls or boys are better at mathematics or that the education system disfavours one gender? Understanding what these gendered differences and similarities tell about the competence of Norwegian students and the education system is not easy. Lack of understanding could contribute to stereotype understanding of the opportunities of boys and girls to learn mathematics and lead to unintended policy decisions. The aim of this paper is to discuss the gender patterns in the results of mathematics assessments taken by Norwegian students, taking into consideration the test constructs, the stakes of the test, the age of the student, and the purpose of the test at a policy level. The purpose of this discussion is to reflect on possible consequences of the tradition to report gender differences by relating the Norwegian situation to the international research on gender differences. Consequently, this paper cannot be termed a traditional empirical paper, as the presented data and analysis are those produced by other researchers or agencies. Rather, it might be termed a policy paper.

## **PRIOR RESEARCH ON GENDER DIFFERENCES**

Schools should provide students with equal opportunities to learn mathematics. Gender differences in learning outcomes are a concern across the world, and such



differences are traditionally reported for tests and exams in many countries. In prior research, gender differences have been found both for overall results and for specific content elements (Liu & Wilson, 2009). As in Liu and Wilson (2009) gender differences are reported mostly in favour of boys, that is, boys are reported to have significantly higher scores on a specific assessment than girls. However, reports of girls scoring significantly higher than boys can also be found (Bjørkeng, 2011; Mullis, Martin, Foy, & Arora, 2012).

Gender differences are often explained by differences in self-efficacy or by boys' more positive disposition to competition or pressure (Hannon, 2012). Boys are, for instance, reported to respond more positively to high-stakes testing. However, it has been reported that when controlling for such factors, test results depend only on ability, in which case gender differences "disappear" (Hannon, 2012). In addition, Cotton et al. (2010) found that for multi-round mathematics tournaments, gender differences could be observed only for the initial round. In later rounds, only ability served as a significant predictor for performance. Other factors that might explain some of the gender differences found in classroom-oriented research are attribution and beliefs about learning mathematics (Shores & Smith, 2011). Boys are more often found to attribute success to their own skills and failure to factors outside themselves, such as the quality of the test or the teaching. For girls, the pattern is reversed.

In a recent meta-analysis, Lindberg, Hyde, and Petersen (2010) found only small effect sizes regarding gender differences. This supports the hypothesis that girls and boys perform similarly in mathematics. Summing up, it is likely that differences in test outcomes are caused by factors other than ability, and it might be likely that the nature of the test construct, the items on the test, the stakes of the test, and the testing circumstances influence test outcomes.

## **METHODOLOGY**

The research design for the analysis discussed in this paper resembles that of a systematic review (Danish Clearinghouse, 2007); however, only a few of the data sources are peer-reviewed, published, "high-quality research." Many of the reported data sources are empirical evidence reported by national agencies. Consequently, it might be more appropriate to label the study a multiple case study (Robson, 2002).

Three cases (tests) are selected. National tests and national exams are part of the national quality assessment system that provides knowledge about the Norwegian education system to schools, school owners, policy makers, or society at large. In addition, the entrance test to beginner college and university students administered by the National Mathematics Council (NMR) is selected because it is administered by an independent agency for policy purposes. This test is the only test except for national exams that has been administered at a regular interval for an extensive period of time in Norway. More importantly, this test is the only test at a national level designed for policy purposes only.

There are several restraints and issues regarding the transparency of the presentations of tests and the comparison of gender differences found in the response patterns to the

different tests. Most importantly, only for the national tests are whole tests released. For national exams, prior exam sets are made available only to teachers and their students, not to the public. Only one item is published for NMR's test. Data are gathered mainly through two approaches: Firstly, information about tests from national web sites or test frameworks is studied to identify test constructs and knowledge about item formats. Second, student outcomes are collected from multiple sources where such outcomes are published: national web sites for national tests and exams. For a few tests, research reports are published, in which case they are considered a primary source (e.g. Bjørkeng, 2011; Nortvedt, 2012). References to the data sources are given in the results section whenever results are reported or referred to. Given these restraints in data sources, the discussion will take into consideration only the overall results, discussing the differences using the available knowledge about test aim, construct, testing time, and stakes.

### **THE ASSESSMENTS, THEIR OUTCOMES, AND THEIR USE**

No term like *accountability* exists in the Norwegian language, and it would be wrong to claim that Norway has an accountability system (Elstad, Nortvedt, & Turmo, 2009), although the Ministry of Education and Research in 2004 introduced a national quality assessment system containing possible building blocks for such a system. The different assessments to be taken at a specified time are identified, as is the purpose of using each of the assessments, including the large international comparative studies TIMSS (Trends in Mathematics and Science Study) and PISA (Programme for International Student Assessment). While some assessments are meant only to provide information at a local school level (e.g. mapping test to screen students' numeracy and reading level), other assessments, like national tests, should in addition provide information at a national level. However, unlike in countries like the US or UK, exams are high stakes only to the individual student. No assessments should be high stakes to the local school (Elstad et al., 2009). However, some school owners, as in Oslo, have systems with what Carnoy and Loeb (2002) term "weak to moderate repercussion". The international comparative studies are used to monitor the level of student competence in mathematics at a national level (NDET, 2011a), as well as to inform education policy. The entrance test to beginner students administered by NMR is not a part of the quality assessment system. Rather, this test has been designed and is owned by an independent entity whose mission is to advise government agencies, the Research Council of Norway, universities, and colleges in matters regarding mathematics education and research policy ([www.matematikkradet.no](http://www.matematikkradet.no)). The purpose of administering the test is to monitor students' prior mathematical level, however, at the lower secondary level (Nortvedt, 2012). Test results are used for policy purposes.

#### **The national tests**

National tests are developed at the Norwegian Centre for Mathematics Education (NSMO) on behalf of NDET, which administers the tests (Ravlo & Johansen, 2012). The tests are computer-based and hence automatically scored. The national tests are designed to measure students' ability to solve pure and applied problems in the content

areas of number, measurement, and statistics without the aid of a calculator (Ravlo & Bondø, 2012; Ravlo & Johansen, 2012). These tests cover parts of the curriculum, however, geometry and functions, for instance, are left out. Testing time is 90 minutes.

National tests are compulsory for students in Grades 5, 8, and 9, who take the test in the autumn, a few weeks after the school year has started. The same test is given to students in Grades 8 and 9 (Ravlo & Johansen, 2012). Tests are supposed to be low stakes to the students and the local school (Elstad et al., 2009). The purpose of the national test is to provide schools, school owners, and the policy level with knowledge about students to inform teaching and policy. Overall test results are published on the public web platform Skoleporten (the school gate, [www.skoleporten.no](http://www.skoleporten.no)). Results are reported at several aggregated levels, but can easily be broken down to the school level. Data are available from the school year 2008–2009 until 2012–2013.

Grade	Gender	2008	2009	2010	2011	2012
5	M	2.1	2.1	2.0	2.0	2.0
	F	1.9	1.9	1.9	1.9	1.9
8	M	3.2	3.2	3.1	3.2	3.2
	F	3.0	3.0	3.0	3.1	3.0
9	M			3.5	3.5	3.5
	F			3.3	3.3	3.4

Table 1: Results of national tests for boys and girls, 2008–2012, based on data published at [www.skoleporten.no](http://www.skoleporten.no)

Test results are given as proficiency levels: three levels for Grade 5 and five levels for Grades 8 and 9 with a maximum of level 3 and 5, respectively. Boys have consistently outperformed girls at the national tests for the whole period. The differences are significant and more boys score at a high level (Ravlo & Bondø, 2012; Ravlo & Johansen, 2012), however, they are not highlighted at Skoleporten, where it is merely stated that boys have higher scores.

### Secondary school exams

The exams in Grades 10, 11, 12, and 13, are designed to assess to what extent students have reached the competence goals of the national mathematics curricula (NDET, 2011b). In Grades 11 to 13, a total of seven mathematics courses are offered. Students can take a maximum of three courses. For all mathematics exams, some students are selected for a paper-and-pencil, five-hour-long exam. Exams for all courses have two parts. Firstly, students should demonstrate both conceptual understanding and procedural fluency by working without a calculator solving more traditional items. Next, they should demonstrate their competence in problem solving and modelling, working either with or without technological tools such as a calculators, CAS tools, or dynamic geometry, in accordance with the curriculum goals of the mathematics course they are taking (NDET, 2011b). All national secondary school exams are developed by

groups of expert teachers on behalf of NDET. Each exam paper is scored by two independently trained assessors who are practicing teachers in secondary school. A detailed guide to help assessors judge the competence of the individual student is developed and published online each year for each exam (See, for instance, NDET, 2012). These guides are made public.

Grade/Course	Gender	2008	2009	2010	2011	2012
10/Compulsory	M	3.0	3.3	3.2	3.1	3.0
Lower secondary course	F	3.2	3.5	3.3	3.2	3.2
12/R1 – Optional course	M	3.3	3.3	3.3	3.4	3.3
Geometry, algebra, functions, statistics	F	3.5	3.6	3.5	3.6	3.4
13/R2 – Optional course	M		3.2	3.3	3.3	3.4
Geometry, algebra, functions, differential equations	F		3.6	3.5	3.5	3.6

Table 2: Average exam results for boys and girls 2008–2012, based on data published at [www.skoleporten.no](http://www.skoleporten.no)

An exam paper is marked on a six-mark scale, with 1 as the lowest score and 6 as the highest. Although boys in Grade 9 outperformed girls on national tests, girls outperform boys on the Grade 10 exam that ends compulsory schooling. Also in upper secondary mathematics courses, girls outperform boys on mathematics exams (Bjørkeng, 2011). Only for one course in Grade 11, selected by a few students, do boys and girls have equal results (Skoleporten.no). Exams are high stakes, and it might be expected, based on recent research (i.e. Lindberg, Hyde, Pettersen, and Linn, 2010), that no gender differences could be observed. However, consistent gender patterns exist (see Table 2).

### **NMR's entrance test for beginner students**

NMR's entrance test is administered to beginner students at study programs including a minimum of 60 study credits in mathematics at Norwegian universities and colleges. The test is designed to assess students' basic mathematical concepts and calculation skills typically included in lower secondary curricula (Nortvedt, 2012). They are, for instance, asked to solve calculation tasks involving decimal numbers and fractions without a calculator. Testing time is 45 minutes. The test is administered by NMR biannually and scored at the local institution. Participation on the entrance test is decided by the mathematics department at the individual college/university. Consequently, the sample that takes the test at each administration varies (Nortvedt, 2012). Usually between 5,500 and 7,000 students take the test. Students can choose to not participate, although results cannot be linked to the institution or to the students. Consequently, the test is considered low stakes. Although sample issues might influence test results, analyses demonstrate a stable gender difference (see Figure 1),

with male students outperforming female students at each administration (Nortvedt, 2012), and in 2009 and 2011, the difference were .53 and .51 standard deviations, respectively.

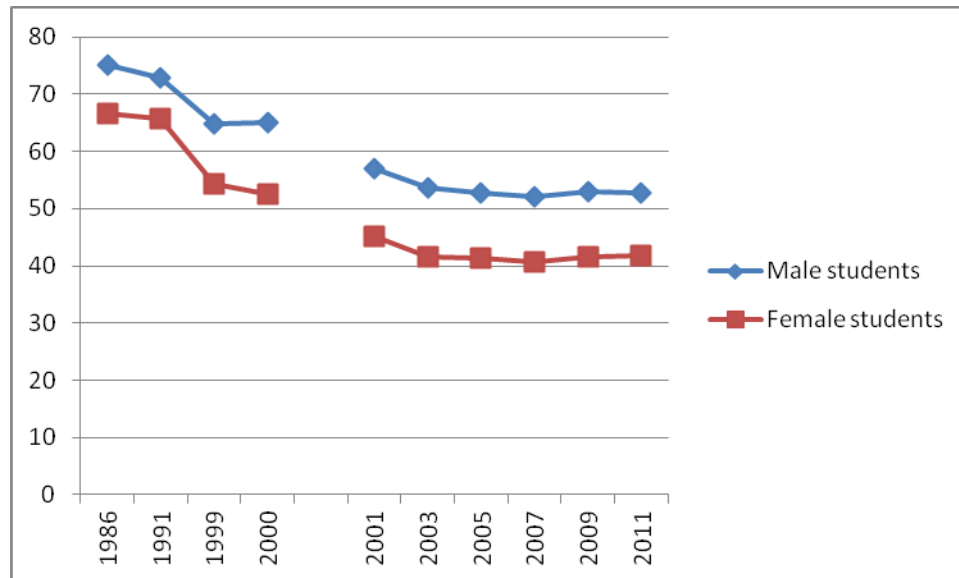


Figure 1: Differences in test results (% of maximum score) between male and female students on NMR's entrance test, based on Nortvedt, (2012). The instrument was changed in 2000, breaking the trend line.

## DISCUSSION AND CONCLUDING REMARKS

Several questions arise from the observed patterns. Can the gender pattern in the test outcomes be explained? Can there be unintentional consequences if the differences are not well understood, but still reported? The observed pattern presented in this paper is gender differences in favour of boys on national tests and the entrance test, while girls have the best results on exams. If these patterns reflect real differences, should we believe that for Norwegian students age 10 to 14, boys are better at mathematics, while for the remainder of secondary school, girls are better? This could be a possible result, but it is unlikely that male students two months after leaving school take the entrance test outscore female beginner students. No gender differences were observed in the last PISA study (Olsen, 2010), and in the TIMSS 2011 study, only small gender differences in favour of Grade 4 boys have been reported (Mullis et al., 2012). No differences were observed for Grade 8. These patterns have been stable for the international comparative studies (NDET, 2011a). The pattern observed for the international comparative studies is what could be expected based on current studies on gender differences (Lindberg et al., 2010). Rather, it might be that the observed differences are caused by differences in what is being measured; that the tests assess different aspects of mathematical competence. The national tests and the entrance test are, in the Norwegian context, short tests on which students have to demonstrate conceptual understanding and procedural fluency, working without a calculator. These tests have low stakes to the individual students, as test results are to be used for policy purposes or to inform the teaching at the local school. The exams, on the other hand, are high stakes to the students, as exam results are listed on their diploma. In addition, tests last for five

hours and students are tested on the full curricula. It is possible that the test constructs, purposes, and testing time contributes to the observed differences. What are possible policy consequences of naively believing in the results? Should we state that boys are better at mathematics because they have higher scores on tests where calculators are not allowed and go “back to basics”, that is, more traditional teaching in mathematics to boost girls’ calculation skills? Or, maybe we should believe that girls succeed better in mathematics because they have the will to work hard? Should we then change schools or exams to better adjust to the needs of boys? Elstad et al. (2009) found government white papers on education to cite the international comparative studies to a growing extent. If Norway moves towards an accountability system and more use is made of the reported data to inform policy, reporting gender differences might lead us to see differences that might or might not “be real” could have large consequences. Introduced interventions might have unintended effects if differences are not caused by differences in ability. Education policy should be built on knowledge about the effectiveness of the education system, its weaknesses and strengths. The national assessments offer insights that are valuable. However, using reported gender differences to inform policy calls for openness regarding what produces the numbers and the will to look beyond the surface of overall results to invest in educational research to understand better these differences.

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# EXPERT MATHEMATICIANS' NATURAL NUMBER BIAS IN FRACTION COMPARISON

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*School students have frequently been found to be biased by natural numbers when they compare the numerical values of fractions (e.g., they believe  $1/4 > 1/3$  because  $4 > 3$ ). Because the natural number bias has also been found in adults, it has been suggested that intuitive processes could be the source of the bias. We studied expert mathematicians on various types of fraction comparison problems to gain further evidence for the intuitive character of the bias. We found that experts are still biased by natural numbers when fraction pairs have common numerators or denominators, but not when fraction pairs have no common components. We discuss these findings and point out implications for mathematics education.*

## THEORETICAL FRAMEWORK

A large body of research has shown that students frequently struggle with rational numbers in general and with understanding the concept of fraction in particular (e.g., Mamede, Nunes, & Bryant, 2005; Sowder, 1988; Stafylidou & Vosniadou, 2004; Vamvakoussi, Christou, & Van Dooren, 2010; Vamvakoussi & Vosniadou, 2004). In this study we focus on difficulties that can occur when students are asked to compare the numerical values of fractions. On such problems, students have frequently been found to base their decision on comparing the natural number components separately, rather than on comparing the holistic fraction values. Exclusively relying on the components leads to correct answers in some cases (such as  $4/5 > 3/5$  because  $4 > 3$ ) but to incorrect answers in other cases (such as  $1/4 < 1/3$  even though  $4 > 3$ ). Thus, fraction comparison problems can be *congruent* (in the first cases) or *incongruent* (in the second cases) with the way natural numbers are processed. Assuming that individuals are biased by the natural numbers of the components, they should perform better on congruent than on incongruent fraction comparison problems, and this has indeed been found in students of different grade levels of primary and secondary schools (Meert, Grégoire, & Noël, 2010; Stafylidou & Vosniadou, 2004; Van Dooren, Van Hoof, Lijnen, & Verschaffel, 2012). This phenomenon has been referred to as the “natural number bias” (Vamvakoussi, Van Dooren, & Verschaffel, 2012; Van Dooren et al., 2012).

The source of the natural number bias is still a matter of debate (Ni & Zhou, 2005). Vamvakoussi et al. (2012) and DeWolf and Vosniadou (2011) explained it within the theoretical framework of dual-processing, an approach that has already been applied successfully to other mathematical domains as well (e.g., Gillard, Van Dooren, Schaeken, & Verschaffel, 2009). Explanations relying on this theory characterise the



natural number bias as an instance of intuitive reasoning rather than a lack of conceptual understanding. Because children have developed and internalized substantial knowledge about natural numbers long before they learn rational numbers, the magnitudes of the natural numbers involved may be processed automatically, and rapidly lead to a (correct or incorrect) representation of the fraction. Next to a cognitive system that rapidly processes these intuitions, there is a second system that is based on analytical reasoning. On incongruent fraction comparison items, this second system needs to be activated to inhibit the initial (incorrect) response. Because this system is slower and draws heavily on working memory, it is possible that the incorrect response is given before the analytical system can intervene. Accordingly, one would not only expect that incongruent items are more error-prone than congruent items (as already mentioned) but also that correct responses on incongruent items take a somewhat longer time than correct responses on congruent items. In fact, Meert et al. (2010) and Van Dooren et al. (2012) showed that this is the case in secondary school students. Although it turned out that the natural number bias decreased with age, Vamvakoussi et al. (2012) and DeWolf and Vosniadou (2011) revealed that it is still present in university students.

The finding that even older students and adults have not completely overcome the natural number bias in fraction comparison problems seems to be supportive of the dual-processing account. However, the abovementioned studies involved children or adults that had no particular expertise in mathematics and were presumably not able to apply sophisticated strategies for solving fraction comparison problems. Thus, it could be argued that overcoming the natural number bias in fraction comparison problems is a matter of mathematical expertise. As even pre-service teachers have been found to have problems with dealing with rational numbers in general (Putt, 1995; Stacey et al., 2001) we studied people with a level of mathematical expertise that is arguably among the highest possible, to test if it is in principle possible to *completely* overcome the natural number bias. Another limitation of the conclusions that can be drawn from previous studies is that we do not know whether the natural number bias occurs in fraction comparison problems *in general*. The fraction pairs that were used in previous experiments were in fact special cases: Each two fractions had either the same numerators or the same denominators (Meert et al., 2010; Vamvakoussi et al., 2012; Van Dooren et al., 2012). In the only study that analysed the natural number bias on comparing fraction pairs without common components (DeWolf & Vosniadou, 2011), an alternative explanation for the response time differences seems reasonable: Fraction pairs used as congruent items had on average a larger numerical distance. Therefore, the shorter response times on these items could reflect a distance effect. Indeed, just like for natural number comparisons, the difficulty of fraction comparison problems has repeatedly been shown to decrease with increasing numerical distance between the fractions (e.g., Schneider & Siegler, 2010). It is therefore still an open question whether the natural number bias occurs on fraction comparison problems in general, that is, also on problems that are not special cases, and when the numerical distance is no longer a confounding factor.

## RESEARCH QUESTIONS AND HYPOTHESES

We investigated whether a very high level of mathematical expertise allows people to overcome the natural number bias on fraction comparison problems, including both the special cases of fraction pairs (with common components) that were used in previous research, and fraction pairs that were not special cases (without common components). We expected that experts would be able to solve all the problems on a very high level of accuracy. However, assuming an intuitive character of the bias, we anticipated that they would still show “traces” of a bias. That means that for correctly solved items, response times would be longer when items are incongruent than when they are congruent, but only when componential comparison strategies are applicable, that is, when fraction pairs have common components (Hypothesis 1). We expected no such differences among fraction pairs without common components (Hypothesis 2), because expert mathematicians would not engage in (generally invalid) componential strategies on such items.

## METHOD

### Participants

Forty-four expert mathematicians (11 female, aged 29.59 years,  $SD = 6.87$ ) participated in the study. They were staff members of a Belgian University at the Department of Mathematics or at the Section of Applied Mathematics and Numerical Analysis of the Department of Computer Science. All of them had an academic degree in mathematics (master or higher) and were actually working as mathematicians. Twenty-six were PhD students, 12 were postdoctoral researchers, and six were professors.

### Comparison problems

To systematically study the occurrence of the natural number bias, we constructed items so that all possible combinations of congruency with natural number processing were covered. Only proper fractions (smaller than 1) and irreducible fractions (the greatest common divisor of the numerator and denominator is 1) were considered. With respect to congruency in the sense described above, there are five different types of items that involve non-equal pairs of fractions and that differ not only with respect to the order of the fractions. Two of these five types are fraction pairs with common components (these items were similar to those used by Van Dooren et al., 2012), while three types of (newly constructed) items do not have common components. Following this distinction, the item types are described in two blocks.

- Items with common components (block CC): If the fractions have either the same denominator (e.g.,  $5/8$  versus  $7/8$ ) or the same numerator (e.g.,  $5/7$  versus  $5/9$ ), it is sufficient to compare the non-equal parts in order to solve the comparison problem. Items containing fraction pairs with common denominators are always *congruent*, whereas items with common numerators are always *incongruent*.

- Items without common components (block WCC): If the fractions do not have common components, the items can be *congruent*, *incongruent*, or *neutral*: If each component of one fraction is larger than the respective component of the other fraction, the item can be congruent (e.g., 11/19 versus 24/25), or incongruent (e.g., 25/36 vs. 19/24). If the numerator of one fraction is larger than the numerator of the other fraction and vice versa for the denominators (e.g., 17/41 vs. 11/57), the item is neutral, because comparison of the components leads to contradictory results.

In none of the items, the numerator or the denominator of one fraction was a multiple of the numerator or the denominator of the other fraction. Very common fractions such as 1/2 or 3/4 were not considered. There were 18 items of each type, resulting in a total of 90 items. The average numerical distance between the fractions of each type was 0.20.

## Procedure

The participants took the experiment in individual sessions in their own office at the university, supervised by the first author. The problems were presented on a laptop and appeared in black colour on a white screen. Response times and accuracy rates were recorded using E-Prime 1.0 software. The items were presented in the two blocks (CC and WCC) as described above. The order of blocks was counterbalanced across participants, and items within each block were presented in randomized order. At the beginning of each block, participants were informed about the types of items that would be presented in the following block (numerators or denominators would be equal/would not be equal). After a fixation cross, two fractions at a time appeared, one on the left and one on the right side of the screen, and participants had to choose the larger fraction by pressing the corresponding key (“f” or “j”) on the keyboard. The correct answer appeared on the left or on the right hand side equally often. There was no time limit but participants were instructed to answer as quickly and as accurately as possible. Two practice items were presented at the beginning of each block.

## RESULTS

As mean accuracies were very high for all the items ( $M = 97\%$ ), only response time data were used for the analysis. Incorrectly solved items and items with response times that deviated more than two standard deviations from the individual mean were considered outliers and therefore excluded from the analysis, separately for each block of items (7% of items of block CC, 9% of items of block WCC). We used a General Estimation of Equations approach to statistically test differences between response times for the different types of items within each of the two blocks.

Within block CC, there was a significant effect of congruency, with larger response times on incongruent than on congruent items,  $\chi^2(1, N = 44) = 61.09, p < .001$ , confirming a natural number bias for items with common components. Within block WCC, there was also a significant difference between item types,  $\chi^2(1, N = 44) = 27.85, p < .001$ , revealing, however, a difference in the opposite direction, namely significantly shorter response times for incongruent than congruent and for

incongruent than neutral items ( $ps < .001$ ), whereas the difference between congruent and neutral items was not significant ( $p = .374$ ). Thus, participants were particularly fast with incongruent items. As the distances between fractions could not be the source of this difference (because they were equal), we analysed all the items of block WCC to identify item features that could explain this unexpected result. It appeared that there were several items that contained one fraction that was very close to 1 (e.g., 24/25), and which had a particular low mean response time. It seems plausible that such items were particularly easy to solve by “benchmarking to 1”. Benchmarking is a comparison strategy, where the fraction values are compared to a fixed value or benchmark – in this case to 1 (see Clark & Roche, 2009). As all the fractions in our experiment were smaller than 1, this was a very effective strategy that allowed to solve a comparison problem by taking into account the magnitude of only the one fraction that was very close to 1. We therefore repeated the analysis after excluding the items that contained a fraction that was very close to 1, defined as fractions with a difference between numerator and denominator of 1 or 2. Thereby, six incongruent, two congruent, and none of the neutral items were excluded. The analysis revealed, as expected, no significant differences between the item types within block WCC,  $\chi^2(1, N = 44) = 2.65$ ,  $p = .266$ . To summarize, the analyses revealed that other than for block CC, congruent items of block WCC were not easier to solve than incongruent or neutral items.

## **DISCUSSION**

For the first time, expert mathematicians were involved in a fraction comparison study, investigating whether they were biased by the natural numbers involved. We further extended previous research by using also fraction pairs that were not special cases and by controlling for the distances between fractions.

We could confirm the results from other studies (Vamvakoussi et al., 2012; Van Dooren et al., 2012) by detecting a natural number bias on fraction comparison problems with common components (Hypothesis 1). For this kind of problems, even expert mathematicians – although being accurate on almost all problems – were faster in giving correct responses when the larger fraction contained the larger natural numbers than when it contained the smaller natural numbers. The manifestation of this “trace” of a natural number bias in response times with simultaneously high accuracy even in expert mathematicians supports the assumption that intuitive processes are the source of the bias, as proposed by the dual-processing approach (DeWolf & Vosniadou, 2011; Vamvakoussi et al., 2012). It seems impossible to completely overcome an initial intuition of judging the fraction with the larger natural number as the larger fraction, even for expert mathematicians. As earlier research has suggested, the numerical values of natural numbers are activated automatically (see Hubbard et al., 2005). This holds even more strongly for the special case problems with common components in our study, because the participants knew that comparing the natural numbers would be sufficient to solve each problem. Thus, it is likely that they focused exclusively on the components of the fractions.

For fraction comparison problems that were not special cases, we did not detect a natural number bias, which is in line with Hypothesis 2 but in contrast to the results reported by DeWolf and Vosniadou (2011). Two arguments may explain our results. First, as mentioned earlier, the difference in response times between congruent and incongruent items in the study by DeWolf and Vosniadou (2011) could be an effect of distance, which was controlled in our study. Second, it is possible that the natural number values could not influence response times because – other than for problems with common components – the participants’ main focus was not exclusively on the natural number values of the fractions. This is again because they were told in advance that the fractions would not have common components, and so the participants certainly knew that focusing only on the components would be an ineffective strategy. Thus, although the natural number values have been found to be activated automatically (see Hubbard et al., 2005), there are presumably other factors influencing response times on such a relatively complex task as comparing fractions without common components. We assume that the main influential factor is the strategy that an individual uses to solve a specific problem. In fact, our initial analysis revealed that for items of block WCC, incongruent items were solved particularly fast, and we have explained this result in terms of a “benchmarking to 1” strategy that could be applied to items including fractions with specific features (namely a very small denominator-numerator difference). There were indeed more items including such fractions within the incongruent than within the congruent or neutral set of items of block WCC.

More generally, one could argue that the absence of the natural number bias in block WCC was due to confounds with other task features, because our item set was controlled for the differences between fractions but not with respect to task features that could have encouraged the use of specific strategies (and these strategies could in turn differ in how difficult they are to apply). Next to items that could be solved easily by benchmarking with 1 (as already discussed), there might have been items for which benchmarking with  $1/2$  was an effective strategy (e.g.,  $15/41$  is smaller than  $13/24$  because  $15/41 < 1/2$  and  $13/24 > 1/2$ ). It is also possible that the participants applied various strategies (such as standard algorithms or cross-multiplication, see Clarke & Roche, 2009) depending on the specific item features. Future studies could systematically analyse the strategy use on fraction comparison problems involving general cases of fractions. Furthermore, a future study could contrast our results to the performance of elementary or lower secondary school students. Because these students have been found not to be very efficient in adapting their strategies to specific task features and may stick to componential strategies also on comparison items without common components to a much larger extent (Clarke & Roche, 2009), we would expect to find a natural number bias in these students also on fraction pairs that do not have common components.

Even though our study was executed with expert mathematicians in an experimental context, it has implications for mathematics education. As the natural number bias seems to be unavoidable in fraction pairs with common components, students may still

struggle with incongruent fraction comparison problems, even when they have developed a sound conceptual understanding of what a fraction represents. As the natural number bias occurs on special cases of fractions even in expert mathematicians, the problem may well be due to the representation itself (i.e., the fact that the larger the fraction's denominator the smaller the numerical value) rather than to a poor understanding of the concept as such. Instruction can thus support students in understanding this representation, but this may never be sufficient to completely overcome the bias. In view of its intuitive character, a promising teaching approach could be to make explicit that people (even experts) are influenced by their intuitions while reasoning mathematically, that these intuitions can be misleading, and that conscious awareness and conscious control is necessary to suppress the natural number bias ("stop and think"; Greer, 2009; Vamvakoussi et al., 2012). This may eventually not help to improve performance by means of response times (which is certainly not the aim of classroom instruction), but could help to increase the correctness of answers.

Our data suggest that expert mathematicians use a variety of strategies on fraction comparison problems that are not special cases. Clarke and Roche (2009) have shown that also students apply a variety of fraction comparison strategies that they have not been taught explicitly at school, but also that they do not always apply the most efficient strategy, and that some of these strategies are indeed invalid. Accordingly, classroom instruction should enhance the appropriate and flexible use of strategies. More evidence is certainly needed to clarify which strategies are used by learners and by experts, and how strategy use depends on the specific problem features.

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# KNOWLEDGE CONSTRUCTION IN A COMPUTERIZED ENVIRONMENT: EIGHTH GRADE DYADS EXPLORE A PROBLEM SITUATION

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*As computers become more dominant in learning settings, studying the impact of computers on knowledge construction and on student performance becomes increasingly important, in particular for designing improved computerized learning environments. We investigated knowledge construction among eighth grade dyads that used GeoGebra to explore a problem situation related to functions. We analyzed the knowledge construction processes through the theoretical lens of the nested epistemic actions model (RBC) for Abstraction in Context (AiC). The tool we used to analyze the interactions was expanded by adding a human-computer channel. The findings indicate that the three dyads constructed the targeted knowledge while interacting with a dynamic and multi-representations environment.*

## INTRODUCTION

The construction of mathematical concepts provides students a firm basis for learning mathematics in school. A major problem that concerns teachers and researchers in the field is how to promote students' understanding of abstract concepts (Vinner & Dreyfus, 1989). Computerized learning environments appear to have the potential to support the learning of abstract concepts among students. Over the past three decades, activities in computerized environments have been developed, implemented and evaluated for further improvements (Tabach, Hershkowitz, Arcavi, & Dreyfus, 2008). Yet student learning processes in a computerized environment still need to be analyzed to support educational developers and decision makers.

The aim of the current study is to analyze the learning processes of eighth-grade student dyads as they solve a problem situation involving three functions in various representations. Abstract knowledge construction has been examined by the RBC model (Schwarz, Dreyfus, & Hershkowitz, 2009). We analyzed the interactions within the dyads and the computerized tool using an extension of the interactions symbolic representation method (Sfard & Kieran, 2001).

## THEORETICAL FRAMEWORK

The theoretical framework comprises (1) the function concept; (2) the meaning of representations in a computerized and dynamic environment; and (3) the RBC model.

### The function concept

The concept of a function is basic and central to mathematics (Leinhardt, Zaslavsky, & Stein, 1990). Understanding a function in one representation does not imply it is understood in other representations. Each representation emphasizes different



characteristics of the function concept. Hence, to fully acquire the concept, students must practice integrating its various representations and flexibly switching between them (Dreyfus, 1991). In Israel, eighth-grade students are required to understand the concept of a linear function in its verbal, symbolic, graphic and tabular representations.

### **Multiple representations**

The transformation from static symbolic representations to dynamic representations of semiotic mathematical objects in a computerized environment helps enhance mathematical explicitness (Bartolini, Bussi, & Mariotti, 2008). Dynamic tools also increase the co-action between the learner and the digital environment, so that learners become actively involved in the development of their mathematical knowledge. Activities that use dynamic representations are important, since they enable students to explore those dynamic figures via interaction that becomes more and more personalized (Moreno-Armella, Hegedus, & Kaput, 2008).

Using multiple representations may help students solve problems (Yerushalmy & Naftaliev, 2011). The capabilities offered by a selected computerized environment enable students to explore functions through their different representations as well as to examine the effect of changes in these functions (Tabach, Hershkowitz, Arcavi, & Dreyfus, 2008). Therefore, in our study we chose to represent the mathematical topic using multiple representations: verbal, tabular, symbolic and graphic.

### **Knowledge construction according to the RBC model**

The process of knowledge construction can be analyzed by using the nested epistemic actions model (RBC) for studying Abstraction in Context (AiC). The RBC model identifies three epistemic actions that occur in the process of knowledge construction: (a) *Recognizing* – the learner recognizes a known element relevant to solving the problem at hand; (b) *Building-with* – an action by which the learner uses recognized elements to achieve a goal, such as to solve a problem or justify a statement; and (c) *Constructing* – the main action in the knowledge construction process, in which the learner vertically reorganizes pieces of known constructs into a new construct (Schwarz, Dreyfus, & Hershkowitz, 2009).

In this study we examine three dyads as they interact in a computerized environment while solving one problem situation. The goals of this study are twofold: (1) to identify knowledge construction among the dyads and (2) to study the interactions between the students and the tool.

## **METHODS**

### **Population and tools**

The study participants comprised three dyads of eighth grade students who had no prior experience of working in pairs or working in computerized environment for learning. All the students were from the same class and, according to their teacher, all had a good background in mathematics. The pairs were selected based upon student friendships. All students reported extensive use of computers for social purposes.

*GeoGebra* was selected as the technological tool since it supports multiple representations that are dynamically synchronized: moving the graph of a function while simultaneously changing its symbolic representation, and vice versa.

### The Flooring Competition Activity

The activity had two parts. The introductory activity (about 25 min.), which was aimed at refreshing students' knowledge about calculating the area of rectangles using numbers and variables, was carried out twice: first in a non-technological environment and then using *GeoGebra*, so that the students could learn to use the tool.

The main activity immediately followed the preparatory activity and lasted around 80 minutes. It involved a problem situation— a competition among three flooring groups (Figure 1, based on Tabach, Friedlander, 2004).

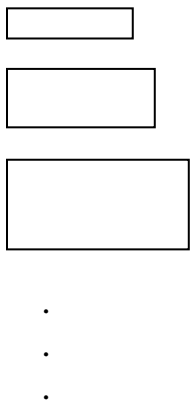
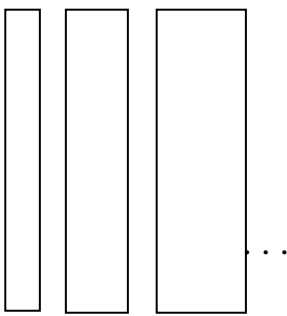
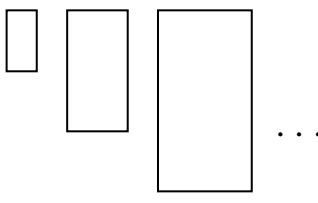
The following sequences of rectangles represent the flooring rates of three construction groups		
<p><b>Group A</b></p>  <p>At the end of the first day, the floor width is one unit, and it increases by an additional unit each day.</p> <p>The length is always three units greater than its width.</p>	<p><b>Group B</b></p>  <p>At the end of the first day, the floor width is one unit, and it grows by an additional unit each day.</p> <p>The length is always 10 units.</p>	<p><b>Group C</b></p>  <p>At the end of the first day, the floor width is one unit, and it grows by an additional unit each day.</p> <p>The length is always twice the width.</p>
At what point during the first ten days does the floor area of one group overtake the area of another group?		

Figure 1. Flooring competition problem situation

The activity involved six questions. In Questions 1-2 the students were asked to hypothesize about the growth of the floor areas. In Questions 3-4 they were asked to organize the data about the rectangle area in tabular, symbolic and graphic

representations. Question 5 asked them to check the hypothesis. To this end, the students had to develop the concept of intersection point ( $C_{\text{Intersection Point}}$ ). Questions 6 asked the students to find the difference between the competitors in three representations and to explain the resulting positive and negative values (a construct we named  $C_{\text{Difference}}$ ). In Question 7 they were asked to move one of the graphs vertically, which caused a corresponding change in parameter  $b$  of the parametric representation  $y=ax+b$  ( $C_{\text{Constant } b}$ ), and to explain the meaning of such a change in the given context ( $C_{\text{Inequality}}$ ).

The students were guided by the first author, who answered mainly technical questions. The work of each dyad was videotaped and transcribed verbatim. In this paper we focus on the students' work on the last three questions (5-7). Each dyad's work was analyzed separately in two ways: cognitive analysis of knowledge construction and interaction analysis.

### Cognitive analysis

The sequence of questions was designed to allow for knowledge construction. Figure 2 depicts the knowledge elements that may be constructed while working on Questions 5-7. The connections among these elements may be understood from the figure. The transcribed work of each dyad on these questions was analyzed for traces of these four knowledge constructs, as discussed in the Findings section.

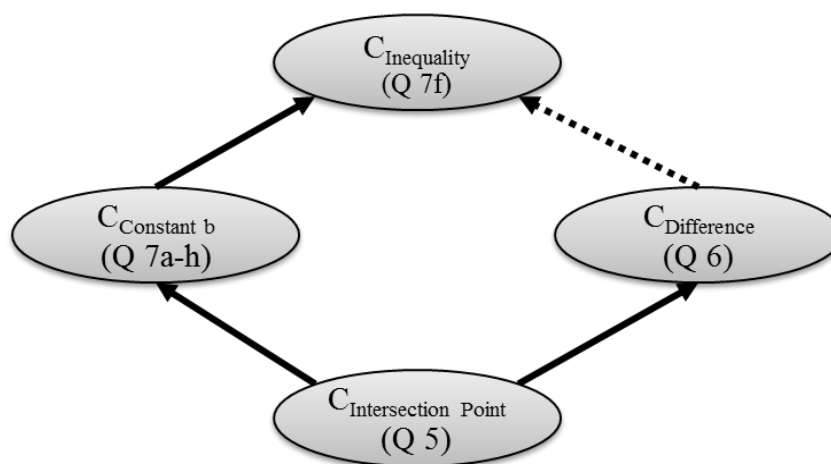


Figure 2: Knowledge elements in Questions 5-7 \*

\* The solid arrows indicate a strong dependency between constructs, while the dotted arrow indicates a loose dependency between them.

### Interaction analysis

Sfard and Kieran (2001) developed a methodology to document and analyze interactions. They defined private and interpersonal communication channels, and used arrows to show reactive or proactive interactions among them (see three left columns in Figure 3). The current study also involved a human-computer channel. Hence, we expanded this model to include the computer interactions in the human-computer channel (Figure 3, right column).



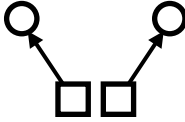
Channel of Type Utterance	Inter-personal	Private	Human-computer
Reactive			

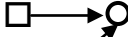

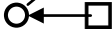



Figure 3: communication channels

Our research revealed reactive interactions in the private and interpersonal channels and in the human-computer channel. The reactive arrow points backwards or upward, starting at the source utterance and pointing to the utterance it reacts to.

## FINDINGS

Due to space limitations, we include excerpts taken from the work of two dyads only, Dave and Leo for Question 5 and Omer and Sam for Question 6. In each case the excerpt is followed by a brief analysis.

The following transcript is taken from the work of Leo and Dave on Question 5. In addition to their utterances, the table includes our representation of the interaction.

- 683 Leo: And the area, it seems to be 18 (see Figure 4) 
- 684 Dave: It's written "18" 
- 685 Dave: We had it written... scroll up to the first (first area where the table was written and points at the table) 
- ...
- 686 Leo: Both intersect... 
- 687 Leo: Then, it is logical 
- 688 Dave: They are on the same point, they have the same area 

In solving Question 5, Dave and Leo first recognized the points on the graph and their corresponding tabular representations (684, 685). Dave directed Leo to the location in the table where the numerical values of the points were recorded. They identified the intersection points on the graphs and in the tables. They expressed their understanding that the areas of rectangles A and C are equal at the intersection point (lines 686-688), and hence constructed  $C_{\text{Intersection Point}}$ .

Dave and Leo cooperated with one another. Dave was more dominant in the interaction and helped Leo construct his knowledge. Leo appeared to feel free to express his answers.

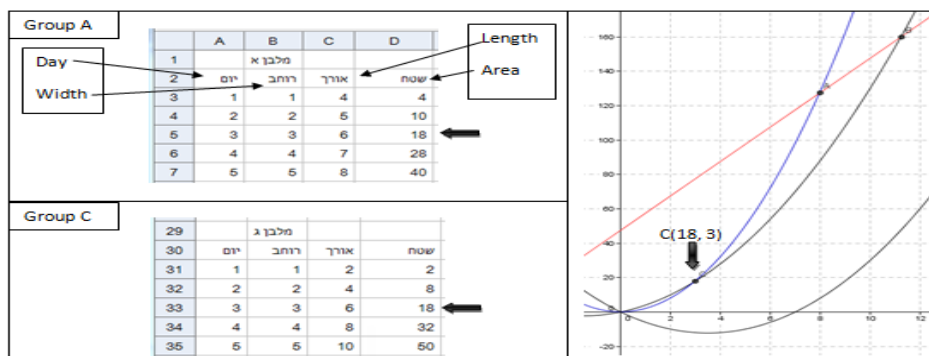


Figure 4: Representations in two tables and in graphs

The following transcript is taken from the work of Omer and Sam on Question 6.

- 896 Omer: (Reads question 6) "Explain how you can compare  
the rectangle areas on different days using the  
graphs"

- 897 Sam: Look, where it is down, it shows the difference as  
negative (points below the X axis).

- 898 Omer: We saw that until the sixth day it (the difference) is negative, also on the graph. On the seventh day, we see it had a point here, exactly on the zero.

...

- 907 Sam: (Sam writes the answer in his notebook, reads out loud and looks at the computer screen). The graph shows the comparison between the rectangles...

- 908 Omer: (Omer looks at the screen and helps Sam phrase the answer.) At the beginning the graph on the first six days has negative (values)

- 909 Sam: (Sam continues writing, looks at the screen and reads out loud.) As we wrote in the (Difference) column, and then we see the intersection point with the zero, and the Difference grows to become positive, as in the (difference) column.

Sam and Omer compared the negative and positive values in two representations, tabular and graphic, in order to justify their answers while constructing  $C_{\text{Difference}}$ . Omer said, "it is positive, it (the difference) is negative, also on the graph" (898). Sam said, "the graph shows the comparison between the rectangles" (907). Omer helped him phrase the answer: "At the beginning the graph on the first six days has negative (values)" (908). Sam went on to write: "as we wrote in the (Difference) column" (909).

Sam and Omer worked in cooperation. They were focused on the task and constantly interacted. Omer summarized their conclusions and Sam documented them with

Omer's help. Omer tended to dominate the work with the computer, and when he himself was not working on it, he instructed Sam what to do.

## SUMMARY AND DISCUSSION

In this study we analyzed knowledge construction and interaction processes among eighth grade students, while exploring a problem situation involving multiple representations in a computerized environment. The goals of this study were twofold: (1) to identify the knowledge construction by the dyads and (2) to study interactions between the students and the tool.

The knowledge construction in a computerized environment was analyzed using the RBC model (Schwarz, Dreyfus, & Hershkowitz, 2009). The analysis showed that all dyads constructed the targeted knowledge elements. To solve Question 5 they constructed  $C_{\text{Intersection Point}}$  by "building-with" two previously learned knowledge elements: marking intersection points and identifying the coordinates of a point on the graph. To solve Question 6, they constructed  $C_{\text{Difference}}$  by using existing constructs along with the construct acquired while solving Question 5. To solve Question 7, the students used  $C_{\text{Difference}}$  and other knowledge elements to construct  $C_{\text{Inequality}}$ . Clearly the students alternatively and simultaneously used the three different computer representations of the same data while constructing their knowledge. Different representations were used to verify or refute their conclusions, as in "we had it written" (685) or "the graph ... as we wrote in the column" (907). This demonstrates the added value of learning the function concept using multiple representations, as Dreyfus (1991) claimed, since each representation emphasizes other characteristics of the function. Therefore it can be said that the computerized and dynamic environment responds to the variance in learning preferences and needs of different students (Tabach, Hershkowitch, Arcavi & Dreyfus, 2008; Kaput, 1999).

We also investigated the interactions between the dyad and the computer. Within each pair of students, one led and supported the other, so that the cooperation between them promoted knowledge construction processes (Sinclair, 2005). The human-computer channel, an innovative contribution of the present study, was used by the students mainly as a verification tool.

The results of our analyses can contribute to mathematics curricula designers and to designers of computer aided learning environments. The dynamic environment supports the construction of abstract mathematical concepts by linking multiple representations of the same data being investigated by the students.

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# IDENTIFYING SITUATIONS FOR FIFTH GRADERS TO CONSTRUCT DEFINITIONS AS CONDITIONS FOR DETERMINING GEOMETRIC FIGURES

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*This study aims to clarify the aspects of the transition from empirical to deductive recognition in terms of how fifth graders can construct geometric definitions as their determining conditions. Through design experiments, we identified five situations in which students enhanced their recognition of such definitions: (1) Understanding the meaning of identifying geometric figures, (2) Constructing examples from non-examples and justifying the constructions via comparisons, (3) Recognizing equivalent combinations, (4) Examining undetermined cases via counterexamples, and (5) Conceiving figures as relations beyond the given actualities. We consider these situations as vital steps that help to advance students' elementary geometry recognition ability along the path toward understanding deductive proof.*

## INTRODUCTION

As a part of a study clarifying the process of bridging the gap between empirical and logical recognition in geometry, we examine how fifth graders construct definitions of geometric figures. The gap between empirical and logical thinking has been identified by the van Hiele model (1986) as a difference between second- and third-level thinking. Battista (2007) explained that in second-level thinking, students can “use and formulate formal definitions for classes of shapes. However, their definitions are not minimal because forming minimal definitions requires relating one property to another using some type of inferential reasoning.” Meanwhile, in third-level thinking, students can “logically organize sets of properties.” Thus, this progression of the defining process becomes an indicator of students' development from empirical to deductive recognition ability.

Some have claimed definitions should not be given but rather constructed by students (de Villiers, 1998). Freudenthal (1973) stated that “most often definitions are not preconceived but the finishing touch of the organizing activity. The child should not be deprived of this privilege.” While several studies have reported student difficulties in recognizing and constructing definitions (Wilson, 1990; Vinner, 1991; Moore, 1994), we note that such studies also included strong influences by irrelevant concept images. We find few studies that have explored the actual processes of constructing definitions in the classroom (see Mariotti and Fischbein (1997) for one important study), especially with respect to elementary students before formal proofs are introduced. In this paper, therefore, we attempt to clarify how fifth graders construct definitions of figures through design experiments. In particular, we focus on what situations allow students to enhance their understanding of definitions and what difficulties they encounter in the learning processes.



## **THEORETICAL BACKGROUND**

### **Several requirements as mathematical definition**

In examining a construction of definitions, we start with an overview of the concept of definition and its roles. First, a definition includes several requirements (Zaslavsky and Shir, 2005; Leikin and Zazkis, 2010): 1) A list of properties of the concept which are necessary and sufficient conditions of the concept; 2) Any example of the concept must fit all the requirements of its definition; 3) A definition is arbitrarily chosen from a set of equivalent statements; 4) All conditions of a definition should coexist; 5) Its meaning should be uniquely interpreted; 6) It must be invariant under change of representation; 7) It should be based on previously defined concepts in a noncircular manner; and 8) It is economical, with no superfluous necessary conditions or information. Second, we find that a definition fills the roles of (1) introducing objects of a theory and capturing the essence of a concept by conveying its characterizing properties; (2) constituting fundamental components for concept formation; (3) establishing a foundation for proof and problem solving; and (4) creating uniformity in the meaning of concepts, which allows us to communicate mathematical ideas more easily (Zaslavsky and Shir, 2005).

Because our current subjects are fifth graders, we should not require a perfect understanding of all conditions. Nevertheless, we consider it important for them to recognize that a definition is a list of properties of a concept which are necessary and sufficient conditions of that concept, because secondary geometry does not function well without an understanding of definitions. We thus think it is effective to learn the multiple conditions necessary for determining a concept. Moreover, because one of the purposes of exploring a definition is to create uniformity among the meanings of concepts, it seems important to establish a common basis for their communication for all students, including fifth graders.

### **Difficulties in understanding definitions**

The difficulties in understanding definitions are frequently explained in terms of the interactions between concept definitions and concept images (Vinner, 1991), especially in cases where it is difficult to reconstruct a concept image successfully. Mariotti and Fischbein (1997) note, “great difficulty is caused by the conflict which often emerges between the need to differentiate, imposed by strong figural structures and the requirement to unify, to generalize imposed by the geometrical conceptualization.” Wilson (1990) found that while sixth and eighth graders have developed strong prototypes of figures, there can be inconsistencies between their definitions and examples, and that even if they recognize such inconsistencies they may not try to change their definitions. These findings indicate that the main difficulties may arise from limited visual images based on prototypical figures.

### **Defining activities for the transition from empirical to logical reasoning**

Let us now consider the statement ‘In a parallelogram, the lengths of the opposite sides are equal.’ While an elementary school teacher may use such a statement to explain

one property of a parallelogram, especially as a visual feature of a drawn or material figure, a secondary school teacher may phrase the statement as ‘If it is parallelogram, then the lengths of the opposite sides are equal.’ de Villiers (1998) characterizes such a transition from (van Hiele’s) level 2 to level 3 thinking as the development of an understanding of the logical structure of “if-then” statements. Specifically, he describes the kinds of definitions at the van Hiele levels 1, 2, and 3, respectively, as follows: 1) Visual definitions, 2) Uneconomical definitions, and 3) Correct, economical definitions. Below, we summarize the transitional stages connecting the empirical and logical thinking stages, in which the first three stages correspond to the three kinds of definitions described by de Villiers.

1. Empirical stage: students visually understand the properties of figures.
2. First transition stage: students understand geometric figure as a set of properties which can be detached from drawn or material figures.
3. Second transition stage: students understand the conditions for identifying figures in which the definitions are differentiated from the remaining properties.
4. Logical stage: students understand a proof as a sequence of statements connecting hypotheses and a conclusion based on definitions, axioms, existing theorems, and logical rules.

In the empirical stage, students’ views of properties may be strongly influenced by the visual characteristics of a shape, such as a parallelogram being conceived as right-learning and having different adjacent angles. Thus, we claim that the first step in the transition of thinking is that the geometric figure being considered needs to be conceived as a set of properties which can be detached from any particular figure being observed. In the logical stage, however, the properties of a parallelogram, for example, must be proved based on definitions. Thus, as a second transitional step, we claim that it is necessary to recognize certain properties, such as the fact that a parallelogram can be identified by its having just two pairs of parallel opposite sides and where any remaining properties are not necessary for a sufficient description of its definition.

## DESIGN EXPERIMENT

We conducted a design experiment with 30 fifth graders in a classroom of a university-attached school, with the collaboration of a teacher with 18 years of teaching experience. The students had already learned the properties of geometric figures in the fourth grade. The experiment consisted of three stages: fostering dynamic views of geometric figures (three lessons), exploring inclusion relations between figures (four lessons), and constructing definitions of figures (three lessons).

In the first stage of the experiment, we aimed at fostering the students’ dynamic views of figures using “operative sheets” of paper, which were made of transparent paper on which pictures of a line, parallel lines, a right angle, and a 60 degree angle were drawn (Fig. 1). The combination of two sheets enables one to make various figures, and if a sheet is moved by translation or rotation, then a group of figures can be constructed. The students explored how one figure could be transformed into another.

The second stage of the experiment was designed to show how students give and develop their arguments in the learning of quadrilateral inclusion relations. As a result, they can enhance their arguments based on similarities or differences between properties, general-specific relations, and consistency among relations along with the convictions of others (Okazaki, 2011). When asked to describe a parallelogram

after learning about inclusion relations, for example, students listed the properties of parallelograms (uneconomical definition). In addition, they noticed the redundancy of some words in the property descriptions, such as in “the sizes of the adjacent angles are not equal,” as de Villiers (1998) noted. However, we believe that even if students are able to recognize the redundancy of partitional definitions, it is still hard for them to abandon their conceptions of partitionality, because the students sometimes reverted to the partitional definitions when referring to the tacit properties of prototypical shapes.

In this paper we explore the third transitional stage. In our experiment, the students were asked to identify the kinds of determining conditions of a parallelogram by combining parts of the following three properties: 1) ‘a pair of opposite sides is parallel,’ 2) ‘the lengths of a pair of opposite sides are equal,’ and 3) ‘the magnitudes of a pair of opposite angles are equal.’ They were also asked to check their ideas using the transparent operative sheets and then describe them.

Our design experiment was conducted according to the methodology described in Cobb et al. (2003). The lessons were recorded with video cameras and supplemented with field notes. We then made transcripts of the video data. We examined our data through both an ongoing analysis in a reflective session after each lesson as well as a retrospective analysis after all of the classroom activities had been completed. We used grounded theory (Glaser and Strauss, 1967) to encode and conceptualize the students’ ideas and explanations. In particular, we focused on those situations in which the students enhanced their recognition of the determining, or identifying, conditions of figures and the characteristics of their understanding in such situations.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
After 1 <sup>st</sup> lesson (N=30)	24	24	21	13	1	10	7	3
After 2 <sup>nd</sup> lesson (N=32)	32	28	29	24	3	16	7	13

Table 1: Identifying conditions recorded by the students: (a)  $AB \parallel CD$  and  $AD \parallel BC$ , (b)  $\angle A = \angle C$  and  $\angle B = \angle D$ , (c) type of  $AB \parallel CD$  and  $\angle A = \angle C$ , (d)  $AB = CD$  and  $AD = BC$ , (e) type of  $AB \parallel CD$  and  $AB = CD$ , (f) (error type 1)  $AD = BC$  and  $\angle A = \angle C$ , (g) (error type 2)  $AB \parallel CD$  and  $AD = BC$ , (h) duplication of same type

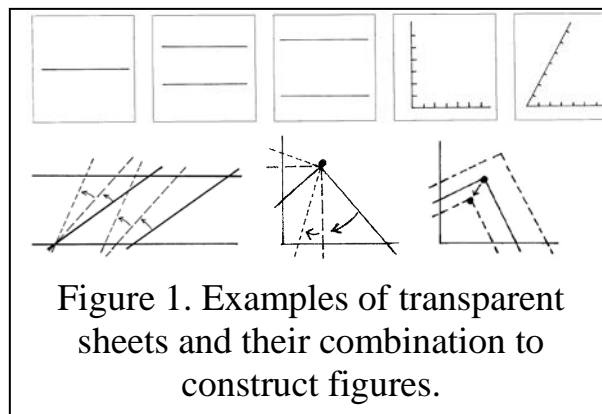


Figure 1. Examples of transparent sheets and their combination to construct figures.

## RESULTS

First we describe the numbers of identifying conditions the students wrote on the response sheets used for recording their ideas (Table 1). The data include the students' own ideas and those of the other students that the teacher wrote on the blackboard.

Next, in the following sections we present the five characteristic situations observed in the data that we considered important in the development of the students' recognition of the identifying conditions of figures.

### First situation: Understanding the meaning of identifying geometric figures

The teacher first instructed the students to list the properties of a parallelogram and clarified the correspondence among those properties, the symbols, and the figures (Fig. 2). After confirming the redundancy of the uneconomical definition, the teacher set the goal of defining a parallelogram with as simple a sentence as possible. The teacher first asked whether only one property (e.g.,  $AB \parallel CD$ ) comprised a parallelogram. The students' responses to this question included answers such as "A rectangle also has the property  $AB \parallel CD$ , so that property is not always a parallelogram." We found that the students did not consider the actual relations between parallelograms and superordinate figures, but rather considered the differences between parallelograms and subordinate figures based on their partitional views of the figures.

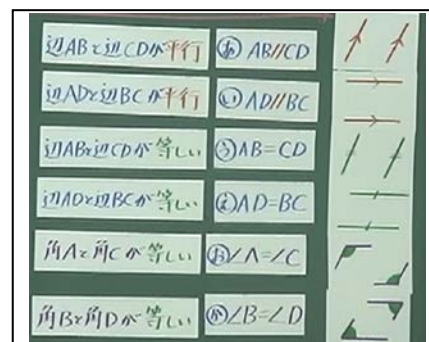


Figure 2: Correspondence among properties, symbols, and figures.

Then, using the operative sheets, the teacher demonstrated how a parallelogram collapses when using only one condition (See Fig. 3, sketched by the author due to faint video images; the same goes for Figures 7-9), and then asked the students what was needed to combine with  $AB \parallel CD$  as the context for constructing a parallelogram. At this point, the students could finally state " $AB \parallel CD$  and  $AD \parallel BC$ " or " $\angle A = \angle C$  and  $\angle B = \angle D$ " as the two determining conditions. The teacher then restated the problem as "The combination of what and what makes a parallelogram?". However, even in this situation a student asked whether it was okay for a parallelogram to be another figure. The teacher responded by reconfirming the basic premise: "Although the figure may become a rectangle or a square when you are making a parallelogram, that is okay because we are still basically constructing a parallelogram."

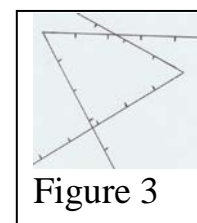


Figure 3

From these results we can conclude that the students had difficulty identifying geometric figures because of their partitional conceptions of those figures, and that they start their activities in the context of 'constructing' geometric figures.

### Second situation: Constructing examples from non-examples and justifying the constructions via comparisons

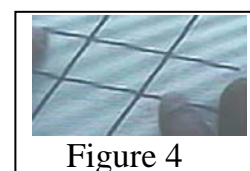


Figure 4

The students were able to identify the combinations of  $AB \parallel CD$  and  $AD \parallel BC$ ,  $\angle A = \angle C$  and  $\angle B = \angle D$ , and  $AB \parallel CD$  and  $\angle B = \angle D$  relatively early; these combinations were checked and confirmed using the operative sheets (Fig. 4). In this situation, the students would explain their decisions by first providing one condition in a case in which the second (necessary) condition was not true (Fig. 5, left); then, they would justify their construction of a parallelogram by providing the second of the two conditions (Fig. 5, right).

In the beginning of the second lesson, when the above combinations were re-confirmed, we observed the students' saying, "Angle B and angle C must be the same because the figure can't be a parallelogram if the angles are even slightly different" (Fig. 6). Also, the words 'determine' and 'identify' came to be used in such explanations.

When a new combination of  $AB = CD$  and  $AD = BC$  was proposed, several students thought that the new combination could not make a parallelogram. From these data we learned that the students were less aware of the condition that opposite sides needed to be equal than the condition that the lines had to be parallel. Thus, it was a surprise for them to realize that a parallelogram could be identified by a comparison between the lengths of its lines. Student A justified the idea as follows.

Student A: Because the lengths of both pairs (of sides) must be the same... If these are not parallel, (that is,) if we move one side in such way (Fig.7, left), then the length of this side is not the same... So, when we make the lengths the same, it becomes a parallelogram (Fig.7, right).

We found that Student A was trying to justify the identifying condition by comparing an example with a non-example beyond the context of constructing the figure. Moreover, their explanations more or less adopted the features of an indirect proof in which after all other cases are confirmed as false, the remaining cases are justified as uniquely true.

### Third situation: Recognizing equivalent combinations

Student B suggested the combination of conditions  $AD \parallel BC$  and  $\angle A = \angle C$ . Student C, however, opposed this combination: "It's the same as  $AB \parallel CD$  and  $\angle B = \angle D$ . They are the same." However, three students indicated that these condition combinations should be regarded as different. Then, Student D said, "We should interpret  $AB \parallel CD$  and  $\angle B = \angle D$  as both two parallel things and similar opposite angles, respectively, because both combinations show that the opposite sides are parallel and that the magnitudes of the opposite angles are equal; thus, I think they are the same." This opinion by Student D then allowed the students to reach a consensus regarding the equivalent condition



Figure 5: Parallelogram non-example (left) and example (right).

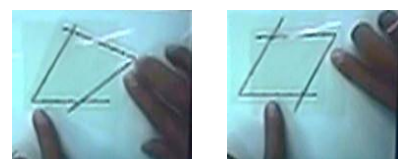


Figure 6

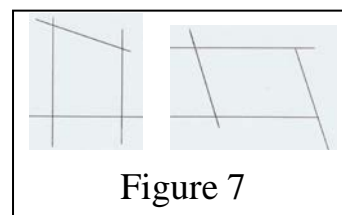


Figure 7

combinations. From these results, we thus concluded that it is effective to confirm the meaning of what is shown by particular symbols using natural language as a way to gain a sense of generality when understanding and expressing conditions.

#### Fourth situation: Examining undetermined cases via counterexamples

In the third lesson the teacher informed the students that one combination of conditions remained unidentified. Students A and B proposed the combinations of  $AD \parallel BC$  and  $AB = CD$ , and  $AB \parallel CD$  and  $AD = BC$ , and explained their decision by noting that one pair of sides was parallel and that the lengths of one pair of sides were equal. However, Student E proposed a counterexample (Fig. 8), with what the other students agreed was a correct counterexample.

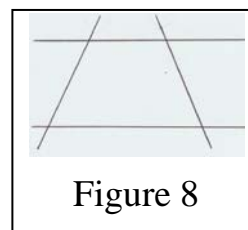


Figure 8

Next, Student E proposed a combination of  $AB = CD$  and  $\angle B = \angle D$ . Many students regarded this combination as being true (16 students at the time), and as a result counter opinions were not readily given. After a while, however, Student A provided a counterexample (Fig. 9).

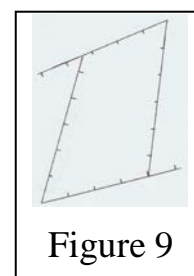


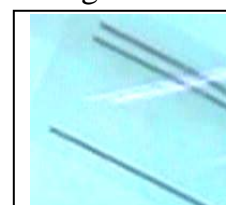
Figure 9

To us, these actions suggest that the students clarified the meaning of identifying a geometric figure by examining undetermined cases through the proposal of counterexamples.

#### Fifth situation: Conceiving figures as relations beyond given actualities

Finally, Student F did suggest the combination of conditions  $AB \parallel CD$  and  $AB = CD$ , and  $AD \parallel BC$  and  $AD = BC$ :

Student F: Even if the lengths of the opposite sides are the same, they may not be parallel... We must add sides that are parallel. If only the opposite sides are parallel, however, it may be a trapezoid. So, I think that both the lengths of the opposite sides must be the same and that the opposite sides must be parallel.



Several students did not agree with this suggestion.

Student A: Only  $AB \parallel CD$  and  $AB = CD$ ? When the sides are only parallel and of the same length, we have just sides  $AB$  and  $CD$ . There aren't any other remaining sides (Fig. 10).

The teacher then demonstrated how one could first take a pair of parallel sides of 2 cm for example and then connect their ends by adding two additional lines (Fig. 11). Seeing this, many students started wondering out loud, "Is that acceptable?" or "Can we be permitted to do that?" From this exchange, we discovered that the students believed that a line could not 'officially' be regarded as a line without it having already been drawn. Thus, it seemed that the students had finally learned that a geometric figure is not necessarily a (pre-drawn) picture, but rather the representation of abstract relations.

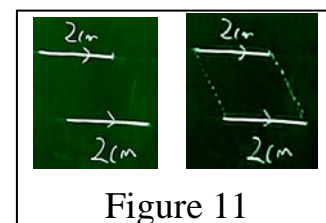


Figure 11

## CONCLUSIONS

Our results suggest five situations in which students can enhance their understanding of conditions necessary to determine figures: 1) Understanding the meaning of identifying geometric figures; 2) Constructing examples from non-examples and justifying the constructions via comparisons; 3) Recognizing equivalent combinations; 4) Examining undetermined cases via counterexamples; 5) Conceiving figures as relations beyond given actualities. We believe such situations are crucial to understand definitions and to transition from empirical to logical geometric recognition.

We have two future tasks: 1) Examine whether situations (1) to (5) constitute an order of emergence; and 2) Explore how students who experienced the learning reported in this paper can further advance to proof learning. We will design new experiments for such students, and through the new experiments we will attempt to clarify a curriculum better suited for the transition from empirical to logical recognition in geometry.

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# CHANGES IN FINNISH MATHEMATICS TEACHERS' BELIEFS DURING 1987-2012

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*The purpose of this study is to investigate what kinds of changes have occurred in Finnish mathematics teachers' beliefs during 1987-2012. The data was collected through a Likert scale questionnaire in years 1987 and 2012, and the analysis concentrates on the comparison between these studies. The information gained from this analysis show what kind of beliefs about mathematics and its teaching teachers' had 25 years ago and how they have changed. Changing teachers' beliefs can help to change teachers' behaviours and in such way improve teaching and learning process.*

## INTRODUCTION

Teachers' beliefs play a significant role in their teaching. They affect on what gets taught, how it gets taught and what gets learned in the classroom (Andrews & Hatch, 2000). Beliefs shape how teachers think and feel about mathematics and its teaching and learning. As teachers' beliefs affect their teaching, it is important to recognize those beliefs.

In the last decades the content and ways of teaching mathematics have changed. Based on the results of the survey called TALIS (Teaching and Learning International Survey) it is evident that teaching style has a big influence on pupils' performance (OECD, 2009).

In this paper the classification presented by Dionne (1984) is used, which introduces three perceptions of mathematics: traditional view (mathematics is limited to calculations and following rules), formalist view (stresses rigorous logic, proofs and exact use of language) and constructivist view (the pupil comes first and the emphasis is on pupil-centered learning methods and intuition.)

Pehkonen (1994) conducted a survey on Finnish mathematics teachers' beliefs about mathematics and mathematics teaching in 1987-1988 using Dionne framework. We have repeated the survey in 2010-2011 and we will report similarities and differences between these two measurements. We expected, that the changes in curriculum and the overall teaching philosophy in teacher education would be reflected also in teachers beliefs.

## Beliefs

Pehkonen and Törner (1998) summarized that an individual's mathematical beliefs are compound of his subjective, experience-based, implicit knowledge on mathematics and its teaching and learning. The spectrum of an individual's beliefs is very large, and its components influence each other.



We base our construction of beliefs and referring terminology on the article of Op't Eynde, De Corte, and Verschaffel (2002), who have strived for making a synthesis regarding previous belief researches. In the article, Op't Eynde and others (2002) define mathematical beliefs to be implicitly or explicitly held subjective conceptions people hold to be true, that influence their mathematical learning and problem solving.

Beliefs may have a knowledge-type nature, (e.g. view of mathematics: “mathematics learning is independent on gender”), which truthfulness can be discussed in social interaction, or volitional nature (individual and subjective; such as “It is important to me to provide good experiences with mathematics”). The latter kind of beliefs' validity can never be judged socially with any “scientific criteria”. Beliefs, as such, are subjective, something that an individual believes to be true, no matter whether the others agree or disagree. (Op't Eynde et al, 2002).

### **Mathematics teachers' beliefs**

Today mathematics teachers' beliefs and their impact are seen as ability and tendency to change (Wilson and Cooney, 2002). Also Lerman (2002) underlines that there is a strong link between beliefs and practices: changing teachers' practices will depend on changing their beliefs and changing beliefs will lead to change in practices. *Teacher change* consist changes in classroom behavior but also in the very art of teaching.

The importance of reflection in changing teachers' beliefs has also being recognized. According to Tobin (1990) reflective thinking about teaching can change the teaching behavior and actions. In addition to reflection, teachers' ability to attend to students' understanding of mathematics and to base given instructions on what and how students are thinking is also important

In the 1990's it was possible to identify two types of mathematics teachers in Finland. The traditional teachers emphasized basic teaching methods and extensive drilling, while the innovative teachers focused more on student thinking and deeper learning. (Kupari,1994). A similar distinction was identified also in an analysis of teacher responses to TALIS survey (Loogma, Ruus, Talts & Poom-Valickis 2009): traditional beliefs (the teaching is first and foremost the direct transmission of knowledge from the teacher to the pupil) and constructivist beliefs (the main emphasize is on the development of thinking and the understanding of the causal connections).

We use in this article the same standardized factor scales as Loogma, Ruus, Talts & Poom-Valickis in 2009: traditional beliefs (the teaching is first and foremost the direct transmission of knowledge from the teacher to the pupil) and constructivist beliefs (the main emphasize is on the development of thinking and the understanding of the causal connections).

### **RESEARCH QUESTION**

What kinds of changes have occurred in Finnish mathematics teachers beliefs' during 1987-2012?

## METHODOLOGY

In 1990 the study on Finnish teachers' beliefs was carried out for comparing Estonian and Finnish teachers of mathematics (Pehkonen & Lepmann, 1994). In 2009 a new cross-cultural survey of mathematics teachers' beliefs was initiated. In this article a study that is a part of this cross-cultural survey of mathematics teachers are presented from the perspective of the comparison. The study focuses on mathematics teachers working in lower secondary school (grades seven to nine). The teacher beliefs are categorized according to Dionne (1984) (traditional, formalist and constructivist).

### Instrument

*Instrument in 1987-1988.* In the beginning of the questionnaire there were questions about teachers' background. This was followed by 54 structured items about different situations in mathematics teaching originating from the research project "Open Tasks in Mathematics (Pehkonen & Zimmermann, 1990). The teachers were asked to rate their views within these statements on a five point Likert scale. Twenty-six (??) of these items were successfully classified into the three dimensions by Dionne (Pehkonen & Leppmann, 1994).

*Instrument in 2011-2012.* The questionnaire was made in connection with an international comparative NorBa study (Nordic-Baltic Comparative Research in Mathematics Education). The questionnaire includes seven modules, one of which includes 24 statements from the 26 'Dionne items' from the questionnaire in 1987-1988. We removed from our analysis two items that we considered unreliable for comparison. One had changed in its wording and the other was originally in a context of several related questions. Theoretical background, development and structure of the questionnaire as well as the sample items for first three modules are described more thoroughly in the previous papers (Lepik & Pipere, 2011; Hannula, Lepik, Pipere & Tuohilampi, 2012).

### Procedure

*Procedure in 1987-1988.* The data was gathered in two different ways: one part of the sample consisted of teachers on in-service courses (N=52) where the questionnaire was filled in at the beginning of the course. And the other group of teachers were reached by mail (N=34), i.e. the questionnaires were sent to them.

*Procedure in 2011-2012.* The data collecting process in Finland took place in two phases: during spring 2010 and between November 2011 and February 2012. First, informative letters and E-mails were sent to schools all over Finland inviting mathematics teachers to participate in the polling. Teachers who wished to participate in the polling filled in applications and sent them back to the university, or used an electronic form to inform about the willingness. Participation in the polling was voluntary. Respondents' identity and records were kept confidential: the report did not disclose teachers' personal data (name, school).

## Sample

*Sample in 1987-1988.* The respondents were 86 Finnish mathematics teachers with different teaching experiences and ages. The youngest teacher was 27 years old and the oldest 57 years old. The average age was 41 years. 60 % of the teachers had a major in mathematics, 21 % in physics and 16 % in chemistry.

*Sample in 2011-2012.* The sampling of Finnish mathematics teachers consisted of 94 teachers from different regions of Finland. The teachers were of different ages, education levels and teaching experiences. The average age of respondents was 41 with range of 25 to 61 years old. The average duration of teaching experience of respondents was 14.5 with range of 1 to 35 years. The majority of respondents hold a master's degree.

## ANALYSES

In both surveys we had 22 identical items. For the older data we used the means and standard deviations that were reported in the research report from Lepmann and Pehkonen (1994). Because we could not confirm the equal variance across samples, we used the Welch's t-test to compare the changes in teachers' beliefs.

## RESULTS

The statistically significant differences in 0.001-0.05 level between teachers' beliefs in 1987 and 2012 appeared in 55% of statements (12 statements out of 22). In the following tables, the mean results of every statement in every category are presented, and the statistically significant differences are marked with \* (0.05 level), \*\* (0.01 level) and \*\*\* (0.001 level).

Item number	Item wording	year 1987	year 2012
20	Mathematics teaching should emphasize logical reasoning	4.31	4.38
12	In teaching, one should proceed systematically above all	4.03	3.83
1	One has to pay attention to the exact use of language (e.g. one should distinguish between an angle and the magnitude of an angle, between a decimal number and a decimal notation)	3.86	3.70
11***	In particular, the use of mathematical symbols should be practiced	2.76	3.41
18	Abstraction practice should be stressed in mathematics	3.06	3.02
4	Working with exact proof forms is an essential objective of mathematics teaching	2.20	2.48
8	The irrationality of the number $\sqrt{2}$ has to be proved	1.92	1.84

Table 1: The formalist view.

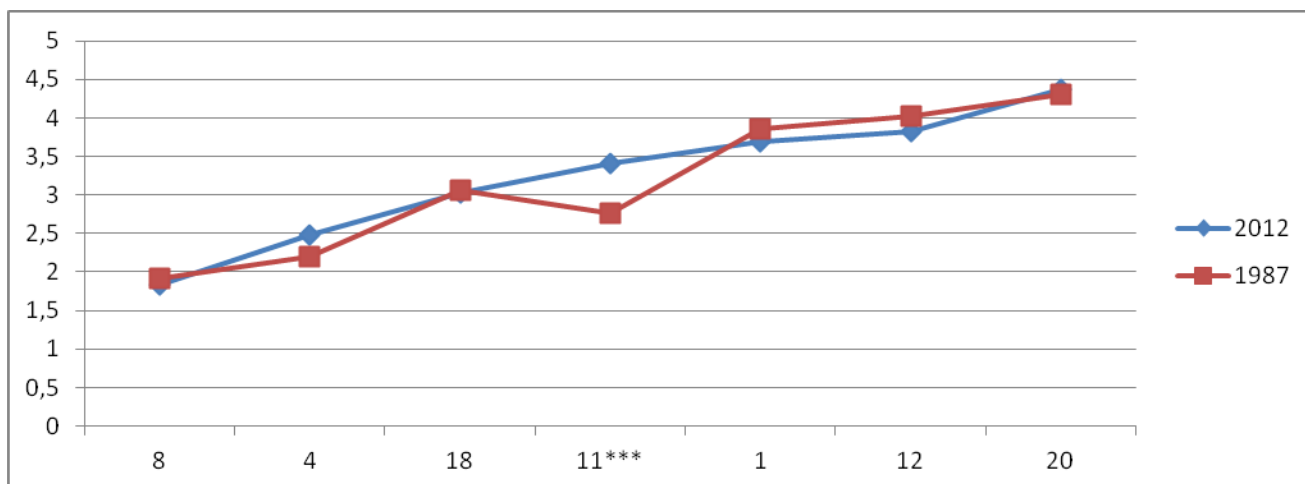


Table 2: Changes in the formalist view.

Item number	Item wording	year 1987	year 2012
13*	The learning of central computing techniques (e.g. applying formulas) must be stressed	4.01	3.71
6***	In mathematics teaching, one has to practice much above all	4.34	3.87
19*	Above all mathematical knowledge, such as facts and results, should be taught	2.86	2.48
2***	In a mathematics lesson, there should be more emphasis on the practicing phase than on the introductory and explanatory phase	4.22	3.48
17	As often as possible such routine tasks should be solved where the use of the known procedure will surely lead to the result	2.83	3.11
14	Pupils should above all get the right answer when solving tasks	2.65	2.41
16	A pupil need not necessarily understand each reasoning and procedure	3.20	3.14

Table 3: The traditional view.

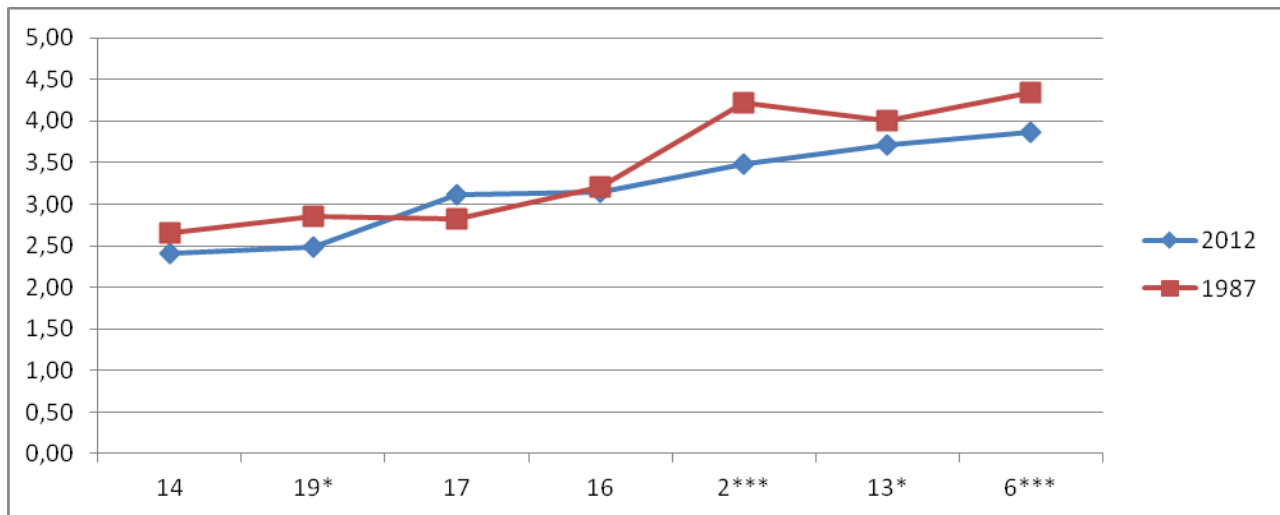


Table 4: Changes in the traditional view.

Item number	Item wording	year 1987	year 2012
21	Pupils should develop as many different ways as possible of finding solutions, and in teaching they should be discussed	3.95	3.88
22***	Pupils should formulate tasks and questions themselves, and then work on them	4.33	3.56
24***	As often as possible, the teacher should deal with tasks in which pupils have to think first and for which it is not enough to merely use calculation procedures	4.43	3.71
15***	Above all the teacher should try to get pupils involved in intensive discussions	2.84	3.89
10*	As often as possible, pupils should work using concrete materials (e.g. cardboard models)	3.66	3.96
5*	Sometimes teaching should be realized as project-oriented (beyond subject limits), and prerequisites for it should be created. (An example of the project: to buy and maintain an aquarium.)	3.42	3.80
9	In mathematics teaching, learning games should be used	3.67	3.87
3**	Mathematics has to be taught as an open system that will develop via hypotheses and <i>cul-de-sacs</i>	3.52	3.12

Table 5: The constructivist view.

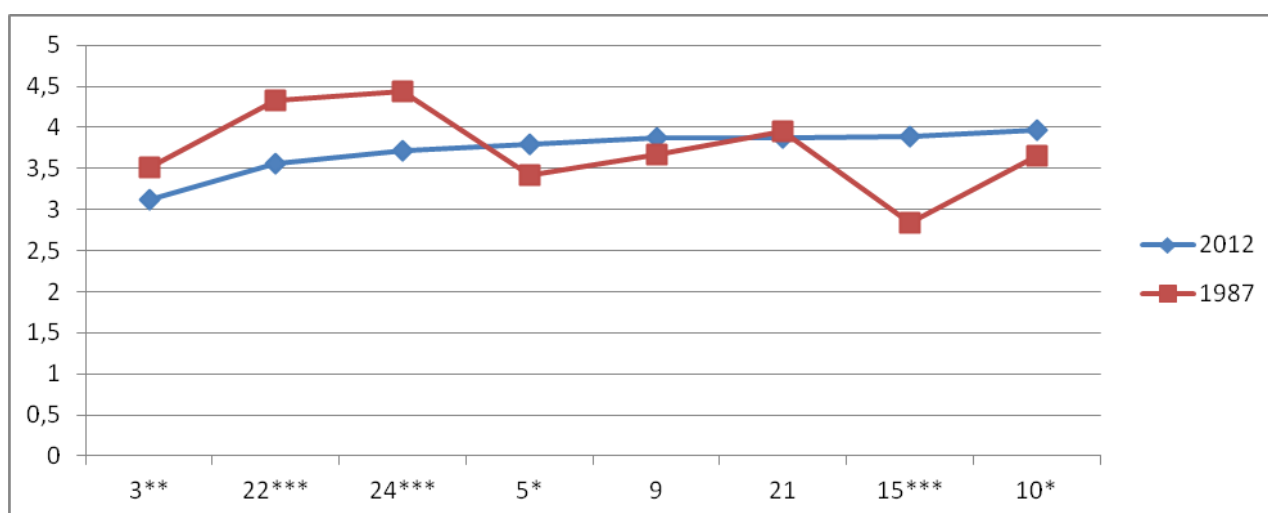


Table 6: Changes in the constructivist view.

## DISCUSSION

As expected, the constructivist way of teaching was in year 2012 supported the most strongly (mean=3.72). The formalistic way (mean=3.23) and the traditional way of teaching (mean=3.17) were supported the least. The directions in the changes are negative in traditional and formalistic view. The changes in constructivist view go to both directions.

When taking the three most agreed statements in the year 2012, the beliefs' of Finnish mathematics teachers can be described as follows: Mathematics teaching should emphasize logical reasoning, as often as possible, pupils should work using concrete materials and above all the teacher should try to get pupils involved in intensive discussions.

It is obvious that the teachers emphasize the importance of logical reasoning and exact use of language and systematizing. But for example sometimes teaching should be realized as project-oriented (beyond subject limits), instead of learning the central computing techniques and solving routine tasks. In the traditional view statistically significant differences were found in four out of seven statements. The support to teach facts and practice a lot of exercises have decreased, and the support to understanding increased.

The importance to use concrete materials has increased. Also the support on using social forms of learning, such as project work and class discussion, has increased. At the same time the independent thinking via problem posing and problem solving has lost some of its popularity amongst teachers.

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# WHAT IS POSSIBLE TO LEARN? USING IPADS IN TEACHING MATHEMATICS IN PRESCHOOL

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*The aim of this paper is to present results from a study investigating the potential of using Ipad when teaching mathematics in preschool. The study explores how the design of applications influences the dialogs that occur between teachers and children and the mathematics that is made possible to learn. The results presented in this paper are from the first investigation where the notions of classification and framing have been used to classify applications. Observations of teachers and children working together with different applications have been carried out and the results indicate that applications with weak framings promote free dialogues withholding mathematics, irrespective whether the classification is strong or not.*

## INTRODUCTION

According to Clements (2002) "there seems to be an increasing potential for children to use computers in early childhood settings" (p.160). The results presented in this paper derive from a Swedish study exploring the potential of using Ipad when teaching mathematics in preschool. Our experience is that Ipad in preschool are often used by children without interaction with teachers but according to Clements (2002), adults play a significant role in successful computer use. Scaffolding is critical when children use computers and it is positive for learning when teachers constantly encourage, question, prompt and demonstrate. Moreover a balance between teacher guidance and children's self-directed exploration has shown to be successful.

Based on previous knowledge about the importance of scaffolding the aim of the present study is to explore how the design of applications used on Ipad influence the participation of children and teachers, the dialogs that occur and the mathematics that is made possible to learn. Inspired by Lange and Meaney (in press) the notions classification and framing are used to categorize the applications used in the study. The results presented in this paper are from an initial investigation in the study.

This paper first presents the background of the study focusing on the context of Swedish preschools, young children's learning of mathematics and use of computers. Then, the theoretical settings regarding classification, framing, participation and dialogue are presented, followed by the design and the results of the study. The paper ends with a discussion and conclusions.

## BACKGROUND

Since the study was conducted in Swedish preschools this section starts with a short description of that context. (However, the implementation of the study and the results are not limited to that context.) Then, preschool children's use of computers and learning of mathematics will be focused on.



In Sweden, preschool is the foundation for lifelong learning and it is offered to children between the ages of one and five. Learning in preschool is expected to take place through children's play and a conscious use of play should be present in all preschool activities to promote the development and learning of each individual child. Children are to be given opportunities to play and explore their surroundings both on their own and together with other children and/or teachers. Multimedia and information technology are to be used in the Swedish preschool and there are several goals to strive for focusing on mathematics. The importance of making children participants in their educational practice is underlined in the preschool curricula (Skolverket, 2010a).

Ipads can be considered as artifacts, manipulatives or as tools. However, "[n]o one would expect a tool to do the work on its own, but an appropriate use of the tool leads to desirable results (Nührenbörger & Steinbring, 2008, p.158). There is limited research about the use of Ipads when teaching and/or learning mathematics in preschool but more is known about using computers for the same purpose. Sarama and Clements (2009) terms objects on the computer screen as computer manipulatives. Computer manipulatives may offer combinations of visual displays; opportunities to explore situations, animated graphics and speech, immediate feedback, while keeping a variety of records. Properly chosen, different computer games can help young children develop mathematical competence.

Young children have the interest and ability to engage in significant mathematical thinking and learning. Young children need to explore mathematics in meaningful circumstances, based on their perspective. Language, words and communication play an important role in younger children's learning of mathematics (Perry & Dockett, 2008; Ginsburg, 2009; Sarama & Clements, 2009). According to Nührenbörger and Steinbring (2008) and Sarama and Clements (2009), manipulatives and mathematics are initially separated and a relation between them does not emerge spontaneously or automatically. Manipulatives are meaningful for learning only with respect to learners' activities and thinking. Similarly, Bartolini Bussi and Maschietto (2008) claim that artefacts become efficient, relevant and transparent through their use.

## **THEORETICAL PREREQUISITES**

In this section the situative perspective of the study will be presented followed by an elaboration of the notions classification and framing. These notions will then be used in the next section to categorise the applications used in the study. Dialogues will also be shortly focused on, as they are part of the forthcoming analysis.

According to Lerman (2000), there "has been a turn to social theories in the field of mathematics education" (p.20). He bases this on the fact that mathematics education research since the late 20<sup>th</sup> century has begun to see meaning, thinking and reasoning as products of social activities and learning, thinking and reasoning as situated in social situations. The term situative refers to a set of theoretical perspectives and lines of research which conceptualise learning as changes in participation in socially organised activities and individuals' use of knowledge as an aspect of their participation in social practices (Borko, 2004). From a situative perspective, the physical and social context

in which an activity takes place is an integral part of the activity, and the activity is an integral part of the learning that takes place within it. “How a person learns a particular set of knowledge and skills, and the situation in which a person learns, become a fundamental part of what is learned” (Putnam & Borko, 2000, p.4).

Emilsson and Folkesson (2006) have used Bernstein’s notions of classification and framing when investigating children’s participation in preschool. Classification refers to the boundaries of activities and framing refers to the form of activities. A strong classification means that subjects and activities are strongly kept apart. A weak classification means the opposite, that is, subjects and activities are less specialized. In a strong framing the transmitter has explicit control over the communication, pacing and the social base, while in a weak framing the acquirers have more control over the communication. Emilsson and Folkesson’s study showed that activities in preschool with strong classification and strong framing reduced the participation of the children, while weak classification and weak framing, on the contrary, increased the participation of the children.

Nührenbörger and Steinbring (2008) stress that possible connections between a manipulative and mathematical knowledge always is produced by a person. In this study the person focused on is the teacher and the physical and social context created by the teachers when using mathematics applications on the Ipad. The character and quality of the interaction between children and teachers in preschool have shown to be of great importance for children’s possibilities to learn. Open questions are one effective pedagogical communication strategy and also a balance between adult direction and child initiatives in dialogues increase quality in communication (Skolverket, 2010b). In different kinds of dialogues children and teachers are assigned different roles. One way to characterize dialogues that occurs between children and teachers is by distinguish between free and directed dialogues. In a free dialogue the communication is informal between (a sort of) equals. In a directed dialogue one person stands for the majority of the speaking and the other participant only speaks when answering questions (Anward, (1983).

## **THE DESIGN OF THE STUDY**

The study presented in this paper is part of an ongoing project focusing on teaching and learning mathematics in preschool. One part of the project has included using Ipad and after an initial explorative phase we began to wonder how the design of the applications influenced the dialogues that occurred between teachers and children and the mathematics that became possible to learn. To investigate this, the above presented notions, classification and framing were used to classify applications into four categories. Interactive and non-interactive in the four categories refer to if the course of events is affected by the actions of the child or not.

	Strong Framing	Weak framing
Strong classification	<i>Non-interactive applications containing strictly mathematics.</i> (SFSC)	<i>Interactive applications containing strictly mathematics.</i> (WFSC)
Weak classification	<i>Non-interactive applications containing integrated mathematics.</i> (SFWC)	<i>Interactive applications containing integrated mathematics.</i> (WFWC)

Figure 1: Four categories into which applications were categorised.

In the initial study, one application was selected characterizing the classification and the framing of each category. (These four applications are presented in the next section.) Then observations were made of teachers as they worked with these four applications together with preschool children.

Due to the ongoing project, both the teachers and the children knew us. Both teachers and children had been working with Ipad before but, as far as we know, not with the applications used in this study. The result presented in the next section is based on observations of four preschool teachers from two preschools. To be able to discover similarities and differences that may be derived from the applications, each teacher was observed together with three children each. The children were four or five years old.

Before the observations the teachers were told that we wanted to observe what is possible for preschool children to learn when working with different applications. The four applications to be used were shortly presented for the teachers without mentioning anything about our classification. The teachers were instructed that they were free to interact with the children as they thought was most suitable in the situations.

During the observations an observation chart was used focusing on the communication that occurred. Did any communication occurred and when it did, on whose initiative? Was the communication that occurred in the form of free or directed dialogue? Was the object of the dialogues mathematics or something else? If the object in the dialogues was mathematics, what was communicated and how?

## RESULTS

In this section the four applications will be presented followed by an analysis of the observations. When starting to work with a new application all teachers used directed dialogues with all children as they introduced what to do. The analysis below is focusing on the time after those introductions. Our question was if different classifications and framings influenced the dialogues that occurred and the mathematics that became possible to learn. As will be shown below that seems to be the case. It was very clear that the teachers acted different based on the four

applications and not based on the children. Below the applications are presented in the same order as they were used during the observations.

### **Strong Framing & Weak classification: (SFWC)**

In this category the application *Nallemix 1* was chosen as an example of a non-interactive application withholding integrated mathematics. In the application the children are shown four to eight items on the screen whereof one doesn't belong together with the others. The task for the child is to identify and click on the item to be removed. When they click on the right item it disappears and a new set of items is presented on the screen. To be able to identify the item to be removed the child has to categorize the items based on their similarities and differences. Examples of categories in the application are flowers, food and furniture. There is only one possible right answer (according to the layout in the application).

Only directed dialogues occurred when using this application and the directions almost never regarded mathematics but either guidance or confirmation. The focus in the guiding were on what the items could be used for or named as, for example “what can you do with this?” or “this is a ...”. The confirmation didn't direct to why an item was removed but instead positive words as “good” or “right”. Sometimes it was visible that the children chose an item based on other criteria than the one predefined in the application but the teachers never asked about the reasons for children's choice when they, according to predefined in the application, did wrong.

### **Weak framing & Weak classification: (WFWC)**

In this category the application *Toy Block* was chosen as an example of an interactive application withholding integrated mathematics. In the application the child can build with three dimensional cubes. They can choose different colours on the cubes and they can rotate their design to view it from different perspectives.

When using this application the dialogue either became free containing a lot of mathematics focusing on shape and position from different perspectives, or there became no dialogue at all. The source to the difference seems to be the teacher as one of the teachers were totally quiet (only when working with this application) while the three others invited the children to talk about their constructions. Some children talked aloud at the same time as they built, describing where they placed their cubes. The three talking teachers used a lot of open questions and notions as cube, high, behind, in front of and perspective were used when talking about the constructions.

### **Weak framing & Strong classification: (WFSC)**

In this category the application *Toca Boca Store* was chosen as an example of an interactive application withholding strictly mathematics. In the application the child and (in this case) the teacher is cashier and customer in a shop. First they are to choose the products that the cashier will offer to sale in the shop. Then the customer buys one product at the time. The cashier sets the price (one to five coins) and the customer pays from a wallet withholding 10 coins at the start. Then this procedure continues until the

customer runs out of money and gets his or hers receipt. (Magical you get extra money if some are missing in the last purchase).

When using this application both teachers and children took initiative in the dialogues which was free and mostly focusing on mathematics. The teachers used a lot of open questions and the children were active when they set prices, counted money and talked about cheap and expensive products. Examples of the mathematics used in the free dialogues were counting up and down, adapt the purchase to what's left, figuring out if there is enough money or how much more money there is needed.

### **Strong framing & Strong classification: (SFSC)**

In this category the application *Fingu* was chosen as an example of a non-interactive application withholding strictly mathematics. In the application a low number of objects (fruits) move on the screen a few seconds and the children are to put as many fingers as objects on the screen. They don't have to put their fingers on the objects, just on the screen. If they do correct some happy figures occurs on the screen, if they do wrong an onion occur.

When using this application all talk was made by the teachers. Only directed dialogues were used and the directions almost never regarded mathematics but either guidance or confirmation. Putting the right number of fingers on the screen was very hard for the children. Even if they could count the objects they didn't manage the motoric part. The focus became on how to arrange the fingers and on how to get the fingers to touch the screen at the same time. Some children did manage this, but even then the dialogues were directed in form of confirmations as "good" or "correct". The children, both the ones who succeeded and the others, wanted to stop using this application quite fast.

## **CONCLUSION AND DISCUSSION**

From a situative perspective, the physical and social context in which an activity takes place is an integral part of the activity, and the activity is an integral part of the learning that takes place within it. This study is only in its initial stage and no conclusions can be made based on the limited sample but some indications can be made to be further investigated. However, before doing these implications some consideration other than the limited sample needs to be made. One regards if the order of the applications in the observations have had an impact on the dialogues that occurred. The applications were used in the order presented above and since they were used in the same order every time we don't know if another order would have made a difference. Another consideration is that the teachers may have developed their knowledge of the applications by using them several times during the study. Such development may have influenced the dialogues that occurred. However, no such development was visible during the observations.

Dialogues are controlled by the social environment (Anward, 1983). Based on this initial study applications seem to play a role regarding the character of the dialogues that occur. The teachers interacted differently with the children and the differences were not linked to the individual children but to the applications. The results indicate

that the classification and the framing of applications influence the dialogues that occur and the mathematics that becomes possible to learn.

Applications with weak framing seem to promote free dialogue and the focus of these dialogues often became mathematics. If the teachers focused on mathematics or not in these free dialogues seemed to be connected both to the applications and to the teachers. When using *Toca Boca Store* (WFSC) the focus of the free dialogues always became mathematics. When using *Toy Block* (WFWF) three of the teachers focused on mathematics while the fourth didn't talk at all. Strong classification may increase the possibility of mathematics becoming a part of the dialogues that occurs, but for that to happen a weak framing seems to be a precondition.

Applications with strong framing seem to promote directed dialogue. Sometimes these directed dialogues were focusing on mathematics but not even when using *Fingu* (SFSC) that was always the case. (Rather the opposite.) Strong framing seems to reduce the possibilities of free dialogues which affect the physical and social context in which an activity takes place. The major part of the directed dialogues, regarded guiding and confirmation focusing on right or wrong, both when using *Fingu* (SFSC) and *Nallemix* (SFWC).

Our initial study implies that framing and classification in applications influence the mathematics that becomes possible to learn. If children are to search for knowledge and develop it through play, social interaction, exploration and creativity, as well as through observation, discussion and reflection as emphasized in the Swedish curriculum (Skolverket, 2010a) especially applications with weak framing are to be used. Of course, there is no guarantee that the dialogues will become free or that mathematics in line with the curriculum will be learned, but weak framing seems to be a precondition. Also a knowable teacher is required since it seems to be depending both on the applications and the teachers if the free dialogues become focused on mathematics or not.

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