



# Proceedings

OF THE 37<sup>TH</sup> CONFERENCE OF THE  
INTERNATIONAL GROUP FOR THE PSYCHOLOGY  
OF MATHEMATICS EDUCATION

*» Mathematics learning across the life span «*

**Volume 4**

PME 37 / KIEL / GERMANY  
July 28 – August 02, 2013

**Editors**

Anke M. Lindmeier  
Aiso Heinze



**IPN**

Leibniz Institute for Science  
and Mathematics Education



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# **RESEARCH REPORTS**

**Pap - Z**



# INFORMAL STATISTICAL INFERENCE ABOUT SAMPLES AND SAMPLING

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*Most of the research on children's reasoning about samples and sampling had primarily focused on understanding current conceptions rather than on developing them over time (Makar, Wells & Allmond, 2011). A teaching experiment was designed to promote understanding of sampling issues in a Grade 6 (11 year-old students) classroom. Since formal statistical inference ideas are beyond the reach of young learners (Ben-Zvi, 2006), the teaching experiment adopted an informal approach to statistical inference. Children participated in data-centered activities, which provided them with opportunities to investigate real world problems of statistics using technology. The study shows how students began to appreciate the need for an adequately large sample size and a random-based sampling procedure.*

## INTRODUCTION

Developing students' informal ideas of inference is a topic of current interest (e.g. Pratt, Johnston-Wilder, Ainley, & Mason, 2008; Gil & Ben-Zvi, 2010). Acknowledging the fact that despite their difficulties with the formal methods of statistical inference students do have some sound intuitions about data that can be refined and moved towards reasoning with inferential qualities, Makar et al. (2011) view informal statistical inference as a more authentic way of teaching statistical reasoning at all educational levels.

The emergence of studies specifically focusing on informal inferential reasoning has begun to shed some light on this important aspect of statistical reasoning (e.g. Pfannkuch, 2010) but research on the topic is still at an early stage. In particular, there exists a gap in knowledge regarding the development of young learners' informal notions of important ideas related to statistical inference. Although the field has produced snapshots demonstrating the promise of engaging primary and middle school learners in informal statistical inference, systematic empirical research of the development of students' inferential reasoning and the factors that promote this development is currently lacking.

Developing students' understanding of the principles underlying sampling is of paramount importance, since the building of connections between sample and population lies at the heart of informal statistical inference (Pratt et al., 2008). As Zieffler, Garfield, delMas, & Reading (2008) point out, informal reasoning about statistical inference is the way in which students build connections between observed sample data and unknown or theoretical populations, and the way they make arguments or use data-based evidence to support these connections.



The present study aims to explore the following question: *How do young children reason about samples and sampling in a project-based learning environment?* The study contributes to the emerging research literature on the early development of informal inferential reasoning by focusing on children's understanding of sampling issues. It reports on a teaching experiment which aimed at providing information about upper elementary school students' developing knowledge and intuitions regarding key statistical concepts related to samples and sampling, the type of informal inferential reasoning and thinking possible for the specific age group, and supportive instructional activities.

## LITERATURE REVIEW

Since formal statistical inference ideas and techniques are beyond the reach of young learners, an informal approach to statistical inference is necessary in the early years of schooling. Makar & Rubin (2009) have identified the following three principles as essential to informal statistical inference: (i) making generalizations (predictions, parameter estimates, conclusions) that extend "beyond the data"; (ii) using data as evidence for these generalizations; and (iii) using probabilistic language in describing the generalizations, including references to levels of certainty about the drawn conclusions.

Sample size and sampling method are the main determinants of the validity of statistical inferences. Because statistical inference is almost by definition imperfect since sampling always introduces some error (Jacobs, 1999), students need to be aware of the potential threats to valid statistical inference (e.g. limitations of small sample size, undercoverage, nonresponse bias, voluntary response bias etc.). Jacobs (1999) investigated Grades 4 and 5 children's informal understanding of sampling issues in the context of interpreting and evaluating survey results. She found that while many of the children acknowledged the advantages of random sampling procedures, they tended to prefer stratified sampling (i.e. specifying the mixture) to simple random sampling.

Watson & Moritz (2000) investigated the characteristics of children's constructions of the concept of sample, and identified two key ideas for developing the sampling concept: appreciation for variation in the population and sensitivity to bias. The authors have found a trend for higher level performance with increasing age. The youngest children in their study (Grade 3, age 8-9) had fairly primitive, idiosyncratic notions of samples and sampling derived from everyday experiences with sample products or medical-related contexts. They were confident in drawing conclusions about the population based on very small samples, with little concern about bias. On the contrary, the majority of the Grade 9 students in the study (age 14-15) had developed an appreciation for variation in the population, and thus for the need to have an adequately large and representative sample. However, they often failed to identify sample bias. Grade 6 children (age 11-12) held a diversity of beliefs about sample size and sampling.

As noted by Makar et al. (2011), most of the research on children's reasoning about samples and sampling conducted in the past had primarily focused on understanding current conceptions rather than on developing them over time. Recent research, however, has shown that young children can demonstrate quite sophisticated levels of informal reasoning about samples and sampling if provided with an interesting and motivating learning context, and appropriate data visualization tools (e.g. Bakker, 2004). Paparistodemou & Meletiou-Mavrotheris (2008), for example, report on how a group of 8-year-old students were able to formulate and evaluate data-based argumentations and inferences about an underlying population based on collected sample data, using the dynamic statistics software TinkerPlots® (Konold & Miller, 2005) as an investigation tool. Similarly, Gil & Ben-Zvi (2010) who studied informal reasoning about sample and sampling among Grade 6 students (age 12), in the context of a collaborative, project- and inquiry-based learning environment, witnessed significant development of children's reasoning about key conceptions related to samples and sampling.

The current trend in education to place informal inference at the centre of both the elementary and secondary school curriculum necessitates a rethink on how to build students' reasoning about samples and sampling (Pfannkuch, 2010). Technology provide a set of tools widely available to young learners in order to give children access to advanced statistical topics including inferential statistics and the broader process of statistical investigation, by removing computational barriers to inquiry. This leads to a shift in the focus of statistics instruction at the school level from learning statistical tools and procedures (e.g., graphical representations, numerical measures) towards more holistic, process-oriented approaches that go beyond data analysis techniques (Makar & Rubin, 2009). The current study contributes to the existing literature by investigating ways to support the development of primary school children's sampling conceptions in the context of making informal statistical inferences.

## **METHODOLOGY**

A teaching experiment was designed to promote understanding of sampling issues in a Grade 6 (11 year-old students) classroom. Nineteen children participated in data-centered activities, in contexts familiar to them, which provided them with opportunities to investigate real world problems of statistics using technology. They posed questions of interest to them, devised and carried out a sample data collection plan, and worked in small groups to formulate and evaluate data-based inferences using the dynamic statistics software TinkerPlots® as an investigation tool.

The teaching experiment was implemented over a period of 4 weeks. There were 3-5 meetings per week, each lasting for 40 minutes. During the study, the research team collected and analyzed a wealth of data to assess students' growth in understanding and reasoning about samples and sampling. Students' learning processes were studied using written assessments, audio-recordings of class sessions, video-records of group sessions, interviews of selected students (the interviewing took place while students were working in groups for analyzing their data), and classroom observations and artifacts.

The videotapes collected during the course were first globally viewed and brief notes were made to index them. The goal of this preliminary analysis was to identify representative parts of the videotapes indicative of students' approaches and strategies when performing specific statistical problem solving tasks. The selected occasions were viewed several times and were transcribed. The transcribed data, along with other data collected in the study, were analyzed in order to investigate children's ways of thinking about samples and sampling while informally drawing inferences from data. We addressed data first by independent coding, and later collaboratively. Collaborative analysis helped us to triangulate our data interpretation (for greater validity) and develop a classroom-practice prospective of students' ways of thinking. The results section shares some of the insights gained from the study regarding patterns and mechanisms of development in children's reasoning about samples and sampling.

## **RESULTS**

The main data source for the activities taking place during the teaching episode was a survey developed and administered by the children, which investigated the community service and volunteerism habits of students in their school. Being sensitized to the importance of voluntary work, the sixth graders in our study conducted a survey in order to investigate the status of school and community service among students in their school. Towards that purpose, they constructed a survey questionnaire. They worked in small groups, and then in a whole class setting, for finding 'important questions' to include in the questionnaire.

Students first completed the questionnaire by themselves and then compared their answers with those of their classmates. They analyzed the data using TinkerPlots® and they drew conclusions regarding the volunteerism habits of students in their class. Next, they devised and carried out a data collection plan in order to obtain information about the volunteering habits of all students in the school. The first reaction of most children was to administer the questionnaire to all students in the school. The teacher/researcher (first author) had to intervene in order to encourage them to consider the possibility of sampling rather than administering the questionnaire to everyone. The fact that students' first suggestion was to take a census rather a sample is interesting, given that throughout the two preceding weeks instruction had revolved around the meaning and practical significance of sampling. Probably, the reason they were suggesting to take a census was the relatively small size of the whole school ( $n=220$  students) which did not make census taking prohibiting.

The children eventually decided that it would be more practical to get a sample of students rather than administering the questionnaire to everyone in the school. The sample selection process was decided after a long class discussion. The following whole-class excerpt shows how students explored the need of obtaining a representative sample from data:

Teacher (T): How are we going to get a sample that represents our school?

Student 2 (S2): Well, we need to have students from all three grades. This is for sure.

Student 4 (S4): Ok. Let's get the five students from the elected committee of each class. The children of the class voted for these students, so let's ask their opinion.

Student 5 (S5): I don't agree with this. It's not fair. These students were selected to represent their class in school's decisions, not to represent what all students think. I didn't want to be in my class committee, but I would like to answer the questionnaire.

Student 6 (S6): I agree.

Student 14 (S14): Well, for having a *fair sample* of students, we also need to have the same number of boys and girls.

T: Why do you think that?

S14: We are different from boys. It is not fair to ask more boys than girls.

T: So, what shall we do?

S14: We need to select children *by chance*. Without knowing...

T: How?

S2: We can get the catalogue of each class. We need to select from each catalogue 5 boys and 5 girls.

Students used the phrase *fair sample* to justify their decisions. Fairness seems to be a big issue for them and this is also the reason for deciding to use a stratified sampling method (stratification by class and gender). The above episode shows also how the real world context influences the idea of obtaining a 'fair' sample *by chance*. S14, for example, seemed very interested in this project. This was one of the few times she took active part in a class activity. Her comments were critical for continuing the discussion with children. Similarly, S2 suggested selecting a stratified random sample. We believe that this came effortlessly to his mind, since children were familiar with class rosters, which were routinely used in class for recording absences and grades. S2 knew that each class had its own roster, so it came as no surprise that he suggested using class rosters to select a random sample of children, stratified by class and gender.

After collecting these real data about themselves and from a sample of students from the whole school, students worked in groups to explore the data, using TinkerPlots<sup>®</sup> as an investigative tool. The class was divided in five groups. Each group got a small number of questionnaires, entered the data in TinkerPlots<sup>®</sup>, analyzed their data, and discussed their findings with each other. The decision to ask students to first work with small datasets before having access to the whole sample dataset was taken with the purpose of giving them first-hand experience of the limitations of drawing inferences from small samples.

In group discussion, students tried to draw conclusions about the data they had at hand. The majority pointed out that their datasets were too small for drawing inferences about the whole school. Children seemed to recognize the important role of sample size as they claimed: "I think that all the questionnaires together give a better result for the whole." On the other hand, it is possible that what some children were aiming for was a higher proportion of the population rather than a larger sample size, per se.

After students had finished with the datacards, the teacher put all cases (n=97) in a single file, and asked students to analyze the data. First though, she initiated a whole

class discussion, during which she encouraged children to make conjectures about what they expected to see in the larger sample:

T: Can we have 80 boys and 17 girls?

S2: No, it's impossible since half of the children in our school are girls

Student 9 (S9): No. The sample is large, so we'll have about the same number of boys and girls.

After analyzing the data, children tested their conjectures by examining graphs such as the ones in Figure 1, and drew some conclusions regarding the quality of their sampling scheme.

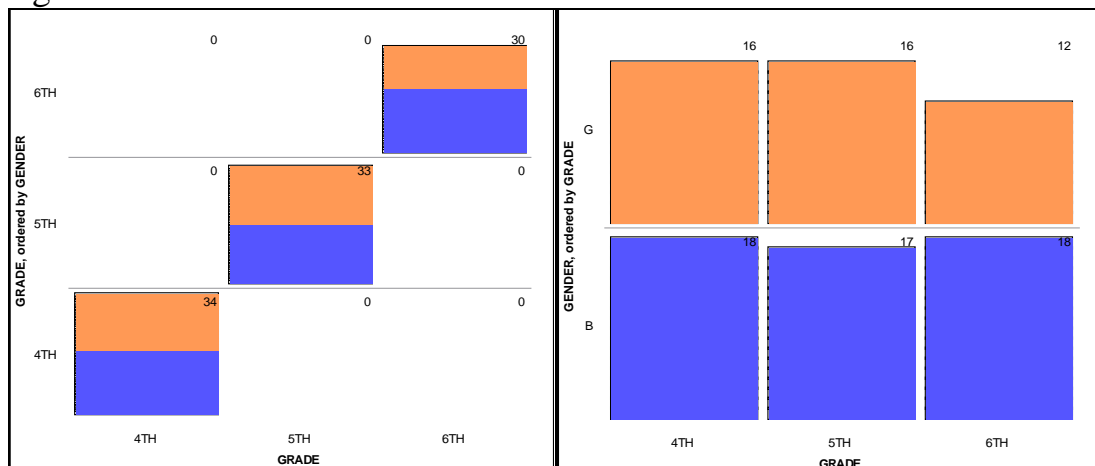


Figure 1: Number of boys and girls ('A'=Boy, 'G'=Girl) in each grade

T: What do you think about our data?

Student 7 (S7): Our numbers are good. In grade 4 we have *almost the same number of students* as in grades 5 and 6. The boys are a bit more...

Student 15 (S15): We have 44 girls and 53 boys in the sample.

S2: Does that look OK?

S4: Yes, it looks fine since the boys are not many more than the girls.

S2: Our method is totally *random*. So, having a bit more boys is ok.

T: Do you think we can draw 'fair' results about our school now?

S4: Yes....Now we have 97 cases out of 220 children. In each grade we have almost the same number of students.

T: So, is this a good sample?

S2: Yes. Because it would be very difficult to collect data from all 220 children. We chose these children randomly. I think they represent our school.

In the above episode, we see that children are satisfied with the sample selection process they had employed. They use phrases like *totally random*, *almost the same number of students* [in each class] to express their satisfaction. Another issue that bothers them again is to ensure that their sample is representative of the population of students in their school. S2 provides a justification for not collecting data from the entire population.

## CONCLUSIONS

This study aimed to investigate children's reasoning about samples and sampling in a project-based learning environment. The 11-year-old students in the study experienced statistics as an investigative, problem-solving process. They formulated questions of interest to them and designed a survey instrument to use for data collection purposes. They began to recognize that, while the colloquial use of the term sample refers to situations where the purpose is to show the homogeneous quality of a product, in statistics the purpose behind sampling is to get a representative picture of a population where there is clear variation among data values (Watson & Kelly, 2006). This realization, in turn, aided children to move beyond their statistically non-normative conceptions of fairness and to start developing basic understanding of the interrelated ideas of sample representativeness, sample size, and sampling method.

This study is an example for improving students' use of statistical reasoning and thinking by embedding statistical concepts within a purposeful statistical investigation that brings the context to the forefront. Our findings illustrate how young learners can begin to reason about sampling issues and other key inferential ideas when their interest in the task is high. Their engagement in an authentic, real world context (Gil & Ben-Zvi, 2010) encouraged students to seek ways to collect sample data that would enable them to draw valid inferences extending beyond their class to the whole school. The children were very much involved with their school project and the conclusions drawn from the data were important for them in order to understand what was happening at their school. Clements & Sarama (2007) argue that children possess an informal knowledge of mathematics that is surprising broad and complex. The current study has illustrated that when given the chance to participate in appropriate instructional settings that support active knowledge construction, children can exhibit well-established intuitions for fundamental statistical concepts related to statistical inference.

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# **SPECIAL EDUCATION STUDENTS' STRATEGIES IN SOLVING ELEMENTARY COMBINATORICS PROBLEMS**

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*This paper reports on a study aimed at revealing special education students' mathematical potential using a dynamic ICT-based assessment environment. The study focused on special education students' (N=84) performance in the domain of elementary combinatorics; a domain which is generally not taught in primary special education. The performance of students in regular education (N=76) served as a reference. The data analysis showed that on average special education students applied a systematic strategy equally often as regular education students. Moreover, we discovered that in both school types a significant increase in the use of systematic strategies occurred.*

## **INTRODUCTION**

In general, research on supporting special education (SE) students in mathematics has a focus on the learning and teaching of basic mathematical operations, like addition and subtraction. Less attention has been paid to higher order thinking processes that go beyond standard procedural skills. This is not surprising because SE students are often behind in their mathematical development compared to their peers in regular education (RE). Nevertheless, some studies have shown that low achieving students in mathematics may have a higher mathematical potential than assumed. For example, SE students turned out to have proficiency in interpreting tables and constructing graphs (Bottge, Rueda, Serlin, Hung, & Kwon, 2007). Even more unexpected was the observation that SE students can solve combinatorics problems in a systematic way without having worked on combinatorics in school before (Van den Heuvel-Panhuizen & Peltenburg, 2008). The aim of the present study was to further investigate SE students' potential in solving combinatorics problems. The strategies of students in RE in solving combinatorics problems served as reference data. The study was carried out in the Netherlands.

## **Revealing SE students' mathematical potential**

An approach to reveal SE students' potential is to present them with mathematical content beyond the regular curriculum, particularly if it requires higher-order skills. As Zohar and Dori (2003) stated, teachers often see higher-order thinking tasks as difficult and highly demanding and therefore do not present such tasks to students they think will find these tasks hard and frustrating. These good intentions lead to a vicious cycle: those students whose thinking skills need to be developed receive less opportunity to do so. We started this study to break this vicious cycle, choosing the domain of combinatorics – which clearly appeals to higher-order thinking, and is not a part of the regular curriculum in SE in the Netherlands. In the study, we used familiar contexts to



make combinatorics problems accessible to students. Furthermore, the problems were presented in a dynamic ICT-based assessment environment that facilitated the students' solution process.

### **Combinatorics in primary school**

Combinatorics is the domain of mathematics that involves systematic listing and counting (NCTM, 2009), based on the so-called 'fundamental counting principle' (DeGuire, 1991). This principle describes how to determine the total possible choices when combining groups of items. If you can choose one item from a group of  $a$  choices, and another from a group of  $b$  choices, then the total number of two-item choices is  $a \times b$ . The principle can also be viewed in terms of the Cartesian product of two given sets,  $a$  and  $b$ , which is the set formed by the combinations produced by pairing each member of  $a$  with each member of  $b$  (English, 2005).

Several mathematics didacticians favor integrating combinatorics in the school mathematics curriculum at all grade levels (e.g., English, 1993; Feijs, Munk, & Uittenbogaard, 2009). An important justification for teaching elementary combinatorics at primary school is that it can help students to develop their reasoning skills, e.g., making conjectures, generalizing and thinking systematically (e.g., English, 2005; Piaget & Inhelder, 1975). Moreover, research has shown that students at primary school age can deal with elementary combinatorics problems. By embedding such problems in rich and meaningful contexts, regular primary school students were found to be able to tackle these problems unassisted (English, 1993; 2005).

However, recommendations to incorporate combinatorics in the primary school mathematics curriculum are often ignored (English, 1993). In the Netherlands, this applies to the regular primary school curriculum. However, combinatorics is completely out of view in SE, where the curriculum mainly covers the four main operations (addition, subtraction, multiplication, and division) supplemented with tasks dealing with measurement, money, time and the calendar.

### **Previous research on primary school students' strategies for solving combinatorics problems**

English' studies (e.g., English, 1993; 1996; 2005) showed that primary school students can use increasingly sophisticated solution strategies for identifying all possible combinations of two- and three-dimensional combinatorics problems. In line with the findings of Piaget and Inhelder (1975), she discovered that these strategies evolve in three stages. According to English (1996), the first or "non-planning stage" comprises random, trial-and-error approaches with no global planning components. Piaget and Inhelder (1975) called this the "empirical combinations stage". English (1996) called the next stage the "transitional stage"; students try to find combinations in a systematic way, but do not succeed in doing so. Piaget and Inhelder (1975) described this stage as "in search of a system" to generate all possible combinations, and it is followed by the final stage in which students "discover a system". According to English (1996), in the third stage students construct the "odometer strategy", which involves keeping one item constant and systematically finding all possible combinations with that item.

After that, a new “constant” item is chosen and the same pattern of finding combinations is repeated. If students are close to this strategy, but deviate slightly from the pattern, English (1996) called this “almost odometer strategies”.

## Research questions

The present study builds on the work of English (1993) who investigated regular primary school students’ strategies in solving two- and three-dimensional combinatorics problems. Our study will investigate whether the strategies of SE students in solving combinatorics problems differ from those in regular education (RE) and how the strategies in both groups change over grades.

## METHOD

### Participants

In total, 84 students from five SE schools and 76 students from five RE schools participated in the study. To enable a comparison of SE and RE students with respect to their mathematics competence level we asked the teachers of each school to choose randomly four students who scored near the 50th percentile on the mid-grade levels M2, M3, M4, and M5 of the CITO LOVS test. The LOVS test is frequently used in the Netherlands and mainly contains items on calculation. Table 1 shows that for each mid-grade level, the average mathematics test scores of the SE students were slightly lower than those of the RE students as confirmed by the small negative effect sizes  $d$ .

LOVS test level	Mathematics scores SE students					RE students					
	N	M	SD	Min	Max	N	M	SD	Min	Max	$d^*$
M2	19	47.7	4.1	38	53	20	49.8	4.5	41	56	-0.14
M3	22	67.5	5.5	53	80	20	71.0	4.5	63	78	-0.24
M4	20	82.0	3.8	76	91	19	85.6	4.3	79	93	-0.25
M5	23	97.2	7.7	83	119	17	100.3	5.4	90	107	-0.24

\*Cohen’s  $d$  was calculated by using the standard deviation of the CITO reference sample in regular education

Table 1: CITO LOVS mathematics scores of SE and RE students

The students in RE were 7-11 year old ( $M=9,4$ ;  $SD=1,3$ ) and the SE students were 8-13 years old ( $M=11,1$ ;  $SD=1,1$ ).

### Data collection

For the data collection, we developed an ICT-based assessment, which included a series of six combinatorics problems. The first three problems have an  $X \times Y$  structure ( $3 \times 2$ ,  $2 \times 3$  and  $3 \times 3$  respectively) and the last three problems have an  $X \times Y \times Z$  structure ( $2 \times 2 \times 2$ ,  $2 \times 2 \times 2$  and  $2 \times 3 \times 2$  respectively).

For each problem the students have an infinite supply of little figures available that can be dressed with different types of clothing items (t-shirts,  $X_1$ - $X_3$ ; skirts,  $Y_1$ - $Y_3$ ; pairs of shoes,  $Z_1$ - $Z_2$ ) presented in different colors. A drag-and-drop function allows moving

both the figures and the clothing items to an empty field. In this field the student can dress the figures and rearrange or remove them.

The students individually completed the ICT-based assessment. For each problem, the researcher asked the students how many different outfits were possible with the available clothing items and how they found their answer. The students' on-screen work and their verbal comments were recorded by screen video software.

## Data analysis

*Coding.* We converted the students' on-screen work into tree diagrams schematizing the identified combinations (c.f., English, 1993). These tree diagrams provide an overview of the combinations that were successfully formed by the students. Based on the tree diagrams, two raters independently coded the students' work as systematic, semi-systematic or non-systematic (98% agreement in coding; Cohen's kappa = .97). A systematic strategy was defined by the use of a cyclic pattern, a constant item, or both. See Figure 1.

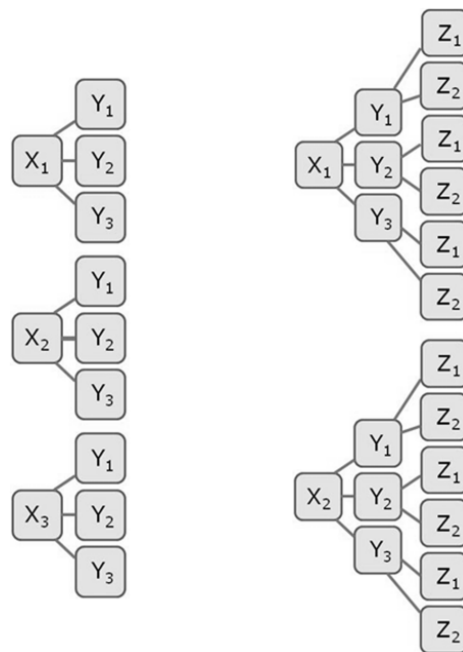


Figure 1: Example tree diagram that reflects a systematic approach for both a two-dimensional problem (3x3) (see left) and a three-dimensional problem (2x3x2) (see right).

A semi-systematic strategy was characterized by using a cyclic pattern, a constant item or both, but in a non-consistent or non-exhaustive manner<sup>1</sup>, whereas the complete absence of a systematic approach was classified as non-systematic.

*Using sample weights.* As the number of students per school type differed per mid-grade level (see Table 1), we used a weighting procedure giving a weighted sample size of 20 students for all the combinations of the mid-grade levels and school types. For example, each of the nineteen mid-grade level M2 SE students had a sample weight of  $20/19 = 1.053$ . All results in the following section are based on analyses using sample weights.

*Analysis of variance.* To investigate differences between SE and RE students, an analysis of variance was carried out at student level. We specified three different models; respectively containing mathematical level and school type (Model 1), age and school type (Model 2), and mathematical level, age and school type (Model 3). All models treated mathematical level and age as linear predictors. In preparing the analysis of variance, we calculated a score for each student reflecting the degree of systematic strategy use. Student scores were obtained in two steps. At the case level, we attributed 1 point to the use of a systematic strategy, 0.5 point to a semi-systematic strategy, and 0 points to a non-systematic strategy. Then, the mean score for solving the series of six combinatorics tasks in the test was calculated for each student, resulting in the interval-scaled variable *strategy use*.

## RESULTS

### Strategy use in solving combinatorics tasks

*Strategy use in SE and RE.* Table 2 shows students' strategy use in both SE and RE. Generally, frequencies of the different types of strategy use differed no more than four percentage points between SE and RE students. In fact, no significant differences were found between the two groups of students in use of systematic, semi-systematic and non-systematic strategies ( $\Phi = .051$ ,  $\chi^2 = 2.485$ ,  $df = 2$ ,  $p = .29$ ).

		Number of cases (%)							
		Strategy type							
		Systematic		Semi-systematic		Non-systematic		Total	
School type	SE	216	(45)	183	(38)	81	(17)	480	(100)
	RE	236	(49)	178	(37)	66	(14)	480	(100)
	Total	452	(47)	361	(38)	147	(15)	960	(100)

Table 2: Cross tabulation of frequencies per strategy per school type

*Strategy use per mathematical level.* Figure 2 represents students' strategy use per mathematical level for both SE and RE. It shows use of systematic strategies increasing per mid-grade level in both school types, while non-systematic strategies decreased. At the M2 and M3 level SE students applied less systematic strategies than RE students. However, at the M4 level SE students reached the same percentage of systematic strategies as RE students. Moreover, at the M5 level SE students applied a systematic strategy more often than RE students. To further investigate differences between SE and RE students regarding their strategy use, we carried out an analysis of variance of which the results are presented in Table 3.

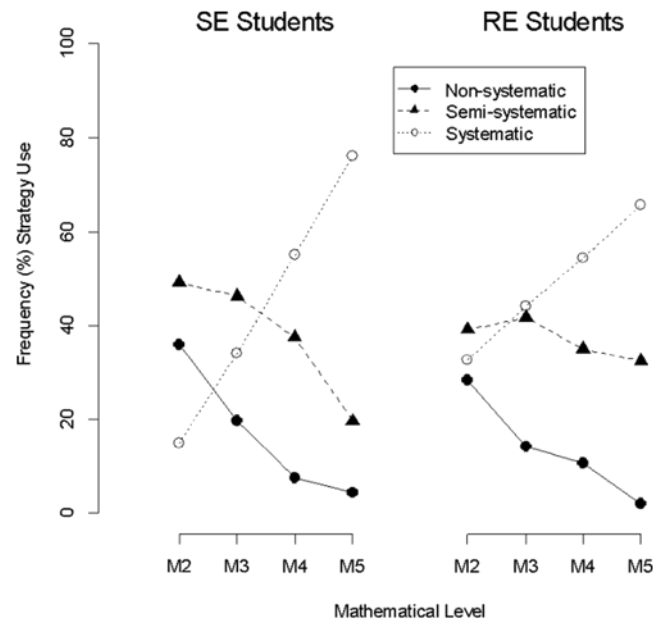


Figure 2: Relation between percentage strategy use on the combinatorics test and CITO LOVS mathematical level for SE and RE students

	Model 1 (Mathematical level, School type)				Model 2 (Age, School type)			Model 3 (Mathematical level, Age, School type)		
	df	F	P	$\eta^2$	F	p	$\eta^2$	F	p	$\eta^2$
Math level	1	63.89	.00	.283				23.94	.00	.106
Age	1				23.26	.00	.130	1.79	.18	.008
School type	1	4.36	.04	.019	.91	.34	.005	.05	.82	.000
Math level*School type	1	3.44	.07	.015				.55	.46	.002
Age*School type	1				.27	.60	.002	.01	.93	.000
$R^2$		.31			.13			.32		

Table 3: Results of analysis of variance of strategy use from different models with Age, Mathematical level and School Type as predictors

In agreement with Figure 2, Model 1 shows that mathematical level, school type and their interaction play a predicting role in use of systematic strategies. Mathematical level ( $F(1,156) = 63.89, p = .00, \eta^2 = .283$ ) clearly appears to be a significant predictor, with school type ( $F(1,156) = 4.36, p = .04, \eta^2 = .019$ ) and the interaction of mathematical level and school type ( $F(1,156) = 3.44, p = .07, \eta^2 = .015$ ) both close to the .05 level of significance. From the results of Model 2 it can be concluded that age is a significant predictor ( $F(1,156) = 23.26, p < .01, \eta^2 = .13$ ) while school type and the interaction between age and school type are not. Finally, in Model 3 only mathematical level appears significant ( $F(1,156) = 23.94, p = .00, \eta^2 = .106$ ).

## CONCLUSIONS AND DISCUSSION

In this study, primary school students worked on a topic that was not part of their mathematics curriculum. It was found that on average SE students applied a systematic strategy equally often as RE students. Moreover, we discovered that in both school types a significant increase in the use of systematic strategies occurred.

Additionally, we found different patterns for strategy use across mathematical levels M2 to M5 by SE and RE students. At the M2 and M3 levels, SE students applied a systematic strategy less often than RE students, while at the M5 level SE students applied a systematic strategy more often than the RE students. An explanation for these different patterns in strategy use between the two school types possibly lies in the different teaching practice in SE and RE. While direct instruction (DI) is popular in SE, this is not so much the case in RE. Characteristic of DI is its systematic step-by-step approach requiring student's mastery at each step. This emphasis on a structured approach, which students at higher grade levels have experienced for longer, might have caused SE students at the M5 level to apply a systematic strategy so often.

Of course, this results of this study should be handled with prudence. We used only a few combinatorics tasks and only of a particular type. Moreover, the selection of students can be criticized. Although we asked the teachers to choose four students at a particular mathematics level at random, there could be some bias because of teacher choice. Another limitation of the study is that we only had a one-shot data collection. Our results would have been more robust with a repeated measurement.

Despite these limitations, the findings of our study convincingly demonstrated the mathematical power of SE students in the domain of elementary combinatorics. Consequently, we would like to recommend investigating the enrichment of the mathematics program in SE, in particular by including activities related to elementary combinatorics. ICT environments such as we developed for this study could be of great value for this.

### Note

1. This classification differs slightly from that of English (1993, 1996). The main reason for our adjustments is that, in our data set of student work keeping one item constant (e.g., item X1) did not necessarily go together with systematically varying an item of another type (e.g., Y1, Y2 and Y3), – the so-called odometer strategy – which is considered a prerequisite in English' classification for coding a student's work as 'most sophisticated'. Because of this mismatch, we redefined the category of most sophisticated strategies (in our classification called 'systematic strategies') by approaches characterized by the use of a cyclic pattern, a constant item, or both.

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# "THIS IS A SWINDLE: I DO NOT NEED TO CALCULATE, I NEED TO THINK ONLY!"

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*"This is a swindle: I do not need to calculate, I need to think only!" said Nea, 25 years old woman, when she realized how to multiply and divide without algorithms. Poor mathematical competencies among adults result difficulties in many common day to day activities. Adults with poor mathematical competencies have often difficulties to get a good education and difficulties in employment. One of the adults of this kind is Nea who has learning disabilities in mathematics, especially in basic mathematical skills. In this study, Nea's story, learning basic mathematics as an adult, is described and examined in the perspective of life-long learning.*

## INTRODUCTION

Basic mathematical skills have been shown to be crucial predictors of individual's life success. Poor mathematical skills are common among adults and result unemployment and difficulties in many common day-to-day activities. (Maloney, Risko, Ansari, & Fugelsang, 2010.) Although, Finland has succeeded in PISA, almost every 20<sup>th</sup> street walker and one student in every school class feels overcoming difficulties in basic calculations (5%-7% of age group on the average). About half these people have wide-ranging learning disabilities and the other half of these people might have problems only in numeracy (Räsänen, 2012). According to Geary's (2012) research review below average mathematical competencies at the beginning of schooling is associated with elevated risk of poor mathematical competencies at the end of schooling. Räsänen (2012) argues that learning disabilities in mathematics are as common as reading disabilities. Unfortunately, those who have poor mathematical skills are not so well supported as those who have reading disabilities. Without early identification of children who are at the risk for long-term difficulties in mathematics will compound into life-long struggles in work and in dealing with the everyday with the demands of the modern society (Geary, 2012). There are no proper tests for the identification of adults' poor mathematical skills. Still, adults' learning disabilities in mathematics can be evaluated by surveying adults' learning history and by applying tests used in schools. (Räsänen, 2012.)

In this article we are describing Nea's story as an example of an adult who has passed the basic schooling system and suffers from the consequences of poor mathematical skills. The aim of this article is 1) to describe and understand Nea's mathematics learning process, and 2) to give ideas how to analyze adults' basic mathematics skills.



## MEANINGFUL MATHEMATICS LEARNING

Traditional mathematics education for students with learning disabilities has emphasized “telling mathematics”, in which mathematics is considered a rule-based comprehensive structure. That means that students were taught to solve each problem in a specific single way, and students’ informal strategies were disregarded. (Tournaki, Bae & Kerekes, 2008.) When learning is focused on “understanding mathematics”, learners become active problem solvers who connect and compare their learning process to a meaningful environment in their own ways of mathematical perceptions (Butler, Buckingham, & Lauscher, 2005). According to Steinbring (2005) mathematics learning and understanding of mathematics is strongly connected to the senses and thus perceivable phenomena and objects. Mathematical objects are ideal structures, not ultimately visible or perceivable. The beginning of learning process of children starts from given concrete and illustrative material as embodiments of mathematical structures. The goal of this is to understand more and more and interpret these given objects as carriers of mathematical concepts and structures. In this way learning is meaningful process where learners are not passive recipients of ready-made mathematics, and mathematics is not presented as an abstraction removed from everyday experiences of the learner (Freudenthal, 1991). This means that concrete material helps in developing ideas from concrete to semi-concrete, semi-abstract, and finally abstract levels. Providing effective manipulatives will significantly help students with learning disabilities to understand mathematical concepts. (Tournaki, Bae, & Kerekes, 2008.) Also Domino (2010) argues that the students who used manipulatives during mathematics instruction had statistically significant higher mathematics achievement than students who were taught by traditional teaching methods. This means that means mathematics learner should have lot of opportunities to use concrete material, and adults with learning disabilities in mathematics should have opportunities to rebuild mathematical concepts through concrete materials.

## METHOD

The data of this study consists of Nea’s results in ALVA (Ikäheimo, 2010) surveying (professional arithmetical skills surveying), observations of Nea’s learning processes and Nea’s interviews. Hannele Ikäheimo has gathered the data of this study. Hannele and Nea has met in the HERO peer group (HERO means Organization of Different Learners in Helsinki) meeting. In the fall of 2010 Hannele surveyed by ALVA the mathematics skills of HERO peer group ( $n = 11$ ). The members of this group have learning difficulties generally. When the ALVA surveying begun, the part of the members of the peer group cried and did, did and cried. *“I have never understood these calculations they are so difficult”* was an ordinary comment. Still, they wanted to make ALVA surveying. The results showed that the basic mathematics was lost. The result of this ALVA surveying was confusing: these adults scored approximately 44 marks of full 90 marks. The decimal system, decimals, unit conversions, fractions and equations were difficult. Hannele decided to help this peer group. She taught with concrete

materials this group once a month about a couple of hours at a time, short a year. Next we describe ALVA surveying and Nea's story during school.

### ALVA surveying

ALVA (Ikäheimo, 2010) is a professional arithmetical calculation skills surveying (Ikäheimo 2010), which involves central concepts of secondary school mathematics core curriculum. The survey is divided into 11 exercise series by which the concepts are mapped. The exercises represent following mathematical issues: The decimal (base ten) number system, Mental arithmetic, The concept of decimals, Calculations by fractions, Decimals and fractions, Units, Measurement and estimation, Conversion of measures, Equations, Percents, and Wholes and parts. The control of these concepts is necessary at the end of secondary school and at the beginning of professional studies. ALVA can be done at grades 8-10 in secondary school, as well as at the beginning of high school studies, vocational school, and University of Applied Sciences.

### Nea's story

*I graduated house painter from vocational school last spring.*

*I had a same teacher in mathematics from the first grade up to class 8 in secondary school. Very often I had to stay at remedial instruction. I was frustrated because I did not learn anything. I had the same teacher at remedial instruction as in the mathematics lessons.*

*When processing of the fractions begun in primary school, the teacher drew a house to the blackboard where diamonds dropped into downstairs. I remember the house because I have been always strongly visual. By the house we practiced the use of decimal point, too. I remember that I learned the use of the house well and how the numbers were in the house. Then we came in to the stage where we had to learn to do things without the house .... in that stage everything went wrong. Although teacher noticed that I cannot manage without the house he did not use the house in teaching situations. I had to learn to manage without the house. After this experience I did not understand mathematics at all. I barely passed or failed math exams.*

*In high school teacher understood which kind of learner I am and how I learn. Mathematics begun to flow ... especially algebra, equations, x and y graphs were even easy. I got very good grades from math exams at high school. Math went on better than ever. When teacher had time he tried to teach me those issues where I had gaps. But always when I realized that I understood something I forgot the issue and I could not keep it in my mind. Former gaps might have caused that understanding and knowledge flew away like water when I tried to stick to difficult issues and to learn.*

*Although I have forgotten high school mathematics I believe that I can recollect it quite easily. In all I have exceeded myself in mathematics because I have learned to perceive and understand issues which I believed that I can never learn in mathematics. For example fractions and percents are difficult issues for me. The fact that I have been able to learn these most difficult issues proves the case that I can learn what I want, it*

*is another thing how much it takes work and time. Although percents have flowed away like water from my mind once again, there is still left of them some idea of understanding and perception. In High school mathematics the issues which went well were fun and challenging enough. Middle school mathematics feels difficult and laborious despite the fact that I have learned so many new issues.*

## RESULTS

Nea, 25 years old woman, was one member of the HERO peer group in the fall of 2010. In ALVA she scored 31 marks of the full 90 marks. This means that she managed to do 34% of the exercises right (in 90 minutes) of the ALVA surveying. One must do 80% of the exercises right in ALVA to pass surveying.

In the fall of 2012 Nea begun to study mathematics with Hannele once a week a couple of hours at a time. At first Nea did ALVA surveying again. Now she scores 45 of the full 90 marks. In other words did 50% of the exercises right, in 105 minutes

(see appendix 1: Nea ALVA+2012 ). She improved her performance with 14 points from last time. Later in the fall of 2012 Hannele surveyed how Nea understands decimal (base ten) number system. In this surveying Hannele found out that Nea does not understand the concept of decimals because she did not understand the decimal (base ten) system either. In the next table 1 there are examples of Nea's mistakes in ALVA in the fall 2012 and Nea's comments of the exercises.

Nea's mistakes in ALVA	Nea's comments in "..."
$0,5 \cdot 0,5 = 5$ $100 \cdot \underline{\quad} = 50$	"Decimals are indefinite, when they are mixed." "Itl disturbs when there are decimals."
$2 + 3 \cdot 5 = 20$ $4 + 10 \cdot 2,5 = 35,0$	
$273 : 100 = 2,7$	Observers' comment: Nea counted all the multiplicatin exercises with algorithms on the help paper when the multiplier was 10 or 100. She also calculated all the division problems with algorithms on the help paper when the divisor was 10 or 100.
$53,25 : 100 = ?$ $2,45 : 10 = 4,9$	
$1 - \frac{1}{4} = 0$ $2 \cdot \frac{1}{3} = \frac{2}{6}$	"Quite okay! I have practiced!"
$\frac{1}{2} = 1,2$ $\frac{3}{4} = 3,4$ $2 \frac{1}{5} = 2,15$ $0,009 = 9/100$	"Little difficult. I knew what is meant. "
	Units and conversion of measures: 6 points out of 22. "Grams are difficult ones. The cereal

	packages are of different sizes. I understand m, cm, among others but $1\text{ m} = \text{--- km}$ is difficult to perceive. Liters and dl are easy. I have used them in baking.”
$x = 0, x - 9 = 0$ $20 - x = 13, x = -7$ $4(x - 2) = 36, x = 28$ $x / 10 = 5, x = ?$	Equations and comparabilities: 6 points out of 12. “I have trained the x calculations. I love them because are easy for me.” Nea did equation exercises at division corner: $35 : 5$ and $24 : 6$ .
1% of the 500 euros is 500 25% of the 400 euros is 1 ...	“You can get 1% of the quantity by dividing it by 1. You can get 25% of quantity by dividing it by 25”.  “I remember when you Hannele explained 25% by fraction cakes in 2011. I cannot do it in practice.”
	The estimations exercises, which were connected to the practice, and wholes and parts 10 points out of 12 points.
	Calculation $10 \cdot 1,5 \text{ €} = 15,00 \text{ €}$ has been calculated with multiplication algorithm!

Table 1: Nea’s performance in ALVA

Nea's individual instruction which has begun in the fall of 2012 has been a couple of hours per week. Every time the working has begun with the help of concrete materials which have helped Nea to understand the basic mathematical concepts. After practicing with concrete material Nea has drawn and has colored all the work with concrete materials in her notebook. *“With the help of colors I perceive and remember more easily.”* In addition to that, Hannele has always sent a few pages of tasks by e-mail and Nea has sent the colored solutions scanned back. Nea felt mathematics learning meaningful and enjoyed it. For example, when Hannele and Nea began practicing with a percentage calculation Nea was very pleased and said: *“As a master builder I need percent calculations and I want to learn them!”*

Because the decimals were difficult to Nea, she was allowed to build numbers to the place value grid with Base Ten materials (including decimal parts) and with teaching money which were easy to draw. *“I have not used to talk aloud my thinking!”* It was difficult for Nea to get used to the fact that she was allowed to use to homework the same methods, tables and drawings as she had used during the teaching. *“I thought that now I already must know without the tables!”* When Nea was at school the teacher had given the permission to use concrete materials only at the beginning of new issues.

After that materials were taken away and the teacher had said: “Now you have to learn without concrete materials!” “This is a swindle, I do not to calculate, I need to think only!” comments tells about joyful inspirations when she learnt how to multiply and divide by 10. In figure 1 you can see Nea’s working.

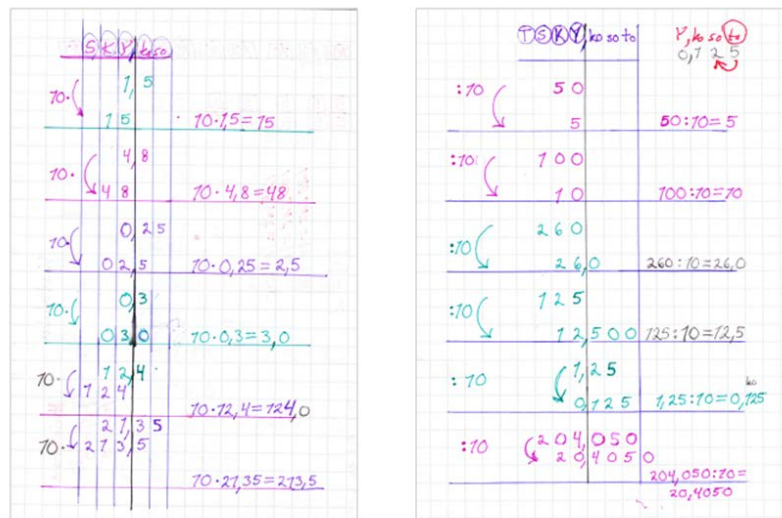


Figure 1: Nea’s worksheets: multiplication by 10 and division by 1

Nea built a number on the place value grid and under that number she build the same number ten times bigger (in Fig 1 on the left). When Nea wrote these numbers in the table she found out: “The decimal point does not move anywhere – it stays in the same place! The numbers are moving!”

## CONCLUSION

If the pupils do not during school years learn to understand the most important basic concepts of the mathematics, they do not possibly know them even as an adult. One can say: “*The age itself does not increase an understanding!*” If the adults want to learn mathematics, it is totally possible.

First, it will be important to find out what they know and what they do not know. After that, suitable learning program for adults should be planned based on surveying, for example ALVA. The learning program should base on the use of concrete materials, aloud thinking, to drawings, and tables (see Domino, 2010; Tournaki, Bae, & Kerekes, 2008; Steinbring, 2005). In the group instruction these concepts can be taught again to the adults. But this is not enough for everyone to reach the permanent learning results. Some of the learners need individual, concrete, meaningful instruction to rebuild mathematical concepts again.

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## Appendix 1: Nea ALVA+2012

Nea's performance in ALVA  
Date 5.9.2012  
Use of time: 105 minutes

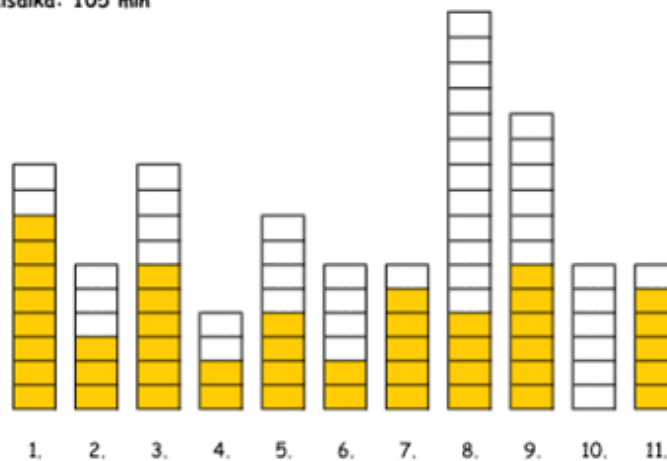
ALVA  
Result 45/90 points  
50% of the exercises  
right  
Limit of acceptance:  
80% of the  
exercises right

### ALVA

#### Opiskelijakohtainen tuloslomake

Opiskelija: Nea  
Luokka: Aikuiset  
Pvm: 5.9.2012  
Kokonaisaika: 105 min

Tulos 45/90 p  
50 % oikein  
Raja 80 % oikein



1. The decimal (base ten) number system
2. Mental arithmetic
3. The concept of decimals
4. Calculations by fractions
5. Decimals and fractions
6. Units
7. Measurement and estimation
8. Conversion of measures
9. Equations
10. Percents
11. Wholes and parts

Opiskelijan osaamien tehtävien lukumäärä on väritetty.

The number of exercise which student did right is colored.

Kommentteja

Helsinki 5.9.2012

opettajan allekirjoitus

# TEACHING METHODS AND MATHEMATICS ACHIEVEMENT IN GERMAN COMPUTER AIDED CLASSROOM TEACHING

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*An analysis of the development of mathematical achievement in German secondary school courses during grades 11 to 12 shows empirical support for the assumption that the presence of mathematical software in school teaching and examinations does not speak against the mastering and understanding of basic mathematical concepts as long as it is accompanied by regular mental maths exercises, repetition and, probably, differentiation.*

## MATHEMATICS ACHIEVEMENT, COMPUTER USE AND TEACHING METHODS

### Computer Use and Achievement in Mathematics

The presence of digital tools in mathematics classroom teaching alone will not lead to a better achievement in mathematics. Studies on the effects of computer algebra systems (CAS) and other tools on mathematical achievement show, consequently, inconsistent results. While e. g. Kieran and Saldanha (2005) report of positive effects of CAS use, Bichler (2010) found only non-significant trends in favor of CAS. As a consequence Weigand & Bichler (2010) suggest to develop a competence model which would also encompass the technical and dynamical aspects of mathematical knowledge that could be related specifically to the use of digital software. Another approach considers the use of digital tools as being one element of the teacher's methodical orchestration of learning processes. The much used metaphor “catalyst for a change in mathematics teaching” (cf. Laborde & Sträßer, 2010) indicates that the presence of computers, handhelds or other digital tools induces the need for change of methodical settings in teaching as a whole. Neill (2009) finds that teachers who welcomed the use of CAS handhelds in their classroom preferred methods that allowed inquiry based learning. Weigand and Bichler (2010) report that most teachers state that they changed their teaching style when they introduced CAS. Ball & Stacey (2004) found that teachers, once they started using CAS in classroom, became more aware and interested in their pupils learning processes. So it seems that for analyzing the effects of digital tools on mathematics achievement we must look at it as one element of the methodical setting in classroom teaching.

### Computer use and diversity of teaching methods

We recently reported on the results of a project that showed correlations between mathematics achievement tests and the methodical organization of class instruction (Pinkernell & Bruder, 2012). The project named CALiMERO 2005-2010 was supported by the state authorities of Lower Saxony and Texas Instruments while the Technical University of Darmstadt provided scientific monitoring. Its main objective



was the implementation and evaluation of a teaching concept for the use of handheld computer algebra systems in German secondary schools from grade 7 to grade 10 (Ingelmann, 2009). The teaching concept, which was developed in Darmstadt, proposed a fixed series of certain methodical elements for each unit that focused on sustainable learning of mathematical competencies and basic knowledge. In fact, the empirical evaluation showed effects, though small, that could be related to the actual methodical organization of class teaching initiated by the teaching concept: Classes that showed what we called a “rich diversity” of teaching methods scored higher than classes with a “poor diversity” of teaching methods. The classification of the project classes into rich or poor diversity was based on quantitative data derived from lesson protocols. An index value was computed by means of a modified Shannon formula of diversity (Voleske, 2007), by which a very high frequency or very low frequency of a given method would lead to a smaller index value. Classes with a high diversity index were characterized by a relatively high occurrence of teacher-pupil-interaction, self-responsible work, working in groups, repetition, frequent mental maths tasks, and provision for different achievement levels – a result that is confirmed by a similar profile of teaching methods that were found in control classes which showed high achievement levels from the beginning of the project (fig. 1). Especially in those classes where the use of the mental maths exercises provided by the teaching concept was frequent, the increase of mathematical achievement was higher than in the other project classes where the material provided by the teaching concept was neglected.

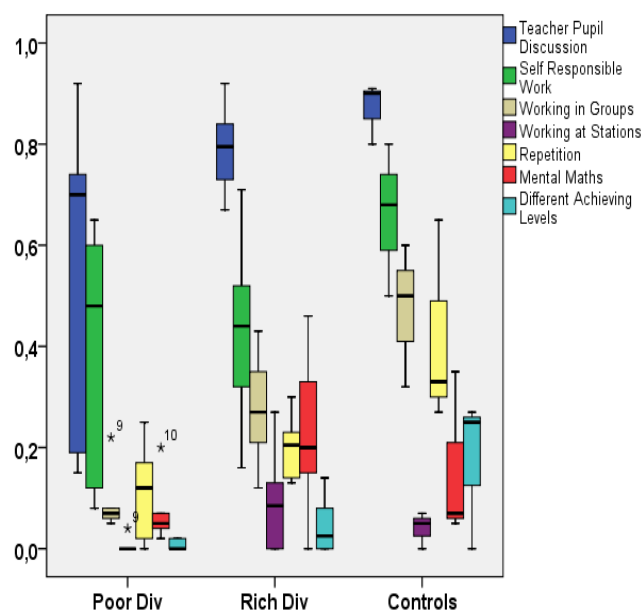


Figure 1: Frequency of teaching methods in experimental classes with a poor diversity of methods, experimental classes with a rich diversity of methods and high achieving control classes for comparison (from Pinkernell & Bruder, 2012)

### Diversity of teaching methods and good practise teaching

In his summary of hundreds of meta-analyses, Hattie (2009) found that classroom management and teaching explains between 20% and 30% of variance in test performance, second only to individual characteristics of the pupils such as prior

knowledge or intelligence. Therefore, the quality of teaching is directly related to the teacher's professionalism: He or she is responsible for classroom management, provision of learning activities and the activation of students' thinking. A variety of teaching methods, among them the use of digital tools, could then be regarded as one visible indicator of the teacher's level of pedagogical professionalism as a whole. Many models of good teaching list a suitable diversity of methods among the criteria (Brophy 1999, Hattie 2009, Helmke 2009). From a theoretical perspective this is seen as necessary in the light of heterogeneous preconditions of learning. When a teacher shows a variety of teaching methods at hand he appears to be able to provide for the many learning styles and levels that can be found in a class. This consideration also has a sound empirical basis. Especially in Germany the question of how good teaching can be modeled has been subject to various large scale studies which repeatedly found correlations between diversity of teaching methods in class and mathematics achievement (Weinert 1997, Helmke 2009).

Among the methodical elements mentioned there, the most prominent are cooperative learning formats like working in groups, which, as Brophy reasons, “creates the potential for cognitive and meta-cognitive benefits by engaging students in discourse that requires them to make their task-related information-processing and problem-solving strategies explicit (and thus available for discussion and reflection)” (Brophy, 1999, 28) However, as Helmke (2006) points out, there is no perfect combination of teaching methods which just could be copied. Hence, in his model of good teaching Helmke (2009) coins the term “learning offers” as a variation of methods, social interaction forms, tasks that is suitable for each class, i. e. its learning preconditions and learning goals. So when we speak of an effective diversity of methods we mean a suitable selection of forms of social interaction as well as adequate variety of tasks and a reflective use of media, among those digital tools.

## **THE STUDY**

The objective of the study CALiMERO 2010-2012 on which we concentrate in this paper is to investigate the relationship between mathematics achievement and methodical aspects of mathematics teaching in classrooms where digital tools are a frequently used. By mathematics achievement we concentrate on the mastery of basic knowledge, often suspected of being impaired by a too frequent use of digital tools.

### **Description**

The project CALiMERO 2010-2012 is a follow-up of the CALiMERO 2005-2010 study mentioned above. It was initiated by the state authorities of Lower Saxony in Germany to support the implementation of a new curriculum for grades 11 and 12 that leads to the final German secondary school examinations. CAS or graphic calculators have been compulsory in secondary school mathematics from grade 7 on for several years, esp. in the final school examinations. Recently, however, the authorities are now planning to add a “technology free part” to the examinations, which would focus on the mastery of basic mathematical skills. Hence the scientific interest in this project lies in monitoring the development of the pupils achievement in the mastering of basic

mathematical knowledge, and how high achievement in this area correlates with the certain methodical elements of teaching.

One method which proved successful for fostering the long-term availability of basic knowledge during the preceding study were so-called regular mental maths tasks (Pinkernell & Bruder 2012). These are sets of ten short problems from different areas of school mathematics from grade 5 on. The areas covered in the sets of this study are basic operations and concepts from early secondary school mathematics (fractions, units, rule of three, functions and equations) and late secondary school mathematics (differentiation and integration, matrices and linear equation systems, basic statistics). The tasks address the mastering of basic operations in these areas as well as conceptual understanding in the form of representational change, interpretation in situational contexts or typical mistakes. We therefore decided to actively encourage the teachers of the new project to make use of mental maths exercises which we would provide in sufficient numbers for the duration of the project.

### **Aim**

The main question of this study is how high achieving maths classes differ from low achieving classes with regard to the teaching methods as perceived by the pupils, among those digital tools and regular mental maths exercises.

### **Design**

The project began with winter term 2010/11 (October 2010) in 43 courses from 6 German secondary schools. These courses started in grade 11 and would lead to the final examinations at the end of grade 12. Among them were 15 courses in which mathematics was taught on a high level and 28 courses of basic level mathematics. The project ended with winter term 2011/12 (February 2011) just before preparations for the examinations began. While many of the participating teachers took part in the preceding project CALiMERO 2005/2010 and were familiar with the teaching concept introduced there, other participants were new on the basis of the school's decision.

Achievement tests which were to be solved without digital means were administered five times during the one and a half years. Though not actually parallelized, all tests were very similar with regard to structure and content. The sets of mental maths tasks were offered to the teachers on a weekly basis during the first year, and later on every two weeks with a note that they could re-use sets from the first year.

We were not able to provide a continuous survey of the teaching. And since numerous applications of tests and questionnaires during the preceding project appeared to be problematic in the same schools we decided to reduce the amount of polling to a minimum. The teachers were asked to fill a questionnaire on their teaching habits and view on mathematics once, during the fourth time of measurement. The use of teaching methods in classroom was reported on by the pupils of each course, by answering additional questions provided on the achievement test sheet. There were four items on teaching methods that concerned the frequencies of forms of social interaction and four that related to the frequencies of tasks specifically to be solved without digital aid,

tasks for different achieving levels, repetition and frequency of use of digital tools. There was not a specific item for the frequency of mental maths tasks but we expected the two items on “tasks without digital means” and “repetition tasks” to cover this. The answers were scaled in a near exponential progression (“every lesson”, “weekly”, “every two weeks”, “monthly”, “less or never”) which linearized the otherwise logarithmic time scale (doing mental maths exercises, e. g., in one more lesson each week should produce a greater effect than doing mental maths in one more lesson each month).

## Results

The completion rates of all achievement test results were Z-standardized. Since the response rate for the fifth was too low for consideration we compared courses based on data from the first and the fourth test. This was done for high level courses (hl) and basic level courses (bl) separately. Only courses that took part in both tests were considered for analysis ( $N_{hl} = 10$ ,  $N_{bl} = 13$ ). Furthermore, both high level courses and basic level courses were each ranked into three groups of equal size, based on their increases between the first and the fourth time of measurement. This gave, for high level courses, a group of 3 courses with highest increases and a group of 3 courses with lowest increases (which were, in fact, negative increases of achievement grades in the two tests), and, for basic level courses, a group of 3 with highest increases and a group of 4 with lowest increases (again, in fact, decreases of achievement grades).

For the methodical settings of each class, we considered the pupils' data from each of all four times of measurement to control the teacher's methodical habits. The frequency of each method, as perceived by the pupils of the class up to the time of polling, is represented by the arithmetic mean of a numerical representation of the pupils' checks in the questionnaire (“every lesson”  $\rightarrow 5$ , ..., “less or never”  $\rightarrow 1$ ).

For identifying methods that could characterize the high achieving courses as compared to low achieving courses we computed, for each method and each time of measurement, Kendall's rank correlation coefficient  $\tau_b$ . Both tables 1 and 2 show that correlations values are predominantly low and vary between times of measurement for most methods. This is not surprising since there was one item only for determining the frequency of each method. However, when the pupils ranking of a method shows correlation values which are considerably high and which constantly point to the same direction one might consider this as an indication of the pupils' view on a necessary frequency of the teaching method.

elements of the methodical setting of in courses taught at a high level (N=6)	Kendall's $\tau_b$ at time of measurement			
	begin of winter term 2010/11	end of winter term 2010/11	mid of summer term 2011	begin of winter term 2011/12
individual work	.086	.535	.000	.602
group work	-.086	0.86	.258	-.258
discussion in class	.430	.258	-.258	.258
explanation by teacher	-.258	.086	.258	.086
tasks without digital tools	.086	-.086	.258	-.086
differentiating tasks	.267	.258	.258	.258
repetition tasks	-.086	-.086	-.258	-.258
use of PC or handhelds	-.356	.258	.000	.267

\* the correlation is significant at the .05 level (2-tailed)

Table 1: Rank correlations between achievement group affiliation and frequency of methodical elements as perceived by pupils in high-level courses

elements of the methodical setting of in courses taught at a basic level (N=7)	Kendall's $\tau_b$ at time of measurement			
	begin of winter term 2010/11	end of winter term 2010/11	mid of summer term 2011	begin of winter term 2011/12
individual work	-.258	-.680	-.086	.323
group work	-.323	-.252	.356	.000
discussion in class	.000	.378	.775*	-.194
explanation by teacher	-.194	-.504	-.430	-.252
<b>tasks without digital tools</b>	<b>.775*</b>	<b>.387</b>	<b>.602</b>	<b>.630</b>
differentiating tasks	.378	.378	.430	.504
<b>repetition tasks</b>	<b>.252</b>	<b>.504</b>	<b>.775*</b>	<b>.756*</b>
use of PC or handhelds	-.126	.000	-.602	-.252

\* the correlation is significant at the .05 level (2-tailed)

Table 2: Rank correlations between achievement group affiliation and frequency of methodical elements as perceived by pupils in basic-level courses.  
Methods with a continual high correlation are highlighted.

In table 1 (high level courses), the “use of differentiating tasks” with a continuous correlation value around .26 appears, if at all, to be characteristic for the high achieving courses. In table 2 (basic level courses) we see two methods – the “use of tasks that are to be solved without the use of digital tools” and “tasks for repeating past contents” – standing out in that the correlation values at the four times of measurement are comparably high. These two methods apparently point to a relatively frequent use of mental maths tasks in class. With smaller correlations but all four having the same sign, the “use of differentiating tasks” could – as it is the case with high level courses – also be considered characteristic for high achieving courses.

questions	teachers of courses taught at a high level (N=10)	teachers of courses taught at a basic level (N=9)
I often use the mental maths task sets in course	$\tau_b = .466^*$	$\tau_b = .719^{**}$
Mental maths exercises are important	$\tau_b = .211$	$\tau_b = .504^*$
My pupils gladly accept the mental maths exercises	$\tau_b = .194$	$\tau_b = .609^*$

\* the correlation is significant at the .05 level (2-tailed)

\*\* the correlation is significant at the .01 level (2-tailed)

Table 3: Rank correlations between achievement group affiliation and level of agreement to questions from teacher questionnaire

With regard to the teachers perspective (table 3), we see high (in basic level courses) and medium (high level courses) correlations between the frequency of use of mental maths sets and achievement group affiliation. In basic level courses we find that the teachers as well as the pupils acceptance of the mental maths sets correlate with achievement group affiliation whereas in high level courses the correlation is low.

### Summary and discussion

From the teachers' perspective, high achieving courses are characterized by a relative frequent use of mental maths exercises. From the pupils' perspective it appears that high achieving courses taught on a basic mathematical level are characterized by a relative frequent use of tasks without digital tools and tasks for repeating contents from past grades, both probably indicating the use of regular exercises in basic mathematics. There is also some evidence that pupils from both high and basic level courses see a relatively frequent use of differentiating tasks as characteristic for high achieving courses. The frequency of digital tools being used in class is not a characteristic for high achieving or low achieving courses. So the presence of mathematical software apparently does not speak against the mastering and understanding of basic maths as

long as it is accompanied by regular mental maths exercises, repetition and, probably, differentiation.

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# STUDENTS' UNDERSTANDING OF THE MONOTONICITY OF EXPONENTIAL FUNCTIONS

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*The purpose of this study is to describe and analyse a model for students' transition from comparing exponents to the monotonicity of exponential functions within the context of operational and structural processes. The study was conducted with 193 11th grade students with the use of a test. The findings suggest that in this transition there are three phases, which are described in terms of Sfard's three - phase theory, interiorization, condensation and reification. Students' various difficulties emerged during this transition which are described and discussed.*

## INTRODUCTION

Exponential function is an important mathematical concept, which has applications in many other disciplines such as Physics, Chemistry, Biology, Economics and Social Sciences. For this reason, it is a central concept in high school and collegiate mathematics. Despite this, there has been comparatively little research on students' learning and understanding of exponents and especially of exponential functions. A number of these studies investigated students' or prospective teachers' understanding of numerical exponents, while other studies looked at instructional programs for exponents and exponential functions (Cangelosi, Madrid, Cooper, Olson, & Hartter, 2013; Kieran, 1992; Sastre & Mullet, 1998; Strom, 2006; Weber, 2002a, 2002b).

In a previous paper (Pitta-Pantazi, Christou & Zachariades, 2007) students' levels of understanding of the concept of exponents were investigated. In particular, students' understanding when comparing exponents with different base or different power was examined. Due to the crucial role of exponential functions in high school, it is important not to simply study students' way of learning and understanding exponents, but also to integrate this understanding with the learning and understanding of exponential functions. Thus, in this paper we investigate a unified developmental model for the understanding of comparing exponents and monotonicity of exponential functions, and provide a theoretical framework that may explain students' understanding and obstacles when engaged in this developmental process.

In the following, we present the theoretical framework of our study, describe our empirical study and close the paper with the discussion and conclusions drawn from our results.

## THEORETICAL BACKGROUND

In mathematics education several theoretical frameworks (e.g., Sfard, 1991; Dubinski, 1991) have been proposed to describe the transition from computational operations to abstract objects. Sfard (1991) analysed different mathematical definitions and representations and postulated that abstract notions, such as number and function, can



be conceived in two fundamentally different ways: structurally as objects, and operationally as processes. According to Sfard (1991), these two approaches are complementary. In the process of concept formation, operational conceptions precede the structural conceptions and there is a deep ontological gap between operational and structural conceptions. In the process of concept formation, Sfard (1991) distinguished three hierarchical phases: interiorization, condensation, and reification. At the phase of interiorization, a learner gets acquainted with the processes, which are operations performed on lower-level mathematical objects which will eventually form a new concept. At the phase of condensation, a learner refers to the process in terms of input-output relations rather than by indicating any operations. At the last phase of reification the learner conceives the notion as a fully-fledged object. In interiorization and condensation the notion is conceived operationally and the changes are rather gradual and quantitative. On the contrary, in the phase of reification the notion is conceived structurally and the transition from the previous faces requires a qualitative shift.

In the process of formation of monotonicity of exponential functions, the students should have the ability to compare exponents and then the ability to understand the monotonicity of exponential functions. Students may encounter certain difficulties during this transition. According to Slavit (1997), a property should be managed by an individual as a single object - in this case the comparison of exponents. However, difficulties may appear when one must define the property in the context of a more general idea, such as one that encompasses numerous and various types of objects and entails an abstract character - in this case monotonicity of exponential functions. The ability to compare exponents is the operational conception, which is conceptualized in the monotonicity of exponential functions.

Taking Sfard's (1991) three-phase framework, interiorization, condensation and reification, the purpose of this study was to identify students' transition from comparing exponents to the understanding of monotonicity of exponential functions. In particular, the aims of the study were:

- A) To investigate whether students' abilities regarding the comparison of exponents and monotonicity of exponential functions can be distinguished in three factors: (a) comparison of exponents with base a real number bigger than one, (b) comparison of exponents with base a real number between zero and one, and (c) monotonicity of exponential functions.
- B) To investigate the structure and relationships amongst these three factors and examine whether they correspond to Sfard's three-phase framework.
- C) To trace group of students that reflect these different types of abilities.

## **METHODOLOGY**

### **Participants**

To provide support for our study, a test was administered to 193 11<sup>th</sup> Grade students majoring in mathematics. All students were familiar with the notion of exponents and

exponential functions, as well as with their notational conventions and properties. Exponents are in essence introduced in Grade 8, where students deal with exponents with power natural numbers. In Grade 10, students deal with exponents with power rational numbers and with the properties of exponents. In Grade 11 students majoring in maths are taught exponents with powers real numbers, exponential functions, their monotonicity and graphs. Data were collected through a test which was administrated to students when the teaching of exponential functions was completed.

### **Test, Procedure and Data Analysis**

The test consisted of 19 tasks. Sixteen tasks required students to compare exponents of the same base. These 16 tasks were separated in two groups: (a) those with base larger than 1 and (b) those with base between 0 and 1. In the cases where the base was larger than one, there were tasks where the base was a natural number or a non-natural number. In particular, there were six tasks which had as a base a natural number, four with base a non-natural number larger than one and six with base real number between 0 and 1. Furthermore, there were also three tasks with exponential functions, where algebraic expressions and graphic representations were given and students were asked to match the algebraic expression with the appropriate graphic representation.

Students were asked to complete the tasks in the test and also provide written justifications for their answers. Students' written responses were analyzed by the researchers. Students' fully correct responses were marked with 1, while incorrect and no response with 0.

Confirmatory factor analysis (CFA) was applied, using MPLUS (Muthén & Muthén, 2004) to assess the results of the study. Our purpose for using CFA was to investigate whether different tasks in the context of exponents can form different factors which reflect different type of abilities. In order to evaluate model fit, three fit indices were computed: The chi-square to its degree of freedom ratio ( $\chi^2/df$ ), the comparative fit index (CFI), and the root mean-square error of approximation (RMSEA) (Marcoulides & Schumacker, 1996). According to Marcoulides and Schumacker (1996), for the model to be confirmed, the values for CFI should be higher than 0.90, the observed values for  $\chi^2/df$  should be less than 2 and the RMSEA values should be close to or lower than 0.08. Latent class analysis (LCA), which is a part of mixture growth analysis (Muthén & Muthén, 2004), was used to trace different groups of students, which exhibit these different types of abilities. LCA is a statistical method for finding subtypes of related cases (latent classes) of multivariate data. Once the latent class model is estimated, subjects can be classified to their most likely groups by means of recruitment probabilities. A recruitment probability is the probability, for a randomly selected member of a given latent class, a given response pattern will be observed.

### **RESULTS**

Initially we examined whether students' abilities regarding the comparison of exponents and monotonicity of exponential functions can be distinguished in three factors: (a) comparison of exponents with base a real number bigger than one, (b) comparison of exponents with base a real number between zero and one, and (c)

monotonicity of exponential functions. We evaluated the construct validity of this model, by examining whether the 19 tasks loaded adequately on each of the above three factors. The confirmatory factor analysis (CFA) showed that this model did not reflect the empirical data because the indices were not appropriate for the acceptance of the model ( $CFI=0.888$ ,  $\chi^2=57.247$ ,  $df=38$ ,  $\chi^2/df=1.51$ ,  $RMSEA=0.074$ ).

We then hypothesised that it was likely that the questions of the first of the above factors could be separated into two factors: (a) comparison of exponents with base a natural number (F1) and (b) comparison of exponents with base a non-natural number bigger than one (F2). The other two factors remained: comparison of exponents with base real number between zero and one (F3) and monotonicity of exponential functions (F4). The confirmatory factor analysis (CFA) showed that this model was better than the previous one ( $CFI=0.934$ ,  $\chi^2=57.247$ ,  $df=38$ ,  $\chi^2/df=1.51$ ,  $RMSEA=0.057$ ).

We then proceeded to investigate the structure and relationships amongst these four factors of abilities. We did this by evaluating the construct validity of various models in order to find the one that best fitted our data. The model that best fitted our data ( $CFI=0.947$ ,  $\chi^2=57.247$ ,  $df=38$ ,  $\chi^2/df=1.51$ ,  $RMSEA=0.051$ ) was the one presented in Fig.1, where F100 is a second order factor composed of F1, F2 and F3. The validated model implies that students grasp F100 only when an understanding of F1, F2 and F3 has been achieved. Finally, for students to achieve F4 they must first achieve F100.

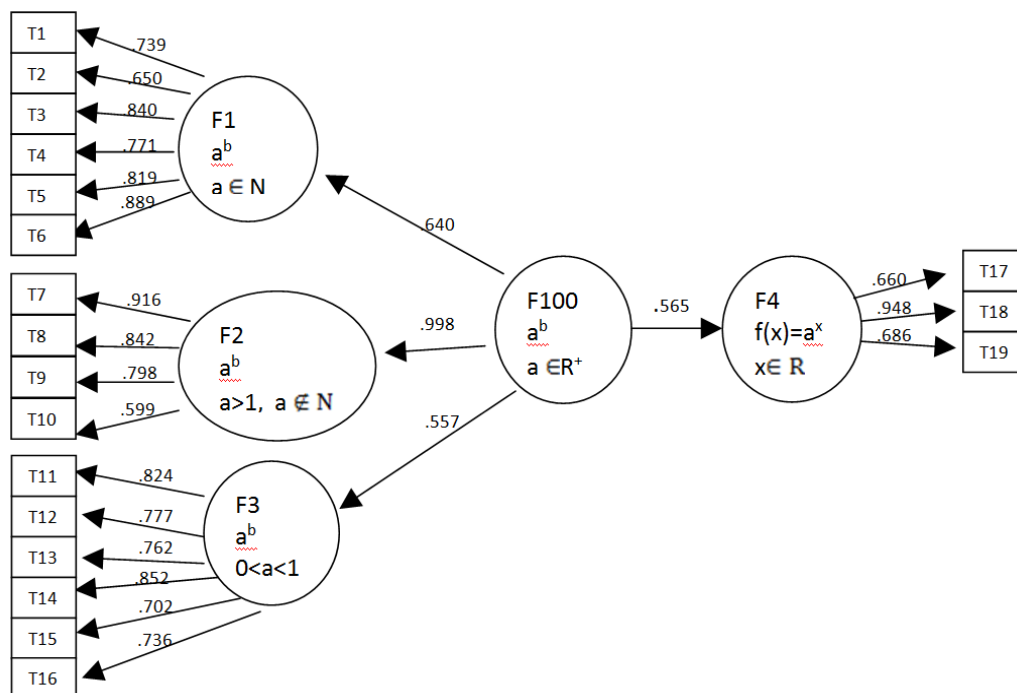


Figure 1: The proposed model.

The second aim of the study was to investigate the possibility that students may be grouped according to their responses to the tasks. Mixture growth modelling was used to answer this question (Muthén & Muthén, 2004) because it enables specification of models in which one model applies to one subset of the data and another applies to another set. The modelling here used a stepwise method that is, the model was tested under the assumption that there are two, three, four, five, and six groups of students.

For instance, it would not have been surprising if only three groups of students were identified, one corresponding to the interiorization phase, one to the condensation and one to the reification phase. However, the best fitting model with the smallest AIC (3208.360) and BIC (3596.620) and largest entropy (0.934) was the one involving six groups: Group 1 ( $n=12$ ), Group 2 ( $n=17$ ), Group 3 ( $n=12$ ), Group 4 ( $n=64$ ), Group 5 ( $n=77$ ) and Group 6 ( $n=11$ ). Taking into consideration the average latent class probabilities which were larger than 94%, we may conclude that these groups are quite distinct. This means each group has its own characteristics and responds to tasks of different levels of complexity. Table 1, presents the factors for which group students had a success rate over 60% for more than 2/3 of the tasks related to each factor.

	Group 1	Group 2	Group 3	Group 4	Group 5	Group 6
F1 $a^b a \in \mathbb{N}$	-----	√	√	√	√	√
F2 $a^b a > 1, a \notin \mathbb{N}$	-----	-----	√	-----	√	√
F3 $a^b 0 < a < 1$	-----	-----	-----	√	√	√
F4 $f(x)=a^x \ x \in \mathbb{R}, a > 0$	-----	-----	-----	-----	-----	√

Table 1: Group students' success rate in the four factors.

Overall students in Group 1 were not successful in any of the factors. Group 2 students were successful in F1, while Group 3 students were successful in F1 and F2, and Group 4 students were successful in F1 and F3. Group 5 students were successful in F100 (F1, F2 and F3), and Group 6 students were succeeded in F100 and F4.

Group 1 students had serious difficulties to compare exponents. They answered correctly only to a small number of tasks of Factors F1, F2, F3 but not to more than 2/3 of the tasks related to these factors. We could argue that these students are just entering the interiorization phase of Sfard's (1991) framework. Group 2 students could compare only exponents with base  $a$  a natural number, Group 3 students could compare exponents with base  $a$  a number bigger than one and Group 4 students could compare exponents with base  $a$  a natural number or a real number between 0 and 1. We could claim that they were familiar with the processes, which were operations performed on lower-level mathematical objects but they did not have the ability to correctly compare all possible cases. We would argue that the students of these groups achieved Sfard's interiorization phase but not the condensation phase. Group 5 students were able to compare any type of exponents but did not conceive the monotonicity of exponential functions. We postulate that these students achieved the condensation phase but not the reification phase. Group 6 students were able to do everything that Groups 5 students could do but they were also able to conceive the monotonicity of exponential functions. We can therefore assume that Group 6 students achieved the reification phase.

## DISCUSSION AND CONCLUSIONS

The purpose of this study was to investigate students' transition from comparing exponents, to the understanding of monotonicity of exponential functions. When this study was conducted, the participants had already been taught exponents, exponential

functions and their properties. The results of the study yielded a developmental model for this transition. This model corresponds to Sfard's three-phase hierarchical framework. Students' achievement placed them across these three phases. Data was collected through a written test. The data analysis showed that the test questions were grouped in four first order factors (F1, F2, F3 and F4), and one second order factor F100 which was comprised by F1, F2 and F3. F1 questions concerned the comparison of exponents which had as a base a natural number, F2 questions concerned the comparison of exponents with base a non natural number bigger than one, F3 questions concerned the comparison of exponents which had as a base a number between zero and one and F4 questions dealt with the monotonicity of exponential functions. Success in F1, F2 and F3 suggests different aspects of operational understanding. We could argue that achieving at the most two of the three factors: F1, F2 and F3, corresponds to Sfard's phase of interiorisation. Students in this phase perform processes and operations on lower-level mathematical objects partially. This means that they are able to perform correctly these operations only in some of the cases of exponents but not in all of them. The condensation phase is characterised by achievement of the second order factor F100 (comprised by: F1, F2 and F3). Students in this phase perform processes and operations in all type of exponents and are able to think of these processes as a whole. We suggest that students who succeed in F100 and F4, become capable of perceiving the monotonicity of exponential functions as a fully-fledged object. According to Sfard (1991), such transition suggests the reification of the concept.

In addition to this, the data analysis showed that six distinct students' groups could be formed based on the students' responses. These six groups of students exhibited variation regarding their abilities in the four factors described earlier on. Moreover, the characteristics of each group of students suggest that each group of students is at a different phase of Sfard's (1991) framework. Group 1 students were just entering in the phase of interiorization, since they were able to answer correctly only to a small number of exponents' comparison. However, they were not able to achieve any of the factors. Group 2 students achieved factor F1; Group 3 students achieved factors F1, F2; and Group 4 students achieved factors F1 and F3. The students of these three different groups were also in the phase of interiorization. Group 5 students achieved factor F100 and we claim that they were in the phase of condensation. While, Group 6 students achieved all factors and were in the phase of reification.

From the results, it emerges that students faced two main difficulties in their transition from the interiorization to the condensation phase. The first difficulty was to move from the comparison of exponents with base a number bigger than one to the comparison of exponents with base a number between zero and one. This finding was also found in a previous study (Pitta-Pantazi et al., 2007). A possible explanation may be that students were affected by the intuitive rule "More A – More B" (Stavy, Tsamir & Tirosh, 2002) since they may have reasoned that "larger power implies larger exponent". The second difficulty was students' transition from the comparison of exponents with base a natural number to the comparison of exponents with base a

number bigger than one. This difficulty was unexpected. We conjecture that this difficulty may have risen due to didactical materials and approaches. It was observed that the examples and tasks related to exponents, in the mathematics textbooks which these students used, were mainly with base a natural number or a number between zero and one. It seems that many students constructed a concept image (Tall & Vinner, 1981) which led to these misconceptions. They seemed to believe that the cases where the base is bigger than one but not a natural number and the cases where the base is a number between zero and one were the same. This explains the failure of Group 4 students in the questions of F2.

The transition from interiorization to condensation requires quantitative changes (Sfard, 1991) and students' difficulties that emerged from this study, are of this type. On the other hand, the transition from condensation to reification requires qualitative changes (Sfard, 1991). Specifically, in the condensation phase students had to deal with properties of real numbers, whereas in the reification phase they had to conceive these properties in a more holistic and comprehensive way, as properties of a function. This ontological shift appears to have caused students serious difficulties (Slavit, 1997).

The results of this study could inform the teaching of exponents and exponential functions in high school. The students' difficulties in comparing exponents and the understanding of the monotonicity of exponential functions arising from this study highlight some crucial points for teaching that teachers should pay special attention.

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# A LONGITUDINAL STUDY TRACING THE DEVELOPMENT OF NUMBER SENSE COMPONENTS

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*Based on a synthesis of the literature, a model for number sense was formulated, and validated. The major constructs incorporated in this framework were elementary number sense, conventional arithmetic and algebraic arithmetic. To trace the development of number sense components a longitudinal study was conducted. One hundred and forty 1<sup>st</sup> grade students were tested individually on four different occasions. Data analysis suggested that the hypothesized components contributed to the latent construct number sense. A latent growth model showed that elementary number sense and conventional arithmetic adopt a linear growth rate, while algebraic arithmetic needs a stabilization period after a significant growth period.*

## INTRODUCTION

In the 21<sup>st</sup> century, children's development of number sense is considered as an important outcome of school curricula. Although, its importance in mathematics curricula is recognized, its usefulness in research is controversial (Verschaffel et al. 2007). This arises from the fact that there is no widely accepted definition of number sense and what this term involves. As Griffin (2004) argues, number sense "is easy to recognize but it is difficult to define and therefore to teach" (p.173). According to Yang (2005), number sense refers to a person's general understanding of numbers and operations and the ability to handle daily life situations which include numbers, while Schneider and Thompson (2000) emphasize that good number sense involves flexibility in thinking about numbers.

The majority of the research studies propose that the number sense is two-dimensional (Jordan et al., 2006). The first dimension refers to basic number skills and the second one to conventional arithmetic situations. However, it seems that this two-dimensional conception of number sense does not contribute to fill in the gap between arithmetic and algebra. Thus, we propose that a new dimension of number sense, namely algebraic arithmetic could enrich number sense conceptualization. Algebraic arithmetic thinking can be conceived a generalized arithmetic of numbers and quantities in which the concept of function assumes a major role (Carraher, Schliemann, Schwartz, 2007). The inclusion of algebraic arithmetic as a component of number sense could be considered as a move from particular numbers and measures towards relations among sets of numbers and measures. In this study we will extend and revisit the notion of number sense by proposing algebraic arithmetic as a new dimension and trace the development of number sense by describing the growth rate of its components. The aims of the study were: (a) to examine the components of number sense through proposing a theoretical model and (b) to describe the development of the components of number sense by validating a unified growth model.



## **THEORETICAL BACKGROUND**

The majority of research studies in number sense analyse its intuitive nature, its gradual development and the ways in which it is manifested (Markovits & Sowder, 1994; NCTM, 2000). The intuitive nature of number sense refers to the general understanding of numbers and operations, the ability and inclination to use this understanding in flexible ways, to make judgments and to develop useful and efficient strategies for managing numerical situations (Reys, Reys, Emanuelsson, Johansson, McIntosh, & Yang, 1999). Number sense is manifested in many ways, including judging number magnitude, appropriate use of benchmarks, using number flexibly when mentally computing, estimating and judging reasonableness of results, understanding relative effect of operations and decomposing and recomposing numbers to solve problems. Understanding number magnitude encompasses the abilities to compare numbers, decide which of two numbers is closer to a third one, to other numbers or to find numbers in a given range (Markovits & Sowder, 1994).

Number sense should result in a view of numbers as meaningful entities. This could be achieved by developing number sense according to the following framework: (a) recognition of relative number size, i.e., implying the recognition of the relative size of numbers, (b) using multiple representations of numbers and operations, i.e., implying the use of different representations to solve numerical problems flexibly, (c) judging the reasonableness of estimates of computed results; i.e., implying the use of estimation strategies without written calculations, and (d) recognizing the meaning and the relative effect of operations on numbers, including flexible computing and counting strategies (Yang, Li & Lin, 2007). Griffin (2004) suggests the term “central conceptual structure for number” to define the core of number sense. That is, the structure of number enables children to conceptualize a broad range of quantitative problems and provides the foundation for the acquisition of more complex number concepts. This makes feasible the transformation of number computations from something that can only be done with manipulative to something that can be done in their own heads. We could conclude that Griffins’ description may be related to the acquisition of pre-algebraic relations. In addition, a number of researchers suggested story problems as a component of number sense. Jordan et al. (2006) claimed that number sense is a two dimensional construct: (a) basic number skills (counting, number recognition and knowledge, number patterns, nonverbal calculation) and (b) conventional arithmetic (story problems and number combinations).

The above studies failed to incorporate the dynamic nature of algebraic relations and the relation among arithmetic and algebra. Researchers emphasized that the artificial separation of arithmetic and algebra deprives students of powerful ways of thinking about mathematics in the early grades and makes it more difficult for them to learn algebra in the later grades (Carpenter, Levi, Berman, & Pligge, 2003). Based on these findings, we propose that algebraic reasoning tasks may also constitute a crucial component of students’ development of number sense.

## **THE PRESENT STUDY**

The purpose of the present study is to propose a model describing the structure and the development of number sense. Specifically, the aims of the study were to (a) to examine the structure of number sense through proposing a theoretical model and (b) to trace the development of the components of number sense by validating an integrated growth model. Based on a synthesis of the literature, we hypothesized that number sense is a general construct that consists of three distinct, but interrelated and complimentary factors. In particular, the contribution of the present study lies on the fact that it revisits and extends the model proposed by Jordan and her colleagues (2006) regarding the parameters of number sense, by adding a third component in the structure of number sense. Thus, in order to capture the nature of first grade students' number sense we theoretically established and empirically validated a measurement model hypothesizing that number sense is a general theoretical construct that consists of (a) elementary number sense, (b) conventional arithmetic and (c) algebraic arithmetic. We hypothesized that the factor elementary number sense refers to key elements of numbers sense (see Baroody & Wilkins, 1999; Jordan, et al., 2006), such as counting and number knowledge by discriminating and coordinating quantities and making numerical magnitude comparisons. It was also hypothesized that the factor conventional arithmetic refers mainly to story problems and number combinations that encompass number transformation situations, such as calculating in verbal and nonverbal contexts. Finally, it was assumed that the third proposed number sense component, algebraic arithmetic, refers to conceptualizing number patterns and number equations. Furthermore, in order to capture the development of number sense components, we hypothesized that a unified latent growth model could predict and describe students' advancements in the three sub-components of number sense.

### **Measures**

The quality of a study evaluating a measurement model depends heavily on the selection of appropriate indicators. Thus the test items were adopted or developed based on previous research studies. The majority of the test items were adopted from the Curriculum Based Measurement (Fernstrom & Powell, 2007). In addition, multiple-indicators were used for each presumed dimension (Kline, 2010). Six types of tasks were used to measure elementary number sense: (a) counting tasks, (b) number recognition, (c) quantity discrimination, (d) number knowledge, (e) enumeration, and (f) non-verbal calculation. In the counting tasks, students were asked to enumerate objects, in the number recognition tasks, students had to read numbers, in the quantity discrimination tasks students were asked to decide which was the largest number, in the number knowledge tasks, students were asked to find smaller and bigger numbers of a given one and in the non-verbal calculation tasks students had to add or delete objects in a given set so the number of objects corresponds to a given number. Conventional arithmetic factor was measured with two types of tasks: (a) story problems, and (b) number combinations. In story problems tasks, students had to select the appropriate number sentence for a list of story problems and in number combinations tasks students had to find mentally the result of addition, subtraction,

multiplication and division combinations. Finally, the proposed component algebraic arithmetic number sense was measured by two types of tasks: (a) number patterns and (b) number equations. In number patterns tasks, students had to extend or complete number patterns, such as 5, 8, 11, ... The ability to solve number patterns implies that a student can conceptualize the relations among numbers to fill in or extend a number pattern. In number equations, students were asked to complete the missing terms of equations, such as  $3+5=4+\square$ . Solving number equations encompasses conceptualizing the equal sign as equity of two quantities and the relations among all its components.

### **Participants and Procedure**

Recruitment forms were distributed to four urban primary schools in Nicosia. Consent forms were returned for 80% of the students, resulting in a sample of 140 students. Each student was interviewed in two sessions of approximately 30 minutes each (5 types of tasks in each session). Students had a time restriction for each type of task (one minute for the majority of tasks). Students were assessed on the number sense measures four times during the period February to June (approximately one administration per month). Students were individually tested in all four occasions. Short breaks were provided between tasks to limit fatigue. All the types of tasks were organised in equivalent parts. The order of the parts was rotated in the four time series.

### **Data Analysis**

The use of confirmatory factor analysis CFA made sense because we wanted to examine the validity of an a priori model, based on past evidence and theory. CFA was conducted by using MPLUS, which is widely popular for its robust parameters (Muthén & Muthén, 2007). To trace the development of number sense components we used growth models. Growth models examine the development of individuals on one or more outcome variables over time. In growth modeling, random effects are used to capture individual differences in development. In a latent variable modeling framework, the random effects are reconceptualized as continuous latent variables, that is, growth factors (Muthén & Muthén, 2007). In order to evaluate model fit, three widely accepted fit indices were computed (Marcoulides & Schumacker, 1996): The chi-square to its degree of freedom ratio ( $\chi^2/df$  should be less than 2); the comparative fit index (CFI should be higher than .9); and the root mean-square error of approximation (RMSEA should be close to or lower than .08).

## **RESULTS**

### **The Structure of Number Sense**

Figure 1 facilitates the conceptualisation of how the various components relate to each other, and presents the structural equation model with the latent variables of the number sense components and their indicators. Confirmatory factor analysis was used to evaluate the construct validity of the model. The descriptive-fit measures indicated support for the hypothesized model ( $\chi^2/df=1.23$ , CFI=.98, and RMSEA=.04), confirming that the observed and theoretical factor structures matched for the data set. The results of the study validated the proposed model indicating that number sense is a

general, higher-order latent construct. As is highlighted in Figure 1, number sense might be described as a synthesis of three dimensions, namely, elementary number sense, conventional arithmetic and algebraic arithmetic. The parameter estimates were reasonable in that all factor loadings were statistically significant and most of them were rather large, ranging from .33 to .82 (see Figure 1). The analysis showed that each of the task type employed in the present study loaded adequately only on one of the first order factors, giving support to the assumption that the three first-order factors could represent three distinct abilities in number sense. The factor loadings of the first-order factors, corresponding to students' elementary number sense, conventional arithmetic and algebraic arithmetic, to the second-order factor were extremely high (.79, .98 and .98 respectively). This indicates that a general construct that refers to number sense could explain students' variances in these types of situations very accurately. In addition, the results revealed that the three first order factors had almost the same prediction validity on the higher number sense factor ( $r^2_{\text{element.}} = .62$ ,  $r^2_{\text{algebraic}} = .98$ , and  $r^2_{\text{conventional}} = .98$ , respectively).

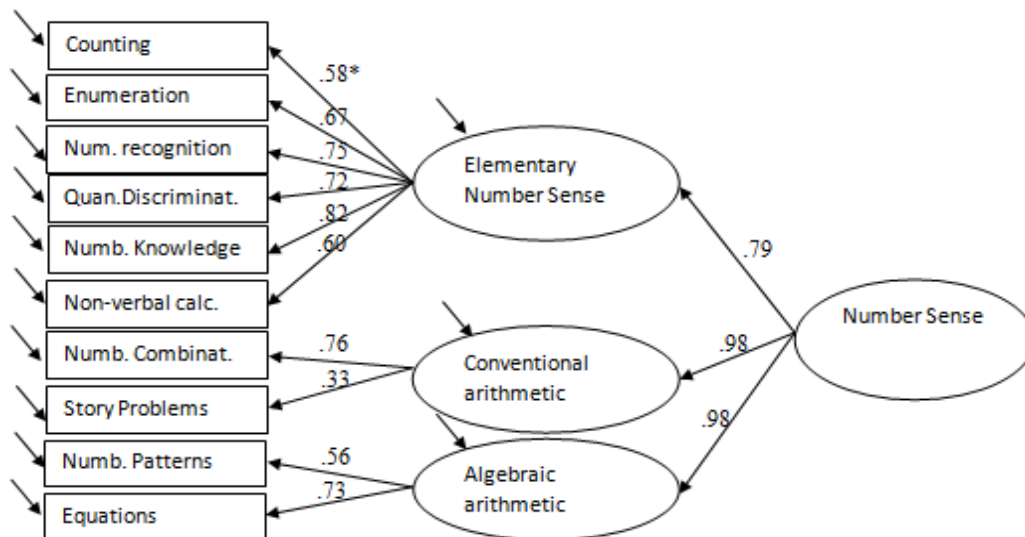


Figure 1: The structure of number sense.

### The Development of Number Sense Components

Longitudinal data in four time waves were used to investigate the development of number sense components. That is, we examined the validity of alternative unified latent growth models, hypothesizing that elementary number sense, conventional arithmetic and algebraic thinking are developed with a linear or quadratic growth rate. The best fitting model (with the smallest AIC and BIC,  $\chi^2/df=1.62$ , CFI=.96, and RMSEA=.07) was the one assuming that (a) elementary number sense develops with a linear growth rate (El\_1@0, El\_2@1, El\_3@2, El\_4@3), (b) conventional arithmetic follows also a linear growth rate (C\_1@0, C\_2@1, C\_3@2, C\_4@3), and (c) algebraic arithmetic develops in steps, i.e., there is a growth between the first and second wave, stabilization in the third wave and then growth in the fourth wave (A\_1@0, A\_2@1, A\_3@1, A\_4@2). Thus, the results of the study showed that three components of number sense do not develop with the same growth rate (see Figure 2). Elementary

number sense and conventional arithmetic are modelled by a linear growth rate, while the development of algebraic arithmetic is more complicated. The adopted growth model indicated that algebraic arithmetic needs a period of stabilization after a growth period. However, the results of the study showed that algebraic arithmetic had the largest mean latent slope (4.44), conventional arithmetic mean latent slope was the second largest (2.12) and elementary number sense mean latent slope was the smallest one (1.80).

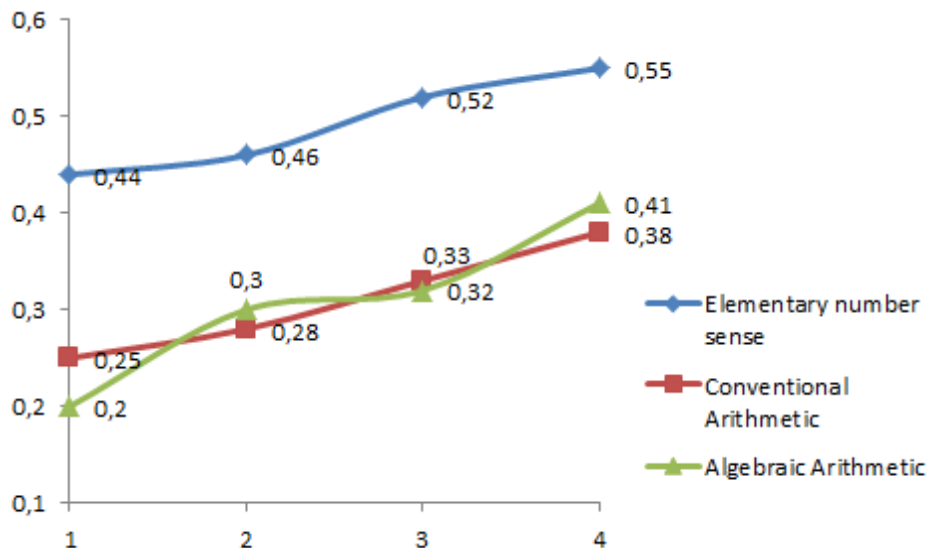


Figure 2: The development of number sense components.

Figure 3 presents the results of the standardized solution of the model. The validation of a unified model for the three latent growth factors makes feasible the study of the relations among the latent factors intercept and slope of three components. The results of the study show that the correlations among the three latent intercepts were statistically significant ( $\underline{r}_{(\text{elem}, \text{conven.})}=.59$ ,  $\underline{r}_{(\text{elem}, \text{algebr.})}=.76$ , and  $\underline{r}_{(\text{algebr.}, \text{conven.})}=.72$ ). In addition, the correlation between the intercept of elementary number sense was highly correlated with the slope of conventional arithmetic ( $\underline{r}=.68$ ) and the intercept of algebraic arithmetic was correlated with the slope of conventional arithmetic ( $\underline{r}=.76$ ).

## DISCUSSION

The contribution of the study lies on the extension of the two-dimensional model proposed by Jordan and her colleagues (2006) to incorporate the dynamic and important dimension of algebraic arithmetic that transforms the “conventional” character of number sense to a more comprehensive construct that could serve as a vehicle to facilitate the development of students’ algebraic reasoning in the future. Defining the components of number sense is extremely important because mathematics teachers should have a deep understanding of the components of number sense and an understanding of the difficulties students encounter in solving problems.

The results of the study show that the proposed “algebraic arithmetic” follows a different growth rate, compared to the other components. This finding exemplifies the need to study in depth what teaching experiences may enhance the algebraic growth after the stabilization period. In addition, the study shows that the latent growth factors

of the three components correlate. For example, this may suggest that the intercept of one component of number sense may affect the magnitude of the growth of another component. Moreover, the study underlies the importance of algebraic arithmetic since it may affect the growth rate of conventional arithmetic.

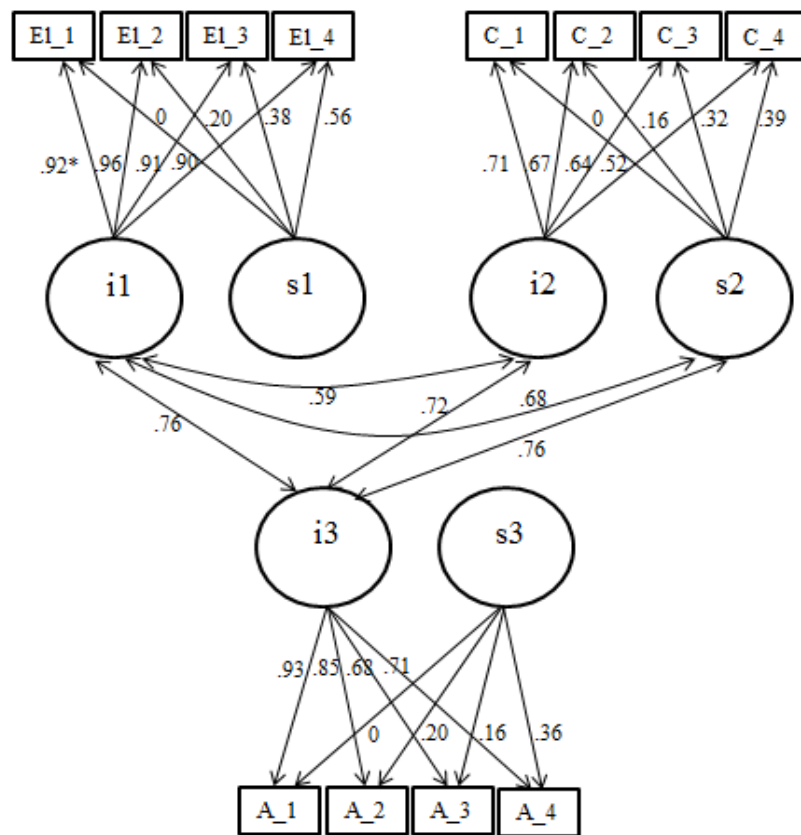


Figure 3: The unified growth model.

Note: i1=Intercept of Elementary Number Sense, s1=Slope of Elementary Number Sense, i2=Intercept of Conventional Arithmetic, s2=Slope of Conventional Arithmetic, i3=Intercept of Algebraic Arithmetic, s3=Slope of Algebraic Arithmetic, E1\_1 to E1\_4 refers to the four measures of Elementary Number Sense, C\_1 to C\_4 to the Conventional Arithmetic ones and A\_1 to A\_4 to the Algebraic Arithmetic ones.

\*:  $p < .05$

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# **FAMILY BACKGROUND OR LANGUAGE DISADVANTAGES? FACTORS FOR UNDERACHIEVEMENT IN HIGH STAKES TESTS**

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*As in many countries, socially disadvantaged learners achieve substantially lower than their classmates in German mathematics classrooms. For the German language context, still little is known about the degree and the mechanisms how multilingualism, SES, immigrant status, language proficiency and other background variables impact the (under-)achievement in mathematics. For specifying the most relevant factors, the presented study explored the data of 1495 high stakes tests. By analysis of variance, academic language proficiency in the language of assessment turned out to be more relevant than other background factors. First explanations for these results can be given by linguistic item analysis and clinical interviews that show typical language demands.*

## **Impact of different background factors**

In many countries, social disparities in achievement raise equity issues when the educational systems fail to decrease disadvantages of underprivileged students. In Germany, social disparities in mathematics achievement have mainly been discussed with reference to two (overlapping) groups of students: multilingual immigrant students and students with low SES (socio-economic status). For both groups, the achievement gap became apparent in different large scale assessments in Germany (Bos et al., 2003; OECD, 2007; Werning et al, 2008; Heinze et al., 2009). That is why the official German education statistics has recently introduced immigrant status as relevant factor (besides nationality, a factor with decreasing explanatory power, cf. Bildungsbericht 2012).

Notwithstanding this dominant German attention to immigrant status and SES as relevant factors, results from empirical studies in *other* countries suggest that language proficiency might be an even more relevant background factor for mathematics achievement (Secada, 1992; Abedi, 2006). As Brown (2005) has emphasized, the language proficiency is especially relevant for those achievement tests that follow a literacy approach (like PISA, cf. OECD, 2007) and provide test items in realistic contexts with high linguistic complexity. However, little is known on the *extent* of impact of language proficiency on mathematics achievement in the German language context, especially for literacy-based high stakes tests.

One important literacy-based high stakes test is the central exam in grade 10 that is conducted in Germany's largest federal state North Rhine Westphalia. As the test determines the final middle school degree in mathematics (eventually combined with



oral exams), it is highly relevant for each student. We therefore investigated the following research questions for the specific German language context:

*Q1. Specification of relevant factors:* Which of the social and linguistic background factors have the highest impact on mathematics achievement in the literacy-based test?

*Q2. Understanding the impact of language issues:* What kind of language demands pose obstacles for students with language disadvantages?

As will be shown, the quantitative part of our mixed methods study on Q1 provides statistical evidence that language proficiency (in the test language German) has a higher impact on mathematics achievement than SES or immigrant status. This hardly surprising result replicates empirical findings in America and other countries where language issues are regularly investigated (Abedi, 2006; Jorgensen, 2011). However, for Germany, it is a politically and educationally important result that hopefully initiates a more consequent shift of attention to language issues. As a statistical analysis can always only find correlations, but no causal connections, the second, qualitative part on research question Q2 intends to offer explanations for this relevance: For *understanding* the detailed mechanisms how language proficiency impacts achievement, a linguistic item analysis and clinical interviews were conducted to reconstruct the language demands not only in terms of test *fairness* (Abedi, 2006) for linguistically disadvantaged students, but also for specifying what exactly students have to learn for being successful in literacy-based tests.

## RESEARCH DESIGN AND METHODS FOR THE MIXED-METHODS STUDY

### Design for the quantitative part

*Sample.* The sample consisted of  $n = 1495$  students from 67 medium streamed mathematics classes, being streamed according to a medium achievement level, in 19 representative comprehensive schools in a metropolitan region in North Rhine Westphalia. In Table 1, we report the composition of the sample with respect to the background variables in view.

Measures for data gathering.

- The dependent variable, *mathematics achievement*, was measured by the score in the high stakes test in grade 10, as summed up from teachers' evaluation sheets. For further statistical analyses, the 27 items were dichotomized and Rasch-scaled for receiving scores on an interval scale (Rost, 2004).
- The *family background* was considered by a questionnaire on students' immigrant status, family languages, age of first contact to German language and other variables. The socio-economic status (SES) was measured by the widely used book scale with pictures (Paulus, 2009).
- *Language proficiency* was measured by two instruments: 1. *reading proficiency* by the reading scales in the parallel central exam of German (in a subgroup of 1066 students being streamed according to medium achievement level in the

language classes), and 2., a more complex construct of *German academic language proficiency* (Cummins, 1986) was assessed by a standardized C-Test (Grotjahn, Klein-Braley, & Raatz, 2002) in a subgroup of 698 students.

*Data analysis procedures.* Different statistical analysis procedures were applied for operationalizing the impact of different factors. The most direct approach is to split the sample into groups of privileged, medium and disadvantaged students with respect to different background factors and language proficiency. The difference of mean scores in the mathematics tests gives a first operationalization for finding the highest impact (see Table 1).

An analysis of variance (one-way ANOVA) was used to test for significant differences in the mean scores between three groups for each background factor (language proficiency, SES, immigrant status...), the privileged, medium and disadvantaged students. The ANOVA was also used to determine the explained variance, i.e. the proportion to which each factor accounts for variance. The explained variances for the interval-scaled factors reading proficiency and German academic language proficiency were also calculated by a regression analysis.

### **Design for the qualitative part of the study**

For understanding how language issues impact the success of item solution, a linguistic item analysis was conducted with respect to lexical, morphological and syntactic specificities and typical challenges (Gürsoy et al., 2013).

In order to reconstruct which of the linguistic specificities pose most obstacles for students, clinical interviews were organized with 20 students with language disadvantages and 20 with high academic language proficiency. In these interviews, students' solution processes were videotaped and accompanied by prompts to verbalize the mental processes for "cracking the code" (Zevenbergen, 2000) of the item texts. The videos were partly transcribed and interpretatively analysed with respect to typical obstacles and ways to overcome them. Due to space restrictions, we cannot show these ongoing analyses in details here, but only sketch selected insights.

## **SELECTED RESULTS**

### **Language proficiency as most relevant factor**

Table 1 shows the composition of the sample according to different background variables and language proficiency. Although achievement gaps appear between the privileged and disadvantaged groups for each factor, the sizes of the gaps vary considerably: The mean score in the mathematics test is slightly higher for the privileged group's mean compared to the disadvantaged group for the factors immigrant status (difference  $46.2 - 40.9 = 5.3$ ), SES (3.8), and age of first contact to German (6.8). In contrast, for German academic language proficiency, the difference of mean scores is nearly a standard deviation (13.0) and slightly lower for reading proficiency (9.7) in the subgroup with higher mean score. The big differences for academic language proficiency translate into an achievement gap between grade 3 and

nearly grade 5 (1 being excellent, 5 being failed): For a score of 37 points, a grade 5 was attributed, for a score of 50 points, a grade 3.

Factor	Specification of groups	Distribution of groups	Mean score (max. 85) m (SD)	Difference
Whole sample	67 medium streamed classes	<i>n</i> = 1495	43.5 (13.6)	
Immigrant status ( <i>n</i> =1480)	1 <sup>st</sup> generation (student immigrated)	152 (10.3 %)	41.3 (13.6)	5.3
	2 <sup>nd</sup> generation (parents immigrated)	623 (42.1 %)	40.9 (13.5)	
	no / 3 <sup>rd</sup> generation	705 (47.6 %)	46.2 (13.0)	
Socio-economic Status ( <i>n</i> =1493)	low SES	509 (34.1 %)	41.9 (14.0)	3.8
	medium SES	488 (32.7 %)	42.9 (12.9)	
	high SES	496 (33.2 %)	45.7 (13.4)	
Age of first contact to German ( <i>n</i> =1486)	first German after age of 3 years	289 (19.4 %)	39.5 (13.7)	6.8
	first German before age of 3 years	538 (36.2 %)	42.2 (13.5)	
	only German monolingual	659 (44.3 %)	46.3 (13.0)	
German Academic language proficiency (C-Test in selected classes; <i>n</i> = 698)	low C-Test	235 (33.7 %)	37.3 (13.4)	13.0
	medium C-Test	233 (33.4 %)	44.2 (12.6)	
	high C-Test	230 (33.0 %)	50.3 (11.4)	
	<i>all C-Tests</i>	698 (100 %)	43.9 (13.6)	
German reading Proficiency (in medium streamed language classes; <i>n</i> = 1066)	low reading proficiency	365 (34.2 %)	40.3 (12.9)	9.7
	medium reading proficiency	405 (38.0 %)	46.6 (12.6)	
	high reading proficiency	296 (27.8 %)	50.0 (12.5)	
	<i>all reading tests</i>	1066 (100 %)	45.4 (13.3)	

Table 1. Distribution and Differences between groups

These results on the initial test scores are confirmed by the analysis of variance with the rasch-scaled measures: For all five factors, the group differences between underprivileged and privileged groups appear as highly significant ( $p < 0.001$  in all 5 F-tests). However, whereas the family background factors (immigrant status, SES, and age of first contact to German) account for only 1 % to 3 % of the variance, the explained variance of German academic language proficiency is much higher with 14 % in the regression analysis. Remark that this value is quite high for a quite homogenous group of students (all being in medium streamed mathematics classes).

These findings show that also in the investigated high stakes test, language proficiency has a higher impact on mathematics achievement than family background alone. The next section gives selected insights into the *quality of language demands* from the ongoing linguistic and interview analysis. Although being only roughly sketched, they offer first clues for explaining the statistical findings.

### “Cracking the connections” as a crucial language demand in many items

On a first sight, the length of the texts or unknown isolated words are the immediately visible characteristics of mathematical item texts in literacy-based tests. However, Zevenbergen (2000) emphasized by the metaphor “cracking the code” that the linguistic and cultural challenges for disadvantaged students are usually more complex than isolated unknown words. In our analysis, one specific aspect of this “code” of academic language was most striking for which we analogize the metaphor to

“cracking the connections”. By this, we mean decoding linguistic means that signify *relevant mathematical or textual relations in the item texts on sentence and text level*.

*On the sentence level*, cracking connections appeared for example in the following item on functional dependencies between fuel consumption and speed of vehicles. The sentence in Item (2) is typical for the academic test language: It is quite short, does not contain many *lexical* demands, but it is *morphologically* and *syntactically* highly complex.

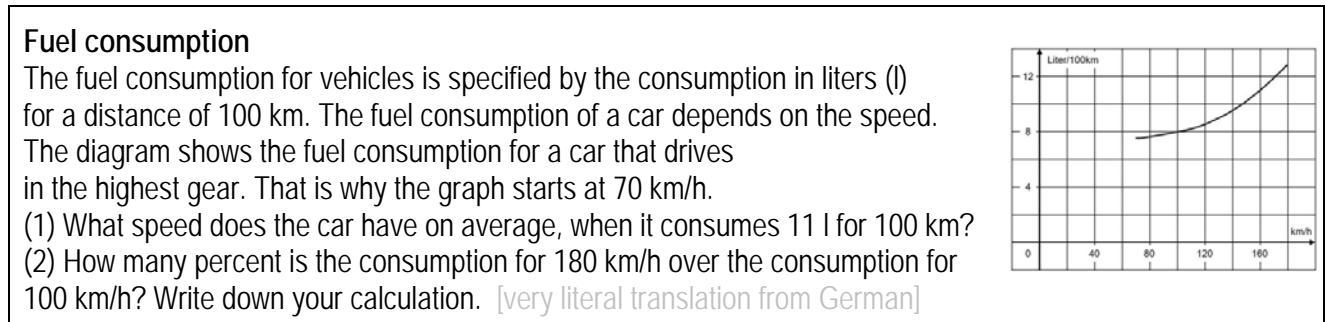


Figure 1. Exemplary test item from the central exam NRW 2012

Three mathematical relations have to be reconstructed: consumption for 180 and consumption for 100 as prompts to find the functional values  $f(180)$  and  $f(100)$  in the graph, and the percentage of growth from  $f(100)$  to  $f(180)$ , i.e.  $(f(180) - f(100))/f(100)$ . These different relations are linguistically encoded in three nested prepositional attributes “for 180 km/h”, “for 100 km/h” “how many percent ... over”. Before finding an adequate mathematization, students have to decode these linguistic constructions. Although the technical demands are not high for grade 10 students, only 12 % of all 1495 students succeeded to decode and solve the item, more specifically 6 % of the group with low academic language proficiency and three times more of those with high academic language proficiency (19.2 %).

The interviews show that even students with high language proficiency first struggle with cracking these connections, whereas students with language disadvantages tend to capitulate and consider only partial information on selected relations, for example only assigning functional values or only calculating percentages. It seems to be this focus on partial information instead of the central relations that hinders them to overcome the obstacle of cracking connections.

In the linguistic item analysis of all items, the relevance of linguistic means for expressing connections was also reflected quantitatively by a significantly higher frequency of prepositions in the mathematics tests (10 % of all words were prepositions) compared to the German language tests (6-7 % of all words). It thus appears to be a specificity of mathematical texts.

*On the text level*, cracking connections appear especially when students have to find relevant information for the questions in other parts of the task (in text, diagram, table,...). Students with language disadvantages seem to have greater difficulties in keeping the track of cohesive connections when signified by linguistic means from the

academic register. With respect to the text level, test fairness could be increased by avoiding too implicit cohesive connections (cf. Gürsoy et al., 2013, for more details)

## Conclusion and Consequences

Not the family background, but the language proficiency matters most for achievement in the investigated literacy-based high stakes test. With respect to the linguistic complexity of the items, this result is not surprising. It is nevertheless important as it serves as motor for further research on details of language demands. Already, the item analysis and first interviews helped to explain these quantitative patterns by reconstructing “cracking connections” as a language demand which seems to be typical for school mathematics. We currently continue the interview analysis and started a DIF-analysis for finding further crucial language demands. Although the study on research question *Q2* is still ongoing, the findings on research question *Q1* call for consequences in three different areas (see below).

*Consequences for test construction.* It is a matter of test fairness to consider each item of a high stakes test or large-scale assessment if it comprises unnecessary language biases and threatens the construct validity (Abedi, 2006; Wolf & Leon, 2009; Martiniello, 2009; Brown, 2005). Sometimes, already small changes in wording can change the characteristic of items, for example by making cohesive connections explicit. However, not all language disadvantages can and should be avoided by test construction. First, some items are difficult for learners with language disadvantages without being linguistically complex, instead, they can contain conceptual demands which the learners do not meet because of longer lasting limits in the acquisition process before the test (Prediger & Wessel, 2013). Second, unnecessary biases must be distinguished from necessary demands of conceptual and linguistic proficiency to which students should get access for the overarching goal of reaching mathematical literacy. This especially applies to complex expressions for mathematical relations that come with items that assess conceptual understanding.

*Consequences for research.* The relevance of language factors has been proven in many international studies (Abedi, 2006; Secada, 1992) and could here be replicated for the German language context. This finding implies the necessity to include language factors consequently into large-scale assessments for grasping social disparities. Of course, the acquisition of language proficiency highly depends on socially determined learning opportunities. In this sense, Cook-Gumperz (1973) is right to emphasise literacy to be a “socially constructed phenomenon, (and) not simply the ability to read and write” (p. 1) and family background does definitely matter, even if mediated by academic language proficiency. But whereas the identification of SES or immigrant status as relevant factors should have important impacts on the level of *policies* for equity, the identification of language disadvantages of a student can lead to substantial consequences in the *classrooms* and are therefore also highly important for fostering equity.

*Consequences for classrooms.* Accepting that language proficiency is also socially determined does not imply that language should be eliminated from mathematics

classrooms, on the contrary: Especially a literacy-based curriculum (as in North Rhine Westphalia, cf. Barzel, Hußmann, & Leuders, 2004) obliges curriculum designers and teachers to consider more consequently how to prepare *all* students (also those with language disadvantages) for the necessary language demands in literacy-based problems and tests. If social disadvantages are linguistically mediated, this finding offers a good access point to enhance equity by fostering students' academic language proficiency. We therefore need enormous efforts of designing language-sensitive teaching strategies and materials (Thürmann, Vollmer, & Pieper, 2010), especially with a focus on expressing connections (Prediger & Wessel, 2013).

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# INFORMAL INFERENTIAL REASONING USING A MODELLING APPROACH WITHIN A COMPUTER-BASED SIMULATION

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*The article investigates how 14- to 15- year-olds build informal conceptions of inferential statistics as they engage in a modelling process and build their own computer simulations with dynamic statistical software. This study proposes four primary phases of informal inferential reasoning for the students in the statistical modelling and simulation process. Findings show shifts in the conceptual structures across the four phases and point to the potential of all of these phases for fostering the development of students' robust knowledge of the logic of inference when using computer based simulations to model and investigate statistical questions.*

## INTRODUCTION

Statistics is becoming increasingly important to all levels of citizenship, with an abundance of data available to inform decision-making. The biggest leaps forward in the next several decades- in business, and society at large- will come from insights gained through understanding data. The *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 2000; Australian Curriculum and Reporting Authority [ACARA], 2011) advocate the increasing importance of student competence in using, analysing, and interpreting data. Other scholars stress the need for statistical reasoning and sense-making in personal judgments and decisions (Franklin et al., 2005). The importance of statistical education coupled with a changing focus in statistics pedagogy--from centering on computation (i.e., the normal distribution and procedures, formulaic hypothesis tests), to the core logic of inference (i.e., chance models, and determining statistical unusualness; Cobb, 2007)--has led to some reconceptualization of teaching statistics. Inspired by Cobb (2007), new curricula have emerged the last two years. Such curricula focus predominantly on using ideas of chance and models, along with computer simulations (i.e., Chance Agents for Teaching and Learning Statistics [CATALST], Garfield et al., 2012) and randomisation-based techniques, to make and understand statistical inferences. CATALST immerses students in the simulation-based approach to inference that requires students to create a model with respect to a specific context, repeatedly simulate data from the model, and then use the resulting distribution of a particular computed statistic to draw statistical inferences.

This paper presents data from a study in Australian schools, focusing on how Grade 9 students develop informal conceptions of inferential statistics as they engage in modelling using TinkerPlots2 computer-based simulations. In this study, the students were introduced to randomisation implicitly through a simulation-based approach, and chance variation as an idea was assessed in an intuitive fashion.



**THEORETICAL FRAMEWORK**

One basic purpose of statistical investigation is to make inferences about unknown populations using samples of data. Research on students' informal inferential reasoning shows many difficulties in understanding and using statistical inference; one of these is the problem of connecting the available evidence with the question under investigation to draw inferences (Pfannkuch, 2005). Other learning difficulties are related to building a schema of interrelated statistical concepts, such as distribution, sampling variability, and representativeness (Saldanha & Thompson, 2003). Many statistics educators advocate that inference should be taught entirely from an empirical perspective through simulation methods that enable students to better think statistically (Cobb, 2007). The empirical study of sampling variability typically focuses on drawing repeated samples from a population, forming a distribution of sample statistics (such as sample means) from those repeated samples, and comparing the observed sample statistic to the empirical sampling distribution. This resampling approach elucidates how probability provides a theoretical structure for statistical inference, as it is based on the notion of considering what would happen when comparing the observed sample statistic to the distribution of sample statistics created under a chance model. Pfannkuch argues (2005) that "the resampling approach to teaching would appear to be the most promising direction, as it could enable students to link probability intuitively with statistical inference" (p. 290). For such an approach, "randomization-based inference makes a direct connection between data production and the logic of inference that deserves to be at the core of introductory statistics" (Cobb, 2007, p.1). According to Cobb the core logic of inference entails: (a) setting up a model that produces randomized data; (b) generate data from the model for a single trial and assess whether the outcomes are reasonable. Specify the summary measure to be collected for each trial and generate data for many trials collecting the summary measure each time; (c) Reject any model that does not accurately represent the phenomenon it was intended to model. Moreover, Cobb suggests that digital technologies provide a natural way to introduce students to computer-intensive simulation-based methods, allowing students to easily create simulation models, and then to interpret the observed outcomes.

Nonetheless, Cobb's conjecture has not been empirically tested. Efforts should be made to fully assess the pedagogical value of computer-intensive and simulation-based methods when teaching the logic of inference, and to investigate the linking of ideas of variation and probability. Such novel methods inevitably bring with them new challenges in how students learn and give rise to research questions about the conceptual development of students who engage in constructing such chance models. It is important to understand how students construct models, run simulations and interpret outcomes, and reason about uncertainty in the context of making informal statistical inferences, as well as understanding the challenges students might encounter in such a pedagogical approach.

Focusing on Cobb's (2007) logic of inference, the researcher conjectures that there are four primary phases of inferential reasoning for students following the statistical modelling and simulation process (table 1), and the students need to understand all of

these phases and the role each of these phases in order to develop robust knowledge of the logic of inference when using computer based simulations to model and investigate statistical questions.

Phases	Description
Phase 1	Specify a model that will generate data to simulate the experiment. Use software interface that relies on signal, variation, and spread of data to create the model.
Phase 2	Use the model to generate a single trial of the experiment, investigate the outcomes from a single trial; Construct an appropriate representation of the outcomes from the single trial; interpret the results of a single trial; Consider possible outliers and other interesting individual cases.
Phase 3	Use the model to generate simulated data for many trials, each time interpreting the results. Examine the empirically observed distributions of the resulting outcomes. Use the observed distribution to assess particular outcomes. Compare the behaviour of the model to observed data; evaluate the model.
Phase 4	Coordinate the actions of phases 1-3 to change the model, interpret the results, draw inferences based on the data at hand.

Table 1: Phases of informal inferential reasoning in the modelling and simulation process.

The research reported here illustrates the conceptual structures that students build about informal inferential reasoning across the four phases as they engage in modelling, using *TinkerPlots2* computer-based simulations.

## METHODOLOGY

Thirty students in Grade 9, ranging from 14 to 15 years in age, from a rural secondary school in New South Wales, Australia, formed the population of this study. The researcher spent 2 sessions (40-45 minutes each) introducing the class teacher and the students to *TinkerPlots2* during regular mathematics lessons. All students were familiarised with the *TinkerPlots2* software, explicitly focusing on learning skills related to *TinkerPlots2*. In the first session, all students watched instructional movies that show how to use *TinkerPlots2* features to build a simulation. In the second session, all students were familiarised with *Tinkerplots2* through a number of introductory activities related to building a data factory that simulates real phenomena. The students also ran a simulation and observed the generation of data, and the distributions of the various data.

Ten average-ability students volunteered to spend a third session, outside of class time, to engage in the task reported in this study. In this session, students were asked to use the tools of *TinkerPlots2* and the “Data Factory” features (see Figure 1) to generate a simulation to investigate the impact of hours spent using Facebook on the school

performance of a group of students. The students created a number of "virtual students," each defined by several variables (gender, hours spent on Facebook per week, school performance) whose values were determined by *Tinkerplots2* using pre-defined probability distributions. After constructing their model, students were asked to run the simulation and interpret the outcomes.

Each session lasted approximately 45-60 minutes and each pair of students worked directly with the researcher. The researcher interacted continuously with the students in order to observe the reasoning they used to explain the data and simulations. The data collected included audio recordings of each pair's voices and video recordings of the screen output on the computer activity using Camtasia software.

In tune with Cobb's (2007) call for teaching the logic of inference and the related theoretical framework to investigate how students might construct models, run simulations, and interpret outcomes, a *qualitative* data analysis was conducted. First, the audio recordings were transcribed and screenshots were incorporated as necessary to make sense of the transcription. The data were then analysed using progressive focusing (Robson, 1993), a process by which the author began with a wide field of focus and gradually narrowed the field by identifying key foci for ensuing study.

This article focuses on one pair of students, George (G) and Rafael (Ra). Although the same insights as reported below were evident in the analysis of the sessions of other pairs of students, George and Rafael provided (in my view) the clearest illustration of how students construct models, run simulations, interpret outcomes, reason when making informal statistical inferences.

## RESULTS

The session began with George and Rafael creating a model to generate data to reasonably simulate the Facebook problem (Figure 1). The students began their attempt by perceiving a holistic entity, such as a student, as consisting of a cluster of pieces of data having attributes such as gender, hours spent on Facebook, impact of hours spent on FB on students' performance, and school grades (Figure 1). When the students drew curves in the *Tinkerplots2* interface to define the probability density functions that would be used to generate the simulation data, they appeared to exhibit Phase 1 reasoning. They talked about the curves with respect to most common values (what we could consider the "signal"), and variation of these values (what we could consider the "noise") (Prodromou, 2012). After the boys had generated 1000 virtual students, they looked at the distributions created in the sampler and the distributions of generated data.

- 1 G: Okay, so these areas here (pointing to the circled areas of the graph on the right of Figure 2). I reckon, they're just spots where, they could just be ugh, smaller populations just happening to do that. It's just, it's hard to explain.
- 2 Re (Researcher): What do you mean?
- 3 G: See, on the graph that we've put here ... there's no spike here (pointing to the graph on the left of figure 2) and yet here there are these spikes

(pointing to the circled areas of the graph on the top right Figure 2) and um, I just think that they're just people who happened to go on for 6 hours.

4 Ra: Yeah... They probably had more free time... or cold weather

5 G: And, there are more people doing that say 6 hours but on the graph (left Figure 3) it doesn't show that.

The students' attention was attracted by slices of prominent features of the observed distribution of hours spent per week by males (see bottom left of Figure 2), such as higher areas of accumulated data compared to the curve that defined a probability density function of the hours spent on FB by males (see left in Figure 2). George seemed to refer to the variation caused by small samples when he referred to "the smaller populations" (line 1). He seemed to recognize that the small sample size introduced variation, but he was unable to explain such vagaries of variation. They attempted to attribute some common cause factors to normal day-to-day variation, for example, people who happened to stay on for 6 hours (line 3), who have free time (line 4), or because of cold weather (line 4).

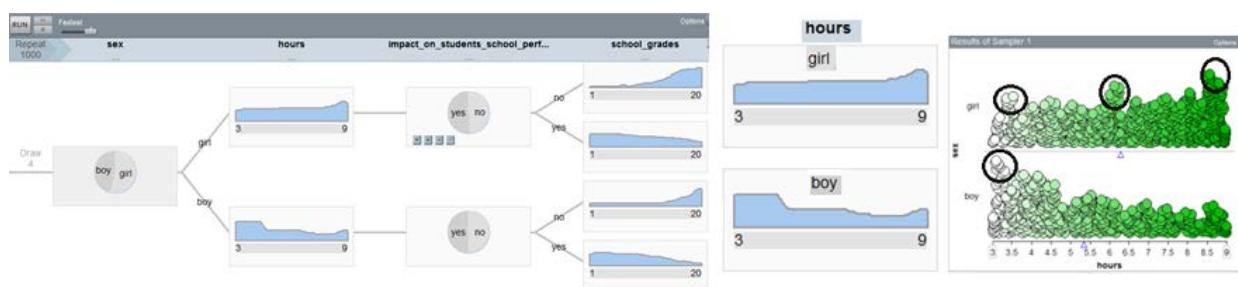


Figure 1 (left). Data Factory that simulates the Facebook Task. Figure 2 (right).

Distributions created in the sampler (left) and distributions of generated data (right).

When the students compared the outcome data distribution of the hours spent per week by students (Figure 2) on FB to the distribution they created in the sampler, they began moving to phase 2, in which they thought about the data in terms of possible outliers, clusters of data and interesting individual cases.

7 Re: What do these graphs show? Which graph shows the actual results?

8 G: These are the actual results, up here ... and it's been put into that because that's how what we think.

9 Ra: That's the results that just come out.

10 Re: Do you think the results show how you think?

11 G+Ra: More or less.

12 G: They could not always be the same. Maybe, sometimes they will be the same.

The above quotes show that Students' reasoning exhibited Phase 2. George appeared to distinguish the distribution of the actual results from the model they created when they drew the density function. He very eloquently articulated that the chance model as presented by the density function as drawn in the sampler shows the "model" they constructed in their mind. He seemed to have a vague sense of the actual results being generated by this model when he mentioned that "it's been put into that". Both boys

articulated colloquial notions of chance when they used expressions such as “more or less”, and “maybe” considering the appropriate phrasing of the statistical questions, that can be attributed to their preliminary sense of chance or variability between the outcomes of different outcomes.

When they observed the distribution of generated data (see left of Figure 3):

- 13 G: It doesn't really seem to have an effect on how much you use it (referring to FB). The grades are still more or less the same.
- 14 Ra: Yet there's no one there and there are gaps.
- 15 G: There are gaps, like here, and there's gaps here...Okay, there's a gap there. Then there's a big bulk here. The bulks, but there's still individual like here. And there's an individual... (while speaking he points to places on the graph, see Figure 3)
- 16 Ra: And they just stand out, there's nothing around them. And there's big clusters of people, where there's circles overlapping other circles, and like here (pointing to the line 0-2.999) and here, everywhere.
- 17 G: It cannot be similar to this (pointing to the graph of school grades in Figure 1). It is confusing.
- 18 Ra: And they just stand out, there's nothing around them. And there's big clusters of people, where there's circles overlapping other circles, and like here and here, everywhere.

The above quotes show George and Raphael using Phase 3 reasoning after they simulated

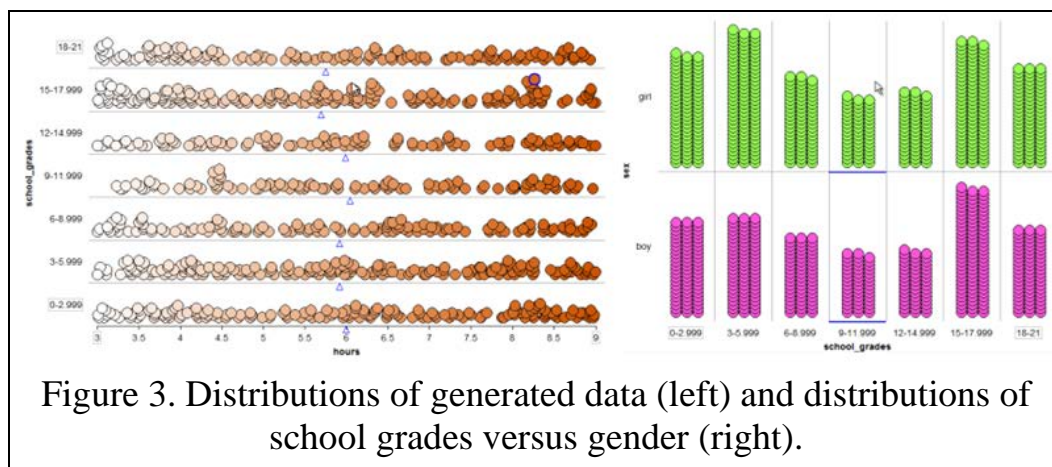


Figure 3. Distributions of generated data (left) and distributions of school grades versus gender (right).

data for many trials At first sight, it seemed puzzling that the boys did not ‘see’ the impact of hours spent on students’ school grade.

However, their extended discussion showed how their attention was, at this point focused on interesting individual cases (lines 16), areas of the graphs that there were no data values and areas where there were big clusters of data values. The boys seemed unable to explain the relation between school’s grades versus hours spent on FB. When George attempted to compare the observed graphs to the density function drawn in the sampler, he found the information displayed in the graph confusing (line 17). He could not even observe that when the model was run a few times, there was stability in the peaks but there was some variation observed in the general details of the shape. When the boys observed the distribution of school grades for each gender:

- 19 G: Girls either averaged like really well, or not very well but boys are sort a just the same.
- 20 Ra: But, there also seems to be a lot of boys that averaged very well.
- 21 G: Fairly well, then there's a couple that do very well. But there are less numbers than this, 9 to 14, 9 to 15 range of um school grades. And that seems to be the same with girls as well
- 22 Re: Do you think that, these are the results what you expected?
- 23 Ra: I reckon girls should've had a bigger bulge just up around there (pointing to the range of 15-20 on the graph of figure 4).

Raphael was expecting to observe an increase in the school grades of the female population (line 23). The boys then suggested to change the model that simulates the empirical grades for girls, increasing the grades for girls. They looked at their previous models from phases 1-3 and also suggested redrawing the density function that generates the boys' school grades:

- 24 G: Make it go down (referring to the curve of the density function of boys' school grades) Make it go down, not as steeply like that.

Students' reasoning is at Phase 4 reasoning, which incorporates revisiting their previous actions and co-ordinating associated actions and generalisations to make adjustments to the models redrawing the density functions in the sampler that generate the empirical data.

## **DISCUSSION**

The results suggest that both the modelling process and the simulation process appear to be appropriate resources for introducing beginning inference to middle school students. The modelling and simulation process challenged students to construct models, interpret empirically observed distributions, compare the behaviour of the models to empirically observed data and evaluate the models used to generate data. Computer-intensive modeling and simulation-based methods reinforce each other, explicitly in terms of the elements of understanding the constructing models (samplers) appropriately to model the statistical problem, generation of simulated data, examination of the empirically observed distribution of the observed outcomes, interpretation of the results, and evaluation of the model used to generate empirical data. Thus these data, though only suggestive, seem to indicate a path by which Cobb's conjecture regarding the value of digital technologies in statistics education can be elaborated. This preliminary research suggests four phases that trace the movement of students' informal inferential reasoning in the modelling and simulation process. The phases need to be refined before trying to apply them to a pedagogical practice.

In terms of assessment, combining the four phases of informal inferential reasoning with a developmental model from cognitive psychology might be promising in documenting students' continuous progress of reasoning. As such, I assume it would shed some light on the structural complexity of making informal statistical inferences.

Given the observation of students' initial difficulties in creating an appropriate model that would generate empirical data, it would appear that the use of a dynamic software

such as *Tinkerplots2* may contribute to developing intuitions about what might be considered “appropriate co-ordination of signal and noise” or “meaningful approximation of real or simulated phenomena” (Prodromou, 2012).

The four phases describe the conceptual structures that students build about informal inferential reasoning and provide a base to trace the cognitive processes involved in inferential reasoning. Future research will investigate more systematically students’ phases of informal inferential reasoning in the modelling and simulation process.

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# STUDYING TRAINEE TEACHER EDUCATORS' DOCUMENTATIONAL WORK IN TECHNOLOGY ENHANCED MATHEMATICS

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*We address the didactical design and corresponding material developed by one trainee teacher educator in the context of an in-service program concerning the use of digital tools in the classroom of mathematics. We analyse the trainee's documentational work carried out for giving lessons to colleagues as part of his practicum. The results indicate that fieldwork activities provided a source of motivations for the development of documents.*

## INTRODUCTION

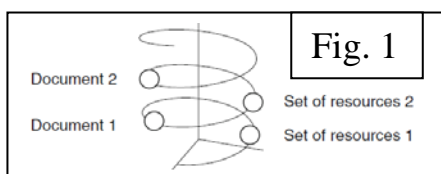
In this paper we study the didactical design and corresponding material developed by one trainee teacher educator in teaching mathematics with the use of digital tools. The trainee drew upon existing resources as he started to teach in teacher education classrooms. The study took place in the context of an in-service program adopting reform-oriented perspectives to train teacher educators into the use of digital tools in the classroom of mathematics. The aim of the program was to provide the participants with methods, knowledge and experience in in-service teacher education and to educate them in the pedagogical uses of digital technologies for the teaching and learning of mathematics. One of the reform aspects of the approach for teacher education (see Kynigos & Kalogeria, 2012) concerned teacher educators' and teachers' active engagement in creating their own didactical design and material as coherent part of their professional development. Taking into account that teacher educators have very few resources to draw on directly (Zaslavsky, 2008), it was critical for the trainees to get used to developing their own material. In this course, the trainees were engaged in designing and generating resources in the form of microworlds and scenarios (i.e. structured activity plans addressing critical aspects of a pedagogically sound use of technology for the teaching and learning of mathematics). A structure [1] for addressing these aspects was developed by Educational Technology Lab (<http://etl.ppp.uoa.gr>), which participated in the design of the course and the corresponding material. The training program took place in specialised University Centres (UC) for 350 hours. The participants were experienced qualified mathematics teachers but the majority of them had no previous experience in the pedagogical use of digital tools. The plan was to employ the newly trained teacher educators in wide-scale 96h courses to groups of teachers in specific Centres for Teacher Education Support (CTES). The trainees were given material by the trainers after each lesson and an official document (called 'Notes for teaching in CTES') containing theory and a set of twelve generic scenarios as a basis for organizing their subsequent teaching in CTES. During the course, the trainee educators gained significant experience with the



pedagogical use of five categories of digital media: Computer Algebra Systems, Dynamic Geometry Systems, Programmable software, Simulations and Data Handling tools. By the end of the course the trainees had to have developed one scenario for each of these categories as well as scenarios for the practicum. Practicum was part of UC official structure provided shortly before trainees complete the course, so as to engage them in field activities and give them the experience of implementing their design in real classroom conditions and reflecting on it. Practicum was divided in two parts: teaching in school and observation - teaching in CTES. Here in focus is the second part consisted of (a) observation of other teacher educators' teaching in CTES, (b) design of a 3-hour lesson for teachers in CTES under the supervision of a mentor, (c) implementation in the classroom, (d) presentation of design and implementation in whole class special reflective sessions, (e) activity report by the trainees.

## THEORETICAL FRAMEWORK

We adopt the documentational approach of didactics according to which the teacher's work is developed *with* and *on* resources in a dialectic process where *design* and *enactment* are intertwined (Gueudet & Trouche, 2009). An implication of this approach is that curriculum material is not conceived as a static body of resources that guides instruction but rather as a set of objects amenable to changes and modifications depending on the teacher's didactical design. Gueudet and Trouche (2009) use the term *resources* to describe a variety of artifacts such as a textbook, a piece of software, a student's sheet, discussions with colleagues etc. Through a class of professional situations and teachers' experience, the existing resources are modified as documents according to the formula: *Document = Resources + Schemes of Utilization*.



Creation of documents is considered as unfolding through a dual process of instrumentation (the resources act on the teachers) and instrumentalization (teachers act upon these resources as they appropriate them). This process gives birth to a new entity, i.e. a

document, which can be further transformed to a new document over time. This process is represented in Fig. 1 by a helix (Gueudet & Trouche, 2009). Describing the nature of the relation between resources and documents, Gueudet and Trouche (2009, p. 206) stress that “documentational genesis must not be considered as a transformation with a set of resources as input and a document as output. It is an ongoing process ...”. However, Kieran (2009) wonders whether or not documentational genesis could be viewed as a set of transformations, albeit interrelated. According to her, a fine-grained zoom-in on a much smaller part of the helix (a point or a short arc) might also contribute to a deeper understanding of when and how instrumentation and instrumentalisation take place and how these processes influence teacher's interaction with resources. In our study a focus on the particular arc of the helix corresponding to the practicum would allow us to gain insight on the processes by which the trainees transformed existing resources to documents for their own teaching in CTES. Another theoretical construct that informs our perspective in this study is *double instrumental genesis (double IG)* (Abboud-Blanchard & Lagrange, 2006). It consists of the *personal*

*instrumental genesis* (leading to the appropriation of a tool as an instrument for mathematical work at the personal level) and the *professional instrumental genesis* (leading to the construction and appropriation of the previous instrument into a didactical instrument for mathematics teaching at the professional level). Although double IG is at the core of teacher education in technology enhanced mathematics, there is very little attention given by research studies at the level of teacher educators. Therefore, we incorporated in our study a focus on the components that may influence the trainees' transition from the personal IG to the professional IG.

The general aim of this study is to shed light on the trainee teacher educators' documentational work as they began to practice teacher education themselves. We particularly explore what are the components involved in the arc connecting existing resources and new documents in these trainees' documentational work. Our focus is on how they act upon existing resources in the process of transforming them into documents for their teaching in real teacher education classrooms (instrumentalization) and how this process shaped their own activity (instrumentation). In this process we were also interested in exploring the trainees' transition from personal to professional IG.

## METHOD

In order to highlight the transition from existing resources to new documents we chose Tom as an exemplary case because his teaching in CTES was based on one of the twelve official scenarios. This allowed us to view comparatively the documents he created to the existing official resources. Our role as academic trainers and mentors in the practicum provided a framework for designing our intervention which was based primarily on the making of links between knowledge-based research and practice in mathematics teacher education. In resonance with Gueudet and Trouche's (2011) principles regarding methodological aspects of research on documentational genesis we chose to (a) analyze Tom's work in time periods in and out-of-class (reflexive investigation principle), (b) address Tom's decisions taken in order to formulate his design through its use (design-in-use principle), and (c) consider his work embedded in and influenced by different collectives (i.e. peers in UC, teachers in CTES) (collective principle). We used data from different periods of time: (a) excerpts from reflective sessions which took place before Tom's design, (b) the official material and its transformations by Tom, as well as the arguments with which he documented his options, and presented in a whole class reflective session (c) Tom's activity report. Coding of discussions of the reflective sessions through constant comparative method (Strauss & Corbin, 1998) lead to a categorization in 'themes' i.e. discussions around a particular issue not necessarily in chronological order. We used parts of these transcripts in conjunction with Tom's activity report as complementary sources for analyzing Tom's design decisions in relation to the existing resources.

## ANALYSIS

Before analysing Tom's documentational work we refer to the discussions that took place during the reflective sessions so as to give a flavour of the context within which

Tom's work evolved. Data analysis revealed that the most frequent discussion theme mentioned in these sessions concerned the dual role of teachers in CTES: "teacher as student" and "teacher of students". In one of the reflective sessions we took the challenge to connect this issue with current mathematics education research through the use of relevant theoretical constructs such as double IG. This idea seemed to have provided a basis for the trainees to rethink their classroom experiences from CTES until that time and interpret them through this perspective as shown in the following excerpt:

TrainerA: What did you observe in the lesson in CTES taking into account double IG?

TraineeA: I think that the teachers acted mainly as students who were trying to learn the tools rather than as teachers who would teach the scenario to their students.

TraineeB: Up to a point you need to act as a student, to press keys, to understand the tools and after that to discuss with your classmates and the teacher educator what you would teach, the teaching sequence etc. There are two stages. You may stay only in the first, but this will happen only in the introductory lessons.

TrainerB: Some scenarios are designed intentionally for the first stage because they reveal how mathematical ideas come to the fore through the use of the software. And it is important for the trainee to acquire such an experience so as to be able to design tasks and activities that facilitate the emergence of these ideas in the surface. I remember in my first lessons in this UC class your intense concentration on the software and nothing else. Now I see that you have started to pose pedagogic and didactic questions.

At the core of the above discussion is the difficulty inherent in teacher education courses that stems from the two roles of trainee teachers and how technology is integrated in each one of them. There are two aspects here: the first concerns the learning of technology itself and its potential for mathematics at the personal level; the second concerns the use of technology as a didactical tool. Double IG seemed to operate as a tool that helped the trainees to address the complexity inherent in teaching mathematics with digital tools as a teacher educator. Trainee B introduces the time parameter regarding the trainees' transition to the didactic level. During their early UC lessons trainees were mainly 'students' concentrated on the learning of technology while at the time of the discussion they had started "to pose pedagogic and didactic questions" (in the words of Trainer B). It seems that trainees' participation in the practicum plays a role in their transition from the personal IG to the professional IG.

### **Tom's documental work**

Tom's lesson in CTES was about the design of scenarios and worksheets. During his observation of lessons in CTES he had noticed that most of the teachers had difficulties in understanding the connection between scenarios and worksheets. He prepared for the teachers a worksheet corresponding to a scenario related to the study of sinusoidal function with Function Probe (FP) [2]. The specific scenario was one of the twelve official scenarios provided for the course. The indicative design of this scenario suggested the following teaching sequence in four phases. Phase 1: Highlighting the

importance of converting degrees to radians for studying  $f(x)=\sin x$ . Phase 2: Creating table of values in  $[-\pi, \pi]$  with step  $\pi/8$ , sending ordered pairs of  $x$  and  $y$  to Graph (first idea of sinusoidal curve), increasing the number of tabular values (development of conjectures), study of properties. Phase 3: Generating and filling columns in the Table with values of  $y=2\sin x$  and  $y=3\sin x$ , comparison to the respective values of  $y=\sin x$ , construction of graphs and study of the corresponding functions, conjectures for the graph of  $y=p\sin x$  and the role of  $p$ . Phase 4: Performing mouse-driven transformation (vertical stretching) of  $y=\sin x$  with the stretch tool, connection between graphs and algebraic representations and extraction of conclusions for the role of vertical stretch magnitude (operator) in the graph with the help of the history window, verification of conclusions through the use of Calculator.

Tom designed an original document-worksheet for the teachers in CTES aiming to teach the sinusoidal function and through this to show one way to connect scenario and worksheets. Besides, he integrated instructions within this artifact so as to facilitate the use of FP. During his presentation in one of the reflective sessions, he explained the decisions underlying his design providing also indications of the instrumentation process underlying his activity: “In my view, a complete scenario needs to have questions or issues that constitute objects for negotiation and activities designed with particular learning aims. Thus, I created a worksheet with two columns. In the left one I inserted the questions of the worksheet. In the second one the rationale, that is to say what I want to achieve with these questions. I marked with grey the questions that could be ignored. During my observation of other teacher educators’ lessons in CTES I also noticed that most of the teachers did not know the software well. I resolved this by inserting footnotes at the bottom of each page.” (see the part of Tom’s worksheet from the Phase 3 of the teaching sequence mentioned above in Table 1).

The structure of this document integrates the issues mentioned by Tom in the previous excerpt and also signifies a process of instrumenalisation through which a document is becoming a didactical tool for CTES. From a didactic/teaching point of view Tom takes a different perspective from the curriculum as regards the use of the stretch tool. The indicated analysis of the teaching sequence provided in the official version of the scenario proposes the use of the stretch tool by the students for manipulating vertically the height of the curve  $y = \sin x$  at random and through this to identify the role of  $p$  through the history window. Tom proposed the design of the three graphs in the same coordination system. Then he suggested that the students should stretch the graph of  $y = \sin x$  until coinciding with the graph of  $y = 2\sin x$ ,  $y = 3\sin x$  and through this to identify the role of  $p$  in the transformations of  $y=\sin x$  kinesthetically. This approach reveals Tom’s conception of the construction of mathematical knowledge according to which dynamic manipulation of mathematics objects can be a precursor mediating the transition to more formal understandings. This was also evident in the next excerpt taken from his presentation in the corresponding reflective session. Tom refers to an episode that took place during his teaching in CTES: “During the lesson we had to study functions in the form  $y=\sin(x+a)$ . One of the teachers said: “We can firstly construct the graph of  $y=\sin(x+\pi/2)$  and then study its relation to  $y=\sin x$ ”. I answered:

“Then there is no reason for using the stretch tool ... I believe that in this case this tool can help students understand the exact relation”. Again Tom expresses his conception regarding the integration of the stretch tool in the teaching sequence. He attaches added value to its use for mediating the targeted relationships. The other teacher’s proposal does not leave space for approaching  $y=\sin(x+\pi/2)$  through dynamic transformation of  $y=\sin x$  which is at the core of Tom’s didactical design.

Questions	Rationale
In the Table window fill in two more columns $z=2\sin x$ and $w=3\sin x$ using different colors.	Study of function $y=p*\sin x$ . What is the role of $p$ . Multiple representations of the relation between $y=\sin x$ and $y=2\sin x$ , $y=3\sin x$ (Table: visually, Graph: visually and kinesthetically).
Send points $(x, z)$ and $(x, w)$ to the Graph window.	
What do you observe as regards periodicity, symmetry, monotonicity and extremes.	
Design the graphs of the above functions together with the graph of $y=\sin x$ in the same coordination system.	
Stretch and contract the graph of $y=\sin x$ through the use of the stretch tool (Note 1) so as to coincide with the new points of $y=2\sin x$ and $y=3\sin x$ . (The magnitudes of the transformation have to be 2 and 3 respectively). What do you observe?	
Experiment with other magnitudes of transformation (i.e. vertical stretch/contract) through the use of the stretch tool. Check your conclusions through the use of the Calculator or the Table.	

Note 1: Before answering this question you have activate the choice “Show transformations”.

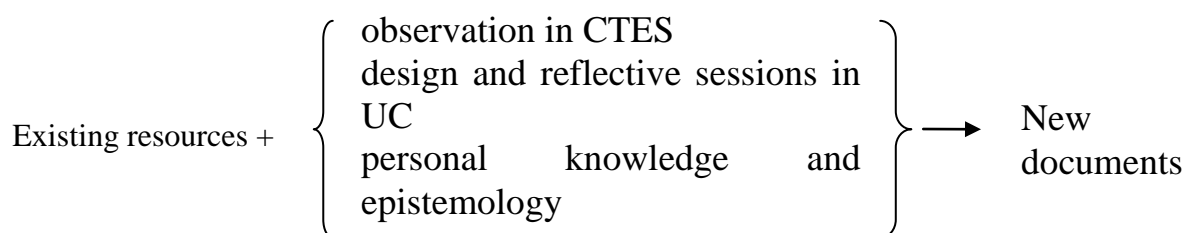
Table 1: Extract from Tom’s worksheet for teachers.

The next step of Tom’s didactical design included the exploitation of the above mentioned document-worksheet for the development of a scenario. The right column of this document brought to the fore the need for an abstract structure depicting the main issues involved in scenario design as well as the main questions that a teacher has to answer before developing a scenario. Thus he constructed a new document-‘structure-for-scenario-design’, which reminds Concept Maps, so as to provide a figural representation of those issues/questions and potential design trajectories. Due to space availability we provide here a brief description of a small part of this document. A cell including the phrase ‘an underlying learning theory’ (i.e. of a scenario) was connected with three other cells entitled (a) ‘potential design’ (connected to five other cells entitled ‘what’, ‘why’, ‘where’, ‘how’, ‘how long’), (b) ‘expected learning outcomes’ and (c) ‘reflection based on the implementation’. Explaining his choices as regards the specific document- ‘structure-for-scenario-design’ in his personal activity report, Tom mentions: “The scenario structure depicted in the map does not follow the official scenario template but it includes all the issues, processes and questions that a teacher has to consider when designing a scenario. At the same time it is closer to everyday teaching practices. It is important for the teachers to understand first what do they have to do and if they grasp it then they can move towards forming their scenario according to the official structure”. Here Tom highlights that the rationale behind the creation of this specific document stems from his need to bring ‘closer’ teachers’ everyday practices to the theoretical aspects of

mathematics teaching addressed by scenarios. Working at the professional level in terms of double IG, Tom develops new didactic tools with the aim to bridge the distance between the language of official resources and the language of practitioners. At the same time he adopts a critical stance towards the official resources which allows him to develop the corresponding schemes of utilization for students and teachers according to his own perspective.

## CONCLUSIONS

Three factors seem to have influenced Tom's development of the two new documents: (1) Teachers' difficulties in developing their own teaching material, (2) Teachers' difficulties in learning the affordances of the software tools and integrating them in activities with added educational value (e.g. microworlds, scenarios, worksheets), (3) Tom's knowledge, pedagogical conceptions and experiences regarding the everyday practice of teachers. The first two factors are directly linked to the observation of other teacher educators' teaching in CTES. During these lessons Tom had the opportunity to detect the above difficulties and to adopt a critical stance towards existing resources. At the same time reflective sessions seemed to have provided trainees with theoretical constructs (i.e. double IG) to reconsider teachers' dual role in CTES. Thus the above processes involved in observation and reflective sessions can be considered as part of the instrumentation aspect of Tom's documentational work. Tom constructed the two documents as didactical tools for addressing those issues and incorporated his own conceptions in his design as regards tool use for engaging students in meaningful mathematical activity. The construction of these documents signifies Tom's passage to the professional level in terms of double IG. Based on these findings we conclude that the components of the arc linking existing resources and new documents at the level of teacher educators are the activities around the practicum (observation, reflective sessions) as well as trainees' personal knowledge and epistemologies. The diagram below represents the evolution of Tom's documentational work over time in the helix:



In our course the practicum was designed to take place shortly before the end of it. Our findings indicate that fieldwork activities motivated trainees to generate documents. A further implication for the design of similar teacher educator courses is that fieldwork activities should be exploited as early as possible in the course.

## NOTES

[1] 1. Title, 2. Scenario's identity (author, subject area, topic), 3. Rationale (innovations, added value by the use of technology, students' learning problems addressed), 4. Context of implementation (grade, duration, location, prerequisite knowledge, social orchestration of the

classroom, goals), 5. Phases of implementation (sequence of activities, roles of the participants, anticipated teaching/learning processes), 6. Possible extension, 7. References.

[2] FP is a multi-representational software with three windows: Table, Graph and Calculator. Function graphs can be produced in a number of different ways, e.g. inserting a formula for the function, “receiving” ordered pairs (x, y) from a table (“x” and “y” columns can be generated). Particular icons allow horizontal and vertical transformations of functions (translations, reflections and stretches) made through direct actions on the graph. Stretching is carried out with the stretch tool that allows mouse-driven horizontal and vertical stretching. The corresponding magnitude of the stretches appears in the upper right corner of the Graph. A history window in the Graph allows viewing the formulas of the transformed functions.

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# WHO IS SWITCHING OUT OF CALCULUS AND WHY

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*A substantial percentage of college students who enrol in Calculus I intending to take more Calculus decide at the end of the semester not to continue with Calculus. This represents a huge loss in terms of the need for more students to pursue a major in one of the science, technology, engineering and mathematics (STEM) disciplines. In this presentation we examine the characteristics of STEM intending students in the USA who begin their post secondary studies with Calculus I and either persist or switch out of the Calculus sequence, and hence either remain or leave the STEM pipeline. The data used for this analysis comes from a unique, in depth national survey aimed at identifying characteristics of successful programs in college Calculus.*

## INTRODUCTION

As detailed in the recent report in the USA from the President's Council of Advisors on Science and Technology (PCAST, 2012), there is tremendous need for more students with degrees in science, technology, engineering, and mathematics (STEM). For example, the PCAST report predicts that, over the next decade, approximately 1 million more STEM graduates above and beyond the current level of STEM graduate production will be needed in order to meet the demands of the national workplace. One strategy for meeting this need is to increase the retention of STEM majors. In fact, the PCAST report predicts that simply increasing the retention of STEM majors from 40% to 50% would go a long way to meeting this need.

As reported by Seymour (2006), students leave STEM majors primarily because of poor instruction in their mathematics and science courses, with Calculus often cited as a primary reason. Therefore, in order to develop more successful retention strategies, the field is in need of a deeper understanding of what separates those who continue with Calculus from those who do not.

The purpose of this report is to examine the characteristics of STEM intending students who begin their post secondary studies with Calculus I and either persist or switch out of the Calculus sequence, and hence either remain or leave the STEM pipeline. The data used for this analysis comes from a unique, in depth national survey aimed at identifying characteristics of successful programs in college Calculus. In this report we answer the following three research questions: (1) What is the profile of students who choose not to continue with Calculus? (2) What are the reasons that students give for switching out of Calculus? (3) What characterizes the behavior of switchers and the behavior of their instructors?

## BACKGROUND

Researchers in Higher Education have extensively studied factors related to student retention at the post-secondary level, often focusing on the effects of student



engagement and integration on persistence (e.g., Kuh et al., 2008; Tinto, 1975, 2004). According to Tinto's integration framework (1975), persistence occurs when students are socially and academically integrated in the institution. This integration occurs through a negotiation between the students' incoming social and academic norms and the norms of the department and broader institution. From this perspective, student persistence is viewed as a function of the dynamic relationship between the student and other actors within the institutional environment, including the classroom environment.

Guided in part by this theoretical and empirical work in higher education, this paper reports on a five-year study of Calculus I instruction at colleges and universities in the United States. The first phase was a large-scale national survey of Calculus I instruction at two- and four-year colleges and universities. The survey was restricted to what is known as "mainstream" calculus, the calculus course that is designed to prepare students for the study of engineering or the physical sciences.

The second phase of the study, which is currently underway, consists of case studies examining Calculus I instruction at sixteen colleges and universities identified as having a notable measure of success with their Calculus I program. Success was defined in terms of both the percentage of students who had successfully completed the course and the percentage of students who maintained or increased their interest in continuing the study of mathematics beyond Calculus I, controlling for the varying academic strengths and interests of the entering students at different institutions.

## **METHODS**

The large-scale national survey of mainstream Calculus I instruction was conducted across a stratified random sample of two- and four-year undergraduate colleges and universities during the fall term of 2010. Preparation for the surveys included a literature review leading to a taxonomy of potential dependent and independent variables followed by constructing, pilot testing and refining the survey instruments (Lodico, Spaulding, & Voegtle, 2010; Szafran, 2012).

A total of six on-line surveys were constructed: one for the calculus coordinator; two for the calculus instructors of which one was administered immediately before the start of the course and the other immediately after it ended; and three for the students of which one was administered at the end of the second week of the course, one just before the end of the course, and the last one year later to those students who had volunteered their email addresses. In addition, instructors reported on the distribution of final grades and submitted a copy of the final exam. All surveys were completed online, and no incentives were given for completing the surveys.

The survey was sent to a stratified random sample of mathematics departments following the selection criteria used by Conference Board of the Mathematical Sciences (CBMS) in their 2005 Study (Lutzer et al, 2007). Following the strategy of CBMS, we separated colleges and universities into four types, characterized by the highest mathematics degree that is offered: Associate's degree, Bachelor's degree, Master's degree, and Doctorate. Within each type of institution, we further divided the

strata by the number of enrolled full time equivalent undergraduate students, creating from four to eight substrata. We sampled most heavily at the institutions with the largest enrolments. In all, we selected 521 colleges and universities: 18% of the Associate degree colleges, 13% of the Bachelor's degree colleges, 33% of the Master's degree universities, and 61% of the Doctoral universities. Of these, 222 participated: 64 Associate degree colleges (31% of those asked to participate), 59 Bachelor degree colleges (44%), 26 Master's degree universities (43%), and 73 Doctoral universities (61%).

There were 660 instructors and over 14,000 students who responded to at least one of the surveys. There is complete data (the first five surveys completed and linked with each other) for 3103 students enrolled with 309 instructors at 125 colleges or universities. However, in order to answer our research questions we did not need to restrict ourselves to the completely linked data set. Instead, we needed either a student end of term survey or follow up survey.

## RESULTS

Depending on a student's initial intention to continue with Calculus and whether they switched or persisted with their intention, we used multiple questions across surveys to classify students into four categories: Culminators, Persisters, Switchers, and Converters. Culminators are those students who began and ended the course not intending to take Calculus II. These students typically only need Calculus I for their major. Persisters were those students who initially intended to take more Calculus and did not change this intention. Switchers, on the other hand, were those students that started Calculus I intending to take more Calculus, but then by the end of the term (or one year later) changed their plans and opted not to continue with more Calculus. Finally, Converters were those students who initially did not intend to take more Calculus but by the end of term changed their mind and wanted to continue taking more Calculus. Out of a total of 7260 students for which we could code in terms of one of the four categories, there were 1,789 Culminators, 4,710 Persisters, 671 Switchers, and 90 Converters.

Persisters and Switchers constitute the two main categories of STEM intending students. For STEM intending students in our sample, we found that 12.5% of them were classified as Switchers. In order to improve retention of STEM majors, we need to understand how Switchers and Persisters are similar and different. The following analysis focuses on comparing Persisters and Switchers.

To address our first research question, we compared Switchers and Persisters across a number of variables. In this report we provide results from gender, ethnicity, career path, and academic preparation. Data for each of these variables was collected on the start of the term survey.

Of the students who reported gender information, 41.5% (1317) of STEM intending students were female and 48.5% (1856) were male. In comparison to males, the percentage of female switchers is significantly higher, indicating that women are more likely to leave a STEM major. Specifically, only 11% of 1856 males were identified as

switchers whereas 20% of the 1317 females were switchers ( $\chi^2$  (df = 1, n = 3173) = 49.14,  $p < .001$ ). Contrary to the observed differences among gender, there were no statistically significant differences by ethnicity ( $\chi^2$  (df = 7, n = 3169) = 3.210,  $p = 0.865$ ).

We also analyzed differences by career path. Switcher rates differed significantly by career choice, with Engineers switching at very low rates (5.9%) and the biological sciences switching at much higher rates than average (24.8%). These results made us interested to know how this was related to differences in gender. As shown in Table 2, the switching behaviors within career choices varied significantly based on gender ( $\chi^2$  (df = 15, n = 3141) = 102.9,  $p < .001$ ). For example, in the biological sciences almost 30% of females switched while only 17% of males did so. Similarly, in the fields of math, physical sciences and computer science women switched out at two to three times the rate of males in the same fields.

Career Choice	Male		Female	
	Persister	Switcher	Persister	Switcher
Biological Sciences	82.55%	17.45%	70.35%	29.65%
Life, Earth, and Environ. Sciences	81.33%	18.67%	82.40%	17.60%
Math and Physical Sciences	95.04%	4.96%	85.88%	14.12%
Engineering	94.45%	5.55%	93.11%	6.89%
Computer Science	88.39%	11.61%	77.27%	22.73%
Math or Science Teacher	86.84%	13.16%	82.98%	17.02%

Table 1: Relation between career choice and gender

We addressed academic preparation by examining coursework taken in secondary school, Advance Placement (AP) pass rates, SAT scores (this is a standardized college admissions test in the USA), self reported algebra skills, and end of term self assessment of preparation. We conjectured that Switchers were less well prepared than Persisters when they began their postsecondary study of Calculus I. In broad terms, this conjecture turned out not to be the case. There was no statistically significant difference between the percentage of Switchers compared to Persisters who took Calculus in high school ( $\chi^2$  (df = 1, n = 2676) = 2.12,  $p = .15$ ). Similarly, the mean SAT score for Switchers ( $M = 642$ ,  $SD = 86.97$ ) was not significantly different than that for Persisters ( $M = 651$ ,  $SD = 75.823$ ),  $t(2710) = 2.233$ ,  $p = .076$ . It was the case, however, that Persisters had significantly higher mean AP Calculus AB scores than Switchers.

To address our second research question (reasons why switchers are choosing to not continue on in Calculus), we looked at Switchers' and Persisters' responses to an end of the term survey question in which students could check off multiple reasons for not continuing with Calculus. The the most frequently given reason for not taking Calculus

II was a changed major, with 38.9% of switchers selecting this option. Because students were allowed to select multiple responses, we were interested to know the overlap between reasons. Specifically, we were interested in the other reasons switchers who changed majors gave for not continuing on in Calculus. As shown in Table 2, this analysis shows that of the switchers who replied that they are not taking Calculus II because they changed their major, 31.4% also replied that their experience in Calculus I made them decide not to take Calculus II.

Reason for not taking Calculus II	
My experience in Calculus I made me decide not to take Calculus II	31.4%
To do well in Calculus II, I would need to spend more time and effort than I can afford	28.7%
I have too many other courses I need to complete	27.6%
I do not believe I understand the ideas of Calculus I well enough to take Calculus II	18.8%
My grade in Calculus I was not good enough for me to continue to Calculus II	11.5%
I never Intended to take Calculus II	6.1%

Table 2: Reasons Switchers give for not taking Calculus II

This result is consistent with Seymour's (2006) finding that students frequently leave their STEM majors because of their experience in Calculus I. This finding necessitates a better understanding about the nature of the Calculus I experience, which leads to our third research question.

For the third research question, we investigated several different variables to understand students' experiences in Calculus I, including student behavior in and out of class and student description of their Calculus I instruction. For example, there was a statistically significant difference ( $p < 0.001$ ) in what Switchers and Persisters tended to do in class. Switchers were less likely to contribute to class discussion, more likely to be lost and unable to follow lecture, and more likely to simply copy what was on the board. This is despite the fact that their mathematical preparation was not significantly different from that of Persisters.

In terms of out of class behavior, we found that similar to Persisters, about 70% of Switchers worked at a job for at most five hours per week. Moreover, the vast majority of Switchers spent about the same amount of time studying calculus as did Persisters. Moreover, a statistically significant greater percentage of Switchers reported visiting their instructor's office hours either weekly or monthly (56.2% versus 48.1%) and going to tutoring on a weekly basis (25.6% versus 15.2%) than Persisters. Thus, this data suggests that Switchers are making the effort to be successful. Compared to Persisters, they do not work more on an outside job, they are studying as much or more, and they are seeking academic help more so than Persisters. All of this, together with the reasons that Switchers give for not continuing on with Calculus, suggests that a closer look at what happens in the classroom is warranted.

To examine student reported classroom instruction, we conducted a factor analysis on the questions on the end of the term survey pertaining to instructor pedagogy. This factor analysis revealed two main factors. The first factor, which we refer to as “Good Teaching,” included questions where students rated their instructor on the extent to which he or she listened carefully to their questions and comments, allowed time for them to understand difficult ideas, presented more than one method for solving problems, asked questions to determine if they understood what was being discussed, discussed applications of calculus, encouraged students to seek help during office hours, frequently prepared extra material, gave assignments that were challenging but doable, graded exams fairly, and gave exams that were a good assessment of what was learned. The second factor, which we refer to as “Progressive Teaching,” included questions where students rated their instructor on the extent to which he or she required them to explain their thinking on homework and exams, required students to work together, had students give presentations, held class discussions, put word problems in the homework and on the exams, put questions on the exams unlike those done in class, and returned assignments with helpful feedback and comments.

Table 3 shows how low and high levels of Good Teaching and low and high levels of Progressive Teaching relate to the percentage of students who were Switchers. The percentages in Table 3 should be compared to 14.8%, which is the switching percentage for the sample of students who responded to the instructor pedagogy questions on the end of term survey.

	Low Good Teaching	High Good Teaching
Low Progressive Teaching	15.3%	14.4%
High Progressive Teaching	13.4%	11.7%

Table 3: Switching rate for low and high levels of good and progressive teaching

As shown in Table 3, the type of instruction seems to make some difference in student retention. “Progressive” teaching, which includes instructional approaches that more actively engages students, is associated with lower switching rates. Indeed, high levels of progressive teaching coupled with high levels of good teaching reduces the switching rate from 15.3% to 11.7%. This findings indicate that Switchers reported having different classroom experiences than Persisters. Their instructors were less likely to actively engage them (working by themselves or with a classmate on problems, having a whole class discussion, asking students to explain their thinking, etc.), they were less likely to contribute to class discussion, and more frequently found themselves lost in class.

Additional studies that include classroom observations are needed to further study the effect of instructional approach on student retention in a STEM major. Nonetheless,

these findings are consistent with prior research summarized in the PCAST report and with the seminal work of Seymour and Hewitt (1997).

## CONCLUSION

Up until now there has been little large-scale data collected on who elects to study Calculus I at a university. Additionally, little is known about the effect of Calculus I on student intention to pursue a career in mathematics, science, or engineering. Even information as basic as the US national success rate and the percentage of students in university Calculus I who successfully complete the course, has not been reported. This large-scale national study is making a significant contribution to what we know about Calculus I (for example, see Bressoud, Carlson, Pearson, & Rasmussen, 2012).

Findings from this report illuminate the types of students who are switching out of STEM majors, as well as their experiences in Calculus I. It is clear that many of the students who intended to take Calculus II but did not were hard working and well prepared. When asked why they no longer intended to take Calculus II, Switchers reported not continuing with Calculus because they changed their major, citing a negative experience in Calculus I and spending too much time and effort in Calculus I as the second and third most responses. When we look more deeply at their experience in Calculus I, switchers and persisters report different experiences. This suggests that instructional variables such as actively engaging students, having students explain their reasoning, etc. may make a difference in retaining STEM majors. While many may have conjectured that such a finding is the case, this is the first large scale, national study in calculus to provide data for this position.

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# **PRESCHOOLERS COUNT AND CONSTRUCT: SPATIAL STRUCTURING AND ITS RELATION TO BUILDING STRATEGIES IN ENUMERATION-CONSTRUCTION TASKS**

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*This study is part of the research project (Y)CUBES K-4, (Young) Children Using Blocks to Express Spatial strategies, which intends to enlighten the elaboration of spatial strategies from kindergarten to primary age and their relationship to constructive activities. In this paper, we focus on the interrelation of spatial structuring in enumerating the number of cubes in three-dimensional arrays and preschoolers' block building strategies. Connecting these strands of research, 22 preschoolers had to cope with different sets of enumeration-construction tasks (count and construct and count). Results of the study lead to the conclusion that awareness of structural elements needs to be fostered by evoking appropriate building strategies.*

## **INTRODUCTION**

Playing with blocks has been a popular activity in early childhood for decades (e.g. Guanella, 1935; Wellhousen & Kieff, 2001). When constructing block buildings, preschoolers encounter geometric concepts like congruence, they distinguish between various solids and compose or transform buildings (e.g. Park, Chae & Boyd, 2008). Meanwhile, they need to reflect on spatial relations, orientations and the structure of three-dimensional arrays.

Although spatial structuring is a cognitive process operating upon internal representations, a person's external articulation (e.g. verbal comments or parallel motor activities) may reveal characteristics of internal processes. Results from psychological and didactic research indicate that children's articulation in drawing activities and products give a clue to their spatial abilities (e.g. Woodrow, 1991; Milbrath & Trautner, 2008). According to this notion, children's drawings are interpreted as an expression of spatial structuring approaches (Mulligan, Prescott & Mitchelmore, 2004; Mulligan & Mitchelmore, 2009). However, the corresponding role of concrete construction activities and their interaction with mental spatial structuring skills are not entirely clear, yet.

## **THEORETICAL FRAMEWORK**

Thurstone (e.g. 1950) distinguished three major spatial ability factors, namely *spatial relations* ( $S_1$ ; including awareness of structural features of a rigid configuration while mentally moving it), *visualization* ( $S_2$ ; ability to mentally manipulate objects and to imagine displacement within a configuration) and *spatial orientation* ( $S_3$ ; imagining spatial relations the observer is part of). Our research primarily refers to this differentiation and to previously detected inter- and intra-individual differences in primary students' spatial strategies and the children's ability to visualize spatial



relations or movement (e.g. Merschmeyer-Brüwer, 2001a; b; 2002; Reinhold, 2007). Results of our studies also indicated that strategy differences are often due to variance in spatial structuring and frequently include the identification of characteristic features or prominent aspects. Hence, other authors also assume that spatial structuring skills influence all components of spatial sense (cf. Pittalis & Christou, 2010; van Nes & van Eerde, 2010).

### **Free construction with blocks**

Preschoolers' building capacity shown in free construction obviously grows in terms of complexity (e.g. Guanella, 1935). Toddlers tend to "construct buildings in the air" – holding two objects in their hands and more or less coincidentally joining them. Later on, they are able to build small and rather shaky "towers" of blocks. From approximately age 2 on, blocks are joined in "one-dimensional" (longer) rows. Progress of these skills includes the ability to construct "two-dimensional" layers of unified blocks and the competence to assemble those components to "three-dimensional" buildings. Elkin (1984) points out that children in kindergarten and grade 1 tend to start a construction in the way they have done previously. This strongly affects the ensuing building sequence. Furthermore, preschoolers seem to prefer closed figures and take symmetry, similarity and continuity into account.

### **Spatial structuring in enumerating and building activities with cubes**

Structuring a geometrical arrangement involves identifying numerical and spatial relations between its constitutive components or among sets of components and the whole arrangement (Battista & Clements, 1996; Merschmeyer-Brüwer, 2001a; b). As individual acts of mental construction, primary students' individual strategies in enumerating the number of cubes in three-dimensional cube arrays range from disorganized activities (double-counting, omitting) and restricted consideration of visible faces to focused attention to columns or layers for elaborate counting or calculation (Campbell, Watson and Collis, 1992; Battista & Clements, 1996). These strategies can be interpreted as the appearance of a general concept that spans early mathematical concepts of number, measurement and space, described as the "Awareness of Mathematical Pattern and Structure" (Mulligan & Mitchelmore, 2009).

Sophisticatedly planned cube (re)constructions can be characterized by constructing layers or columns that are brought together only in a second step (McFarlane, 1925). In contrast, younger children tend to neglect the fact that buildings consist of several layers (Piaget & Inhelder, 1971), which is an analogue observation to problems in the enumeration tasks mentioned above. According to Piaget and Inhelder, they may also identify specific parts of the configuration but attach them incorrectly. Global similarity of the construction stands in contrast to buildings with incorrect details (e.g. wrong orientation of segments, erroneous number of cubes).

Even if the construction activities succeed, a correct enumeration of these cubes may not be guaranteed. Voulgaris and Evangelidou (2004) found that only 8.9 % of fifth or sixth graders improved their enumeration performance after constructing a cube building shown in a picture. These students moved their strategies, thus indicating an

increasing awareness of the structural arrangement. Moreover, there is greater evidence of an increasing ability in spatial structuring during the whole interview. Voulgaris and Evangelidou hypothesize, according to Ben-Haim, Lappan and Houang (1985) and Battista and Clements (1996), that this improvement particularly relies on physical activities with cubes. Details concerning the impact of preschoolers' constructive activities on the elaboration of their spatial structuring shown in enumeration are scarcely available. Hence, conscientious analyses of preschoolers' related capacities are required to base the design of fostering learning environments.

## RESEARCH QUESTIONS

The research project *(Y)CUBES –(Young) Children Using Cubes to Express Spatial Strategies*<sup>1</sup> is deeply interested in investigating the elaboration of spatial strategies and its interrelation with children's building activities. Several contributory studies serve to enlighten this field and accentuate various related research questions covering intense work with children aged 5 to 10. This paper reports the part of our current work that focuses on case studies with preschoolers and that is interested in the following:

- What kind of strategic elements can be found in preschoolers' building activities when (re-)constructing three-dimensional cube buildings?
- How are the identified building strategies connected to ways and outcomes of enumerating the number of cubes in a three-dimensional cube building before and after a construction process and after a kindergarten mathematics program?

## METHODS

In the winter of 2011 and the spring of 2012, 22 preschoolers (aged 5 to 7) dealt with geometric tasks in a one-on-one setting with one child per interview. The videotaped interviews were conducted twice – before and after the children attended a kindergarten mathematics program. This was carried out by an external teacher who designed the program to improve children's basic concepts of number and geometry (including block play activities which are not discussed in this report).

Four task sequences were designed to capture the nature of preschoolers' building strategies in relation to their spatial structuring abilities in this study: (I) ordering cube buildings and constructing the following, (II) enumerating cubes in a building, copying them and enumerating again, (III) constructing same sized buildings by using blocks of different shapes and (IV) copying cube buildings by correcting a given composition. For example, task sequence II included two buildings made of concrete glued cubes and two buildings shown in their cavalier-perspective drawings (see fig. 1). Children had to enumerate the cubes presented before they copied the buildings using additional concrete cubes and then had to enumerate the cubes in their own buildings.

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All children were tested with a standardized arithmetic test called TEDI-MATH (Kaufmann et al., 2009) before pre- and post-test to assess pure arithmetic skills. As qualitative analyses provide a deeper understanding of difficulties and strategies, all enumerating and building processes were captured in sequences of drawings and were combined with transcripts of the scenes. Analyses were supported by ATLAS.ti.

No.	Concrete cubes		Cavalier-persp. draw.	
	II.1	II.2	II.3	II.4
Presented and producing structure				

Fig. 1: Task sequence II: Enumerating and copying cube buildings

## FIRST RESULTS

Firstly, TEDI-MATH revealed high arithmetic competencies in verbal counting and enumerating, i.e. the children were able to enumerate sets of 15 or more items – even in the pretest. Unexpectedly, the participants were very good at copying buildings, too. Most of them could interpret cavalier-perspective drawings and used them to construct a correct copy with ease. Enumeration rarely improved after construction and appeared to be much more complex than the construction of an identical building (see fig. 2). From pre- to post-test, we detected progression in both first and second enumeration.

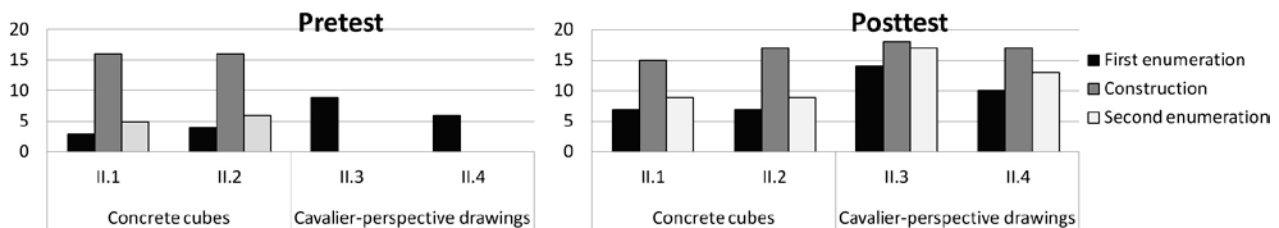


Fig. 2: Number of correct solutions in II (in pretest construction only in II.1 and II.2)

The analysis of solving strategies in task sequence II provided further insight into connections between enumeration and building processes. Like many other children, Julia (5;11) started with difficulty in enumerating cube buildings (see fig. 3).

First enumeration	Construction				Second enumeration
	a.	b.	c.	d.	
	e.	f.	g.	h.	

Fig. 3: Julia's enumerating and building process in pretest, II.1

Instead of integrating faces to a three dimensional object, she primarily recognized the faces as single units with double-counting of two columns (cf. Merschmeyer-Brüwer, 2001a; b). In II.1, Julia and others counted the faces of exactly three side views. They focused on front views, rotating the buildings twice by 90°. The result of this strategy was a count of 18 (Julia had 16; two further errors in one-to-one correspondence).

A net diagram of two dimensions for arithmetic strategies and four dimensions for spatial structuring was created to describe and categorize the children's competencies. This tool records arithmetic strategies of enumeration and children's consideration of part-whole relationships and includes a differentiation of spatial structuring aspects:

- **structuring complexity:** consideration of only one layer, orientation in faces, focus on single components, local and global structuring
- **coordination of structuring:** omitting, overlapping, completely correct
- **attention to intended structures:** handling of loose components, arbitrary, partly guided by structural elements, discerning dot configurations like on dice, discerning rows or columns, discerning layers
- **decoding of depth** (mainly for decoding drawings or more complex structures)

Julia's first enumerating process (see fig. 4a) can be described by a structural complexity with orientation in faces which is accompanied by a coordination of structuring with omitting and overlapping parts of the building. This counting is only partly guided by structural elements. It is marked by counting by ones and neglecting part-whole relationships.

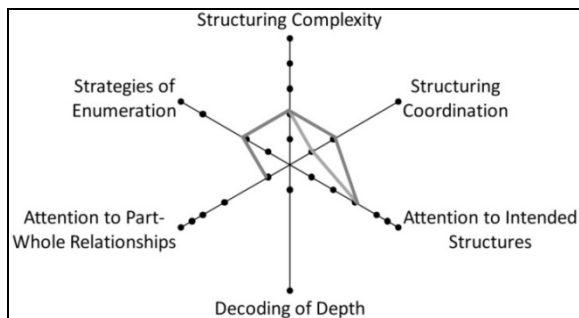


Fig. 4a: Julia's first enumeration in pretest, II.1

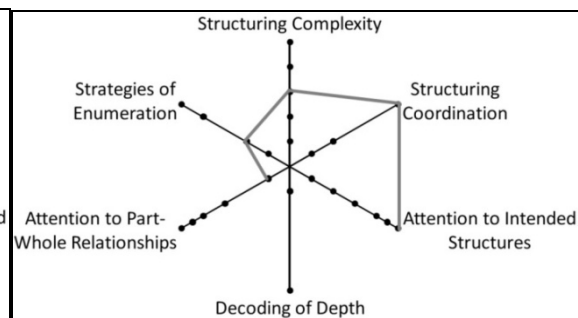


Fig. 4b: Julia's second enumeration in pretest, II.1

Julia's following building process showed a growing awareness of the building's structure. This can be described by means of the model displayed in fig. 5 which is derived primarily from peer-analyses of the recorded interview scenes with preschoolers. It is also based on former Grounded Theory research on primary students' construction strategies in mental rotation of cube buildings (Reinhold, 2007). As many others did, Julia started her construction arbitrary. She grasped three single cubes (TC) and then divided the set into two portions (HC) including a column (DC) for copying the front wall. Immediately, she seemed to recognize the structure of two columns (from trial and error to local approach) which became apparent when she constructed two columns simultaneously by putting one cube per hand (TC) on both tops (HC, DC) before assembling them (HC). After completing the front view wall (DC) by filling in (HC) one cube (TC) she continued to put single cubes and a domino (TC) behind it. Again, she took a bearing on columns (DC) with a local or even a global approach.

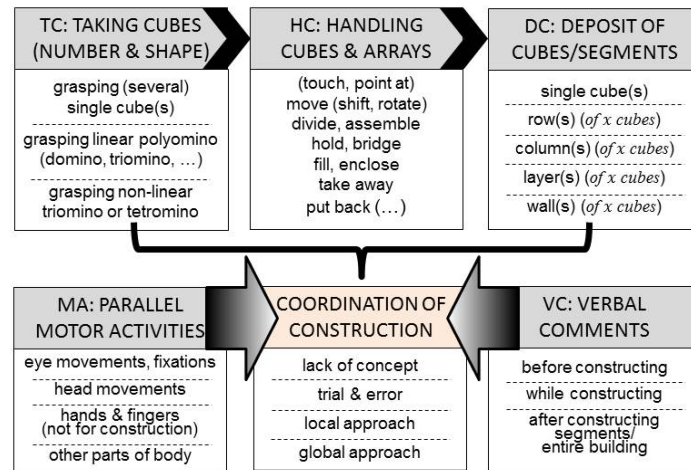


Fig. 5: Characteristics of cube construction

The strategy Julia applied on enumerating cubes in her own building is analogous: she counts the front view (in rows) before counting the rear columns. This gives the impression of imitating her building sequence – starting at the bottom. She focused on single components, her coordination was completely correct and she paid attention to layers (see fig. 4b). However, without further instruction she counted the cubes of the presented building once more. As in her first counting, she rotated the building twice by  $90^\circ$  – now accompanied by double counting of only one column. Her awareness of structure was not stable enough to be applied to the presented building.

The enumeration and construction processes in the posttest half a year later (see fig. 6) showed that Julia considered structural elements with more sophistication as she constructed separate columns. Her counting process proceeded according to the concrete construction.

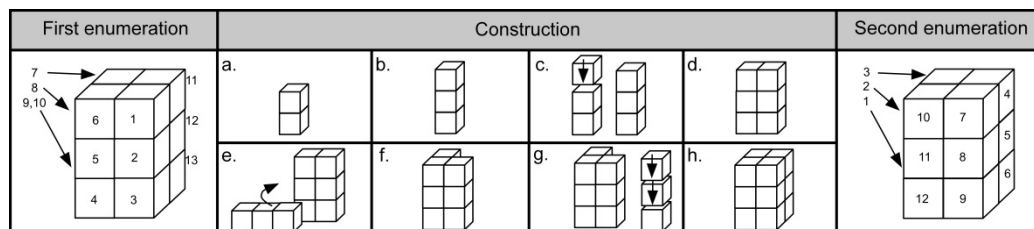


Fig. 6: Julia's enumerating and building process in posttest, II.1

## DISCUSSION

The results of our study give an impression of qualitative variance in preschoolers' spatial structuring during enumeration and construction of three-dimensional cube arrays. A typical problem in enumeration is the coordination of different views of three-dimensional objects – hence, in the ability to reconstruct appropriate representations of spatial relations. Paying attention to intended structural elements (counting in rows or columns) does not guarantee an awareness of the building's structure. Although children often start to copy front views, problems in coordinating views during construction are very rare. Probably, problems may increase while copying bigger buildings with hidden cubes (cf. Revina, Zulkardi, Darmawijoyo & van Galen, 2011; Olkun, 2003). In our study, children gain insight into structural elements

and change trial and error building strategies into orientation in structural elements. On the other hand, there is little quantitative improvement in enumeration after having constructed the building once.

In order to foster spatial structuring abilities, group discussions with proposals on enumeration strategies and the design of constructions might help children to become aware of structures and to generate successful mental structuring strategies (cf. van Nes & Doorman 2011). Intervention studies (Merschmeyer-Brüwer, 2009) demonstrate significant support by using complex cube-blocks, e.g. linear polyominoes of two, three or four cubes, small convex layers of four and six cubes and cubes consisting of eight smaller cubes. Furthermore, the children become more capable of coordinating complex compositions like rows, columns or layers with regard to symmetry, congruence and similarity in order to enumerate the cubes more effectively. Therefore, ongoing studies of the **(Y)CUBES K-4** project deal with preschoolers' enumerating and structuring of more complex and heterogeneous spatial arrangements of cubes – e.g. including buildings with hidden cubes or constructions of polyominoes that require mental movements. These studies are expected to improve structural skills in the applied kindergarten program.

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# CHARACTERIZING PROSPECTIVE TEACHERS' KNOWLEDGE IN/FOR INTERPRETING STUDENTS' SOLUTIONS<sup>1</sup>

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*This paper focuses on primary prospective teachers' mathematical knowledge for teaching (MKT) when interpreting and making sense to students' answers in a fraction problem. We designed a suitable set of tasks in order to inquire this particular kind of knowledge and clarify its features and dimensions. Using the MKT conceptualization we found a special role of prospective teacher's Common and Specialized Content Knowledge sub-domains in their interpretation, making sense and evaluation of student solutions.*

## INTRODUCTION

Nowadays teachers are required to be able of complex and nuanced judgments concerning the teaching and learning of mathematics. One important aspect of teaching is promoting students' reflection upon the effectiveness of their own (and others') reasoning and representations chosen to solve a problem. This task requires, among others, the teachers' capacity to make sense and provide feedback to students' solution processes and to support them in their developing knowledge of mathematics. It is also widely acknowledged how this skill differs from those associated with a traditional view of mathematics as merely a set of definitions and rules. For example, Ball (1993) argues that in the "reform" view of mathematics teaching, students/children are viewed as being unpredictable thinkers. To make sense of and support mathematical thinking for such unpredictable thinkers requires a precious and flexible mathematical sensitivity from teachers.

These peculiar features of mathematical knowledge needed for teaching are well caught and framed by the Mathematical Knowledge for Teaching (MKT) conceptualization (Ball, Thames & Phelps, 2008). Among other things, teachers' knowledge must include knowing the topics in a way that allows them to understand students' answers – mainly in those cases grounded in non-standard reasoning (for a particular perspective see Borasi, 1994). Such understanding of students' answers implies also being able to evaluate if a solution can be considered mathematically valid and/or generalizable to other situations; and working on the given solutions (even the incorrect ones) in order to explore them in a mathematically valid and significant way. For this reason, teachers' knowledge should include a broad range of strategies and

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<sup>1</sup> The three authors contributed equally to the preparation of this article.



representations for problem solving that could help them to make sense of students' productions, reasoning, and strategies.

Here we report a specific work developed for identifying particular features and dimensions involved in prospective teachers' MKT while interpreting and giving sense to students' productions. With this perspective in mind, we have designed a sequence of tasks focused on the emergence and the discussion of perspective teachers' knowledge involved in this interpretative/evaluative work. These tasks can be framed as professional learning tasks, or PLTs (Smith, 2001).

## THEORETICAL FRAMEWORK

Teacher knowledge, and in a broader sense, teacher cognition (knowledge, beliefs, and goals) play a key role in practice. From among the different conceptualizations of teacher knowledge, the Mathematical Knowledge for Teaching (MKT) (Ball, et al., 2008) effectively captures the particular features of mathematical knowledge needed for teaching (aiming at developing students' mathematical knowledge and understanding) in comparison with the mathematical knowledge involved in other settings. MKT is knowledge that serves as a resource for specifically addressing the mathematical demands of teaching (Ball, et al., 2008), and several researches all around the world are demonstrating the effectiveness of a teacher education centred on this specific knowledge (for a list of references, see Jakobsen, Thames, & Ribeiro, 2013).

One of the tasks of teaching is to make sense of students' solutions and help them to develop their mathematical knowledge. The knowledge involved in such a task (we will call it *interpretative knowledge*) owns a peculiar and specific nature, and that also motivates our use of the MKT framework to investigate about this particular prospective teachers' knowledge. In particular in our inquiry about *interpretative knowledge* we focus mainly on the Common and Specialized Content Knowledge (CCK and SCK) sub-domains of MKT. Indeed, in evaluating and giving sense to students' solutions of a mathematical task the CCK, used both in teaching and in other settings that uses mathematics related at that specific mathematical topic, is of course fundamental. But in the same time this teachers' interpretative/evaluative work needs a specific SCK that does not only concern Pedagogical Content Knowledge (PCK) and also differs from the mathematical knowledge commonly (CCK). Indeed, besides knowing a definition of a concept, how to solve a problem, or how to perform a certain calculation, it is essential that teachers also have knowledge allowing them to understand mathematical reasoning behind such calculation, definition, or problem-solving processes (Thames & Ball, 2010). In this sense teachers should possess a rich and ample knowledge of examples, strategies, and representations for problem solving (Chapman, 2012) that allows them to make sense not only of solutions similar to their own, but also of students' answers, reasoning, and strategies even very different from their own. In this study the interpretative knowledge is perceived as part of SCK and intertwined with the ability of noticing, meant as the capacity to increase the range and decrease the grain size of relevant aspects accordingly with specific

educational goals (Mason, 2002). This involves teachers' ability to make informed choices in contingency moments (Rowland, Huckstep & Thwaites, 2005) and to respond to situations as they emerge, in ways that supply sustainable mathematical knowledge to students.

Our preference for the MKT conceptualization over others in this work derived also from the nature, focus, and aims of the work we are developing. Indeed, the content of the sub-domains of the MKT are considered as a relevant starting point for designing tasks for the mathematical preparation of teachers, and for doing research on what inputs to teacher training and teacher knowledge produce effects on practices and students. Such tasks can be perceived as professional learning tasks – PLTs (Smith, 2001). In our perspective, by considering the importance of a practice-based approach, we assume also as essential that prospective teachers experience the same kind of situations they will encounter in practice (Magiera, van den Kieboom, and Moyer, 2011) as well as expected situations they will explore with their pupils, thus allowing them to develop their SCK.

In this study, our interest concerns teachers' interpretative knowledge in a problem consisting of sharing and fractions in primary schools. The pervasiveness of fractions in many other specific mathematical domains (e.g., operations, probability, and measurement) makes them a crucial and strategic mathematical topic of inquiry. Moreover, fractions are among the most complex mathematical concepts that children encounter in their primary education years (Newstead & Murray, 1998). According to Kieren (1995), students' difficulties with fractions can be traced to the fact that fractions comprise a multifaceted construct and, as particular representations of rational numbers, they present many links with other ways of representing the same entities. These well known difficulties, together with the teacher's role in students' learning, justify the importance of improving teacher training about the teaching of fractions and our choice to place our inquiry in this specific mathematical topic.

## **METHOD**

Our sample is composed of prospective elementary teachers taking mathematics courses at universities in each of the authors' countries (Portugal, Norway, and Italy), where we served as lecturers. An open questionnaire with a set of tasks was designed and translated in the three languages. The tasks aimed at accessing (and discussing) prospective teachers' interpretative knowledge, in a problem involving fractions and were grounded in some mathematically-critical situations previously identified (Ribeiro & Jakobsen, 2012). Although data was gathered in parallel (November 2012), the intention was not to do a comparative study, but rather to use the diversity of contexts as one element contributing to a richer understanding of prospective teachers' knowledge of the subject at hand. However, in this study, due also to space limitations, we focus only on data from 108 prospective teachers in Italy.

The nature and focus of the questionnaire is an important part of this study, therefore some of its characteristics deserve discussion. It starts by describing a situation in which a teacher gives the following problem to her pupils in order to explore with them

the concept and nature of fractions: *What amount of chocolate would six children get if we divide the five bars equally among them?*

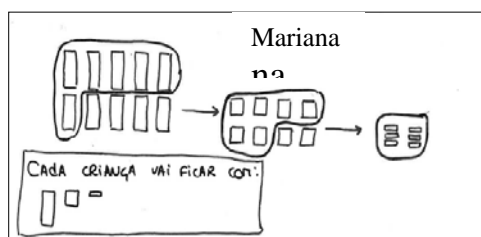


Figure 1: Student solution  
"Each child gets"

In the first part of the questionnaire, we ask prospective teachers to solve the problem by themselves. In the second part of the questionnaire, we present seven different student solutions to the problem (see Ribeiro & Jakobsen, 2012), which showed different possible ways to divide the chocolate bars using graphical representations.

Mariana (in Figure 1) used a graphical representation involving only the division into halves; Ricardo, after a text explanation of the subdivision, presented a solution in terms of natural numbers ("we divide each bar in six squares, each child gets five squares"); in Madalena's solution a decimal representation was used ( $0,8\bar{3}$ ). The answers of the remaining four students (Sofia, Ines, Isabel, Flor) contained the use of fractions  $5/6$  and  $1/6$ : Flor used them in an appropriate way; while in the remaining cases (Sofia, Ines and Isabel) after different graphical divisions of chocolate bars there was a mathematically inadequate use of the answer  $5/6$  to the problem, such as "each child gets  $5/6$  of each bar." Our choice of these solutions is justified by the possibility to access (and to understand) prospective teachers' interpretative knowledge, analysing the ways they evaluated, in terms of mathematical correctness: i) different possible subdivisions of the bars; different ways of expressing these subdivisions (by means of graphical representations or texts); iii) different kinds of numbers (natural or rational); and iv) different representations of rational numbers (decimal or fractional). Moreover, we want to see how perspective teachers make sense of and interpret less obvious and explicit divisions of the chocolate bars (such as in Figure 1).

Besides commenting and reflecting upon these seven answers and their mathematical correctness (or adequacy), prospective teachers were asked also to think about how they could, as teachers, lead students to detect eventually inconsistencies in their reasoning and to reach adequate possible answer(s). These tasks (as PLTs; Smith, 2001) were used both as a tool to observe and deepen perspective teachers' MKT mobilized by this prompt, and as a tool in our lectures to support the prospective teachers' development of MKT. Indeed, after applying these tasks, two lessons (audio recorded) were dedicated to discuss and work on it with the goal to develop prospective teachers' MKT.

In the next section we offer a quantitative analysis of prospective teachers' answers to the fraction problem and a first qualitative analysis of their interpretations of student solutions. We will focus on some particular perspective teachers' comments as an opportunity to gain a deeper understanding of the features of their interpretative knowledge and its links with CCK and SCK.

## RESULTS AND DISCUSSION

We start by first presenting and discussing prospective teachers' answers to the fraction problem. The answers contain rich evidence of their knowledge and understanding of fractions. Then we discuss their comments and reflections while interpreting to student answers.

In the first part of the task, the majority (55 out of 108) of the prospective teachers presented a solution using only natural numbers. These solutions correspond to a graphical subdivision of each bar into six pieces. They concluded that each child would get five pieces (see Figure 2). This fact appears even more relevant if we consider that text accompanying the problem explicitly says: "The teacher uses this problem to explore fractions with her pupils." Despite this, more than one half of the prospective teachers chose to remain within the domain of natural numbers, revealing evidence of a

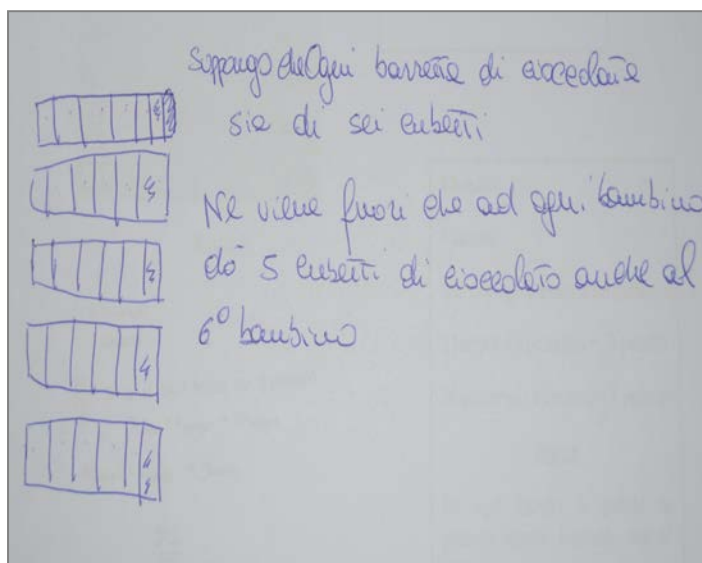


Figure 2: Student solution "I suppose that each chocolate bars be composed of six cubes. It comes out that I give 5 chocolate cubes to each child also to the sixth child".

lack of familiarity with the mathematical notion of fraction. In some other answers they opted for different uses of fractions in some cases combining the reasoning about the distribution of pieces with the expression of the pieces in terms of fractions. Among the prospective teachers who used fractions to answer the problem (46 out of 108), we found some using them in an inappropriate way. Typical wrong answers in terms of fractions are "each child gets  $5/6$  of each bar" (11 out 46) or "each child gets  $1/6$  of one bar" (3 out 46) revealing difficulties in understanding the role of the whole. Similar results were obtained in a previous study (Ribeiro & Jakobsen, 2012). We also found answers that used decimals (8 out of 108). The most frequent decimal answer was: 0.83. This first quantitative analysis about teachers' own solutions is useful to frame our particular sample as characterized by a poor CCK about fractions. As confirmation of this we could observe that in the following task about evaluation of the correctness of student solutions, a really little number of perspective teachers (4 out 108) is able to detect the mathematical incorrectness of the expression "each child gets  $5/6$  of each bar" as solution of the problem presented in three student solutions.

Concerning the prospective teachers' comments and making sense of students' solutions to the problem, we find that many of them find difficulties in interpreting children's solutions different from their own solution, similar to what was expressed by Kuhlemann (2013). In reference to the graphical subdivisions of the bars, teachers' comments included: "Sofia's solution appears disorderly, all those signs make the idea confused", "Ines has just complicated her life," "Mariana follows a more intricate

*path*". What it is interesting here is that if from one side they are not able to detect the mathematical inadequacy of the expression "*each child gets 5/6 of each bar*", they are ready to disapprove their subdivision of the bars that, at a more careful mathematical eye, match with two different sum of fractions of the right amount of chocolate each child gets. Comments as these reveal a big difficulty to leave one's own space of solutions, a space evidently consisting of a single element, making thus impossible to appreciate and understand different student solution strategies and to fruit them to support children deeper fractions knowledge development. If in the above comments this negative propensity toward a variety of approaches and reasoning is only visible in transparency, some other prospective teachers explicitly said that a correct answer should necessarily be similar to what they do or think. An example of such feeling is in the following:

I consider ISABEL-SOFIA-RICARDO-FLOR correct [capital letters are in the original text]: to put it better, I mean that they translate my own THOUGHT in the problem solution [...] I consider Marianna's solution mathematically inadequate at the level of understanding, I mean that it is not understandable at a large range, at first not to myself. A priori I would not surely fail her, I would skip it now and analyse it in a second moment.

This comment - "*I mean that they translate my own THOUGHT*" supports the conclusion that prospective teachers are not able to interpret student solutions when they are different from their own reasoning paths. Of course this seems a very natural human attitude (and to some extent we could accept it from someone who acts in contexts other than teaching, or who is not going to become a teacher), but we all agree that a teacher, as well as a practitioner, needs to overcome it by developing particular sensitivity and insight – skills which are linked to the specificity of teachers' mathematical knowledge. In this sense we believe that prospective teachers need to work and learn and develop their knowledge in order to be able to interpret children's solutions. On the other hand, one could think that this is the kind of knowledge that will be developed over time with practice. However, evidence shows, and our data confirm, that this is not completely true:

I'm a teacher since ten years and I do think that these solutions are very confusing. I think that Ricardo's solution is the simplest and clearest, as the mine. Indeed, I always try to make the visual images as clear as possible and I try to lead my pupils to do the same. In this case the reasoning paths are very disorderly and lead to confusion.

It is meaningful to notice that this prospective teacher (although she could be considered an expert teacher, nevertheless she still lacks her proper degree), keeps interpreting as correct only those reasoning that are similar to her own. In this sense we can say that teachers' special knowledge and their ability to interpret and make sense of student answers do not naturally develop over the course of years of work experience, but are something that requires a special attention from an educational point of view.

We also addressed the issue of how teachers could help students understand their mistakes, and then develop correct reasoning and adequate justification and argumentation. We found that the prospective teachers' options often consisted of

showing their own solution to pupils, what can be paired with their difficulty in making sense of others' answers:

Mariana's solution is not understandable so the first question would be: what does this representation mean? After I have listened her answer, I will try to show her my own representation and we will together get the solution.

Some prospective teachers, who revealed a good CCK in the fraction problem, perceive student solutions, even the inadequate ones, as a possibility and a good starting point to work with the class. They see the wrong answers as opportunities to develop students' knowledge and awareness on subdivisions and fractions:

The first thing that I would do is to invite the children to copy the solutions of Isabel, Sofia, Ricardo and Flor onto one poster, and then copy the solutions of Madalena, Mariana and Ines onto another one. I will let children notice that the different graphical representations in the first poster are essentially similar. Things are different in the second poster and require much care from the teacher, who has to decode what the pupil meant and to help children give sense to different solutions.

These answers reveal very precious and valuable pedagogical insights. We can observe a certain link between these proposals and Mason's idea of noticing: "To increase sensitivity to notice opportunities to act, while at the same time, to have come to mind in the moment when they are relevant, a range of possible appropriate actions" (Mason, 2002, p. xi).

## CONCLUSIVE REMARKS AND FUTURE PERSPECTIVES

The quantitative analysis on prospective teachers' answers to the fraction problem showed evidence a somehow poor CCK about fractions. Moreover the designed tasks gave us the opportunity to access also other sub-domains of MKT. In particular in the qualitative analysis, the observed prospective teachers' difficulties to leave their own space of solutions and to appreciate and understand different student solution strategies, revealed also their lack of SCK. One of the possible future research roads is to deepen the links between CCK and SCK (and its content) in teacher interpretation knowledge/capacity. Finally we want to underline that, after administering the questionnaire, the designed tasks opened space of and for reflection in which we, as lectures, could work with the prospective teachers on their MKT and beliefs about mathematics and its teaching-learning processes. We believe that the proposed tasks could represent a good example of PLTs to explore with prospective teachers and in this sense further analysis on these recorded lessons will be conducted.

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# PATTERN GENERALIZATION PROCESSING OF YOUNGER AND OLDER STUDENTS: SIMILARITIES AND DIFFERENCES\*

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*We compared the pattern generalization processing of US elementary and middle school students on similar tasks before and after a long-term exposure to a multiplicative-driven mathematics curriculum. Results show that both groups learned to express structural function-based generalizations as a result of their firm understanding of multiplicative relationships. However, the elementary group consistently processed inductively in nonsymbolic algebraic terms, while the middle school group processed the same tasks deductively in symbolic algebraic terms.*

## INTRODUCTION AND RESEARCH QUESTION

In this paper we resolve the following research question:

How do we characterize the pattern generalization (PG) processing of US elementary and middle school students on similar tasks before and after a long-term exposure to a multiplicative-driven mathematics curriculum?

*Patterning* generally involves searching for mathematical regularities and structures. In this study, we further characterize individual students' *PG* in terms of their ability to mutually coordinate their visual and non/symbolic inferential abilities that enable them to *construct* and *justify* a plausible function-based algebraic structure (Rivera, 2013).

Following Duval (2002), we assume that visual is different from mere vision. While vision sees what it sees and experiences and needs physical action in order to completely apprehend an object, visual is neither physical nor mental but semiotic (pp. 320-322). Thus, a visual inferential skill toward patterns utilizes both epistemological and synoptical performance, where the epistemological aspect involves constructing an appropriate signifier (e.g. formula) for a given signified and its class (e.g. the known stages in a pattern) and the synoptical aspect involves analytically seeing them through a sequence of focusing actions producing relations and organization of relations between them (ibid). In PG we assume that a synoptic grasp of an interpreted structure involves both theoretical (i.e. abductive and/or deductive) and empirical (i.e. inductive) justifications.

Following Heeffer (2010), algebraically structures are either symbolic or nonsymbolic. Both require analytical methods in which case the unknown quantities in a pattern task are represented abstractly in some way (e.g. the use of variables). However, nonsymbolic algebraic structures remain at the level of notational representation, while symbolic algebraic structures begin with and proceed to systematically manipulate the corresponding abstract representation (pp. 88-89).



## RECENT RESULTS ON PATTERN GENERALIZATION

Overall our interest in PG as a research activity stems from our view that patterning activity democratizes students' access to structures, in general, and functions, in particular. Looking at the two tasks shown in Figure 1, both the figural pattern and the function problem require the same mathematical analysis despite appearing to have two different contexts. Both tasks ask students to obtain values for near generalizations (cases 9 and below), far generalizations (cases 10 and above), and a formula that works for any case and can conveniently predict future outcomes.


Figural Pattern				Function Problem											
Stage	Stage	Stage	Stage	The data in the table show the cost of renting a bicycle by the hour, including a deposit.											
1	2	3	4												
				<table><tr><th>Hours (h)</th><th>Cost in \$</th></tr><tr><td>1</td><td>3</td></tr><tr><td>2</td><td>5</td></tr><tr><td>3</td><td>7</td></tr><tr><td>4</td><td>9</td></tr></table>		Hours (h)	Cost in \$	1	3	2	5	3	7	4	9
Hours (h)	Cost in \$														
1	3														
2	5														
3	7														
4	9														
Assume the pattern resembles a growing L shape. How many squares would there be in stage 6? 8? 12? $N$ ?				How much does it cost to rent a bicycle for 6 hours? 8 hours? 12 hours? $h$ hours?											

Figure 1: Two Patterning Tasks that Require the Same Structural Generalization

Three additional factors informed our decision to coordinate students' multiplicative-driven understanding and their PG (see Rivera (2013) for a detailed synthesis). First, students' verbal descriptions of generalizations in past studies were consistently categorized as nonfunction-based. Second, they usually had difficulty identifying what stayed the same and what changed in their constructed patterns. That even if some students were able to state interesting properties, however, the descriptions were too difficult to be converted algebraically. Third, given the sequential manner in which patterns were oftentimes presented to students, recursively additive generalizations (i.e.  $\text{Next} = \text{Current} + \text{Common Difference}$ ) were favored most of the time and that naïve induction approaches in cases of covariational-correspondence relations were most prevalent. Figures 2A and 2B illustrate the two "weak" pattern processing relative to the tasks shown in Figure 1. In the case of Figure 2A, research has shown that many students were unable to convert recursively additive expressions in function form. In the case of Figure 2B, research has indicated that many students were unable to justify naïve induction (or other pattern spotting) approaches despite their success in converting a structure in direct formula form. In multiplicative-driven responses on PG tasks, as exemplified in Figure 2C, students initially stipulate and justify an abducted (i.e. hypothesized) common unit that they then inductively test (i.e. verify) over several more cases and generalize to any stage in the pattern.

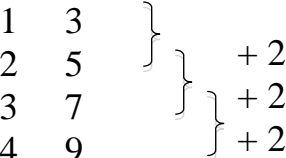
A. Recursively Additive Generalization	B. Naïve Induction Generalization	C. Multiplicative-Driven PG
 <p>“Add 2.”</p>	$1 \times 1 + 1 = 2$ . But I need 3. $1 \times 2 + 1 = 3$ . Then $2 \times 2 + 1 = 5$ . $3 \times 2 + 1 = 7$ and $4 \times 2 + 1 = 9$ . I think the formula is $S = n \times 2 + 1$ .	The L-shape pattern has two growing legs and a corner square. So: Stage 1 has 1 corner square and 2 groups of 1 square. Stage 2 has 1 corner square and 2 groups of 2 squares. Stage 3 has 1 corner square and 2 groups of 3 squares. Stage 4 has 1 corner square and 2 groups of 4 squares. Stage $n$ has $1 + 2n$ squares in all.

Figure 2: Three Generalization Processing Relative to the Figure 1 Task

### EMERGING PG PROCESSING OF GRADE 3 STUDENTS

We resolve our Research Question in three sections. In this section we present Year 2 pre-post results drawn from a two-year longitudinal study that started when the students were in Grade 2. For four consecutive weeks the second grade class of 21 students formally learned about the concept of multiplication through representations that conveyed the set and array models. Skip counting by 2, 5, and 10 was also a significant part of this unit. The class consisted of 14 males and 7 females; the majority were of Hispanic origins. In third grade, they continued to learn multiplication involving larger numbers, committed to memory the multiplication table from 1 to 10, multiplied up to 4 digits by a single factor, and worked through arithmetical problems whose products maxed to 100000. Figural patterning activity was not a stipulated recommended activity in the CA mathematics standards for second and third graders at the time of the study. Hence, in second grade the students pursued patterns as an enrichment activity for two weeks toward the end of the school year after state testing. In third grade, they dealt with patterns only during the two clinical interviews, which were the sources of the pre-post results shown in Table 1. The study utilized a pre-post repeated measure design. A pre-post instrument involving 3 figural tasks, see Figure 3, was administered during the individual clinical interviews. Pre-interviews were conducted in early October 2010 and post-interviews took place in mid-April 2011. The interview protocol was as follows: Before the first pattern was presented, the interviewer asked each student to recall how he or she expressed multiplicative expressions by going over a series of 3 visual tasks (e.g. Counting Stars Task in Figure 3). The student was then shown a copy of the pattern stages and asked to reconstruct them either by using the blocks on the table or by drawing them on paper. He or she then responded to the questions listed in Figure 3.

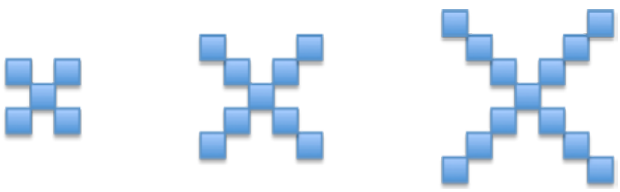



<p>Cross Pattern (CP)</p> 	<p>Two-Row Squares Pattern (TRP)</p> 
<p>Flower Pattern (FP)</p> 	<p>Counting Stars Task</p> 
<p>1. Show me how stage 4 might appear to you. Use either the blocks on the table or draw it on paper. 2. Show me stage 5. Use either the blocks on the table or draw it on paper. 3. A friend asks you to describe your pattern. What do you say? 4. Can you describe stage 10 for me? Stage 15? Stage 100? 5. Will you describe your pattern in terms of groups?</p>	

Figure 3 Tasks Used in the Grade 3 Pre-Post Assessment

For coding and analysis, verbal descriptions for each figural task were categorized along the following three levels: Additive and recursive (AR); Additive but composite-driven (Level I; AC); and Multiplicative (Level II; M). Numerical points were assigned, as follows: 1 point for AR; 2 points for AC; and 3 points for M. Totals were obtained for each student and then a paired t-test was conducted.

As indicated in Tables 1 and 2, extensive exposure to a multiplicative-driven mathematics curriculum had a significant positive effect on the third-grade students' ability to obtain direct formulas beyond additive recursive forms. The paired-samples t-test show that the scores were significantly higher on the posttest ( $M=7.3$ ,  $SD=2.0$ ) than on the pretest ( $M=4.6$ ,  $SD=1.7$ ;  $t(18)=4.6$ ,  $p < .001$ ,  $d=1.5$ ).

## EMERGING PG PROCESSING OF GRADE 8 STUDENTS

In this section we present two sets of data drawn from a three-year longitudinal study that started when the students were in Grade 6. For six consecutive weeks the sixth grade class of 29 students formally learned about figural PG using several units from the Mathematics in Context curriculum. While the students learned about integers and integer operations prior to learning figural PG, the number operations activity was implemented based on research implications then about the value of having students obtain a firm grasp of the arithmetical operations prior to any patterning activity. Also, patterning activity in sixth grade focused on the students' ability to visually infer structures on patterns since research implications then articulated the students' tendencies toward numerical- and recursively additive-based patterning at the expense of understanding and function-based formulas. The shift toward a multiplicative-thinking approach to PG took place when the students were already in eighth grade. In eighth grade multiplication was central to how the students learned Algebra 1 concepts.

	Pre	Post
Mean	4.58	7.32
SD	1.74	2.00
N	19	19
SEM	.40	.46

*Statistically significant; mean of pre minus post equals -2.74; 95% CI of this difference – from -3.98 to -1.49*

Pretest Frequencies					Posttest Frequencies				
	C P	TR P	F P	%		C P	TR P	F P	%
AR	10	12	9	.54	AR	2	6	5	.23
AC	7	6	9	.39	AC	5	5	7	.30
M	2	1	1	.07	M	12	8	7	.47

Table 1 Grade 2 Pre-Post Results

Table 2 Grade 2 Pre-Post Frequencies

The sixth grade students (mean age of 11 years; 12 males, 17 females; majority of Southeast Asian origins) participated in a pre-post assessment consisting of six tasks that were administered to them as homework problems over three consecutive days (2 tasks per day). The tasks were very similar to the ones shown in Figure 3, but they were simply asked to draw or explain. Figure 4 shows a representative sample pair (1 was figural, the other one was function-based). The design of the study utilized pre-post repeated measures. Each task was assigned the following points for formula construction: 0 for additively recursive expression or formula; and 1 point for an algebraically useful formula. The mode of justification for each task was also categorized separately as follows: recursive explanation (e.g. “I kept adding ...”); and structural explanation (i.e. based on the context of the problem). Totals were obtained for each student and then a paired t-test was conducted.

1. Consider the growing ladder below.

```

|-|  |-|  |-|
    |-|  |-|
        |-|

```

a. Fill in the table.

X	1	2	3	4
Y	3	?	9	?

b. How many sticks are needed to form a 5-step ladder?

c. If a ladder has  $n$  steps, how many sticks are there altogether? Explain your answer.

2. The data below shows the cost,  $C$ , in dollars of painting a wall and the number of cans of paint,  $n$ , needed to paint the wall.

$n$	1	2	3
$C$	42	49	56

a. How much does it cost to paint the wall with 7 cans?

b. How about  $n$  cans? How do you know? Explain your answer.

Figure 4 Pre-Post PG Tasks Administered to Grade 6 Students

As indicated in Table 3, the paired-samples t test indicate that the scores were significantly higher on the posttest ( $M=4.9$ ;  $SD=2.2$ ) than on the pretest ( $M=1.9$ ,  $SD=.8$ ;  $t(28)=7.2$ ,  $p < .001$ ,  $d = 1.9$ ), which meant that the students successfully expressed their structural generalizations in the posttest in algebraically useful terms. However, as shown in Table 4, the mode of justification remained numerical and very few employed structural understanding based on how a direct formula might make sense by inferring them on the stages. A recursive justification meant that a student

simply added or subtracted until the correct value emerged, while a mere appearance match justification meant that a student simply substituted an input value to a constructed direct formula as a way of checking whether the formula was correct (e.g. “My direct formula for Figure 3 item 1 was  $y = 3x$ . I plugged in 2 for  $x$  and the result was  $3 \times 2 = 6$ , which had the same value of  $y$  on the table, so my formula made sense to me”).

	Pre	Post
Mean	1.86	4.93
SD	.83	2.17
N	29	29
SEM	.15	.40

*Statistically significant; mean of pre minus post equals -3.07; 95% CI of this difference – from -3.94 to -2.20*

Pretest Means		Posttest Means		
Recursive	Structural	Recursive	Mere Appearance Match	Structural
5.3	0.7	1.2	3.6	1.2

Table 3 Construction Pre-Post Results

Table 4 Justification Pre-Post Results

In seventh grade, the students who continued to participate in the Year 2 study exhibited the same findings that we obtained on the posttest of the preceding year. Consequently, all their constructed direct formulas relative to linear figural PG took the constructive standard form  $y = mx + b$  that were mostly justified by a mere appearance match. In eighth grade, when the students acquired a multiplicative understanding of algebraic concepts, the nature of their PG forms also changed. Table 5 shows the percentages of correct PG by type over three years based on the post clinical interviews, which also indicates the different types of multiplicative-driven direct formulas that they generated. Figure 5 illustrates these types in relation to the Cross Pattern task shown in Figure 3. A deconstructive PG involves figural processing via decomposition with overlapping parts, which explains the converted formula consisting of multiplication and subtraction. An auxiliary-driven PG involves figurally seeing wholes first by adding auxiliary units (small squares), which explains the converted form involving the operations of multiplication and subtraction. A constructive nonstandard PG is an additively composite relation that simplifies into a constructive standard PG. A transformation-based PG involves the mereological process of reorganizing a figure into something that is familiar. In Table 5, we see that some PG processing types are not as frequently used as others.

## PG PROCESSING OF GRADES 3 AND 8 STUDENTS

The results in the preceding two sections indicate similarities in the PG processing of younger and older students before and after a long-term exposure to a multiplicative-driven mathematics curriculum. Before formal instruction, their initial formulas were recursive in form and perhaps this is due to either their prior experiences or the sequential format of the tasks. Interestingly enough, both groups successfully transferred their understanding of multiplication to PG contexts that enabled them to

construct and justify different types of direct formulas that were mostly constructive standard perhaps due to the limited nature of patterns that they dealt with (i.e. linear patterns). The transfer, in fact, occurred with very minimal formal training. Further, like their older counterparts, the younger group produced a significant number of additively composite (i.e. constructive nonstandard; early multiplicative) generalizations (see Table 2 pre-post frequencies).

Year	Recursive	Constructive Standard	Constructive Nonstandard	Deconstructive	Auxiliary-Based	Transformation-Based
Grade 6 (n=29)		100%				
Grade 7 (n = 8)		100%		100%		
Grade 8 (n = 14)		100%	36%	86%	21%	14%

Table 5 Percentages of Correct PG by Type Over 3 Years Based on the Post Interviews

<i>Constructive Standard</i> $S = 1 + 4n$ 1 middle square plus 4 groups of $n$ squares	<i>Constructive Nonstandard</i> $S = 1 + n + n + n + n$ 1 middle square plus $n$ squares on each of 4 sides	<i>Transformation-Based</i> Move the squares so that you see 4 rows of $n$ plus an extra square, so $S = 4n + 1$ .
<i>Deconstructive</i> $S = 2(2n + 1) - 1$ 2 diagonal sides each of length $2n + 1$ minus a middle square that has been counted twice	<i>Auxiliary Driven</i> $S = (2n+1)^2 - 4n^2$ Larger square is $(2n+1)^2$ , take away the added small squares consisting of 4 groups of $n^2$	

Figure 5 Different Multiplicative-Based Structures for the CP in Figure 3

One remarkable difference between the two groups is the algebraic nature of their PG. Results of the post clinical interviews indicate that the third grade group consistently employed inductive generalizing across task, unlike the eighth grade group that consistently exhibited deductive-driven generalizing, which they already started to manifest, in fact, in sixth grade on the basis of the post clinical interviews. Below are two representative post-clinical interview responses from Rudy (Grade 3) and Tamara (Grade 6) in relation to the Cross Pattern task shown in Figure 3. Tamara in lines 1-3 below ignored the standard step-by-step questions and proceeded immediately to establish a variable-based generalization that she then used to deal with any near and far generalization tasks.

- 1 For stage 1 there's 5, then 9, 13, 17. Then 1, 2, 3, 4 [stage numbers]. So there's 4 in
- 2 between. So 1 times 4 + 1 [for stage 1], 2 times 4 plus 1.... Yeah, it would be  $S = n$  times
- 3 4 plus 1.

Rudy (R) initially drew stages 4 and 5. He then claimed that “the rule is adding 4.” When the interviewer (I) asked him to obtain an alternative “expression involving

*multiplication and/or multiplication that does not involve adding 4,” he focused on stage 5 first and saw that it had “5 on each side.” When asked to further describe stages 10, 15, and 100, he said: “[For stage 15], it’s  $15 + 1$ , then 15 on [each remaining side]. ... [For stage 100, it’s]  $100 \times 4 + 1$ .”*

4 *I: So what does that mean? What does each number mean?*

5 *R: This one [4] is for the sides, this one’s [100] for the number on each side, plus the*  
6 *one square in the middle.*

## DISCUSSION AND CONCLUSION

Results of the two longitudinal studies conducted with the two groups of learners yield two insights that were not evident and clearly articulated in earlier reported studies on pattern thinking. *First*, we offered sufficient empirical evidence of the strength of coupling PG processing and multiplicative thinking together. Since multiplication as a concept fundamentally draws on common unit understanding (i.e. thinking in (equal) groups of some common unit), the same cognitive action applies to objects in patterning contexts. Across the two groups, we saw that students who held a firm understanding of multiplication overcame the gap between representational (figural) processing and (visual-to-algebraic) conversion. *Second*, due to the rather large samples and the longitudinal nature of the interventions, we also managed to capture stable PG actions over time that enabled us to infer similarities and differences in the two groups’ generalizing performance on linear patterns. The differences, especially, should help us further characterize the nature of algebraic content that younger children are capable of exhibiting over time perhaps not in terms of early algebra versus something else but in terms of nonsymbolic and symbolic algebra. In closing we recommend that studies are necessary that can determine how early and what factors in elementary students’ learning experiences might they be able to process structural generalizations in a deductive context.

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# KNOWLEDGE FOR TEACHING MATHEMATICS WITH TECHNOLOGY – A NEW FRAMEWORK OF TEACHER KNOWLEDGE

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*Knowledge for Teaching Mathematics with Technology (KTMT) is a theoretical model that seeks to articulate previously existing models on professional knowledge and the conclusions that the investigation around the integration of technology has achieved. KTMT is a dynamic knowledge, informed by the practice, that develops from the knowledge on the base domains (Mathematics, Teaching and Learning, Technology and Curriculum), evolving as knowledge in the base domains interacts and as this promotes the development of inter-domain knowledge, which continue to interact, strengthening relations and leading to the development of an integrated knowledge, where knowledge on the base domains and on the two sets of inter-domains appears deeply integrated into a global knowledge.*

## INTRODUCTION

The difficulties documented in research with respect to the problems that teachers face when trying to integrate technology into their practice (Goos & Bennison, 2008; Rocha, 2008) and different gaps in teacher knowledge that seem to be associated with such difficulties (Doerr & Zangor, 2000; Guerrero, 2005), highlight the importance of a specific attention to technology and to the professional knowledge necessary for its proper integration in practice.

In this paper I introduce a new conceptualization of professional knowledge where technology is an important element. I start from an analysis of the existing models of knowledge and I try to characterize each of the proposed domains of knowledge. I conclude with an analysis of the differences and agreements of this model with respect to others and with the presentation of some examples intended to illustrate the three stages of knowledge intrinsic to the model.

## THE FRAMEWORK

This is a model that arises in the course of an investigation about teachers' integration of graphing calculators into their professional practice. It is essentially a theoretical model, that finds its genesis on my belief that we need a conceptual framework that allows integration and coordination between the characterizations that have been developed around the professional knowledge and the conclusions reached by the investigation regarding the integration of technology. In general, these are two separate research areas and, in my opinion, this is one of the main reasons for the difficulties concerning the technology integration widely documented in the literature (e.g., Goos & Bennison, 2008).



**Base knowledge domains**

Shulman's work (1986) and in particular its notion of Pedagogical Content Knowledge (PCK) constituted a starting point for a whole set of investigations that have sought to characterize the teacher's professional knowledge. In recent years many authors dedicated themselves to this theme, trying to clarify concepts and to propose new characterizations. A look at those characterizations allows the identification of a set of domains that, with greater or lesser emphasis, appear to be consensual.

Shulman (1986) proposes seven knowledge domains but highlights two of them: one related to content knowledge and other related to pedagogical content knowledge. Fennema and Franke (1992) value the concept of PCK and highlight the importance of taking into account the influence of teacher beliefs, arguing that knowledge for teaching includes four components: Knowledge of Content, Knowledge of Pedagogy, Knowledge of Students' Cognition, and Teachers' Beliefs. Ball, Thames and Phelps (2008) build on Shulman's framework, attempting to clarify the distinction between Content Knowledge and PCK. The authors propose a model dividing each domain into three categories where Mathematics, Students, Teaching, and Curriculum are taken into account. Angeli and Valanides (2009) emphasized the importance of taking technology into account and propose five knowledge domains: Learners, Pedagogy, Technology, Content, and Context. Recently, on a synthesis of models on teacher mathematical knowledge, Petrou and Goulding (2011) consider three domains: Subject-Matter Knowledge, Curriculum Knowledge, and Pedagogical Content Knowledge (where the authors include a reference to Knowledge of Teaching and Students). Besides that, the authors emphasize the teacher's context, assuming that "the context in which the teachers work is the structure that defines the components of knowledge central to mathematics teaching" (p. 21).

A global analysis of all these characterizations suggests that the main domains of professional knowledge are mathematics, teaching and learning and curriculum. Being still important to consider the influence of teacher beliefs, as well as the influence of the context in which the teacher works. In this sense, the base knowledge domains of the Knowledge for Teaching Mathematics with Technology (KTMT) are Mathematics, Teaching and Learning, Curriculum and, of course, Technology.

Knowledge of Curriculum encompasses the knowledge and understanding of the teacher in relation to the purposes and objectives of mathematics teaching and also in relation to program guidelines, taking into account the contents but also the methodologies, types of tasks, materials and students' assessment guidelines. The curriculum is, however, conceptualized in a transversal way, and therefore influential over all other domains. It is the curriculum that determines what content is taught and, to a large extent, the sequence of that teaching. Influence thus, for example, the mathematical knowledge that a student can mobilize during the resolution of a certain question. It is also the curriculum that characterizes the aims and objectives of mathematics, as well as the methodological options, and then influences the options in terms of teaching and learning, valuing certain approaches over others. Similarly, it is

the curriculum that advocates the technologies to be used, promoting, or not, a more comprehensive use. All these influences are felt within the context where the teacher works and within their beliefs and conceptions, which stand as a backdrop and a filter for all the action and inherent decisions.

The Mathematics base domain includes knowledge of concepts and theories and procedures of the disciplinary area, also involves knowledge of the structure that organizes and connects ideas within the area, knowledge of rules of evidence and knowledge of the nature of the mathematics. It includes not only knowledge of the truths accepted in the knowledge area, but also knowledge of how to explain why a certain result or assertion is valid, why it is worth knowing it and how it relates to other outcomes within and outside the area of knowledge.

The Teaching and Learning base knowledge involves knowledge about how students think and learn and, in particular, how this happens in the case of specific content. Includes an understanding of the processes commonly used by students, the difficulties usually associated with a specific content and the ability to anticipate these problems and deal with them. This knowledge is thus directly related to knowledge concerning teaching, including the choices made by the teacher in the different phases of teaching (planning, implementation and evaluation) and the inherent knowledge about aspects such as the sequence of activities, types of tasks, classroom structure and how students work.

The technology base knowledge involves the capacity to operate with certain technology and consists essentially in knowing how it works or, in other words, what it does and how it does (from an operational point of view).

### **Inter-domains knowledge**

In addition to knowledge base domains, this model particularly values knowledge developed at the confluence of more than one domain. This importance given to a knowledge that goes beyond a particular domain is somehow recognized by Hill and Ball (2009), which consider in their model, for instance, knowledge such as Knowledge of Content and Students. Koehler and Mishra (2006) adopt a similar view in their model, considering also knowledge domains which consist of the mutual influence among other domains. But the first author who actually considers more than some influence among domains of knowledge, understanding this "influence" as a new domain of knowledge that is added to the initial ones is Shulman (1986). Here I adopt a similar perspective. I consider two sets of knowledge, that I call inter-domain knowledge: the Mathematics and Technology Knowledge (MTK), and the Teaching and Learning and Technology Knowledge (TLTK). The first focuses on how technology influences mathematics, enhancing or constraining certain aspects. Similarly, the second focuses on how technology affects the teaching and learning process, enhancing or constraining certain approaches. In both cases the curriculum is considered to be a transversal influence, always present.

Like Rowland, Huckstep and Thwaites (2005), in this model, rather than make a characterization of teacher professional knowledge, I seek a conceptualization that can

not only contribute to the understanding of the teacher's knowledge, but ultimately provide clues to this knowledge development. Since this model somehow finds its genesis in the problems identified in technology integration, it seemed appropriate to consider the key points that research has highlighted in relation to these issues. An analysis of the existing literature (e.g., Cavanagh & Mitchelmore, 2003; Doerr & Zangor, 2000; Dunham, 2000; Farrel, 1996; Ferrara, Pratt & Robutti, 2006; Goos & Bennison, 2008; Heid & Blume, 2008; Hoyles & Lagrange, 2010; Laborde, 2001; Penglase & Arnold, 1996; Zbiek et al., 2007) suggests a body of knowledge associated with the use of technology that goes beyond the knowledge in each base domain, and that seems to be required for the development of KTMT. Thus, the first inter-domain knowledge, MTK, necessarily includes:

- Knowledge of technology's mathematics fidelity, i.e., knowledge of the level of agreement between the results of the Mathematics and the results of the mathematics of the technology;
- Knowledge of the new emphasis that technology puts on the mathematical content (e.g., more intuitive approaches encouraging or requiring a different domain of the influence of the values represented in the coordinate axes on the shape of the displayed graph);
- Knowledge of new sequences of content;
- Representational fluency, involving knowledge of different representations, of how to relate and move between them, and of the contribution that each representation can bring to better illustrate or justify claims.

The second inter-domain knowledge, TLTK, necessarily includes:

- Knowledge of new issues that technology requires students to deal, including the difficulties they face when using technology and that arise from such use;
- Knowledge of mathematical concordance of the proposed tasks, i.e., the alignment between the mathematics the teacher intended the students to work on and the mathematics the students effectively worked;
- Knowledge of the potential of technology to the teaching and learning of mathematics, including knowledge of different types of work and teacher roles that technology becomes possible, knowledge of ways of articulating them and knowledge of the contribution they can bring to mathematics learning.

### **Integrated Knowledge**

Integrated Knowledge is a knowledge held by the teacher that articulates simultaneously the knowledge of each of the base domains and the two sets of inter-domain knowledge. It is a knowledge that develops from the interactions between all domains and is characterized by his global and comprehensive nature but, at the same time, by his particularity, in the sense that it is that knowledge that maximizes the specific potential of technology to provide better mathematics learning. It is this knowledge that is the true essence of KTMT.

## **Knowledge for Teaching Mathematics with Technology**

KTMT is established as the knowledge that encompasses the knowledge of each of the domains (base, inter-domains and/or integrated) possessed by the teacher. It is knowledge with a dynamic and interactive nature, as advocated by Fennema and Franke (1992), which reveals itself in practice, but is also informed by it. One feature that other authors, such as Grossman (1995) and Ma (2009), also ascribe to professional knowledge.

The structure of KTMT model has implicit three stages, one centered in the base knowledge domains, another in inter-domain knowledge arising from the integration of the base knowledge, and another in the integrated knowledge resulting from the integration of all knowledge domains (base and inter-domains). This structure encompasses a categorization of the knowledge held by the teacher. A teacher that on his practice rests only on base knowledge domains, has a KTMT which is still in an early development stage; a teacher who already has some inter-domain knowledge, is at an intermediate stage of his KTMT development; and a teacher who has integrated knowledge, articulating the different base knowledge and inter-domains knowledge, has a KTMT already at an advanced stage of development.

It is this way of conceptualizing knowledge, based on a pyramidal structure that deepens and develops progressively from bottom to top, that is a major difference of this model compared to others. Another major difference relates to the connection in one single model of the contributions from research conducted around the professional knowledge and the research on technology integration and the inherent problems. It is the simultaneous connection of all these aspects that turns the KTMT model into a characterization of professional knowledge from which it is possible to consider not only the teacher's knowledge but also his development, fostering an effective integration of technology by meeting the specific needs and characteristics of each teacher. In fact, this is a model with a close link to professional practice, like the one of Ball, Thames and Phelps (2008), but also with a strong appreciation of the contribution it can make to this practice, by developing the teacher professional knowledge, as advocated by Rowland, Huckstep and Thwaites (2005). This model makes possible to characterize the teacher's knowledge and consider the best training that will enhance his development, in agreement with Kissane (2003). This author points to the mismatch between the training available and the needs of the teacher as one of the main causes for the problems surrounding the integration of technology. According to his perspective, it would be useless, for example, to provide training focused on the integration of inter-domain knowledge to a teacher who does not have that knowledge or that is still developing his base knowledge about technology.

### **SOME EXAMPLES**

A teacher proposes to his students a task where the emphasis is placed on the use of a graphing calculator to plot a graph of a function on the standard window, and then on the use of the machine to prepare a table, putting the focus on technology and what it can do and not on the learning of mathematics. This may be a teacher who has not yet

developed inter-domain knowledge, lacking the knowledge to consider the implications of this technology on teaching and learning and on Mathematics. That is, it may be a teacher whose KTMT is still at an early stage of development.

It is however important to note that this is a model designed for a global analysis of teacher practice and not for an analysis based on a single task and independent of the reasons underlying the options assumed by the teacher. That is why I say the teacher might be at an early stage of development of his KTMT. Indeed, a teacher may develop a task as described above for students who have never used the calculator, intending to show them how to graph a function and draw a table with this technology, and this does not mean that the teacher cannot be on an advanced stage of development of his KTMT. Nevertheless, a teacher who usually uses the calculator as described above, is at an early stage of development of his KTMT.

A teacher proposes to his students an investigation task on a family of functions using a graphing calculator but, when the students show some difficulties in managing the work on an open task, he decides to truncate the exploratory nature of the task, directing their work in order to ensure that they achieve all the intended conclusions. This teacher recognizes the contribution that technology can bring to the implementation of certain kinds of tasks but, when he thinks that the mathematical learning he had in mind can be compromised, he chooses to focus on the mathematical content and abandons his initial intention. This may be a teacher who already has some inter-domain knowledge but that still has difficulty in simultaneously articulate his knowledge in all the domains. That is, it may be a teacher whose KTMT is still at an intermediate stage of development.

A teacher proposes to his students an investigation task using a graphing calculator. The teacher discusses the number of examples that the students should consider in the development of their conjecture, analysing the difference between conjecture and demonstration, highlighting the role of the latter in Mathematics and eventually asking for the demonstration of the conjecture formulated by the students. This teacher is not only considering the potential of this technology to perform certain types of work, but also considering the implications of this on the core aspects of Mathematics. This may be a teacher who not only possesses knowledge on base domains and inter-domains, but who can also articulate this knowledge, having already an integrated knowledge. That is, it may be a teacher whose KTMT is already at an advanced stage of development.

## **CONCLUSION**

KTMT is a conceptualization of the knowledge needed for teaching with technology. It allows a characterization of the knowledge held by the teacher and the subsequent identification of key aspects to be taken into account when considering the professional development of that teacher. But this model is more than that. Constitutes the basis for the development of a unifying theory, incorporating in an articulated and connected manner the results from the studies conducted around the technology and its integration. And this unifying theoretical framework is essential for the full

understanding of what is at stake when we think the integration of technology in mathematics education. It is therefore important to refine the model, detailing in particular the characterization of inter-domain knowledge, and analysing the contribution it can bring to the understanding and development of teacher's professional knowledge and to the evolution of how technology is integrated by teachers into their practice.

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# COMPARISON OF EXPERT AND NOVICE PROBLEM SOLVING AT GRADES FIVE AND SIX

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*Comparing expert and novice performance is a fruitful approach in research on mathematical problem solving, but is there something like expertise in children aged 10 – 12? The problem solving processes of 10 “pupil experts” (successful participants of mathematical competitions) are compared to those of 45 “novices” (regular pupils). The “experts” show superior performance in all of the three tasks that were chosen for this study as well as higher “mental flexibility” in executing the processes which seems to distinguish them from the novices beyond advance in practice.*

## BACKGROUND

**Problem Solving:** As an important part of mathematics, problem solving is fundamental for school mathematics (cf. Schoenfeld 1992, p. 334 ff.). The terms “problem” and “problem solving” have differing meanings ranging from working routine tasks to solving perplexing or difficult situations (ibid., p. 337 ff.) of which I refer to the latter interpretation as in the following definition:

“When you are faced with a problem and you are not aware of any obvious solution method, you must engage in a form of cognitive processing called problem solving. Problem solving is cognitive processing directed at achieving a goal when no solution method is obvious to the problem solver [...]” (Mayer & Wittrock 2006, p. 287)

It is important to note that the attribute “problem” does not depend on the task itself but on the solver. A perplexing situation (sensu Schoenfeld) for one pupil or student can be a routine task for another (e.g., more experienced) one. Thus, research on problem solving should focus on processes.

Important factors in such processes are *control* and *heuristics* (resources and beliefs being other factors, cf. Schoenfeld 1985, p. 44 f.). The term “control” refers to “the question of resource management and allocation [...] [specifically] major decisions regarding planning, monitoring, and assessing solutions on-line” (ibid.). Whereas “heuristics” are “rules of thumb for effective problem solving” (ibid.) or “methods and rules of discovery and invention” (Pólya 1945, p. 112) like *working backward* or *looking for a related problem*. Bruder and Collet (2011, p. 34 ff.)<sup>1</sup> claim that successful problem solvers often show intuition and mental flexibility and that a lack of such flexibility can be compensated (up to a certain degree) by learning heuristics.

Research results show that missing control leads to failure in problem solving attempts. Schoenfeld (1985, ch. 9), for example, shows in a study with more than 100 students, that successful problem processes contained a significant amount of self-regulation,

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<sup>1</sup> See Bruder (2003) for an English version: <http://www.math-learning.com/files/learnmethod.pdf> (12.12.2012).



whereas the majority of the unsuccessful processes covered almost no planning or regulation activities (so-called “wild goose chase” processes). See Schoenfeld (1992, p. 354 ff.) for a review of control related research. Schoenfeld (ibid.) also summarizes several studies indicating that “students’ use of heuristic strategies was positively correlated with performance on ability tests, and on specially constructed problem solving tests; however, the effects were relatively small.” (ibid., p. 352 ff.)

**Expertise Research:** The comparison of experts and novices is a research approach with the goals of understanding “how [experts] perform in their domain of expertise” (Chi 2006, p. 21) and “how experts became that way so that others can learn to become more skilled and knowledgeable.” (ibid., p. 23) There are several criteria to consider persons as experts, the two most frequently used are nominal (academic titles, awards, ...) or performance-based, of which the latter one implies stable or reproducible (rather than occasional) superior performance by the expert in his or her domain of expertise. Analyses of expertise have proved themselves to be fruitful for research on problem solving, often professors of mathematics are used as experts (e.g., Schoenfeld 1985; Silver & Metzger 1989).

Results in expertise research, which are surprisingly constant for several domains, show that (among others) experts (a) excel mainly in their own domains, (b) perceive large meaningful patterns (“chunks”) in their domain, (c) are faster than novices at solving problems in their domain and do so with little error, (d) spend more time analyzing and planning problems than novices, and (e) have more accurate monitoring skills (cf. Glaser & Chi 1988; Chi 2006). Studies on the acquisition of expertise identified a “10-year rule” for several domains as the required period of intense preparation to reach expert performance, see Ericsson, Krampe, and Tesch-Römer (1993) for a review of related research. Ericsson et al. (ibid.) present two studies for the acquisition of expertise with violinists and pianists and introduce an according framework. They argue that – with few exceptions like height in sports – the differences in performance of experts and novices are not immutable or genetically prescribed. Instead, those differences result in a life-long period of deliberate practice, which is described as personalized training and activities that maximize improvement in contrast to playful interaction, or paid work in the domain (cf. ibid., p. 368).

**Research Questions:** A research gap is the development and performance of younger children – in problem solving (cf. Heinze 2007) as well as in expertise (cf. Ericsson et al. 1993; Ericsson 2006). Because of the “10-year rule” it cannot be expected to find “real experts” in younger children, but there might be equivalents.

- Is there something like expertise in children at the age of 10?
- Granted that the first question can be answered positively, what can we learn from or about those “student experts”?

## DESIGN OF THE STUDY

To explore pupils’ problem solving processes, we videotaped fifth and sixth graders (aged 10 to 12) working on mathematical problems. The pupils did so without interruptions or hints, because we wanted to study their uninfluenced problem solving

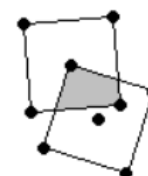
attempts. We decided not to use an interview or a think-aloud method, so as not to interrupt the students' mental processes. To get an insight into their thoughts, we let the children work in pairs to interpret their communication in addition to their actions.<sup>2</sup> Three tasks were selected for the comparison presented in this paper:

### Beverage Coasters

The two pictured squares depict coasters. They are placed so, that the corner of one coaster lies in the center of the other.

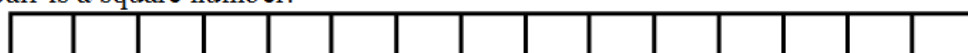
Examine the size of the area covered by **both** coasters.

[Idea: Schoenfeld (1985, p. 77): *Mathematical Problem Solving*. Orlando, Florida, Acad. Press.]



### Marco's Number Series

Marco wants to arrange the numbers from 1 to 15 into the caskets so that the sum of every adjoining pair is a square number:



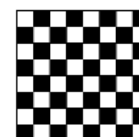
For instance, if there are the numbers 10, 6, 3 in three consecutive caskets, the 6 adds up to a square number with its left ( $10+6=16$ ) and its right neighbor ( $6+3=9$ ).

How could Marco fill-up his 15 caskets?

[Source: F rther Mathematikolympiade, 2005/06, 1. round (www.fuemo.de)]

### Squares on a Chessboard

Peter loves playing chess. He likes playing chess so much that he keeps thinking about it even when he isn't playing. Recently he asked himself how many squares there are on a chessboard. Try to answer Peter's question!



[Idea: Mason, Burton, & Stacey (2010, p. 17): *Thinking Mathematically*. Dorchester, Pearson, 2. ed.]

**“Novices”:** The so-called novices were pupils from secondary schools in Hanover that took part in our support and research program MALU<sup>3</sup> for fifth graders, which started in November 2008. These pupils came to our university once a week for 1.5 hours and worked on problems for about half of this time (see Rott 2011; 2012 for details).

**“Experts”:** The so-called pupil experts were successful participants of mathematical competitions, namely of the final round of the German Mathematical Olympiad<sup>4</sup> in 2009/10 (8 pupils of grade 5 and 6) and price winners of the Mathematical Kangaroo<sup>5</sup> in 2009 (2 pupils of grade 6). We visited the venue of the Olympiad and the school of the Kangaroo winners and asked the pupils to take part in our problem solving activity; the pupils did so voluntarily and have not worked with their video-partners beforehand.

## METHODOLOGY

**Product Coding:** The pupils' products were graded in four categories of success: (1) *No access*, when they showed no signs of understanding the task properly or did not

<sup>2</sup> After comparing several methods of expertise research – especially the often used biographic interview – Ericsson (2006, p. 237) concludes: „Protocol analysis of thoughts verbalized during experts' superior performance on representative tasks offers an alternative to the problematic methods of direct questioning and introspection.“

<sup>3</sup> Mathematik AG an der Leibniz Universit t which means Mathematics Working Group at Leibniz University.

<sup>4</sup> The German Mathematical Olympiad consists of four rounds: (1) tasks to be solved at home to qualify for (2) a written tests at schools (180 minutes for grades 5 and 6). The best 200 students of all grades qualify for (3) the final round of the federal state which takes place at a central place. And the 12 winners of those state finals are invited to the final round of all German states – but this round is only for students of grade 8 and higher.

<sup>5</sup> An international competition which consists of 30 tasks (75 minutes); 5 to 6 % of the German participants receive prices.

work on it meaningfully. (2) *Basic access*, when the pupils mainly understood the problem and showed basic approach. (3) *Advanced access*, when they understood the problem properly and solved it for the most part. And (4) *full access*, when the pupils solved the task properly and presented appropriate reasons, if necessary.

This grading system was customized for each task with examples for each category. Then, all the products were rated independently by the author and a research assistant. After calculating Cohen's kappa ( $\kappa > 0.85$  for each task), we discussed and recoded the few products with differing ratings, reaching consensus every time.

**Process Coding – Episodes:** The pupils' processes were parsed into episodes using an adapted version of the framework for the analysis of videotaped problem solving sessions by Schoenfeld (1985, ch. 9). An episode is "a period of time during which [...] a problem-solving group is engaged in one large task [...] or a closely related body of tasks in the service of the same goal" (ibid., p. 292). Schoenfeld introduces six types, namely *Reading*, *Analysis*, *Exploration*, *Planning*, *Implementation*, and *Verification*, which resemble Pólya's (1945) phases of the problem solving process (cf. Rott 2011).

All videos were coded independently by three researchers and interrater-reliability was computed as described in the TIMSS 1999 video study by applying the "percentage of agreement" approach<sup>6</sup> for determining the starting and ending points of episodes ( $P_A > 0.7$ ) and for labelling those episodes ( $P_A > 0.85$ ) into the types. Afterward, all differing codes were recoded together (*consensual validation*).

**Process Coding – Heuristics:** Appearances of heuristic techniques (like *drawing a figure* or *examining special cases*) were coded using a manual that was designed on the basis of a framework by Koichu, Berman, and Moore (2007) (cf. Rott 2012 for details).

Similar to the episodes, the videos were coded independently by several researchers with interrater-reliability  $P_A > 0.7$  for identifying points of time in the processes with heuristics and  $P_A > 0.85$  for characterizing the heuristics. After calculating the reliability, we attained agreement by recoding together all differing codes.

## RESULTS

Preliminary note 1: To forestall an expected result, the experts were significantly more successful than the novices. To be better able to interpret the results, I did not only look at the totals of both groups but also at equally successful subgroups by comparing arithmetical means of respective product categories (see Table 1).

Preliminary note 2: Ten of the 19 novices that worked the "Squares on a Chessboard" task misinterpreted its formulation and answered "64 squares" within less than 3 minutes (no expert did). So to say, these pupils worked another task – a routine task – that is solved quickly and without the need to use heuristics. To take this into account,

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<sup>6</sup> Chance-corrected measures like Cohen's kappa are not suitable for this calculation, as there is no model to calculate the agreement by chance for a random number of heuristics distributed randomly over the course of the process.

I additionally calculated group totals for the “Chessboard” task after excluding these 10 processes; these results are marked with an asterisk (\*) in the text and in the tables.

**Statistics:** Table 1 summarizes statistical data from both samples. The column “count (%)” shows the number of pupils for each of the four product categories. It can easily be seen in the totals of column “succ.” (success) that the expert group was more successful in each of the three tasks; all these differences are statistically significant.<sup>7</sup> Column “time” shows the mean amount of time (in decimals) required by the pupils. The experts were faster in total and in every product category (except “Coasters”, cat. 4); these differences are significant for the “Number Series” and “Chess\*” tasks. In column “heu” are the mean numbers of heuristics coded in the processes for each product category and in total. The novices use more heuristics compared to equally successful experts (significant only for “Chess\*”). For the novices, there is a medium correlation between the use of heuristics and success (see Rott 2012 for details).

NOVICES

Beverage Coasters				Marco's Number Series				Squares on a Chessboard									
	succ.	time	heu	count (%)		succ.	time	heu	count (%)		succ.	time	heu	count (%)			
$\bar{x}_{cat1}$	=	1.00	17.89	1.64	<b>14 (43.8)</b>	$\bar{x}_{cat1}$	=	1.00	16.18	0.14	<b>7 (21.9)</b>	$\bar{x}_{cat1}$	=	1.00	2.73	0.00	<b>10 (52.6)</b>
$\bar{x}_{cat2}$	=	2.00	16.53	2.89	<b>9 (28.1)</b>	$\bar{x}_{cat2}$	=	2.00	17.01	0.60	<b>5 (15.6)</b>	$\bar{x}_{cat2}$	=	2.00	9.75	1.50	<b>4 (21.1)</b>
$\bar{x}_{cat3}$	=	3.00	9.14	4.00	<b>6 (18.8)</b>	$\bar{x}_{cat3}$	=	3.00	32.89	2.69	<b>13 (40.6)</b>	$\bar{x}_{cat3}$	=	3.00	19.38	2.80	<b>5 (26.3)</b>
$\bar{x}_{cat4}$	=	4.00	5.92	4.00	<b>3 (9.4)</b>	$\bar{x}_{cat4}$	=	4.00	23.38	2.43	<b>7 (21.9)</b>	$\bar{x}_{cat4}$	=	---	---	---	<b>0 (0.0)</b>
$\bar{x}_{tot}$	=	1.91	14.74	2.66	<b>32 (100)</b>	$\bar{x}_{tot}$	=	2.63	24.68	1.75	<b>32 (100)</b>	$\bar{x}_{tot}$	=	1.74	8.59	1.05	<b>19 (100)</b>
$\sigma_{tot}$	=	1.01	12.41	1.77		$\sigma_{tot}$	=	1.07	12.40	1.63		$\sigma_{tot}$	=	0.87	7.71	1.31	
												$\bar{x}_{tot}^*$	=	2.56	15.09	2.22	<b>9 (47.4)</b>
												$\sigma_{tot}^*$	=	0.53	6.41	0.97	

EXPERTS

Beverage Coasters				Marco's Number Series				Squares on a Chessboard									
	succ.	time	heu	count (%)		succ.	time	heu	count (%)		succ.	time	heu	count (%)			
Cat. 1/2	---	---	---	<b>0 (0.0)</b>	Cat. 1/2	---	---	---	<b>0 (0.0)</b>	Cat. 1/2	---	---	---	<b>0 (0.0)</b>			
$\bar{x}_{cat3}$	=	3.00	6.95	2.75	<b>8 (80.0)</b>	$\bar{x}_{cat3}$	=	---	---	---	<b>0 (0.0)</b>	$\bar{x}_{cat3}$	=	3.00	5.33	1.00	<b>2 (25.0)</b>
$\bar{x}_{cat4}$	=	4.00	9.50	3.00	<b>2 (20.0)</b>	$\bar{x}_{cat4}$	=	4.00	11.92	1.70	<b>10 (100)</b>	$\bar{x}_{cat4}$	=	4.00	6.00	1.17	<b>6 (75.0)</b>
$\bar{x}_{tot}$	=	3.20	7.46	2.80	<b>10 (100)</b>	$\bar{x}_{tot}$	=	4.00	11.92	1.70	<b>10 (100)</b>	$\bar{x}_{tot}$	=	3.75	5.83	1.13	<b>8 (100)</b>
$\sigma_{tot}$	=	0.42	3.41	1.69		$\sigma_{tot}$	=	0.00	5.63	1.25		$\sigma_{tot}$	=	0.46	2.74	1.25	

Table 1: Statistical data of the processes for both groups.

Table 2 shows the number of “wild goose chase” processes (which are processes that consist – besides *Reading* – only of *Exploration* or *Analysis* and *Exploration* episodes; cf. Rott 2011) – as a sign of missing process regulation – which is strongly correlated to products of categories 1 and 2 for the novices (cf. Rott 2011). Table 2 also shows the number of processes with *Planning* episodes for each product category as well as miscellaneous process which are neither “wild goose chases” nor contain *Planning*. The novices used “wild goose chases” in 32 of 83 processes (39 %) whereas the experts showed this kind of behaviour in only 4 of 28 processes (14 %). Planning activities were coded in 41 of 83 novice processes (49 %) whereas 24 of 28 expert processes (86 %, all but the 4 “wild goose chases”) planned their problem solving attempts.

<sup>7</sup> Because of the small sample size, the non-parametric Mann-Whitney-U-test was used for all the tests of difference between totals (two-sided test, 5 % significance level).

EXPERTS NOVICES	Beverage Coasters					Marco's Number Series					Squares on a Chessboard				
		WildG	Plan	Misc.	total		WildG	Plan	Misc.	total		WildG	Plan	Misc.	total
	Cat. 1	13	1	0	14	Cat. 1	5	0	2	7	Cat. 1	0	10	0	10
	Cat. 2	8	1	0	9	Cat. 2	1	2	2	5	Cat. 2	0	4	0	4
	Cat. 3	4	2	0	6	Cat. 3	1	11	1	13	Cat. 3	0	2	3	5
	Cat. 4	0	1	2	3	Cat. 4	0	7	0	7	Cat. 4	0	0	0	0
	total	25	5	2	32	total	7	20	5	32	total	0	16	3	19
		WildG	Plan	Misc.	total		WildG	Plan	Misc.	total		WildG	Plan	Misc.	total
	Cat. 1/2	0	0	0	0	Cat. 1/2	0	0	0	0	Cat. 1/2	0	0	0	0
	Cat. 3	2	6	0	8	Cat. 3	0	0	0	0	Cat. 3	2	0	0	2
	Cat. 4	0	2	0	2	Cat. 4	0	10	0	10	Cat. 4	0	6	0	6
	total	2	8	0	10	total	0	10	0	10	total	2	6	0	8

Table 2: Results of the episode coding for both groups.

**Interpretations:** For “Marco’s Number Series” every integer has (at least) two possible neighbors – with the exception of “8” or “9” who have to start or end the row. Most of the novices worked this problem by trying, that is starting the row with any number and continued it by adding numbers to the right until they got stuck and started with a different number. Three pupils solved this task by *looking for patterns*: they listed the possible neighbors for each integer and realized this way that they had to start with “8” or “9”. Three other pupils solved it by showing *mental flexibility*: they added numbers to the right as well as to the left when working on either side came to a stop.<sup>8</sup> The experts did not have to use the heuristic *looking for patterns*; they all proved to be *mentally flexible* by adding numbers on both sides of their rows naturally.

The same is true for the “Beverage Coasters” task. Only a few of the novices were able to rotate the squares mentally around each other. The others had to sketch figures of the squares in different positions and one even used a cut-out paper square to imagine their possible positions. Most of the experts, on the other hand, claimed to “see easily” that the area covered by both squares is one fourth of a square in every possible position.<sup>9</sup>

For the “Squares on a Chessboard” task, only 5 of 19 novices realized that there are squares of every size from 1x1 to 8x8. Those who did were not able to see a pattern in the number of squares or to count them correctly. All the experts realized almost immediately that there are squares of all sizes and all but one pair (who did a “wild goose chase” on this task) found a pattern<sup>10</sup> to count those squares quickly.

These findings support the claim of Bruder and Collet (2011) that a lack of mental flexibility can be compensated by the use of heuristics. All the experts showed *mental flexibility* in each of the tasks without the need to use as much heuristics as the novices.

<sup>8</sup> The seventh novice who solved this task correctly (as well as one of the experts) added the number “8” at the left side of her row after realizing that it was the only missing number (starting with “1” by chance).

<sup>9</sup> Seeing it so easily, most experts did not seem to realize a need to reason this claim, reaching only product category 3.

<sup>10</sup> There is/are 1<sup>2</sup> square of size 8x8, 2<sup>2</sup> squares of size 7x7, 3<sup>2</sup> squares of size 6x6, ..., for a total of 204 squares.

## DISCUSSION AND CONCLUSIONS

To answer research question 1, the ten participants of mathematical competitions can be considered as some kind of expert according to nominal (reaching final rounds of difficult competitions) as well as performance-based criteria by being significantly more successful in less time, and showing more self-regulatory and planning activities.

According to the “framework for the acquisition of expert performance” by Ericsson et al. (1993, p. 368 ff.), these ten pupils seem to be in the second phase, “an extended period of preparation [that] ends with the individual’s commitment to pursue activities in the domain on a full-time basis.” (ibid., p. 369) Whereas the first phase “begins with an individual’s introduction to activities in the domain and ends with the start of instruction and deliberate practice” (ibid.) and the third phase “consists of full-time commitment to improving performance and ends when the individual either can make a living as a professional performer in the domain or terminates full-time engagement in the activity.” (ibid.)

One of the main differences between the novices and experts we worked with – to answer research question 2 – is the constant appearance of *mental flexibility* in the experts’ processes, which could be an explanation for their superior performance, the less amount of time required as well as the lower number of heuristics needed to solve the problems. This could be an ability similar to “Situation Awareness” described by Endsley (2006) as “the perception of elements in the environment [...] [and] the comprehension of their meaning” (ibid., p. 634) which “is a significant contributor to the high levels of performance exhibited by experts in many domains” (ibid., p. 649).

There surely is a difference between both groups in exercise and dedication, but mental abilities seem to be the main point here. According to Bruder and Collet (2011), a lack of this kind of flexibility can be compensated by the use of heuristics, but it remains an open question, if this kind of flexibility can also be learned. And it especially remains an open question if children could have mastered this ability by the age of 10 already or if it is an inherent ability of theirs. In contrast to the claim of Ericsson et al. (1993), becoming an expert could be more than just practicing deliberately. For mathematical problem solving being mentally flexible could be the equivalent to being tall for becoming an expert basketball player or high jumper. Whether mental flexibility is the main difference between novices and experts of this age and whether this kind of ability can be learned or just be compensated needs to be addressed in future research.

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# A MODEL TO INTERPRET ELEMENTARY-SCHOOL STUDENTS' MATHEMATICAL ARGUMENTS

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*The goal of this study was to develop and use a modified coding system and model of argumentation to investigate and characterize fourth-grade students' mathematical arguments during an eight-lesson, whole-class teaching experiment regarding the arithmetic number properties. Sixty-seven instances of student arguments were identified using a model adapted from Toulmin's (1958) model of argumentation and Stylianides' (2007) elements of argumentation. In this paper, I document the development of the modified coding scheme and model, present the relevant elements of the model of argumentation, and briefly interpret the characteristics of students' early arguments, which will be further discussed in the presentation.*

## PURPOSE OF THE STUDY

Many students are progressing through school with the false impression that mathematics is answer-driven and static rather than a dynamic subject area open for discovery. Computing seems to be an instructional priority despite the fact that the field of mathematics goes far beyond procedures. Mathematics involves exploring, conjecturing, and justifying, which are also important components of other subjects (e.g., science, language arts, and history). There have been calls for conceptual understanding (Kilpatrick, Swafford, & Findell, 2001) and mathematical argumentation in recent American mathematics education documents, which require going beyond computation at the elementary level.

It is well recognized by researchers that at the elementary level, students are capable of going beyond computation in order to justify mathematical concepts (e.g., Stylianides, 2007; Keith, 2006; Carpenter, Franke, & Levi, 2003; Ball, Lewis, Thames, 2008). Although justifying and arguing are seen by many researchers in the field to be vital aspects of elementary school curriculum, "little research has focused on the issue of understanding and characterizing the notion of proof at the elementary level" (Stylianides, 2007, p. 1). We need information about early reasoning and proof if we are to help students transition along the continuum from early mathematics and informal proof toward grade levels requiring formal proof. If educators aim to include more argumentation in the elementary classroom in order to teach for conceptual understanding, they need to know what characterizes children's mathematical arguments. In this study, I taught lessons about the arithmetic properties to a classroom of fourth-grade students with instruction that promoted mathematical argumentation in order to develop a model highlighting the characteristics of students' early mathematical arguments for number properties. In this paper, I document the development of the modified coding scheme and model, present the relevant elements



of the model of argumentation, and briefly describe the characteristics of students' arguments.

## BACKGROUND

Students' use of empirical evidence is common and even at the college level, students are willing to accept examples as proof (Harel & Sowder, 1998). Consistent with research on empirical evidence, Keith (2006) noticed all of her second-grade students using examples to justify in November before shifting to a "generalized argument" (p. 64) later in the school year. Keith used Carpenter, Franke, and Levi's (2003) levels of justification (appeal to authority, justification by example, and generalizable arguments) to categorize students' justifications and showed that over the school year there was a shift away from examples toward generalizable arguments. In Keith's (2006) study the students wrote conjectures, but the analysis is mostly limited to the justifications since that was the focus of her research study. In studies regarding elementary school students' justifications, the claim is often an implicit part of the argument even though "in mathematics, the making of claims is central" (Ball, Lewis, & Thames, 2008). Ball, et al. investigated the process of making conjectures as part of three essential practices: "naming and using names, making and interpreting claims, and evaluating mathematical assertions" (p. 41). In another study, Stylianides (2007) analyzed third-grade students' arguments about even and odd numbers in the context of four elements of argumentation (foundation, formulation, representation, and social dimension) in order to determine if the arguments could qualify as proofs. In his study, and similarly with other studies, the students explored a problem chosen by the teacher and investigated conjectures together as a class and thus, did not need to state the claims explicitly since everyone was part of the same conversation. My research extends the importance of claims in mathematical argumentation while investigating and characterizing the other vital components of a mathematical argument at the upper-elementary school level.

## THEORETICAL PERSPECTIVES

I view mathematics learning as a construction of ideas through social interactions wherein novices are brought in to a larger mathematics culture and community. Consistent with this view, the *emergent perspective* (Cobb, 2000; Yackel & Cobb, 1996) provided an overarching framework for this research, in which I taught lessons and analyzed the students' arguments. Following the emergent perspective, learning can be examined along two dimensions: the social and the psychological. The social aspects informed my development of the lessons and my understanding of the students' learning through the sociomathematical norms and classroom mathematical practices that developed during a whole-class teaching experiment. I examined the psychological aspects through the lens of argumentation that was informed by Toulmin's (1958) model of argumentation and Stylianides' (2007) four elements of argumentation, which I will describe below.

### **Toulmin's Model of Argumentation.**

Toulmin (1958) attempted to identify the universal and field-dependent elements of an argument and describe the layout of all well-constructed arguments. He wrote that all well-constructed, rational arguments consist of three main interrelated elements: the claim, data, and warrant. The claim is the conclusion being justified and the data consist of the facts that build the foundation for the claim. The warrant is the element that builds a bridge from the data to the claim, which shows that the conclusion is valid. He proposed three auxiliary components (modal qualifier, backing, and rebuttal) that may be present in arguments, although they are not essential elements. Toulmin's model can be used to both construct a claim and examine the claims of others and it has been used across disciplines in order to identify, create, and evaluate arguments in science education, mathematics education (e.g., Pedemonte, 2007), and language arts. Specifically for mathematics education, Pedemonte found that using Toulmin's model was an essential tool for comparing the structure between students' arguments and mathematical proofs.

### **Stylianides' Four Elements of Argumentation.**

Stylianides (2007) was interested in the characteristics that determine if an argument can count as a proof at the elementary level and he defined the four elements of an argument as the following: (a) foundation, (b) formulation, (c) representation, and (d) social dimension. Stylianides developed his model based on his observations of the development of formal mathematical proofs. According to Stylianides, proofs are developed using definitions to form a solid foundation. Next, logical arguments are formed and arguments are represented using a mathematical language. Finally, a proof being accepted as such relies on the opinions of the mathematical community.

## **METHODOLOGY**

A basic qualitative research methodology was appropriate for this study, where the focus was on collecting, analyzing, and characterizing students' arguments about number properties during a sequence of instruction that promoted mathematical argumentation. Specifically, the study employed a whole-class teaching experiment (TE) methodology set in the fourth-grade mathematics curriculum, which was focused on some of the arithmetic properties, in order to promote instruction that included mathematical argumentation. The goal was to model and characterize the students' mathematical arguments. Without capturing and describing what the individuals constructed with regard to mathematical argumentation after interacting with their peers and teachers, "there would be no basis for coming to understand the powerful mathematical concepts and operations students construct" (Steffe & Thompson, 2000, p. 267).

The participants of the study included a fourth-grade classroom with 22 students at a small, suburban elementary school in the midwestern United States. There were 463 students enrolled in the elementary school. The majority of the students at the school were white/non-Hispanic (96.1%) and about a quarter of the students (24%) received free and reduced lunch. I volunteered in the classroom before the teaching experiment

began and the students were thus familiar and comfortable with me as their teacher for the TE lessons.

### **Teaching Experiment Lessons.**

I taught the eight, non-consecutive TE lessons over a period of five weeks and utilized the existing textbook and lesson plans with modifications drawn from the literature on promoting argumentation and teaching number properties. In the textbook, fourth-grade students are introduced to properties of multiplication (commutative property, identity property, zero property, and the associative property) as vocabulary words and diagrams, are given examples to examine, and then assigned exercises to practice the specific property addressed in each section. The properties in the textbook chapter are taught along with multiplication facts in order to show efficient ways of multiplying (e.g., that changing order does not matter and to generate fact families), and are not taught in their own right as key elements of mathematics.

The TE lessons were related to the classroom textbook, but included more opportunities for argumentation and justification. Modifications to the textbook lessons included (a) supplementary problems, number sentences, and counter-examples and (b) an emphasis on explaining and exploring the arithmetic properties using the language of argumentation. Promoting a discussion about number properties was an important component of the modifications and true/false and open number sentences provided opportunities for discussing numerical equality.

### **Data Collection and Analysis.**

In order to analyze and characterize the students' arguments, I audio- and video-recorded each of the TE lessons, transcribed them, and highlighted instances of arguments. I eliminated sections of the lesson transcripts that related to organizing and gathering the class to the front of the room, daily classroom dialog unrelated to mathematical arguments, behavior issues, and computational tasks that did not require justification. Keeping the models from Toulmin (1958) and Stylianides (2007) in mind, I developed a modified coding model that better described student argumentation at the Grade 4 level while still maintaining the overall structure of their analytical tools. I discuss this modified model and coding scheme in detail in the next section.

Using the coding structure, I identified 67 student arguments and noted what was happening in each clip and the recurring themes. Characterizing what was occurring in the students' mathematical statements involved looking at the statements in context rather than looking at lines of transcript unconnected to what was happening in the classroom.

Each argument was coded using the key words and definitions related to the elements of argumentation that I identified empirically when open coding for themes in the highlighted arguments. After modifying the definitions and adding an example for each keyword, I randomly selected eight arguments to check with another coder in order to see how another person interpreted the keywords and definitions. The other

coder and I compared codes, I revised the definitions of the keywords, and then I randomly selected 20 arguments to code and compare again. In the final round of coding and comparing, I randomly selected 20 arguments from the arguments that had not been coded. After all the inter-rater reliability coding, two people analyzed 48 (or about 72%) of the arguments. All of the clips were interpreted using the model developed through the coding and revisions, and then the codes were transferred to a spreadsheet so that the collection of 67 arguments could be sorted and analyzed.

### Revised Coding Scheme.

When using Toulmin's model of argumentation, Klumpp (2006) wrote that some researchers have difficulty because it is common for them to neglect the role and context of statements. My goal was to investigate all parts of fourth-grade students' arguments and the interactions between them while also capturing the social dimension of the arguments. The social interaction is an essential part of an argument and I wanted this to be prominently represented in the model and analysis. Toulmin's (1958) closed model had an element in which challenges could be predicted, but there is not a place for actual challenges to be placed within the model to record the conversation that unfolded between students. Stylianides (2007) included a social element but the conversation and challenges did not play a prominent role. The students in my study did challenge each other and so I replaced Toulmin's *exceptions* with challenges to reflect what the students were doing.

As I looked at the open coding themes, I saw similarities between the arguments emerge that became the essential components of the final model used for coding and analysis. Examples, used for two purposes, were frequently used and so needed to have a place in the model. The term *background information*, rather than what

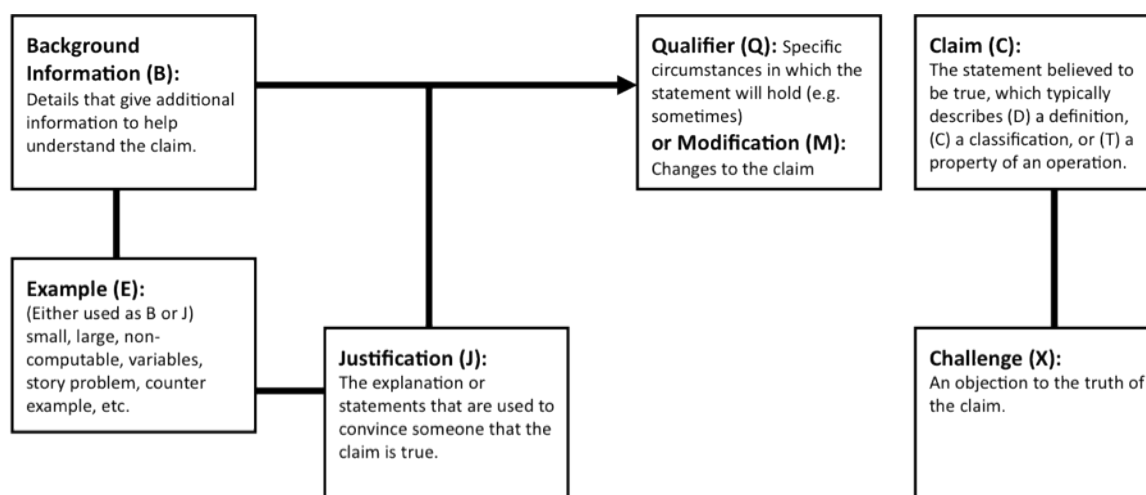


Figure 1: The modified model of argumentation

Toulmin called data, captured the students' language and thinking about this element in the study. The students used it to elaborate and clarify the claims. I also saw that qualifiers were relevant to this age group and went beyond the phrase "I think". In addition to qualifying their claims and the claims of others with phrases like "sometimes" or "when  $a$  and  $b$  are the equal", the students also modified the claims.

The six elements captured in the model (See Figure 1), used for coding and analysis, represent the students' arguments from my research and reflect current research about students' mathematical arguments.

## GENERAL RESULTS RELATED TO ARGUMENTATION

In addition to the development of the coding scheme and model, there were other findings related to students' mathematical argumentation. From the data, I found that the students included many of the six elements within a single argument (claim, justification, qualifier, modification, challenge, background information, and examples). Even though this model has six main elements, only two out of the 67 student arguments contained all six in a single argument, yet every argument contained a claim. Both of these arguments occurred in the sixth lesson, when there was more small group-work time. This could suggest that either the small-group work promoted more elements to surface or that it took some time for the students to consider all of the elements in an argument. The data from this study provides a snapshot of students' early attempts at argumentation when it is emphasized in classroom lessons and so it is possible that students do not consider all of the elements until they have been exposed to argumentation for a longer period of time.

Specifically related to *claims*, out of the 83 claims in this research study, 17 were "classifications or definitions" and 66 were "properties of operations". About 95% of the claims that I provided as the teacher were properties of operations, yet about 63% of the students' claims were properties of operations and about 37% were classifications or definitions. As the teacher, I emphasized properties of operations, but the students found classifications and definitions valid claims for which to provide an argument. The three most common types of *justifications* used by the students were examples with small numbers (30% of all justifications), justifications with reasoning (about 28%), and examples with large numbers (about 12%). Only eight students out of 22 in the classroom provided *background information* as part of their mathematical argument. About 69% of the time, a student used background information to clarify his or her own claim. Out of the 67 student arguments, 14 arguments included at least one *challenge*. Two challenges came from the teacher and students made 18 challenges. The challenges were mostly directed at property of operation claims that dealt with properties of addition and multiplication.

## DISCUSSION

From the data, I identified elements of fourth-grade students' mathematical argumentation (claim, justification, qualifier, modification, challenge, background information, and examples). By characterizing the elements of an argument, I was able to elaborate on arguments at the fourth-grade level and develop a more field-specific model. Particularly, I elaborated on the current understanding of the purpose of examples and their use as either justification or background information. Past research studies on students' justifications, at different levels of school, have highlighted the use of empirical data as a way to convince (Carpenter, Franke, & Levi, 2003; Ball, Lewis, & Thames, 2008; Keith, 2006; Chazan, 1993; Harel & Sowder, 1998). I found that the

students not only use examples as justification, but also as background information when stating a claim. This finding contributes to the research literature by broadening what we understand to be the students' use of examples. There is evidence that students use examples in a role other than justification.

Claims are an important component of arguments (Toulmin, 1958; Ball, Lewis & Thames, 2008) and of this model of argumentation. Researchers have known that students in elementary school can propose claims and conjectures (Keith, 2006; Ball et al. 2008, Schifter et al., 2008), but the findings from this study add to the research by describing the claims that students develop more specifically. I identified two types of claims in the analysis of the data: (a) properties of operations and (b) classifications or definitions.

## CONCLUSION

The empirical model adds to the research literature about upper-elementary students' arguments by detailing the elements of mathematical arguments and the roles the elements play in an argument at this grade level. Two of the analytical tools that already existed in the literature were used as a basis for this revised, more specific model. Research has shown that students use examples in their justifications (Chazan, 1993, Harel & Sowder, 1998, Carpenter, Franke, & Levi 2003), and the empirical model extends research on models of students' arguments by incorporating a place for examples and specifying the role that the examples play. Using the revised model I could represent an argument as it occurred between more than one person. I wanted to model a dynamic argument because arguments are naturally a social endeavor and that was not apparent in Toulmin's universal model of argumentation. Using the model, I was also able to look at specific elements of argumentation, characterize them, and explain how they were used by fourth-graders. The model was helpful in describing where fourth-grade students with limited exposure to argumentation begin. Even without prior instruction emphasizing argumentation, students bring relevant knowledge regarding arguments to the discussion that can be used to help students transition to more formal proof. A recommendation for future research includes trying to characterize student arguments with the model with other mathematical concepts and grade levels. Finding data that does not match the model, adapting the model, and the application of the revised model have the potential to strengthen the model and extend the research literature on upper-elementary school students' mathematical argumentation.

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# A COMPETENCY FRAMEWORK FOR ANALYSIS OF MATHEMATICAL PRACTICE

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*Despite an increased interest in the processes involved in mathematical practice, the interplay and relations among these processes are seldom studied. For this purpose, a framework for analysing how mathematical competencies are exercised is put forward. Examples of results from two case studies are presented, one looking at preschoolers' arithmetics and one looking at university students' construction of a proof, and the benefits and drawbacks of the framework are discussed.*

## INTRODUCTION

Within both the research and teaching communities, there is an increasing interest in mathematical practice and a growing recognition of its complexities. As a result, there is an international trend of reformulation of curriculum goals in terms of processes and abilities, rather than content knowledge (Kilpatrick, Swafford and Findell, 2001; NCTM, 2000; Niss & Jensen, 2002). Those who advocate a reform of mathematics education describe mathematical proficiency, or competence, as a multifaceted phenomenon trickling through all topics of mathematics. However, even though the functions and properties of specific aspects of competence have been studied, e.g. the handling of representations, reasoning and communication, research has not addressed the total interplay among them. Though the different competencies are assumed to depend on and support each other, this has not been thoroughly analysed and confirmed by empirical investigation.

Though mathematical practice naturally takes different forms, depending on the mathematical domain, educational level and social setting, it is believed that mathematical competence has essential aspects which can be generalised across these variations. The framework presented here can be used for analysis of data from any mathematical practice, describing how competencies are exercised by practitioners. The analytical tools are exemplified by application to two case studies: five-year old Nina handling doubles and a group of university students working on a calculus task.

## FRAMEWORK

The framework is an adaptation of the Mathematical Competency Research Framework (Lithner et al., 2010; Säfström, manuscript) for analysis of the exercising of competencies in a mathematical practice. General competence in some domain of personal, professional, or social life is defined as the ability to handle essential aspects in that domain. The essential aspects are called competencies. Within the mathematical domain, five such competencies are identified: representations, procedures, connections, reasoning and communication. Across these competencies run a



productive and an analytic aspect. The productive aspect ranges from imitative use to creative construction, while the analytic aspect comprise meta-level activities such as reflection and evaluation

The five competencies contribute to mathematical competence in different ways, and they relate to one another pairwise differently. Theoretically, one may see them as successively building up mathematical competence from the notion of entities. A mathematical entity is an abstract mathematical object, a building block of mathematics. Such abstract entities are represented in various ways, and the construction, use and evaluation of such representations constitute the representation competency. A *representation* of an entity is both a concrete replacement of the entity and a mapping between the entity and its replacement. There may be multiple levels of representation where one object may be the representation of one entity, as well as the entity represented by another mapping.

The competency to handle *procedures* concerns solving tasks. A procedure is always linked to one or more abstract entities, and often also to specific representations of entities. One part of handling procedures is to apply accepted algorithms to standard tasks, but this competency also embraces constructing and evaluating procedures. Procedures are linked to both entities and representations, since they are based on the properties of entities, and often rely on specific representations. Between entities, representations and procedures mathematical *connections* reside. Connections may go between entities, parts of entities, different representations of an entity, representations of different entities, parts of a representation and procedures. However, the representation mappings and the links between entities or representations and procedures are not seen as connections.

*Reasoning* is defined as explicitly justifying choices and conclusions by mathematical arguments. One can reason about entities, representations, procedures and connections. Reasoning often takes the form of *communication*, but can also be done internally. It is possible to communicate by means of speech, gestures and various forms of symbols, and with the intention to convey or to construct meaning (Truxaw & DeFranco, 2008). One does, however, communicate about something in terms of entities, representations, procedures or connections. The relations between competencies are summarised visually in the competency graph, which is further described in the next session, and exemplified in figure 1 and 2.

## METHOD

The analysis aim to categorise actions in terms of competencies and their sub-aspects. To help this process, a series of questions with auxiliary explanations is used, called the analysis guide. Each main question has a number of subquestions to enable a detailed explanation of the phenomenon categorised. The questions are organised by the competencies and linearly ordered. To begin with, the concepts dealt with during a sequence are listed, and the entities singled out. After this, one naturally proceeds by looking for representations and procedures, indicating how they relate to entities. Next, the constructed and used connections between entities, representations and procedures

are identified. Connections include associating objects, processes and situations by common properties, as well as extending processes and arguments to new cases (Ellis, 2007), but also differences are seen as a form of connections. With these elements in order, reasoning is easier to pinpoint and follow. The imitative–constructive spectrum is spanned by the actions: reiterate, redefine, expand and initially construct (Mueller, 2009; Mueller, Yankelwitz & Maher, 2012). When reaching communication, the matter at issue should already be clear from the previous results, and the risk of confusing means and message minimised.

The data is divided into short sequences and analysed competency by competency, according to the guide. Then, the analysis is consolidated to form a picture of the interplay between competencies in the form of a narrative and a graph. The narrative offers depth to the graph by describing how the competencies are exercised. To capture the use of utterances, gestures and material objects, the narrative is a combination of excerpts and accounts of events.

The competency graph gives an overview of what competencies are exercised and how these relate to one another. It also provides a visual representation of how the competencies relate to each other as concepts, and thereby their very nature. The graph is built up in levels, with the most general entities at the top. Below, follows one or more levels of representations, with representation mappings drawn as white, double arrows. Procedures are written inside clouds with thin lines, indicating how they relate to entities and representations. The connections are drawn among the levels as black connectors. Reasoning is marked around what it reasons for. The scope of communication is visualised by a dotted enclosure (see figure 1 and 2).

## RESULTS

The sample results are taken from two studies where the framework was applied (Säfström, manuscript; Säfström & Pettersson, manuscript). In the first one, four pairs of five-year old children were interviewed around a set of tasks dealing with whole numbers, represented by lego bricks and written numbers. The tasks were designed as to give rich opportunities to exercise competencies. The example presented here shows how Nina exercises competencies in her handling of doubles. The second one is a case study, where four university students collaborate to construct a proof by induction for a statement in calculus. The data has previously been analysed with the help of other frameworks by Pettersson (2004, 2008) and Ryve, Nilsson and Pettersson (2012). The excerpt analysed here is taken from the beginning of the group session, where a sub-task is solved.

### The case of Nina

In the following excerpts, Nina is associating lego bricks with the numbers 1-15 written on paper cards. She represents a number by lego bricks with that number of studs, alternating between attending to quantity and attending to shape. She has already identified the 2x3-brick as a representation of “six”, when she comes across a 1x6-brick.

Nina: Hm. Six again. One two three. One two three. Six. 'Cause if you break it apart ((claps hands together)) it becomes like this ((points at the 2x3-brick))

Later, she is looking for a brick with ten studs, and makes use of a connection between ten and twenty, as well as between quantity, shape and length. She has previously found and tallied a brick with two rows of ten studs, but since there was no card with 20, she puts it aside. Instead, she mistakenly picks up a brick with two rows of eight studs, and compares it to another brick:

Nina: ((Holding 2x8 and 1x8 together)) Same length. Ten... I know that ten plus ten is twenty, so it's ten.

Interviewer: Mhm. That's right... hm.

Nina: Ten plus ten is twenty. ((Putting 2x10 and 1x8 together)) Though it's probably te...

Interviewer: No, oh, this one, aha, it was maybe shorter than this ((holding 2x10 and 2x8 together))

Nina: Mm. What about that one then? ((Putting 1x8 next to 2x10 and 2x8)) (inaudible) twenty. Eight is this one. One...look here now: one two three four five six seven eight ((counting the studs of 1x8))

Interviewer: Mhm. Can we put it here? ((Putting 1x8 on the 8-card, next to 2x4)) Can you see that they are the same in some way?

Nina: Yea, one two three...four. Four. And four more plus four. Plus four plus.

In the case of Nina, the specific numbers are all representations of quantity, and the bricks are spoken of and used as representations of these numbers, giving examples of both the productive and analytic aspect of the representation competency. The other abstract entities, shape and length, form the bases for procedures, which determines representations of quantities through the connections between quantity, shape and length. The connection between three and six is used and communicated, but it is on the brick level that this connection supports the reasoning for the conclusion that the two representations are the same.

Two procedures determine the representations and support the reasoning: verbal counting and seeing and reshaping. These procedures are also helpful when extending the reasoning to the two representations of eight by connecting to four. A similar way of reasoning justifies the representation of ten, but in this case it is supported by the measuring procedure, and the connections between ten and twenty, and between shape and length. All three – three and six, four and eight, and ten and twenty – are examples of the productive aspect of reasoning where both aspects of connections and procedures come into play.

When another representation of twenty is added, new connections between the different representations of twenty and the representations of ten are used to revise the reasoning in the last case. As a result, the 1x8-brick is changed from representing ten to representing eight. Again, both aspects of the representation competency are exercised.

In the graph, merely the entity quantity lies outside the scope of communication, which is related to the frequent exercising of the analytic aspects of the competencies.

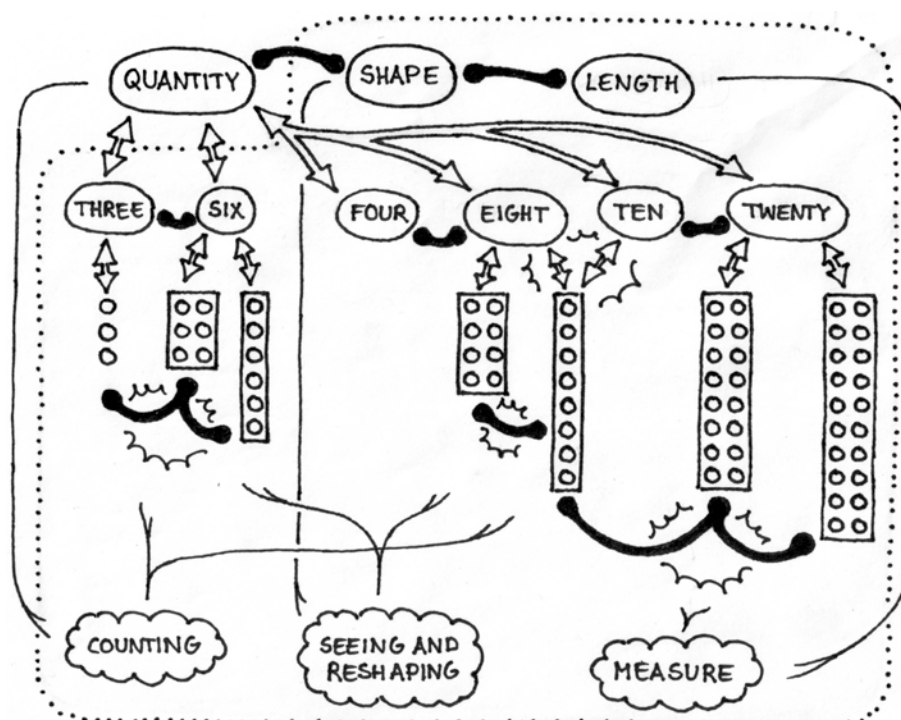


Figure 1: Nina's competency graph.

#### Four university students engaged in construction of a proof

The group is trying to answer the question: If  $f''(x) \neq 0$ , what can be said about the number of zeroes of the function? Immediately before this excerpt, they have concluded that if the derivative is non-zero, the function can have at most one zero.

Carl: Then  $f$  double prime  $x$  is nonzero.

Beth: Mm...

Diana: Then the same thing holds that...

Beth: Is it the case that, that's what I was also thinking about, 'cause can...

Diana: That  $f$ , that is, 'cause then it's, if  $f$  double prime is nonzero, that the derivative is increasing or decreasing.

Beth: Mm.

Diana: Then it means that  $f$  double prime is nonzero, that the derivative is increasing or decreasing.

Carl: Mm.

Beth: Mm...

Diana: Which means.

Carl: Keep talking.

- Diana: Which means that the derivative is allowed to change sign once at most, or change sign, yeah change sign once.
- Alex: But it doesn't have to change sign.
- Diana: No, thus, once at most, that's what we're talking about here.
- In chorus: Mm. ((A nods))
- Diana: And,...
- Beth: If it changes sign at most once then, how...
- Diana: ...and then the derivative looks like this ((draws and points at the left drawing)) and then it's minus here and plus here ((adding + and – to the drawing))
- Alex: Mm.
- Beth: Mm, and then it could have?
- ...
- Diana: ((draws the right drawing)) Or that's the way, isn't it? Or...
- Alex: Yes.
- Beth: Then it becomes...
- Diana: ...at most...
- All: ...two.

The whole system of entities, representations and connections supporting their reasoning for the conclusion that a function having a non-zero derivative has at most one zero, is considered an entity. This entity is connected to the whole setting with “non-zero second order derivative”. The monotonicity of the derivative is now connected to its sign, allowing for at most one change. A drawn representation of the extreme case of the derivative is constructed and then used to construct a drawn representation of the function, displaying the numbers of its zeroes. The group thus conclude that the function has at most 2 zeroes.

As they build their argument they make use of each other's ideas. Alex redefines Diana's formulation of the sign change of the derivative, which is then reiterated by both Diana and Beth. Diana then expands the argument to a visual representation of the statement. Everything in the graph is communicated about, except the property “monotone”. Once again this is linked to the frequent use of the analytic aspects of the competencies. Representations, connections and reasoning are all discussed and evaluated. In addition, the productive aspects of representations, connections, reasoning and communication are exercised. However, the procedure competency is absent.

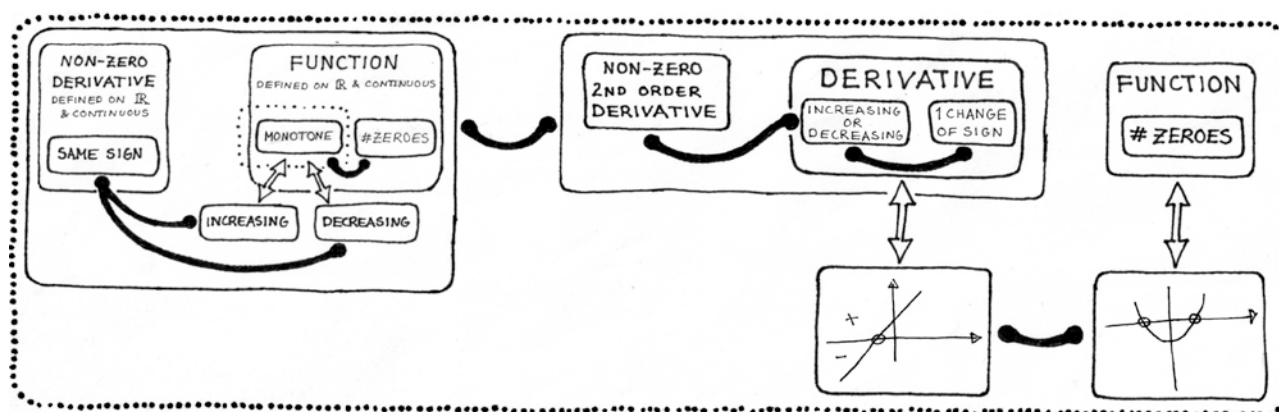


Figure 2: Competency graph for the group of students. The two function graphs are taken from Diana's own notes.

## CONCLUDING REMARKS

It has been shown that this framework can be used to analyse the exercising of competencies in various mathematical practices, highlighting the interplay and interdependence of competencies. Representations are found to support reasoning, as Diana's drawings and Nina's hand gestures. The same holds for connections, as seen when the group of students build up their argument for the number of zeroes. However, the relationship is reciprocal, as reasoning is used to construct and determine representations, as Diana's function graph and Nina's ten-brick, as well as connections, as the one between the non-zero derivative and the zeroes of the function. Nina's use of procedures is intertwined with representations and the connections between them and her counting justifies these representations. In both examples there is also strong correlation between communication and the analytic aspect of other competencies. The competency graph becomes a powerful tool for visualising the relations between competencies, forming a structured image of the content in the activities and how it is addressed. It captures interesting features of mathematical practice, such as multiple representational levels, links between procedures and entities and "entitification" of whole systems as the focus of reasoning shifts.

However, the framework is still under development and has been influenced by the applications of these sets of data. Without doubt, the analysis of other mathematical activities will further inform the framework, clarifying the constructs and sharpening the analytical tools. While the strength is the overall picture and the interaction between competencies, this leads to less precision when it comes to specific competencies. Though research focused on representations, connections, procedures, reasoning or communication has had an impact on the framework, this analysis can not capture the full complexity of every competency.

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# MATHEMATICAL REASONING WITH ANIMATED WORKED-OUT EXAMPLES

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*This paper examines how the processing of interactive and animated worked-out examples in the domain of fractions is affected by the application of self-explanation prompts. 56 German sixth-graders were videotaped while they were working pairwise in an individual learning environment and especially while they were processing the worked-out examples. In contrast to previous research papers that focus on learning strategies and learning success, the present work analyses the occurrence of self-explanations and mathematical reasoning activities. First results reveal that working with animated worked-out examples elicits spontaneous mathematical reasoning activities. In particular, self-explanation prompts amplify this connection.*

## **THEORETICAL FRAMEWORK**

### **Animated worked-out examples**

Worked-out examples consist of a task and its solution. Many researchers showed the benefits of those examples in the early knowledge acquisition-phase in structural domains like mathematics (e.g. Sweller et al. 1985). In contrast, worked-out examples lose their effectiveness with increasing expertise (Kalyuga et al. 2003).

The results of studies examining animated and interactive worked-out examples however are controversial and varying from context to context (Betrancourt 2005). A meta-review of Tversky et al. (2002) describes the benefits of animations only in concrete domains where phenomena are observable. Scheiter et al. (2009) argue that also in abstract domains like mathematics animations can have positive effects.

The construction of worked-out examples is a crucial factor for their helpfulness in learning activities. A lot of study results provide guidelines for the construction of worked-out examples. Most of their arguments are based on Cognitive Load Theory (Sweller et al. 1998).

Apart from those guidelines, the mathematical contents of the examples often remain on a calculatory level. Many studies use algebra and probability theory examples (e.g. Sweller 1985, Große 2005 etc.). Furthermore a lot of studies do not publish the used worked-out examples although their construction could have huge influence on future empirical studies (Betrancourt 2005).

### **Self-Explanations**

Another crucial point for the effectiveness of worked-out examples is the learner's activity. Chi et al. (1989) demonstrate that students who actively process the example by combining information within the material, filling in missing information or comparing new information with existing knowledge have a higher learning outcome than students who do not show such self-explanations. However, spontaneous self-explanations occur rarely and often worked-out examples are processed passively



or superficial (Renkl 1997). Several researchers have found ways to elicit these activities, for example self-explanation prompts or self-explanation trainings (Chi et al. 1994, Wong et al. 2002).

Nearly all studies dealing with worked-out examples and self-explanations focus on a single learner processing the material. There are only a few studies that concentrate on pairs or groups working with worked-out examples (e.g. Retnowati et al. 2010).

Learning outcomes are often measured with post-tests or comparable methods – the way, how animations are explored or processed and how reasoning is affected, is rarely described (Betrancourt 2005).

## **Fractions**

Research has shown that German students have great problems with fractions (e.g. Padberg 2009). These problems are mainly located in the application of mathematical knowledge to contexts that arise from innermathematical or real world tasks. A lot of studies show that missing, fragmentary or deficient concepts could be one major reason for the poor performance of German students e.g. in PISA in the domain of fractions (Wartha & vom Hofe 2005).

The transition from natural to rational numbers claims the transformation of several concepts. A well-known concept (e.g. antecessor and successor, one unambiguous symbol for one number) often works for natural numbers but fails for fractions (Padberg 2009). A fraction can describe a part of a whole, a set or an abstract quantity, a ratio, a division, an operator, etc. (Hefendehl-Hebeker 1996). The understanding of these different meanings and the connection to the mathematical symbols, definitions and rules are necessary for a successful handling of fractions.

The fractions course used in the present study includes basic concepts and operations with rational numbers, namely fractions as parts of a whole or a set, as mixed numbers, as parts of arbitrary quantities, pictorial expansion or reduction of fractions as well as addition and subtraction of fractions with common denominators. The course and the materials focus on the construction of concepts with regard to real world contexts, meanings of fractions and the integration of textual, pictorial and symbolic representations in order to develop a map of ideas and concepts that serves as a fundament for further syntactic and semantic work with fractions.

## **Reasoning**

The NCTM-Standards from 2000 and the German standards from 2003 emphasize the importance of process competencies besides the mathematical main ideas (NCTM 2000, KMK 2004). One of these competencies is mathematical reasoning. It includes activities like expressing and shaping suggestions, investigating given argumentations, creating reasoning statements and comparing them to other statements (NCTM 2000).

## **Self-Regulated Learning**

In a self-regulated learning process the learner initiates, monitors and evaluates his or her own learning activities by self-controlling (some of) the cognitive, metacognitive and volitional parameters as well as his or her own behaviour (Schiefele & Pekrun 1996).

The facilitation of self-regulation can be a way to deal with heterogeneity in class. The teacher does not have to supervise the students at every step of their learning processes, but delegates responsibility of the learning progress to each student (vom Hofe 2011). The ability to manage one's own learning activities is an important competency going far beyond the learning in school. With that presumption the construction of learning environments is a crucial point for successful self-regulated learning at school.

## CONCEPTUAL FOCUS & RESEARCH QUESTIONS

This paper compares two ways of dealing with interactive and animated worked-out-examples as instructional materials in self-regulated learning environments: self-explanation training and self-explanation prompts.

To get data as ecological as possible the study was integrated into the regular fractions course of a middle school. Because there are many studies with high school and university students, the chosen students were between 12 and 14 years old. The qualitative data were not collected with the think-aloud-method because that would have been a major interference in the usual behaviour of the students.

To minimize expertise-reversal effects and to give the students the chance to choose the materials they want to work with, the intervention phase was designed as an individual learning environment that delegates lots of decisions concerning the learning activities to the students.

Negotiation processes between peers play an important role during learning processes. To take account of this component the students were grouped in pairs. This is an alternative way to get valid qualitative data instead of using think-aloud- procedures that would disturb the class-routine.

The paper focuses on spontaneous mathematical reasoning processes and occurring self-explanations, thus every computer was equipped with a webcam to record the students' activities and dialogues. To control prior knowledge a pre-test was carried out. A parallel-constructed post-test was conducted to get an impression of the knowledge about fractions after the intervention,

The following research questions were of special interest:

- To what extent does working with animated worked-out examples foster mathematical argumentation and reasoning activities?
- What kind of spontaneous dialogues and mathematical reasoning-activities can be observed?
- How do the different ways to elicit self-explanations affect the processing of the worked-out examples?

## METHOD

### Subjects

The subjects were 85 students at the age of 12 to 14 from three sixth-grade-classes of a German secondary school (Realschule). The prior domain knowledge of the participants was supposed to be very small because there is no systematic course dealing with fractions before the sixth grade in primary or secondary schools.

## Setting

The study was integrated into the fractions-curriculum of three classes. As preparation for the intervention-phase each teacher designed his or her 18-lessons-course based on common guidelines that were defined by the teachers and the researcher. The normal extent of such a course is about 23 lessons. To realize the reduction of the number of lessons, the exercise-parts of the course were shortened and the extra time was used for a 180-minutes exercise-period – the intervention-phase. This phase was meant to be an individual training-session before the class-test after the study. So all contents of the intervention were part of the preliminary course.

During the whole study, each class worked in its own classroom. At first, a 30-minutes pre-test, consisting of multiple items concerning various aspects of the introductory fractions-course, measured the prior knowledge of the class members. Then an introduction into the materials of the intervention-phase was given.

The following intervention setting varied slightly in the three classes. In this paper only two of them are considered (the self-explanation prompt class and the training class, 56 subjects), the third class worked without animated worked-out examples. Students of the prompt class received worked-out examples accompanied with self-explanation prompts. No prompts were presented to the students of the training class. They passed a self-explanation training during the introduction before the intervention phase instead. In the first 90 minutes of the intervention-phase the students dealt with three selected animated worked-out-examples and two incomplete examples. In addition, they completed a self-assessment questionnaire and worked individually about twenty minutes. That means they could decide on their own which tasks or examples they wanted to work with. During the second 90 minutes the students concentrated on two further selected examples and could use the rest of that time for individual training. The same instructor, who was unknown to the students, conducted the intervention phase of each class. To quantify the learning outcomes, a post-test – designed matchable to the pre-test – was carried out.

## Materials

During the intervention-phase the students of the two classes worked with a computer-supported learning environment and a workbook. The computer environment provides 18 different interactive animated worked-out examples related to the sections of the fractions course. The workbook contains the self-assessment questionnaire, instructions for the animated worked-out examples (accompanied by self-explanation prompts in the self-explanation class), incomplete examples and tasks.

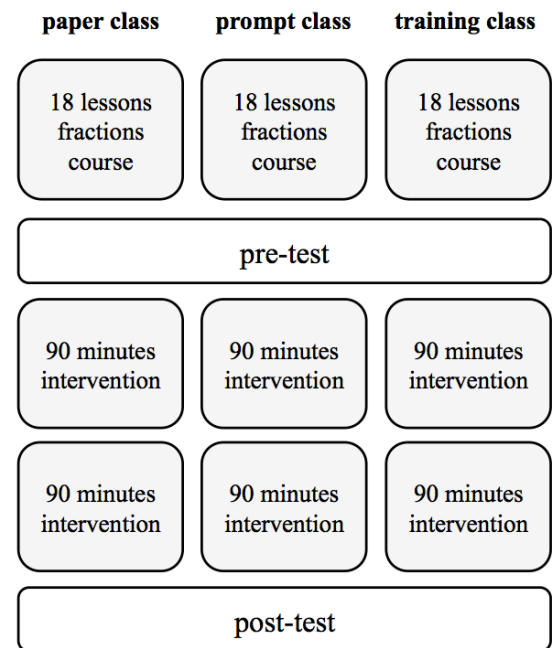


Figure 1: setting of the study

The examples used in this study were created based on guidelines that reflect the current status of research in instructional psychology and didactics of mathematics. Furthermore, these examples were evaluated in several classes. Every example was equipped with a bar of five interactive control-buttons (next step, skip step, step back, still, reset).

### **Collecting and coding of video data**

Every pair of students worked with a desktop-computer. No matter what kind of material they processed (animated worked-out examples or other tasks in the workbook), a camera on top of the computer-screen recorded the pairs' behaviour during the intervention. Simultaneously the screen was recorded. Both data were synchronized and then coded with the usability testing software MORAE®.

The video data was coded on two levels. On the first one, all recordings were divided into disjoint time intervals based on 21 categories. These periods classify the processed task-form (worked-out-example, self-explanation prompt, incomplete example, task or miscellaneous) and the students' behaviour (quiet processing, content-related dialogues with group-members, content-related dialogues with the teacher and distraction). "Class instruction" constitutes another single category.

On the second level, the worked-out-example- and self-explanation-prompt-periods were examined more specifically. Three main categories were coded with markers: metacognition, self-explanations and reasoning. Three distinct sub categories for the coding of metacognition were distinguished: planning, monitoring and regulation. The self-explanations were divided into two categories: high self-explanations and anticipation. For the analysis of reasoning three different categories were established: describing mathematical facts, questioning of given facts and reasoning. The reliability for both coding procedures exceeded 90%.

### **EXEMPLARY RESULTS**

The collected data allow analyses on several levels. On the global level, the students' prior knowledge and their productivity during the intervention phase can be estimated by the pre- and post-test-results, the sum of all distraction-periods per class and the spent time per example. Results of the pre- and post-test show significant increase in learning in both classes. On a more specific level, analysing the time-intervals against the background of the different task-forms or the different kinds of behaviour during the processing of animated worked-out-examples or self-explanation prompts gives hints for answering the research questions.

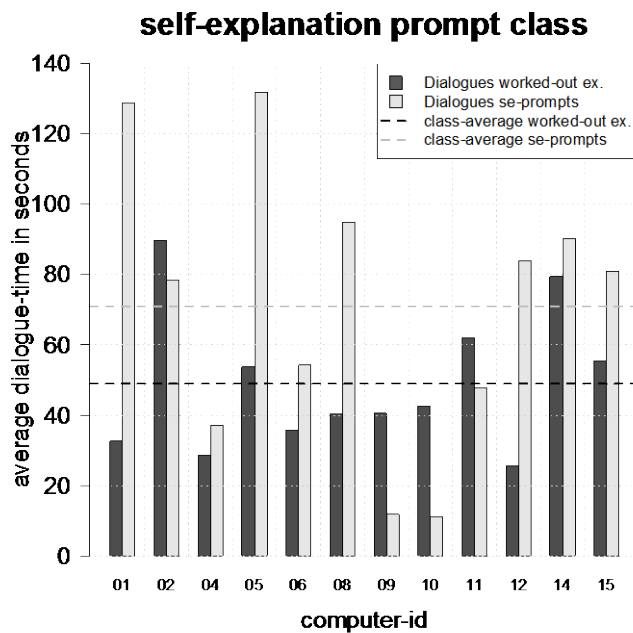


Figure 2: Comparison of length of dialogues

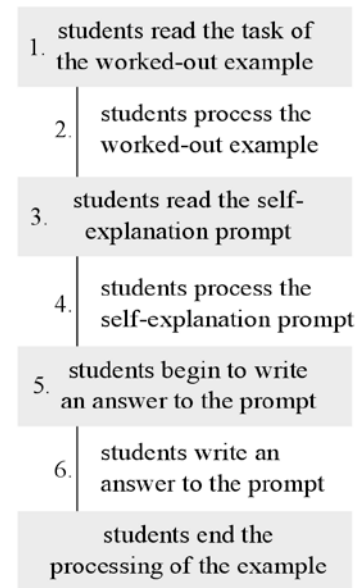


Figure 3: phase model

A lot of spontaneous substantial dialogues concerning mathematical argumentation and reasoning occurred in both classes. In the self-explanation prompt class these dialogues are distributed among worked-out-example periods and self-explanation prompt periods. Figure 2 shows a comparison of the length of substantial dialogues during the processing of worked-out-examples and self-explanation prompts.

These facts give hints for the assumption that self-explanation prompts can foster the appearance of content-related dialogues. At this point the structure of the dialogues is analysed, especially how many occurrences of self-explanations and acts of mathematical reasoning could be observed. Results and examples of this analysis will be presented.

To get insight into the content of the dialogues a comparison of the total number of markers in the categories concerning metacognition, self-explanation and mathematical reasoning was conducted. An extraction of the ways how students worked with the different examples allows the construction of a phase model to structure that process (Figure 3). In more than 90% of the cases the processing of the worked-out examples and the self-explanation prompts fit into this linear progress. For a detailed analysis the different markers were attributed to the 6 sections of this model. Results of the marker analysis and examples of transcripts will be shown in the presentation.

## DISCUSSION

This paper demonstrates the role of animated worked-out examples and self-explanation prompts with regard to process competencies such as reasoning and communicating. With regard to the research questions first answers can be stated. The results give evidence to the fact that mathematical reasoning-processes occur spontaneously during the work with interactive animated worked-out examples. Furthermore, it outlines the perspective of animated worked-out examples as elements of individual learning environments. From previous analysed video data it can be

assumed that contents of the enunciated dialogues and reasoning activities refer to central concepts of fractions. Particularly, self-explanation prompts structure the processing of the worked-out examples and let the students discuss and focus on key concepts of a task, even during the writing of the answers to the prompts.

The complex design of the present study and the small number of subjects do not allow quantitative statements about the relation of animated worked-out examples, self-explanation prompts and learning outcomes. However, qualitative data offer a detailed view on this relation and the learning process. On-going analyses shall sharpen this view and identify archetypes of reasoning- and argumentation- behaviour. Altogether, the results show that animated worked-out examples accompanied with prompts can serve as useful elements in individual learning environments in competence-oriented fraction courses.

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# **INSTRUMENTED ACTIVITY AND SEMIOTIC MEDIATION: TWO FRAMES TO DESCRIBE THE CONJECTURE CONSTRUCTION PROCESS AS CURRICULAR ORGANIZER**

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*We document part of the process through which conjectures produced by students, with the aid of the dynamic geometry software Cabri, when they solve proposed geometric problems, become a curriculum organizer in the classroom. We first focus on characterizing students' instrumented activity recurring to utilization schema (Rabardel, 1995; in Bartolini Bussi and Mariotti, 2008), and then describe the teacher's content management through which the ideas produced by the students become key elements of knowledge construction.*

## **INTRODUCTION**

During the years 2010 and 2011, we carried out a formal study<sup>1</sup> of pre-service mathematics teachers' conjecture production and the content organization of their second semester plane geometry course (Universidad Pedagógica Nacional, Colombia) based on these conjectures. This academic effort continues our research interest which, since 2004, consists of delving into issues related to teaching and learning proof in geometry at tertiary level. In this paper we want to present the tool we use to analyze how the formulated conjectures become an organizer of the implemented curriculum, favoring student participation in its construction. Initially, we present our framework that includes not only two theoretic references on which our analysis is based, instrumental approximation and teacher semiotic mediation, but also what we mean by "curricular organizer". We then present the research methodology, and we briefly describe the experimental device. Following, we give a succinct description of the categories that we constructed to analyze students' instrumented activity and the teacher's mediation activity. We then present an example in which the categories were used to analyze the activity concerning a particular problem. Finally, we discuss the achievements and projections of our study.

## **THEORETIC FRAMEWORK**

It is our intention to try to articulate, imitating what Arzarello and Paola (2008) do, two theoretic references that are useful to deepen into issues that concern teaching and learning proof: instrumental approximation that becomes a lens to provide information about the use of dynamic geometry as an effective instrument for student conjecture production; and the teacher's semiotic mediation that gives insight to the teacher's role

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when the intention is to make students' productions, personal meanings, evolve towards mathematical meanings them to obtain the statement they propose.

### **Instrumented activity**

We follow the widely accepted premise that learning is mediated by artifacts. Interpreting Rabardel (1995, in Bartolini Bussi and Mariotti, 2008), we consider an *artifact* to be any material or symbolic object created by man with a specific aim.; From our point of view, artifacts, as a product of human activity, are perceptible, can be analyzed, and have physical existence independent of the situation from which they originate, be it as an object itself or through a register.

So that an artifact can intervene in knowledge generation it must become an instrument for the person learning. In Rabardel's theory, the notion of *instrument* includes an artifact (or part of an artifact or some artifacts) together with utilization schemes, that is, an observer's interpretation of actions "relative to the management of the characteristics and particular properties of the artifact" or that are "means for the achievement" of a task (Rabardel, 2011/1995, p. 171). According to the researcher, making an artifact an instrument requires the articulation of two processes, in what he calls *instrumental genesis*. These are: (i) *instrumentalization* that is, progressive acknowledgement of the possibilities and limitations of the artifact and its different components; and (ii) *instrumentation*, or appearance and development of utilization schemes. Our interest in the theory of instrumented activity lies in that the utilization schemes can be seen as signals of mathematical activity carried out by the students when they solve problems, aided by the use of Cabri, and propose conjectures as a result of their work. They provide us information about the personal and contextualized experiences with which the students give meaning to the problem's statement and the objects involved in the situation, propose a strategy to solve the problem, and formulate conjectures as the solution. As the artifact is used varied signs are produced the capture (encapsulate) the actions of the instrumented activity and the ideas that sprout during these or associated to these. These signs evidence personal meanings with which the teacher can carry out the semiotic mediation of the geometric content that students should understand and learn.

### **Teacher's semiotic mediation**

In our study, the analysis of the teacher's semiotic mediation is focused on the treatment of the signs derived from the use of the artifact Cabri when solving problems for which students have to produce a conjecture. These signs include geometric Cabri figures and statements with mathematical content, related to the conjecture, produced by them. The signs are considered as entities that represent something for someone; they reflect internal cognitive processes and are mental tools for doing particular tasks.

According to Mariotti (2012), the principal characteristic of the signs derived from the use of an artifact, in the mathematics education field, is its strong link with the actions done with it. We recognize that the artifacts have *semiotic potential* (Bartolini Bussi and Mariotti, 2008) that is evidenced in so far the *utilization schemes* and the *signs* are used intentionally for mediation. The teacher's semiotic mediation is based on the

profit he/she can get from the signs produced by the students to propitiate, foment and affect the relation between the students and mathematical knowledge.

Therefore, the teacher's responsibility is to design strategies that connect the individual and social perspectives, and to act on the cognitive and metacognitive levels. The semiotic mediation related to the production of conjectures in proving activity can be evidenced in actions centered on: I. The conceptualization of objects and relations; II. The understanding and use of conditional statements; III. The conjecture as solution to the problem; IV. The theorem looked for; V. The conformation of the notion of theorem, postulate, definition. In all of these, the teacher's semiotic mediation process relies on the experience lived by the students, on his/her own observations and formulations, to give meaning to the mathematics statements that emerge.

### **Problems-conjectures-theoretic system as curricular organizer curricular**

Rico (1997, p. 45) introduces the notion of curricular organizer to refer to "the knowledge we adopt as fundamental component to articulate the design, development and evaluation of didactic units". For the author, a necessary condition for accepting a type of knowledge as a curriculum organizer must be its objective character and the diversity of options it generates. It must offer a conceptual framework for teaching mathematics and a space for reflection that shows the complexity of the transmission and construction of mathematical knowledge processes, and criteria to approach and control that complexity. In our methodological approach for teaching proof in the tertiary level, we have used a composition of three elements as curriculum organizer which we denote by *problems-conjectures-theoretic system*. To characterize it, we begin by detailing its elements and the relations among them.

The problems are of a geometric nature; to solve them empiric exploration is required for which the use of dynamic geometry is permitted, and a conjecture must be formulated; one or more possible theorems of the theoretic system underlie each problem. The problems proposed to the students are of two types: of *suggested construction*<sup>2</sup> and of *creative construction*<sup>3</sup>. Student's conjectures are the result of exploring the situation given in the problem, using dynamic geometry software, and have a high probability of being acceptable proposals from a theoretic point of view. The theoretic system is the framework for the problem's solution; the problem can be represented and tackled with the available geometric content. The conjectures that originate from solving the problem extend the theoretic system if they become theorems. In our case, the guideline of the theoretic system is Birkhoff's model (1932) for Euclidian geometry, in which "the facts embodied in the scale and protractor" are introduced (p. 329).

<sup>2</sup> The representation of the situation described is based exclusively on the construction of the objects that satisfy the conditions given in the problem itself and the search for invariants is based on the direct exploration of the objects represented (constructed).

<sup>3</sup> They require auxiliary constructions that provide the necessary geometric conditions to determine, via explorations, the existence of an object (generic or specific). Therefore, the representation of the situation is based not only on the construction of objects that satisfy the given conditions but also those that permit solving the problem.

Given the above description it is relatively easy to see that in the composite *problems-conjectures-theoretic system* the geometric content occupies an outstanding position; it is natural to imagine that if it is used as a prescriptive curriculum organizer it will not present any novelty with respect to the traditional way of presenting and/or studying mathematical content. But, as implemented curriculum, what is actually done as response or reaction to the situated events and a specific time and place, our proposal is different from other organizers because the students' voices are the key element during the collective study of their conjectures.

## RESEARCH METHDOLOGY

Our research study is framed in the teaching experiment methodology (Cobb, 2000): events of experimental teaching are analyzed using a certain theoretical reference with the purpose of identifying phenomena of interest. The research design has as starting point the transcriptions of some Plane Geometry classroom sessions, course developed during the first semester of 2007.

To study the instrumented activity with Cabri, we identified, in the students' interventions, issues that became analysis categories (instrumentalization, instrumentation-utilization schemes, artifact semiotic potential, geometric figure-signs, and statement-signs). The knowledge we had of the program and, based on frequent observation of students' performance when using Cabri to solve problems, we proposed, a priori, a set of utilization schemes with respect to constructions and the use of the dragging function; other schemes arose as we carried out our analysis. With the objective of analyzing the teacher's semiotic mediation in the conjecture treatment in class, and this way document the process through which the ideas produced by the students become key elements of knowledge construction, the teacher's actions were coded and grouped into the five categories previously mentioned.

### Analysis examples

To illustrate how we did the analysis, we present two examples, one relative to the students' activity and the other to the teacher's mediation.

#### Students' activity

We present the analysis of part of a pair of student's (Nan and Ing) activity when solving a *creative construction problem*<sup>4</sup>. The theorem we expected to obtain with this problem is the existence of a perpendicular line to a given line through a point of that line. It results as a theoretic necessity to be able to construct the complementary angle and justify its existence. Even though an exploration based solely on measuring and dragging is useful to notice that point  $E$  exists, we expected students to geometrically characterize the point in their conjecture. That requires an auxiliary construction that provides the sufficient geometric conditions for such an existence.

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<sup>4</sup> The problem statement is: let  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$  be opposite rays and  $\overrightarrow{AD}$  another ray. Does there exist a point  $E$  in a half-plane determined by  $\overrightarrow{AB}$  in which  $D$  is found such that  $\angle CAE$  and  $\angle DAB$  are complementary?

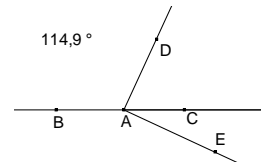
The group's activity can be divided into six phases: (i) construction of opposite rays, (ii) construction of  $\overline{AD}$ , (iii) dragging  $\overline{AD}$  to explore, (iv) construction of  $\angle EAC$ , (v) dragging  $\overline{AD}$  to verify the construction, (vi) writing the conjecture. In these phases, the following construction and dragging utilization schemes were inferred:

Scheme	Code	Phase	Description
Opposite rays construction	C7	i	They construct a line, use the Ray option and construct opposite rays with the same origin point on the line.
	C11		They construct two rays in opposite directions that perceptually seem to be colinear.
Ray construction	C5	ii	They use the Ray option, click in two different places on the screen and determine a point on the ray.
Construction of complementary angle	C18	iv	They measure the initial angle, calculate the difference between that measurement and 90, construct a ray and rotate it around the endpoint according to the calculated difference.
AETO		iii	They drag a point or object to sweep out and explore a more or less complete region.
AVCXMO		v	They drag a point or object to place them in extreme situations and verify the construction
AVCEMO			They drag a point or object to place it in special situations and verify the construction.

Table 1: Construction and dragging schemes: Nan and Ign

The following transcription corresponds to phase v. Once the students constructed  $\angle CAE$  (C18), they use dragging to verify that this angle is always the complement of  $\angle BAD$ . The students' dialogue during this phase is the following one:

- 1 Ign: Ready. And it satisfies the dragging test. Yes. Let's drag. Okay?
- 2 Nan: Drag  $\overline{D}$ .
- 3 Ign:  $\overline{D}$ ? Ah! Yes. Because  $\overline{E}$  is the dependent one. [Drags  $\overline{AD}$  bringing it close to  $\overline{AB}$ .]
- 4 Nan: Make it more than ninety [referring to the measure of  $\angle BAD$ .]
- 5 Ign: [Rotates  $\overline{AD}$  in the negative direction in such a way that  $\overline{E}$  ends up in the other half-plane.]



Initially, Ign's dragging corresponds to the utilization scheme AVCXMO, because he moves  $\overline{AD}$  to positions close to  $\overline{AB}$  [3]. Later, to satisfy Nan's request, he drags the ray until it is in a position for which  $m\angle BAD$  is more than ninety, that is, he considers a special position and therefore uses scheme AVCEMO [5]. They do not realize that when they drag, point  $\overline{E}$  is in a different half-plane than where  $\overline{D}$  is fact that does not correspond to the problem's statement. They have not yet understood the conditions

stated in the problem because they consider angles  $\angle DAB$  that are not acute. The utilization scheme used has semiotic potential to inquire about their concept of complementary angles. In this fragment, we observe that the instrumentalization stage in which Nan and Ign are permits them to easily recognize the independent and dependent points in their construction. Moreover, Ign's intervention "And it satisfies the dragging test" [1] is another evidence of his level of instrumentalization. The conjecture they finally express (mathematical sign) is: *If you have  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$  opposite and  $\overrightarrow{AD}$  another ray such that the measure of  $\angle BAD$  is acute, and  $\angle CAE$ , with  $E$  in the same half-plane in which  $D$  is, so that they are complementary, then  $\angle EAD$  is right.*

### Teacher's mediation

To analyze the teacher's presentation of the conjectures produced by the students and the respective commentaries, we fragmented her intervention in moments. For each fragment we specified the intention that we could discern or that she makes explicit, and also the actions through which she manages her intentions. She finds in each conjecture some element worthy of an analysis that will contribute something to understand the situation itself, some associated theoretic element, or mathematical issues. We present part of the analysis of Fragment 14 as an example of the use of the codes –teacher's actions. These will be written in *italic* and in parenthesis the Roman numeral that indicates the general category in which the action is found.

The teacher analyzes the conjecture of one group. She points out that they propose a conjecture that is different from those analyzed in previous fragments<sup>5</sup>, because their conclusion is not that complementary angles but that  $\overrightarrow{AE}$  is perpendicular to  $\overrightarrow{AD}$ . To analyze the conjecture she compares it with the one obtained as a consequence of the analysis of the previous conjectures<sup>6</sup>, from which the expected theorem arises something she *emphasizes as a contribution to the theoretical system* (IV). The teacher indicates she has *identified that the conjecture reported is in agreement with what was constructed and with the conclusion* (II) and at the same *realized theoretic control* (III) in so far as she mentions that the student's report of the conditions they constructed is valid. She *emphasizes that there is an imprecision* (IV) in the conjecture and a member of the group immediately recognizes the data that is missing in the hypothesis: not mentioning that point  $E$  is in the same half-plane, determined by  $\overrightarrow{AB}$ , in which  $D$  is. We conclude that student's sign evolved maybe due to the teacher's mediation process carried out with the previous conjectures.

<sup>5</sup> Given two complementary angles, coplanar, that share the same vertex and one of its sides is the opposite ray of one side of the other angle then the angle formed by the other two rays is right.

<sup>6</sup> The conjecture is: If  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$  are opposite rays and  $\overrightarrow{AD}$  is another ray, and let  $\overrightarrow{AE}$  such that  $m\angle EAD = 90^\circ$  then  $\angle DAB$  is complementary to  $\angle EAC$ .

The teacher starts an analysis by comparing the conditions exposed in the hypothesis and thesis of both conjectures, as resource to aid in the comprehension of the difference between them. As she indicates that the thesis and one of the conditions included in the hypothesis are interchanged, she *emphasizes the fundamental elements of a conditional statement* (II). She then classifies Nan and Ign's conjecture, where the conditions are complete, as a theorem indicating with this that she approves the statement because it reports the dependency evidenced in Cabri and a result that can be proved within the theoretic system they have on hand. The teacher emphasizes that this conjecture *indicates the constructed properties and those discovered* (II).

## FINAL COMENTARIES

The detailed study of the transcriptions of the students' work was a useful because it permitted us to describe and analyze their actions to solve a problem and formulate conjectures; infer and categorize utilization schemes related to the artifact Cabri that students bring into play, according to the type of problem; identify different types of signs produced by them; amplify Mariotti's proposal (2012) to determine the semiotic potential of the artifact via the schemes; and establish a correlation between the schemes and the personal signs produced by students when solving a problem. Also, since the teacher was committed to the curriculum organizer *problems-conjectures-theoretic system*, the transcriptions of the socialization, guided by the teacher, where the conjectures are discussed, permitted us to identify the teacher's semiotic mediation actions directed towards producing a mathematical sign (theorem statement) that corresponds to the one that she wants to include in the theoretic system once it is proven. Besides, we were able to establish a categorization and a frequency count of the teacher's semiotic mediation actions referred to the central object of mediation: conceptualization, conditional statements, problem solution, theorems and metamathematic notions. With the proposed semiotic mediation categories, we contribute methodologically to the corresponding study when the teacher centers on mathematical content derived from the problem solving process. This categorization differs from the one proposed by Mariotti (2012), since hers is based on actions realized with the artifact. The semiotic potential of the artifact, via the utilization schemes identified, is very close to the mathematical.

Also, the classification of the utilization schemes permitted us to define and group them into two categories: construction schemes and dragging schemes for exploration or verification. With our analysis we could establish links between the theory of semiotic mediation and instrumental genesis, but it was not possible to use them to extend the connection, due to time limitations. Our next step is to carry out another experimental design to determine if the products we have so far are useful to propose a model that ensures getting the most out of the semiotic potential of Cabri and ensuring the legitimate participation of the students in constructing knowledge during the teacher's semiotic mediation process.

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# RECOGNISING DIFFERENT ASPECTS AS A KEY TO UNDERSTANDING: A CASE STUDY ON LINEAR MAPS AT UNIVERSITY LEVEL

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*Learning linear algebra is a major challenge for students in the transition from school to university level mathematics. The construction of knowledge seems to be very difficult even for basic notions like linearity. In this paper a detailed analysis of the mathematical concepts that must be recognised by learners in order to solve standard tasks for linear maps is given and an empirical analysis based on interviews with learners solving these questions using the RBC-model is presented. The results hint at the problematic role of formal calculations and the importance of looking at the different contexts of epistemological actions.*

## INTRODUCTION

In the transition from school to university level mathematics linear algebra is a source of many problems. As Dorier, Robert, Robinet and Rogalsiu (2002) put it

For a majority of the students, linear algebra is no more than a catalogue of very abstract notions that they represent with great difficulty. In addition, they are submerged under an avalanche of new words, new symbols, new definitions, and new theorems. (Dorier et al., 2002, p. 95)

The authors call this “the obstacle of formalism”. In many countries the transition involves the shift from a little axiomatic or non-axiomatic geometry of vectors in school to a fully axiomatic linear algebra in the style of Bourbaki. It has been a subject of studies for many years and there have been many curricular reform actions and teaching experiments in this area. The reader may refer to Dorier & Sierpiska (2002) for an overview.

The authors summarise

The learning of linear algebra is a long process, requiring maturation of thought and an evolution of points of view. Short-term teaching experiments are likely to pass over some important factors of the learning of the subject. (Dorier & Sierpiska, 2002, p. 271)

Dreyfus (2002, p. 18) finds that in linear algebra “students tend to avoid this high level of abstraction by performing actions on a purely formal level” and concludes that “... even the most basic notions of linear algebra, including the very essence of linearity are often poorly mastered and understood by the students.” (ibid.).

Thus, it seems reasonable to investigate this connection further. Why do students fail to recognise certain mathematical structures like linearity? We approach this from an epistemological perspective utilizing the theory of abstraction in context. In the next section the theoretical background is explained, before it is made more concrete for the special learning environment.



## THEORETICAL BACKGROUND

### Abstraction in context

Hershkowitz, Schwarz and Dreyfus (2001, p. 202) define abstraction as the activity of vertically reorganising previously constructed mathematical structures into a new one. This is based on Davydov's notion of abstraction and is rooted in Leont'ev's view of activity theory (Leont'ev, 1981).

The authors model the processes of knowledge construction in mathematics in three stages: First there has to be a phase of need for a new mathematical construction, followed by a construction phase and completed by a phase of consolidation of the new construct. Their notion of abstraction does not view it as decontextualizing mathematics, but it is more like seeing common facets inside a context which includes, among other things, the history of the learner, the setting given by the task, and social relations as far as they are relevant to the learning.

They propose to study the second phase by studying epistemic actions, more specifically they postulate a model of nested epistemic actions for the phase of the actual construction. It comprises *recognising* a mathematical construct in a setting, *building with* the recognised mathematical constructs, and *constructing* as the three epistemic actions. These actions are nested in the sense that building-with requires a successful recognition and construction requires both recognition and building-with to have taken place.

Recognition is the action of recognising a known mathematical structure. It can happen by analogy or by specification (Dreyfus, Hershkowitz, & Schwarz, 2001) or by direct recognition of patterns (Schäfer, 2010).

As in (Schäfer, 2010), Oerter's theory of action (Oerter, 1982) is used to distinguish between different layers of relation to the object for the epistemic actions of recognition. Each action is performed on a specific *layer of relation to an object*:

Layers	Description
Singular	object is only existent in the course of the action
Contextual	object is discernible, but only inside its specific context of use
Formal	formal structure of the object without the use

Table 1: Layers of relation to an object (Oerter, 1982) as used in (Schäfer, 2010)

Thus, a recognising might be situational or contextual. To understand how these layers work, consider the following example:

A child might play a builder and use a stick as “drill” and forget about the drill after playing is finished. This use of drill would be singular. If playing goes on and other children start picking up sticks to use them as drills or the same child repeatedly uses sticks as drills, the relation to the object reaches the second layer. A “stick drill” object is created as a durable object beyond the momentary actions. But this stick drill still is

bound to a certain usage (playing builder), i.e., a contextual usage in a set of similar situations.

When the object is no longer tied to a certain context of use, but instead the formal structure is the only thing left, this object is at the formal layer. We define this layer to be the equivalence classes of the uses in different contexts. Many mathematical objects can be thought of as being in this layer, e.g. the notion of a triangle.

## **CASE STUDY**

Two pairs of students in their second semester were interviewed for about 90 minutes. They were subsequently given the three tasks described below. For each task the interviewer would present it to the students and try not to disturb their solution process. If the students explicitly asked for help, minimal help was given. After they were finished with the task the interviewer gave them a printout of a solution of the problem, ask the students to read it, explain the solution and transfer it to an example with different numbers. In this way it is possible to see what concepts in the problem they recognise for themselves and which (additional) concepts they can recognise in a solution.

The students had completed a first course in linear algebra which requires them to solve weekly tasks as homework. Each of the tasks had been part of one of those home exercises such that the task format cannot be considered as new.

The interviews were videotaped, transcribed and analysed according to the RBC-model combined with Oerter's layers. The two guiding questions were

1. Which constructs do the students recognise at which level?
2. Are there any patterns connecting the recognition or non-recognition to the layers of object relation?

## **TASKS**

The following three tasks were chosen for the interview and the study because they are typical tasks in linear algebra courses at German universities. They cover different aspects of linear maps.

The first task probes the connection of linear maps and systems of linear equations. The difficulties are that the system of linear equations is given by the rows in the table and only a vector with non-negative components is an acceptable solution.

1. Three metal alloys are given that comprise copper, silver and gold in the following percentages:

	copper	silver	gold
M1	20	60	20
M2	70	10	20
M3	50	50	0

Is it possible to mix these alloys, such that a resulting alloy is produced, that consists of 40% copper, 50% silver and 10% gold?

2. Is there a linear map  $F: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ , such that  $F(2,0)=(0,1)$ ,  $F(1,1)=(5,2)$  and  $F(1,2)=(2,3)$ ?

This task was presented in a sequence of subtasks, i.e., whether it would be possible to find a linear map that satisfies the first condition, next if it was possible to satisfy to first two conditions and finally all of them. A detailed analysis is given in the next section.

3. Let  $V$  and  $W$  be real vector spaces and  $A=(v_1, v_2)$  be a basis for  $V$  and  $B=(w_1, w_2, w_3)$  be a basis for  $W$ . A linear mapping  $F: V \rightarrow W$  is given by the matrix  $M=((3,2), (-2,-2), (4,0))$ .

Additionally let  $A^*=(v_1+v_2, v_2)=(v_1^*, v_2^*)$  and  $B^*=(w_1, w_1+w_2, -w_1+w_3)=(w_1^*, w_2^*, w_3^*)$  be bases of  $V$  and  $W$ , respectively. Determine the matrix representations of  $F$  for  $A^*$  and  $B$ , for  $A$  and  $B^*$ , and for  $A^*$  and  $B^*$ .

In order to solve this task the students need to understand how a change of basis influences linear map.

Certain mathematical concepts must be recognised by the students, depending on the possible way of solving. To give a more detailed description we focus only on the second task for the rest of the paper.

Possible solution A:	Recognise that $(2,0)$ and $(0,1)$ are linear independent and that $(1,2)=2*(1,1)-0.5*(2,0)$ . Recognising the linearity condition for $F$ and building-with it $F(4,3) = 2*F(1,1)-0.5*F(2,0)$ . Recognising that this validates the condition, because: $2*(5,2)-0.5*(0,1)$ is not $(2,3)$ .
Possible solution B:	Recognise the matrix representation of linear map $F$ and building three pairs of linear equations for the coefficients of the matrix with it by inserting the given values. Solve the system of linear equations to see the contradiction between them.

Possible solution C:	Recognise that $(2,0)$ and $(0,1)$ are a basis of $\mathbf{R}^2$ and that $F$ is uniquely determined by the images of a basis. Calculate the matrix representation of $F$ in the standard basis by solving the linear equations for the coefficients or alternatively use the matrix representation in the basis given by $(2,1)$ and $(0,-2)$ and the standard basis, i.e. writing down $(1,1)$ and $(4,2)$ as columns and calculate the coordinates of $(4,3)$ in the new bases and recognise that the image of this point contradicts the assumption.
	Other solutions are possible, but unnecessarily complicated and artificial, like changing both bases in the domain and codomain and calculate from there.

Table 2: Possible solutions for task 2 with all three conditions simultaneously

## SUMMARY OF THE INTERVIEWS RESTRICTED TO TASK 2

**Group A:** The first group starts by solving the problem for  $F(2,0)=(0,1)$  with help of writing down an abstract  $2 \times 2$  matrix representation of  $F$  in 4 unknowns and finding a solution by setting the second column to zero. Asked by the interviewer what the linear map is, they struggle. They say it is a linear map and the map is given by matrix multiplication, but fail to recognise that the product “matrix times vector” is a linear operation. Checking the linearity condition they manage to prove the linearity for this special matrix.

At the presentation of the problem to have  $F(2,0)=(0,1)$  and  $F(1,1)=(5,2)$ , the initial reaction is that the matrix mentioned above is the only possibility and so there is no such  $F$ . In the following they speculate that, since the first component of the image of  $(2,0)$  is zero, the first component of the image of every vector under  $F$  must have a zero as its first component.

After an explicit counterexample for this hypothesis by the interviewer the group manages to find the correct solution  $F(a,b)=(5b, 0.5a+1.5b)$  and write it down in matrix form. They argue that the function is linear, since the calculations were the same. Asked by the interviewer if the different numbers would change anything, one student says “Yes, that still does not change anything, erm, I would say, it works.”

The students do not solve the full problem of task 2 and are presented a detailed version of solution A as described above by the interviewer. The students are able to use this solution to work out the same problem with different vectors, but fail to recognise the uniqueness condition, i.e., that the images of a basis completely determine a linear map in this context.

**Group B:** This group starts by giving the equation  $F(x,y)=(x-2,y+1)$  for the initial problem. They work out that this function is not additive. Next, they try the function  $F(x,y)=(0,1)$  and find this also not being additive. Finally, they start with a system of linear equations and come to the solution that the first column of the matrix is  $(0, 0.5)$  and that the second column can be chosen arbitrarily. They remember that every linear

map can be represented by a matrix and that the representation would be given by the images of the basis of the domain space. Asked if the matrix would yield a linear map, one student answers “Erm, because in a linear mapping you have this one with the scalar multiplication, because one cobbles this together in the right way in the matrices and vectors.” Asked to make this more explicit, they give the formula and recognise the matrix multiplication is always linear and state “matrices are linear maps.” Moreover, they recognise that this process gives only one possible matrix for a linear map with respect to the standard base vectors.

Solving the equation for the parameters in the second column of their solution matrix they find that  $F$  is completely determined by the images of  $(2,0)$  and  $(1,1)$ . Next, they find that this linear map does not map  $(1,2)$  to  $(2,3)$ , which they recognise as a contradictory requirement. In studying the printout, they also recognise that the linear independence of  $(2,0)$  and  $(1,1)$ , and, thus, that the vectors are a basis of  $\mathbf{R}^2$ . Presented a printout of solution C one of them (student 2) does not understand why the images of the points become the columns of the matrix. Student 1 recognises that he had used the standard basis in their solution. He then explains

SB1            We know that the two points are linear independent. A basis has to be linear independent, therefore we know: this is a basis for the first  $\mathbf{R}^2$ . This basis is mapped onto those two. One writes them columnwise and then one gets the matrix of the linear map.

SB2            The matrix of the linear ... that is the matrix.

Finally they recognise that the matrix depends on the choice of base and are able to explain solution B and apply it.

## FINDINGS

As reported in (Dreyfus, 2002) the students fail to see that if a map is given by matrix multiplication is linear. They give a proof, too. While they are not working circularly, i.e., both groups start with linear equations and get the coefficient matrix, the students have not constructed and consolidated the concept of the matrix representation for linear maps with respect to given bases sufficiently.

Group A mainly acts in the contextual layer where the context is “concrete vector and matrix operations”, i.e., they recognise things that relate to this layer like the necessary dimensions of matrices and vectors, which linear equation one has to solve, etc. When asked what the linear map is, they can relate it to the product of “matrix times vector”, but are unable, to state why it is linear. But as student 2 states

SA2            a physicist would say it is linear, because there is not written a “square”, but this is a funny answer in someway

Thus, the recognition is contextual, and they cannot apply this to give a general proof of linearity or remember the fact from their lectures. Instead they remember that there should be “three axioms for linearity”, but are unable to give a formal definition without a subtle hint from the interviewer. After that they can prove by direct vector and matrix operations, that the  $2 \times 2$  matrices they look at give rise to linear maps.

When they try to understand solution A to the full problem they recognise that it is important to express to third vector in the domain as linear combination, but do not understand why. They can recognise the concrete operations:

SA2: In principle everything is done here with linearity. They have pulled out the factor before the vector somehow and expanded it.

They are able to transfer the structure of the matrix and vector operations to an example with different numbers, but as student 2 says “I know that it does not work, but I cannot give an argument.”

Group B acts mainly on the contextual level, too, but they act in two different contexts. One context is given by concrete matrix and vector operations, but they are able to change contexts and also recognise properties of sets of vectors, e.g., linear independence or the representation of a vector with respect to a basis.

While their knowledge about properties of linear maps is not so broad, e.g., they have to calculate whether the map given by  $F(x,y)=(0,1)$  for all  $x,y$  is linear, where in contrast group A explicitly states that a linear map must satisfy  $F(0,0)=(0,0)$  from the beginning. Group B remembers the formal definition and is able to apply it without mistakes. Moreover, they construct new knowledge or at least reproduce and consolidate knowledge, e.g., the concept of the matrix representation for linear maps with respect to given bases.

## RESULTS

Both groups achieve some understanding of the situation and are able to solve the initial problem and at least transfer to solution of the full problem to an example with different numbers.

In order to understand the solutions A and C it is necessary to recognise the linear independence of a pair of vectors in the domain and find an expression for the third vector. Furthermore, it is necessary to recognise either the linearity property for solution A or the recognise that the images of a bases uniquely determine a linear map. Group A does recognise that the vectors  $(2,0)$  in the domain and  $(0,1)$  in the image of  $F$  are linear independent, but they never come back to the notion of linear independence again. They never recognise bases in the interview (restricted to task 2). Once their epistemic actions are in the contextual layer of concrete vector and matrix operations, they almost never leave that context. They only do when presented the solution of type A, but since their concept of linearity seems not well consolidated they cannot build-with the information given. Group B on the contrary manages to get to this construction, although their concept of linearity is not well consolidated, but, because they are able to switch the contexts, they can relate the different concepts.

## SUMMARY AND OUTLOOK

Since this is a case study based on two interviews there is very little room for reasonable speculation of general effects. As known in the literature, students lack understanding of basic notions in linear algebra, even after the course in linear algebra. But it could be seen that this failure of understanding may stem from failed recognition

of important concepts, thus breaking the process of abstraction before it even could start. A key factor for the groups studied was that one group stay within one specific contextual layer which made it harder from them to recognise important concepts outside this layer, although they could recognise them in different situations before and after.

Understanding what contextual layers there are in linear algebra and how to help students not to restrict themselves to one context, but to be open to recognise structures from different contexts seem important problems for further study.

## ACKNOWLEDGEMENT

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# EQUATIONS IN ELEMENTARY SCHOOL

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*Although it is generally recognized that algebra has a role to play in the early mathematics curriculum, the issue of whether elementary school children are developmentally ready to use algebraic notation and to understand the syntactical rules of using algebra for solving equations is still a matter of debate. We examine, at the end of a third- to fifth-grade (8 to 11 years) intervention using a functional approach to algebra, how students try to solve word problems they represent as equations containing the same single variable on each side of the equals sign.*

## INTRODUCTION

We describe how elementary school students solve a word problem through an equation with variables on both sides of the equals sign, at the end of a three-year early algebra classroom intervention program.

The difficulties middle and high school students have with algebra often arise from their interpretation of the equals sign, the meaning of literals and algebraic expressions such as “ $3a + 7$ ”, and the presence of variables on each side of the equals sign (see review by Kieran, 1985). Some have attributed their difficulties, which frequently mirror stumbling blocks in history of mathematics (e.g., Collis, 1975; Filloy, Puig, & Rojano, 2008; Sfard, 1995), to constraints of cognitive development. Others have suggested that the difficulties stem from the computational focus of elementary mathematics as presently conceived (Davydov, 1991; Booth, 1988; Carraher, Schliemann, & Brizuela, 2000; Kaput, 1998; Schliemann, Carraher, & Brizuela, 2007; Schoenfeld, 1995).

There is considerable evidence (see reviews by Carraher & Schliemann, 2007 and by Rivera, 2006) that elementary school children can understand the logic underlying the symbolic manipulation of equations, even before they are familiar with algebraic notation. For example, they may understand that the same transformations applied to equal quantities result in still equal quantities (see interview studies in Schliemann, Carraher, & Brizuela, 2007). However it remains unclear, with few exceptions (e.g. Bodanskii's, 1991; Brizuela & Schliemann, 2004), whether young children can learn to solve word problems by means of written equations with a single variable on each side of the equals sign. We suspect that students can learn to understand the syntax for solving equations if they are given the opportunity to examine and compare quantities at various moments during the equation solving process.

We report on partial results of a third- to fifth-grade intervention with 22 students in two classrooms, where we adopted a functional approach to algebra, based on



understanding relations between quantities and numbers. (The study was supported by grant #0310171 from the National Science Foundation.) These results complement our previous analysis of their work with variables and functions (Schliemann, Carraher, & Brizuela, 2012). Here we describe how, at the end of fifth grade, students produced, solved, and interpreted an equation with a variable on each side of the equals sign.

### **Our approach**

Equations may not constitute the ideal entryway into algebra insofar as a variable can easily be misunderstood as standing for a single missing value, rather than a huge, possibly infinitely large, set of values (the domain). For this and other reasons, the concept of function may provide a more suitable pathway to algebra (e.g., Schwartz & Yerushalmy, 1992). Adopting a functional approach, we view algebra in elementary and middle school as a generalized arithmetic of numbers and quantities and the introduction of algebraic activities as a move from computations on particular numbers and measures towards thinking about relations among sets of numbers and variables (Carraher, Schliemann, & Schwartz, 2007; Schliemann, Carraher, & Brizuela, 2007).

Traditionally, instruction on equations aims at “solving for  $x$ ,” where  $x$  stands for an unknown number. But when equations are introduced in the wake of functions, the expressions on each side of the equals sign can be viewed as functions that vary over all values of  $x$  from the domain (even though many of them produce untrue statements). For example,  $5 + x = 8$  sets equal the functions  $f(x) = 5 + x$  and  $g(x) = 8$ . There is only one solution to the equation; all other values of  $x$  lead to syntactically correct but false statements.

A functional approach to equations promotes the shift of students’ attention from particular (e.g. the view that  $x$  has a determined value) to the much larger and less tangible set of possible cases. At first, natural language, graphs of events, and some combination of language and tables serve as the media for representing variables and expressing generalizations. Even though these means are typically less succinct than algebraic notation, because they are relatively more accessible to young learners, algebraic notation can initially piggyback on them and on the meaning of the situations being modelled.

Our pedagogical approach, rooted in constructivist theories of learning and cognitive development, also attributes a central role to instruction and access to new representational tools and procedures (Carraher & Schliemann, 2002). Because young students typically do not draw mathematical inferences directly from algebraic expressions, elementary school instruction needs to be grounded in familiar contexts and extra-mathematical situations. There is reason to believe that, within a functional approach to early algebra, students may be able to make headway if the symbols are related to the quantities and actions regarding the situation at hand.

To evaluate the impact of a functional approach to early algebra on students’ work with equations, we will briefly describe the main activities implemented in an early algebra longitudinal intervention. This will be followed by the analysis of students’ ways of solving a verbal problem by representing the problem as an equation.

## The intervention

The three-year intervention was carried out by researchers at an inclusion school, that is, a school where children with disabilities are placed in regular classrooms, in Boston, MA. The lessons highlighted the tension between algebra as a means for expressing properties and relations about worldly situations, on the one hand, and as a closed symbol system used to make statements about idealized mathematical objects, on the other. The lessons in third and fourth grades focused on functions and their representation through natural language, function tables, graphs, and algebraic notation. Students gradually learned to use letters to stand for unknown amounts and to establish correspondences across various representations of a given problem (Brizuela & Earnest, 2007; Carraher, Schliemann, & Schwartz, 2007; Schliemann et al., 2003). Algebraic notation was introduced first to help them express what they already knew and later to allow them to derive new insights (Brizuela & Schliemann, 2004; Carraher, Schliemann, & Brizuela, 2000; Schliemann, Carraher, & Brizuela, 2007; Kaput, Blanton, and Moreno, 2007). Later, we guided students to derive conclusions directly from mathematical representations such as graphs or equations and, ultimately, to solve equations using the manipulation rules of algebra.

In fifth grade, the researchers implemented ten 90-minute lessons, including discussion and embedded assessments, on the representation of verbal problems as equations. Each lesson was followed by a 45-minute homework review session by the students' regular teacher. The following is a brief summary of the lessons:

*Lesson 1* covered the representation of statements in a problem as two functions: “Anna went to the arcade with some money. She first spent five dollars playing video games. Then she won a prize where they doubled her money. Bobby went to the arcade with ten dollars. Then his mother gave him thirty more dollars. Afterwards, he spent half of all of his money playing video games. Represent Anna’s and Bobby’s money at the arcade at the end of the day.”

*Lesson 2* entailed representing and solving a problem involving a similar situation. The students used the structure shown in Figure 1, which called for operations to be noted in the oval shaped areas and results in the rectangles. This template was adapted for different equations produced by the students throughout the lessons.

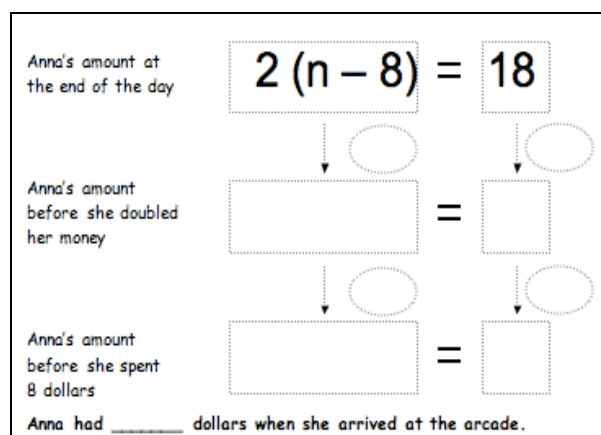


Figure 1: The template for solving equations

In lesson 3, students were given loose candies and unknown amounts of candies in closed containers. They were asked to determine how many candies were in each tube after being told that the number of candies in 3 boxes, 1 tube, and 6 candies was equal to the number in 1 box, 1 tube, and 20 candies [ $3x + y + 6 = x + y + 20$ ].

During lessons 4 and 5 students answered a written assessment and discussed their answers to the assessment.

In lessons 6 and 7, starting with a solution [e.g.,  $x = 5$ ], students produced new equations by proposing equal changes to  $x$  and to 5, checking at each step if replacing  $x$  with 5 led to same results on both sides.

Lesson 8 aimed at proficiency in solving equations with students solving a series of equations with variables on both sides of the equals sign.

During lesson 9 the students represented and solved a verbal problem leading to an equation with a variable on both sides of the equal sign [ $x+40 = 5x$ ] and related the solution to the story context.

In Lesson 10 the students were given a second assessment.

## RESULTS

We analyse students' attempts to represent and solve the following problem at the end of the school year, more than eight weeks after the 10 lessons on equations we described, using a template similar to the one in Figure 1:

Claudia and Adam have been playing with numbers. They each created a rule for changing any positive number you give them. Claudia's rule: I triple the number and then add 5. Adam's rule: I double the number and then add 12 to it. Write their rules with algebra. Write an equation [in the template] that shows that Claudia's rule would give the same number as Adam's rule and solve the equation. Explain what the solution means.

### Written Assessment Results

Fifteen students (68.2% of the 22 students) correctly represented Claudia's rule ( $3 \times n + 5$ ), Adam's rule ( $2 \times n + 12$ ), and the equation  $3 \times n + 5 = 2 \times n + 12$ . Three students produced operational rules like  $\times 3 + 5$ , with no explicit placeholders for variables. One student wrote  $3 + 5$  and  $2 + 12$ , and three students produced unclassifiable expressions.

Ten students (45.5%) correctly solved the equation using the rules of algebra. Only one student attempted to apply different operations to the two sides of the equation. Of the five students who correctly generated the equation but did not solve it, four proposed equal operations for each side of the equation, but failed to correctly implement them.

### Interview Results

A week after the assessment, in the individual interview, students were asked to discuss their answers to the problem and, if they had failed to do so before, to solve the equation and to explain what the solution  $n = 7$  meant.

Solving the equation: The ten students who had solved the problem in the written assessment were again successful in the interview setting; eight additional students now achieved the correct solution, increasing the number of correct answers to 18 (82%). Because we viewed the interviews as further opportunities for participants in the study to learn, the initial answer provided by each student to any of the questions was registered but the interviewer asked further questions which could help students produce more advanced answers. In the interview, one of the eight students solved the equation with no help from the interviewer, four needed a small amount of support, and three needed substantial support. The categorization in terms of a small or substantial amount of support was determined by two researchers who separately examined the videotaped interviews, with discrepancies between the two evaluations resolved through discussion with the research group.

The meaning of the solution: Out of the 18 students who solved the equation in the interview, nine (50%) were now able to explain the meaning of the solution  $n = 7$  saying, for example, “For the rules to be equal,  $n$  has to equal 7” or “That at seven, um, Claudia’s rule will equal to Adam’s rule.” Five other students simply stated that  $n$  was equal to 7 and four gave wrong or uninterpretable explanations.

### **Students’ Difficulties**

Although each student’s interview was unique, most of their difficulties, occurring in isolation or combined, could be classified in two main groups: those related to the representation of the unknown quantity (six students) and those emerging from operating on multiple values of the unknown quantity (nine students), with three students presenting both kinds of difficulties.

## **DISCUSSION**

Students’ performance at the end of the three-year early algebra intervention was generally encouraging. In the written assessment, more than two thirds of the fifth graders in the study could produce the equation representing the problem and nearly half of them solved the equation, which included a variable on both sides of the equals sign. In the follow up interview, in interaction with the interviewer, the percentage of students finding the correct solution to the equation increased to 82%. This is no trivial achievement if we consider previous studies on middle and high school students’ difficulties with algebra (e.g., Filloy & Rojano, 1989). In the interview, only four students ultimately failed to solve the equation, displaying various types of difficulties. Moreover, half of those who solved the equation could explain its meaning in relation to the verbal problem they were given.

While previous research has shown that even high school students do not realize that the left and right sides of the equation are no longer equivalent when an incorrect value is used in an equation, seven of our fifth graders explicitly stated that using a value other than the solution to the equation would destroy the equality.

A few students were not initially successful in writing expressions containing an unknown. Some of them attempted to deal with this by trying to instantiate the

problem with specific numbers. This indicates that they are still in the process of accepting that one can work with an unknown quantity using the same arithmetical rules that one applies to numbers. However, when the problem was structured with the unknown in place, some of these same students were able to solve the equation.

Previous research has highlighted the difficulties that older students, including adolescents, have in dealing with equations. Moreover, the transition from the semantics of the problem to the syntactic rules of algebra is a rather challenging task that our project tried to address, even though much more research is needed to understand how students can be guided towards learning, using, and understanding the syntactic rules of algebra. Despite these difficulties, our preliminary data show that young students can use algebraic notation to represent verbal problems and attain some degree of success in manipulating symbols to solve equations with a single variable represented on both sides of the equals sign. Their difficulties may be related to the arithmetic curriculum students are exposed to in elementary school, not to cognitive-development constraints. Our data suggest that a functional approach to early algebra and opportunities to discuss the logic implicit in the syntactic rules of algebra can help elementary school students to learn basic algebraic representations and procedures. As we showed in a follow-up study of students in this project (Schliemann, Carraher, & Brizuela, 2012), participating students were better prepared to expand their understanding of algebra in middle and high school than their control group peers.

We hope our work will provide resources for the implementation of algebra in elementary schools, as recommended by the National Council of Teachers of Mathematics (2000), and will contribute to a balanced view of what young children can do regarding the understanding of functions and solution of equations.

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# FROM ‘ARMCHAIR PEDAGOGY’ TO EXPERIMENTAL RESEARCH AND TO CASE STUDIES

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*The transformation of studies on the teaching and learning of mathematics into the scientific discipline mathematics education depended largely upon the switch from ‘armchair pedagogy’ (Kilpatrick) to experimental research providing empirical data and facts instead of opinions. It is studied here how respective developments within neighbouring disciplines became received and integrated into the emerging mathematics education and in particular within PME. Using textual analyses of characteristic documents, the profiles of experimental researches in the early times of PME are reassessed. The results show more restricted notions than expected.*

## INTRODUCTION

In a research report for PME 35, it were studied, for the case of Germany, the empirical roots leading to the unfolding of the research activities of PME (Schubring, 2011). It focused in particular the period at the beginning of the 20<sup>th</sup> century to unravel connections with the emerging field and community of mathematics education. In fact, the development of psychological and educational research on learning mathematics in Germany in this period was there shown to have become integrated with the first manifestations of research in mathematics education. Then, “experimental pedagogy” – as represented in particular by Ernst Meumann, a former student of the famous psychologist Wilhelm Wundt – had differentiated from psychology and propagated the necessity of experimental foundations for the “didactics” of the various school disciplines, especially of mathematics.

One is led to compare these results with the seminal study by Kilpatrick on the history of research in mathematics education, which is largely based on research published in the United States (Kilpatrick, 1992). Searching, too, the roots for such research, he looked for disciplines constituting such roots. Basically, it were just two, mathematics and psychology. While mathematicians provided some ideas, reflecting upon their own thinking modes – in particular on intuition - and upon problem solving behaviours, psychology used to be increasingly interested in mathematical thinking, psychologists showed a systematic interest in mathematics. On the international level, analysed by Kilpatrick, and in particular for the United States, where the bulk of related studies were published, differentiations analogous to Germany can be observed. At first, it was psychologists who investigated in their laboratories issues of mathematics. The most remarkable case was the French Alfred Binet (1857-1911), who is commonly held to be the inventor of intelligence tests. But as Kilpatrick aptly pointed out, Binet’s aim was to assess mental ability – not for ranking and rather for diagnosing; for this, he had developed “complex” tasks: “Binet devised his IQ scale for the limited purpose of identifying those students whose performance suggested that they might be in need of



some special education“; he rejected innate, fixed and hereditary concepts of intelligence (Kilpatrick, 1992, 8). Later on, Binet turned to educational psychology; then, he argued for replacing “a priori assertions by precise results based on data“ (ibid., 7). Given that such research could not be undertaken within a laboratory, it required classroom studies and thus he called for joint investigations with teachers. Yet, experimental pedagogy in France remained basically restricted to Binet’s laboratory. A collaborator at this laboratory was, in 1920, the recently graduated Jean Piaget, developing later on a genuine methodology for experimental research on psychology of mathematics education, with a truly international impact.

On the other hand, Binet’s concept of intelligence tests was transmitted to the United States, but became there transformed into a radically different conception, a hereditary theory of IQ, due in particular to Thorndike and in general to the dominant behaviourism. Research in psychology on mathematical thinking used to be characterised by

the heavy use of elaborate statistical procedures - such as correlation analysis, regression analysis, and factor analysis - which ordinarily require the questionable assumptions that relationships are linear and effects additive (ibid., 9),

reducing thus its significance for mathematics education.

Here, too, an educational psychology, studying teaching and learning mathematics, became differentiated from psychology,. The methodology of the latter, too, was not well adapted to educational processes:

In this tradition, research undertakes to examine the “effects” of instructional “treatments.” Teaching is taken as a treatment and learning as an effect. In an analogy to research in agriculture or pharmacology, the effects of various treatments are studied by systematic variation in the treatments followed by careful measurement of the presumed effects (ibid., 9).

Furthermore, within education as a science, numerous studies were undertaken on teaching and learning mathematics, since the end of the 19<sup>th</sup> century. Despite their considerable number, they can hardly be called to constitute experiments, and results were often contradictory (ibid., 13).

In his retrospect on the state of art before systematic research on mathematics education, Fischbein observed a separation into two types of relevant publications:

Some papers expressed the authors’ suggestions concerning the teaching of mathematics: new topics, new examples, new ways of teaching. No empirical data were, generally, invoked for supporting the authors’ ideas.

A second type of publications was focused on applications of psychological concepts and theories to mathematics (Fischbein, 1990, 5).

## **THE TAKE-OFF OF RESEARCH BY MATHEMATICS EDUCATORS**

As Kilpatrick has remarked, the early practitioners of mathematics education, which emerged at a low degree by the turn to the 20<sup>th</sup> century, did not themselves much research (ibid., 12). And as Schubring observed, by the beginning of the international

activities of IMUK/CIEM, related research used to be performed in the neighbouring disciplines (Schubring, 2011, 123 ff.). Yet, for the following inter-war period, there is more activity than reported by this study: Gustav Rose, having studied mathematics and psychology in Göttingen, thus in the context of Felix Klein, obtained his PhD, in 1914, for an experimental study in psychology and published in 1928 empirical research results, based on systematic classroom observations (Rose, 1928; Bauersfeld, 2012, 75).

But how did the various strands of research performed in neighbouring disciplines become integrated into the emerging scientific discipline mathematics education? Schubring's study seems to assume that it was by PME that the relevant research unred to be assessed, adapted and improved. Although such an assumption seems to be correct at a first instance, we will see whether it is justified.

The *first* decisive change to be observed was a change in the paradigms of psychology. While it used to be split up in a number of different schools, following a proper (pre-) paradigm, and not communicating with each other – like Gestalt psychology, associativity psychology, “Denk” psychology, attention psychology, etc., the *cognitive move* became initiated, with Piaget's genetic epistemology as its first major paradigm. These new directions lent themselves not only more directly to research in mathematical thinking, but they were also received very actively by the now constituting researchers in mathematics education.

Actually, it was a rather informal group, which was eager to assess new conceptions into ongoing research: the *Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques* (CIEAEM), founded in 1950 by Caleb Gattegno. ICMI still being dormant, this group was, with its yearly meetings – in always another country –, the first truly international forum of communication between mathematics educators. Already at its first meeting, Gattegno had expressed his intention to invite Piaget. In fact, Piaget participated at the third and the fourth meeting, at Herzberg 1951 and Melun 1952 (Félix, 1986); this turned out to be the starting point of a close cooperation of Piaget with mathematics educators, resulting among others in the book elaborated jointly with leading figures of CIEAEM like Beth, Dieudonné, Lichnérowicz, and Choquet: *L'enseignement des mathématiques* (1955). ICMI, after having been revived, was eager, too, to integrate Piaget's conceptions as foundation for psychology of mathematics education (Schubring, 2011, 128).

The *second* decisive change, often not taken into account, is constituted by the change from small scale research – performed by isolated scientists or typically by PhD dissertations, thus resulting in the piecemeal, incoherent and often contradictory results – to large scale research, enabled by funding agencies. In fact, it was by a huge grant by the National Science Foundation in 1958, in the wake of the Sputnik shock, that the School Mathematics Study Group (MSG) became established, with Ed Begle as its director. MSG turned out to be the best developed curriculum project of *new math*, with systematic and extended testing of the elaborated textbook versions in classrooms and ever new rewriting according to the experiences made. In a similar vein, Heinrich

Bauersfeld received in 1968 a one million German marks grant from the *Volkswagen-Stiftung* and was thus able to start the important Frankfurter Project, developing the innovative *alef*-textbook series for primary grades, testing preliminary versions, with control groups, in the entire *Land Hessen* (Bauersfeld, 2012, 77).

Begle developed himself on the one hand to the most sophisticated practitioner of experimental research in mathematics education. An integral component of SMSG became his accompanying experimental studies on achievements and abilities, published as the NLSMA: the National Longitudinal Study of Mathematical Abilities

examining the mathematics attainment of over a hundred thousand students. It had begun gathering data in September 1962 and had tested students in the autumn and spring each year for 5 years. The purpose was to ascertain the effects of the new-math curriculum revision efforts (Kilpatrick, 1992, 29).

A list of its 33 volumes is given in (Begle, 1979, xxiv-xxvi). On the other hand, Begle now acted as the most prolific and energetic advocate for empirical research on mathematics teaching and learning. It is not as well known that he participated in 1959 at the two key events for curriculum reform: at Bruner's conference in Woods Hole, and at the OECD seminar on modern mathematics in Royaumont; there, his plea for empirical research remained practically without effect. At the first International Conference on the teaching of mathematics, ICME 1 at Lyons, in 1969, his key address "The role of research in the Improvement of Mathematics Education", however, was well heard. His provocative claim was often quoted:

I see little hope for any further substantial improvements in mathematics education until we turn mathematics education into an experimental science, until we abandon our reliance on philosophical discussion based on dubious assumptions and instead follow a carefully correlated pattern of observation and speculation, the pattern so successfully employed by the physical and natural scientists. [...] We need to start with extensive, careful, empirical observations of mathematics teaching and mathematics learning. Any regularities noted in these observations will lead to the formulation of hypotheses. These hypotheses can then be checked against further observations, and refined and sharpened, and so on (Begle, 1969, 110).

In his book *Critical variables* of 1979, which summarizes the essential of his empirical research endeavours, Begle emphasized even more critically that various of the most firm convictions about mathematics teaching prove to be wrong when scrutinized empirically:

Much of what goes on today in mathematics education is based on opinions that are so firmly held that the thought of doubting them crosses very few minds. Yet most of these opinions have no empirical substantiation, and in fact many of them are, if not wrong, at least in need of serious qualifications. [...] These erroneous opinions are often the cause of inefficiencies in our educational programs. Until we learn to recognize them and until we start to pay more attention to facts than to opinions, no matter how plausible the latter may be, we can do nothing to eliminate these inefficiencies (Begle, 1979, xv f.).

Given his pleas and being him the pioneer of large-scale empirical research, Begle could be counted as a father of PME. While participating at all the first three ICMEs

(1969, 1972, 1976), which prepared the founding of PME, his premature death in 1978 prevented any involvement. On the other hand, his methodology was exactly of the type, which Kilpatrick had critically related as the heavy use of statistical procedures, with all kinds of regression analysis, etc. And although admitting that, for obtaining valid results, the numbers of students and of teachers “need not be as great as those we have dealt with in our SMSG studies”, he underlined his sticking to large-scale empirical studies: “Nevertheless, to restrict ourselves to small scale observations would be to sacrifice the generality of our theories” (Begle, 1969, 111).

Commenting on critical appraisals of Begle’s research methodology, Kilpatrick has rightly pointed out that by then approaches were still dominating, which he aptly coined “armchair pedagogy”, i.e. proposals for curricular and teaching practice by people who had never taken a careful or systematic look at actual mathematics classrooms (Kilpatrick & Sierpinska, 1998, 531). Yet, the low degree of use of this enormous amount of empirical data collected is striking.

### **WHAT RESEARCH ENTERED INTO PME AT ITS BEGINNINGS?**

Given the many strands of research having developed so far and in particular the evolution of experimental studies, it is now the question to see what of them did enter into the first works of PME.

At ICME 1, in 1969, upon proposal of its president Freudenthal, a round table took place, “devoted to the psychological problems of mathematical education”, organized by Ephraim Fischbein and assisted by Lee Shulman. Due to the success of this spontaneous action, a working group on this subject was organized for the second ICME, in 1972, now with invited papers. Presided again by Fischbein, it proved to be “the most popular group” at Exeter, so that at ICME 3, in 1976, there took place a second such workshop; there, it was decided to constitute a permanent body, first called IGPME – International Group for the Psychology of Mathematics Education -, and later abbreviated to PME.

The first proper PME Conference took place at Utrecht, in 1977, and was organized, without much preparation, by Freudenthal, on request by Fischbein, then still in Romania. Unfortunately, no Proceedings of this first of the yearly Conferences were ever published. The only published elements are Freudenthal’s opening address (Freudenthal, 1978) and an excellent report (in German) by a German participant (Schmidt, 1978). However, most of the papers in the same issue of ESM as Freudenthal’s address, are practically papers from PME 1 (Bessot & Comiti, Laborde, Streefland). There were 84 participants, basically from Europe; strangely, nobody from the USA, but five from Canada (Nicol et al., 2008). In his opening address, Freudenthal constantly spoke of PME as “Professor Fischbein’s group” – apparently to distinguish it from Zoltan Dienes’s group “The International Study Group for Mathematics Learning”.

In retrospect, Fischbein characterized the emphasis of the early PME conferences as follows:

More and more, psychological problems inspired by school reality captured the interest of researchers: computation, numbers, fractions, proportional reasoning, elementary arithmetical operations, visualization, geometry, the use of symbols, mathematical proofs, computer environments, and so forth (Fischbein, 1990, 5).

There is no particular emphasis on experimental methods and approaches. Actually, Freudenthal presented himself as an outsider: “I am not a psychologist, and I could hardly define in which way I am interested in psychology” (Freudenthal, 1978, 1) – although, according to Schmidt’s report, commenting many of the presentations.

At Utrecht, a few presentations applied the methods of statistical analysis (for instance de Leeuw (Netherlands), K. Hart (England), R. Rees (England), Cohors-Fresenborg (W.-Germany)), thus following what was then standard methodology in educational psychology; other papers seem to have applied a proper approach to classroom observations (see Schmidt, 1978).

Fischbein himself, in a paper together with Dina Tirosh, reported on interviews with students regarding their intuition of the infinite. As is well known, intuition was his main research focus. His paper at ICME 2 presented a methodological discussion of the notions of intuition, of structures, and of problem solving – not focusing on experimental methods (Fischbein, 1973).

Richard Skemp, regarded as a co-founder of PME, presented at Utrecht in his paper “*Relational understanding and instrumental understanding*” a theoretical model for interpreting student’s achievements (Schmidt, 1978, 177). It is revealing to analyse under this point of view Skemp’s famous and influential textbook *The Psychology of Learning Mathematics* (1971), given his double formation in mathematics and in psychology and his year-long lecturing of psychology. The book provides a completely theoretical analysis of mental activities involved in learning mathematics, developing in particular his notion of ‘schema’. The book is entirely “self-contained” in the sense of not relating to any other literature or conceptualizations. There is not the least reflection on experimental methods, no empirical data or evidence are ever presented.

Given the profiles of experimental research in these early periods of PME, Bauersfeld sharply criticized the methodological approaches of the beginnings:

From the very beginning I was unhappy with the exclusive concentration on Psychology only, which meant focusing on the individual and neglecting the social dimensions of the complex teaching-learning processes. Research on the complex problems of learning/teaching-processes and of teaching teachers to teach mathematics will not arrive at helpful constructive information as long as such vast domains as language, human interaction (not the usual psychological interaction of variables!) and rich case studies are neglected and/or treated by inadequate research methods. Usual refusal sounded like: “We have enough to do with psychology!” Freudenthal was interested, and used to ask me accidentally: “What shall I read about it?” But at that time there was neither easy access available nor were there convincing overviews from sociological perspectives (Nicol et al., 2008).

In his chapter on “future perspectives” for PME research, in the volume *Mathematics and Cognition* (1990), devoted to the first 13 years of PME activities, Balacheff in fact rather gave a retrospect on the research profiles of these 13 years. And, as if giving an answer to this later remark by Bauersfeld, he included as one of four directions of future Research “the social dimension of teaching-learning phenomena” (Balacheff, 1990, 136). In the respective section of his paper, Mathematics Learning from a Social Perspective, he postulated: “it is imperative that researchers try to interpret mathematics learning from a social perspective” (ibid., 139), but he tried to show that there had already been performed relevant such research. Actually, a part of quoted relevant publications were journal papers or books, made outside of PME, and those quoted from PME Proceedings constituted rather traditional classroom studies.

What should become decisive for progress in empirical research on the psychology of learning mathematics was the change from quantitative methods to qualitative methods.

In fact, it was the methodology of *case studies*, substituting the emphasis on studies dealing with large-scale groups, which proved to become decisive for researching on mathematics learning understood as a social process. Bauersfeld, one of its pioneers in mathematics education, had to organize workshops at meetings of mathematics educators, trying to overcome deeply grounded doubts whether this methodology might yield valid results (Bauersfeld, 2012, 77 f.; 82 f.).

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# PLANNING, MONITORING AND MULTIPLE SOLUTIONS WHILE SOLVING MODELLING PROBLEMS

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*In a quasi-experimental study that was carried out in the framework of MultiMa-project, we compared two groups of 9<sup>th</sup> graders from six middle track classes. In one group modelling tasks, where the solutions do not demand making assumptions about the missing data, were treated for five lessons. In the other group students solved similar modelling problems, where different assumptions were possible and students had to develop two and more different solutions five lessons long. Before and after the teaching unit students answered questionnaires about planning and monitoring their activities while solving problems. The analysis showed the positive influence of treating and developing multiple solutions on students' planning and monitoring activities.*

## INTRODUCTION

Development of multiple solutions by students is an important part of curriculum in different countries (NCTM, 2000). However, we do not know much about the influence of treating multiple solutions on students' learning. In the MultiMa-project (Multiple solutions for mathematics teaching oriented towards students' self-regulation) the impact of teaching multiple solutions on students' performance, affect and metacognitive activities was investigated. In this paper we focus on treating multiple solutions in the classroom, on developing multiple solutions by students, and also on students' planning and monitoring activities while solving modelling problems.

## THEORETICAL BACKGROUND

In this section we report on three theoretical issues: (1) planning and monitoring as metacognitive activities, (2) developing and treating multiple solutions, as well as (3) modelling problems and multiple solutions.

### Planning and monitoring as metacognitive activities

Metacognition and cognition are important for the performance in cognitive tasks. The relationship between cognition and metacognition is explained by Garofalo and Lester (1985, p. 164) as "... cognition is involved in doing, whereas a metacognition is involving in planning and choosing what to do and monitoring what is being done". "'Metacognition' refers to one's knowledge concerning one's own cognitive processes and products or anything related to them ..." (Flavell, 1979, p. 232). Metacognition includes among other things active monitoring, planning and consequent regulation of cognitive processes in order to achieve goals.

In the discussion about how to solve mathematical problems successfully, planning and monitoring are considered to be important activities. Polya's (1948) description of solving problems consists of four steps: (1) understanding the problem, (2) devising a



plan, (3) carrying out the plan, and (4) looking back. The second and the fourth step refer mainly to the planning and monitoring activities. Garofalo and Lester (1985) also included planning solution and checking results in their list of persons' activities that help to solve complicated problems.

Most research results from correlational and interventional studies support the importance of metacognition for students' performance (see summary by Schneider and Artelt (2010)). German 15-year-olds from the high academic track not only outperform students from the low academic track, but also know more about metacognitive activities (Schneider & Artelt, 2010). A part of the program for improvement in metacognition (IMPROVE) that was developed and evaluated in Israel includes stimulation of planning and monitoring activities with a help of questioning (Kramarski, Mevarech, & Arami, 2002). The following questions were among others: What strategy, tactic, or principle can be used to solve the problem or complete the task and why? Does it (the result) make sense? How can I verify the solution? The analysis showed the positive impact of metacognitive instructions on low and higher achievers from the 7th grade (12 years old).

### **Developing and treating multiple solutions**

In the last decades, the principle focus of research in the field of developing and teaching multiple solutions has mainly been on students' performance. Whereas in the high achievement countries, such as Japan, teachers demand to develop multiple solutions of a problem, German and American teachers are often highly satisfied with one solution only (Hiebert et al., 2003). Teachers believe that presentation of multiple solutions confuse students and do not stimulate their development in mathematics classrooms (Leikin & Levav-Waynberg, 2007).

In the domain of mathematics, several experimental studies showed positive effects of treating multiple solutions on performance and cognitive flexibility by students with sufficient prior knowledge (Große & Renkl, 2006; Rittle-Johnson & Star, 2007). In these studies the main principle for teaching multiple solutions was stimulation of connection between different solution methods. This principle is based on the constructivist theories of learning, which argue that developing different solutions and representations helps students acquire multiple representation of the subject matter and improve their performance.

Conceivably, the treatment and development of multiple solutions stimulates planning activities. While developing multiple solutions of modelling problems, students might identify the missing data and think about possible assumptions that allow to develop two results before they begin with solving the task. Therefore, they might plan their solution. Further, they can compare the results and control their activities frequently if they have to develop more than one solution.


Rittle-Johnson and Star (2007) have investigated, whether comparing two solution methods of the same problem or presenting two solution methods using different problems effects students' procedural flexibility. As students of the first group were more flexible in the choice of the appropriate solution method, we assumed that

metacognitive abilities, like planning and monitoring can be improved due to the treatment and development of multiple solutions. In this study, we aim to prove this assumption.

### Modelling problems and multiple solutions

Students' improvement in ability to solve problems with close connection to reality is an important goal of mathematics education. The core of activities while solving modelling problems is the demanding transfer processes between reality and mathematics (Blum, Galbraith, Henn, & Niss, 2007).

We distinguish between three types of solutions while solving modelling problems. First, multiple solutions can be constructed due to the variation in mathematical solution methods. The second type of multiple solutions can be developed if students have to make assumptions about the missing data and, thus, get different outcomes/ results. The third one includes the variation in mathematical solution methods as well as in different outcomes/ results. In this paper we report on the study carried out in order to explore the effects of treating the second type of multiple solutions on students' learning.



**Parachuting**

When “parachuting”, a plane takes jumpers to an altitude of about 4000 m. From there they jump out the plane. Before a jumper opens his parachute, he makes free fall of about 3000 m. At an altitude of about 1000 m the parachute opens and the sportsman glides to the landing place. While falling, the wind carries the jumper away. Deviations at different stages are shown in the table below.

Wind speed	Side deviation per thousand meters during free fall	Side deviation per thousand meters while gliding
Light	60 m	540 m
Middle	160 m	1440 m
Strong	340 m	3060 m

What distance does the parachutist cover during the entire jump?

Figure 1: Modelling task “Parachuting”

While solving the modelling task “Parachuting”, among various assumptions, also those about the wind power in the respective falling stage have to be taken (see Fig. 1). Depending on the assumed wind power, students get different results using Pythagoras' Theorem as a mathematical solution method.

The influence of treating multiple solutions while solving modelling problems on students' self-regulation was investigated in the study by Schukajlow and Krug (2012). In this study planning and monitoring activities were used for the conceptualisation of self-regulation. The self-regulation of learning was measured on the basis of the statements that refer to setting goals, making a plan for attainment of these goals, and monitoring the attainment of the goals set. The results showed that the group, in which multiple solutions were treated, reported significantly more often on their self-regulation in post-test than the group, where students were instructed to develop one solution only, under control of self-regulation in pre-test. This finding points out that treating multiple solutions can have positive influence on students' metacognitive

activities such as planning and monitoring ones. In order to prove this assumption, we analysed this study's data on the improvement in both activities.

Another important question is what role the number of solutions really developed by the students play in students' learning. A recent study showed that treating multiple solutions in the classroom does not always result in development of multiple solutions by all students: 4% of students could not find any solution, 38% found one and 58% two and more solutions (Schukajlow & Krug, 2012).

As the treatment and development of multiple solutions can stimulate students' reflection on the questions: *How can different solutions be developed?*, *Do the results of multiple solutions significantly differ from each other?* and *Do the results make sense?*, we expected the positive effects of both factors on planning and monitoring activities.

### **Research questions**

The research questions of the study were:

- Do students' planning and monitoring activities differ according to the possibility to develop multiple solutions? In particular, whether treating multiple solutions while solving modelling problems results in more frequent planning and monitoring activities?
- Does the development of multiple solutions influence students' planning and monitoring activities positively?

## **METHOD**

### **Design and sample**

138 German ninth graders (42.8% females; mean age = 15.2 years) were asked about their planning and monitoring activities while solving complicated word problems before and after a five-lesson period (see Figure 2). Three schools with two middle track classes each participated in this study. Each of six classes was divided into two parts with the same number of students in a way that the average achievements in the both parts did not differ and there was the approximately same number of males and females in each part. In one part of each class students were instructed to develop multiple solution of modelling problems (group "multiple solutions") and in the other part to develop one solution of these problems (group "one solution"). The students of groups "multiple solutions" and "one solution" were taught in different classrooms.

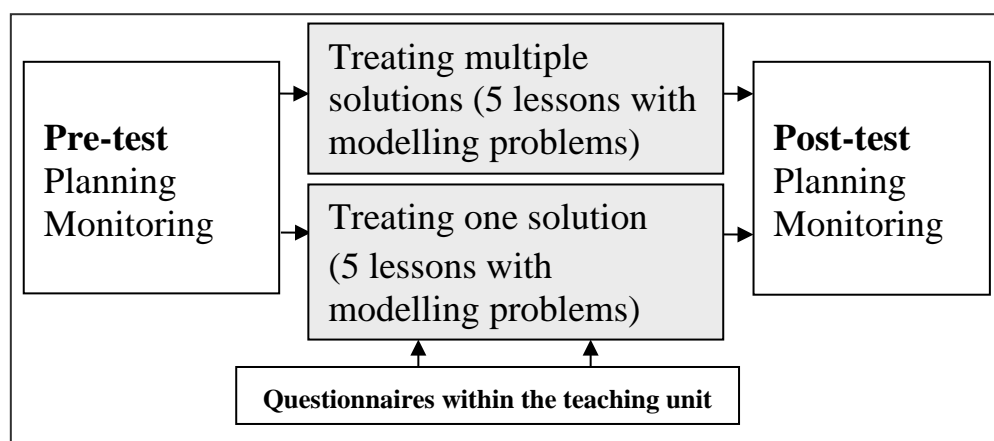


Figure 2: Overview of the study design

Four teachers that participated in this study received instructional manuals with all the tasks to be treated in each group, with the solutions of the problems, and with a detailed plan of the teaching unit. Further, all teachers were instructed about specific ways to promote the development of multiple and one solution while treating modelling problems. As each teacher instructed the same number of student groups in the “multiple solutions” and “one solution” conditions, the influence of a teachers’ personality on students’ learning did not differ between both conditions. In order to observe the implementation of the treatment, one member of the research group was present in each lesson.

### Treatment

The student-centred learning environment from DISUM-project (c.f. Schukajlow et al., 2012) was taken as a base of the teaching method that we applied in the recent study. Elements of “directive” instruction complemented this teaching method. In both experimental groups the same methodical order was used. Students solved a modelling task according to a special kind of group work (alone, together and alone) and then a teacher presented the solution (or different solutions) or otherwise students discussed their solution (or different solutions) in the whole group in the classroom. The teacher summarised a lesson and reflected on the key points of each experimental group. In the “multiple solutions” condition, the teacher emphasised the development of different results by estimating the missing data and made the connection between different solutions a subject of discussion. In the group “one solution”, the teacher focused on the development of one solution only.

In order to stimulate the construction of multiple solutions in one experimental group and to prevent the development of more than one solution in the “one solution” condition, two similar versions of each task were developed. Each problem in the group “multiple solutions” required the construction of two solutions. In the task “Parachuting” (see Figure 1), the following question was posed: “What distance does the parachutist cover during the entire jump? Find *two* possible solutions”. Students in the group “one solution” solved similar versions of the problems that had to be solved by the students from “multiple solutions” group. But unlike them, they had to deliver only one solution. The main data that are needed to solve these versions of the

problems were specified. In the one-solution version of the problem “Parachuting” the main data were the wind velocity and altitude in which the jumper opens his parachute.

### Measures

Students’ planning and monitoring activities while working on problems with connection to reality were measured using a 5-point Likert scale (1 = not at all true, 5 = completely true) before and after a five-lesson teaching unit (see Figure 2). The sample items were for scale “planning” (4 items) “If I solve a complicated word problem ... I make a plan” and for scale “monitoring” (8 items) “If I solve a complicated word problem ... I prove at the end, whether a result fits the problem approximately”. Both scales were adapted from the study that was carried out by Rakoczy et al. (2005) and already used in other studies (see e.g. Schukajlow & Leiss, 2011). The reliability values (Cronbach’s Alpha) were in both pre- and post-test .66 and .74 for planning and .82 and .84 for monitoring.

The number of solutions that were developed by students during the teaching unit was measured using students’ questionnaires. After every lesson, the students were asked about the number of solutions they developed for each modelling problem in this lesson. For example: “While solving the problem “Parachuting” I developed today ... (0: no solution; 1: one solution; 2: two solutions; 3: more than two solutions)”. Students’ answers were summarised to the mean score, which we used for further analysis of the data.

## RESULTS AND DISCUSSION

In order to control the implementation of the treatment, all lessons were observed by at least one member of our research group. The observations confirmed the correct implementation of instructions in both treatment conditions (c.f. also Schukajlow & Krug, 2012).

### Impact of treating multiple solutions on planning and monitoring activities

First, the ANCOVA with “treatment condition” as independent measure, “planning” in post-test as dependent measure, and “planning” in pre-test as covariate was conducted (see the means in Table 1). The analysis shows that students of the group “multiple solutions” reported in post-test to have planned their activities while solving word problems more often than students of the condition, where only one solution was treated ( $F(118, 2)=3.5$ ,  $p=.07$ , effect size  $(\eta)^2=.03$ ). The similar analysis with “monitoring” in pre-test as dependent measure and “monitoring in post-test” as covariate also reveals a positive impact of treating multiple solutions on students’ monitoring activities ( $F(118, 2)=9.8$ ,  $p<.01$ , effect size  $(\eta)^2=.08$ ).

	Treatment of multiple solutions		Treatment of one solution	
	Mean	SD	Mean	SD
planning in pre-test	3.11	.74	3.08	.83
planning in post-test	3.48	.81	3.23	.83
monitoring in pre-test	3.56	.68	3.54	.81
monitoring in post-test	3.90	.65	3.54	.78
number of solutions within the teaching unit	1.72	.54	1.17	.34

Table 1: Students' planning, monitoring and number of solutions

These results confirm our assumption that treating multiple solutions has positive influence on students' metacognitive activities. If a teacher encourages students to develop more than one solution in a way applied in our teaching unit, students make plan and control their solutions more often.

### **Influence of the number of developed solutions on planning and monitoring**

As the number of developed solutions is a continuous predictor variable, we have used a linear regression with two predictor variables to answer the second research question. The predictor variables for explanation of the variance in planning activities in post-test were planning measured in pre-test and the number of solutions developed by the students during the teaching unit. Apart from planning in pre-test ( $\beta=.42$ ,  $p<.01$ ), the number of solutions has a significant influence on students' self-reported planning activities in post-test ( $\beta=.24$ ,  $p<.01$ ). The analysis of students' monitoring activities was conducted in the same way and shows the similar result. The monitoring activities measured in pre-test and the number of solutions have a significant impact on monitoring activities in post-test (monitoring in pre-test:  $\beta=.54$ ,  $p<.01$ ; number of solutions:  $\beta=.19$ ,  $p=.01$ ). Students who developed more solutions during the teaching unit planned and controlled their solution more often than students who developed fewer solutions.

One important limitation of the recent study is using questionnaires in order to measure students' metacognitive activities and the number of developed solutions. A validation of these measures in future studies is essential.

The results of this study point out that treatment as well as development of multiple solutions have a positive impact on students' metacognitive activities. Thus, processing of the tasks that required the development of multiple solutions can foster not only performance (Rittle-Johnson & Star, 2007) but also other learning outcomes (Schukajlow & Krug, 2012, 2013). An open question is however, how metacognitive activities, performance and development of multiple solutions link to each other. Development and verification of theories which specify the impact of treating multiple solutions on cognitive and metacognitive variables are an important future research issue.

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# CHANGING MATHEMATICAL CONTENT-RELATED DOMAINS – A GENUINE MATHEMATICAL ACTION?

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*This paper deals with one aspect of the endeavour to generate a theory of the development of mathematical thinking of children of ages 3 to 10: the situationally emerging change between mathematical domains. By this, the designed learning expectation might change, too. What are the developmental effects on the learning of mathematics if such changes turn into a continuous movement of oscillating alternations among different mathematical domains? Theoretically, we attempt to implement these developmental paths in a concept of an “interactional niche in the development of mathematical thinking”(cf. Krummheuer 2011a). Central is the differentiation between the “learning offerings” presented by a group or society and the situations that emerge from these “learning offerings” in an interactional process.*

## INTRODUCTION

In earlier analyses Krummheuer created a conceptual framework for this theory that he named the “Interactional Niche in the Development of Mathematical Thinking” (NMT; Acar Bayraktar & Krummheuer, 2011; Krummheuer, 2011a; Krummheuer, 2012a; Krummheuer, 2012b). In this paper we will employ the NMT concept to describe the relationship between mathematical content area and the early development of mathematical thinking. We will focus on children of late-kindergarten<sup>1</sup> age 5 to 6. The research question that the following will address is:

What is the relationship between mathematical content-related domains and the development of mathematical thinking in children of preschool age?

An interdisciplinary theoretical framework is required to address the research question. We therefore focus on both social-interactional and individual aspects in order to trace development in various interconnected mathematical areas. Interactional-theoretical and social-constructivist approaches, (cf. e.g. Krummheuer/Brandt 2001, Sfard 2008) as well as approaches from the usually psychologically characterised area of mathematical concept research (Vosniadou 2007, Vogel/Huth 2010) are taken into account.

The first section will briefly categorise the methodology of the experiment that forms the basis for this paper. In the second section we will briefly explain the necessary theoretical elements. An example of analysis is presented in the third section. The aim of this paper is to illustrate the empirical foundation of the study and then present at the

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<sup>1</sup> In Germany, the word “Kindergarten” describes the institution children attend from ages 3 to 6 before they start primary school.



conference a hypothesis on the development of mathematical thinking based on more extensive analysis, which will be presented in the final section of the paper.

## RESEARCH BACKGROUND OF THE PROJECT

In 2008 the Centre for **I**ndividual **D**evelopment and **A**daptive Education of Children at Risk ([www.idea-frankfurt.eu](http://www.idea-frankfurt.eu)) was founded in Frankfurt am Main, under the framework of the “LOEWE” initiative (State Programme for the Development of Economic Excellence in Hessen). The IDeA Centre is an interdisciplinary research centre that was set up as a cooperative project of the German Institute for International Pedagogical Research (DIPF), the Sigmund Freud Institute (research institute for psychoanalysis and its applications, SFI) and the Goethe University. It brings together qualitative and quantitative approaches. The IDeA Centre’s principal task is to research and describe conditions for children’s development together with so-called risk factors in different disciplines, and to develop means of support suitable for long-term practical application (cf. Brandt et al, 2011)

The present paper is grounded on research results from one of the Centre’s projects – the *early Steps in Mathematics Learning* (erStMaL) project. The aim of this project is to develop elements of a theory of processes of mathematical thinking between the ages of 3 and 9, which takes all the relevant mathematical content-related domains of early mathematical development into account. Relevant areas are: *Numbers and Quantitative Thinking, Geometry, Measurement, Pattern and Algebraic Thinking and Data Analysis* (cf. Sarama/Clements, 2008).

The erStMaL project is conceived longitudinally. In the first permitted period (July 2008 to June 2011), 144 children took part in the experiment in 12 day care centers in the Frankfurt area, with data being collected over four phases. On average about twice a year, video recordings of the following situations in the familiar Kindergarten environment were made:

- Play and discovery situations in pairs or small groups, in which children worked with developed stimulus materials with the support of an adult
- Teacher situations, in which children in pairs or small groups worked according to the directions of the teacher on a specific mathematical topic.

In addition, in the first year of data collection the basic cognitive ability of a small group of the children was also assessed.

## THE FUNDAMENTAL THEORETICAL ORIENTATION

Our aim is to develop a theory within the scientific framework of mathematics education. By this remark we want to stress that that we understand the subject matter of mathematics to be a constitutive dimension of developmental theory. It is not seen as a case of application of a psychological theory of development that typically claims to be universal with regard to the content-related domains. Previous studies attempt to describe mostly linear processes within specific mathematical areas, for example arithmetic or geometry. We seek to attain a more universal view of mathematical development processes by observing interconnecting relationships between different

mathematical areas. With reference to e.g. Sarama and Clements (2008), complex inter-relations emerge between individual mathematical areas.

The theoretical perspective on the generation of mathematical thinking taken here is that of socio-constructivism. This perspective encompasses two research traditions: the first is based on the phenomenological sociology of Alfred Schütz (Schütz & Luckmann, 1979) and its expansion into ethnomethodology (Garfinkel, 1972) and symbolic interactionism (Blumer, 1969)<sup>2</sup>; the other tradition refers to the cultural-historical approach of Vygotsky and Leont'ev, among others (see Wertsch & Tulviste, 1992 and Ernest, 2010).

With respect to the child's development of mathematical thinking, we refer to the concept of "developmental niche" (Super & Harkness, 1986):

"The developmental niche .... is a theoretical framework of studying cultural regulation of the micro-environment of the child, and it attempts to describe the environment from the point of view of the child in order to understand processes of development and acquisition of culture" (p. 552).

This consists of the

- "learning offerings" provided by a group or society, which are specific to their culture and will be categorized as aspects of "allocation";
- situationally emerging performance occurring in the process of the negotiation of meaning, which will be subsumed under the aspect of the "situation" (cf. Krummheuer 2011a)

These two aspects (allocation and situation) are then subdivided into three components: "content", "cooperation", and "pedagogy and education" – a further development of the original components of the developmental niche of Super and Harkness especially taking into account the domain specificity of the mathematical content.<sup>3</sup>

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<sup>2</sup> Surprisingly, Ernest (2010) does not mention this research tradition, which is usually subsumed under the term "micro-sociology". For its reception in mathematics education see Bauersfeld (1995), Krummheuer (1995), as well as more recent studies, for example that of, Schütte (2009).

<sup>3</sup> Super & Harkness, 1986 defined "their" development niche by the three components "the physical and social settings in which the child lives", "culturally regulated customs of child care and rearing" and "the psychology of the caretakers" (ibid. p. 552). They conducted anthropological studies without focusing on the situational aspects of social interaction processes.

NMT	component: content	component: cooperation	component: pedagogy and education
aspect of allocation	mathematical domains; body of mathematics tasks	institutions of education; settings of cooperation	scientific theories of mathematics education
aspect of situation	interactive negotiation of the theme	leeway <sup>4</sup> of participation	folk theories of mathematics education

Table 1: interactional niche in the development of mathematical thinking  
(Krummheuer 2011 a)

Table 1 expresses the momentary status of the development of the concept of NMT, which is empirically grounded in the research on the erStMaL project. Subsequent research may necessitate modifications and potentially the supplementing of additional components, for example the planned individual development component as an additional line. Acknowledging it to be a temporary artefact at this stage of the research, in the following we further explicate the details of the table:

*Content:* Children are confronted with topics from different domains of mathematics in their everyday life. The following data was gathered in the research project erStMaL and in everyday mathematic classroom situations. These mathematical topics are usually presented in the form of a sequence or body of tasks, which are adapted with respect to their content and difficulty according to the assumed mathematical competencies of these children.

*Cooperation:* Beside this content-related component, the children participate in culturally specific social settings which are variously structured as in peer-interaction or small group interaction guided by a nursery teacher or primary mathematics teacher, etc. These social settings do not function automatically; in fact they need to be accomplished in the joint interaction. Depending on each event, a different leeway of participation for the children will come forward.

*Pedagogy and education:* The science of mathematics education develops theories and delineates – more or less stringently – learning paths and milestones for children’s mathematical growth.

NMT refers to the table as a whole. Because of its situational aspect an NMT has to be accomplished in the process of interaction by the participants of the situation; it is not “just there”. In this paper we do not refer to all categories of the NMT table.

<sup>4</sup> Leeway taken here in the colloquial meaning of “room for freedom of action” (see Webster, 1983, p. 1034), originally the notion of “Partizipationsspielraum” in Brandt, 2004.

## A SHORT EXAMPLE FOR ILLUSTRATION

### The Junebugs Task


The Kindergarten children Marie und René participate in this task. They are 4 years old. Both sit at a table, on which lies a round mat, with which they are already familiar. The supporting adult, a colleague from the research team, offers both children a range of cards of dimensions 5 cm x 5 cm, on which ladybirds are depicted in three different colours (red, green, yellow) with different respective markings (circle, triangle, square) and of different respective number.



First of all the children familiarise themselves with the play and discovery situation by sorting the ladybirds, for example by colour. This raises the question of which cards are more numerous. This leads to the following exchange:

362	RK 39	There are so many of these \ find the green junebug with a square and place it in
363		the / shall we do that / do you want to put yours - shall we take yours - shall we
364		compare / put the green junebug together in a pile again - the junebugs /
366	B	<i>In agreement</i> Mhm do that \
367	RK 39	Which are - put a green junebug in the middle - put yours next to it /
369	RK 40	Put a red junebug next to the green one

After a while, the following arrangement of cards on the table is completed:

449	RK 39	Put the last green junebug right at the bottom in the row next to the red ones
450		<i>Done\</i>
450.		
1		
451	B	Aha\ So what do we see now /
452	RK 40	Hey - they are the same as mine \ <i>points to the row of cards</i> The same number \
453	B	How do you know that /
454	RK 40	Because that's 1 2 3 4 5 6 7 8 9/ <i>Whilst counting moves hand down the lines of</i>
455		<i>red junebugss; at first the speed of the gesture corresponds to the counting; from</i>
456		<i>5, however, the gesture becomes faster and the distance between index finger</i>
457		<i>and junebugs becomes larger 1 2 3 4 5 6 7 8 9- moves hand over the green</i>
458		<i>junebugs, similarly to before; gesture and counting however go out of synch with</i>
		<i>the number 3</i>
461	B	Mhm\
462	RK 39	So now we just have to put the yellow ones into a row \

## PRELIMINARY ANALYSIS AND RESULTS HYPOTHESES

### Short Analysis

In this scenario the task that emerges is to ascertain how many of each type of card there are; that is, which kind of card is most numerous. This task could be categorised in the mathematical content-related domain of Numbers and Quantitative Thinking; however, the children solve the task in a different way. The children, seemingly quite naturally, reframe the mathematical domain in such a way that their solution would be better categorised in the domain of Measurement. They compare the length of the rows that emerge when all cards of the respective ladybird families are laid next to each other. They arrive at the correct result – that there is the same number of cards of each ladybird family. Their attempt to count off the ladybirds at the end shows that they may not have been able to solve the task whilst remaining only in the domain of Numbers and Quantitative Thinking, since their attempts at counting always tended to confuse rather than provide support for their obtained result.

### First Hypotheses

The following formulates a first hypothesis in relation to the “content” aspect. On the allocation level the children are presented with a problem whose design can be categorised in the domain of Numbers and Quantitative Thinking and which would therefore lead us to assume that the solutions reached would also correspond to this domain. Seen from a situational perspective, the solution requires negotiation of meaning in the situation, which is an open-ended process and cannot sufficiently be defined on the allocation level in advance. In a similar context, Krummheuer (2012) mentions a “non-canonical” solution that was accomplished by the children whereas the necessary reaction of the participating adult can be described as “improvisation” of his/her mathematical solving routines for such tasks. Possibly, this improvisation might function as a support for the development of the children’s mathematical thinking. One also can find situations, however, in which the adult rather attempts to retrace the children on the initially intended, “canonical”, mathematical domain – needless to say: more or less successful

In terms of NMT one can reformulate, that the pair of “children’s non-canonical solving approach and the adult’s mathematical improvisation” resembles one possible kind of an appearance of NMT. In such a NMT, during the situational negotiation of meaning the children appear to reveal an ability to interpret and comprehend a mathematics subject matter across several mathematical domains. The effectivity of such a NMT relies and the children’s competency to draw links between different mathematical content-related domains, an ability that is rather expected of pupils in secondary schools or even later, for university students in mathematics. From our first results, however, this competency seems to be a genuine mathematical one that potentially does not have to be taught in school, but simply is prevented from coming to light thanks to differentiation of mathematical topic areas and a teaching style that serves to constrict behaviour. Therefore, it can be hypothesised that in early learning processes for children it is possible to change content-related domains used in solving

mathematical problems, when the accomplished NMT supports the production of not canonical solutions and process of negotiation of meaning does not restrict related mathematical activities or even targeting precisely this canonically understood solution. We will present more complete analyses and a further differentiation of our hypotheses at the PME conference.

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# ASSESSING VALUES IN MATHEMATICS EDUCATION

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*Values in mathematics education have traditionally been assessed with qualitative data or with data which have been qualitisied. With the knowledge accumulated of what are valued in mathematics and its pedagogy, it is timely now for values in mathematics education to be assessed accurately by teachers for a multicultural respondent pool. This paper explains how the 'What I Find Important (in mathematics education)' questionnaire was constructed, and reports on its validation.*

## VALUES THROUGH MATHEMATICS, AND MATHEMATICS THROUGH VALUES

Research into the role of values and valuing in mathematics learning and teaching began with Alan Bishop's proposal of three pairs of complementary values for 'Western' mathematics in the seminal book, 'Mathematical enculturation: A cultural perspective on mathematics education' (Bishop, 1988). Our readings of the academic literature had led us to see that what an individual values defines for him/her a window through which s/he views the world around him/her. Values are the convictions which the individual has internalised as being the things of importance and worth. They regulate the ways in which a learner utilises his/her cognitive skills and emotional dispositions to learning.

Being extremely internalised and stable (see Krathwohl, Bloom & Masia, 1964), values are usually not acquired overnight. Our valuing of *friendship*, *honesty*, and *creativity*, for examples, is normally developed over the years as relevant experiences provide the context for these values to be considered and accepted. Indeed, Rath, Harmin and Simon (1987) had proposed a seven-step valuing process which is made up of three stages, that is, choosing, prizing, and acting.

Values such as those mentioned in the previous paragraph, however, have always been espoused and transmitted in the education system. Teachers' roles have always been involved with the teaching of values (Veugelers & Kat, 2000), even though such teaching is often implicit with even the teachers themselves not aware of the process (Clarkson, Bishop, FitzSimons & Seah, 2000).

These educational values might be taught through dedicated school subjects labelled as 'civic education' and the like. Most of it, however, is transmitted via the teaching of other subjects, including the languages, history and the sciences. Thus, the valuing of *peace* or *diplomacy* can be taught in history lessons, while the valuing of *conservation* or *precision* can be represented by teachers of science. Mathematics too has the potential to be a medium through which such values are taught (see, for examples, Andersson & Valero, in press; Gutstein, 2006).



The focus of this paper, however, is not on the values that can be taught through mathematics. Rather, the intention is to highlight the various values that optimise mathematics pedagogy. In particular, this paper reports on the use of the questionnaire method to assess values which operate in mathematics classrooms.

## VALUES IN THE MATHEMATICS CLASSROOM

Bishop (1996) organised the values that function in the school mathematics classroom into three groups, these being mathematical (the valuing of *openness*, for example), mathematics educational (*fun*), and general educational (*respect*).

The category of mathematical values refers to the convictions that have been emphasised in the tradition of 'Western' mathematics. Alan Bishop had proposed three pairs of complementary mathematical values, namely *rationalism* and *objectism*, *control* and *progress*, as well as *mystery* and *openness* (see Bishop, 1988, for details).

Mathematics educational values are expressed through the pedagogical practices of the subject in schools. Data analysis by the international Third Wave Project group has identified such values continua as *ability* and *effort*, *wellbeing* and *hardship*, *process* and *product*, *application* and *computation*, *facts* and *ideas*, *exposition* and *exploration*, *recalling* and *creating*, as well as *ICT* and *pen-and-paper*.

## RESEARCHING VALUES IN MATHEMATICS EDUCATION

The researching of values in the mathematics classroom has traditionally been approached using the research methods of questionnaires, observations and/or interviews. There has been a predominant focus on collecting and analysing qualitative data; quantitative data were either qualitisied (Tashakkori & Teddlie, 1998), or analysed using descriptive statistics.

In the Australian 'Values and Mathematics Project' in the early 2000s, teacher valuing was examined through a large scale administration of questionnaires, through which prospective teacher participants for the next phase of the study were identified. In this second phase, teachers' professional practice was observed several times, each of which was followed by a teacher interview. The data were analysed qualitatively.

The Taiwanese 'Pedagogical Values in Mathematics Teacher Education Project' (Chin & Lin, 2001; Leu & Wu, 2001) was stimulated by the Australian study outlined above. As such, the research methodologies adopted were the same.

Seah's (2005) research into what immigrant teachers valued likewise followed the 'questionnaire plus observation-interview cycles' methodology, although the questionnaire data were reported using descriptive statistics. Given that the questionnaire was administered to a large pool of teachers, this reflected his attempt at mapping the scene, although the sampling size was not large enough for any inferential statistical analysis to be conducted.

However, research such as Clarkson, Bishop, FitzSimons and Seah (2000) has highlighted that direct means of gathering data (e.g. through interviews) might not be effective. The respondents themselves may not know that certain values are personally

embraced, which affects the extent to which they are able to report or discuss particular values. Indeed, investigating the socio-cultural nature of values calls for an innovative research approach to facilitate identification and analysis (Keitel, 2003).

By the late 2000s, values were also identified through content analyses of artefacts such as photographs and drawings, often followed by participant interviews which served to clarify initial findings or questions.

Seah and Ho (2009) examined what were being valued by primary school students when they learn mathematics well. These values were inferred from the drawings made by the students in class, immediately after they were asked to reflect on moments when they learn mathematics effectively. The reasoning behind the use of participant drawings was not just a consideration of the students' ability to describe phenomena, but also an acknowledgement that drawings of mental images would capture what might be valued implicitly or subconsciously.

Under the coordination of the Third Wave Study (a research consortium which is interested in better understanding the role of values in mathematics pedagogy), students' valuing of effective mathematics learning was investigated by research teams across several countries. The researchers acknowledged that valuing is embedded in multiple contexts, and that a holistic representation of what are valued (and at the same time, not valued) would be best captured through visual images of classroom practice. Instead of working with drawings generated by the student participants, however, representations of the classroom contexts were made in photographic form.

While photographs might capture the classroom contexts more authentically than drawings, there were different preferences for who the photographers should be. It was rather natural to assume that the student participants were to be the ones taking the photographs whenever they feel that they are learning mathematics particularly well. With appropriate briefings to the students, the photographs they take should represent an authentic documentation of what the classroom learning context looks like from the students' perspective at the moments of effective learning. The photographs would show clearly where, at these moments, the teacher is positioned, where the peers are, what the teacher is doing (and to whom), how the classroom furniture is arranged, and so on. However, some participating schools and teachers were concerned that students holding the cameras in their hands might be distracted from the proceedings of the lessons, or that the presence of the cameras in the hands of students might be too distracting to their peers. Accordingly, the research teams concerned responded in a variety of ways to address the potential ethical concerns. These include stationing a research team member at the back of the class, armed with a camera, and taking a snapshot of the class from that location at the signal of any student who thought that they were learning well at a particular instant (see, for example, Law, Wong & Lee, 2011). Another way is to make a video recording of lessons, which then serves as a tool for stimulated recall when student participants are interviewed after the lesson. In all these approaches of generating photographs or videotapes, the richness of the data is governed by the student participants' memory: remembering to take a photograph or to

signal for it to be taken during the ‘moments of effective learning’, or remembering what these moments are when reviewing videotapes of the lessons.

### **THE ‘WHAT I FIND IMPORTANT (IN MATHEMATICS EDUCATION)’ APPROACH**

The research methods discussed above have meant that the identification of what are valued by stakeholders and institutions in mathematics education has been in the main a slow process, requiring repeated visits to research sites and time-consuming transcriptions and content analyses (of transcriptions and artefacts such as photographs). The time involved, as well as the research skills required, in using these methods have meant that classroom teachers may be discouraged from engaging in systematic mappings of what their students value. A more user-friendly way of assessing students’ values is needed if these values are to be harnessed to optimize mathematics learning and teaching. For the research community, the qualitative nature of these research methods has meant that sampling size needed to be small, thus preventing meaningful generalisations to be made regarding the utilization of values in enhancing mathematics learning and teaching.

A questionnaire survey that is designed in such a way as to yield quantitative data might help us to address both these concerns. The production and validation of such a data-collection instrument is the inspiration for the ‘what I find important (in mathematics learning)’ [WIFI] research study. Conceptualised in 2012, this study has brought together research teams from eleven different countries / regions such as China, Hong Kong, Japan, Singapore and Sweden. In addition to producing a questionnaire which encourages teacher use and which facilitates large scale testing, the WIFI study also aims for the questionnaire to be valid and reliable when used across cultures, not just to enable cross-cultural comparisons, but more so to ensure that the questionnaire remains valid when used in the increasingly multicultural makeup of the student population in many a school classroom today.

The questionnaire method can be used to accurately assess values (Johnson & Christensen, 2010; Pang, 1996; Reichers & Schneider, 1990). Indeed, in research areas beyond mathematics education, several values questionnaires exist. Examples include the ‘survey for terminal and instrumental values’ (Rokeach, 1967) and ‘survey for personal and work values’ (Senge et al, 1994). The values questionnaire that is being developed by the WIFI study, however, is being about the first known questionnaire developed to facilitate further understanding in mathematics pedagogy. When fully developed, it would be the first ever resource for classroom teachers to assess values with, as well as being a data-collection tool with which to map values across a bigger segment of a population. Equally if not more significantly, the questionnaire has been designed for administration across diverse cultures and for bodies of multicultural respondents.

### **Adopting an Essentially Quantitative Approach**

There has been a tendency in prior research studies to employ methods such as interviews, observations and content analyses to identify values, often leading to the

use of qualitative data analysis processes. The predominantly qualitative approach had been particularly useful in a research context in which values studies were relatively new, when it was not known what the scope of values were, and indeed, what they looked like. At the same time, this research approach has its own constraints, such as the time and skills that are needed to investigate and analyse the values respectively.

Over the last decade or so, this research tradition has resulted in the explicit validation of the relevance of mathematical values (see Bishop, 1988) and the identification of mathematics educational ones, as listed above. In this context, the WIFI questionnaire harnesses this knowledge to construct a tool with which relevant values can be assessed conveniently and accurately in schools. Whereas some ten years ago, it was difficult to nominate particular mathematics educational values for a large group of participants to respond to, now we have the language/expression and the understanding to use the questionnaire items to find out the extent to which any mathematics educational value is regarded as important by any cultural group.

### Selecting Items for the Questionnaire

The item types as well as the number of each item type were guided by current academic knowledge about values. For example, the WIFI team has been aware that teacher participants in earlier values research in Australia and Taiwan found it difficult to identify what values they were espousing. After all, values are the invisible threads of culture (Henderson & Thompson, 2003). The ability then of (higher) primary school students to do the same was doubtful. As a result, instead of posing questionnaire items asking respondents to state their valuing of particular convictions, such as *creating*, an inference for this valuing would be made through responses to a phrase which is embodied by it. In the WIFI questionnaire, the valuing of *creating* is thus inferred by the phrase ‘students posing maths problems’. Yet, this same phrase can be the embodiment of other values as well, which has meant that several other phrases which represent the actualisation of *creating* needed to be included in the questionnaire.

The questionnaire items were firstly chosen from four existing sources, namely the ‘values and mathematics project’ (Clarkson, Bishop, FitzSimons & Seah, 2000), a questionnaire used by Alan Bishop in a Thai workshop, the ‘mathematics education values questionnaire’ (Dede, 2011), and the findings of a prior international collaboration study in the ‘Third Wave Project’ (see Seah, 2011). A diverse range of items that span across the three categories of values in the mathematics classroom – mathematical, mathematics educational, and general educational – was sought for.

In terms of item type, Likert-scale items with five choices allow each questionnaire respondent to indicate the extent to which s/he finds something important. That is, how much a conviction is being valued by him/her. For example, item 10 of the WIFI questionnaire asks respondents to indicate if they personally find ‘relating maths to other subjects / disciplines’ absolutely important, important, neither important nor unimportant, unimportant, or absolutely unimportant. For this item, a respondent’s choice will indicate the extent to which s/he values the mathematical value of progress,

as adapted from its use in the ‘mathematics education values questionnaire’ (Dede, 2011). There are 64 such items in the WIFI questionnaire.

Bishop’s (1988) conception of the mathematical values as complementary pairs, as well as Hofstede’s (1997) idea of value continua, have prompted the next set of ten items, in which respondents mark on a horizontal line their relative valuing of the complementary values at both ends of the lines. For example, item 71 asks for a respondent to show his/her relative valuing of *exposition* and *exploration*, the former represented by the phrase at one end of the line, ‘Someone *teaching* and *explaining* maths to me’, and the latter, by a phrase at the opposite of the line, ‘*Exploring* maths myself or with peers / friends / parents’.

Of course, the 74 items cannot possibly cover the range of values that operate within the mathematics classroom. Thus, the next section is made up of four contextualized, open-ended items which encourage respondents to write down what they themselves value, given a common scenario of the production of a magic pill the ingestion of which makes one excel at mathematics.

Thus, the approach adopted by the WIFI study has been that of assessing quantitatively what people value. The questionnaire is designed to facilitate use by teachers, generalisations in particular cultures, and deployment across different cultures.

### **Asking the Questions in and for Different Cultures**

If cultures are defined by or characterised by values, then any values assessment instrument should be expected to be able to elicit these different values. This applies whether the instrument is to be administered across different cultural settings, or if it is being used in a setting that is populated by different cultures. The latter is increasingly prevalent across the world, given the scale of human migratory movements in the current socio-politico climate.

It was explained earlier in this paper why the questionnaire items need to be actions, such that what the student respondents value would be inferred from their respective responses. Different cultures, however, may see different messages in the same action. For example, the item ‘going up to the front of the class to explain my solutions’ was originally drafted to reflect the valuing of *process*. In some Asian cultures, however, teacher calling particular students out to the front of the class is a classroom management strategy. In these cultures, a student is thus more likely to associate being called out to the front of the class with public shaming. In view of this potential for the item to be interpreted differently, the phrasing was later changed to the following, so as to better assess respondents’ valuing of *process*: ‘explaining my solutions to the class’.

### **VALIDATING THE WIFI QUESTIONNAIRE**

The WIFI questionnaire was first validated on-site by the group’s Hong Kong team. A total of 1081 (367 Grades 5/6 and 714 Grades 8/9) students, of which 486 were male and 595 were female, from several schools in the urban areas of Hong Kong had filled in the questionnaire.

The construct validity of the questionnaire was assessed with a Principal Components Analysis [PCA] with a Varimax rotation. The significance level was set at .05, while a cut-off criterion for component loadings of at least .45 was used in interpreting the solution. An exploratory factor analysis [EFA] was conducted to validate the various mathematics and mathematics education value dimensions. Guided by the scree plot and the interpretability of the factors, a nine components orthogonal solution was accepted after the extraction of principal components and a Varimax rotation. A final confirmatory factor analysis was carried out, following the elimination of psychometrically 'poor' items. Variables loading on more than one factor were eliminated. Nine factor components with eigenvalues greater than one explain 57.20% of the variance, with almost 12.32% attributed to the first factor – *exploring*. The Kaiser-Meyer-Olkin (KMO) is 0.96 and Bartlett's test of sphericity (BTS) is significant at the 0.001 level and so factorability of the correlation matrix is assumed.

Reliability analysis yielded satisfactory Cronbach's alpha values for each of the nine components, ranging from 0.70 to 0.91, indicating a strong or acceptable degree of internal consistency in each subscale.

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# AN INVESTIGATION OF STUDENTS' CONCEPT IMAGES AND INTEGRATION APPROACHES TO DEFINITE INTEGRAL: COMPUTER ALGEBRA SYSTEM VERSUS TRADITIONAL ENVIRONMENTS

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*This research is a part of a wider project which is concerned with students' understanding of the first-year calculus. In this study we focused on the question "What is the role of Computer Algebra System (CAS) for concept images of definite integral and integration approaches". In this experimental investigation which is based on qualitative data, the students in the experimental group were subjected to CAS enhanced teaching whereas the students in the control group were subjected to the traditional centered teaching approaches. The findings demonstrate that the students in CAS group, in comparison to the students in traditional group, gained richer images in relation to a definite integral concept. In this investigation, it has also been discussed how teaching environments and concept images influence knowledge and performance in integration approach.*

## INTRODUCTION

After Calculus Reform Movement, which put forward proposals as an alternative to the traditional approach to enhance the curriculum with different representations and to supplement it with the use of technology (Goerdts, 2007), there have been many researches conducted by comparing instructional designs (reform-based versus traditional) in relevant literature (Porzio, 1999; Camacho & Depool, 2003). These researches show that the reform-based instruction, in contrast to the traditional one, do better in interpreting function and derivative graphs, and that different images relating to the derivative concept improve in Computer Algebra System (CAS) enhanced environments (Porzio, 1999). There are also many researches in the literature looking at the difficulties experienced in understanding integral, which is another important topic in calculus, students' misunderstandings and the pedagogical difficulties arising from the teaching of the concept.

There are only a limited number of researches in which definite integral has been investigated from the concept image perspective (Rasslan & Tall, 2002; Rösken & Rolka, 2007). Enhancing students' conceptual knowledge of integral can be achieved by raising their awareness of different images relating to the concept (Orton, 1983). Otherwise students may see definite integral only as a calculation tool and remain unaware of different definitions of this concept (Sevimli & Delice, 2011). However the relation definite integral has with anti-derivative method via Fundamental Theorem of Calculus (FTC) demonstrates its algebraic meaning (Thompson, 1994) while the relation definite integral establishes with the method of applications of integration via



area, arc length and volume calculations shows its geometric meaning (Orton, 1983). Moreover, the relation definite integral has with accumulated change method via limit of a sum approach demonstrates its numerical meaning (Thompson & Silverman, 2007). In the context of this investigation, the problem statement is “how a CAS enhanced teaching environment set up according to the reformist approach influences the students’ concept images of definite integral and what the relation between the different meanings of integral and these images is”.

## **THEORITICAL FRAMEWORK**

Calculus is a subject taught at various levels of higher education and one that students have difficulty interpreting. The difficulty in understanding the concept of integral which is covered in calculus courses is universally acknowledged (Orton, 1983; Rasslan & Tall, 2002). The findings of the relevant researches reveal that students suffer from huge shortcomings in carrying out routine operations such as calculating the definite integral or the derivative of a given function, in interpreting the calculation, and in acquiring the skills to utilize the rules and formulae (Camacho & Depool, 2003). The fact that, while answering questions where rules, formulae and operations are successfully applied, students fail to comprehend the mathematical thinking behind this process, and that they experience difficulties in interdisciplinary relations is a common problem statement for researchers. When one looks at the literature regarding the investigations in the field of integral, one sees researches, which take theory and practises into account, consider negative area misconception (Orton, 1983), knowledge and awareness of FTC (Thompson, 1994), preferences of multiple representation (Sevimli & Delice, 2011) and definition-image for the integral concept (Rasslan & Tall, 2002). In this context, calculus reform movement takes a critical approach towards dynamics in the traditional classroom setting. The reform movement also argues that students who are successful in problem solving that require operational knowledge have difficulties in their relational understanding because of limitations in concept definitions and images. Enrichment of images relating to a concept can be achieved by establishing relations between the concept and cognitive structures (Tall & Vinner, 1981). At this point, apart from the formal definition of the relevant concept, the individual’s own definition, the mental image of the concept, its features, in short the whole internal processing plays an important role in conceptual meaning making. In this respect, many researches reach the same conclusion that the meaning attached to the definite integral needs to be enriched (Rasslan & Tall, 2002; Rösken & Rolka, 2007). Rasslan and Tall (2002), following from the research by Tall and Vinner (1981) on concept definition and concept images, look at the relation between the definition of the definite integral and its images, and point out that students who do not understand the definition of definite integral have difficulty in interpreting area calculation problems and establishing interdisciplinary relations. Likewise Rasslan and Tall (2002) proposed the adoption of new approaches in higher education to end students’ calculation based approaches and instead to encourage approaches that can reveal the relation between integral and the topics of limit and derivative. Following from these

proposals, students' concept images and integration approaches of definite integral have been investigated through CAS and traditional environment.

## METHOD

### Research Design and Group

In this investigation, the effectiveness of a teaching experiment in terms of improvement of concept images has been assessed, and in this respect a mix-method approach which combines experimental design and qualitative data support has been adopted. The study was carried out in Calculus II during the 2011-2012 spring term. The participants of this study consists of 84 undergraduate calculus students at a state university; out of these students two groups have randomly been assigned, one as experimental group ( $n=42$ ) and the other as control group ( $n=42$ ). When assessing whether experimental and control groups are equal, their marks in Calculus I in the previous term have been taken as a criteria. It has been established that both groups are equal to each other in terms of their scores in Calculus I.

### Settings

The treatments in experimental and control groups in Calculus II are carried out during six weeks. In this period the effectiveness of two teaching approaches in the context of concept images are tested. Both approaches have been followed by the researchers. Out of these two approaches, CAS enhanced teaching has been adopted by the experimental group to teach the topic of integral. CAS enhanced instructions were: Using Livemath software embedded textbook which was adjusted as per calculus reform and emphasizes translations between/within representations (Hughes-hallet et al., 2009). Livemath software has been preferred because of its easy syntax and because it also enables the multiple representation-based instruction. In the CAS enhanced approach, the theory part of calculus course (4 modules) has been delivered in a teacher centered and technologically supported way; the practical part of the course (2 modules) has been covered in a practice centered way and in an environment where every student has access to the software. In the control group, where the course has been delivered in the traditional approach, the course notes from previous students have been made use of, and a traditional calculus textbook which generally emphasizes symbolic representation and focuses primarily on definition, theorem and proof processes has been used.

### Data Collection Tools

Data collection techniques were test and interviews. Concept Definition Questionnaire (CDQ) used for defining students' concept images of definite integral, and task-based interviews conducted for evaluating students' problem solving performance in terms of the use of their concept images.

### *Concept Definition Questionnaire (Pre& post test)*

The questions students mostly used in calculus are used to try to determine the students' images of definite integrals. In this context first the general picture of concept images is studied according to the groups, and then interviews with the students are

conducted. CDQ includes verbal questions to reveal the richness of students' images as well as certain problem types that are aimed at analyzing the use of images in the process of problem solving. In the CDQ, students are asked the following questions : “What does  $\int_a^b f(x)dx$  mean to you?”, “Tell more about other meanings of this expression, if any, in order of importance.” “Draw a concept map for the topic of integral. The test was found to have face and content validity after the analysis made by three experts in mathematics (education). CDQ, CAS and traditional groups were given pre and post tests.

### **Task-based Interviews**

Task based interviews were aimed to assess students' knowledge regarding numeric, graphic and algebraic methods and their performance in the use of these methods while solving tasks. After administering post-CDQ, task-based interviews were conducted with four respondents for a deeper understanding of the role of concept images in terms of problem solving performance on definite integral. These four participants in the interviews were selected using the purposeful sampling technique. Main selection criteria were that each participant taught by different teaching approach (CAS or traditional) and that they had the different concept image domain (Dominant/Multiple). Respondents chosen for the interviews were given numeric, graphical and algebraic tasks. While selecting interview questions a pilot study was carried out and expert opinion was consulted. Below, an example problem from task is presented, the integration approach of which is numeric.

- *What does numerical approach mean in integral?*
- *Does numerical approach an important proficiency in integral calculus?*
- *Please solve the task given below.*

**Task 1:** Values for a function  $f(t)$  are in the following table. Estimate  $\int_a^b f(t)dt$

$t$	0	2	4	6	8
$f(t)$	2	6	8	11	12

Figure 1: Task-based interview for numerical approach

### **Data Analysis**

Pre & Post CDQ's data was first assessed according to image domain in terms of dominance and multiplicity. Students who gave an explanation centred on a single image for definite integral were coded as dominant whereas students who expressed different meanings of the same concept were coded as students with multiple images.

When the questions, which are assessed in the category of multiple concept images, contained two or more concept images, the sum of total images will be more than 100%. In interview data, when determining respondents' knowledge about integration approaches and their performances, sample quotes from the interviews were used, and correct solutions and explanations were categorised and coded as S (Successful), correct solutions or explanations as P (Partially Successful) and wrong solution and incorrect explanations as U (Unsuccessful). The respondents, whose views on integration approach were consulted, were assessed in terms of their views whether the

relevant attainment needs to be included in content, and were coded under must be (m), could be (c) and not necessarily (n) categories.

## FINDINGS

### Pre-Concept Images of Definite Integral

According to the Pre-Concept Definition Questionnaire (Pre-CDQ) findings, the type of image which both CAS and traditional group students have most of in definite integral is that of area (CAS: 60%, traditional: 67%). 43% of CAS and 36% of traditional group students made use of reverse image of derivative when explaining  $\int_a^b f(x)dx$  function. The percentage of students who stated that the integral can be understood as a calculation tool is 14% in CAS and 10% in traditional group. Prior to the teaching practice, the percentage of students who possessed multiple images was 17% in CAS and 13% in traditional group. Pre-CDQ findings demonstrate that both groups had single and dominant images for definite integral concept.

### Post-Concept Images of Definite Integral

By the end of the teaching practice, the types of images for definite integrals as used by the groups were assessed in post-CDQ (Table 1). The post-CDQ findings show that 58% of CAS and 24% of traditional group were in the multiple image domain. Again the findings reveal that the percentage of students with multiple/area images in CAS group is higher than those in traditional group (CAS: 43%, traditional (Tra): 19%).

<u>Images</u>	<u>Example</u>	<b>Dominant</b>		<b>Multiple</b>	
		<u>CAS</u>	<u>Tra.</u>	<u>CAS</u>	<u>Tra.</u>
Area	area defined by x axis and f(x) function graphic	21	24	43	19
Anti-derivative	a certain calculation used in the operation $\int_a^b f(x)dx = F(b) - F(a)$	14	45	21	12
Accumulated change	The limit of totals as expressed in this function $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$	7	-	31	7
The others	Reverse of differentiation/An operation	-	7	10	7

Table 1: Common categories set up as per definite integral concept.

Dominant/anti-derivate image was more frequent among traditional group. 45% of the students in this group defined definite integral via dominant/anti-derivative whereas this number was 14% in CAS group. The percentage of the students who took into consideration the numeric nature of definite integral and referred to accumulated change image was 38% in the CAS group (dominant+multiple), whereas in traditional group this was 7%.

### Respondents's Knowledge and Performance on Integration Approaches

Task based interviews with respondents have been conducted in three strands. In the first strand, respondents' knowledge about different integration approaches was studied. In this context, some sample sentences which reflect students' knowledge

levels are given in Table 2. The respondents in the CAS group attempted to explain algebraic approaches via FTC and by referring to the relation between derivative and integral. On the other hand, the respondents in traditional group put more emphasis on integration techniques when defining algebraic approaches. In traditional group the students with dominant/anti-derivative image attempted to explain graphical and numerical approaches by referring to input representations. The respondents in the CAS group who expressed numeric approaches as a type of approximation explained these problems via estimation. The second strand in the interviews has to do with views about the inclusion of integration approaches in calculus. The respondents in the CAS group expressed positive views about all three integration approaches (Table 2). In contrast to this, the respondents in traditional group expressed a “not necessary” view about numerical approaches and a “could be” view about graphical approach. In the final strand of the interviews, the performance of the respondents in numeric, graphical and algebraic tasks was assessed. In algebraic tasks all of the CAS and traditional group students who were interviewed were successful. The respondents in the CAS group were more successful than those in traditional group with regards to graphical and numerical tasks. Particularly in numerical tasks neither of the two respondents in traditional group was successful. One of the reasons behind the failure of these respondents in traditional group was their insistence on using algebraic solution methods while in fact different integration methods were required.

	<u>Image</u>		<u>The type of Task</u>	
	<u>Domain</u>	<u>Numerical</u>	<u>Graphical</u>	<u>Algebraic</u>
CAS	<b>Dominant/ Area</b>	[P→m]“...an attempt to find the best estimate, like the total change”	[S→m]“...interpreting integral on geometric milieu, like area”	[S→m]“...using FTC to make connections between derivative and integral”
	<b>Multiple/ Area</b>	[S→c]“...an approximation with Riemann sums”	[P→m]“...area under the curve or volume of the solid”	[S→m]“...an operation to find the reverse of derivative”
Traditional	<b>Dominant/ Anti-derivative</b>	[U→n] “...giving values of a function by tabular”	[P→c]“...representing integral equations by graphics”	[S→m]“...calculating integration by parts or substitution”
	<b>Multiple/ Area</b>	[U→n] “...methods of integration like Simpson rules”	[U→c]“...area of plane regions”	[S→m]“...flexibility to use integration techniques”

Table 2: Students’ views and performance on numerical, graphical and algebraic tasks

## DISCUSSION

In the education system of many countries, students cover the topic of integral for the first time in the last year of secondary education. At this level the relations between derivative and integral may not be covered in sufficient detail. For this reason students come to see derivative and integral as each other’s opposites. The focus of this study was to see how teaching approaches influence students’ pre concept images. It was observed that in traditional approach, students whose pre image was reverse of

derivative changed this in their final image to anti-derivative. Using a definite textbook, lecture notes with emphasis on the algebraic representations of definite integral and working on blackboard without any other supporting resources such as technology might influence this change in concept images from an algebraic representation to another algebraic representation. As it is seen dominant concept image is algebraic as anti-derivative and is not visual as graphs. In this respect when students in traditional group established a relation between integral and derivative, they used FTC as a bridge (Thompson, 1994). Apart from this finding, there was not any other significant differentiation between the pre and final images of traditional group. However in CAS group, prior to the treatment, dominant/area image was frequently observed, and following the practice, images showed variety. In this group particularly accumulated change image was used together with other images to explain definite integral. Enhancing image variety is also important in raising awareness for integration approaches. However it can be argued that task performances are more linked to teaching approaches. For example, although traditional respondent with multiple/area image was able to explain the numerical approach, he/she was unable to use it numerical approach in problem solving. The attraction point in CAS group looks like the well known influence of the using technology which is enhancing students visual ability and presenting wide variety of the graphs. So that in addition to some algebraic representations, visual representations are dominant to form the concept image of definite integral. In addition to teaching approaches in both CAS and traditional groups, the criticism levelled at traditional textbooks by the calculus reform movement also makes the point that while operational domains are emphasized, conceptual domains are not covered enough.

There was not a clear distinction among the respondents in terms of their knowledge about and attitudes towards algebraic and graphical approaches. On the other hand, the respondents in traditional group stated that they did not see numerical approach as an important attainment that justified a place in integral. Sofronas et al. (2011) aimed to establish the objectives of calculus courses to understand the goals of the first year calculus through perspective from experts. Their findings show that the percentage of experts who believe that integration techniques which require algebraic approaches should be included in teaching of integral is quite high, whereas the percentage of experts who refer to numerical approaches is low. However numerical approach is a process which should not be ignored as it is required to understand the definition of integral in terms of Riemann (Thompson & Silverman, 2007) and to disclose the conceptual relations between limit and derivative. It is believed that students who are unaware of the meaning of approximation in integral tend to think in terms of objects rather than processes and fail to see the relation between concepts.

In conclusion, this study shows that CAS supported teaching approach enabled students to become aware of different concept images of definite integral and to interpret more than one image at the same time. It is observed that in this respect the students in CAS group, in comparison to their peers in traditional group, develop richer images with regards to definite integral. This study reveals that teaching approach and

concept images influence the knowledge and solution performance of different integration approaches. The study also emphasises that traditional approaches remain ineffective in raising awareness towards numerical approaches, and states that the students who are aware of different meanings of integral have a higher performance rate throughout problem solving.

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# SEMIOTIC CHAINING AND REIFICATION IN LEARNING OF SQUARE ROOT NUMBERS: ON THE DEVELOPMENT OF MATHEMATICAL DISCOURSE

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*This paper reports part of a theoretical and empirical study into the discursive approach to mathematical reification. It focuses on the semiotic chaining process in the teaching and learning of square root numbers. The purpose of this study is to characterize the development of discourses concerned with the addition of square root numbers in mathematics lessons in ninth grade classroom, in terms of theories of semiotic chaining and reification. To achieve this objective, classroom activities are observed and analyzed. The results suggest that the change of meta-discursive rule, which is an essential aspect of the reification of new signifier, is inevitable for the transition from the template-driven use of signifier to the objectified use of symbol. Some additional theoretical and practical implications are also discussed.*

## THEORETICAL PERSPECTIVES

### Theories of semiotic chaining and reification

The *theory of reification*, invented by Anna Sfard, is a well-known model of mathematical concept formation. According to this theory, the same mathematical notion can be conceived in two fundamentally different ways: *operationally*, as a process, and *structurally*, as an object (Sfard, 1991). Although there is a deep ontological gap between operational and structural conceptions, they are in fact complementary. A basic tenet of the theory is that the operational conception emerges first, and the structural conception develops afterward in three phases: *interiorization*, *condensation*, and *reification*. This transition is a long-term process, and reification in particular can be a rather complex phenomenon.

The *theory of semiotic chaining* in mathematics, introduced by Norma Presmeg, is mainly based on Peirce's semiotics (particularly, the triadic model) and/or Saussure's semiology (particularly, the dyadic model). In the case of Saussure's dyadic model, the *sign* is defined as a combination of a *signified* together with its *signifier*. In a semiotic chaining process, a signifier in a previous sign combination becomes the signified in a new sign combination (Presmeg, 1997). Presmeg refers to "reification" together with "metonymy" in semiotic chaining, such that "I see this chaining process as involving metonymy – as indeed all signifiers are metonymic in a semiotic model, since they are 'put for' something else – and also reification, since each signifier in turn is constructed as a new object" (ibid., p. 275). She also remarks, "In reification, at each node in the chain a new process is encapsulated as an object which stand in a signified-signifier relationship with its characteristic symbolism" (ibid., p. 276). Thus, the reification can be considered as an essential phase in a semiotic chaining process.



Such a semiotic or linguistic view by Presmeg (1997) underlies the following remarks by Sfard (1997):

Since transferring templates from discourse to discourse is a case of *metaphor*, we may say, once again, that what we call “mathematical objects” are metaphors resulting from certain linguistic transplants. [...] introduction of a new mathematical symbol is often enough to spur such ontological shift in the discourse – and to bring about reification. (Sfard, 1997, p. 358)

This statement has important implications for the theoretical consistency of “metaphor” and “reification”. In Sfard (1997), the linguistic transplant of a new symbol into a well-known phrase is called an act of *metaphorical projection*; here we do not make a clear distinction between the terms “metaphor” and “metonymy” in the theoretical considerations. Therefore, an act of metaphorical projection can be a key notion, both in the theories of semiotic chaining and reification.

### **Learning mathematics as developing discourse**

Previously, Sfard (1991) used the term “reification” to describe the shift from the operational conception to the structural conception that refers to the cognitive activity of an individual. In Sfard (1997, 2000), she extends the notion of operational/structural duality to include a discursive perspective: “The distinction between structural and operational signifiers is relative to the discursive context in which they are employed, and the same symbol may sometimes be used as operational, and sometimes as structural” (Sfard, 2000, p. 49). Further, the notion of reification is reconsidered from the discursive point of view:

[T]he new signifiers did fulfill the mission for which they were meant: they helped to thematize the former discursive processes and turn them into the focus of the mathematical conversation. In this way, they catalysed a new discursive formation and raised the discourse to a metalevel where mathematical processes previously merely executed became a reified object of study. (Sfard, 2000, p. 53)

Sfard (2000, 2001) conceptualizes learning mathematics as developing mathematical discourse in terms of a two-phase process: (1) template-driven use of new signifiers and (2) objectified use of symbols. The first phase is characterized by inflexible use of the new signifier that is based on substituting new signifiers into old discursive templates: metaphorical projection. The second phase is characterized by a more flexible use of the signifier with it being signified via reification, and the signifiers will begin to be referred to as representations of other entities. As far as the discursive approach to reification is concerned, the two developmental phases can be schematized in terms of Presmeg’s (1997) semiotic chaining. Such a hypothesis makes it possible to interpret the places of metaphorical projection and reification in developing discourse, using the following diagram.

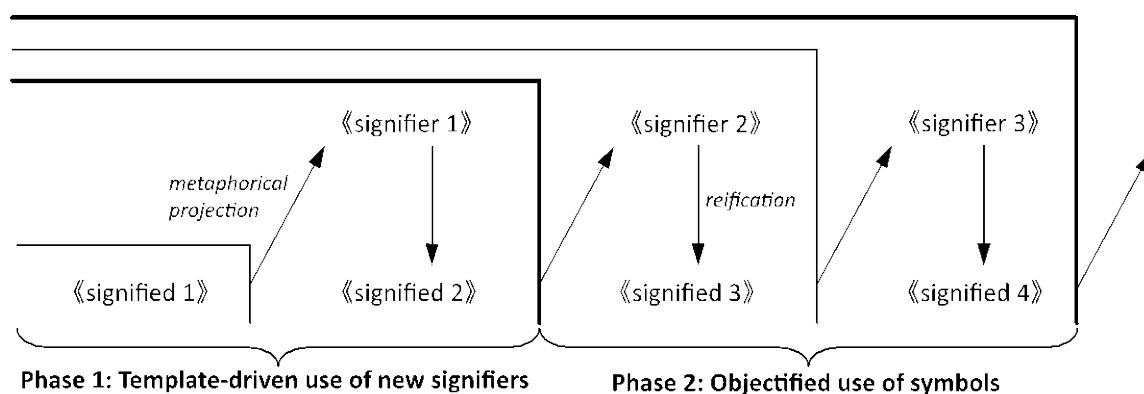


Figure 1: Hypothetical model of the semiotic chaining and reification

In the present study, we attempt to describe the two-phase process of discourses on square root numbers using this hypothetical model.

### METHODOLOGICAL CONSIDERATIONS

As is noted in Sfard (2000), “object mediation would be very slow to develop” (p. 58); thus, we need to take a relatively long-term perspective in order to analyze the development of a mathematical discourse. The target of the analysis in this report is the ordinary mathematics lessons of “square root of numbers” in ninth grade mathematics classrooms, in a lower secondary school attached to a certain national university in Japan. A series of lessons, which consisted of 14 planned and conducted lessons, were observed and videotaped. The outline of the teaching unit “square root of numbers” is listed below. In a general sense, L1 to L6 are concerned with discourses on the existence of square root, and L7 to L14 are concerned with discourses on the calculations of square root numbers.

- L1: Quadratic equation
- L2: Existence of square root
- L3: A number that cannot be expressed as a ratio of integers (fraction)
- L4: Root sign ( $\sqrt{\phantom{x}}$ )
- L5: Ordering the square root of numbers
- L6: Magnitudes of square root of numbers
- L7: Prime number and prime factorization
- L8: Multiplication and division of square roots (1)
- L9: Simplifying the expressions including square roots
- L10: Rationalizing the denominator
- L11: Multiplication and division of square roots (2)
- L12: Addition and subtraction of square roots (1)
- L13: Addition and subtraction of square roots (2)
- L14: Various calculations including square roots

According to Sfard (2001), there are three dimensions for the two-phase analysis of mathematical discourse: its *vocabulary*, the *visual means* with which the communication is mediated, and the *meta-discursive rules*. The extension of vocabulary (e.g., introduction of new terms such as “square root”, “rational number”, and “irrational number”) and the addition of visual means (such as a new operational symbol “ $\sqrt{\phantom{x}}$ ” and an extended (real) number line) are changes in object-level discourses.

On the other hand, the meta-discursive rules are changes in the metalevel dimension, and these rules are defined as the ones that “navigate the flow of communication and tacitly tell the participants what kind of discursive moves would count as suitable for this particular discourse, and which would be deemed inappropriate” (Sfard, 2001). Although the change of meta-discursive rule cannot always occur explicitly, we attempt to analyze this aspect as a key to the transition from the phase of template-driven use to the phase of objectified use. Figure 2 shows an analytical framework for the changes of mathematical discourse, based on the discussions in Sfard (2000, 2001). In this report, we would like to characterize discursive changes in terms of the semiotic chaining and reification by focusing on the discourses on the addition of square root numbers.

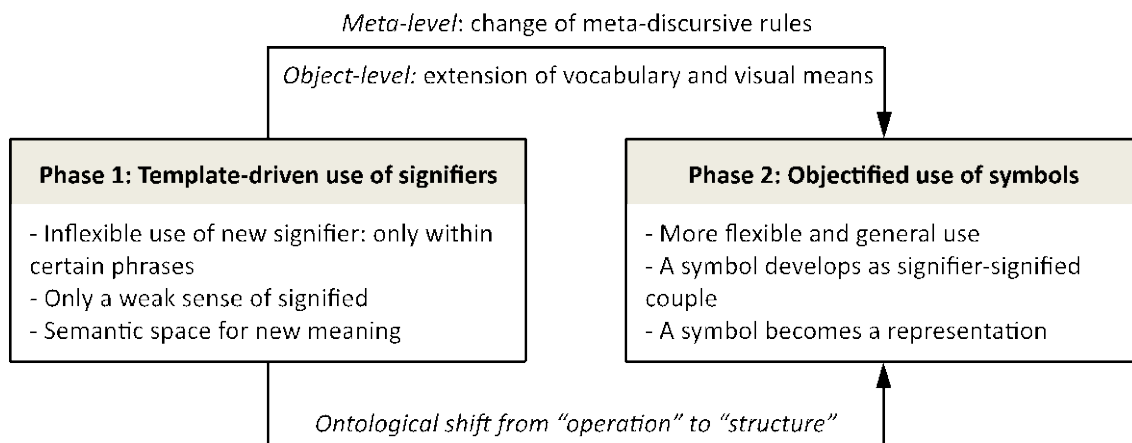


Figure 2: Analytical framework for the changes of mathematical discourse

## ANALYSIS OF MATHEMATICS LESSONS

### Metaphorical projection with introduction of a new signifier: Phase 1

A new signifier “ $\sqrt{2} + \sqrt{3}$ ” was introduced in L12. Students have already learnt the multiplication and division of square root numbers. At the beginning of the lesson, a teacher posed a simple question “Is  $\sqrt{2} + \sqrt{3} = \sqrt{2+3}$  true?” Because this question reminded the students of the calculation “ $\sqrt{2} \times \sqrt{8} = \sqrt{2 \times 8}$ ”, it brought about a kind of discursive conflict in students’ interactions, as follows:

- 1 S1: It raises to the power of two, it will be whole numbers..., it will be  $2+3 = 5$ , so, square root of 5 is  $\sqrt{5}$ .
- 2 T: The idea of “square” has been used many times. You applied this idea. Okay.
- 3 S2: No...,  $\sqrt{2}$  is about 1.414.  $\sqrt{3}$  is about 1.732. The sum is 3.146. But  $\sqrt{5}$  is about 2.236. So, I think it is different.
- 4 T: S2 says there is a big difference between 3.1 and 2.2. What do you think?
- 5 S3: S2’s idea is not so clear. Because there must be computational errors, there are thousands of decimal places. So, when calculating...
- 6 T: You know  $\sqrt{2}$  and  $\sqrt{3}$  are infinite decimal numbers. So, the addition of them might be indefinite.
- 7 S3: It might be.

- 8 S4: Even if there are errors, they will be very small, such as 0.00001. It does not influence the fact that the sum is 3.146.
- 9 T: Okay. How about you?
- 10 S5: S1 said the expression “ $\sqrt{2} + \sqrt{3} = \sqrt{5}$ ” becomes “ $2+3=5$ .” But in the case of integers, it will not be “ $(2+3)^2 = 5^2$ .” So, I think “ $\sqrt{2} + \sqrt{3} = \sqrt{5}$ ” is wrong.

Superficially, this discursive conflict can be seen as a kind of miscalculation by multiplication. However, it seems reasonable to suppose that S1’s argument is based on substituting the new signifier “ $\sqrt{2} + \sqrt{3}$ ” into the ready-made discursive template, namely the discourse on the multiplication of square root numbers (see Figure 3).

$$\begin{aligned}
 (\sqrt{2} \times \sqrt{5})^2 &= (\sqrt{2} \times \sqrt{5}) \times (\sqrt{2} \times \sqrt{5}) \\
 &= \sqrt{2} \times \sqrt{5} \times \sqrt{2} \times \sqrt{5} \\
 &= \sqrt{2} \times \sqrt{2} \times \sqrt{5} \times \sqrt{5} \\
 &= (\sqrt{2})^2 \times (\sqrt{5})^2 \\
 &= 2 \times 5 \\
 \text{Because } \sqrt{2} \times \sqrt{5} &\text{ is the square root of } 2 \times 5, \\
 \sqrt{2} \times \sqrt{5} &= \sqrt{2 \times 5}
 \end{aligned}$$

Figure 3: An existing discourse posed by the teacher in L8

Although Figure 3 shows the teacher’s writing on the blackboard in L8, in order to justify the computational form of “ $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$ ”, here it functions as an existing discursive template for the metaphorical projection by inflexible use of a new signifier. S1’s approach, that uses square calculation, is a case of unsuccessful attempt: a metaphorical *overprojection* (Sfard, 1997). On the other hand, S3’s approach, that uses an approximate value, is a case of successful attempt. Such metaphorical uses of new signifiers can be characterized as a particular action on Phase 1, where the new signifier “ $\sqrt{2} + \sqrt{3}$ ” is operationally used as a computational process; it is not more than a term that comes to complete the “unfinished” mathematical propositions “ $\sqrt{2} + \sqrt{3} = \dots$ ”. Therefore, in this phase the signifier has only developed as far as signified by the existing discourse.

### Reification in the semiotic chaining: Phase 2

In the excerpt that was described above, S5’s discourse suggests that “ $\sqrt{2} + \sqrt{3} \neq \sqrt{5}$ ”. Subsequently, the teacher used geometrical representation as a visual means for the explanation (see Figure 4). As a result, “ $\sqrt{2} + \sqrt{3}$ ” starts to involve “the length of a side of given square” as a new signified. Then teacher posed a new question, such as “If  $\sqrt{2} + \sqrt{3}$  is not  $\sqrt{5}$ , what can  $\sqrt{2} + \sqrt{3}$  be?” This question implies the need for a new semantic space of the signifier. Interestingly, a student explained that “ $\sqrt{2} + \sqrt{3}$  can be expressed as  $\sqrt{5+2\sqrt{6}}$ ”, because the area of the given square is  $5+2\sqrt{6}$ . The signifier “ $\sqrt{2} + \sqrt{3}$ ” is gradually objectified through their interactions. Thus, the teacher’s question was modified to become “What is ‘ $a$ ’ (*rational number*) for  $\sqrt{2} + \sqrt{3} = \sqrt{a}$ ?”

Although this question could not be solved by the students, the teacher concluded that “‘ $a$ ’ for  $\sqrt{2} + \sqrt{3} = \sqrt{a}$  is not a rational number”, by using an indirect proof (Figure 5). In this argument, the signifier “ $\sqrt{2} + \sqrt{3}$ ” can be structurally used as an objectified symbol, rather than as a computational process.

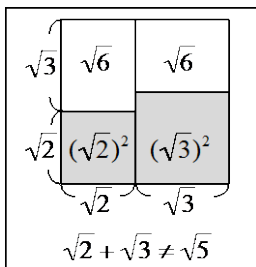


Figure 4: The length of the side of a square

We suppose that  $\sqrt{2} + \sqrt{3}$  is expressed as  $\sqrt{a}$  ( $a$  is *rational*)  
 $\sqrt{2} + \sqrt{3} = \sqrt{a}$   
 Square both sides,  
 $5 + \sqrt{6} + \sqrt{6} = \sqrt{a}$   
 It means,  
 “irrational number” = “rational number”  
 It is contradict with the supposition. Therefore  $a$  is not rational number.

Figure 5: The argument of “ $\sqrt{2} + \sqrt{3} \neq \sqrt{a}$ ”

The development of the discursive process on the addition of square root numbers, which we showed above, can be characterized in terms of the hypothetical model using the following schema:

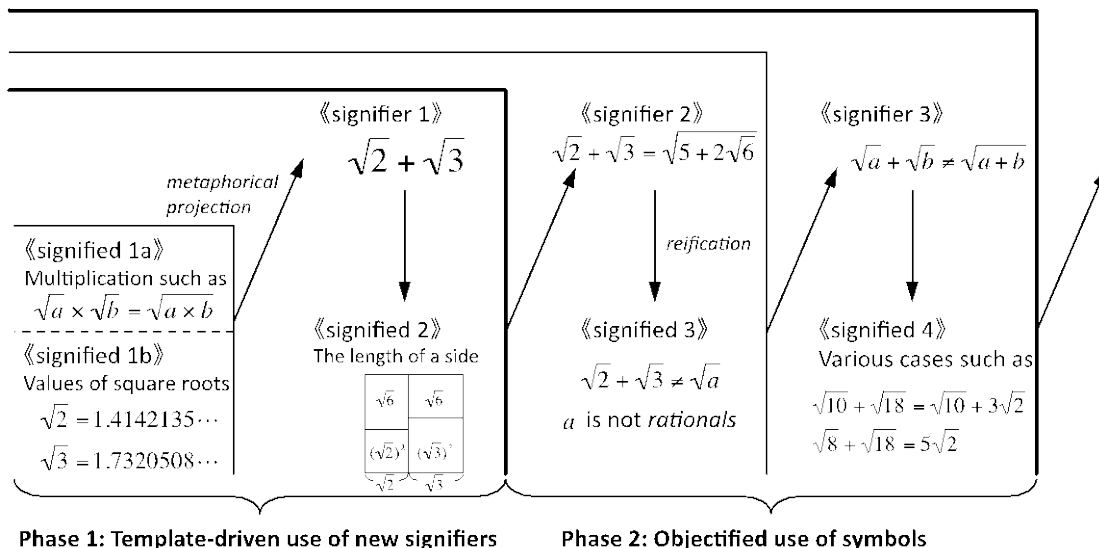


Figure 6: The semiotic chaining on the addition of square root numbers

Now let us explain the place of reification in Figure 6. In Figure 5, “ $5 + \sqrt{6} + \sqrt{6} = \sqrt{a}$ ” is objectified as “the irrational = the rational”. This objectification might undergo the change of meta-discursive rule. In the Phase 1, a signifier such as “ $A \circ B = C$ ” (including square root) has been dealt with as a process-product schema. That is, “ $A \circ E$ ” was considered as a computational process, and “ $C$ ” - as a computational product. Although this tendency, or difficulty in learning algebra, has been repeatedly pointed out from a cognitive point of view (e.g., Kieran, 1990; Herscovics & Lincheveski, 1994; Gray & Tall, 1994; Sfard & Linchevski, 1994), we would like to mention such a process-product duality (using the terminology from Gray & Tall (1994), a “*procept*”) from a discursive point of view. Pointing out “the irrational  $\neq$  the rational” requires the signifier “ $5 + \sqrt{6} + \sqrt{6}$ ” to represent *one irrational number*. In other words, the arguments in Figure 5 are endorsed by the new meta-discursive rule which states that

the signifier “ $A \circ E$ ” such as “ $5 + \sqrt{6} + \sqrt{6}$ ” represents both a computational process and a product. In this situation, the signifier “ $\sqrt{2} + \sqrt{3}$ ” can be a reified object as a representation of irrational number. Thus, in L13, the discourses on the additions of square root numbers become more flexible to be used in various calculation cases, such as “ $\sqrt{8} + \sqrt{18} = 2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$ ”.

## AS A WAY OF CONCLUSION

Let us reflect on some theoretical and practical issues associated with the present study. Presmeg (1997) characterizes a mathematical semiotic chaining such that “At each link in the chain, characteristic mathematical symbolism provides the new signifier which signifies the new reified object in a metonymic relationship, which constitutes the sign” (p. 276). In this report, we based on a two-phase discursive development of Sfard (2000, 2001), and attempted to modify the characterizations of metonymy relation, which we call “metaphorical projection”, and of reification in the chaining process. As a result, we characterized the act of metaphorical projection as a beginning feature of template-driven use of new signifier (Phase 1), and placed reification not only as an essential process of the objectified use of symbol (Phase 2) but also as a key to the change of meta-discursive rule that can be the characteristic mechanism of the transition from the Phase 1 to the Phase 2.

In earlier research on the reification (Sfard, 1991), the transition from operational to structural conceptions was an essential feature of mathematical concept formation in the theory. In the discursive approach to the reification, the operational/structural use of signifiers depends on the developmental phase of the discourse. These different theoretical approaches are based on different epistemological paradigms (Sfard, 1998). Nevertheless, reflection on previous analyses will clarify that the reification can be still an essential notion, which can be used to characterize the mechanism of *structuralizing the operation* in developing mathematical discourse. In this respect, further theoretical discussions and empirical studies might be needed.

In the series of lessons that we observed, there are many crucial “questions” posed by the teacher, such as “Is  $\sqrt{2} + \sqrt{3} = \sqrt{2+3}$  true?” or “If  $\sqrt{2} + \sqrt{3}$  is not  $\sqrt{5}$ , what can  $\sqrt{2} + \sqrt{3}$  be?” These questions made the learning of students orientated and activated. However, the teacher’s question does not always initiate arguments by students. For example, in L8, most students accepted the fact that  $\sqrt{2} \times \sqrt{8} = \sqrt{2 \times 8}$  without any justification when the teacher posed the question “Is  $\sqrt{2} \times \sqrt{8} = \sqrt{2 \times 8}$  true?” We think that this didactic point is concerned with the notion of “*devolution*”, suggested by Brousseau (1997). The devolution means “the act by which the teacher makes the student accept the responsibility for learning situation or for problem, and accepts the consequences of this transfer of this responsibility” (p. 230). Although this is not a focus of the present study, as a future task, we need to analyze social interactions between the students and the teacher in terms of the relevant theoretical framework.

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# TRACING ARIEL'S GROWTH IN ALGEBRAIC REASONING: A CASE STUDY

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*We report the results from a case study of Ariel, a middle-school participant in a 3-yr longitudinal study of the development of understanding of mathematical ideas. We focus on Ariel's use of arithmetic knowledge in finding a rule for his solution to a problem task that takes the form of a composite function. Over a year later, after being introduced to the technical algebra language and formal notation, Ariel revisits the problem and offers a general, closed-form solution. When re-solving the problem his language is more precise as he connects meaning to the symbols. Viewing a video of his earlier solution, Ariel acknowledges the correctness of his earlier work and indicates that the earlier solution was not generalizable.*

## INTRODUCTION

In this study, we examine the development of algebraic reasoning of Ariel, a participant in a 3-year, after-school, informal mathematics program. We focus on Ariel's engagement with early algebra ideas, particularly his work on the Ladders Problem (Davis, 1964). The task requires that the student construct a rule to predict the number of Cuisenaire rods needed to build a ladder with varying number of rungs. We report Ariel's problem solving in sessions, fifteen months apart - grade 7 and the end of grade 8 solving the same problem. The research questions guiding this study are: (1) How does Ariel make use of his knowledge of arithmetic and algebra to build his solutions to the Ladders Problem? and (2) What change, if any, was there in Ariel's problem solving from grade 7 to 8?

## THEORETICAL FRAMEWORK

While research has shown that understanding the concept of a function is essential for success in other areas of mathematics (Carlson, 1998; Rasmussen, 2000), students continue to struggle learning the concept (Vinner and Dreyfus, 1989). Yet, there is promising work that supports the claim that young children, who are engaged in problem-solving activities designed to elicit justifications for their solutions, can develop understanding of fundamental algebraic ideas such as function (Maher, Powell & Uptegrove 2010; Kieran, 1996; Yerushalmy, 2000; Kaput, Carraher, & Blanton, 2008). As early as 1985, Davis advocated the introduction of algebra to elementary age students, some even as young as grade 3. He argued that the idea of function can be built intuitively by learners as they engage in explorations of problems that called for the identification of increasingly more challenging patterns, and that students can build the conceptual idea before formal notation is introduced. In his work, Davis has offered sets of tasks for student exploration, capturing on video the problem solving of young children as they successfully find solutions that can be expressed with linear, quadratic and exponential functions (Mayansky 2007; Giordano, 2008). Also, examples of



children providing verbal expressions of algebraic function before learning to write the rules in symbolic form have been reported by Bellisio and Maher (1998). This study seeks to extend earlier work by examining in detail how one student builds an understanding of the linear function concept and represents his understanding of the basic algebra ideas underlying its construction

## METHODOLOGY

### Setting

This study is a part of a larger three-year, research project, Informal Mathematics Learning (IML), a partnership program between the Robert B. Davis Institute for Learning (RBDIL) and an economically depressed urban school district in the USA. The case study was conducted with middle school aged students (11 to 13 years). A primary goal was to investigate how mathematical ideas and ways of reasoning by students were developed over time in an informal, after school environment. The content strand reported here is algebra. Ariel and other members of his cohort, were introduced to functions by engaging in Guess My Rule activities

The algebra strand began at the end of the second year of the program. Ariel participated in six group algebra sessions over 3 weeks and one interview about his experiences in the algebra domain 15 months later. Each session lasted 60- to 80 minutes. At all of the sessions, the student participants worked on open-ended problem solving tasks that challenged them to build the rule that described the function.

### Task

For this report, we focus on a task that asks students to determine how many light green Cuisenaire rods (Figure 1) would be needed to build a ladder with different number of rungs. The shortest ladder has only one rung and can be built with 5 light green Cuisenaire rods. A two-rung ladder would be modelled using 8 light green rods. It was of interest to see if students could provide a general solution to the problem. The problem was presented as follows:

The Ladders Problem: Build a rod model to represent a 3-rung ladder. How many rods did you use? How many rods would you need to build a ladder with 10 rungs? How could you represent the number of rods needed if you were to build a ladder with any number of rungs? Justify your solution.



Figure 1: Picture of Cuisenaire rods and a ladder with one rung built with the rods.

## Analysis

All sessions were videotaped and prepared for storage as a part of the RBDIL video collection that is hosted on an online video repository, the Video Mosaic Collaborative (<http://www.videomosaic.org>). The video sessions were first described, transcribed, and coded for critical events. (based on Powell, Francisco, and Maher, 2003). The videotapes were then analysed across all sessions to identify themes about Ariels' algebraic reasoning. The Ladder Problem was chosen to analyse for this paper because it represented both his early and later algebra knowledge and enables us to trace the details of how he builds his solution.

## RESULTS

### Event 1: Developing a composite function (7th Grade)

The first event occurs during the 7<sup>th</sup> grade problem-solving session. After introducing the problem, the researcher asks Ariel to find how many Cuisenaire rods are needed to build a ladder with ten rungs. After building ladders with varying number of rungs he offers the answer of 32 and is asked about how he solved the problem. He was then asked to write up his solution (Figure 2).

Ariel: I take half of the number if it has a half, I will multiply it by two and subtract two.

Researcher: So like for ten, how would you do that?

Ariel: I think a ladder of five, then count the rods, multiply that by two and subtract two.

Researcher: And subtract two. How would you do that for nine?

Ariel: For nine, I would do eight, then I would multiply the number by two and then I would subtract two and then I would add three.

As indicated in Figure 2, Ariel offered a composite function depending on whether the number of rungs in the ladder was odd or even. He justified his rule for four cases and was satisfied that it addressed the conditions of the problem for those cases.

For odd numbers I go to the nearest even number take  $\frac{1}{2}$  of that even #, count the rods for a ladder with that many steps multiply it by 2 subtract 2 and add 3

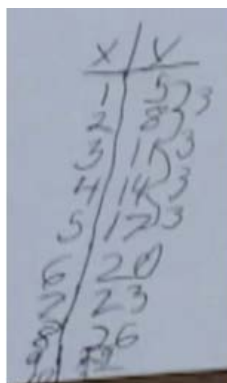
For even numbers I take  $\frac{1}{2}$  of that number and make a ladder with that many steps. Then I multiply the # of rods of that ladder by 2 then I subtract 2.

Figure 2: Student work - Ariel's rule for even and odd numbers.

### Event 2: Using First Difference (8th Grade Interview)

Fifteen months later, an interview was conducted with Ariel in which he again was given the Ladders problem. When asked to determine how many rods are needed for a ten-rung ladder, he immediately built a table (Figure 3) and provided the following justification.

Ariel: How many rods would you use to build a ten-rung ladder? Well, *I'm not going build it, I'll just do a X, Y table*. Um, that the difference between the Y, the Y, um, variables, was from 5 to 8 it increased by 3, from 8 to 11 it increased by 3...so you would just keep on increasing by 3, so until and then, it'll keep on going. *And then it shows that it's linear obviously because in a linear equation the first difference is always the same*. So, it's linear.



X	Y
1	5
2	8
3	11
4	14
5	17
6	20
7	23
8	26
9	29
10	32

Figure 3: Student work - Ariel's X, Y function table.

During this session, although Ariel had available Cuisenaire rods to build his ladder, he chose not to use them and instead he constructed an X Y table as indicated in Figure 3 above. Notice that to the right of his Y column he noted the first order difference of 3 between each value.

Ariel completing his table, the researcher asked him how he could build a ladder with any number of rungs and Ariel *immediately* responds with his solution.

Ariel: Um, how could you represent the number of rods needed to build a ladder with any number of rungs? That would be... hmmm... Oh! That's easy. Y equals 3X plus 2.

The researcher invited Ariel explain where his formula came from.

Interviewer: Wait... where'd that come from?

Ariel: Cause uh... how could you represent the number of rods needed to build a ladder with any number of rungs. So, you'd get the number of rungs from multiplying the, the...ladder which it is, like if it was the first ladder, second ladder, third ladder...multiply by 3, like on this one...it would be nine...and plus two is eleven. So, substitute the number, for, in each X, it would be 3 times 3 plus 2...it would be 9 plus 2... it's 11. And it works out, for every one.

The researcher then asked Ariel where the plus two came from, and Ariel responded.

Ariel: Because... *I just looked at it* and if... if you multiply each by three... it's gonna be, um, m plus the y intercept, which is gonna be 2. *Cause if it's adding three each time, if you reverse this to when it was at 0, it would be a 2 right there*. Wait... yeah, it'd be a 2 right there. And then, *this [indicating 3] would be your slope of 3, and your y intercept of 2 [indicating the value of (0,2)]*. And then it's a linear equation.

Ariel pointed out that his rule “works for every one”. He made use of the first difference of 3 to identify the slope and noted that the y intercept was 2 to produce the rule of  $3X+2$  where X is the number of rungs in the ladder.

### Event 3: Reflecting on past problem solving (8th Grade Interview)

After re-solving the problem the researcher invited Ariel to watch the video of his earlier work on the Ladder Problem in the 7<sup>th</sup> grade and to compare how he solved it in 7<sup>th</sup> grade to how he just solved it minutes earlier.

Interviewer: So now that you saw how you did it back in the day and how you did it now, what can you tell me about the two solutions? Like how can you compare them?

Ariel: Well, like I said before, *that way would be the long way* and then now it's like you just combine it together to make an equation all together. *Like, back then I didn't really know that much about equations, so it would just be like separate steps, and I didn't know, instead of combining them.* Like there, that's say, like you know how an equation could have the factored form and the expanded form. So this would be the expanded and that would be the factored. That would be the long way, *like all the different pieces all separate instead of just putting them into one.*

Ariel described his earlier work as “the long way”, and indicated that he “didn't really know that much about equations”. He also pointed out the efficiency of his new approach.

Interviewer: Oh, I see. So you're saying that you could take all these steps here and then just combine them?

Ariel: Yea, probably. But it would be more difficult to explain the part about going to an even number in the equation. Well, actually, it would be, cause if your x is going to equal nine then you just do, it'd be x minus one, you get the eight, which is eight. *But it wouldn't work every time...* cause then you would get an even number, wait, *no it would work almost every time.*

Ariel continues to explore other cases and concludes that the solution is basically a check.

Ariel: Say, well this *would be basically a check up*, in a way, to see if it matches that, if the formula could help you match the number of rungs to the number, I mean, the number of rungs to the number of pieces. So then, here you just take the number of rungs and multiply it and then you add two. So, yea they were both effective in a way, in their own little way. Yea, cause here it would be like a check, a good check right there.

Ariel maintained that both solutions were “effective”, suggesting that he was open to more than one approach to establish the correctness of his reasoning.

## DISCUSSION

Ariel began his exploration by identifying two rules to account for the number of rods required to build ladders of finite number of rungs. He partitioned his solution into two

cases: odd and even. In so doing he built a composite function that provided a solution of ladders for certain numbers of rungs. It is interesting that Ariel first described his solution in words, using language that was familiar to him from his arithmetic learning. This approach contrasts starkly to his later problem solving in which he chooses to construct a function table, indicate the first finite difference to identify the slope and justify the value of the y intercept to produce an elegant, general solution to the same problem. Confronted with his earlier work by viewing the video, he shows amusement in the video clip as he acknowledges the correctness of his earlier work, yet recognizing that he now produced a general solution in the form of a linear equation, whose components are connected to the function table he created. His use of the formal mathematics language, elegantly stated, shows his growth in connecting the meaning to the language to the symbolic notation. Ariel recognizes that both approaches have merit; they provide a check for him, connecting the informal with the formal. He has shown that the visual model created with the rods from his earlier problem solving was no longer needed. In his later work, he can imagine and represent his ideas with numbers and symbols, as well as with formal language. Ariel's case is important in that it underscores the need for providing opportunities for students to make use of their personal representations of the problem, using what is already known, and building on that knowledge.

Early, informal open-ended problem solving opportunities provide the ground work on which learners can build their knowledge. These investigations are not to be viewed as activities that should be given to students merely as a follow-up to procedural instruction; rather, these explorations are not only the beginning but the *essence*. Teachers must find the time in their classrooms to provide students with the opportunities to explore, examine, revisit and connect ideas and concepts through investigations that enable the learner to build strong intuitions of the problem conditions. Engagement in activities, such as the Ladder Problem, are needed to build a strong foundation for gaining insights and deeper understandings..Ariel had these opportunities and built his algebra knowledge on solid ground. His success is indicated in the elegance of the solution he provided, the understanding he had of his earlier work, and the confidence he had in offering his providing clear justifications for his work when asked..

## ENDNOTE

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# PROMOTING REVERSIBILITY: BUILDING ON LEARNING THROUGH ACTIVITY

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*Our research program is generating a theoretical framework that characterizes the process by which students make mathematical abstractions from their mathematical activity in the context of carefully designed sequences of mathematical tasks. The framework provides a foundation for instructional designs that actively promote the development of particular mathematical understandings. Students' development of the reverse of recently learned schemes is a well-documented challenge. In this report, we describe and demonstrate how we use our framework to design for promoting reversibility and to explain learning that can be understood in terms of reversibility. This is a theoretical paper, in which data are used for illustrative purposes.*

## REVERSIBILITY

“Piaget determined that the key characteristic distinguishing logico-mathematical operations from other mental actions is reversibility” (Norton & Wilkins 2012). Reversibility is important at all levels of mathematics learning (e.g., addition-subtraction of whole numbers, differentiation-integration in calculus). Inhelder and Piaget (1958, pp. 272-273) defined reversibility as “the permanent possibility of returning to the starting point of the operation in question.” They went on to explain,

From a structural standpoint, it can appear in either of two distinct and complementary forms. First, one can return to the starting point by canceling an operation which has already been performed – i.e., by inversion or negation. In this case, the product of the direct operation and its inverse is the null or identical operation. Secondly, one can return to the starting point by compensating a difference (in the logical sense of the term)-i.e., by reciprocity. In this case, the product of the two reciprocal operations is not a null operation but an equivalence.

The work reported here focuses on the development of reverse schemes. Once a scheme has been developed, the development of a reverse scheme is neither automatic nor unproblematic (Lamon, 2007). Steffe built his definition of reversibility on von Glasersfeld's (1995) tripartite model of a scheme. The model consists of

1. recognition of a certain situation;
2. a specific activity associated with that situation; and
3. the expectation that the activity produces a certain, previously experienced result. (p. 65)

Steffe and Olive (2010, p. 125) defined reversibility as “taking the results of the scheme as input for producing a situation of the scheme.” In other words, if we think of a scheme as anticipation of a resulting situation given an initial situation and an



appropriate action on that initial situation, reversibility is the anticipation of the initial situation given the resulting situation. We use this idea in our research as well.

Steffe (1994, cited in Ramful & Olive, 2008) described the process of moving from a scheme to a reverse scheme in the context of the development of number:

I understand the construction of reversibility as a product of reinteriorization of the initial number sequence that occurs when a child takes an associated verbal number sequence as material in a review of counting. After constructing the number sequence at two levels of interiorization, the child can take the contents of the first level as the material of the operations at the second level, which is the constitutive operation of reversibility. (p. 16)

Our work described in this paper offers an approach to fostering the *reinteriorization* to which Steffe refers.

## **REVERSIBILITY VERSUS SEPARATE REVERSE SCHEMES**

Key to our work on reversibility is our understanding of the relationship between the scheme and the development of the reverse scheme. We understand the development of reversibility as *constructing the reverse scheme from the scheme* (reinteriorization). This represents a higher level of understanding. I highlight this point, because instruction does not always follow this principle. For example, students are often taught whole number or fraction division independent of their learning of multiplication. This can be done, because one can build up division from physical actions on objects without an understanding of multiplication. The result of this independent development, however, is that the students may not understand the relationship between multiplication and division. In the example from the MARN Project (below), constructing the reverse scheme on the basis of the scheme is demonstrated.

## **BUILDING AN INSTRUCTIONAL APPROACH BASED ON LEARNING THROUGH ACTIVITY:**

Where mathematics instruction has moved away from a predominantly lecture model, instruction often takes the form of presenting problems whose solutions are the mathematics to be learned. Students work these problems individually and/or in small groups and then solutions are shared and discussed in a class discussion.<sup>1</sup> Some of the students are able to solve the problems and some of the remaining students may understand the successful solutions of others. Of this latter group, some may be able to later work similar problems. In contrast, we have been developing an instructional design model, that does not replace problem solving in mathematics classrooms, but which provides an additional design tool, not dependent on problem solving<sup>2</sup>, for helping students learn intractable concepts.

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<sup>1</sup> A notable exception to this model is Realistic Mathematics Education (Gravemeijer, 1994), which involves instructional design constructs that are used to engineer development of formal mathematics from informal initial approaches.

<sup>2</sup> By problem solving we use the commonly accepted definition of working on a task for which the learner has no solution method at the outset.

The space available does not allow a review of our work on learning through activity (see Simon et al, 2010; Simon, 2011). The work is based on Piaget's (2001) construct of reflective abstraction. That is, we take the development of a new mathematical abstraction (in any situation) as a product of reflection on one's own activity. This perspective and the research deriving from it has led to an instructional approach in which we design tasks that promote the particular activity and the needed reflection to construct the new abstraction. In this way, the engineering of the task sequences is more intricate and deals with the learning paradox (Bickhard, 1991; Smith, diSessa, & Roschelle, 1993). That is, how can someone who does not have the knowledge to be learned solve a problem that requires the knowledge to be learned? Whereas no mathematics instruction is deterministic of particular learning and no instruction is successful for all students, our goal is to go beyond creating opportunities for learning (as in the problem solving lessons) and engineer (to the extent possible) the generation of the raw materials of the conceptual learning intended. One symptom of our design perspective and commitment is that if a student in our one-on-one teaching experiments cannot solve a task, we take it as a sign that we have skipped steps or have a faulty path towards the concept. We modify our design of tasks rather than helping the student, giving hints, or asking leading questions.

Because of this approach in which we promote new understandings rather than occasioning them, we were interested in understanding *how to promote reversibility*. Not only does there seem to be no existing literature on this question, the question itself tends not to be asked. Rather, the question derives from our task design model in which promoting learning has the particular meaning described in the last paragraph. There are a number of examples of posing problems that require the reverse of the learned scheme (e.g., Hackenberg, 2010), but we mean something more when we speak of *promoting* reversibility.

## ONE APPROACH TO PROMOTING REVERSIBILITY

### Description of the Approach

Our current Measurement Approach to Rational Number (MARN) Project<sup>3</sup> has two goals: (1) further explication of learning through activity and (2) development of useful task sequences for promoting deep fraction and ratio understanding in upper elementary (9 – 11 years old). It is in this context that we encountered the challenge of promoting reversibility. We describe here an approach which emerged after some less successful empirical work.

Two principles guided our exploration of ways to promote reversibility (both discussed above):

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1. *Promoting understanding* means posing tasks that foster those activities that will lead to reflective abstraction of the new understanding (not posing a task that requires the new understanding).
2. *Developing reversibility* involves constructing the reverse scheme from the scheme.

Based on these principles, we developed an approach to promoting reversibility that included the following sequence.

1. We first pose tasks that require the scheme that was learned previously.
2. We pose a reverse task as finding the input that was used to generate the given output. (E.g., if a student understands converting a mixed number to an improper fraction, we would ask, “What was the mixed number from which this improper fraction was made.” This is not the same task - in terms of its demand – as “Convert this improper fraction to an equivalent mixed number.”)
3. If the student is not able to solve the reverse task, we encourage guess and check.
4. We pose reverse tasks of this type until the student no longer requires guess and check (i.e., can go directly to the answer and justify it).

### Example of Use of the Approach

The following example is taken from a one-on-one teaching experiment with Kylie, a 4<sup>th</sup> grade (10 years old) student. Kylie had previously learned to solve missing-denominator equivalent fractions tasks in which the missing value was part of the fraction with the *larger* numerator (e.g.,  $8/7 = 24/?$ ). To begin to promote the reverse scheme, we posed a few missing-denominator tasks of the type she was already able to do. For the task  $3/25 = 12/?$ , Kylie explained that since the three 25ths became 12 pieces, each of the three 25ths must have been cut into 4 pieces, and cutting a 25th into 4 parts would produce 100ths. (We had established previously that she had no difficulty justifying this last statement.)

We then gave her the task  $4/? = 36/63$  (reverse task).

K: I don't know.

R: Okay. So. Try, try a number and see if you're right.

K: ...I'm gonna try nine.

R: Okay. Any particular reason you're trying nine?

K: Nope, that's not gonna be right ...

R: ... First of all, why did you pick nine?

K: Cause I was thinking about the top number

R: Okay, what about the top number?

K: That it's thirty-six, and I know that four times nine equals thirty-six.

R: Okay. Alright. And you put nine in there, and how'd you know nine wasn't gonna work?

K: Cause then I tried it and that would be seven [referring to the number that would need to be multiplied by 9 to get 63], and ... seven times four is twenty-eight [referring to splitting each of the 4 9ths into 7 parts, the result would be 28 pieces, not the 36 given in the task].

R: Okay.

K: That wouldn't be right, so now let's try eight. ... Mm ... Uh no, seven, seven, seven [showing excitement], okay, seven...nine...Yes! Nine, seven! seven!

R: Seven?

K: Yeah

R: Okay. How do you know it's seven?

K: Cause, if I put it there and I divided - I know that seven times nine equals sixty-three, so I divide the one seventh into nine pieces, and then. ... I know that nine sixty-thirds, is equal to one seventh. So, uhh, mm... then nine times four is thirty-six, so that's correct.

This type of problem and solution, soon led Kylie to be able to anticipate the input value, with no need to guess and check, and justify her solution.

### **Theoretical Basis for the Approach with Reference to the Example**

Here I explicate the theoretical basis of the four steps of the approach discussed above. Before doing so, I call attention to one additional aspect of our theoretical framework, the distinction between a participatory level of a concept and an anticipatory level. (See Tzur & Simon, 2004, for a more complete discussion of these constructs.) The distinction builds on our work focused on abstracting new ideas (developing an anticipation) from one's activity. The key point for this discussion is that the first level in learning a concept is the *participatory level*. At this level, students develop a new anticipation based on the activity that they are engaged in. That is they no longer need to go through the activity sequence; they can anticipate the effect of the activity. The new concept is at a participatory level, because the student does not yet know to call on the activity, and thus the anticipation, in response to tasks that do not cue them for the activity. When the student can call on the activity and related anticipation, the concept is at the *anticipatory level*.

This discussion of our method of promoting reversibility focuses on development of the participatory level of the reverse scheme. Our theoretical work on promoting the anticipatory level of a scheme in general is still relatively primitive.

*Steps 1 and 2: Pose tasks that require the original scheme. Pose a reverse task as finding the input that was used to generate the given output.*

In the example, Kylie was given tasks that required the original scheme and asked to justify her solutions. The second step was accomplished by using the same form for the reverse-scheme tasks. That is, the fraction with the smaller numerator remained on the left. For Kylie this seemed to be sufficient to suggest finding the input based on the

output. In another situation, it might have been necessary to explicitly invite determination of the input as in the example of converting improper fractions above.

*Steps 3 and 4: If the student is not able to solve the reverse task, we encourage guess and check. Continue until the student can go directly to the answer and justify it.* This approach is a way of structuring the task so that the student engages in an activity involving the recently learned scheme. Even though steps 1 and 2 cue for the original scheme, the learner is unlikely to know how to use the scheme to complete the task. As the student engages in trying a value and checking (using the learned scheme), she comes to the point of anticipating the correct value based on an anticipation of how that value will be transformed (during the forward or checking process) to produce the given value. This is an example of reflection on activity applied to development of the reverse scheme.

## **USING OUR APPROACH TO PROMOTING REVERSIBILITY TO ACCOUNT FOR LEARNING**

In this section, I demonstrate how we used the above approach to reanalyse learning described in an earlier paper. In so doing, I offer a second example of the application of the approach and demonstrate its usefulness in explaining learning that was not originally conceived of as the development of reversibility.

In Simon et al (2004), we used an example of learning to ground our discussion of learning through activity. Whereas the empirical work upon which the example was based and that article preceded the approach to promoting reversibility described here, the approach allows us an additional tool for explicating the learning involved. I will first describe the learning and then give the new explication. (Please see the original article for a fuller description.)

Micki (9 years old) does not yet understand that equal partitioning produces parts of a particular size relative to the whole. Micki is working in a computer microworld, *Sticks*<sup>4</sup> (Steffe & Wiegel, 1994) in which she can draw sticks of various lengths (line segments), copy them, and iterate them multiple times to make a longer stick. Micki is asked to cut a given stick into 5 equal parts. She draws a small stick under the large stick, iterates the small stick 5 times and compares the result to the original stick. She compares the resulting stick to the original stick. Seeing that the stick she produced is not the same size, she adjusts the size of the small stick and repeats the process. She continues this process until she has produced a stick the size of the original. Doing tasks of this type with sticks of varied length and varying number of parts leads her to abstract that equal partitioning results in a part of a particular size.

In Simon et al (2004), we explained the learning as involving Micki's composite unit scheme. That is, she had an understanding that a unit iterated a certain number of times produced a composite unit of a particular size in relation to the unit. Our recent work on reversibility allows us to interpret Micki's learning as development of a reverse

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<sup>4</sup> Sticks was refined and is now available as JavaBars.

scheme for her composite unit scheme. Whereas the composite unit scheme anticipates the creation a new, proportional unit through iteration, the reverse scheme anticipates the creation of a new proportional unit through partitioning. Micki's learning can be seen as parallel to Kylie's learning equivalent fractions (above). Micki's strategy of iterating and a small stick and then adjusting the small stick based on the result of the iterated stick can be seen as the "guess and check" aspect that we described in the design for Kylie's learning. By trying a small stick of a certain size and iterating it, Micki was using her scheme for composite units in two ways. First, she was able to anticipate that the small stick iterated would produce a composite stick of a particular size. Second, she was able to anticipate the direction of the adjustment needed for the small stick based on the result of the comparison of the composite stick with the original stick. This use of the composite unit scheme in service of developing the reverse scheme led to Micki's abstraction that partitioning the original stick a certain number of times created a part of a particular size.

## CONCLUSION

I have described an emerging approach to promoting reversibility of a scheme. We consider it to be consistent with Steffe's notion of reinteriorization, because students use the anticipation developed in the scheme as part of their activity (including guess and check) that they interiorize to develop a new anticipation, the reverse scheme.

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# IMMIGRANT MATHEMATICS TEACHER STUDENTS' SUBJECT POSITIONING

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*In this paper we present an analysis and discuss implications of immigrant mathematics teacher students' subject positioning, in discourses of language and cultural concerns. Discourses that became obvious in the analysis of interview data and ethnographic fieldwork in a Swedish teacher education programme. We conclude that there is a need to take into account how discourses of language and mathematics work within the educational programme and hence, discuss how to meet different needs without omitting the core of mathematics teaching.*

## INTRODUCTION

An increasing interest in research addressing social justice and equity is important for meeting the needs of mathematics education in the 21st century. The socio-political turn (Gutiérrez, 2010) signals a theoretical shift where mathematics practices are seen as nothing but neutral and where concepts as identity and power are interwoven and constituted within social discourses. However, because inequities relate to the marginalised as much as they do to the powers that sustain the inequities, sociolinguistic research in less diverse settings or in a range of settings is also important. Positioning is always at work in mathematics classrooms, no matter the range or depth of diversity in the context (Wagner & Herbel-Eisenmann, 2009).

Our main research focus is concerns raised by mathematics teacher education students. However, we found that immigrant students addressed special issues that we feel an obligation to address. This specific group consists of students who are born outside Sweden and moved to Sweden as adolescents or adults which implicates that they have done a large part of their schooling outside Sweden; and accordingly that Swedish is not their first language. The research question we specifically discuss in this paper focus the revealed concerns that were unique for this group; hence not brought to the fore by other students. The concerns related to being immigrant, having experience of different school cultures and also to what we have labelled institutional constraints within the educational system. We take the concerns one step further and show, through a socio-political analysis of discursive positioning, how these teacher students' enacted subject positions were characterized; and their relation with the available discourses in the university courses. Posed in other words: How do immigrant student teachers' subjects positioning reveal power relations in discourses, available in mathematics teacher education? We introduce four students, Nadia, Evelyn, Theresa and Rita in conjunction with the main part of the paper which is dedicated the analysis of becoming mathematics teachers' discursive positioning.



## **THEORETICAL FRAMEWORK**

### **Positioning – a dynamic and relational concept**

Positioning is a conversational phenomenon, where the actors can negotiate and choose subject positions in discursive practices. Actors are constantly positioned and re-positioned by themselves or by others (see e.g. Davies & Harré, 1990). Harré and van Langenhove (1999) stress the importance to grasp the dynamic character of positioning within conversations and that people involved in a discursive practice can negotiate and act new positions. This implicates that a speaker, who takes up a position by opening or interacting in a conversation, can change positioning during the communication. “This act does not and indeed could not preempt the future structure of the conversation” (Harré & van Langenhove, 1999, p. 28). Wagner and Herbel-Eisenmann (2010) draw attention to feelings, attitudes and values with regard to mathematics that were communicated through positioning. They conclude that there are positionings that are particular to mathematics classrooms in its characteristics and strongly connected to student’s experiences of mathematics.

Positioning is in this study understood as relational (Harré & van Langenhove, 1999) and as “actively negotiated and achieved” (Tan & Moghaddam, 1999, p. 187) in discursive practices. We thus have our focus “on intergroup relations as a process” (p. 186). This dynamic understanding of positioning implies that positions can and do change; and thus, we need to use positioning as verbs that describe a person “acting on or with another person or community” (Wagner & Herbel-Eisenmann, 2009, p. 9). By using positions and positioning as verbs we can easier see that positions change “because the act of positioning implies a move to change the positioning” (p. 11). From this point of departure we see positioning as fluent and constantly changing within discursive practices.

### **Discursive positioning**

Discourse is in this study understood and defined in line with Gee’s (2011) definition of the little “d” discourse: as “any instance of language-in-use or any stretch of spoken or written language” (p. 205). But in addition to Gee’s definition we need to explain a little more about our special focus. Discourses are central in the sense that positioning always is “associated with particular rights, duties and obligations for speakers and hearers” (Tan & Moghaddam, 1999, p. 184). Language-in-use is always political (Gee, 2011), and so are mathematics language (Gutiérrez, 2010). According to these statements we claim that also talk about mathematics and mathematics teaching is political. This socio-political approach requires awareness and a fluent understanding of discourse. Wagner, Herbel-Eisenmann, & Choppin (2012) wrote:

We see the word ‘discourse’ being used to describe how contexts, such as mathematics classrooms, are structured in order to broadly consider how language exchanges embody the diverse social, political, cultural and socioeconomic positions at play (p. 4).

This imply that discourses change and develop within mathematics education contexts, as shown by Andersson (2011), and also that several discourses at the same time can

play and interfere in mathematics classrooms. We hence assume that several discourses could interfere in talk about mathematics and mathematics education and that discursive positionings within mathematics teacher education can reveal student teachers' empowerment and disempowerment within the discourses.

Harré & van Langenhove (1999) point out that positioning, over time, will change due to social forces of what can be said and done. This is what we will refer to as discursive positioning.

*/.../ the catalogue of kinds of positions that exist here and now will not necessarily be found at other places and times. In so far as the content of a position is defined in terms of rights, duties and obligations of speaking with respect to the social forces of what can be said, and these 'moral' properties are locally and momentarily specified, positions will be unstable in content as well (Harré & van Langenhove, 1999, p. 29).*

The discursive positioning is hence relational (van Langenhove & Harré, 1999), ephemeral (Wagner & Herbel-Eisenmann, 2009) and dynamic (Harré & van Langenhove, 1999), which in this analysis imply that the student teachers' positioning is *enacted in relation* to someone or something; that positioning can be enacted *within single utterances* and then "disappear"; and that positioning *can and do change* over time. Hence, our starting point is that subject positioning is dependent on the discursive practices and can change to adjust the actual discourse.

## METHODOLOGY

### The research context and methods

Through ethnographic fieldwork, Kicki participated as observer during the two first years in a four-year mathematics teacher education programme. As emphasised by Andersson (2011), field notes became the important tool for understanding the students' actions and interactions within different contexts and discourses. Semi-structured interviews (Bryman, 2008) were conducted in the beginning of the university based courses, after the first period of practice and after one and a half, respectively two years. The interviews differed in length and content, but the red thread through all interviews though, were mathematics, mathematics teaching and learning and their teacher education. Kicki posed follow-up questions that were connected to what the students talked about, which made each interview unique and sometimes had a character to be more like a conversation. This uniqueness created on-going storylines between the interviewer and the individual participants<sup>1</sup>.

### Analysis

In the analysis we focus on specific concerns that emerged during interviews. 13 interview transcripts from seven students were analysed since these transcripts indicated specific concerns related to language, culture and specific institutional constraints. Notable is that every student in this group is immigrant. None of the native

<sup>1</sup> All interviews were transcribed and analysed in Swedish. Thus the transcripts below are not translated verbatim, but key words, pronouns and verbs that we found important are thoroughly translated. The participants were given pseudonyms.

Swedish speakers mentioned any of these issues. This group is thus generated from how discursive positionings are constructed in relation to concerns that emerged in the data. This dynamic way of grouping emerging phenomena and letting the studied group alter makes possible to understand concerns within the whole student group instead of grouping and categorizing individual students.

In the analysis we draw from the mutually determining triad (Harré & van Langenhove, 1999, p. 18) and relate each positioning to the actual storyline and the social force of what can be said and done. But, as stated by Wagner and Herbel-Eisenmann (2009), “we can interpret any situation within different storylines” and because perspective differs, “there is no way of establishing correct storylines or positionings” (p. 9). There may be several ways of interpreting each situation and thus different conclusions could be drawn from the data. The presented storylines and positionings are interpreted from how the students use personal pronouns and verbs in relation to the specific concerns that were brought forward. Within another storyline – could be just some moments later or after a year – the positionings can be completely different in relation with the actual context and available discourses.

Central in the analysis is becoming mathematics teachers’ discursive positioning, and in this special case *intergroup* and *personal* positioning. Intergroup positioning, following Tan and Moghaddam (1999) is “fundamentally achieved through the use of linguistic devices such as ‘we’, ‘they’, ‘us’, ‘them’, ‘I’ [and] ‘you’ (as a member of a certain group)” (p. 183). And as Herbel-Eisenmann, Wagner and Cortes (2008), we see personal pronouns like “I” and “you” as strong markers for personal positioning.

## THE IMMIGRANT STUDENTS’ CONCERNS

Evelyn and Nadia moved to Sweden six years ago, Evelyn as an adult and Nadia as an upper secondary student. They both enjoy mathematics and chose mathematics as their main subject when applying to the teacher education programme.

All the first interviews were opened up by the question “How do you feel about the first weeks at the mathematics teacher education programme?” Hence the storyline was as becoming mathematics teachers at university. This initial positioning was in several interviews first accepted and the students answered accordingly. Then some students challenged this position and changed storyline in line with their concerns. Let us follow how Evelyn and Nadia’s positioning brought their language concerns to the fore by positioning themselves through personal and intergroup positioning<sup>2</sup>.

Evelyn: How I feel? Right now I’m okay. Two days ago I thought about jumping off. Uhm... The first two weeks were very tough and maybe it was because of language, I don’t know. It felt like this was not at all what I thought it would be and... I decided to not become a mathematics teacher /.../ I felt like this last week and now I think this course is okay. But I don’t know how I will feel down the line /.../ I hope it will get better and better.

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<sup>2</sup> All interviews were conducted in Swedish, but Evelyn partly speaks English during the interviews, which for instance explains the metaphorical utterance “down the line”.

Evelyn took up the position of feeling something in relation to the educational programme. However, she challenged the storyline of herself as becoming teacher and a shift in position took place. The new storyline concerned her language difficulties and hence Evelyn positioned herself within an available discourse of language when describing her experiences during the first two course weeks. Evelyn's personal positioning within the educational discourse [Right now I'm okay /.../ I think this course is okay] shows empowerment. The shift in storyline indicates that her language concern is strong and within the working language discourse her disempowerment is clear. These two different positionings make her seem ambivalent in relation to mathematics teacher education.

Also Nadia took up the position of feeling in relation to the educational programme; especially of "being in this class":

Nadia: Honestly, it feels quite bad... being in this class.

Kicki: Okay.

Nadia: What happened was that eh... immigrants were grouped together with immigrants and the Swedes with them. This will lead to that we will not develop our language skills. And especially for us who want to become teachers and... because we are immigrants too. So we will kind of be... we will be assessed as students, not... as immigrants. But I think they mostly will focus on our language. How we speak Swedish, how we work together. I have difficulties in the Swedish language. Not only mathematics [is important] as the teacher told us. The language is the most important... tool. If one look at this, I don't think we will succeed...

Nadia initiated a storyline that focused on institutional concerns, related to the group of immigrant students' language difficulties and the particular differentiation between native Swedish speakers and immigrants leading to [that we will not develop our language skills]. Nadia's intergroup positioning within the working language discourse shows her experienced disempowerment both in relation to the institutional steering (including the teachers' assessment) and to language difficulties that will sustain. Nadia's disempowering intergroup positionings is also an example of that immigrant students not only are concerned with their own language difficulties, but also include peers with whom they share this problem. However there is an important and empowering intergroup positioning in the midst of her talk about language concerns. She says "especially for us who want to become teachers", which we interpret as an empowering intergroup positioning expression. They intend to become mathematics teachers, despite the language concern.

Theresa, a young woman from the Middle East, is concerned about her lack of Swedish language skills. These became central when neither her mathematical or didactical knowledge were taken into account by her supervisor who had told her she would not succeed due to language difficulties. In an interview after the first period of teaching practice she referred to an interaction between herself and the supervisor:

Theresa: One bad thing happened, eh... it was about dates [she planned a lesson focusing time and date]. I showed her one way and said "It was this way." She said to me "It couldn't be. That's not correct". I said to her "Yes, it is correct". She said "No". I said "Please, look in the book". She said "Aha!"... If you can find it in the [math]book... She believed in the book more than in my words. I think it is because of my language. Do you see what I mean? They... they don't trust me. I don't have the language skills; I don't know how to say... I don't think that is good.

Through personal positioning within the mathematics teaching discourse, Theresa initiated a storyline about herself as proficient in planning mathematics teaching. As the supervisor did not agree, Theresa eventually showed how the concepts were demonstrated in the book, which the supervisor then accepted. Theresa then changed storyline towards explaining why this happened "I think it is because of my language". This intergroup positioning within the working language discourse is directed to people or institutions she does not specify. She also changes her personal positioning from being engaged in mathematics education to addressing her specific language problems "They... they don't trust me. I don't have the language skills". However, Theresa's positioning within the mathematics education discourse shows empowerment in relation to her subject knowledge. But on the other hand her subject positioning expresses disempowerment since it became obvious that her language difficulties obscured her proficiency in mathematics.

Rita is a young woman from south East Asia who moved to Sweden six years ago. She has experienced mathematics teaching in Swedish schools during several practice periods. Rita is concerned about how differently mathematics teaching is conducted and the different attitudes to mathematics compared with her home country:

Rita: Well, in my country it was different compared to Sweden. There, one should read books and then read... learn different rules by heart without understanding anything. And I was quite good at managing such things. And I always got good grades, even though... I lack in... thinking.

Through personal positioning within the mathematics teaching discourse at the university, in which teaching based on pupils' understanding is emphasized, she initiated a storyline about differences in teaching practices and about herself as able to manage learning "different rules by heart without understanding anything". Through her positioning Rita shows awareness of the differences and even if it seems like she distances herself from the way she was taught, Rita shows empowerment in the sense that she was able to manage and to get good grades. In the interview she made further comparisons between her two home countries:

Rita: Since, you know, the way mathematics is taught in my country, most pupils had problems in mathematics. And because of the way it was taught they lost all interest in learning, and maybe... I don't know what happened later, but my experience was that most pupils hated mathematics, in my country. They, who couldn't master the formulas or understand what it was all about [inaudible] lack of teaching, maybe,

I don't really get why most Swedes have problems in mathematics. Well, they have so good education and they are... The teachers, I think, are really good at explaining the rules. But anyway... they've got mathematics problems. So I don't know... what is the reason? Why is it so? If this have been in my country – and I am 100 % sure, well maybe 99 % of the pupils would get good grades, or like mathematics if they... was taught the way you do in Sweden.

Within the storyline about differences in teaching practices, Rita positions herself within what we have named a “we/you” discourse. For example by saying “the way mathematics is taught in my country” and “the way you do in Sweden” her personal positioning as immigrant is empowering and makes possible for her to ask “why most Swedes have problems in mathematics” despite good teaching. Through intergroup positioning within the mathematics teaching discourse Rita also discuss the consequences mathematics teaching has in her first home country. The storyline focusing on differences in mathematics teaching and attitudes to mathematics developed towards questioning why Swedes are not successful in mathematics when the education is so good. Rita's personal positioning within both the “we/you” discourse and within the mathematics teaching discourse shows empowerment in the way she talks about having experiences from different cultures and contexts.

## CONCLUDING REMARKS

We have here shown how immigrant student teachers' subjects positioning reveal power relations in discourses, available in mathematics teacher education. First and foremost we want to address the dominating language discourse, within which enacted personal and intergroup positioning shows disempowerment. What is striking is how the language discourse sometimes impede subject positioning within discourses of mathematics whereas the opposite was not found. This implicates that for immigrant students, language issues become more important than mathematics. We argue that language issues should be taken into account, both within teacher education and during teaching practice. But we need to consider how to develop teacher education to meet these needs without omitting the core of mathematics teaching.

However, it is not necessarily disempowering being immigrant student and we argue that awareness of this is essential in mathematics teacher education. Although we have shown that subject positioning within discourses of language and mathematics education reveal power relations, and that the working discourse of language is disempowering for immigrant students, we have also shown that subject positioning can challenge discourses through changed storylines, as in the example with Rita. Gutiérrez (2010, p. 3) wrote that “it is from the views of subordinated individuals and communities that we will learn how to rethink mathematics education” We agree, and realise that if we take these students concerns seriously, and open up possibilities for other discourses, probably all students will benefit.

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# YOUNG CHILDREN'S ADDITIVE AND MULTIPLICATIVE REASONING

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*This paper describes a study on young children's (4-6-years-old) additive and multiplicative reasoning. It tries to have an insight on kindergarten children's understanding of additive and multiplicative structure problems. The study addresses three questions: (1) How do young children understand direct problems of additive structure and of multiplicative structure?; (2) How do they understand inverse problems of additive and of multiplicative structure?; and (3) What type of arguments they present when solving these problems? Results suggest that kindergarten children can solve some additive and multiplicative structure problems with understanding. Inverse problems are more difficult for them than the direct ones, in both additive and multiplicative structure problems.*

## FRAMEWORK

This paper focuses on young children's understanding of additive and multiplicative structure problems. Concerning additive structure problems, Carpenter and Moser (1984) conducted a research on primary school children to analyse their solution strategies according to the type of problem presented. The authors argue that the processes that children use to solve addition and subtraction problems are intrinsically related to the structure of the problem. This idea that addition and subtraction word problems differ both in semantic relations used to describe a particular problem situation and in the identity of the quantity that is left unknown is also supported by other researchers (see Carpenter & Moser, 1982; Riley, Greeno & Heller, 1983) who argue that addition and subtraction problem types are related to fairly systematic differences in children's performance at various grade levels.

According to Nunes et al. (2005), children's ability to solve problems involving an additive structure develops in three phases: first children can solve simple problems; then they can solve the inverse problems; and finally they can solve static problems. The addition and subtractions simple problems are those in which children are asked to transform one quantity by adding to it or subtracting from it (e.g., Joe had 5 marbles. Then he gave 3 to Tom. How many marbles does he have now?). These types of problems involve relations between the whole and its parts. The inverse problems are those in which the situation presented in the problem relates to a schema, but the correct resolution demands the inverse schema. For example, in the problem "Joe had some marbles. Then he won 2 more marbles in a game. Now Joe has 6 marbles. How many marbles did Joe have in the beginning?" (Nunes & Bryant, 2009), subtraction appears as the inverse of addition; the quantity increased and the final one are given, and the initial quantity is unknown. The addition and subtraction static problems are those in which children are asked to quantify comparisons. For example, "Joe has 8



marbles and Tom has 5. Who has more marbles? (an easy question) How many more marbles does Joe have than Tom?” (a difficult question) (Nunes & Bryant, 1997; Nunes et al., 2005).

For Nunes and Bryant (1997) the difficulty of the problem is determined not only by the situation but also by the invariants of addition and subtraction that have to be understood by the children in order to solve a particular problem, and these invariants change according to the unknown parts of the problem. Nunes and Bryant (1997) also point out that the success in addition and subtraction tasks for young children is also determined by the resources that children are using to implement computational procedures, the system of signs. For the authors problems that involve relations are more difficult than those that involve quantities. The literature about additive reasoning has been giving evidence that compare problems, which involve relations between quantities, are more difficult than those that involve combining sets or transformations. Carpenter and Moser (1984) refer that many children do not seem to know what to do when asked to solve a compare problem.

Nunes et al. (2005) conducted a research with primary school Brazilian children, from grades 1 to 4, to analyse their performance when solving problems of additive reasoning. Their results indicate levels of success above 70% for the children of all grades when solving simple problems of part-whole relations involving addition and subtraction; 60% of the first graders and more than 80% of the 4<sup>th</sup>-graders succeeded in inverse problems. These results support the idea that the development of children's additive reasoning is progressive, but also suggest that children are able to solve many of these problems before they receive any formal instruction on addition and subtraction.

Literature gives evidence that kindergarten children are able to solve some addition and subtraction problems (see Fuson, 1992; Nunes & Bryant, 1997), but that does not mean that they understand all the relations in the context of additive reasoning problems. The children's understanding of addition a subtraction is progressive and develops over a long period of time. To understand more about children's additive reasoning, it becomes relevant to analyze younger children's ideas of addition and subtraction.

Vergnaud (1983) and Nunes and Bryant (2009) support the idea that additive and multiplicative reasoning have different origins. For Nunes and Bryant (2009) “Additive reasoning stems from the actions of joining, separating and placing sets in one-to-one correspondence. Multiplicative reasoning stems from the action of putting two variables in one-to-many correspondence (one-to-one is just a particular case), an action that keeps the ratio between the variables constant.” (p.11).

Multiplicative reasoning involves two (or more) variables in a fixed ratio. Thus, problems such as: “Joe bought 5 sweets. Each sweet costs 3p. How much did he spent?” Or “Joe bought some sweets; each sweet costs 3p. He spent 30p. How many sweets did he buy?” are examples of problems involving multiplicative reasoning. The former can be solved by a multiplication to determine the unknown total cost; the later

would be solved by means of a division to determine an unknown quantity, the number of sweets (Nunes & Bryant, 2009). Research has been giving evidence that children can solve multiplication and division problems of these kinds even before receiving formal instruction about multiplication and division in school.

Frydman and Bryant (1988) showed that 4- and 5-years-old children could share fairly a quantity based on sharing, relying on a strategy based on “one for A, one for B, one for A, ...”, which indicates that children can understand the sharing process based on one-to-one correspondence, and this understanding can be the foundation for the arithmetical operations (Bryant, 1997). Also Becker (1993) analyzed 4-5-years-old children solving multiplicative structure problems based on one-to-many correspondence. The problems involved 2:1 and 3:1 correspondence. Results showed that children performed better for 2:1 correspondence than for 3:1, and 5-years-old children succeeded in 81% of the problems. Carpenter and colleagues (1993) reported 71% of success when observing kindergarten children solving problems involving 4:1 correspondence.

Nunes et al. (2005) analysed primary Brazilian school children performance when solving multiplicative reasoning problems. When children were shown a picture with 4 houses and then were asked to solve the problem: “In each house are living 3 puppies. How many puppies are living in the 4 houses altogether?”, 60% of the 1<sup>st</sup>-graders and above 80% of the children of the other grades succeeded. When children were asked to solve a division problem, such as: “There are 27 sweets to share among three children. The children want to get all the same amount of sweets. How many sweets will each one get?”, the levels of success for 1<sup>st</sup>-graders was 80% and above that for the other graders (2<sup>nd</sup> to 4<sup>th</sup>-graders). Kornilaki, refereed by Nunes et al. (2005) analysed 5- to 8-years-old children performance when solving multiplicative reasoning problems, presented to them using only pictures. She presented multiplication and division problems of two types, direct and inverse problems. In the direct problems children can reach the solution using directly correspondence and distribution to solve multiplication and division problems, respectively. In the inverse problems this cannot be done immediately. In an inverse multiplication problem such as “It’s Charles birthday. Each friend that is coming to his party will get 3 balloons. He bought 18 balloons. How many friends are there in the party?”. Kornilaki’s results showed that 30% of the 5-years-old and 50% of the 6-years-old children succeeded in this problem. In the inverse division problem “It’s Ana’s birthday and she is going to share cookies among her friends. She prepared small bags with 3 cookies each to share between her friends. She used 18 cookies to prepare the bags. How many bags did she make?”, 40% of the 5-years-old and almost 68% of the 6-years-old children succeeded. Again, literature is giving evidence that children possess some informal knowledge of multiplication and division.

This paper investigates whether the additive and multiplicative structure problems influences children’s performance in solving direct and inverse problems. The study was carried out with kindergarten children who had not been taught about additive and multiplicative structure problems.

Following the work of Fuson (1992), Nunes and Bryant (1997) and Nunes et al. (2005), it is hypothesised that children's performance when solving direct problems involving additive structure will be better than when inverse problems are involved; and that when additive structure problems are easier for children to solve than the multiplicative structure ones, as in the former children rely on one-to-one correspondence procedures and in the last they rely on correspondence of one-to-many. Although some research has dealt with additive structure problems (Carpenter & Moser, 1982, 1984; Fuson, 1992; Nunes & Bryant, 1997) and with multiplicative structure problems (Carpenter *et al.*, 1993; Nunes et al., 2005) separately, there have been no comparisons between the two structure problems in research on young children. This paper compares children's performance when solving additive and multiplicative structure problems using the right controls. Research is giving evidence that children can solve multiplicative reasoning problems before being taught in school about it and before achieving all the additive reasoning development. Nevertheless, there is more to be known about the way 4-6-years-old children master the different types of additive and multiplicative reasoning problems. How do young children understand direct problems of additive and of multiplicative structure? How do they understand inverse problems of additive and of multiplicative structure? What type of arguments they present when solving these problems?

## METHODS

### Participants

Individual interviews were conducted to ninety 4-6-years-old kindergarten children, from Viseu, Portugal. These children were distributed into two groups: one group (n=45) solved additive structure problems, the other group (n=45) solved multiplicative structure problems. In each group there were 15 children of 4-, 5- and 6-years-old, respectively.

### The tasks

Each interview comprised 8 problems (4 direct and 4 inverse problems). The problems were an adaptation of those previously documented in the literature by Nunes et al. (2005). Tables 1 and 2 give some examples of additive and multiplicative structure problems, respectively, presented to children.

Type of Problem	Example
Direct	Kate's mum gave her 4 pencils. Later she gave her 2 more. How many pencils does Kate have now?
Inverse	Paul had 5 candies. He ate some and now he has 2 candies. How many candies did Paul eat in the beginning?

Table 1: Problems of additive structure presented to the children.

Type of problem	Example
Direct	In this street there are 3 houses. In each house are living 2 rabbits. How many rabbits are living in the houses altogether?
Inverse	It's Bill's birthday. He is going to offer 3 balloons to each friend in his party. He bought all these balloons to offer (Showing a bowl with 15 balloons). How many friends are in the party?

Table 2: Problems of multiplicative structure presented to the children.

The problems used in this study were presented by the means of a story and material was available to represent the problems. Each interview lasted approximately 15 minutes. After each resolution, the child was asked “Why do you think so?” in order to reach a better understanding of his/her reasoning. Material to represent the problem situation was available to the children. Data were collected using video recording and field notes.

### Design

The children were randomly assigned to either the group of additive structure or group of multiplicative structure problems. The direct and inverse items were randomly ordered and presented to each child in the interview session. This sequence was the same for all the children. No feedback was given for any of the test items. The numerical values were controlled for across type of structure problems.

## RESULTS

A descriptive analysis of children's performance when solving additive and multiplicative reasoning problems was conducted. Table 3 summarizes this information for each type of problem, when solving problems.

Mean (s.d.)	Additive structure		Multiplicative structure	
	Type of Problem		Type of Problem	
	Direct	Inverse	Direct	Inverse
6 yrs (n=15)	3,53 (0,83)	2,53 (1,25)	2.87 (0.99)	1.67 (1.29)
5 yrs (n=15)	3,75 (1,36)	1,80 (1,27)	2.47 (0.92)	1.33 (1.29)
4 yrs (n=15)	2,13 (1,25)	1,47 (1,30)	1.53 (1.36)	1.00 (0.93)

Table 3: Mean (standard deviation) of children's correct responses of additive and multiplicative structure problems, by age.

A three-way mixed-model ANOVA was conducted to analyse the effects of age (4- to 6-years-old) and type of problem structure (additive vs multiplicative) as between-subjects factor, and tasks (direct vs inverse) as within-subjects factor. There

was a significant main effect of type of problem,  $F(1,84) = 51.67$ ,  $p < .001$ , indicating that children's performance when solving additive structure problems is significantly better than when solving multiplicative structure problems. There was a significant task effect,  $F(1,84) = 8.08$ ,  $p < .05$ , indicating that children were significantly better solving direct problems than solving inverse problems. There was also an age main effect,  $F(2,84) = 9.36$ ,  $p < .001$ , indicating that children of 5-6-years-old performed significantly better than children of 4-years-old. There were no other significant effects.

An analysis of children's arguments was conducted among those who solved the problems correctly, in order to have an insight of their reasoning when solving the tasks. These arguments relied on children's verbal explanations when asked "Why do you think so?" after solving each problem. Four categories were considered in both additive and multiplicative structure problems: valid arguments (V), comprises the explanations that attend correctly to all the quantities involved in the problem (e.g. in an additive problem, "4 plus 2, it's 6" explains using his/her fingers "four...1, 2, 3, 4 plus two, is 5, 6"); partially valid arguments (PV), comprises explanations in which a child attends only to a part of the problem producing an incomplete argument (e.g., in a multiplicative problem, "There are 9 carrot cakes. The rabbits are 3, thus there are 3 carrot cakes"); no argument (NA), comprises the absence of arguments; and invalid arguments (I), comprises the explanation that were not understood. Table 4 resumes the type of argument given by the children when solving additive and multiplicative reasoning tasks correctly, according to the age. Many children succeeded in their explanations presenting valid arguments for their correct resolutions, revealing a clear understanding of the problems. In many cases the partial valid arguments were presented using material, representing the situation correctly, in spite of the difficulty in the verbal communication. To present an explanation is not a simple task as many children of all age groups presented invalid explanations, in spite of solving the problems correctly.

		Children's arguments (%)							
		Additive structure				Multiplicative structure			
Problem		V	PV	I	NA	V	PV	I	NA
Direct	6 yrs (n=15)	71,7	0	7,5	20,8	65.1	7	18.6	9.3
	5 yrs (n=15)	42,2	20	17,8	20	46	5.4	24.3	24.3
	4 yrs (n=15)	53,1	3,1	9,4	34,4	47.8	8.7	34.8	8.7
Inverse	6 yrs (n=15)	73,7	2,6	18,4	5,3	64	12	16	8
	5 yrs (n=15)	63	7,4	7,4	22,2	45	0	35	20
	4 yrs (n=15)	54,5	0	31,8	13,6	40	20	40	0

Table 4: Percentage of type of children's argument when solving additive and multiplicative structure problems, by age.

## DISCUSSION AND CONCLUSIONS

Kindergarten children can solve additive and multiplicative structure problems relying on their informal knowledge. This idea is also supported by previous research (Becker, 1993; Carpenter *et al*, 1993; Kouba, 1989; Nunes, Bryant & Watson, 2009). Direct problems are easier for the children than inverse problems, and this idea is also supported by Nunes *et al.* (2005) regarding additive structure problems. But this study gives evidence that this can be extended to the multiplicative reasoning problems. The inverse multiplicative structure problems are more difficult for the children than the inverse additive structure; in agreement with Greer (2012), the inverse relation from division to multiplication is more complex than the inverse relation from addition to subtraction.

The differences in children's performance according to the age suggest that not only addition and subtraction is progressive and develops over a long period of time (Fuson, 1992; Nunes & Bryant, 1996), but also multiplication and division.

The results suggest that additive structure problems are easier for children than the multiplicative structure ones. The scheme of actions involved in the additive and multiplicative reasoning problems are different, but children do not need to achieve all the development of the additive reasoning before achieving the multiplicative one. This study reveals that some children of 4-years-old can solve both additive and multiplicative reasoning problems, suggesting that both reasoning develops at kindergarten level.

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# STUDENT TEACHER NOTICING DURING MATHEMATICS INSTRUCTION

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*Teacher noticing is central to student-centered instruction. Although teacher education programs have increasingly incorporated activities to support teacher noticing, and these efforts have been found to be effective in the short term, little is known about how they influence what teachers notice during the act of teaching. This paper reports on the results of a study in which six student teachers were asked to document their in-the-moment noticing during mathematics instruction using a self-mounted camera. The data revealed a strong student focus to the participants' noticing, with some focus on the details of student thinking. Barriers to noticing were also identified. Implications for teacher education programs are discussed.*

## INTRODUCTION AND BACKGROUND

The value of student-centered mathematics instruction is widely acknowledged within the U.S. mathematics education community (e.g., Ball & Cohen, 1999; National Council of Teachers of Mathematics, 2000). Such instruction requires that teachers attend to and make sense of student ideas that surface during instruction and consider how these ideas can be used to advance students' mathematical learning. Because the enactment of this type of instruction has been found to be difficult for teachers (e.g., Berliner, 2001; Davies & Walker, 2005; Peterson & Leatham, 2009; Sherin, 2002), many teacher education programs have incorporated activities intended to develop the dispositions and skills necessary for such instruction (e.g. Stockero, 2008; Leatham & Peterson, 2010; Santagata & Guarino, 2011).

Efforts to develop these skills and dispositions in methods courses have been found to be effective in the short term—that is, at the conclusion of a teacher education course. Stockero (2008), for example, found that using video cases in a mathematics methods course developed a focus on making sense of student thinking and cultivated an evidence-based approach to analyzing practice. Similarly, Santagata and Guarino (2011) found that video-based activities helped prospective teachers attend to the details of student thinking and teacher moves that help make student thinking visible, and to consider how teacher moves impact student learning. What is not known, however, is what teachers who have been through coursework focused on student thinking attend to in the act of teaching.

Such research is situated within a growing body of work around *teacher noticing* (e.g., Sherin, Jacobs, & Philipp, 2011)—what teachers do or do not attend to during instruction. In general, noticing is a key component of expertise in a range of professions (National Research Council, 2000), and teacher noticing is considered foundational to responsive instruction, since teachers cannot respond to that which they do not notice (e.g., Schoenfeld, 2011). Studies over the last decade (e.g., Sherin &



van Es, 2005; Jacobs, Lamb, & Philipp, 2010) have documented that noticing is a skill that can be learned, both by preservice and inservice teachers.

This research builds on prior work on teacher noticing by documenting what novice teachers who have completed a teacher education program that emphasized attention to student thinking notice during the act of teaching. Although others have documented what teachers deem important in their own instruction via post-lesson reflection (e.g., Peterson & Leatham, 2009) or analysis of video recordings of lessons (Santagata & Guarino, 2011; Sherin & van Es, 2005), the act of noticing is quite different when done in-the-moment versus when there is more time to reflect on events. Better understanding novice teachers' in-the-moment noticing can provide a foundation for thinking about the extent to which methods course activities cultivate the nuanced real-time noticing that is necessary to implement instruction that builds on student thinking in meaningful ways.

## **THEORETICAL PERSPECTIVES**

The work is grounded in a particular vision of teaching—one in which teachers continuously build on student thinking in ways that are responsive to their current understanding (e.g., NCTM, 2000). This involves the teacher carefully listening to students' ideas, analyzing the mathematics underlying them, and then making in-the-moment decisions about whether and how these ideas can be used to develop students' understanding of important mathematical ideas. The construct of *teacher noticing* (e.g., Sherin et. al, 2011) is generally defined to include these three components and thus, is considered to be foundational to student-centered instruction.

Although instances of student thinking that occur during instruction are central to noticing, this work is also grounded in the perspective that not all instances of student thinking are equally important to notice in terms of their potential to help achieve the goal of supporting students' mathematical learning. Some instances, for example, may be important for affective reasons, such as developing student confidence, while others may be important for pedagogical reasons, such as cultivating norms for working in small groups. This research project aims to promote *mathematical noticing*—noticing of important mathematical student ideas that surface during instruction and have the potential to support students' understanding of important mathematical ideas (e.g., Leatham, Stockero, Peterson, & Van Zoest, 2011).

## **CONTEXT AND METHODOLOGY**

The study participants were six prospective mathematics teachers completing a semester-long student teaching experience during the final semester of a secondary-school level teacher education program. They were members of the comparison group in a larger study focused on developing prospective teachers' abilities to notice and capitalize on mathematically important moments that occur during instruction. As such, they had enrolled in courses that were a normal part of the teacher education program, but did not participate in any additional experiences that might further develop their noticing skills. In short, the program focused on developing student-centered instruction grounded in inquiry and sense-making. The participants

had all completed a mathematics methods course with a strong focus on listening to, making sense of, and considering how to use student thinking during instruction. Course activities included studying Smith and Stein's (2011) *5 Practices for Orchestrating Productive Mathematics Discussions*, analyzing student thinking in written and video cases of instruction, and working with small groups of secondary school students on tasks to elicit and build on their thinking to support their mathematical understanding (see Van Zoest & Stockero, 2008 for a description of a course after which this was modeled).

Each participant was observed and video-recorded teaching a mathematics lesson three times during their 15-week student teaching experience. During each observation, the participant was asked to document important instances that they noticed while teaching by wearing a self-mounted camera that allowed them to capture a 30-second segment of video when they felt an important instance occurred. The definition of what might constitute an important instance was left open-ended to allow the researcher to understand what participants viewed as important during their teaching. Following each lesson, the participant engaged in an interview focused on discussing self-documented instances, undocumented instances that were deemed to be mathematically important by the researcher/observer, and the participants' general ideas about what might be important to notice during a lesson. At times during the interviews (two or three times per participant), the participants discussed instances they noticed as important, but had failed to document because they forgot to activate the camera at the right moment. In these cases, the instance was included in the analysis only if the participant discussed a specific classroom instance in detail (i.e., where the noticed instance was evident). If they made a general statement about what they meant to document (e.g., "students were becoming frustrated during the lesson"), no related instance was included in the analysis.

Data included the lesson video recordings, the participants' documentation of important instances they had noticed, and video recorded post-lesson interviews. A research team that included the author, a graduate student and an undergraduate student researcher analyzed the data. Post interview videos were segmented into conversations about instances that were discussed and general ideas about noticing. Each discussion of a noticed instance was coded for agent (Who was the focus of the noticing?), topic (What was the noticing centered on?) and the level of specificity of the mathematics discussed (How grounded was the noticing in mathematical ideas?). Some instances were coded for primary and secondary topics if the participant discussed two key ideas (e.g., a teacher move and its effect on student thinking). Discussions of undocumented instances were coded for participant-identified barriers to noticing (What prevented them from noticing a particular moment?). Numerical coding summaries were compiled across both participants and coding categories; written participant summaries were compiled across all three observations. The written and numerical summaries were analyzed across participants to characterize their individual and collective real-time noticing during instruction.

## RESULTS

In sum, the participants identified 80 important instances during their teaching, ranging from 6 (Eric) to 20 (Rich) instances per participant (Table 1). There was an average of 4.4 identified instances per lesson.

Table 1 summarizes the agent coding by participant. In the table, *Teacher/Student* and *Student/Teacher* both indicate a focus on interactions between the two, with the first agent being the primary noticing focus. This data indicates a strong student focus to the participants' noticing, with 87.5% of all instances including some focus on students and 67.5% of instances having a primary focus on students. In contrast, 53.75% of instances included some focus on the teacher, with 32.5% having a primary teacher focus. About 46% percent of all instances focused only on students as the agent, either individually (38.75%) or as a group (7.5%). This agent focus varied by participant, however, with Les and Rich always demonstrating some focus on the student and Eric always having some focus on himself as the teacher. Although the previously-documented novice teacher tendency to focus on self (e.g., Berliner, 2001) was present to some degree, the participants' noticing was generally reflective of the program's student-centered focus.

	Les	Eric	Rich	Ally	Adam	Heidi	Total
Teacher [T]	0	2	0	3	4	1	10 (12.5%)
Teacher/Student [T/S]	0	1	5	2	5	3	16 (20%)
Student/Teacher [S/T]	0	3	4	1	6	3	17 (21.25%)
Student (individual) [S(Ind)]	8	0	11	4	1	7	31(38.75%)
Students (group) [S(Grp)]	4	0	0	2	0	0	6 (7.5%)
Total identified instances	12	6	20	12	16	14	80

Table 1: Agent coding by participant.

In terms of specificity, the participants' noticing was primarily centered on specific mathematical ideas (58 instances, 72.5% of total), rather than on general mathematical ideas or non-mathematical concerns (11 instances, 13.75% of total for each). For example, "*The student answered incorrectly*" is general in nature, while "*The student asked a question about the order [of the points] in finding slopes*" is specific. The specificity was relatively consistent across agents, and for all of the participants except Heidi, for whom only 43% of instances were related to specific mathematics. Overall, the findings indicate that the participants were primarily attending to the teaching and learning of the mathematics of the lesson, rather than to affective issues or making broad claims about student understanding.

The topic coding was more difficult to draw conclusions from due to the larger number of topics identified and the small number of occurrences of some topics; recall also that some instances were coded for both primary and secondary topics, which resulted in 112 topic codes for the 80 instances. The most prevalent topics of noticing (those with at least five documented instances) are shown in Table 2.

	Video 1		Video 2		Video 3		Total	
	Pri	Sec	Pri	Sec	Pri	Sec	Pri	Sec
Teacher move	5	2	6	4	7	6	18	12
Student thinking	3	2	6	2	7	0	16	4
Generalize/connections	2	2	2	1	3	0	7	3
Affective	1	0	2	2	3	0	6	2
Correct answer	2	1	3	1	2	0	7	2
Student interactions	0	0	4	0	3	0	7	0
Question/confusion	1	0	1	2	3	0	5	2
Math issue	0	0	1	0	4	0	5	0

Table 2: Primary and secondary topic coding by video.

The most prevalent noticing topic was teacher moves, coded as a primary (18) or secondary (12) focus in 30 of 80 instances. This was a consistent focus throughout the student teachers' experience (Table 3), with the most instances (10) documented by Adam and no instances documented by Les. Considering what participants were attending to as they considered teacher moves provides further insight into their noticing. It was found that only eight of the instances, documented by Adam (4), Ally (3) and Eric (1), were focused on the teacher alone; the remaining instances were focused on how teacher actions affected student learning, or how student comments/ideas influenced what the teacher did during the lesson (Table 3).

Student thinking was also a prevalent focus, coded in 20 of 80 instances (16 primary, 4 secondary). Twelve of the instances (15%) were focused on making sense of an individual student's thinking (Table 3), a major focus of the mathematics methods course. Six instances were focused on how what the teacher did may have affected student thinking or how the teacher responded to student thinking, and only two were focused on making claims about the thinking of the class as a whole.

	T	T/S	S/T	S (Ind)	S (Grp)
Teacher move	8	14	8	0	0
Student thinking	0	2	4	12	2
Generalize/connections	0	0	1	7	2
Affective	0	2	3	3	0
Correct answer	0	1	2	6	0
Student interactions	0	1	4	2	0
Question/confusion	0	0	4	3	2
Math issue	1	0	1	1	0

Table 3: Topic coding by agent.

Because attention to individual students was a goal of the program, it is also interesting to note what participants attended to other than student thinking when focused on individual students. Over a quarter of instances with individual students as the agent focused on whether students were getting correct answers or seemed confused, while another 15% were related to non-mathematical issues, such as affective concerns and how students interacted with one another. This indicates that nuanced noticing of student ideas was not the norm, and is something that may require further work.

To understand barriers to mathematical noticing, two types of instances were analyzed: those that a participant failed to notice and those that they attended to but in which they failed to notice the mathematical importance of what a student said. The most common barrier to noticing was unanticipated student responses—those the teacher had not previously thought about. This barrier was documented in five instances in four participants' lessons. When an unanticipated response was given, the teacher typically acknowledged the response, but did not capitalize on it. This particular barrier seems to be related both to mathematical knowledge and to careful listening.

Participants' preconceived ideas about what they should be noticing was also a barrier, as it often resulted in a failure to notice events that were beyond their focus. This seemed to be a particularly important factor for Ally, Eric, and Les, and could be an explanation for the fewer number of topics coded in their noticing; this will be further examined in subsequent analyses. Other documented barriers included perceived time constraints that caused participants to rush through portions of the lesson, assumptions of student understanding that caused them to dismiss potential misconceptions, affective concerns about how much to push students, and a concern that student ideas would take the lesson in an unproductive direction.

## **CONCLUSION**

The results suggest that a strong focus in teacher education coursework on student-centered instruction and, in particular, on building on student ideas, has the potential to affect what novice teachers attend to during instruction. The data showed a primary focus on students in two-thirds of all instances. In addition, evidence documented in 15% of instances that participants were beginning to attend to individual student ideas—a key component of student-centered instruction—is encouraging. The participants' most frequent focus was on the teacher, but much of this focus was on the interaction between their moves and student learning, suggesting an attempt to implement instruction that was responsive to students.

Although the strong student focus was encouraging, fine-grained analysis indicates more work must be done, as nearly half of instances with individual students as the agent were focused on correct answers, general evidence of confusion, and non-mathematical concerns. These findings, along with the identified unanticipated response barrier, indicate a need to help prospective teachers attend to the nuances of student ideas and what they might indicate about their mathematical understanding.

In general, the findings provide a foundation for thinking about the extent to which methods course activities cultivate the type of noticing necessary to implement instruction that builds on student thinking in meaningful ways. Methodologically, it was found that novice teachers are able to document their own noticing during instruction; thus, the techniques can be adapted to other work focused on understanding novice teacher attention. Future work will focus on studying how targeted noticing activities in teacher education affect noticing during instruction.

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# EMBODYING THE CONVERGENCE OF THE RIEMANN ACCUMULATION FUNCTION IN A TECHNOLOGY ENVIRONMENT

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*This case study is one in a series of studies regarding the learning of the integral concept. Its focus is on the processes involved in the convergence of Riemann accumulation function (RAF) to the accumulation function (AF). The study is guided by the objectification theory which considers learning to be a process of becoming aware of the knowledge which exists in the culture. Through a task performed with a designed technological tool, the students were asked to suggest ways to make the Riemann accumulation graph and the accumulation graph converge. A semiotic analysis of the learning process of a pair of students serves as an example of evolutionary processes of the personal meaning toward the cultural mathematical meaning consist of two successive processes: a) attention to the differences between the RAF graph and the AF graph, and b) the actions performed by the student to make the two graphs converge.*

## INTRODUCTION

This case study is one in a series of studies regarding the learning of the integral concept and the fundamental theorem of calculus when the integral concept is constructed based on the idea of accumulation among high school students. The aim of the research project is to analyze the processes involved in conceptualizing the Riemann accumulation function (RAF), the accumulation function (AF), and their usages through applying the fundamental theory of calculus (FTC) when they are learnt in an environment which places more emphasis on the graphical and numerical aspects of the integral concept and less on the use of algebraic notations. Our research project is situated in what Tall (2010) called the sensible approach to calculus, in which we develop the ability to formulate sophisticated ideas through our perceptions and the use of language.

To improve the cognitive basis of learning the integral concept, it has been proposed that the idea of the integral be constructed based on the idea of accumulation (Thompson & Silverman, 2008). The usefulness of accumulation in conceptualizing the idea of the integral does not make it easy to learn. On the contrary, Thompson and Silverman (2008) note two points of possible difficulty:

First, students find it is hard to think of something accumulating when they cannot conceptualize the “bits” that accumulate. ... Second, the mathematical idea of an accumulation function represented as  $\int_a^x f(u)du$ , involves so many moving parts that it is understandable that students have difficulty understanding and employing it. (Thompson & Silverman, 2008, p. 43)



We previously analyzed the processes involved in making sense of the RAF graph in a dynamic and multi-semiotic technological environment (Swidan & Yerushalmy, 2009; Yerushalmy & Swidan, 2012). It is dynamic in that the students have the ability to change any variable mentioned in the accumulation function above. It is multi-semiotic in that its interface includes graphical and numerical signs referring to function, products, and the sum of products (accumulation). In those studies, we identified the foci involved in making sense of the (RAF) graph while the student changed any variable but kept  $\Delta x$  fixed. The result evidenced that keeping  $\Delta x$  fixed focused the student's attention on the "bits" that accumulated, the role of the varying variable (the lower and upper limits), and the role it played in the conceptualization of the accumulation function. We ask whether a dynamic and multi-semiotic environment (like that suggested in Yerushalmy & Swidan, 2012), where  $\Delta x$  and the upper limit vary while the rest of the quantities (the lower limit, the function variable), which are variables that are traditionally kept fixed, create the opportunities for learning the core idea of the definite integral as convergence of the RAF graph to the AF graph. The convergence of Riemann sums to a definite integral as a *number*, in technology environments, has been studied in previous works (e.g. Robutti, 2003). Constructing the idea of the integral based on the idea of accumulation creates a different reality concerning the convergence concept to be taken into account.

Little is known about the processes of learning about the convergence of a family of Riemann accumulation functions to the accumulation function when the learning is done using multi-semiotic and dynamic technological tools. Our case study intends to contribute to emerging interest in the learning the integral concept by identifying the processes involved in the convergence of RAF to AF. The aim of this study is to analyze the processes involved in the process of convergence of RAF to get the AF when this learning is done graphically and numerically in a dynamic and multi-semiotic environment.

## THEORETICAL FRAMEWORK

The theoretical framework which guided this study depends on the embodied view of cognition and semiotics mediation. Learning mathematics with technological tools allows the integration of embodied action with semiotic activities (Botzer & Yerushalmy, 2008). Embodiment is a cognitive movement that grants the body a central role in shaping the mind. This cognitive movement supports the idea that bodily activities may be involved in conceptualizing processes (Wilson, 2002). According to this view of cognition, the perception and the action of the students are considered essential in the learning process and concepts are not analyzed on the basis of "formal abstract models, totally unrelated to the life of the body, and of the brain regions governing the body's functioning in the world" (Gallese & Lakoff, 2005, p. 455). The analysis of the concept must consider the multi-modality of our cognitive performances (e.g. verbal language, gestures, interaction with artifacts, glance, sound ...) (Arzarello & Paola, 2007).

The relation between artifact and knowledge is expressed by signs, which are culturally determined. The relation between the artifact and the accomplishment of a task is expressed by signs such as gestures, speech, and drawing. Signs in general and mathematical signs play two roles. Radford, Bardini, Sabena, Diallo, and Simbagoye (2005) define these roles as "social objects in that they are bearers of culturally objective facts in the world that transcend the will of the individual. They are subjective products in that in using them, the individual expresses subjective and personal intentions" (2005, p. 117). Berger (2004), who studied the functional use of mathematical signs, suggests a two-fold interpretation of the meaning of signs and objects: personal meaning, "to refer to a state in which a learner believes/feels/thinks (tacitly or explicitly) that he has grasped the cultural meaning of an object (whether he has or has not)," and cultural meaning, "to the extent that its usage is congruent with its usage by the mathematical community" (2004, p. 83).

Learning in this setting means participating in an active process that leads to making sense of the elements and bringing about an encounter between personal and mathematical meanings. In other words, to learn something, the learner must attend to the existence of the knowledge within the culture and become aware of its existence Radford (2003). The processes of attention to and awareness of an existing mathematical object require engagement in a mathematical activity to grant meaning to the object. Radford (2003) called this process an objectification process. Objectification requires making use, in a creative way, of different semiotic tools such as words, symbols, and gestures that are available in the universe of the discourse. Semiotic tools play a central role in the objectification process. Using these tools, students make the transition from the embodied meaning associated with concrete objects to a disembodied meaning which is generally associated with abstract concepts (Radford. et al, 2005).

## THE DESIGN OF THE STUDY

The main study question is: How do the personal meanings of high school students that arise through their interaction with a dynamic and multi-semiotic environment evolve into the cultural meaning of the definite integral as a convergence of RAF to AF? To answer this question we analyzed approximately three hours of learning by Aroob and Nimat, one pair<sup>1</sup> of 17-year-old students in the math class taught by the first author. The experiment took place in school. The students volunteered to participate in five after-school meetings that aimed to teach the ideas of the integral concept. In this study we concentrated on the third meeting, which dealt with learning the AF concept as a convergence of the RAF. The students who participated in this study had prior knowledge of the concepts of function and derivative but were not familiar with the integral concept. They were familiar with the function graph software, which was part of their previous studies on functions and derivatives. The two students shared a single

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<sup>1</sup> For reasons of space, we decided to perform the micro-analysis of the learning process with one pair of students from the 10 students participating in the research project.

computer, and the author briefly introduced them to the interface and functionality of the three major tools of the artifact: graphing, accumulating, and controlling.

The students were video-recorded and their computer screens were captured. The video recording was achieved by software which captured the footage in two different windows: the computer screen and the student's body. The first author was present as an observer. Despite the continuous presence of the math teacher throughout the learning process with the artifact, we made it clear that we would attempt to minimize any intervention. The data were then analyzed according to Radford's categories of attention and awareness (2003).

## THE ARTIFACT USED IN THE EXPERIMENT

The artifact used in our study is the computer software *Calculus UnLimited* (CUL; Schwartz & Yerushalmy, 1996; Fig. 2). The software CUL consists of three main components: 1) the function component, which contains the symbolic expression which is entered by the user in the "Function List" box and the graph of the function when its symbolic expression is entered; 2) the accumulation component, which contains the rectangles, the graph in the lower Cartesian system, the value table, and the summation icons; 3) the value control component, which allows the user to control the upper and lower boundaries of the integral and the size of the rectangle as well. Each of these components may be considered as a sign signifying an abstract mathematical concept. So we are considering the components of CUL as signs that bear accepted meanings in mathematical culture, and all the signs together are called the semiotics system. The theoretical assumptions assert that through the interaction of the students with the artifact their personal meanings may evolve to reach the accepted cultural meaning, that is, the idea of the integral.


- ✓ To work on the given task, you will use the Integral tools of CUL. Your task is to come up with a conjecture and explanation about the mathematical relations between the upper and lower graphs and the values table.
- ✓ To create the upper graph, enter the function  $f(x) = x^2$ , set as a default the value -3 in both controlling boxes and set  $\Delta x = 0.5$ .
- ✓ To create graphs in the lower graph window, select the right rectangle and the integral representation icon.
- ✓ You are asked to increase, step by step, the parameter value in the control box by discretely pressing on the icon  up to reach get the value 4 in the control box.
- ✓ Following each increase try to explain the differences and the similarities between the two number columns in the table of values and both graphs in the lower windows.
- ✓ Your task will end when you are able to explain how to cause both graphs to converge, and to make the right rectangle and integral numbers equal in each row.

Fig. 1: The task was presented to the students

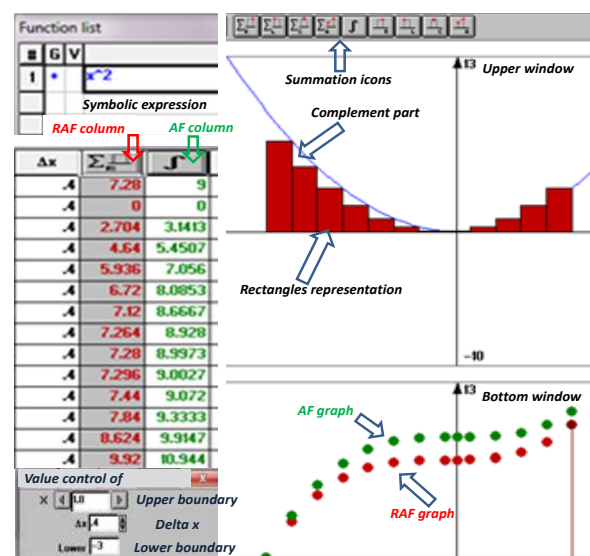


Fig. 2: CUL interface - the option of showing rectangles between the integral boundaries in the upper window

## DATA ANALYSIS

We used *attention* and *awareness*, Radford's categories of objectification of knowledge (2003), to analyze the processes of evolution of personal and mathematical meanings. We identified *attention* as a declaration about the existence of a mathematical relationship between objects in the semiotic system. In the present study, declarations about the existence of mathematical relationships tended to be based on visual considerations. Students' justifications and interpretations based on mathematical considerations of relationships that students had noticed were defined as *awareness*.

### The first interpretation: noticing the difference between the RAF and AF graphs

Through the second session the students become aware that each point in the lower Cartesian system graph represents the accumulated products of the rectangles (Yerushalmy & Swidan, 2012). On the computer screen appears a semiotic system which includes two graphs, the RAF graph and the AF graph.

- 1 Aroob: We concluded that the value of this point [pointing to the computer screen] is equal to the rectangle area [Fig. 3 appears on the screen].
- 2 Namat: This one [pointing to the lower point in the RAF graph and checking its value; Fig. 3]
- 3 Aroob: Yes ... It's the same here [pointing to the square with a value of 3.1 in the table of values]. The value of the green point is 3.7 [pressing the integral icons; Fig. 4 appears on the screen]. I have an idea of how to make these two values [the RAF and AF values] closer, for instance if we have a rectangle and add a triangle above of it. Therefore we can compute both areas.

The students notice the difference between the RAF and AF graphs when looking at the lower windows (which include two graphs) and the table of values (which includes two columns of numbers). Their utterances and gestures suggest [1, 2, 3] that they have signified the red points on the RAF graph and the red numbers of the RAF in the values table and endowed them with the meaning of an area of the shaded rectangle that they observe in the upper window. Aroob's utterance [3] together with the appearance of the whole bounded region between the function and the x-axis in the interval whose size is  $\Delta x$  (the green area) on the screen suggests that they connect the value numbers of AF and the points of the AF graph with the meaning of the area of the whole bounded region in the specific interval. The last utterance in [3] shows that the students consider the shaded region as a combination of the two regular geometric shapes – triangle and rectangle.

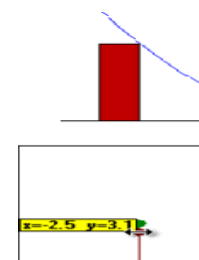


Fig. 3

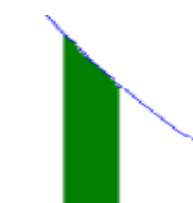


Fig. 4

## Second interpretation: the awareness of the difference between the RAF and AF graphs

- 4 Namat The red [pointing to the RAF column values; Fig 5] is the sum of the products [pointing to the RAF graph; Fig. 5] of the rectangles achieved [using the mouse, she draws rectangles on the bounded region]. The green [pointing to the AF numbers] is the sum of the products of the rectangles [drawing rectangles on the bounded region with the mouse] and the empty parts.

The students explain the differences between the two graphs in the lower system. Their utterances and gestures suggest a division of the objects appearing on the screen into two groups: the first group includes the RAF numbers, the points of the RAF graph, and the rectangles which do not appear on the screen but are presented to the students through the use of the gestures they make with the mouse. These three embodied signs have been signified by the students to achieve the mathematical meaning of the sum of products of the rectangle's dimensions. The second group includes the AF numbers, the points of the AF graph, and the whole bounded region. Namat's last utterance [4] and the drawing of the rectangles on the bounded shaded region with the mouse suggest that these signs signify discovering the meaning of the whole which is composed of its parts – the sum of products of the rectangle's dimensions and the complementary parts.

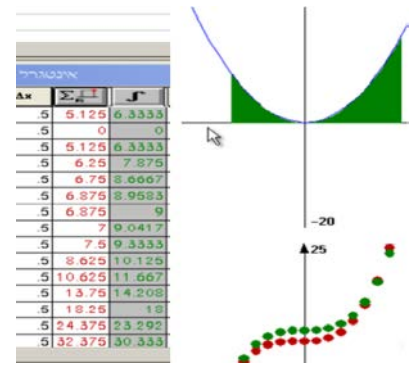


Fig. 5

## The third interpretation: awareness of the convergence of RAF to AF

- 5 Aroob: When do we get two overlapping graphs? When the empty parts are gone? When delta x [gestures to  $\Delta x$ ; Fig. 6] gets smaller and smaller.
- 6 Namat: When delta x becomes smaller [gestures to  $\Delta x$ ; Fig. 7].
- 7 Aroob: [Writing with a pencil on a piece of paper.] Take for example a function with a big delta x. See these empty parts which cause the difference between the green and the red points [the RAF and AF]. As delta x approaches zero, both graphs are now overlapping.
- 8 Nimat: As delta x approaches zero the red sum of the products [pointing to the RAF column] and the green one [pointing to the AF column] becomes equal.

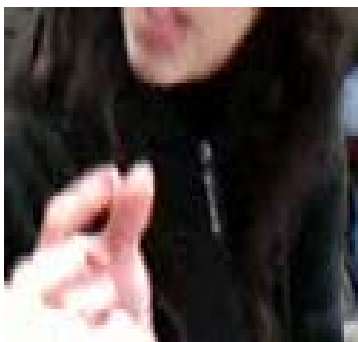


Fig. 6

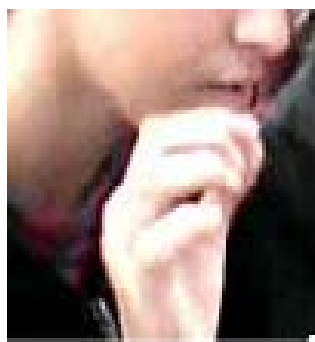


Fig. 7

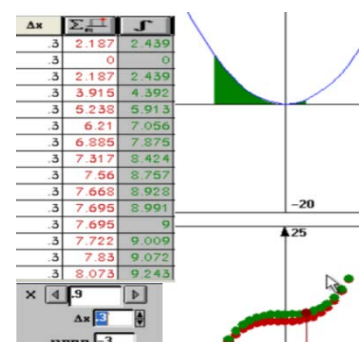


Fig. 8

In this excerpt, the students distinguish between the task they were asked to perform – to make the two graphs converge – and the operational method of performing this task. The students choose the complementary parts to act on it. Their action asked to eliminate these parts, which differentiate between the two graphs [7]. Their expressions “empty parts” and “delta x gets smaller and smaller” together with the appearance of the semiotic system on the screen with the whole shaded region (Fig. 8) suggest that the students are seeing the whole region composed of rectangles with a very small width. In addition, their utterances “delta x gets smaller and smaller” and “delta x becomes smaller” and the gestures they make with their fingers suggest that the students are seeing the shaded region as a result of a dynamic process of reducing the width of the rectangles. Furthermore, their utterances about reducing delta x in line [6] evolve to become compatible with the mathematical expression “delta x approaches zero” in line [7].

## CONCLUSIONS

The study reveals that the evolutionary process of the personal meaning toward the cultural mathematical meaning consists of two successive processes: a) attention to the differences between the RAF graphs and the AF graph, and b) the actions and techniques performed by the students to make the two graphs converge. The first process is characterized by considering the region under the function curve and its numeric value as a whole consisting of two parts – the rectangle part and the complementary part. The ability of the students to endow the numbers in the table of values with their mathematical meanings assists the students in attending to the complementary part, whose numeric value is the difference between ‘the whole’ and the parts. Actually, the presence of the complementary parts explains the differences between the RAF and AF graphs and the process of their elimination explains the process of convergence of the RAF toward the AF. Their conclusion concerning the elimination of the complementary parts emerges after they reduce the width of the rectangle using the software and identify that the graphs become closer than before. The act of reducing the rectangle’s width and the appearance of the RAF together with the AF in the same Cartesian system strengthens the students’ feeling that the convergence process is possible and the action leading to that is the reduction of the width of the rectangles. This interactive reduction of the width supports the students’ tendency to view the continuous shaded region (the whole) as consisting of rectangles whose width is undergoing a continuous process of reduction. For this reason, we conclude that the students were first aware of the difference between and similarity of the two graphs by attending to the visual and numerical linked representations and then, by attending to the variation of delta X, became aware that the continuous shaded region under the curve is no more and no less than the accumulative multiplicative quantities. This learning achievement is important for students who aim to conceive the cultural accepted meaning of the ‘area under a curve’ as area and more so as a quantity other than area (Thompson & Silverman, 2008). We end with two questions arising from this study and are of interest: What will change in the learning processes mentioned in this study when the students meet situations in the graph where the



rectangle's area is the whole while the area under the curve is a part? Another important issue is that this study places more emphasis on the aspects of learning the integral ideas graphically and numerically and less on the symbolic aspect. It will be necessary to investigate the role of graphic and numeric aspects in learning the idea of the integral symbolically among high school students and the influences of using these ideas when solving calculus problems.

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# PRE-SERVICE TEACHERS DESCRIBE GEOMETRICAL FIGURES: THE ‘BROKEN PHONE’ REVISITED

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*In this study we focus on students’ evaluations of verbal descriptions of two-dimensional composite geometrical figures. A quantitative study was conducted with pre-service primary school teachers (N=110) investigating their evaluations of different descriptions (topological, geometrical, everyday) of four composite objects (familiar and unfamiliar). In most cases and in contrast with our expectations the participants of this study appeared to prefer the geometrical or the topological descriptions of the provided figures, regardless of their familiarity or of the students’ attainment. The implications of these findings are discussed.*

## INTRODUCTION

Geometry is considered to pose difficulties to students at all ages, mostly due to the significant role of the visual representations involved. Although visual representations are important in most mathematical fields, in geometry they become the object of learning by themselves and the student is expected to interpret, manipulate and communicate spatial information (Gorgorió, 1998). The importance of spatial ability is acknowledged, even from the early years of schooling (Clements, 1999), with visual representations appearing in everyday situations and various scientific domains. Moreover, the aforementioned considerations are reflected in the mathematics curricula worldwide, according to which the students at all grades are expected to be able to analyse the properties of two- and three-dimensional geometric shapes and to use visualisation and spatial reasoning to solve problems (NCTM, 2000).

Consequently, a considerable amount of research has concentrated in, amongst others, recognising two-dimensional shapes by young children (Clements, 1999) and in analysing the visualisation processes in three-dimensional shapes of students (Gutiérrez, 1996) or teachers (Malara, 1998). Moreover, the students have been found to encounter difficulties either when asked to visually represent a three-dimensional object given a verbal description, or when asked to verbally describe a three-dimensional object given its visual representation (Ben-Chaim, Lappan & Houang, 1989; Mitchelmore, 1983; Parzysz, 1988). Following these, in previous studies (Tatsis, 2007; Tatsis & Goutsi, 2011) we observed the effect of coding and decoding visual and verbal information related to composite geometrical objects. In the present study, we focus on the students’ ability to interpret and evaluate verbal information related to two-dimensional geometrical objects, addressing the question: *Which are the relationships between pre-service primary school teachers’ evaluations of verbal descriptions of two-dimensional composite geometrical figures and the characteristics of these figures?*



## VISUAL AND VERBAL ASPECTS OF GEOMETRY

Mathematics is characterised by the different representational systems involved, together with the importance attached to the ability to communicate and move amongst these representations (Duval, 2006). Sáenz-Ludlow (2006) identifies the following abilities related to mathematical knowledge: a) to represent in order to communicate, b) to deal simultaneously with several semiotic systems, c) to recognise a mathematical object in a representation without conflating the object with its representations, d) to transform representations of mathematical objects within and between representational systems, and e) to construct and to interpret meanings mediated by signs.

In geometry, the combination of two representational systems, a verbal one and a visual one, is most of the times necessary, “even if only one of them can be explicitly highlighted according to the mathematical activity that is required” (Duval, 2006, p. 108). For example, a figure can be ‘sufficiently’ defined verbally, but no transformation would be possible without the use of its visual representation. Duval (2008) speaks of two types of semiotic changes: *conversion* and *treatment*. Conversions take place during the move, for example, from a verbal description to an image, while treatments occur within the same semiotic registry. Moreover, unlike treatments, conversions have no explicit rules for them, while a conversion in one direction may have no cognitive link with the conversion in the opposite direction (Duval, 2008). The special character of conversions and treatments, as well as the requirement of *spatial processing ability* (Gorgorió, 1998) may be responsible for students’ difficulties in geometry. A spatial processing ability may include the various spatial transformations, “the *ability to interpret spatial information* [...] the *ability to communicate spatial information*” (Gorgorió, 1998, p. 210).

Thus, a geometrical task is particularly challenging for the solver, requiring the solver to know more than just the terminology and/or the theorems. The solver is expected to perform some mental (and physical) activities, including separating the visual aspects of the figure from the properties stated, discriminating the figural units and performing some visual treatments (Duval, 2008). In our previous research (Tatsis, 2007; Tatsis & Goutsi, 2011), we focused on these activities and the aforementioned conversions by engaging students of different age groups in a game called ‘broken phone’. During that game the students were asked to convert visual representations of composite geometrical objects into verbal descriptions of them and vice versa. The findings of these studies revealed the students’ reliance on everyday language, their inability to identify topological and/or geometrical features of given figures and, more generally, their inability to discriminate the figural units of the composite objects provided. Following these, in this study, we investigate the phenomenon of switching amongst different representations, focusing on the students’ evaluations of different types of verbal descriptions related to different types of composite geometrical figures. Drawing upon our previous projects (Tatsis, 2007; Tatsis & Goutsi, 2011), we identified that the given figures could be classified based on their expected resemblance to familiar objects: *familiar* or *unfamiliar*. Moreover, those findings suggested that the students’ descriptions could be qualitatively differentiated amongst

*topological descriptions* (bearing only topological information), *geometrical descriptions* (containing both topological and geometrical information), and *everyday descriptions* (including a mixture of everyday and geometrical or topological information). Furthermore, we hypothesised that the students' descriptions would be affected by their mathematical attainment. Following these, we designed a quantitative study in order to delineate the relationships between the students' evaluations of different descriptions (topological, geometrical, everyday) and the familiarity of the provided composite geometrical figures, taking into consideration the students' mathematical attainment.

## METHOD

This study was conducted in a Greek university with 102 first-year students (N=110; female=81, male=21; 18 years old) following a 4-year BA in elementary education forming the pool from which elementary school teachers derive. Their mathematical knowledge is of high-school level, including two years of geometry (15-16 years old).

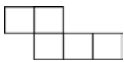
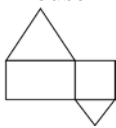
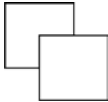
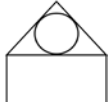
	Topological	Geometrical	Everyday
<p>'Squares'</p> 	<p>Draw a box (middle). Draw another one next to it, on the right. Draw another box one next to it on the left. Draw another one next to it and another one.</p>	<p>Draw five equal squares. Three of the squares are next to each other and on top of the first there is one more and another one next to it on the left.</p>	<p>Draw two stair steps. The lower stair has three squares one next to each other and the upper stair step has two equal squares.</p>
<p>'House'</p> 	<p>Draw a triangle and under it draw a rectangle. On the right of the rectangle draw a square and under a square and square and not on top of it. underneath a triangle pointing down.</p>	<p>Draw a rectangle and above the rectangle draw a big triangle. Then, right next to the rectangle draw a square and under it draw a triangle with a side equal to the square and not on top of it.</p>	<p>Draw a house, with no windows nor doors and roof without tiles and an upside down house the same (without windows, doors, and tiles), on the right of that house – something like a garage (upside down)</p>
<p>'Tiles'</p> 	<p>Draw a square. With this square on its down right side draw another square.</p>	<p>Draw a big square. And on the one quarter of the square that is in the middle of the square draw a square. That is, the way the square is, it is divided in four little squares, isn't it? From one of the little squares that are divided draw another square. From the middle of the square, from the inside of the square, draw a line and extend it outside of the square.</p>	<p>Draw a square, somewhere in the page. Next to it in the middle of the square draw another square a little bit on the right, like two playing cards one over the other.</p>
<p>'Attic'</p> 	<p>Draw a rectangle and over it a triangle and inside the triangle draw a circle.</p>	<p>Draw a triangle. Inside the triangle put a rectangle that exactly matches its side. Right under the triangle put a circle.</p>	<p>Somewhere in the middle of the page draw a house. A rectangle with a roof somewhat big (closed roof). In the middle of the roof draw a circle.</p>

Figure 1: The 'Evaluating the descriptions' questionnaire (tasks and descriptions).

For the purpose of the study, the 'Evaluating the descriptions' questionnaire was constructed, including twelve (three different descriptions about four different composite two-dimensional shapes; see Figure 1) seven-point Likert type questions

(ranging from ‘not at all well’ to ‘extremely well’). Drawing upon our previous projects (Tatsis, 2007; Tatsis & Goutsi, 2011), we included two *familiar* (‘house task’ and ‘attic task’) and two *unfamiliar* shapes (‘squares task’ and ‘tiles task’). The verbal descriptions (topological, geometrical, everyday) were adapted from actual data (Tatsis & Goutsi, 2011) and although they may be characterised as partially complete, care was taken so that they contained correct information about the given figure. Moreover, we collected data about their pre-university general and mathematical attainment, as possible factors affecting their evaluations. All data was collected through a web-based questionnaire, administered in a university laboratory.

Non-parametric statistical analysis was conducted with SPSS 17. Friedman’s ANOVA was employed, for the identification of intra-task type or intra-description type differences. The post-hoc analyses included Wilcoxon’s signed rank tests (Bonferroni correction applied). The relationships of the various students’ evaluations and their links with the students’ attainment were investigated through Kendall’s tau.

## RESULTS

The students’ evaluations are diagrammatically outlined in Figure 2.

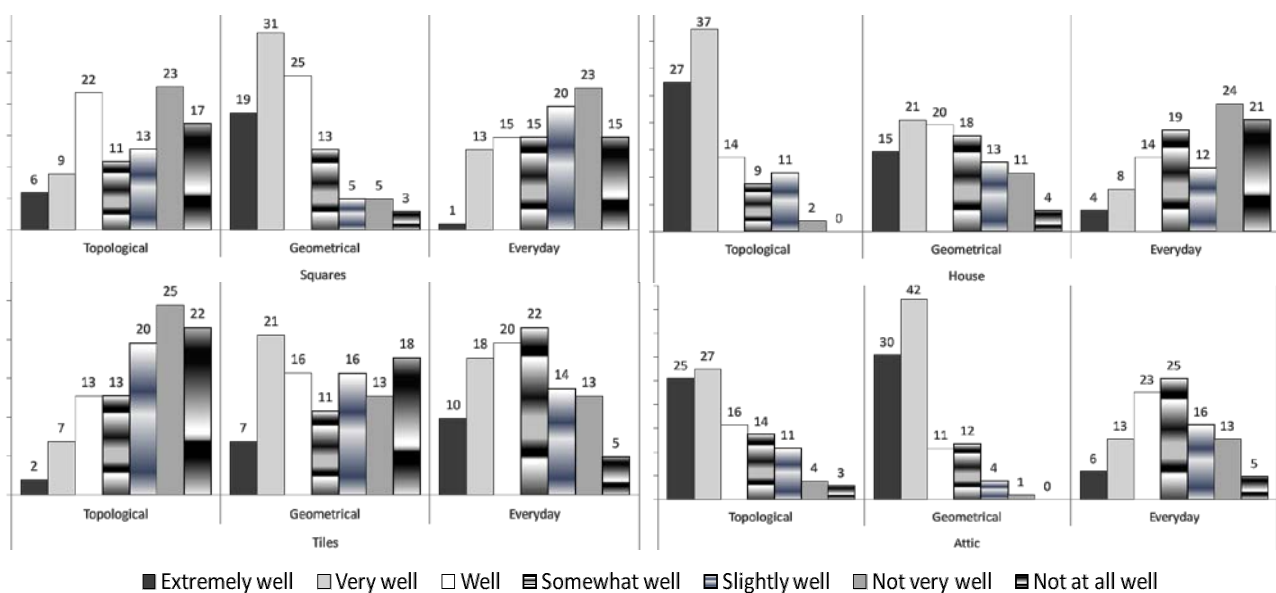


Figure 2: The students’ evaluations of the descriptions for each task.

Statistical analyses were conducted to further investigate these results (outlined in Table 1). The intra-task comparisons of the students’ evaluations for each type of description were found to be statistically significant for all four tasks. We followed these results with post-hoc analysis, revealing statistically significant differences in almost all the possible pairs of descriptions (topological-geometrical, topological-everyday, geometrical-everyday), except for the topological-everyday contrast in the ‘Squares’ task and the geometrical-everyday contrast in the ‘Tiles’ task. Moreover, we wish to note that: a) in the ‘Squares’ task, the students’ evaluations were *high* for the geometrical description and *medium-low* for the other descriptions, b) in the ‘House’ task, the students’ evaluations were *high* for the topological description and the geometrical description and *low* for the everyday description, c) in the ‘Tiles’

task, the students' evaluations were *high* for the everyday description, *medium* for the geometrical description and *low* for the topological description, and d) in the 'Attic' task, the students' evaluations were *high* for the geometrical description, *medium* for the topological description and *low* for the everyday description.

Squares	Top	2.25 <sup>a</sup>	Top-Geo <sup>d</sup>	$z=-6.278, r=-0.44^e$
	Geo	1.41	Geo-Eve	$z=-6.724, r=-0.47^e$
	Eve	2.34	Top-Eve	<i>ns</i>
	Intra <sup>b</sup>	$\chi^2(2)=61.3^c$		
House	Top	1.44	Top-Geo	$z=-5.088, r=-0.36^e$
	Geo	1.96	Geo-Eve	$z=-5.643, r=-0.40^e$
	Eve	2.60	Top-Eve	$z=-7.901, r=-0.56^e$
	Intra	$\chi^2(2)=78.1$		
Tiles	Top	2.35	Top-Geo	$z=-3.246, r=-0.23^f$
	Geo	2.01	Geo-Eve	<i>ns</i>
	Eve	1.63	Top-Eve	$z=-5.182, r=-0.36^e$
	Intra	$\chi^2(2)=61.3$		
Attic	Top	1.97	Top-Geo	$z=-4.507, r=-0.32^e$
	Geo	1.52	Geo-Eve	$z=-7.237, r=-0.51^e$
	Eve	2.51	Top-Eve	$z=-4.636, r=-0.33^e$
	Intra	$\chi^2(2)=60.3$		
Topological	Squ	3.08	Post-hoc comparisons statistically significant ( $p<0.001$ or $p<0.05$ ) except for 'Hou-Att'	
	Hou	1.71		
	Til	3.35		
	Att	1.86		
Geometry	Intra	$\chi^2(3)=142.4$		
	Squ	2.32	Post-hoc comparisons statistically significant ( $p<0.001$ or $p<0.05$ )	
	Hou	2.66		
	Til	3.18		
Everyday	Att	1.83		
	Intra	$\chi^2(3)=69.4$		
	Squ	2.75	Post-hoc comparisons statistically significant ( $p<0.001$ or $p<0.05$ ) except for 'Squ-Hou' & 'Att-Til'	
	Hou	2.91		
	Til	2.12		
	Att	2.23		
	Intra	$\chi^2(3)=30.8$		

<sup>a</sup>Mean ranks, the higher mean rank indicates a lower evaluation, <sup>b</sup>Intra-task or Intra-description comparisons, <sup>c</sup> $p<0.001$ , <sup>d</sup>Inter-task or Inter-description comparisons, <sup>e</sup> $p<0.001$  (Bonferroni correction), <sup>f</sup> $p<0.05$  (Bonferroni correction)

Table 1: Comparison of students' evaluations (outline of results).

Subsequently, we focused on the inter-task comparisons to identify differences affected by description type, revealing that: a) with respect to topological descriptions, all the students' evaluations differed statistically significantly for all tasks except for 'House' vs. 'Attic', b) with respect to geometrical descriptions, all the students'

evaluations differed statistically significantly for all tasks, and c) with respect to everyday descriptions, all the students' evaluations differed statistically significantly for all tasks except for 'Squares' vs. 'House' & 'Attic' vs. 'Tiles'. Furthermore, we wish to note that: a) in the topological descriptions, the students' evaluation scores were *lower* for both the 'familiar' tasks ('Squares' & 'Tiles') in comparison with the 'unfamiliar' ('House' & 'Attic'), b) in the geometrical descriptions, the students' evaluation scores were *very low* for the 'Tiles' task and *very high* for the 'Attic' task, while the other two evaluation lied in the middle of these evaluations, and c) in the everyday descriptions, the students' evaluation scores were *lower* for the 'Squares' and 'House' tasks in comparison with the 'Tiles' and 'Attic' tasks.

Finally, we focused on students' attainment, revealing that their evaluations: a) were *not* statistically significantly correlated with the National Exams points, b) were significantly *negatively* correlated with their overall grade in the last year of high school in the 'Attic' topological description ( $\tau=-0.147$ ,  $p<0.05$ ), and c) were significantly *negatively* correlated with their mathematics grade in the last year of high school in the 'House' everyday description ( $\tau=-0.185$ ,  $p<0.05$ ).

## DISCUSSION AND CONCLUDING REMARKS

In this study, we investigated the students' evaluations of verbal descriptions of two-dimensional composite geometrical figures. Drawing upon our previous studies, we hypothesised that the students' evaluations would be affected by: the type of the provided description, the type of the composite figure and the students' attainment. The given descriptions were categorised according to the mathematical information included (topological or geometrical) and on whether or not they incorporated reference to everyday activities or objects (everyday). We conjectured that the students would prefer 'everyday' descriptions to geometrical or topological, based on the findings of our previous studies. With respect to the figures, we conjectured that the familiarity of the given figure would attract the students' more positive evaluations for everyday descriptions. Finally, we expected the students with a stronger mathematical or academic background (as indicated by their mathematics grade and their overall grade in the final year of high school, as well as their overall points in the National Exams) would show a stronger preference for geometrical descriptions.

Our findings in most cases appeared not to reflect these expectations. With respect to the description types, in contrast with our conjecture, the students showed in most tasks their stronger positive evaluations for geometrical descriptions, followed by weaker positive evaluations for topological descriptions. These were accompanied with their relatively negative evaluations for everyday descriptions. The situation was different for the 'tiles' task, in which the students seemed to prefer the everyday description to the topological or the geometrical ones. Considering the students' mathematical or general attainment, we did not find it to have a statistically significant effect on their evaluations; in fact, although most of our students had average or low marks in mathematics at school and in the national exams, they showed a clear preference for the geometrical descriptions and then the topological ones. These findings are not in

line with our previous research, when students at a younger age were asked to produce their own descriptions, thus revealing their tendency to use more everyday language in their descriptions. This could be a result of the fact that in the present study the students evaluate rather than produce a description and/or that the participants of this study have had bigger school experience. The fact that mathematical or general attainment did not appear to be a decisive factor may suggest that it could be an ‘evaluate vs. produce’ phenomenon. Nevertheless, in order to further investigate this, a relevant study has been carried out to investigate the students’ productions of descriptions.

Furthermore, our hypothesis that the task familiarity would affect the students’ evaluations appeared not to be confirmed (in most cases), since the everyday descriptions were not evaluated as sufficient enough for the included familiar tasks. Nevertheless, their evaluations for topological descriptions were negatively linked with the familiar tasks. These complex findings raise further questions about what constitutes a familiar figure, as well as about the decisive factors that render an everyday description to be considered as ‘sufficient’.

In conclusion, in this study we considered the complex phenomenon of switching amongst different representational systems by focussing on the relationships between the students’ evaluations of verbal descriptions and the type of the object described. We posit that the quantitative perspective adopted allowed posing further questions about the factors affecting the students’ evaluations (description type, task familiarity, attainment), as well as about the complex interplay between producing and evaluating a verbal description of a composite geometrical figure.

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# **FACILITATING PROSPECTIVE TEACHERS' KNOWLEDGE OF STUDENT UNDERSTANDING: THE CASE OF ONE MATHEMATICS TEACHER EDUCATOR**

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*As a field, we know little about the practices of mathematics teacher educators, especially in relation to developing pedagogical content knowledge (PCK), as these practices are not widely researched or disseminated. This study investigated the actions and related purposes of what one mathematics teacher educator said and wrote during whole group instruction across three years in her elementary mathematics content/methods course to provide opportunities for prospective teachers to develop knowledge of student understanding. Findings from this study contribute to the literature on practices of teacher educators that can inform the design and implementation of teacher preparation programs.*

## **INTRODUCTION AND BACKGROUND**

During the past 25 years, there has been increased attention to conceptualizing the knowledge bases that mathematics teachers need to develop in order to effectively teach mathematics. Shulman (1986) identified a variety of knowledge bases and introduced the idea of pedagogical content knowledge (PCK). Subsequently, other researchers have built on this work in a variety of disciplines, including English (e.g., Grossman, 1990), science (e.g., Magnusson, Krajcik, & Borko, 1999), and mathematics (e.g., Ball, Thames, & Phelps, 2008).

An, Kulm, and Wu (2004) have argued that enhancing prospective teachers' PCK "should be the most important element in the domain of mathematics teachers' knowledge" (p. 146). Thus, it follows that a significant focus of mathematics teacher educators should be to provide opportunities for prospective teachers to develop PCK. There has been great attention to conceptualizing and delineating the components of PCK, yet there is little research that investigates how mathematics teacher educators facilitate its development in prospective teachers. In fact, we know little about the practices of mathematics teacher educators as these practices are not widely documented or disseminated (e.g., Bergsten & Grevholm, 2008).

There have been some research efforts related to the work of mathematics teacher educators (henceforth referred to as teacher educators). For example, teacher educators have conducted self-studies of their practice (e.g., Tzur, 2001), investigated professional development opportunities for teacher educators (e.g., Even, 2008), described specific activities that they use in their methods courses (Goodell, 2006), and examined a collaboration between novice teacher educators and a more experienced teacher educator (Van Zoest, Moore, & Stockero, 2006). Although these efforts represent a good start, additional research and development work are needed in order to accumulate a useful knowledge base for mathematics teacher education.



To better understand the teaching of university mathematics content/methods courses, this research study aimed to investigate the research questions: (a) What actions does an experienced and U.S. nationally recognized mathematics teacher educator enact during whole group instruction in an elementary mathematics content/methods course?; and (b) For what purposes does she use the identified actions? *Actions* are defined as what the teacher educator said and wrote (i.e., on the board, overhead, or document camera) while instructing prospective teachers. In this paper, empirical data is presented in relation to actions one teacher educator employed to provide the opportunity for prospective teachers to develop knowledge of one PCK component—*knowledge of student understanding*. Whether prospective teachers’ developed *knowledge of student understanding* was not the focus of the study—rather, the focus was on understanding one teacher educator’s practices as she provided the opportunity for prospective teachers to develop this knowledge.

### **THEORETICAL FRAMING FOR THE STUDY**

Within the context of mathematics content/methods courses, teacher educators employ various actions aimed to improve prospective teachers’ knowledge bases (e.g., content knowledge, PCK, general pedagogical knowledge, etc.). One significant focus in such courses is often providing the opportunity for prospective teachers to develop what Shulman (1986) called PCK. Building on Shulman’s work, Grossman (1990) identified *knowledge of students’ understanding, conceptions, and misconceptions of particular topics in a subject* as one of four central components of PCK. Magnusson, Krajcik, and Borko (1999) further modified this component by including *teachers’ knowledge of student misconceptions, approaches, and strategies when subject specific concepts are addressed*. Ultimately, this PCK component—which is the focus of this study—includes knowledge prospective teachers must have about students in order to help children develop specific mathematical knowledge. This includes strategies that will help prospective teachers identify students’ conceptions and misconceptions for solving various types of mathematical problems, as well as strategies that will help them aid students in understanding mathematical concepts and learning about specific mathematical topics.

### **METHOD**

A single case study design was used because the research questions were exploratory in nature and enabled the author to try “to illuminate a decision or set of decisions” (Yin, 2003, p. 12) regarding actions specific to teaching prospective teachers that one teacher educator used in her mathematics content/methods course. This design was used not because of an interest in the specific case participant, but in understanding actions specific to teaching prospective teachers that teacher educators use to provide the opportunity for prospective teachers to develop PCK. In other words, the case is examined to provide insight into a larger issue (Stake, 2005).

This study was conducted over a three-year period in the same course at a Midwestern U.S. university that was the second of a two-course sequence required for certification in elementary education (grades 1-6). The focus was on the content and complexities

of teaching geometry, measurement, probability, and statistics. The course met for 75 minutes twice a week for 15 weeks, and prospective teachers enrolled in the course during their final semester of coursework prior to a year-long student teaching placement. The average class size for the three years was 24 students (22 females and 2 males).

## Participant

Leah (a pseudonym), a former elementary school teacher and the teacher educator chosen for the case, taught the methods course six times over her seven years as a teacher of teachers by the conclusion of the study. She was purposeful about what she said and did in her classroom—devoting a tremendous amount of time each week to reflect on how her instruction influenced her students' participation and performance. Leah had received national recognition for her teaching, service, and scholarship since receiving her doctoral degree. In addition, her students recognized her efforts by consistently rating her teaching performance at the highest levels in course evaluations. Leah was willing to open her classroom and her teaching for others to learn from as evidenced by her mentoring of doctoral students who taught the same course. She frequently invited these students into her classroom to observe her teach and engage in weekly planning sessions, and to ask her questions during debriefing sessions related to what she said and did during her class. Her mentoring experience, as well as her experience as an established teacher educator, made her particularly suitable to explore the research questions.

## Data collection and analysis

Data included videotaped mathematics content/methods course lessons taught by Leah in Spring 2007 and field notes taken by the author in Spring 2008 and Spring 2010. This data was initially collected for another purpose. Snapshots of Leah's practice (i.e., what she said and wrote) were identified in the data. In the videotapes, a snapshot ranged in length from 10 seconds to 3 minutes and entailed: (a) an entire speaking turn from Leah, (b) a segment of a lengthy speaking turn from Leah, or (c) dialogue between the prospective teachers and Leah around a single mathematical concept. Snapshots identified in the field notes included an image of: (a) what Leah wrote or drew on the board, or (b) what Leah said related to providing an opportunity for prospective teachers to develop PCK. Additional sources of data included interviews where Leah provided commentary on three extended video segments from Spring 2007 about her purpose(s) and/or what she was hoping to address in class about learning to teach, as well as video/field note based interviews where Leah discussed her purpose(s) for employing actions identified in preselected snapshots.

The author conducted all analyses of the data. The HyperResearch qualitative data analysis software program (ResearchWare, 2007) was used to code Leah's PCK-related actions captured in the video and field note snapshots. Snapshots tagged during the initial coding were categorized (and re-categorized) using Magnusson et al. (1999) conceptualization of *knowledge of students' understanding* as including teachers' knowledge of student misconceptions, approaches, and strategies when

subject specific concepts are addressed. Through the categorization process, themes of actions within *knowledge of student understanding* began to emerge and descriptions of those themes were written. Through several iterations of sorting the snapshots, a coding dictionary was created from the data to define and illustrate each action. The researcher collaborated with Leah to refine descriptions of specific actions, as well as the categories of actions she employed to provide the opportunity for prospective teachers to develop *knowledge of student understanding*. Consistency of coding was verified with three other researchers.

## RESULTS

In the study, seven actions were identified, grouped into four major categories, aligning to what Leah said or did to help prospective teachers develop *knowledge of student understanding*. Table 1 summarizes the actions. Below, one of the most prevalent actions (bolded in Table 1) along with corresponding purposes, is elaborated on to provide the reader with specific images of Leah's practice.

<i>Categories of student understanding</i>	<i>Actions</i>
Sample grade 1-6 student mathematical answers	States atypical grade 1-6 student answers/thinking to mathematical concepts under discussion
	States incorrect grade 1-6 student answer to mathematical concept under discussion that relays the student's incorrect mathematical understanding about the topic under discussion
Predicting grade 1-6 student mathematical responses	<b>Prompts prospective teachers to predict mathematical answers/strategies grade 1-6 students generate/how grade 1-6 students will solve posed mathematical problems (which were posed to the prospective teachers or prospective teachers discussed as whole class)</b>
Grade 1-6 student mathematical misconceptions or error patterns	Shares misconceptions and/or error patterns grade 1-6 students (and teachers) have regarding mathematical concept(s) under discussion
Mathematical concepts that are abstract or confusing for grade 1-6 students	Articulates mathematical concepts that are abstract for grade 1-6 students
	Articulates language issue grade 1-6 students may have that interfere with grade 1-6 students' understanding the mathematics under discussion Describes examples of mathematical connections that grade 1-6 students may not make

Table 1: Summary of teacher educator actions identified to facilitate the development of prospective teachers' knowledge of student understanding

Throughout the semester, one of Leah's most prevalent actions was to ask prospective teachers to *predict what grade 1-6 students will say* to a mathematical problem or concept currently under discussion in the class. This action was enacted in a variety of ways. Sometimes, Leah had her students engage in the mathematical task themselves and then predict typical grade-level responses for the mathematical situation. For example, Leah gave the prospective teachers three squares and asked them to create all of the different possible shapes with the three squares where the side of one square must be flush with the side of another square (i.e., the corner of one square may not

solely touch the corner of another square nor may just part of a side of one square solely touch a part of the side of another square). Then the prospective teachers worked in groups to find the number of different shapes they could make with four squares (given specified constraints). Leah then prompted them to make a prediction for how many different shapes were possible using five squares. She said,

Ok so now we [haven't done] this problem, but I want a prediction. So kids often are looking for patterns, so we are going to have some guesses for how many shapes we are going to find with five squares. What are your guesses and predictions? So you are not actually finding them yet, you are just guessing. [Video Day13]

At this point, there was a whole class discussion where prospective teachers made predictions about how many different shapes could be formed using five congruent squares. The prospective teachers started by stating their predictions, but then Leah prompted them to provide answers they thought elementary students might pose. The predictions, with justification, they provided included: (a) 10 (doubled five); (b) six (difference of one); (c) eight (difference of three between the two and five in the “number of different shapes” column, so add the difference of three to the five); and (d) seven ( $2 + 5$  in the “number of different shapes” column). However, the prospective teachers did not provide all of the answers and describe all of the patterns that Leah had observed children provide. The prospective teachers were still missing two responses that grade 5 students gave when she had posed this problem to them. Leah decided that she would state these two additional patterns. She commented,

You are missing a couple other ones that kids come up with... There are two right answers in my brain that you are trying to figure out... I will just tell you. Ok,  $3 + 2 = 5$ , so  $4 + 5 = 9$ . Ok? This is a sophisticated child; let me see if I can remember it— $2 \times 2 + 1 = 5$  so  $5 \times 2 + 1 = 11$ . Yes, sign that kid up. That kid is doing some thinking. So, this one wasn't in the chart and says if I double two and add one I get five, so I am going to double five which is 10 and add one which is 11.

Leah also asked prospective teachers to predict grade 1-6 student answers in other contexts. For example, she asked prospective teachers to predict what a student would say when she engaged them in analyzing student work—specifically, “What do we hope here” when the subtraction problems  $25 - 21$  and  $103 - 99$  were posed [FN2008 Day13]. Here, Leah shared that she would expect a child to solve these two subtraction problems without using the traditional algorithm. She also shared the results of a research study where children were given two rectangles and asked to find the perimeter of each figure. One rectangle had two adjacent sides labelled while the other rectangle had all four sides labelled. Leah asked the prospective teachers to predict what elementary students would do [FN2010 Day14].

Occasionally, when Leah asked her students to *predict what grade 1-6 students will say*, after they worked on mathematical tasks, analyzed student work, etc., she reiterated the potential student answer suggested by a prospective teacher. An example of this was seen when she engaged prospective teachers in the game Roller Derby—a game where two dice are rolled, the sum is computed, and if an individual had a counter on the number of the sum that was rolled, one counter was removed. She asked

the prospective teachers to predict where elementary students would put their 12 counters on the game board that had the numbers 1 through 12. She stated,

Leah: So let me ask you this question, I have now done this game in Kindergarten, first, second and third grade. What do you think the number one answer is? ...How kids put their distribution down?

Student: One on each.

Leah: They put one on each. [Video Day2]

Other times, Leah carried out the action by answering her own question before the prospective teachers had an opportunity to do so, as seen in the next example.

Leah: Some of the big measurement ideas that you need to have, these are kind of your goals and things you are assessing. First you must include a number and a unit...You may compare two measurements if the same unit is used. So, if I say that this table is this line here and that is 30 inches and this table here is five feet. Which one is longer? The five feet or 30 inches?

Student: Five feet.

Leah: But what are the kids going to say? They are going to say this one because the number is bigger. Now, I cannot make those comparisons because my unit is different. These are the big ideas of measurement that take a lot of time to develop. [Video Day19]

Leah purposefully *prompted prospective teachers to predict what grade 1-6 students will say* because she wanted them to keep thinking about children when they plan and teach lessons. Leah admitted that many times when she asked her students to predict what grade 1-6 students would say, the question was rhetorical, but she kept posing the questions because they need to keep thinking about children. Leah elaborated on her observations of lessons that prospective teachers teach in their university field experience in local elementary schools while they are enrolled in her course:

They're so focused on themselves that they don't think about kids. And they focus on what they're going to do. And I try and get them to think about the kids. And so don't just think about the questions you pose, think about the answers you're going to get and how are you going to respond to those answers. Now with no teaching experience, I recognize that they're not very good at this. They don't know what to expect...but I keep throwing [those questions] out there. [EVT#3]

Leah recognized that prospective teachers lack experience in elementary classrooms, but she reiterated that she wanted them to consider students in addition to their own teaching practice, emotions, or beliefs. A second purpose Leah communicated for sharing predictions grade 1-6 students articulated was that she "want[ed her students] to know that kids are very creative and come up with lots of interesting things that [the prospective teachers] won't anticipate" [VDFNInt1].

As the examples above illustrate, Leah provided opportunities for prospective teachers to predict grade 1-6 student responses in various ways. Her overarching purpose for employing each of the observed strategies was to encourage her students to think about

children and what children will say and do (i.e., prospective teachers' knowledge of approaches and strategies grade school students use) so that they would be more apt to be responsive to students during their own instruction.

## CONCLUSIONS

The results discussed above detail one of the seven identified actions, and two corresponding purposes, that Leah demonstrated to provide the opportunity for prospective teachers to develop *knowledge of student understanding*. Five different themes were identified across three years regarding how Leah carried out this one action in her classroom, highlighting the complexity of the work of supporting mathematics teacher learning. Although the identified actions and purposes may not be exhaustive and may vary across different settings (e.g., at other grade levels such as middle and secondary), based on the experiences of teacher educators (e.g., teaching experience at the level in which they are preparing teachers to teach, etc.), they provide a foundation on which other studies can build. Thus, this study joins others (e.g., Even, 2008; Van Zoest, et al., 2006) in providing new insights regarding practices that might enrich the preparation of mathematics teachers—practices teacher educators could draw upon in order to enrich the learning experience of prospective teachers.

Specifically, findings from this study build on research focused on identifying components of PCK that prospective teachers need to develop to identify actions teacher educators can use to develop that PCK. The seven actions listed above may help teacher educators plan instructional activities to engage prospective teachers in opportunities to foster their *knowledge of student understanding*. Additionally, the examples of how one of the actions was implemented may help teacher educators consider additional ways to challenge and expand prospective teachers' initial understandings of what it means to effectively teach mathematics to grade 1-6 students. The highlighted purposes are representative of Leah's thinking about one action, which begins to develop a sense of why one might demonstrate specific actions in teaching prospective teachers. Finally, this research serves as a model for extending work on PCK from teaching at the K-12 student level to the teacher education level, conceptualizing actions and purposes teacher educators need to consider to develop PCK.

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# A STUDY OF MAGNITUDES AND MEASUREMENT AMONG BRAZILIAN INDIGENOUS PEOPLE: CROSSING CULTURAL BOUNDARIES

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*The main aim of this article is to discuss how traditional indigenous knowledge can play a role in mathematics teacher education focusing indigenous teachers. I take as an illustrative example lessons on magnitudes and measures among a group of Brazilian indigenous undergraduates who wrote a textbook on that topic to be used in work in their schools. This experience was analysed through a cultural-historical lens, focusing on teacher students' activity. The analysis shows how the tensions that emerge in the activity originate from an asymmetry of power between the people involved in it on the basis of the cultures to which they belong. When it is possible to achieve a balance of power among these people, the teacher students get more agency and become more able to cross cultural boundaries and acquire new knowledge.*

## INTRODUCTION

In my experience as a mathematics professor education of indigenous Brazilian students to be mathematics teachers,<sup>1</sup> I have observed several powerful examples of how to bring cultural aspects of traditional indigenous knowledge into the classroom. In this article, I aim to discuss the role of this knowledge in the study of magnitudes and measures among indigenous Brazilian undergraduates belonging to the Xakriabá, Pataxó, and Tupinikim peoples, many of whom were already teachers. This study led the teacher students to write a textbook<sup>2</sup> for the teaching of magnitudes and measures that is intended to be used in indigenous schools. Furthermore, the textbook would respect their traditional knowledge and help address the dominance of non-indigenous forms of knowledge in schools, which was seen in this study to lead to tensions in the classroom.

According to Baturo and Lee (2007) research in indigenous mathematics education should focus on improving the capacity and life chances of these people. Given the complexity of this issue, I think that a theoretical approach is needed that emphasises the central role of the culture in the practices of education of indigenous teachers. In light of this, the cultural-historical perspective of *activity theory* (Leont'ev, 1978; Engeström, 1987) allows us to look at activity that involves indigenous people through

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<sup>1</sup> This education program to indigenous people is a regular course at Universidade Federal de Minas Gerais. The work team program is formed by professors and post-graduate students and I am one of professors. Thus, I refer to them all as 'teachers' and indigenous undergraduates as 'teacher students'.

<sup>2</sup> Tomaz, V.S. (Ed.). (2012) *A Matemática e os saberes indígenas dos povos Xakriabá, Pataxó e Tupinikim*. Belo Horizonte: PIBID/FAE/UFMG. (Coleção Pibid Faz[Pibid Collection])



several different lenses: actions, language and mediating artefacts, values, power relations, identity, and ethico-moral dimensions.

## THEORETICAL FRAMEWORK

The activity theory perspective is still not very widely used in research on classrooms of the indigenous people, but as expressed in Tomaz and David (2011) and in David and Tomaz (2012), where the role of drawings in classroom geometry activities with non-indigenous elementary students is discussed, I think that this framework is also appropriate to mathematics education for to analyse activities of indigenous teachers education that will teach mathematics in their schools. It allows us to analyse the tensions that evolve in indigenous classroom activity and the changes that can result from them.

According to Leont'ev (1978), the concept of *activity* represents a specific form of social existence that includes crucial changes to social reality. Leont'ev explains that activity develops out of *necessity*, in that the latter drives the emergence of *motives* towards a related object. To satisfy these motives, actions are needed. These, in turn, are accomplished in accordance with *conditions* that determine the operations related to each action.

Engeström (1987) added new components to Leont'ev's model, taking an *activity system* as the basic structure and pointing out that the analysis of its constituent components and actions should be done *historically*, so that one activity system can be connected to others through some components. This also means that if one of these components changes, other changes must take place to adjust the whole system.

In Engeström's structure, the *subject* is an individual or group of individuals working toward a unique goal, whose agency is the focus of the analysis; the *object* is the 'space problem', or the problem of which direction the activity should develop in; *artefacts* are mediating tools and signs; *community* refers to the people who share the same object; *division of labor* is the division of tasks according to the status of the members of the community; and the *rules* are the implicit and explicit norms and conventions that regulate actions and interactions within the activity system.

The unit of analysis was thus expanded by Engeström (1987) to a minimum of two interacting activity systems; also, a prominent role emerges for *contradiction* as a driving force for transformation of a system. Contradictions in this context are more than problems or conflicts; they are rooted in structural tensions within and between activity systems that have accumulated historically. However, these tensions may not lead to a contradiction; instead, they may trigger small changes in the activity, when are situated at the level of short-time action. For Engeström an activity is always understood as a collective phenomenon within a community; individuals can only perform actions inside a larger system of collective activities. When discussing indigenous teacher students training, according to this theory, like Engeström, I thus put primacy on learners as members of communities and on the *hybridisation of cultural contexts* to allow the *creation/expansive transformation of culture*.

As people participate in multiple activities, they constantly change and create new objects for these activities. The object of collective activity has an inherent ambiguity; it is an invitation to interpretation, personal sense-making, and cultural transformation, and can be seen as a *generalised* or *specific* object. The generalised object is connected to societal meaning while the specific object is jointed with personal sense. Engeström and Sannino (2010, p.6) assert that the increasing tensions between multiple activities' components can lead to a deterioration of the situation or changes in which all parties together generate a new shared object and concept for their shared activity.

The concept of activity is thus the locus of complex interrelations between the individual subject and his/her community. In this paper, I focus my analysis on the tensions that evolve in one components of the activity, the *community*, considering its role in the emergence of a potentially shared object, a book about magnitudes and measures written by indigenous teacher students learning Mathematics to be a teacher. I consider the focal role in this process of the traditional knowledge possessed by these teacher students, recognising that it is a driving force of a 'horizontal expertise where practitioners must move across boundaries to seek and give help, to find information and tools wherever they happen to be available' (Engeström & Sannino, 2010, p.12)—a practice which can be described as *boundary-crossing*. They argue that crossing cultural boundaries is largely dependent on the employment of appropriate tools and the expansion of agency. It requires negotiation and re-coordination of actions towards a horizontal or sideways developmental dimension.

Thus, in this paper, the unit of analysis is an activity system that can be referred to as *the writing of a book about magnitudes and measures by indigenous teacher students*. The work was motivated by the general desire of the group of the teachers to contribute to the development of an intercultural approach to indigenous teacher students' mathematics education. With the approach outlined above, I was able to understand the complexity of the phenomenon as observed in its natural context.

### **THE ACTIVITY SYSTEM: THE WRITING OF A BOOK ABOUT MAGNITUDES AND MEASURES BY INDIGENOUS EDUCATION STUDENTS**

As mentioned above, the writing of the book took place as part of an indigenous teacher education in mathematics education to 32 teacher students. The course is organized in semester and each one is organised in three stages: in the *intensive* stage, the teacher students participated in lessons at the university, in the *intermediate* stage, the teachers developed lessons and other activities within indigenous territories and *final stage*, which again occurred inside of university when the activities developed in the previous stages are concluded. The discussion of magnitudes and measures thus began at the intensive stage, mainly on the basis of the guidance of the already existing class materials. However, the students wished to have a textbook of another new kind that would reflect and express in a school setting the value of their traditional knowledge; this became their main motive to write a new book about magnitudes and measures.

I consider at least three moments in the writing of the book to constitute key parts of the activity system; this system, in turn, emerges from a constellation of related activity systems, and I analyse this process historically.

At first moment, the discussion about magnitudes and measures was introduced in classroom and the teacher students decided to write a textbook. The traditional activities implemented consisted only of the illustration of learning points by the introduction of everyday situations for the indigenous groups to which the students belonged. In other ways, the materials, methods, and assumptions were the same that would be used in any such class; for instance, the SI system of units, whose principles are taught in school maths classes, was the main system for the expression of magnitudes and measures. Thus, the activity system could initially be characterised as having as an object the exploration of measurement procedures in agreement with SI, making SI an *object*, and the traditional activities as *artefacts*, in the terms of activity theory. Supporting the actions of the subjects were two communities: from one side were the indigenous elders, who impart traditional knowledge, and from the other, the mathematicians, mathematics teachers, and curriculum developers, who represent scientific/school knowledge. The power relationship between these communities is thus asymmetric: indigenous elders have less authority to drive the teacher students' actions than their teachers and the edifice of Western science.

The second moment occurred during the *intermediate stage* when the teacher students were asked to search the everyday life of their people to find activities that demanded some kind of measurement. They developed a list thereof since a *socio-cultural calendar*<sup>3</sup> of when it would be best to observe or engage in these activities and conduct measurements. The students wrote about the activities but did not emphasise the mathematical notions. Following the arguments of Owens and Outhred (2006), the teachers' assumption was that practical measurements using informal units can show the principles of measurement and the relationship between any linear units and a formal scale; it was also understood that the organisation of the units is fundamental to understanding the measurement of magnitudes. The teacher even intended that the teacher students could lead the traditional practices to the schools.

Following in the actions' sequence, when I analyse them historically I can also consider another activity system, whose purpose is to represent the magnitudes and measures found in the everyday traditional activities of those indigenous people. Thus, in contrast to, the traditional measurement procedures and concepts employing them were the *object*, once the students were focused on detailing the traditional activities. I consider that this shift happened because of the class's focus on traditional practices what provoked changes in power relations between communities that gave to the indigenous elders a position of greater authority.

Given this occupation of the position of highest authority by the elders, the actions of the teacher students were redirected to a new object containing ambiguities: the

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<sup>3</sup> The socio-cultural calendar is a powerful tool used by Gasché (2004) in his intercultural inductive method of investigation to draw an intercultural curriculum.

expression of measurements and magnitudes related to traditional activities in their communities according to the SI. This ambiguity creates tensions among the subjects and between subjects and their communities. It seems that to the degree that the teacher students emphasised the expression of traditional procedures of measurement, they had not got to terms with the use of SI standards. Likewise, when SI standards are emphasised by teachers as a way to translate traditional standards, the teacher students seem to feel that the indigenous identity of the book is being lost, which is contrary to their motive in writing it.

As the focus of the teacher students was on other aspects of the practices, extraneous to the definition of SI, the elders acquired greater authority to driving the activity, SI standards moved to the position of *artefacts*; this ensured a focus on further description of the traditional activity, but without any mention of magnitudes or measures. There was a strong tension between the subjects and the community, seen as an activity's component, because, at that moment, the most important thing to the teacher students was recording their traditional activities and knowledge and passing them on to future generations. Despite the best efforts of the teachers, the goal of connecting this activity to the development of a textbook teaching SI was not being achieved. These tensions lead us to think about other ways to produce the book.

The third moment occurred in the following year's *final* and *intensive* stages, when the teachers decided to change the direction of the work. We perceived that the tension that had emerged in the activity could lead to a fragmented object and split the activity system. Thus, adopting a new direction, we asked the teacher students to write a textbook detailing the traditional activities that they had recorded, without worrying about whether the principles of measurement and magnitude and the procedures used were being highlighted. The group of teachers, in turn, when following the teacher students' writing, identified places where these principles and procedures were mentioned, but interfered as little as possible, only in the case that distortion of the principles of the SI was introduced in the descriptions or to evaluate the accuracy of the descriptions of traditional methods where they did not refer to the SI..

In addition, after the completion of the first draft of the text, the students did oral presentations for each other, exchanging information and clarifying the final structure of the text. In these presentations, we were introducing a new *mediation artefact*, in line with the oral transmission of knowledge by indigenous elders. The teacher students wrote ten different texts, assembling in chapters each one of traditional activities<sup>4</sup>. At the end of each chapter, the students presented a table where they identified the measured magnitudes, corresponding SI units, and tools (see Table 1).

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<sup>4</sup> The eleventh chapter was written by the professor who did the relation between the measurement procedures presented in the other chapter and SI units.

Activity	Measurement activity	Magnitude	Unit of measure <sup>5</sup>	Tools	Measure (examples)
Building of the Pataxó canoe	Measuring the canoe's dimensions	Length	The flat of the hand of an adult man	The hand of an adult man	5 flats of the hand
Building of the Tupinikim hut	Measuring roof of the hut	Volume	Sheaf of straw	Cypó (natural fiber)	15 sheaves of straw
Growing of corn and beans by the Xakriabá	Demarcation of the ground for planting	Area	Quantity of seed to fill up a Indigenous plate	Salamim (a wood box as a rectangular prism)	2 Indigenous plates of seeds
Growing of cassava by the Pataxó	Measuring the height at which to cut the cassava	Length	Distance between an adult's foot and knee	An adult's legs	½ of the leg of an adult
Handicraft: seed necklace	Trading the seeds	Volume	Indigenous Litre	Tin of oil with capacity of 900 ml	1 litre of seeds

Table 1: Traditional activities of indigenous teacher students and measurement procedures used in them<sup>6</sup>

This third moment can also be seen as an activity system containing an *object* (the mathematical knowledge arising from indigenous practices) shared collectively by crossing cultural boundaries. As observed by Engeström (2009), the construction of a shared object between two or more activity systems is a challenge, but in this case, the teacher students and teachers, albeit with different motivations and goals, guided their actions to such a shared object, coming to act as a collective subject or group of individuals. Until this confluence, the activity was not correctly orientated toward the object, since two different communities were driving the actions of the subjects, leading to fragmentation of the object and an activity seemed not to exist. To Leont'ev (1978) there is no activity without an overt object, but to Engeström (2009), there can be objects that remain dormant, invisible, or unseen for lengthy periods of time before bursting into the open in the form of acute crises or breakthroughs' (p. 304). As Engeström argues, 'the object is persuasive and its boundaries are hard to draw'.

The construction of this shared object and activity system became possible when we can perceive a new distribution of power within a community, achieving a minimal

<sup>5</sup> These indigenous people, in particular, often use some indirect methods to do measures, e.g., a quantity of the sheet to demarcation of an area. The indigenous plate/litre units used in Table 1 are referring to only the indigenous people mentioned in this paper.

<sup>6</sup> Besides the activities listed in Table 1, the students also wrote about the building of houses, harvesting of Pequi (a popular Brazilian fruit), forestry, and fishing.

balance of agency between the two groups or communities that polarised the subject's actions. Indeed, In this activity it is possible to assert that the first outcome is the construction of a community; Taylor (2009, p. 238) says that 'for any activity, human or other, to reproduce itself and display continuity in its activities, beyond the immediate response to its environment, it must have the means to re-create the conditions of its own survival and perpetuation as a system', that is, it must produce its own community.

As all shared objects contain ambiguities, the collective actions of subjects that come from different cultures introduce *new rules* to a related activity allowing it to be accepted by community members. For example, the 'new' units of measures (see Table 1) and the procedures for measurement arising from indigenous traditions those people, even though not adopted by SI, can be accepted by the community of its representatives in the context of the activity system. Thus, the traditional procedures and measures attain the status of a recognised system of measures, structured by the hybridisation of cultural context, and with mathematical rigour commensurate with the type of knowledge that the teacher students want to introduce in indigenous schools.

## CONCLUSION

The tensions that have evolved in the activity system, referred to as *the writing of a book about magnitudes and measures*, were resolved when the power relations inside the *community* became more horizontal and authority more distributed. As the teacher students' everyday traditional activities are incorporated in their ways of teacher Mathematics, the (non-indigenous) school mathematics is unable to impose itself, within a indigenous teacher education program. Then, we can see the importance of community component in this activity system and in shaping its products.

Finally, it is important to highlight the perception of the teachers (professors and postgraduate students) for the ethico-moral dimensions of this activity and their crucial role, to achieve positive changes in task coordination, giving a new direction to the subjects' actions, but not to play the role of sole authority. According to Engeström (2009), *authority* is foundational to sustain the existence of a community but 'coordination is not exactly the same as authority. However, the achievement of coordination is a central manifestation of authority' (p. 315). The teachers' actions in the present study seem to be the main impetus for the reconfiguration of the division of labor and the rules, and for promoting negotiation across horizontal and vertical boundaries between communities (indigenous and non-indigenous). This negotiation process, required whenever the object of activity is unstable and resists attempts at control and standardisation, helped to achieve the success of the school activity described here by increasing teacher students' agency as they secured the support of the united community.

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# NORMS AND PERSPECTIVES IN PRE-SERVICE SECONDARY MATHEMATICS TEACHERS' DISCOURSE

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*This study is part of wider research that seeks to investigate the existence (or not) of socio-didactic-mathematical norms in the context of solving a didactic-mathematical task. In addition, we try to analyse whether any of these norms could have relationships with future teachers' perspectives related with their role in the classroom. The data for our study consists mainly of the transcriptions of the verbal dialogues the students maintained when solving a didactic-mathematical task put to them in the classroom. Based on our analysis, we have been able to identify five socio-didactic mathematical norms in our study. Three of them were in some way related to the mathematical content and its learning. The other two are related to teachers' role, providing information about characteristics that the future teachers associate with said role. Furthermore, we have identified features of determinate perspectives through the latter two norms.*

## INTRODUCTION

The last few years have seen a growing number of studies related to the different norms that can be identified in discourse. Authors such as Tatsis & Koleza, (2008) have focused on social and socio-mathematical norms, contemplating them in the same sense as Yackel & Cobb (1996, p. 461) as 'normative aspects of mathematics discussions specific to students' mathematical activity'. Other authors (Gorgorio & Planas, 2005) have used the construct socio-mathematical norms with a heavier social weight than in Yackel and Cobb's interpretation, focusing on the different re-interpretations of the same norms in multiethnic classrooms. Sánchez & García (2011) have considered the socio-mathematical and mathematical norms that arise in the interaction between primary student teachers when they solve a mathematical task, incorporating the mathematical topic addressed as a new variable.

Along with this, Simon and colleagues (Simon, Tzur, Heinz & Smith, 2000; Tzur, Simon, Heinz & Kinzel, 2001) have utilised the term *perspective* to 'to postulate a broad pedagogical structure composed of a multiple conceptions that collectively organize some aspects of a teacher's practice' (Tzur et al., 2001, p. 228). Taking into account the importance of identification of these perspectives that underlie teacher's practice, in other occasions we have considered the existence of relationships between teachers' perspectives and the relational architecture that is established (Escudero & Sánchez, 2008), and the way in which the construction mechanisms of knowledge are modelled by teachers in classrooms (García, Gavilán & Llinares, 2012).

Nevertheless, how some norms could encourage the development of ideas or feelings related to the adoption of a teacher's determinate perspective have not been given



in-depth consideration. The relationships of the perspectives with some norms that are socially shared by future secondary mathematics teachers become our object of interest in this work.

## **THEORETICAL FRAMEWORK**

In our research the way of considering norms is based on the point of view of Sfard, who considers norms to be ‘metadiscursive rules that are widely endorsed and enacted within the discourse community’ (Sfard, 2008, p. 300). Furthermore, without minimising the importance of other norms, we focus on socio-didactic mathematical norms (SDMNs) that arise in the interaction between groups of secondary student teachers when solving a task related to a teacher’s specific professional activity: analysing textbook tasks related to functions. We hypothesize that, in the colloquial discourse that arises between a group of future secondary mathematics teachers when solving a didactic-mathematical task, a discourse linked to socio-didactic mathematical aspects could exist. These aspects come from the manner of considering mathematics as a subject matter, which has to be taught and learnt in a school context. In addition to these specific norms, we are aware that other norms exist, but they are not addressed in this part of our study.

Furthermore, in our theoretical framework, we include the three different perspectives characterised by Tzur et al., (2001): traditional, perception-based and conception-based. They emphasize that these perspectives are their characterization of teachers’ practice, and not how the teachers themselves would describe their practices. Focusing on the characteristics closer related to teachers, traditional perspectives to teaching can be characterized by ‘teachers’ attempts to transmit particular mathematical ideas to students’ (Tzur et al., 2001, p. 247). As Tzur points out, in this perspective ‘Teachers feel responsible for logically organizing and clearly presenting the mathematical content’ (Tzur, 2010, p. 56). This content, for him, has ‘crystallized’ through millennia. A perception-based perspective considers ‘the teacher’s primary role is not to directly transmit the intended ideas to students, but to orchestrate conditions that engage students in actively seeing and connecting those ideas’ (Tzur et al., 2001, p.247). Finally, in the conception-based perspective teachers feel responsible for engaging learners in realistic tasks, orienting learning reflections through a reorganization of previously established schemes (Tzur, 2010).

In our case, these perspectives allow us to characterize the previous ideas of future secondary mathematics teachers about mathematics teachers. Here we focus on how socio-didactic-mathematical norms which arise in the interaction between groups of secondary mathematics student teachers when they are solving a didactic-mathematical task could be related with the distinct aspects that characterise the different perspectives.

The research questions behind this study are:

- Is it possible to identify socio-didactic-mathematical norms in the context of solving a didactic-mathematical task?
- Could these norms have any relationship with future teachers' perspectives related with their role in the classroom?

## **METHOD**

### **Participants**

In the part of the research reported here, participants were 20 of the 28 future secondary mathematics teachers enrolled in the Master's Degree in Secondary Education Teacher Training at a large university in Spain, and specifically in the 'Mathematics learning and teaching' course. This Master's is a postgraduate course of 60 credits ECTS (European Credit Transfer System), recently implemented in Spanish universities. It plays an essential role: features the necessary professional requirements that enable an individual to become a teacher at secondary school level.

The students participated voluntarily in the study. Although they came from different specialities, all had a university degree related to mathematics or other scientific specialities. On the course, the students worked in small groups (namely G1, G2, etc. on our research) in two-hour sessions per week. There were two 5-student groups, two 3-student groups, and three 4-student groups. The study was developed with five of these groups at seven sessions.

### **The research instrument**

Taking into account that in the commognitive framework of Sfard the mathematical discourse of students is the unit of analysis, researchers have used different learning environments to access students' thinking (Wille & Boquet, 2009). The data for our study consists mainly of the transcriptions of the verbal dialogues the students maintained while solving a didactic-mathematical task proposed in the classroom. This task followed the theoretical ideas of situated cognition (Brown, Collins & Duguid, 1989; Sánchez & García 2009). It was very different to traditional tasks that are usual in the scientific fields in which these students had obtained their degrees. Specifically, the task was a didactic-mathematical situation in which a teacher's professional task was put to them (analyzing school tasks taken from secondary school textbooks) and a mathematical content (functions). Some conceptual tools (articles from specialised journals) were provided to enable solving of the task, and the groups were given total autonomy with respect to their way of working. The students had to write a final report featuring the findings of their analysis.

### **Data analysis**

Once the dialogues were transcribed into written text, in a the first step we analysed the discourse on the basis of the four properties identified by Sfard that "can be considered as critical in deciding whether the given instance of discourse can count as mathematical" (Sfard, 2008, p.133): Mathematical words, Visual mediators, Endorsed

narratives and Routines. In our study, the mathematical words were replaced by didactic-mathematical words, with these words taken to be those linked to the teaching /learning processes.

In a second step, we focused on narratives. When a narrative was endorsed on several occasions, we analysed its meaning in an attempt to identify norms. In particular, SDMNs were inferred by identifying on the basis of different features linked to aspects coming from the way of considering mathematics as a subject matter to be taught and learnt. Afterwards, the SDMNs were analysed (third step), selecting those related to teacher's role.

We here-below present a brief example from a group of students (Group 7), to show how we identify SDM norms.

Representative examples of endorsed narratives	Identified features	Inferred norm
<p>(G7, Page 63) [Students discuss the information provided in relation to the translations between modes of representation]</p> <p>....</p> <p>1619: D: <b>We are going to call him (the teacher) to see if the theory is the ...</b></p> <p>1620: A: What?</p> <p>1621: D: <b>That we are going to call him, right?</b></p> <p>1622: A: <b>Yes, but if this is all clear to us, the question is the most ...</b></p> <p>1623: M: <b>But we are going to ask him the doubts....</b></p>	The teacher confirms results and clarifies doubts	'The teacher validates the knowledge and solves the doubts'
<p>(G7, Page 67) [Students try to decide the best procedure to solve problem]</p> <p>1723: A: Solving an equation, exactly, okay, I think that's done, right?</p> <p>1724: M: <b>Now we call him (the teacher) to check, now when he comes we show him the answer to question 1</b></p> <p>1725: A: <b>Ok</b></p>	The teacher confirms what has been done	
<p>(G7, Page 75) [Students have finished three sections of the task]</p> <p>1911: A: [addressing the teacher as the representative of the group] <b>We have done three exercises if you'd care to take a look...</b></p> <p>1912: S: <b>To see what you think ...</b></p>	The teacher validates the response to allow them to continue	
<p>(G7, Pages 81-82) [Students are discussing the interpretation of a graph]</p> <p>2058: M: .... the interpretation, ...yes I've thought about it, <b>it is very important but I don't know if it is an element</b></p> <p>2059: A: <b>Okay then put it as a mathematical element and we can ask him afterwards when he comes</b></p> <p>2060: M: <b>I'm going to write it in pencil and then we will</b></p>	We can put whatever and then confirm with the teacher	

know that we have to ask about it.		
(G7, Pages 84-85) [Students discuss whether the problem they are analyzing presents a local or global situation]		Guide about the way forward
2127: A:	<b>I think that the problem is in ...</b>	
2128: M:	<b>I would ask him</b> [referring to the teacher]	
2129: A:	I think that the problem is the solution; is it global or local? ... because you say, well yes, I know that this is the maximum but because I take a look to all..	
2130: M:	Right	
2131: A:	<b>Exactly, but I think that it is more of the solution, but we are going to ask him</b>	
[...]		
2133: A:	I think that's what it is, but well <b>now we can ask him...</b>	

Table 1: Example from a group of students

In the above table, the successive inclusion in the discourse of endorsed narratives that emphasized different features of the teacher's role lead us to identify the socio-didactic-mathematical norm 'teacher validates the knowledge and clarifies doubts'.

Finally, in the fourth step, these norms were considered from the different characteristics of the perspectives that have been described for the above-mentioned authors (Simon et al., 2000; Tzur et al., 2001; Tzur, 2010). In our example, we can say this norm fits with a traditional perspective.

## FINDINGS OF THE STUDY

Based on our analysis, we have been able to identify five SDM norms in our study.

Three were in some way related to the mathematical content and its learning. For instance, from the different narratives identified in Groups 1 and 6, we were able to identify the norm "There are some representation modes that are more necessary than others" (Norm 2). This SDMN emphasises the importance that is given in our context to the use of the table as a way to obtain a graphical representation, minimising the use of certain properties (cutting with axes, vertices) that can allow a better picture of the situation. Other narratives identified in Group 4 lead to the norm "A mathematical result is (or is not) correct depending on situation" (Norm 4). This norm shows how a solid training in scientific content helps to link mathematics results to the posed situation. Precisely, this solid training can lead to the norm "Explanations in the answers to the tasks are not necessary because time is wasted" (Norm 5), in which communication is not considered a relevant mathematical process.

Finally, two SDMNs were linked to the teacher. One of them (Norm 1: 'The teacher validates the knowledge and clarifies doubts') was identified in all the groups and has been detailed in the analysis section. The other (Norm 3: 'To introduce a mathematical content the teacher should always follow an established sequence') was identified in two groups (G1 and G4), on the basis of narratives such as those by Group 4 identified in the following table:

Representative examples of endorsed narratives	Identified features	Inferred norm
<p>(G4, Page 114) [Students discuss the order in which to present the problem to their future students]</p> <p>2784: D: Look I propose the following <b>first a table, okay a table, afterwards the representation of one of the variables that we can call height with respect to the other one</b> which is boiling temperature, ...</p> <p>2785: M: Okay</p> <p>2786: D: Then we already have <b>here that we can present, to the pupils, first the data processing to the table and then a graphical representation</b> of the problem ...</p>	Students identify what appears to them to be a correct sequence that is assumed without posing more alternatives	'To introduce a mathematical content the teacher should always follow an established sequence' (Norm 3)
<p>(G4, Page 116) [Students discuss the order in which to present the problem to their future students]</p> <p>2845: J: <b>First mathematical element the representation in table of values, and its construction</b> by means of a specification, you know, the process of, we will take advantage of that every thousand down, <b>then the second mathematical element is the graphical representation of the linear function as a relative line in the plane, the following is, rule of proportionality ...</b></p>	Students identify what appears to them to be a correct sequence that is assumed without posing more alternatives	

Table 2: Identification of norms

If we focus on Norm 1, we can infer that future teachers need a teacher's presence and opinions to validate their work. Two important aspects emerge: Doubts are not discussed thoroughly in the groups, and possible alternative answers are not discussed. Students think that accurate solutions and alternatives come from the teacher. If we focus on Norm 3, a previous content already established should be taught through a 'correct' sequence. A teacher must follow that sequence to transmit the content, being a 'person who transmits mathematical knowledge'; this knowledge is perfectly structured both in its way of teaching and learning. Both norms are related to some characteristics identified in a teacher's traditional perspective.

## DISCUSSION AND FURTHER CONSIDERATIONS

Our results extend the work of authors who have dealt with different types of norms (Tatsis & Koleza, 2008; Gorgorio & Planas, 2005; Sanchez & Garcia, 2011), incorporating the definition, identification and study of social norms related to didactic-mathematical contents. We have been able to identify some of these norms in future secondary teachers' mathematical discourses related to different aspects that intervene in the teaching/learning processes. Some of them could be closely related with Spanish educational context that emphasizes the use of some representation modes over others and over translations between modes; in addition, it does not promote mathematical communication as an important element in students' mathematical education. Other SDMN's are related to their way of considering

teacher's role, providing information about characteristics that these future teachers associate with this role.

Furthermore, we have identified features of determinate perspectives through these latter norms. To transmit, to validate and to solve are verbs that can describe actions, or conditions that characterise a teacher's role for these students (we do not try to generalize to other students). These aspects are also part of the features of the above-mentioned traditional perspective. Two important aspects arise from these results. First, the relationship between norms and perspectives can be reflected in student teachers' future professional work. Second, the identification of SDMN's such as these are linked to a way of understanding the generation of knowledge and teacher's role might be considered as something shared. This could indicate a cultural feature of the society in which future teachers find themselves, feature that could be related to the social consideration of teachers' work. We need to extend this work to other students to further these results.

Other questions are related with the task. We wonder whether different tasks might give rise to different SDMN's that are related to other characteristics of other perspectives. Furthermore, in our case, the lack of coherence between the theoretical frameworks related to teaching/learning that these students have experienced and the fact of having to solve a task situated in a different theoretical framework can favour the emergence of the norms identified here.

To sum up, if we take into account that future secondary mathematics teachers are key elements in the improvement of mathematics education, norms and perspectives become relevant elements that should be taken into consideration in the teaching/learning processes.

### **Additional information**

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# PERIODICITY IN TEXTBOOKS: REASONING AND VISUAL REPRESENTATIONS

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*This study aims to explore how the visual and the verbal components of texts from different subjects in specific topics related to the notion of periodicity interplay in the reasoning process. To this end, we developed an interdisciplinary framework on reasoning in texts. By comparing and contrasting the argumentation in two grade 11 texts from the subjects of mathematics and science we get evidence on how the reasoning is developed and on how this could affect students' conceptualization. Our analysis identifies common reasoning behaviours in the different educational fields while different routes in reasoning could broaden and enrich students' perceptions. Furthermore, our analysis illustrates the 'flexible' character of the visual components inside and across texts that could contribute to the formation of the invariant notion of periodicity. \**

## INTRODUCTION

Love and Pimm (1996) highlight the role of argumentation developed in the school textbooks in the meaning-making process by denoting that although the implied relation between the reader and the text is inherently passive, “*the most active invitation to any reader seems to be to work through the text to see why the particular ‘this’ is so*” (p. 371). In this direction, in the science context, Chi and her colleagues (Chi, deLeeuw, Chiu & LaVancher, 1994) point out that students generate self-explanations in order to fill in substantial details in textbook argumentation. Even though realizing the logic of the presented knowledge in school textbooks is central in developing meaning few research studies have focused on the argumentation developed in textbooks. These studies identify empirical inductions and deductions as modes of reasoning (Stacey & Vincent, 2009) while many times these arguments occur presumably in conjunction with achieving a better understanding (Cabassut, 2005). More to the point, while a number of mathematics educators are investigating the use of visual representations in reasoning and proof (for an extended literature review on visual reasoning see Hanna, 2000) we do not have enough evidence on how the visual and the verbal components of a school text interplay in the reasoning process.

In this paper we address the above issue in different educational fields around the same concept, the concept of periodicity. Tall and Vinner (1981) argue that students'

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different experiences on a common notion are the major constituents of their concept image creation. The concept of periodicity plays a central role in the school curriculum and is expressed in different educational fields where it acquires practical importance. The students form this concept in different school subjects like science, mathematics, technology, astronomy. Our overall goal is to investigate through the analysis of the textbooks the relation between argumentation and conceptualization in the above school subjects and how they are interwoven in the texts. Particularly, in this paper we focus on two components of the text (the visual and the verbal), aiming to explore how these are linked in the reasoning process in texts on the notion of periodicity. Specifically, our research questions are: How is the reasoning process in school texts differentiated in the mathematical and scientific context? What is the role of visual representations in this process?

## THEORETICAL FRAMEWORK

We adopt Vergnaud's (2009) theory of conceptual fields where a concept is a triplet of a set:  $C=(S, I, L)$  where  $S$  stands for the set of situations which give sense to a concept (*the referent*);  $I$  stands for the set of operational invariants associated to the concept (*the meaning*);  $L$  stands for the set of linguistic and non-linguistic representations which allow for the symbolic representation of a concept, its attributes, the situation to which it applies and the procedures it nourishes. In this paper, we expect to identify elements of the invariant notion of periodicity across subjects by analysing two parts of the above triplet in school texts: the situations (i.e. the thematic units where the concept of periodicity appears in school texts); and (b) the interplay of linguistic and non-linguistic representations (i.e. the role of verbal-visual relation) in the argumentation process.

*Argumentation and reasoning in different contexts:* Daily life reasoning is characterized as *informal* since people draw inferences from uncertain premises. Scientific reasoning may be either *deductions* based on a set of a priori premises; or *inductive* generalizations based on laws; or inferences to the best explanation as in Darwin's development of evolutionary theory (Szu & Osborne, 2012). Furthermore, the modes of reasoning in science text could be (a) *logical* when they are based on the finished products of science (i.e. laws, principles, models, theories and mathematical and algorithmic procedures); and (b) *empirical* when they are based on experiments and intuitiveness. On the same direction mathematical reasoning on school texts can be categorized as (a) *deductive* (by using a model, or a specific or a general case); (b) *empirical* reasoning (e.g. experimental demonstration); and (c) metaphorical reasoning (Stacey & Vincent, 2009). However, the outcomes of inductions and deductions (i.e. the general conclusions of reasoning) and the examples and images provided in the text are also important on the concept image formation (Tall & Vincent, 1981). Additionally, while empirical reasoning in mathematics has an informal purpose in the scientific context it has a validating intention which leads to generating scientific knowledge. Having taken all the above issues into consideration we have started developing an interdisciplinary framework on reasoning in texts (Triantafillou, Spiliotopoulou & Potari, in press) with the following categories: Nomo-logical;

Logical-mathematical; Logical-empirical & Empirical. In the present study subcategories were formed and their interrelations were recognised by matching our emerging classification to our data.

*The visual-verbal relation in text analysis:* The importance of the multimodality approach has been argued by Kress & van Leeuwen (2006). They consider that visual and verbal elements in texts are two items of information which are interrelated as follows: by elaboration (when an item elaborates on the meaning of another by further specifying or describing it) and by extension (when an item extends the meaning of another by adding something new to it). In this study, the function of the visual representations in the reasoning process is investigated thoroughly.

## METHODOLOGY

A grounded theory research approach (Strauss & Corbin, 1998) is adopted, while our methodological framework is based on the qualitative inductive content analysis of both verbal and visual elements of the text. By restricting our analysis to topics that are related to periodicity we analyzed 11 textbooks from the subjects of Mathematics, Physics, Astronomy and applied technologies (Electrology, Electronics and Informatics). These textbooks are used in Greek lower and upper secondary General and Vocational schools.

*Analysis of data:* Initially, we define the conceptual thematic units that have an independence from the rest of the text, are characterized by one thematic content (e.g. “Define Linear Harmonic oscillation”), and produce an argumentation from the particular way they are organized. The process of argumentation in each text is realized as a sequence of interdependent and logically connected statements. So, the main unit of analysis is defined by either one sentence or a sequence of sentences and the accompanying visual representations (VRs) that produce a type of reasoning and support the generation of argumentation developed in the thematic unit. The identified types of reasoning are called ‘modes of reasoning’. Moreover, after analyzing a lot of textual units, we identified the role of VRs in the argumentation process. The systematic qualitative content analysis of all the units of analysis (92 units) has led us to the production of two schemes of categories: one of the modes of reasoning and one of the roles of visual representation inside each mode of reasoning. The structure of the category schemes on argumentation is finalized after a number of reconstructions checking the categories through data.

## FINDINGS

In the first part of this section we present the extended interdisciplinary framework of categories of modes of reasoning developed, as well as the scheme of categories on the visual-verbal relations in the reasoning process. In the second part we exemplify our analysis in grade 11 mathematics and science texts. Such kind of analysis is didactically important since both texts are addressed to the same student. In the two selected examples we illustrate the way we characterize the modes of reasoning as well as how the verbal and visual elements are interwoven in the presentation of new knowledge.

*The categories of modes of reasoning:* (i) *Nomo-logical*: (ia) when the reasoning of the text is based on axioms or theories or previously established statements and these are the basis for further reasoning in the text (N1); (ib) when a definition, a generalization or a law emerges as a result of previous generalizations (N2) (it is usually recorded in the text in a distinctive way e.g. in bold letters). (ii) *Logical-Mathematical*: Applying mathematical relations and techniques (LM). (iii) *Logical –Empirical*, when experiences are either related to logical conclusions or linked to general statements with examples and specific situations. This mode of reasoning is further discerned in the following categories: (iiia) *Application reasoning* that starts from a general idea of logical type and ends up in implementing it in certain empirical or specific situations (LE1). (iiib) Reasoning that starts from specific situations or empirical data and ends up in general phrasings, meanings or conclusions (LE2). The case of deductions based on generic examples fit in this category. (iiic) *Explanatory reasoning* which aims to explain theoretical ideas or exploit invented situations to explain phenomena (LE3). (iv) *Empirical reasoning*: (iva) *Recalling experiences* from everyday life (E1). (ivb) *Describing enactive experiences* either of everyday life or in an experimental activity (E2). The case of naive empiricism and crucial experiment (Balacheff, 1988) or experimental demonstration (Stacey & Vincent, 2009) fit in this category.

*The categories of visual-verbal relations on reasoning:* (a) *illustrative* (the image adds to the verbal without being embedded in the reasoning); (b) *exemplifying/explanatory* (the image gives an example or explains a reasoning process); (c) the image is the *starting point* on which reasoning is developed; (d) the image is the *fundamental* tool in a mode of reasoning; (e) The image is the *product* of a mode of reasoning; (f) the *organizational* character tool (the image organizes the outcome of a mode of reasoning); and (g) *Complementary* (the content of the image complements to the reasoning of the verbal text).

### **The text examples**

The mathematical text is from the subject of trigonometry and its thematic content is: "Graphing the  $\sin x$  function". The science text is from the subject of Oscillations and its thematic content is: "Defining the Linear Harmonic oscillation". Specifically, both texts study the function that models a specific periodic phenomenon. Particularly, in the mathematical text the periodic phenomenon is the rotation of a point M that moves counterclockwise on the unit circle (VR1m) while in the science text is a body that oscillates with the help of a spring (VR1sc). The function that models both phenomena is the sinusoidal function. In Table 1 we illustrate the visual components in each text.

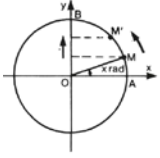
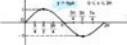
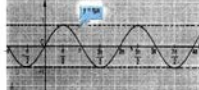
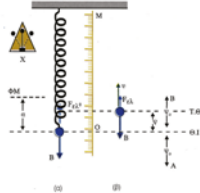
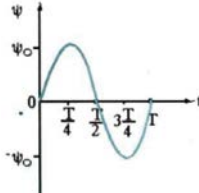
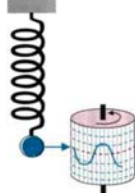
THE VISUAL COMPONENTS OF THE MATHEMATICAL TEXT																																																									
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	<p>Τα συμπεράσματα αυτά συνοψίζονται ως εξής:</p> <table><tr><td>x</td><td>0</td><td><math>\frac{\pi}{2}</math></td><td><math>\pi</math></td><td><math>\frac{3\pi}{2}</math></td><td><math>2\pi</math></td></tr><tr><td>sinx</td><td>0</td><td>1</td><td>0</td><td>-1</td><td>0</td></tr><tr><td>cosx</td><td>1</td><td>0</td><td>-1</td><td>0</td><td>1</td></tr></table>	x	0	$\frac{\pi}{2}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$	sinx	0	1	0	-1	0	cosx	1	0	-1	0	1	<p>Για να ελέγξουμε τις σχέσεις αυτές μπορούμε να υπολογίσουμε τους αριθμούς που δίνονται στον πίνακα παρακάτω:</p> <table><tr><td>x</td><td>0</td><td><math>\frac{\pi}{2}</math></td><td><math>\pi</math></td><td><math>\frac{3\pi}{2}</math></td><td><math>2\pi</math></td></tr><tr><td>sinx</td><td>0</td><td>1</td><td>0</td><td>-1</td><td>0</td></tr><tr><td>cosx</td><td>1</td><td>0</td><td>-1</td><td>0</td><td>1</td></tr></table> <p>Παρατηρούμε ότι η περίοδος του ημιτονικού είναι <math>2\pi</math> και του κοσινονικού είναι <math>2\pi</math>.</p> 	x	0	$\frac{\pi}{2}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$	sinx	0	1	0	-1	0	cosx	1	0	-1	0	1	<p>Εάν θέλουμε να ελέγξουμε τις σχέσεις αυτές μπορούμε να υπολογίσουμε τους αριθμούς που δίνονται στον πίνακα παρακάτω:</p> <table><tr><td>x</td><td>0</td><td><math>\frac{\pi}{2}</math></td><td><math>\pi</math></td><td><math>\frac{3\pi}{2}</math></td><td><math>2\pi</math></td></tr><tr><td>sinx</td><td>0</td><td>1</td><td>0</td><td>-1</td><td>0</td></tr><tr><td>cosx</td><td>1</td><td>0</td><td>-1</td><td>0</td><td>1</td></tr></table> <p>Παρατηρούμε ότι η περίοδος του ημιτονικού είναι <math>2\pi</math> και του κοσινονικού είναι <math>2\pi</math>.</p> 	x	0	$\frac{\pi}{2}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$	sinx	0	1	0	-1	0	cosx	1	0	-1	0	1
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	<p>Μπορούμε επίσης να μετρήσουμε τους χρόνους για τις διαδρομές ΑΟ, ΟΒ, ΒΟ και ΟΑ και να διαπιστώσουμε ότι είναι ίσοι μεταξύ τους (άρα ο καθένας είναι ίσος με T/4).</p> <table><tr><td>Ψ</td><td>t</td></tr><tr><td>0</td><td>0</td></tr><tr><td>Ψ<sub>0</sub></td><td>T/4</td></tr><tr><td>0</td><td>T/2</td></tr><tr><td>-Ψ<sub>0</sub></td><td>3T/4</td></tr><tr><td>0</td><td>T</td></tr></table> <p>Εκτ. 4.1-8. Πίνακας τιμών της απομάκρυνσης σε χαρακτηριστικές χρονικές στιγμές.</p>	Ψ	t	0	0	Ψ <sub>0</sub>	T/4	0	T/2	-Ψ <sub>0</sub>	3T/4	0	T																																												
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Table 1: The visual components in each text.

We can identify two models of periodic motions (the rotational on the mathematical text (VR1m) and the oscillatory on the science text (VR1sc); three value charts (VR2m, VR3m & VR2 sc); three sinusoidal curves (VR4m, VR5m & VR3sc); and one scheme which integrates the above two periodic motions and the sinusoidal curve (VR4sc).

By analysing the verbal mode we could identify four common actions in the development of the presented conceptual field throughout each text. Action 1 refers to how the period T is defined while Action 2 refers on how each periodic phenomenon is studied. Action 3 refers to the generalized outcomes of Action 2 and Action 4 refers to how they apply the new knowledge on specific situations. Action 1, 3 and 4 are comprehended through one mode of reasoning (not always the same in each text) while Action 2 is accomplished through two modes of reasoning in the mathematical and one in the science text. In the following table (Table 2) we present the four actions in the argumentation process, we comment on the modes of reasoning developed and on the visual-verbal relations.

THE VERBAL COMPONENT IN EACH TEXT AND OUR ANALYSIS	
MATHEMATICAL TEXT	SCIENCE TEXT
Action 1: Define the interval T to study the periodic motion	
"Since the function $f(x)=\sin x$ is periodic with period $2\pi$ it is sufficient to study it in an interval that has length $2\pi$ , e.g. $[0, 2\pi]$ ".	"In order to study the oscillation that an object conducts with the help of a spring we need an ideal spring, a compact object, a timer and a tape measure. With the help of a timer we find the T period of the oscillation counting the time for every 'cycle' of route (e.g. AOBOA or OBOAO) and note that it remains constant".

<p>We consider this mode of reasoning as <i>Nomo-logical</i> (N1) since a previously known property of the sine function consists the basis for defining the period to study the phenomenon.</p>	<p>We consider this mode of reasoning as <i>Empirical</i> (E2) since it describes an enactive experience of an experimental activity. The justification proposed is made with the help of empirical measurements.</p> <p>The visual representation (VR1sc) is the <i>starting point</i> on which this mode of reasoning is developed.</p>
<p>Action 2: Study the periodic motion on the interval of period T.</p>	
<p>By reminding the reader that <math>\sin x</math> represents the y-coordinate of the point M(x,y) on the unit circle continues: "<i>We notice that as x values from 0 to <math>\pi/2</math> the point M moves from A to B. Therefore, the y-coordinate increases, thus the function <math>\sin x</math> is strictly increasing in the interval <math>[0, \pi/2]</math>. Similarly, we find that the function is strictly decreasing in the interval <math>[\pi/2, \pi]</math>. [...] Moreover, the function has a maximum value on <math>x=\pi/2</math> (<math>\sin x=1</math>) and a minimum value on <math>x=3\pi/2</math> (<math>\sin x=-1</math>). The results are summarized in the following table</i>" (VR2m).</p> <p>Reasoning in this case is based on a model (the unit circle) and since it starts from actual data (monitoring the y-coordinate of the point M) and ends in general phrasing as presented in VR2m we classify it as logical-empirical (LE2).</p> <p>The role of VR1m in this mode of reasoning is <i>fundamental</i> since the reader must reflect on this VR throughout the thinking process. The table representation (VR2m) <i>organizes</i> the 'steps' of the reasoning.</p>	<p>On the basis of experimental measurements defines the time interval T/4 and invites the reader to record the displacement y(t) in each point. The recorded data are presented in an abstract form on VR2sc. "<i>If we want this information to be more precise we can use chronophotography where the body motion has been photographed several times in different positions during one period. Thus, the value chart is quite thorough so as for the <math>y=f(t)</math> curve (VR3sc) to be designed continuously and assume that it is very close to the real one. This curve has a sinusoidal form which is the characteristic feature of the linear harmonic function</i>".</p> <p>Although the reasoning is based on experimental methods, there is an obvious intention to generalize these outcomes in the last sentence. Moreover, we can trace generalized semiotic elements in the VRs (in the value chart and in the graph representations e.g. <math>\psi_0</math>). These elements led us to discern this reasoning from the empirical ones and characterize it as <i>Logical- empirical</i> (LE2).</p> <p>VR2sc presents how the reader could <i>organise</i> the experimental outcomes while the curve VR3sc is the <i>product</i> in this mode of reasoning.</p>
<p>The text continues by employing mathematical relations (presented on VR3m) in order to sketch the graph of <math>\sin x</math> in the interval <math>[0, 2\pi]</math> (i.e. VR4m).</p> <p>We characterize this mode of reasoning as <i>logical - mathematical</i> (LM) while the tables VR2m and VR3m are the <i>starting point</i> in this reasoning and the sinusoidal curve (VR4m) is the <i>product</i> of this mode of reasoning.</p>	

Action 3: Generalization (Nomo-logical (N2) mode of reasoning since the definition emerges as a result of previous inferences)	
<p><i>"Since the function <math>f(x)=\sin x</math> is periodic, with period <math>2\pi</math>, the curve has the same shape in the intervals <math>[-2\pi, 0]</math> [...]. So, we have the following graph which is called sinusoidal function (VR5m)".</i></p> <p>Now VR4m is the initial situation, the starting point, while VR5m is the product.</p>	<p><i>"Linear Harmonic oscillation is the oscillation that an object performs when its orbit is on a straight line and its displacement is a sinusoidal function of time"</i></p> <p>In this case VR4sc has an illustrative character.</p>
Action 4: Applying the new knowledge in specific situations (LE1 mode of reasoning)	
<p><i>"We know the opposite angles have opposite sines. Hence, for every <math>x \in R</math> <math>\sin(-x) = -\sin x</math>. This means that the function is odd and hence the graph has a point symmetry on <math>0(0,0)</math>".</i></p> <p>In this mode of reasoning VR5m has an explanatory role.</p>	<p>Supports the definition by providing another example of a modified experiment as follows: <i>"If we modify the experiment, we can directly see the curve that we have previously formed if we adjust a stylus to the object [...]" (VR4sc).</i></p> <p>VR4sc exemplifies the mode of reasoning developed.</p>

Table 2: The Verbal component and our analysis

## CONCLUDING REMARKS

Our analysis on the one side identifies common modes of reasoning (when empirical data are the basis for logical conclusions or when exemplifying these conclusions on specific situations). Even though, these common modes of reasoning are impended in the goals of each subject they could establish common reasoning behaviours in the different educational fields. On the other side, we identified modes of reasoning in the mathematical text (Nomo-logical (N1) and Logical-mathematical (LM)) that are absent in the science text. This absence influences the argumentation process since additional examples are provided to persuade and convince the reader on the produced knowledge. These differences in argumentation are not only identified in the specific examples but throughout our extended data. Besides, the presence of different routes in reasoning when defining the period T (nomo-logically (N1) i.e. relying on previously established statements and experimentally (E2) i.e. making genuine links to the real world) could motivate and allow students to broaden and enrich their perception of this notion.

By analyzing the visual-verbal relation on each mode of reasoning we recognize that common type of Visual Representations serve common purposes in different modes of reasoning (e.g. the sinusoidal curves as reasoning products or the tables as organizing tools). Furthermore, our analysis illustrates how the visual representations change their

character in subsequent modes of reasoning (e.g. from product to the starting point of reasoning). This changing character on the one hand contributes to the cohesion of the text argumentation while on the other highlights the 'flexible' character of the visual components. Moreover, the visual product of argumentation in mathematics (the sinusoidal curve) acts as a prototypical image of the sine function in the science text. This piece of evidence establishes the freedom of the mathematical visual images to travel across subjects and possibly contribute to the formation of the invariant notion of periodicity as described by Vergnaud (2009).

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# TEACHERS' REFLECTING ON THE NATURE OF THE MATHEMATICAL KNOWLEDGE UNDER CONSTRUCTION IN THE CLASSROOM

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*In this paper we examine a primary teacher's mathematics knowledge for teaching as evidenced in the way she manages the epistemological features of the subject matter in the context of her teaching as well as in her reflections emerging during an interview focusing on this management. The results of the analysis reveal important aspects of how this management together with the teacher's teaching concerns shape her mathematics knowledge for teaching.*

## THEORETICAL CONSIDERATIONS

The need for teachers to know mathematics differently from mathematicians and other professionals is well recognised. The relevant research in mathematics education has been shaped by Shulman's ideas (1986), who attempted to link subject matter knowledge with the knowledge needed for teaching. A considerable body of this research has dealt with teachers' subject knowledge distinct to teaching and its limitations (wrong procedures or definitions, inappropriate metaphors, or other mathematical mistakes), while another significant part concentrated on the impact of this knowledge on teaching (Hill et al, 2008).

Despite the fact that Shulman's work provided a broad framework of making sense of the studies focusing on the nuances of mathematical knowledge in the teaching practice, there has been criticism for overlooking "the social production of mathematics for teaching" (Adler & Huillet, 2008, p. 195) implying that it might be time for abandoning it in favour of "a more grounded and expanded theorization of mathematical knowledge in practice" (Ruthven, 2010, p.92).

Adopting a practice-based approach, Ball et al (2008) attempted to specify the mathematical knowledge needed for teaching (MKT), relating it to the quality of the latter. According to them this special knowledge combines PCK (knowledge of content and students, content and teaching, content and curriculum) with SMK (entities associated with content knowledge, specialized content knowledge and horizon content knowledge - an awareness of how mathematical topics are related in the curriculum-). Remaining within a practice oriented framework, Rowland and his colleagues employed a grounded theory approach to analyse teachers' mathematics content knowledge as evidenced in their teaching (e.g., Turner and Rowland, 2010). This led to the identification of four broad dimensions of this knowledge, which they named 'knowledge quartet', a construction used to further explore this type of knowledge in practice.



A considerable part of the research focusing on MKT carried out in the last decade concerns teachers' development of this knowledge (Wilson et al, 2012). For example, Silverman and Thompson (2007) argue that this happens when a teacher uses a developmental understanding of a mathematical idea. Similarly, Silver et al (2007) advocate that the use of practice-based materials promotes teachers' MKT. In general, the issue of MKT attracted much attention as it is closely related to teachers' professional development, particularly in the light of recent reforms around the world. However, the mathematical meaning shaped in the classroom is also related to matters only indirectly covered by the above approaches. For example, how a teacher connects the specific to the general, how s/h differentiates a mathematical object from others, how s/he allows for the properties of a mathematical entity to emerge and consolidated within the classroom, and so on. It might be argued that the approaches concerned with the mathematical quality of instruction, e.g., Hill et al (2008), somehow incorporate these aspects by being sensitive to matters of richness and rigor of the lesson, presence of mathematical explanation and justification, mathematical representation, etc. However, the exact content of these facets of the mathematical knowledge under classroom construction as well as teachers' relevant understanding need further specification. Although the work briefly discussed above «offers an important view of MKT, it has not led to consensus regarding the nature of this knowledge» (Chapman, 2012, p.118), leaving the question of its content and its understanding by the teachers open to further exploration.

Mathematics is a discipline of special nature. Its concepts are theoretical objects related to one another. It uses definitions to identify and differentiate these objects; it studies attributes and relations and uses theorems to present them. It also follows certain processes as means of management of objects and relationships and produces new objects. Pupils gradually become acquainted with these objects and their properties, whereas their teacher's classroom management attaches meaning to all these. In a series of studies we examined this management aiming at identifying its impact on the epistemological status of the knowledge under construction in the classroom. The results showed that teachers tended to replace definitions with morphological descriptions, reducing arguments to properties and altering a solution process to a course of operations. As a result, the activity developed in the classroom had none of the epistemological features characterizing mathematics, thus affecting the quality of mathematics development (e.g., Kaldrimidou, Sakonidis & Tzekaki, 2008). In the present paper, on the basis of the above, recognizing that teachers' classroom management of the epistemological features of mathematics constitutes a significant learning site for both pupils and teachers and taking into account the centrality of reflection in teachers' learning and generally development, we embark to analyse teachers' mathematical knowledge for teaching through a parallel examination of their practice and reflection on it.

## **THE STUDY**

The data utilized in the study come from a large project following the development of a new mathematics program of study promoting active learning and using mathematics

to understand and critically advance personal as well as social life. More than 150 schools were selected from all over the country by the Ministry of Education to pilot the new program. Here the focus is on a small group of primary teachers chosen on the basis of their teaching experience and personal professional development profile implementing units of the new syllabus over a school year.

The subject of the case study constituting the focus of the present paper is Antigoni, a female mathematician and primary teacher with 10 years of teaching experience, professionally highly active, who teaches in one of the pilot schools, in fact, an experimental primary school in the northern part of the country. Over the year, she discussed, designed, implemented and evaluated a series of lessons in collaboration with her colleagues and under the supervision and support of an advisor/consultant. The lessons, the meetings as well as a number of interviews were taped and transcribed providing the data for the study. For the purposes of this report, one out of four transcribed lessons on fractions taught by Antigoni to her fourth grade class is utilized as well as a semi-structured interview on aspects of her teaching management.

The research problem pursued was to identify specifics regarding teachers' classroom management of the epistemological features of mathematics and examine the way they make sense and explain these aspects of their management.

A combination of content analysis and grounded theory approach was used to analyse the transcribed lesson and the discussion developed in the interview. The former provided the latter with characteristic management episodes which were used to encourage the teacher's reflection on them targeting at revealing her own MKT.

## **RESULTS**

The lesson focuses on approaching fractional (rational) numbers through specific situations suggested by the teacher, where these numbers express parts of a whole. To this purpose, the students are first invited to negotiate the problem "can  $\frac{1}{2}$  of a pizza be less than  $\frac{1}{3}$  of another pizza?" and then, using the same representations (same pizzas – same circles), to construct fractional numbers via the repetitive addition of fractional units, the comparison of fractional units and later of fractional numbers.

The analysis of the transcribed lesson concentrated on sections/episodes where mathematical entities were under negotiation (definitions, procedures of proving/justifying and generalizing), in parallel with the teacher's explanations and connections exploited with regard to two special epistemological characteristics: specific/general and definition/properties relationships. Due to lack of space, we focus here only on episodes related to the management of definitions and generalisations.

In the following, some indicative episodes are presented, followed by the teacher's comments and reflections on them provided in the context of the interview.

### **1. Managing definitions episodes**

Antigoni offers an introduction to fractions in the following way.

T(eacher). ... Tell me, what is the difference between fractions and natural numbers? ...  
How do they differ? ... Are they the same numbers?

S(tudent). The fractional numbers ...can be ... That is, we have a cake and we cut it in six pieces and take one ... This is  $\frac{1}{6}$ . The natural numbers are 1, 2, 3, ... up to infinity!

T. Good! ...

Here the student presents the number with reference to a specific example (cake) accompanied by descriptive characteristics rewarded by the teacher with no reference to other features. Commenting on this management in the interview, Antigoni argued that:

*“I attempt a connection with the pre-existing knowledge, that is, with the natural numbers, because I believe that ... it is very important, to see if the students recourse to this knowledge, which is clearer” (comment related to the use of descriptive characteristics and the connection with natural numbers). She added that she wants her students “to feel safe ... and they find safety in drawings in their body... This particular student couldn’t deal with the question on a general level; the familiar was there, in cakes, in pizzas”.*

It is clear then that the choice of specific situations and representational means is a conscious one and the criterion behind this choice is the shaping of a familiar and safe environment for the students.

This is also the case when Antigoni works later on comparing fractions.

T. How do we call this in mathematics, same pizzas? How do we call it? ... We have said it before!! The fractions, in order to compare them, they must be pieces of ... of...?

S. Same number!

T. Same number! How do we call this number?

S. Fraction!

T. Fractions! They should be pieces of the same ... whole...That is, the whole which gives us the piece... The whole must be....?? (Draws on the board) ... I believe I drew them the same. Eh? Ok! There exists a small difference! ... In any case, we compare fractional numbers when they have the same size!

There is a confusion here related to the specific-general relationship: the fraction must be part (i.e., specific) of the same number (i.e., general) and of the same whole. So, fractional numbers should have the same size. As a consequence, the number changes to quantity). Thus, the reliance on descriptive elements to define fractions (mathematical object) leads to presenting them mainly as parts of a whole (quantity), i.e., to distorting the nature of the fractional concept.

This particular management, which limits conceptually the notion of a rational number, is not only due to Antigoni’s concern to create a ‘safe’ learning environment for her students reported above. It appears to also reflect her own relevant conception as indicated in her commenting on the episode in the interview:

*“the activity was dealing with the concept of fraction, which is the whole and how we compare... that the fraction is essentially related to something, says nothing on its own ... and when this quantity changes [...]. I do not know if it is right or wrong... I thought you need a model that students can see, this sharing into 8 pieces, in the restaurant, at home...*

*To first become clear that a whole is shared ... These primordial ideas of a fraction (laughs)... to start from there”.*

Following a similar way and reasoning, Antigoni tries to define fractional units below.

T. So, using ‘1’, we generated the natural numbers. All of them?

Students: Yes!!!!

T. Yes!...Why do we call these fractional units?

S1. Because they are fractions!

T. Because they are fractions!

S2. The ‘1’, with ‘1’ ...

T. Do you mean to say how ‘1’ came up? (the student waves ‘yes’). We decided to call the whole ‘1’. Ok? And with this we generated these numbers, which we called? ... Naturals! What did we decide here? That we will call  $\frac{1}{2}$  fractional unit and, using this fractional unit, we can construct all fractional numbers (gives cards with fractional numbers). Let’s see which fractional numbers we can make.

Antigoni encourages the presentation of the ‘fractional unit’ via the use of ‘1’ and the construction of the fractional numbers in accordance with the construction of natural numbers. This way, the definition of fractional units acquires procedural characteristics and the connection to the natural numbers reinforces conceptions which present difficulties to students’ dealing with fractions: the discrete structure, the existence of a preceding and a following number, and so on. These are features characterizing natural numbers, differ epistemologically different from those in power for the set of fractional numbers.

Reflecting on her management of these features, Antigoni seems to be aware of their functioning arguing:

*“(Natural numbers) are more familiar to the pupils, they constitute a measurable, discrete set, etc... All these count... Fractional numbers constitute a very hard set of numbers, it is dense, includes different concepts... involves classes of equivalence. How can you deal with all these? It was difficult to negotiate all these ... We needed a frame of reference, something like this... This was more familiar”.*

It is clear that Antigoni knows well the rational numbers and she is aware of their conceptual complexity. However, she returns to the need for the existence of a (conceptually) familiar setting for the students, confirming her relevant view.

## **2. Managing generalizations episodes**

In closing the lesson, the teacher attempts to lead pupils to generalize and come up with a conclusion.

T. But there must be something in order to be able to compare! What have you noticed?  
How did I place the fractions in order to be able to compare? What is common in each case?

S. Either the denominators or the numerators are the same!

T. When the numerators are the same, which fraction is larger?

S. When the numerators are the same, you eat more when there are fewer pieces!

T. Listen to what Spyros says! When the denominators are the same, when do you eat more?

S. When the denominator is smaller!

T. Smaller!! Whereas, when the denominators are the same, when do you eat more?

S. You look whether the numerator is bigger!

Until the end of the lesson, both the teacher and the students remain faithful to referring to pizza and to the quantity “we eat” (specific and procedural). Antigoni comments in the interview:

*“Hmm... Spyros took for granted that the denominators are the same! ... In general, at the beginning, they could not look at both terms, they were getting confused. We had grasped the case where both numerators and denominators were the same: we had to simply divide the same thing in a different way”.*

There is also a morphologically intense management of fractions in the above extract, placing emphasis on ‘the top’ (numerator) and ‘the bottom’ (denominator) numbers, which promotes dealing with them as independent entities and not in terms of their interrelationship which characterizes fractional numbers. Antigoni, reflecting on this matter in the interview, claims:

*“I am trying to introduce both terms in the discussion, but they keep one as given ... So, ‘you look at the numerator, if it is bigger’... If I had the time (laughs)... How do we compare natural numbers? It is known. I was expecting them to tell me how to compare fractional numbers in the two cases, when the numerators are the same and when the denominators are the same. That’s all I wanted”.*

It is evident that for her this was a rather straightforward situation. This is possibly why she did not encourage a more general negotiation, which is attempted in concluding the lesson:

T. I want you to tell me, then... Are fractions numbers? Can we see which one is bigger and which one is smaller?

Antigoni is trying to increase the level of generalization achieved by reversing the question originally set in dealing with comparing fractions to “*are fractions numbers?*” (not that we can compare them, because they are numbers, but because we can compare them, they are numbers). However, this leads children nowhere. She comments in the interview:

*“No, I didn’t expect them to tell me in general, but only for the two specific cases, the ones we worked on. If they could have at least reached this”.*

Summarizing the epistemological features of the mathematical entities emerging in Antigoni’s didactic negotiation of fractional numbers, it is noticeable either an emphasis on procedural-constructive-morphological features or a continuous reference to the specific situation or even an inversion of the relationship definition/property

resulting from the definition. Her teaching management choices do not seem to be due to mistaken or imperfect mathematical knowledge. However, they occasionally indicate the lack of awareness of the importance and complexities involved in proving and generalizing procedures for the construction of the mathematical meaning. This can be attributed either to an ignorance of the significance of this construction for students' learning or is driven by her intentions and persistence to create and maintain a "familiar environment of reference, where students feel safe", as she argues in her interview.

## CONCLUDING REMARKS

We believe that the analysis of the data on the basis of the teacher's management of the epistemological features of mathematics contributes to further specify MKT and appreciate the mathematical quality of a lesson (Ball et al, 2008; Hill et al, 2008; Silverman & Thompson, 2008; Silver et als, 2007). According to the results of this analysis, Antigoni first makes an effort to connect the specific to the general and to explain by attaching mathematical meaning to situations. Following an epistemological management (EM) perspective, we deepened into the specific epistemological features providing meaning to the conceptual construction of fractional numbers (their definition and the process of generalisation). This analysis allowed us to recognize that the epistemological management of the teacher promoted viewing these numbers as an expansion rather than as a re-organization of the set of natural numbers.

Antigoni's answers on her management practices of the epistemological features (in the interview) revealed occasionally lack of concern in relation to the meaning emerging in the classroom and in other times the conscious use of situations which, although too specific, make sense to the students. Her choice of explanations, connections and situations to deal with as well as of desirable procedures seems to result from her own conception related to the need of ensuring a safe and familiar environment for the students. This goes along Silverman and Thompson's (2007) finding that a teacher takes into account what students might understand about the idea and the ways that new understanding positions students to approaching other mathematical ideas. Such an approach leads to the distortion of the nature of the entities under construction in the mathematics classroom, with serious implications for students' learning, thus constituting a crucial issue for teachers' development that deserves systematic study.

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# ARGUMENTATION IN UNDERGRADUATE MATH COURSES: A STUDY ON DEFINITION CONSTRUCTION

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*The purpose of this study is to analyze the complex argumentative structure in undergraduate mathematics classroom conversations during definition construction by taking into consideration students' and teacher' utterances in the classroom using field-independent Toulmin's theory of argumentation . The analyses contributed to an emerging body of research on classroom conversations.*

## INTRODUCTION

Definitions express the properties that characterize the 'objects' of the theory and relate them within a net of stated relations (Mariotti & Fischbein, 1997). In constructing definitions arguments must be brought to support the thought processes in constructing them. In mathematics education literature the argument concept is used in the sense of justifying a conclusion based on a data (Toulmin 2003; Mejia-Ramos & Inglis, 2009; Knipping, 2008). On the other hand, argumentation is a verbal and social activity of reason aimed at increasing (or decreasing) the acceptability of a controversial standpoint for the listener or reader, by putting forward a constellation of propositions intended to justify (or refute) the standpoint before a rational judge' (van Eemeren et al, 1996).

As Toulmin's model is intended to be applicable to arguments in any field, it has provided researchers in mathematics education with a useful tool for research, including formal and informal arguments in classrooms (Knipping, 2008). Studies using Toulmin model focused on analyzing students' arguments and argumentations in proving processes in a classroom (Knipping, 2002, 2008; Krummheuer, 1995) and, individual students' arguments in proving processes (Pedemonte, 2007). Toulmin himself noted that his ideas has no finality. Indeed his model has been reshaped in various ways, his claims have been contested by some and in response reformulated by others, and some but not all aspects of his approach have been incorporated in applications in different domains (Hitchcock & Verheij, 2006).

Having established these facts, the goal of our research is to study the argumentation in undergraduate mathematics classrooms during definition construction using Toulmin's theory of argumentation. Specifically, the aim is to analyze the structure of the arguments accomplished in the course of interaction where the teacher and students involvement in this accomplishment. This study is part of a wider study investigating the argumentation generated in undergraduate mathematics classes while proof generation (see Ubuz, et al, 2012), definition construction, and problem solving. Here we concentrate only definition construction because of page restrictions. This paper suggests a method by which complex argumentation in defining processes can be reconstructed and analyzed. Analyzing students' and teacher' utterances in the



classroom according to Toulmin model allows us to reconstruct argumentations evolving in the classroom talk since arguments are produced by several students together with the guidance of the teacher.

## THEORETICAL FRAMEWORK

In the following sections we will expose some theoretical considerations on the Toulmin model, and the definition construction process.

### The Toulmin Model

According to Toulmin, an argument is like an organism. It has both a gross, anatomical structure and a finer, as-it-were physiological one (Toulmin, 2003). He is interested in the finer structure. The Toulmin model is differed from analysis of Arisitotle's logic from premises to conclusion. First, we make a claim(C) by asserting something. For the challenger who asks “What have you got to go on ?”, the facts we appeal to as foundation for our claim is called data (D) by Toulmin. After producing our data, we may being asked another question like “How do you get there ?”. He notes, at this point we have to show that the step from our data to our conclusion is appropriate one by giving different kind of propositions like rules, principals, inference – licenses or what you will, instead of additional items of information (Toulmin, 2003). A proposition of this form Toulmin calls a warrant (W). He notes that warrants are of different kinds and may confer different degrees of force on the conclusions they justify. We may have to put in a qualifier (Q) such as “necessarily”, “probably” or “presumably” to the degree of force which our data confer on our claim in virtue of our warrant. However there may be cases such that the exceptional conditions which might be capable of defeating or rebutting the warranted conclusion. These exceptional conditions Toulmin calls as rebuttal (R). For our challenger may question the general acceptability of our warrant: “Why do you think that?” Toulmin calls our answer to this question our backing (B) (Hitchcock & Verheij, 2006). The diagram of the Toulmin model is as follows :

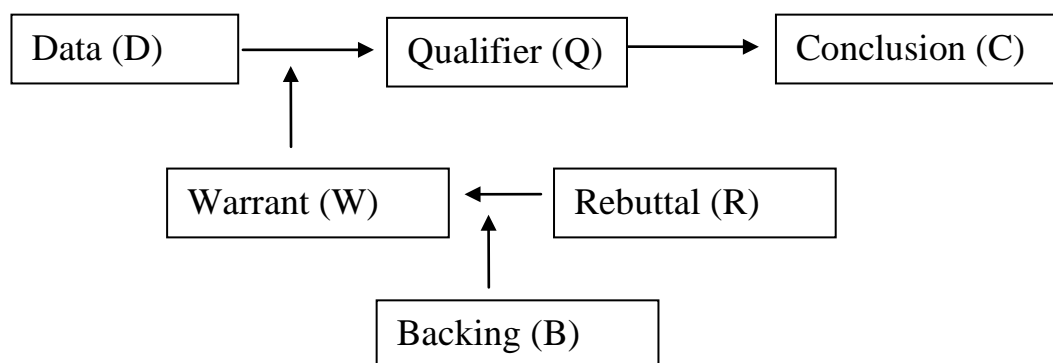


Figure 1: The Toulmin Model

Reconstructing and analyzing the complex argumentative structure in classroom conversations follow their own structure. For example, careful analyses of the types of warrants (and backings) that students and teachers employ in classroom situations allowed two distinctions in the justifications: visual and conceptual (Knipping, 2008). The warrants and backings based on conceptual aspect or deductive are mathematical

concepts or mathematical relations between concepts, and make reference to theorems, definitions, axioms and rules of logic. The warrants and backings based on visual or figural aspect make reference to figures as part of the argumentation.

### **Definition Construction Process**

Defining is a basic component of mathematical knowledge, and learning to define is a basic problem of mathematical education (Mariotti and Fischbein, 1997). Ouvrier – Buffet (2006) reported that Situation(s) of Definition Construction gave the opportunity to work on scientific processes (construction of definitions and proof in particular). “Scientific processes are constituted by students’ experiments with different cognitive attitudes: doubting, conjecturing, refuting, generating new counter-examples, testing etc” (Ouvrier – Buffet, 2006; p.279). Ouvrier – Buffet (2006) and Mariotti and Fischbein (1997) have pointed out that the intervention of the teacher plays a central role in guiding the discussion and mediating the definition construction process. In addition, Ouvrier – Buffet (2006) have emphasised that active connections have to be underlined in defining processes.

### **METHODOLOGY**

Data were collected through nonparticipant observations that were videotaped. Observation was conducted 2009-2010 spring semesters in real analysis course for eight weeks, and 2010-2011 spring semesters in advanced calculus course for six weeks, offered to mathematics education student at the third and second years, respectively. These courses were selected as both formal and informal argumentations were at the focus of these courses. In these courses the number of students were 45 and 40, respectively. Formal proof approaches are given to the students at the “Abstract Mathematics I - II” courses provided in the first year. In these courses, students learn what a proof is and how to prove theorems. That is, they learn how to argue mathematically, justify their claims and encounter the cases named “counter example” for the first time which rebuttals their claims.

The analysis of the observations is based on the transcripts. As Toulmin(2003) noted, “an argument is like an organism. When set out explicitly in all its detail, it may occupy a number of printed pages or take perhaps a quarter of an hour to deliver; and within this time or space one can distinguish the main phases marking the progress of the argument from the initial statement of an unsettled problem to the final presentation of a conclusion” (p. 87). Based on this explanation, eleven argumentations were determined and two of them were on definition construction. These two argumentations were observed in real analysis course.

Observations were conducted by the second author. He analyzed the transcripts by marking the progress of the argument from the initial statement to the final conclusion through using Toulmin model components. He noticed that some aspects of observed argumentations were overlooked. He modified the Toulmin model by integrating guide – backing and guide – redirecting additional components which were observed in almost all argumentations. We called an approval given by teacher to the warrants, backings or intermediate conclusion as guide – backing. When the argumentation does

not start from a right point or students get stuck on an argument point, teacher intervenes with an example, a question or a suggestion to arrange the argument. We called such intervenes as guide – redirecting.

Having discussed with the first author who is a full professor in mathematics education and doing research on proof, it was decided that observed argumentations could be considered into three classes: proof generation, definition construction, and problem solving. She also noted that some components could be classified in itself. After re-analyzing observed argumentations, *warrant* component were divided in two categories: *deductive warrant* and *reference warrant*. Students appeal reasoning like numerical computing, applying a rule to an inequality, creating new ideas from a definition, a theorem or a rule in producing their warrants. We called this kind of warrants as *deductive warrants* as Inglis et al. (2007) did. When a warrant referred to a theorem, a definition, a rule or a problem, we called such a warrant as *reference warrant*. *Guide – backing* was divided into three categories: *approval*, *reference* and *terminator*. When teacher just approve the students' warrant, backing or conclusion by saying “good, fine, great, well done” and does not use any mathematical phrase, we called this kind of guide backing as *approval guide backing*. When teacher approve the students' warrant, backing or conclusion by referring a definition, a theorem or a problem recently solved, we called this kind of guide backing as *reference guide backing*. Argumentations come to an end when teacher or students reach the final conclusion to be achieved. In case, teacher reaches the final conclusion, students convince that the conclusion is legitimate. In case, students reach the final conclusion, teacher serves a backing. This backing shows the final conclusion and we called it as *terminator guide backing*. One important point that must be noted here is that argumentations were not analyzed according to their mathematical correctness.

Finally, full transcriptions together with analysis model components explanation are provided to an external auditor who is a researcher in mathematics education field. After a week, the auditor completed her analysis and a complete consensus was reached on analysis of argumentations.

## RESULT

Two definition construction cases constituted two different argumentation context. In this paper only one case is considered as an example because of page restrictions. Here we analyze a transcript of an argumentation in which deductive warrant, guide – redirecting, approval guide – backing and terminator guide – backing appear. The following argumentation occurred when defining distance from a point to a set.

- 1 Teac : Well...Our goal is how to define the distance from a point to a set . At first let's say our work is on  $\mathbb{R}$ . Let  $A$  be a set on  $\mathbb{R}$ , say  $A = [-2, 2)$  and  $P$  be a point in the exterior of  $A$ . Well...Fatma, what is your comment on the distance from  $P$  to  $A$ .

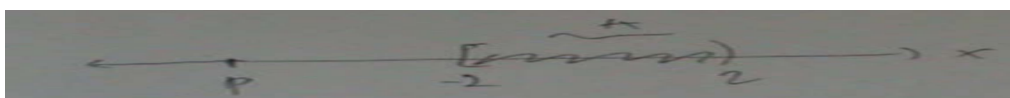


Figure 2 : The point  $P$  and the set  $[-2, 2)$

- 2 Fatma : I think, one is from  $P$  to  $-2$  and the other one is from  $P$  to  $2$ .

- 3 Teac : Yes
- 4 Fatma : Umm...I am thinking...The distance we are looking for should be between these two values but not predicting its exact value.
- 5 Teac :Are you saying “I am not predicting it” or something symbolic ?
- 6 Fatma : Well, I am taking the absolute value of  $2 - P$  and  $-2 - P$ .
- 7 Teac : You are focusing on  $|2 - P|$  and  $|-2 - P|$ .
- 8 Fatma : I think that the distance we are looking for should be one of them.
- 9 Teac : Which one then ?
- 10 Fatma : Umm...I think that the distance we are looking for should be between them.
- 11 Teac : Your thoughts are essentially true. Well, you calculate  $|2 - P|$  and  $|-2 - P|$  and according to you one of them is the distance we are seeking. So, which one would you think?
- 12 Fatma : It could be the small one or the big one or between them since A has elements between -2 and 2.
- 13 Teac : Let’s take an example from real life. Suppose that we are travelling to Istanbul. Seeing the city boarder on our way then you would say that “we are in Istanbul”, right?
- 14 Class : The closest one.
- 15 Teac : The closest one ? What does it mean in mathematics ?
- 16 Ahmet : The smallest one.
- 17 Teac : Fine, nice. Well, what if our set A is in that form (see Figure 3)



Figure 3 : The point P and another set

- 18 Zeynep : Calculate distance from P to every point of A which means taking the absolute values. Then the minimum of these values is the distance from P to A.
- 19 Teac : Well done! That’s it! Her thought is correct.

In line 2, Fatma considered the distances from P to 2 and from P to -2 as her data. Based on these data, in line 4 she concluded that the desired distance from P to A must be between these distances. The teacher, in line 5, gave a guide – redirecting by expecting a symbolic statement. Thereupon Fatma modified her data by using absolute value concept and pointed out in line 8 that the desired distance must be one of the absolute values, but in line 10 she returned to her old conclusion. In line 13, the teacher gave first but weak approval guide – backing for Fatma’s modified conclusion in line 11 and a guide – redirecting by asking her what it happen to be. In line 12, Fatma seems a bit confused. She combined her old conclusions in line 8 and 10 and produced a new conclusion. That A has elements between -2 and 2 and used it as a deductive warrant for this new conclusion. In line 13, teacher gave an example from real life as a guide – redirecting. Hereafter most of the students reached the closest point to P as conclusion in line 14. Teacher gave another guide – redirecting by asking the meaning of ‘the

closest' in line 15. Ahmet, in line 16, responded it as the smallest absolute value which means his conclusion. In line 17, teacher gave an approval guide – backing by saying 'nice, fine' and a guide – redirecting by asking another form of A. In line 18, Zeynep gave the desired definition and got terminal guide – backing in line 19. The diagram corresponding to the argumentation above is provided in Figure 4.

## **CONCLUSIONS**

The intervention of the teacher plays an important role in definition construction process as pointed out by Ouvrier – Buffet (2006) and Mariotti and Fischbein (1997). Construction of definitions and proof in particular is a basic component of mathematical knowledge. Teacher acts as a guide who exactly knows the path to follow i.e. where to start and to end the argumentation. During the argumentation if students follow the wrong path, get a false intermediate conclusion or get stuck in a point, teacher intervene the students to put them on the path in which they have to follow.

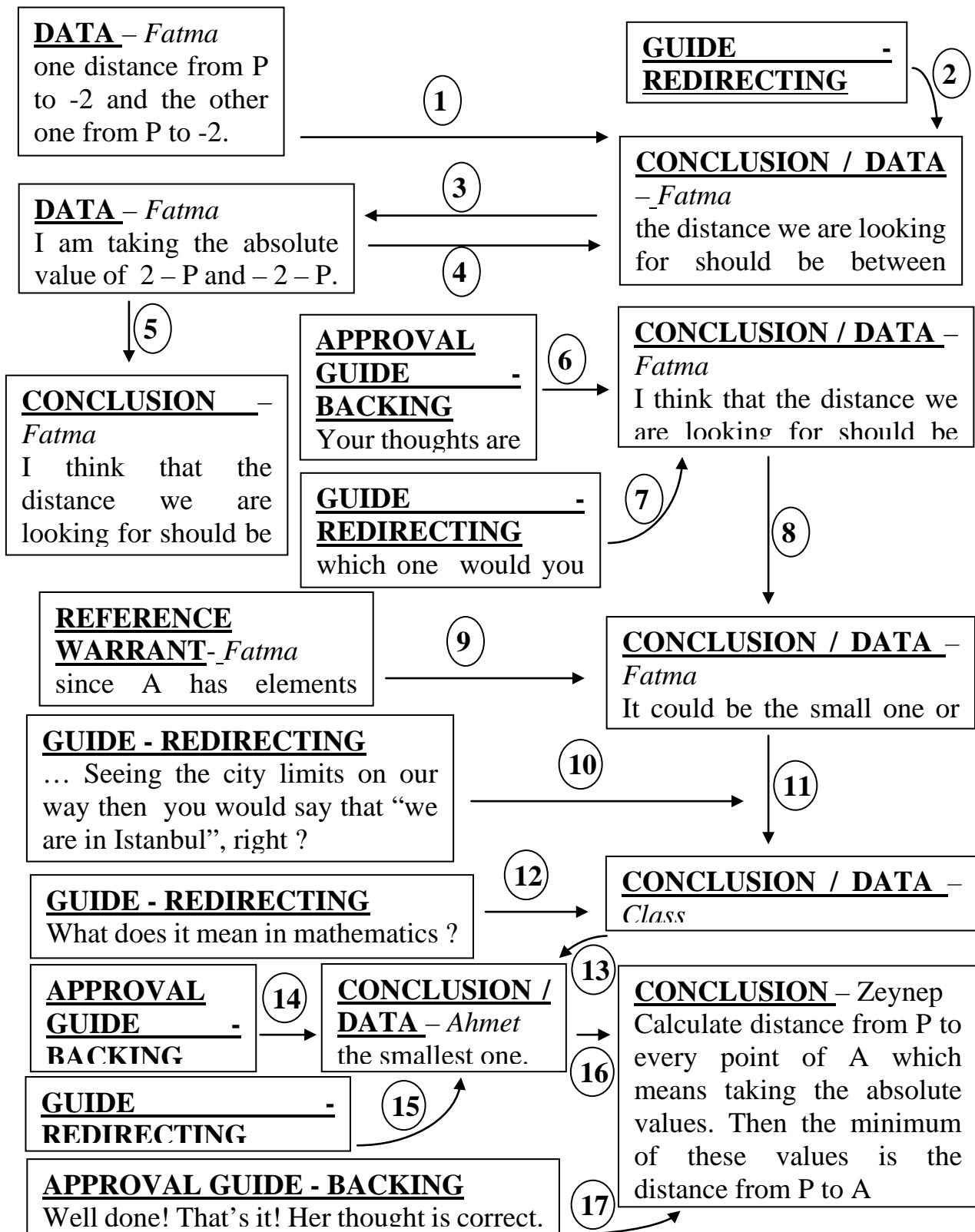


Figure 4: The Toulmin model of the argumentation

Based on our observations, the teacher played a role in argumentations by doing guide – backing and guide – redirecting. Mathematical process of defining incorporated two of the three categories of guide – backing component: approval and terminator. Careful analyses of the types of warrants that students and/or teachers employ in defining process allowed us to identify and classify only deductive warrant but not reference

warrant. On the other hand, reference guide backing and reference warrant are encountered frequently in mathematical process of prove generation (Ubuz, et al, 2012).

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# THE ROLE OF EXTERNAL REPRESENTATIONS IN SOLVING MULTIPLICATIVE PROBLEMS

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*Four groups of third graders solved two kinds of multiplicative problems: equal group and Cartesian product problems. A first group solved the problems without any external representation, a second group was asked to create an external representation, a third and fourth group received representations that showed the semantic structure underlying the problem, whereby the representations in the fourth group were less adequate than those in the third group. For Cartesian product problems, pupils benefitted from the provided representations (regardless of adequacy), while asking pupils to create a representation even had an opposite effect.*

## THEORETICAL BACKGROUND

### Multiplicative reasoning

Research (e.g., Davydov, 1991) has repeatedly shown that multiplication is considerably more difficult for young children than addition and subtraction. This is particularly the case when children are brought in situations in which the mathematical model underlying the problem is not straightforward (for instance, when solving a word problem), and they have to choose themselves an arithmetical operation. Children then often approach multiplicative situations in additively (e.g., Hart, 1981). There is some dispute about whether children initially go through an additive stage before developing the ability to reason multiplicatively (Nunes, Bryant, & Watson, 2009), but repeated addition is anyhow considered the primitive model of multiplication held by many children (Fischbein, Deri, Nello, & Marino, 1982).

Repeated addition is a model that describes very well multiplicative situations in which equal groups are involved. However, this is only one of classes of situations in which multiplication can occur. Several authors (e.g., Greer, 1982; Vergnaud, 1983) highlight a wide range of such classes: Besides equal groups, they distinguish equal measures, rate, multiplicative comparison, multiplicative change and Cartesian product problems. As will be explained below, the study in this paper will focus on only two types, namely equal groups problems and Cartesian product problems.

“5 children each have 7 marbles. How many marbles do they have altogether?” is an equal groups problem. It describes a situation in which there is a certain number of groups, each consisting of the same number of elements. The two numbers in the problem have a different role: The number of groups (children, 3) acts as the multiplier, and acts on the number of elements in each group (marbles, 7), i.e. the multiplicand. For this reason, the situation can be considered asymmetrical (Greer, 1992).



“Marc can choose between 3 shirts and 7 ties. How many combinations can he make?” is a Cartesian product problem. In this multiplicative situation, the product is defined by the number of unique pairs that can be composed by an element of the first collection (shirts, 3) and an element of the second collection (ties, 7). In this case, both collections take the same role, and thus the situation can be considered as symmetrical. It is therefore not meaningful to discern a multiplier and a multiplicand (Greer, 1992).

### The role of external representations

When solving mathematical modelling tasks such as word problems, the creation and/or use of an external representation of the problem situation can be a beneficial heuristic in order to reach a solution. As Van Essen and Hamaker (1990) point out, creating a schematic representation of a word problem may lead to a reduction of working memory load afterwards, because some information is embedded in the representation. Moreover, the representation may reorganise the information in the problem in such a way that the mathematical relationships underlying the problem are made more explicit. Cox (1999) stressed that for these elements to be effective, the external representation needs to be a correct display of the problem situation.

Specifically for multiplicative situations, there are many representations that may help the problem solver to discover the multiplicative nature of the situation and to arrive at the correct solution to the problem. Among the representations mentioned in the literature are the group diagram, bar representation, tree diagram, roads model, number line, and rectangular array (Goffree, 1982; Greer, 1992).

Given that our study focuses on equal groups and Cartesian product problems, the group diagram and rectangular array (for examples, see Figure 1) are particularly relevant representations. The group diagram is generally considered to be a good way to show the semantic structure of equal groups problems: The repeated addition model of multiplication is clearly visible in the group diagram (Anghileri, 2000), and the asymmetry between the multiplier (the number of groups) and the multiplicand (the number of elements in a group) is very prominent (3 groups of 7 items look very different from 7 groups of 3 items). A rectangular array more adequately represents the structure of a Cartesian product situation, and particularly the symmetry that exists between the two dimensions involved in the situation (Skemp, 1986). Both dimensions in the representation are interchangeable and the array representation would not look substantially different when  $7 \times 3$  instead of  $3 \times 7$  would be represented.



Figure 1: Examples of a group and rectangular array diagram for  $3 \times 7$

### Focus of the current study

As mentioned above, our study focuses on equal groups and Cartesian product problems, and on the group and array representations. We had two important reasons.

First of all, there is a pronounced difference in terms of semantic structure between the two problem types, as was already explained above: Equal groups problems are asymmetric in nature while Cartesian product problems are symmetric. It was also already explained that some external representations –such as the group diagram– are inherently asymmetric, while others –such as the rectangular array– are symmetric. So, it can be argued that a group diagram is more adequate to represent the situation described in an equal groups problem, while a rectangular array more adequately represents the situation in a Cartesian product problem<sup>1</sup>.

A second difference is that for younger children equal group problems are easier than Cartesian product problems. The structure of equal groups problems closely fits to the primitive model of repeated addition that children typically have of multiplication. It is also this type of problems that often appears in the earliest years of primary education in order to illustrate to children situations in which multiplication is applicable (Greer, 1992), and performance on such problems is most often rather good (Nesher, 1992). Cartesian product problems are more difficult to match to children's available repeated addition model. Obtaining the result by exhaustively listing all possible combinations of one element from a first set and one element from the second set can be rather difficult without the help of an external representation. This type of problems moreover receives considerably less educational attention, especially in the early years of primary education, and performance on Cartesian product problems typically is much worse. An error that frequently occurs in children is that they add the given numbers in a Cartesian product problem instead of multiplying them (Nesher, 1992).

## RESEARCH QUESTIONS AND HYPOTHESES

The goal of this study is to clarify the role external representations can play when third graders (who are not yet experienced in multiplicative reasoning) solve multiplicative word problems. The literature suggests that creating or using an external representation may be beneficial in solving a word problem, but the conditions under which this is the case for multiplicative problems solved by young children are not so clear.

First of all, we wondered whether the beneficial effect of an external representation differs when it is provided versus when children are invited to construct a representation themselves. It can be suspected that providing a representation is more beneficial because when children are merely invited to construct one, they may either not follow this suggestion, or construct a representation that does not adequately show the semantic structure underlying the problem situation (*hypothesis 1*).

Second, we wanted to investigate whether –if providing a representation is beneficial for pupils' performance– the adequacy of a representation matters. Given that rectangular arrays do not represent the asymmetric nature of equal groups problems very adequately and that group diagram do not adequately represent the symmetric

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<sup>1</sup> By making claims about the adequacy of representational matches, we do not want to propose an absolutist view on the link between representations and tasks. Various student and context characteristics may also play an important role that we extensively address in other research (see, e.g., Acevedo Nistal, Van Dooren, Clarebout, Elen, & Verschaffel, 2009).

nature of Cartesian product problems, one may suspect that providing a representation that does not adequately match a specific problem type will be less helpful than when the provided representation does match the problem type (*hypothesis 2*).

Third, we wondered whether the effect of making or providing external representations depends on the problem type. One can assume that particularly for problems where pupils experience difficulties in unravelling the semantic structure underlying the problem situation, a good representation might be helpful. Therefore, we expect that the beneficial effect of making or providing an external representation would be more pronounced for Cartesian product than for equal group problems (*hypothesis 3*).

## METHOD

Participants in this study were 233 third grade pupils from 8 schools in Flanders, Belgium. In Grade 2, the multiplication tables up to 100 had already been extensively trained, but pupils had little experience yet in solving multiplicative word problems.

All pupils received a word problem test containing 6 multiplicative word problems, distributed over 2 different types: 3 equal group problems and 3 Cartesian product problems. Additionally, 6 buffer items dealing with addition and subtraction were included in the test, to make sure that the participants would not notice the focus of the study. Examples of word problems can be found in Table 1.

Multiplicative problems	
<i>Equal groups</i>	Tom has 3 boxes of cookies. Each box contains 8 cookies. How many cookies are there?
<i>Cartesian product</i>	An ice cream bar has 5 flavours of ice cream and 4 types of cones. How many kinds of ice cream can be made?
Buffer problems	
<i>Addition</i>	John has 13 marbles. Tina has 7 marbles more than John. How many marbles does Tina have?
<i>Subtraction</i>	Sarah has 3 strawberries on her plate. Mama gives her some more strawberries. Now she has 9 strawberries. How many did mama give her?

Table 1: Examples of multiplicative problems and buffer items

The 12 word problems were offered in different randomized orders in the test. They were made as comparable as possible with respect to length, reading difficulty, and technical calculation difficulties (e.g. avoiding identical multipliers and multiplicands, avoiding multiplications by 1, 2, and 10), and by creating test different versions so that the numbers that appeared in the equal groups problems in one version appeared in the Cartesian product problems in another test version.

Pupils were randomly assigned to one of four conditions, that differed in terms of the instructions and/or the representations that were provided:

- Control (C) condition ( $n=57$ ): Pupils received the word problems without external representations and without specific instructions to solve them.
- Make representation (M) condition ( $n=60$ ): Pupils received the instruction on the first sheet to create, for each problem, a sketch or drawing that could help in solving it.
- Provided adequate representation (P-A) condition ( $n=60$ ): Pupils received a representation accompanying each word problem, and they were told that the representation could help them in solving the word problem. The representations were considered adequate in the sense that equal groups problems were accompanied by a group diagram and Cartesian product problems by a rectangular array.
- Provided less adequate representation (P-LA) condition ( $n=56$ ): This condition was identical to the PA condition, but the representations were considered less adequate, as equal groups problems were accompanied by a rectangular array and Cartesian product problems were accompanied by a group diagram.

Answers that were obtained by correctly multiplying the given numbers were scored as accurate. Accuracies to the experimental word problems were analysed by conducting a repeated measures logistic regression analysis, using condition (C, M, P-A, and P-LA) and type of problem (equal groups and Cartesian product) as predictors.

## RESULTS

Table 2 provides a summary of the accuracies obtained in the different conditions and for both problem types. As can be seen in the table, there was a large difference in accuracy between equal groups and Cartesian product problems, which was significant, *Wald Chisquare* (1,  $N= 233$ )=370.418,  $p<.00015$ . In line with what is reported in the literature, the third graders in our study were rather good at equal groups problems (90% correct), but the accuracy on Cartesian product problems was much worse (36% correct).

	Equal groups	Cartesian Product	Total
C condition	84	38	61
M condition	83	16	50
P-A condition	93	48	71
P-LA condition	95	49	72
Total	90	36	63

Table 2: Percentages of correct responses to equal groups and Cartesian product problems per condition

Table 2 also shows different accuracies for the four conditions, which were also significant, *Wald Chisquare* (3,  $N=233$ )=31.846,  $p<.00015$ . As can be seen, the highest accuracies were achieved in the P-A and P-LA conditions where a representation was provided to the pupils (71% and 72% respectively), which was

significantly higher than the C condition (61%), which was in its turn significantly higher than the M condition where pupils had to make a representation (50% correct).

Finally, the results indicate that particularly for Cartesian product problems, providing representation leads to higher accuracies. There was a significant condition  $\times$  type of problem interaction effect, *Wald Chisquare* (7,  $N=233$ )=620.445,  $p<.00015$ . Pairwise comparisons point out that for the equal groups problems, there is no significant difference between the C condition and M condition, and no significant difference between the P-A and P-LA condition. The significant difference between the C and M conditions on the one hand, and the P-A and P-LA conditions on the other hand is moreover rather small. For the Cartesian product problems, significant differences were found between the P-LA and P-A condition on the one hand and the C condition on the other hand, and between these three conditions and the M condition.

## CONCLUSIONS AND DISCUSSION

First, in line with *hypothesis 1*, external representations may positively affect the accuracy of third graders' solutions to elementary multiplication problems. Pupils who were provided a representation performed better than pupils in the control condition who did not receive any representation. When pupils were requested to make a representation themselves, the beneficial effect was not observed. For Cartesian product problems, an opposite effect of requesting to make a representation was even found. Ignoring the instruction to make a representation could explain why pupils in the M condition did not perform *better* than those in the C condition. An analysis of the response sheets showed that this happened in only about 20% of the cases. But this does not explain the *worse* performance on Cartesian product problems in the M condition compared to the C condition. The explanation can probably be found in the representations that pupils actually made. An analysis of those representations that were constructed showed that for equal group problems, most pupils constructed a group diagram, which represented the semantic structure underlying the problems. For Cartesian product problems, however, pupils often constructed decorative pictures of the objects that were involved, or a representation that reflected an additive rather than multiplicative relation, so it is not surprising that these representations often resulted in pupils merely adding the two given numbers instead of multiplying them.

Second, an unexpected observation (not in line with *hypothesis 2*) was that pupils performed equally well when the provided representation was adequate or less inadequate. Even when a rectangular array accompanied an equal group problem or when a group diagram accompanied a Cartesian product problem, pupils performed equally well as when an adequate representation was shown. An explanation may be that each of the two multiplicative representations (group diagram and rectangular array) always showed the solution to the multiplicative problem, even when the semantic relations in the problem were represented less adequately in the P-LA condition. So, once pupils knew how to correctly interpret the representations as such, they could in principle respond correctly without even reading the problem. It may be that the distinction in terms of adequacy of representations is more relevant when

pupils construct the representation themselves, as the activity of constructing can give pupils the insight in the semantic relations underlying the problem, while this was not longer necessary in our conditions where representations were provided.

Third, as expected (*hypothesis 3*), providing external representations is only really effective for problems where pupils experience difficulties in understanding the problem situation, which in our study were Cartesian product problems.

Taken as a whole, when a multiplicative problem is accompanied by a representation, this has a beneficial effect on third graders' accuracy, particularly for Cartesian product problems, which are typically more difficult for this age group. If a representation is provided, at first sight it does not even matter whether it represents the semantic structure underlying the problem situation adequately. However, while in those cases a pupil can calculate the correct response using the representation at hand, he will not necessarily gain insight in the underlying semantic and mathematical relations. Asking pupils to create a representation themselves even resulted in worse performance than not asking to do so, as pupils often created representations that did expressed no or a wrong semantic structure. Given the multitude of classes of multiplicative situations (Greer, 1992) that need to be addressed in instruction, and the variety of external representations that may match them better or worse, pupils' understanding might benefit from devoting explicit attention to matching multiplicative situations to provided external representations, and to constructing representations that adequately show the semantic structure themselves.

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# IN SEARCH FOR THE NATURAL NUMBER BIAS IN SECONDARY SCHOOL STUDENTS WHEN SOLVING ALGEBRAIC EXPRESSIONS

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*Dealing with rational numbers causes difficulties for many children, even though it is an essential part of mathematical literacy. The natural number bias is considered a major source of this difficulty. By means of two studies we investigated if and to what extent students who are just introduced into expressions involving literal symbols suffer from the natural number bias when interpreting algebraic expressions that address the effect of the four basic operations. Evidence for the natural number bias was found in the significantly higher accuracy levels on congruent items (where interpreting the letter as a natural number leads to a correct answer) than on incongruent items (where interpreting the letter as a natural number leads to an incorrect answer).*

## THEORETICAL BACKGROUND

An essential part of mathematical literacy is a good understanding of rational numbers. However, in recent years a growing body of research provided evidence that children make systematic errors when they solve rational number tasks where relying on the natural number knowledge leads to an incorrect answer (Moss, 2005; Ni & Zhou, 2005; Smith, Solomon, & Carey, 2005; Vamvakoussi & Vosniadou, 2010). On the other hand, when the same children solve tasks with rational numbers that are in line with reasoning about natural numbers, the accuracy rates are much higher (Nunes & Bryant, 2008; Van Hoof, Lijnen, Verschaffel, & Van Dooren, in press).

The general trend in the literature has been to suggest that a major source of these errors on tasks with rational numbers is the inappropriate application of natural number knowledge, a phenomenon known as “the natural number bias” (Ni & Zhou, 2005; Van Dooren, Van Hoof, Lijnen, & Verschaffel, 2012). Even before instruction, children have already formed an idea of what numbers are and how they behave. This idea is mainly based on the experience with and knowledge of natural numbers (Vamvakoussi & Vosniadou, 2010). In the first years of instruction, then, this natural-number based knowledge is systematized. Indeed, both in daily life and in the first years of their school career, children come across rational numbers far less frequently than natural numbers (Greer, 2004). From these daily experiences with natural numbers in and outside the school context, children create those beliefs about what numbers are and how they should behave. As a result, once the mathematical concept of rational numbers is introduced in the classroom, problems and misconceptions occur when children (and students later on) encounter situations with rational numbers in which the



rules for natural numbers are no longer applicable (Gelman, 2000; Smith et al., 2005; Vamvakoussi & Vosniadou, 2010).

There is a substantial body of literature on this topic, which reports that there are four main aspects on which natural numbers differ from rational numbers and lead to systematic errors. The first aspect of the difference between natural and rational numbers is density: While natural numbers are discrete (you can always point out which number comes next), rational numbers are dense (you cannot point out which number comes next, because between any two rational numbers there are always infinitely many numbers). This leads to the common mistake made by children who think that there are no (or finitely many) numbers between two pseudo-successive numbers (for example 1.2 and 1.3) (Vamvakoussi, Christou, & Van Dooren, 2010). The second aspect where rational numbers differ from natural numbers is number representation: While natural numbers have a single symbolic representation, rational numbers have a multitude of possible symbolic representations. Research has shown that children do not see fractions and decimals as representations of the same number (Vamvakoussi et al., 2012), and consider fractions as two (natural) numbers instead of a number as such (e.g. Stafylidou & Vosniadou, 2004). The third aspect on which rational numbers differ from natural numbers is the way the number size can be determined. Research indicates that errors in size comparison tasks are made for instance due to the fact that children wrongly assume that “longer decimals are larger” and “shorter decimals are smaller” (Resnick, Nesher, Leonard, Magone, Omanson, Peled, 1989). Because children have troubles seeing a fraction as one number instead of two separate numbers, they further tend to wrongly assume that a fraction’s numerical value always increases when its denominator, numerator or both increase (Mamede, Nunes, & Bryant, 2005; Meert, Grégoire, & Noël, 2010). The fourth aspect of difference concerns operations with numbers. A number of rules related to operations with natural numbers are no longer applicable in the domain of the rational numbers. For example, children learn, in the domain of the natural numbers, that multiplication and addition will always lead to a larger outcome and that division and subtraction will always lead to a smaller outcome. Even though most often, this rule is not stated explicitly in instruction, children can deduce it from the multitude of experiences where this is indeed the case when doing natural number arithmetic. However, in the domain of rational numbers, these properties do not longer hold, leading to mistakes where children for example think that  $5 \cdot 0.99$  will lead to an outcome larger than five (Hasemann, 1981; Vamvakoussi, Van Dooren, & Verschaffel, 2012). In this research, we chose to conduct a study with a focus on the aspect of operations.

Next to the four different aspects wherein the rules for natural numbers may no longer be applicable for rational numbers, a second body of research points out that students have a tendency to substitute literal symbols in algebra only with natural numbers. Students further tend to have difficulties to understand that these literal symbols can stand for more than one number (Christou & Vosniadou, 2012).

While the natural number bias has amply been studied in elementary school children and adults, this phenomenon remains –to the best of our knowledge– unexplored in the

age stage between the elementary school childhood, where the children are just taught in rational numbers and still make a lot of mistakes, and adults, who make less mistakes but who still reveal the natural number bias through their reaction times (e.g. Vamvakoussi et al., 2012).

In this paper we combined the two domains mentioned above (rules related to operations and substitution of literal symbols) and investigated in two separate studies if the natural number bias could also be found in secondary school students when interpreting algebraic expressions that address the effect of operations.

## STUDY 1

### Method

In the Flemish curriculum, algebraic expressions are introduced for the first time in the first year of secondary education. To ensure that the participants of our study were in principle capable of interpreting algebraic expressions, we chose to carry out the study with students of the second year of general secondary education, an age-group where the natural number bias is still being underexplored. Twenty-two students from a secondary school in a middle-sized city in Flanders participated in this study.

They received a paper-and-pencil test consisting of 40 algebraic expressions offered in a random order. The task was to interpret the algebraic expressions and judge whether they can be true or not. When the student thought that the expression could be true, he had to tick a box next to the item, if not, he had to leave the box next to the item blank. These judgments had to be made for algebraic expressions consisting of a number and an unknown term (one letter that could stand for any number, as was also clarified to the students). Examples are provided in Table 1.

Type of test	Congruent item	Incongruent item
Can this expression be true? (study1)	$n > n-2$	$a:4 > a$
Is this expression always true? (study 2)	$9+c > c$	$2*m > m$

Table 1: Example items per type of test and per type of congruency

The 40 expressions were composed by making every possible combination of congruency (congruent and incongruent), operation (addition, subtraction, multiplication and division), position of the unknown term and the fact that the algebraic expression was indeed true or not. This led to a combination of 24 congruent items (where interpreting the letter as a natural number leads to a correct answer) and 16 incongruent items (where interpreting the letter as a natural number leads to an incorrect answer), divided over the four operations: addition ( $N=12$ ), subtraction

( $N=12$ ), multiplication ( $N=8$ ) and division ( $N=8$ ). Finally, different letters of the alphabet were randomly chosen as unknown term and the given numbers took values from zero to ten. There was no time limit to complete the test, but all students finished within twenty minutes.

### **Research questions and hypotheses**

The central research question in the study was whether and to what extent students who are just introduced into expressions involving literal symbols suffer from the natural number bias while interpreting algebraic expressions (research question 1a). Our hypothesis was that when a student solved an item, he would be biased by his natural number knowledge and would make more mistakes on incongruent than on congruent items (hypothesis 1a).

If a bias indeed was found, our second research question was whether the kind of operation (division, multiplication, subtraction and addition) would have an effect on this natural number bias (research question 2a). We had no specific hypothesis for this research question.

### **Analysis**

The data were analyzed using SPSS 17. Because there were multiple measurements per subject, we analyzed the data using the Generalized Estimation of Equations (GEE) approach that corrects for repeated (and therefore probably correlated) categorical measures within subjects (Liang & Zeger, 1986). Due to the dichotomous outcome of the dependent variable, logistic regression was used.

Next to the main effects of congruency (congruent and incongruent items) and operation (addition, subtraction, multiplication and division), the interaction effect between them was also analyzed.

### **Results**

A significant main effect of congruency,  $X^2(1, N=880)=43.936$ ,  $p=.000$  was found. Students' accuracy levels were significantly higher on congruent (90.2%) than on incongruent items (34.4%), which is in line with hypothesis 1.

The results also showed a significant main effect of operation,  $X^2(3, N=880)=12.906$ ,  $p=.005$ . The pairwise comparisons indicated that division had the lowest accuracy (53.4%) and differed significantly from addition (76.1%,  $p=.001$ ) and multiplication (64.2%,  $p=.009$ ), but not from subtraction (71.6%,  $p=.067$ ).

Finally, results showed a significant interaction effect between congruency and operation,  $X^2(3, N=880)=7.959$ ,  $p=.047$ . The pairwise comparisons revealed no significant differences between operations for the congruent items. For the incongruent items, however, there were differences: Division (18.2%) was significantly more difficult than multiplication (33.0%) and subtraction (37.5%), whereas addition (48.9%) was significantly easier, which gives an answer to our second research question.

## STUDY 2

While the results of the first study already shed a light on the natural number bias in students of secondary education, we chose to conduct a second similar study. In the first study, the students needed to judge whether an algebraic expression *can* be true. So, they only needed to find one example to confirm that it indeed can be true. In our second study, we changed the assignment and the students now needed to judge whether an algebraic expression *is always* true. To solve this task correctly, the students needed to find a counterexample to reject a generally applicable rule, which may elicit quite a different reasoning process in checking the algebraic expressions, and therefore resulting in a different manifestation of the natural number bias.

### Method

The method of the second study was very similar as the method in the first study: In a paper-and-pencil test 40 algebraic expressions needed to be judged by 22 students of the second year of secondary education. There was one main difference with the other study: While the assignment in the first study was to judge whether an algebraic expression can be true, the assignment in the second study was to judge whether an algebraic expression is always true.

The items were composed in the same way as the first study, examples can be found in Table 1.

### Research questions and hypotheses

The central research question in this study was also whether and to what extent students who are just introduced into expressions involving literal symbols suffer from the natural number bias while interpreting algebraic expressions (research question 1b). Our hypothesis was that a natural bias would be found in the higher accuracy levels on congruent than on incongruent items (hypothesis 1b).

Our second research question was whether the kind of operation (division, multiplication, subtraction and addition) would have an effect on this natural number bias (research question 2b).

### Analysis

As in the first study, the data were analyzed using the Generalized Estimation of Equations (GEE) approach (Liang & Zeger, 1986), using logistic regression due to the dichotomous outcome of the dependent variable.

Next to the main effects of congruency (congruent and incongruent items) and operation (addition, subtraction, multiplication and division), the interaction effect between them was also analyzed.

### Results

A significant main effect of congruency was found,  $X^2(1, N=880)=29.123$ ,  $p=.000$ . Accuracy levels were significantly higher on congruent (87.3%) than on incongruent items (42.9%), which is consistent with hypothesis 1b.

Differences in accuracy between the four operations were found, but in contrast to the previous study, the main effect of operation was not significant,  $X^2(3, N=880)=2.431$ ,  $p=.488$ .

Finally, as an answer on the second research question, no significant interaction-effect was found between congruency and operation,  $X^2(3, N=880)=1.725$ ,  $p=.631$ .

## CONCLUSION AND DISCUSSION

The natural number bias has amply been investigated in elementary school children and adults (Moss, 2005; Ni & Zhou, 2005; Smith, Solomon, & Carey, 2005; Vamvakoussi & Vosniadou, 2010). The gap in the literature was in the middle age group between those two age groups. We investigated this group to find out if the natural number bias could also be found in students of secondary education who are just introduced into algebraic expressions with literal symbols.

In two closely related studies, students needed to judge the correctness of algebraic expressions. Our results extend previous research by providing evidence for the fact that students from the second year of secondary education are also hampered by the natural number bias in solving tasks with rational numbers, which address the effect of arithmetical operations. While the assignment in the first study was to judge whether an algebraic expression can be true, the assignment in the second study was to judge whether an algebraic expression is always true. A different reasoning process was needed in the two studies. While in the first study the students only needed to find one example to confirm that an expression indeed can be true, the students in the second study needed to find a counterexample to reject a generally applicable rule.

The main research question in both of the studies was whether lower secondary school students would be hampered by the natural number bias while interpreting algebraic expressions. Our hypothesis (hypothesis 1a and 1b) in both of the studies was that this would indeed be the case and our findings did confirm this: Higher accuracy levels on congruent (whereby relying on natural number knowledge leads to a correct answer) than on incongruent (whereby relying on natural number knowledge leads to an incorrect answer) items were found twice. This finding led to an additional question, namely whether the kind of operation would have an effect on this natural number bias (research question 2a and 2b). The results of the first study (where students needed to judge if an expression *can* be true) showed that a difference could be found: For the congruent items there was no difference between the operations, but for the incongruent items, 'division' had the lowest accuracy level and differed significantly from subtraction and multiplication, whereas addition was significantly easier. One could say that the source of this finding is that the operation 'division' is in itself more difficult than the other operations (and that addition is in itself the easiest one). However, if this was the case, significant differences should also be present in the congruent items, which was not the case. Interestingly, in the second study, no significant effect of operation on the natural number bias was found. To find out what are the underlying reasons for our results, we plan to add interview data to further reveal students' reasoning processes and more specifically to shed a light on which

numbers are used by students to check whether an algebraic expression can be/is always true.

In summary, a good knowledge of the rational numbers is an essential part of mathematical literacy, but the research literature indicates that rational numbers are a big source of difficulty not only for children but also for secondary school students. From previous research we know that a source of these difficulties in elementary school children is the natural number bias. The results reported in this article contribute to the existing literature by providing evidence that the natural number bias can also be found in lower secondary school students.

This finding leads to the implication that not only elementary school teachers, but also secondary school teachers should be aware of this problem and that sufficient education is needed to (continue to) suppress this natural number bias, e.g. by making the students more aware of the fact that there are many differences between natural and rational numbers (for example through the use of examples that point out the systematic errors students make). This study further indicated that at the moment when students learn how to solve algebraic expressions involving literal symbols, more attention should go to the fact that these literal symbols can stand for any number (natural numbers, but also whole numbers and rational numbers).

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# THE IMPLEMENTATION OF CONTEXTS IN DUTCH TEXTBOOK SERIES: A DOUBLE DIDACTICAL TRACK?

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*Research on the development of algebraic proficiency of Dutch pre-university students indicated that students make progress, both cross-sectionally and longitudinally; however, the development of the majority of students does not encompass important conceptual aspects. Since textbooks play an important role in Dutch mathematics education, a textbook analysis was performed to investigate to what extent these offer means of support to reach this higher level. Using emergent modeling as theoretical background, we found that textbooks hardly provide this kind of support. Moreover, we found two distinct tracks of presenting mathematics: one track that follows the RME approach, and one track with a more traditional approach, in which new concepts are introduced as ready-made mathematics.*

## INTRODUCTION

A study on the development of algebraic proficiency of Dutch students in pre-university education revealed that students made little progress, both cross-sectionally and longitudinally (Van Stiphout, 2011). The students mastered simple tasks, but tasks became too complicated for them rather quickly, and the range in which the development of the majority of students took place did not incorporate the tasks involving important conceptual skills (Van Stiphout, 2011). These conceptual skills, which are in the heart of the aforementioned research, have been described in many different ways by many different researchers. For example, relational understanding (Skemp, 1976), conceptual understanding (NRC, 2001), flexible manipulation skills and the ability to read through symbols as behaviors of symbol sense (Arcavi, 1994), ‘proceptual’ view (Tall & Thomas, 1991), reification (Sfard, 1991), relations between relations (Van Hiele, 1986). We use the term *conceptual proficiency* to denote this kind of conceptual skills in which three aspects are essential: the ability to recognize and flexibly use algebraic structure, the ability to deal with the ambiguous nature of mathematical concepts, and the ability to see the relations between mathematical concepts.

This report reflects on which means of support Dutch textbook series offer to help students to develop conceptual proficiency. We choose to focus this analysis on topics linear relationships because this topic is central in Dutch lower secondary education. Dutch textbook series are strongly influenced by the theory of Realistic Mathematics Education (RME). As a consequence, linear relations are introduced by means of contexts and models. The role of models within RME is described in the theory of emergent modeling (Gravemeijer, 1999) on which we elaborate below.



## THEORETICAL FRAME: EMERGENT MODELING

Emergent modeling is an instructional design heuristic which is an element of the theory of Realistic Mathematics Education (Gravemeijer, 1999). RME is a domain-specific instruction theory that has its origins in the early 1970's, and is based on Freudenthal's view on mathematics. He argues that "what humans have to learn is not mathematics as a closed system, but rather as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics" (Freudenthal, 1968, p. 7). In Freudenthal's view, students should not be confronted with ready-made mathematics. Rather, they should be enabled to learn mathematics by mathematizing both reality and their own mathematical activities. In RME, the focus is on teaching the *activity* of mathematizing instead of teaching the *results* of the mathematizing activities of others. The latter would lead to an *anti-didactic inversion* of the way the mathematics was invented.

Modeling activities play an important role in mathematics. By translating a contextual problem into a mathematical problem, the problem solver makes the problem amenable to mathematical procedures. From an RME perspective, students should not be confronted with ready-made models. Rather, these models and their meaning should emerge from their own mathematical activities. The idea of emergent modeling is that students start with modeling experientially real problem situations. Then, in the following learning process, the model gradually develops from a *model of* the students' own mathematical activity to a *model for* more formal mathematical reasoning. The use of models in the latter form lies close to the intended use of ready-made didactical models, which are meant to make formal mathematics more accessible to students.

From a constructivist perspective, the problem with this kind of didactic models is that in order to interpret the model correctly, students should already have acquired the mathematical knowledge that is intended by the model. As an alternative, emergent modeling concentrates on rooting the development of the model in the experiential reality of the students. In this way, models can support mathematical growth. While working with the models, students start to come to grips with the mathematical relations involved. In this way, the model starts to function as a model for mathematical reasoning. The transition from *model of* to *model for* concerns a shift in students' thinking. This shift concerns thinking about the modeled contextual situation to thinking about mathematical relations. Gravemeijer (1999) discerns four levels of different activities: task setting, referential, general, and formal.

Based on these levels, we can describe an *ideal* instructional sequence. From this point of view, the introduction of linear relations has to start with contextual situations of linear relations. Activities at the task setting level have to invite students to reason and calculate within these contexts. Examples of such activities are discovering patterns in contextual situations and informal reasoning and calculating. In this way, some contexts—such as contexts about fixed and variable costs—may become paradigmatic for linear relations. These relation then may be modeled with word formulas. Next, the attention of the students has to shift to the mathematical characteristics and the

mathematical relations as such. In this manner, the model gradually develops into a model that derives its meaning from a network of mathematical relations that is being construed in the process and students' reasoning loses its dependency on situation-specific features. In this manner, a linear relation becomes an object. This object does not necessarily need its context, but instead has meaning in and of itself. This object is incorporated in a network of relations. At this general level, linear (word) formulas start to function as *models for* linear relations.

## METHOD

The data set is constructed from two Dutch textbook series, *Moderne Wiskunde* (MW) and *Getal & Ruimte* (GR), which together have an estimated market share of over 95%. From these textbook series, chapters, and sections within them, are selected that concern linear relations. Theory on these topics is presented in the textbooks for grades 7, 8, 9 and 10. In grade 10, the student population is split into a social stream and a science stream. In this paper, we focus on students in the science stream because especially for these students it is important to develop conceptual proficiency. As units of analysis we choose tasks and fragments of texts on theory that belong to the main sections of the chapters. Besides these main sections, chapters of both textbook series also contain sections concerning pre-requisite knowledge, information technology, diagnostic test, summary etcetera. We decided not to include these additional sections in the analysis, because they do not directly contribute to the introduction of new concepts.

The four levels of activity described in the theory of emergent modeling cannot be directly translated into categories of activities in the textbooks for two reasons. The first reason is that the levels describe the students' mental activity, not the tasks or the models per se. The second reason is that in the theory of emergent modeling, the shifts between levels are considered essential. We therefore developed categories of tasks, which we deem important for fostering the process of emergent modeling. That is, tasks that address, respectively, the contextual basis, the shift from contextual meaning to mathematical meaning, and the reification of the object.

Activities in the first category, referred to as C1, concern the investigation of contextual situations. Figure 1 represents an example of such an activity of a grade 7 textbook. In this task, students have to reason about the number of chairs depending on the number of tables. Each table provides four chairs, excluding the ends. The relationship between the number of tables and the number of chairs is linear, but the term 'linear' is not introduced in this grade 7 textbook. At this stage, the focus is on capturing the relation in a word formula and using this word formula to calculate the number of chairs when the number of tables is given, see part **d** and **e** of the task in Figure 1. In this example, students are not asked to explicate patterns or characteristics of the context in a formal manner. They have to be aware of them, however, to carry out the calculations within the context.

**4** Een caféhouder heeft rechthoekige tafels. Hiernaast zie je hoe hij ze neerzet.


**a** Hoeveel mensen kunnen er zo bij vier tafels zitten? En hoeveel bij vijf tafels?

**b** Hoeveel stoelen staan er steeds per tafel, de uiteinden niet meegerekend?

**c** Hoeveel stoelen komen er nog bij aan de uiteinden van de rij?

**d** Schrijf de volgende regel voor het berekenen van het aantal stoelen op: 'Het aantal tafels keer ... plus ... is gelijk aan het aantal stoelen.'

**e** Bereken hoeveel stoelen er in deze opstelling moeten staan bij zeven tafels.



Translation:

**4** A café owner has rectangular tables. On the side, you can see how he places them.

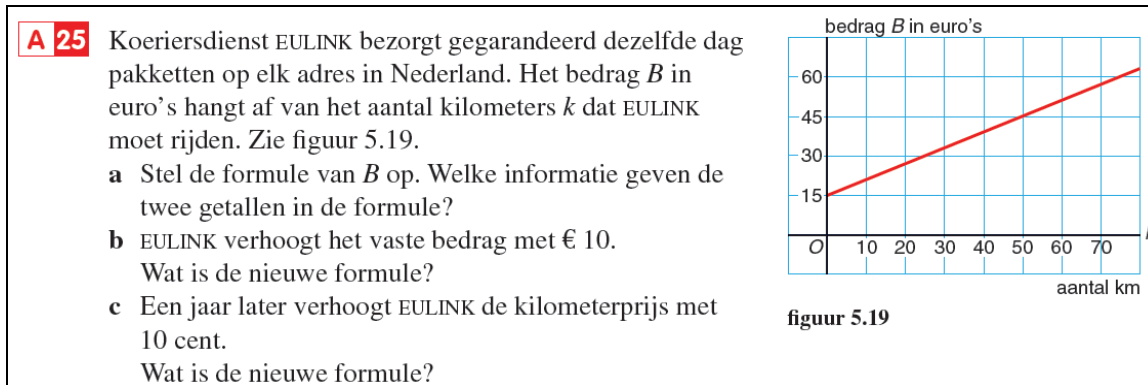
**a** How many people can be seated at four tables? And how many at five? **b** How many chairs are there per table, the ends excluded? **c** How many chairs do the ends of the row have? **d** Write the following line down for the calculation of the number of chairs: “The number of tables times ... plus ... equals the number of chairs.” **e** Calculate how many chairs are needed in this setting with seven tables.

Figure 1: Example of a task setting activity in the first category of a grade 7 textbook.

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The second category (C2) consists of activities in which the focus shifts from the context itself, to properties and characteristics of the context. A representative example of an activity at this level is presented in Figure 1. In this example, the graph is given of the linear relation between the costs of a courier service and the number of kilometers traveled. Based on this graph, students are asked to construct the corresponding formula. Next, students have to explain the meaning of the numbers in the formula. At this point, students have to make the connection between characteristics of the context (the fixed costs and the variable costs) and the numbers in the formula. In this way, the focus shifts from the context itself to its properties.

In the third category (C3), we included activities that help students to develop conceptual proficiency. Activities at this level require students to see and treat the linear relationship as an object that does not necessarily derive its meaning from the context. In activities in this category, the linear relation has become an independent entity, an object, that has meaning on its own. An example is an exercise in which three formulas of linear relations are given:  $y = 0.5x - 4$ ,  $y = 2x + 2$ , and  $y = 0.5x + 2$ . The question is which of these formulas intersect the  $y$ -axis in the same point, and how you can see this from the formulas. In a similar manner, steepness may be addressed. In this way, the linear relations are seen as objects with certain properties. These properties are the subject of investigation.



Translation:

Courier service EULINK guarantees to deliver packages in one day to any address in the Netherlands. The cost  $B$  in euros depends on the number of kilometers  $k$  that EULINK has to drive. See Figure 5.19. **a** Construct the formula for  $B$ . What information do the two numbers in the formula provide? **b** EULINK raises the fixed costs by €10. What is the new formula? **c** After a year, EULINK raises the costs per kilometer by 10 cents. What is the new formula?

Figure 2: Representative example of a second category activity of a grade 8 textbook. Reprinted from Reichard et al. (2005, p. 179) with permission of the publisher.

During the analysis we found that the activities in Dutch textbook series did not fit nicely into the above categories. This is due to the fact that the introduction of the general formula for a straight line in both textbook series is not related to the contextual situations discussed in the preceding chapters. Instead, the definition of a linear formula is given by means of examples of formulas, and that of linear relation by means of equal growth of variables. In both textbook series, the introduction of the general formula for a straight line  $y = ax + b$  seems just a new track of activities that does not fit in the categories already described.

Because of the missing link between the contextual situations of linear relations and the introduction of the general formula of a linear relation, activities in which students get acquainted with this formula do not fit in the emergent modeling approach. These activities are not part of a process of progressive mathematization. Rather, these activities seem to start a new track of activities, which are not grounded in contextual activities in the first and second categories. Therefore, we introduce a residual category (C4) of activities concerning the bare introduction of the linear formula  $y = ax + b$ .

An example of an activity in this residual category is presented in Figure 3. In this fragment of text, a definition of linear relation is given, based on the equal growth of variables. In preceding tasks, students have had to calculate the equal growth from tables. So superficially, there seems to be a good connection between this fragment of theory and preceding tasks. However, a more profound analysis showed that in this fragment, no connection is made between this equal growth of the variables and the structure of the linear formula. The latter is presented in the example on the left-hand side by means of examples and non-examples. In these examples, no connection is made to the text about the equal growth. In the example on the right, the equal growth

is demonstrated. However, the formula  $b = 4n + 3$  is not given. Also, the relation between the equal growth and the formula is not discussed. In our view, the lack of these connections makes this definition of a linear relation one that dangles in the air.

Because this text is not related to contextual situations, it does not fit into the first and second category of activities. Also, this text does not contribute to the development of conceptual proficiency due to the lacking connections. Therefore, we consider this text as an example of the fourth category.

**THEORIE**

Als in een tabel in de bovenste rij opeenvolgende gehele getallen staan en in de onderste rij de toename steeds hetzelfde is, dan is er sprake van een **lineair verband**. De toename in de onderste rij getallen kan positief of negatief zijn. De bijbehorende grafiek is een rechte lijn. Zo'n grafiek heet een **lineaire grafiek**. Een formule waarvan de grafiek een rechte lijn is, noem je een **lineaire formule**.

**Voorbeeld**

Lineaire formules zijn bijvoorbeeld  $7x - 90 = y$  en  $B = 34 + 75t$ .  
Niet-lineaire formules zijn bijvoorbeeld  $h = t^2 - 6$ ,  $g = 3a^2 + 7a$  en  $m - m^2 = d$ .

**Voorbeeld**

In de bovenste rij van de tabel hiernaast staan opeenvolgende gehele getallen. In de onderste rij van de tabel is de toename steeds hetzelfde. Tussen  $n$  en  $b$  bestaat een lineair verband.

$n$	0	1	2	3	4	5
$b$	3	7	11	15	19	23

$+1 \quad +1 \quad +1 \quad +1 \quad +1$   
 $+4 \quad +4 \quad +4 \quad +4 \quad +4$

Translation:

If the upper row of a table contains subsequent integers, and in the bottom row the increase is constant, then there is a linear relation. The increase in the bottom row of numbers can be positive or negative. The corresponding graph is a straight line. Such a graph is called a linear graph. A formula whose graph is a straight line, is called a linear formula.

Example [left] Examples of linear formulas are  $7x - 90 = y$  and  $B = 34 + 75t$ . Examples of non-linear formulas are  $h = t^2 - 6$ ,  $g = 3a^2 + 7a$ , and  $m - m^2 = d$ .

Example [right] In the upper row of the adjoining table are subsequent integers. In the bottom row of the table, the increase is constant. Between  $n$  and  $b$  there exists a linear relation.

Figure 3: Representative example of a fourth category activity of a grade 8 textbook.

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## RESULTS

Using the categories defined above, we classified units of analysis in GR and MW. Table 1 provides an overview of the numbers of units of analysis in GR and MW in each category for grades 7, 8, 9 and 10. The selected chapters and sections consisted of 123 (GR) and 140 (MW) units of analysis. From these units, 29 of GR and 68 of MW concerned activities from the first category; 17 of these units of GR and 14 of these units of MW concerned activities from the second category; 2 of these units of GR and 6 of these units of MW concern activities from the third category.

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Grade	Getal & Ruimte					Moderne Wiskunde				
	C1	C2	C3	C4	Total	C1	C2	C3	C4	Total
7	26	5	-	-	31	63	1	-	9	73
8	1	8	2	27	38	3	12	6	27	48
9	2	4	-	33	39	2	1	-	10	13
10	-	-	-	15	15	-	-	-	6	6
Total	29	17	2	75	123	68	14	6	52	140

Table 1: Numbers of units of analysis in each category (C1, C2, C3, C4) in the Dutch textbook series Getal & Ruimte and Moderne Wiskunde

These results indicate that both textbook series include a considerable amount of activities of the first and second category. So, both textbook series pay attention to the development of contexts to models of linear relations. However, both textbook series have a small number of activities in the third category. This means that both series provide only few activities that offer students support to develop conceptual proficiency with respect to linear relations.

The most important finding is the need for a fourth category that emerged during the analysis. As a consequence, the instructional sequence of linear relations can be seen as composed of two different tracks: one track starts with activities in contextual situations and gradually, in a process of progressive mathematization, more formal mathematical knowledge is constructed. The other track introduces mathematical concepts by means of giving the formal definition, without linking this definition explicitly to the theory discussed in earlier chapters of the textbooks.

From an emergent modeling perspective, since there is no link in the textbooks between the contexts and the general formula, the chain of the development of contexts as *models of* linear relations to contexts as *models for* linear relations is broken.

## CONCLUSION AND DISCUSSION

Summarizing, both textbook series pay extensive attention to phenomenological exploration, with a considerable amount of activities that focus on the exploration of contextual problems. However, the most crucial step in the formation of concepts, in which students use their understanding of ‘linear’ contexts to construct mathematical relations and mathematical objects, is hardly supported.

Instead, we found two distinct didactical tracks: one track that follows the RME approach, and one track with a more traditional approach, in which new concepts are introduced as ready-made mathematics. The RME track mainly contains activities at the task setting and referential levels. We found only a few activities that support the switch from contexts to formal mathematics. The other track does not build on the contexts and models, thus fails to make the link to the student's prior knowledge. Because of these two distinct tracks, we concluded that these two Dutch textbook series do not offer a consistent instructional sequence according to the emergent modeling theory, thus offering students little support to develop conceptual proficiency.

The research presented in this report covers only linear relations, but we conjecture that the problem of a double didactical track is more widespread. A similar analysis on linear equations yielded similar results: one track following the RME approach and one track with a more traditional approach in which new concepts are introduced as ready-made mathematics (Van Stiphout, 2011).

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# CONCEPTUALIZING MATHEMATICALLY SIGNIFICANT PEDAGOGICAL OPENINGS TO BUILD ON STUDENT THINKING

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*The mathematics education community values using student thinking to develop mathematical concepts, but the nuances of this practice are not clearly understood. We conceptualize an important group of instances in classroom lessons that occur at the intersection of student thinking, significant mathematics, and pedagogical openings—what we call **Mathematically Significant Pedagogical Openings to Build on Student Thinking (MOSTs)**—and introduce a framework for determining when they occur. We discuss how the MOST construct contributes to facilitating and researching teachers’ mathematically-productive use of student thinking through providing a lens and generating a common language for recognizing and agreeing upon high-leverage student mathematical thinking.*

Research in mathematics teacher education suggests the benefits of instructional practices that build on student thinking (e.g., Fennema, et al., 1996; Stein & Lane, 1996), but such practices are complex and difficult both to understand and to enact (Ball & Cohen, 1999; Sherin, 2002; Silver, Ghouseini, Gosen, Charalambous, & Font Strawn, 2005). Often opportunities to use student thinking to further mathematical understanding either go unnoticed or are not acted upon by teachers, particularly novices (Peterson & Leatham, 2009; Stockero & Van Zoest, 2012). Despite a growing number of teachers who are convinced of the value of student thinking and the need to encourage it, neither teachers nor those who educate them have a clear understanding of what thinking can best be used to develop mathematical concepts (Peterson & Leatham, 2009; Van Zoest, Stockero, & Kratky, 2010). We address this issue by providing a conceptual framework for thinking about the instances of student mathematical thinking that emerge while teaching. We refer to high-leverage instances of student thinking—those that have the most potential to increase student understanding of important mathematical ideas—as **Mathematically Significant Pedagogical Openings to build on Student Thinking (MOSTs)**.

We focus attention on the MOST construct because of its potential to contribute to the work of facilitating and researching teachers’ mathematically-productive use of student thinking. Although this paper is about characterizing and recognizing MOSTs (as opposed to using them), they are better understood if the reader has a sense of our vision of how they might be productively used to support student learning. A teacher may respond to MOSTs in a variety of ways, from inserting a teacher explanation to asking follow-up questions to orchestrating a class discussion. When a teacher sees a MOST as an opportunity to step in and explain, it could be classified as naïve use (Peterson & Leatham, 2009) in that the teacher may be using the MOST merely as a trigger to lecture about the mathematical topic, rather than to build on the student



thinking. A more productive use of a MOST is to orchestrate a discussion around the mathematics at hand. This orchestration could be done, for example, by posing questions that focus the class on connections between the mathematics of the observed student thinking and other concepts that are related to the mathematical goals of the classroom.

The MOST construct contributes to research on productive use of student mathematical thinking primarily through providing a lens and generating a common language for recognizing and agreeing upon high-leverage instances of student mathematical thinking. Specifically, it contributes to the work of facilitating teacher learning by providing guidance for identifying the characteristics of students' mathematical thinking that are most productive to focus on in preservice teacher coursework and inservice teacher professional development. It also provides a framework and language for conversation among teacher educators and teachers about high-leverage student thinking. Similarly, it contributes to researching teachers' use of student thinking by providing a lens to focus classroom discourse analysis on student mathematical thinking and tools to assess which student mathematical thinking is high-leverage. In this paper we describe the characteristics of MOSTs and introduce a framework for identifying them.

## **MATHEMATICALLY SIGNIFICANT PEDAGOGICAL OPENINGS TO BUILD ON STUDENT THINKING**

Although skilled teachers and teacher educators often recognize when important mathematical moments occur during a lesson and can readily produce ideas about how to capitalize on them, the literature reveals a construct that is neither well-defined nor explicitly articulated. While not the focus of extant literature, such instances are mentioned in a number of different ways. For example, Jaworski (1994) referred to "critical moments in the classroom when students created a moment of choice or opportunity" (p. 527). Davies and Walker (2005) used the term "significant mathematical instances" (p. 275) and Davis (1997) used "potentially powerful learning opportunities" (p. 360). Schoenfeld (2008) referred to moments that contained "the fodder for a content-related conversation" (p. 57), "an issue that the teacher judges to be a candidate for classroom discussion" (p. 65) and the "grist for later discussion or reflection" (p. 70). Schifter (1996) spoke of "novel student idea[s] that prompt teachers to reflect on and rethink their instruction" (p. 130).

It is clear from the literature that these instances, whatever they are called, are important to mathematics teaching and learning. In studying such references to these instances and drawing on our own classroom and research experiences, we have identified three critical characteristics of these moments: student thinking, significant mathematics, and pedagogical openings.

### **Student Thinking**

Because the MOST construct is designed to help articulate productive use of student mathematical thinking, we begin by defining what we mean by *student thinking*. We recognize our inability to access directly the thoughts of students. Instead we make

inferences based on our observations of what they say and do. Teachers (and researchers) must “listen to the student, interpret what the student does and says, and try to build a ‘model’ of the student’s conceptual structures” (von Glasersfeld, 1995, p. 14). Thus, when we use the phrase *student thinking* we refer to observable evidence of student thinking, which we define as any instance where a student’s words or actions provide sufficient evidence to make reasonable inferences about their thinking. In the classroom setting, this evidence most commonly is visible in verbal utterances, gestures, or written work (including on the board).

Note that we make a distinction between *observable* and *observed*. There are many cases, particularly with novice teachers, where student thinking is observable, but not observed by the teacher (e.g., Peterson & Leatham, 2009; Stockero & Van Zoest, 2012). One explanation for this phenomenon is *inattentional blindness* (Simons, 2000)—described in the psychology literature as a failure to focus attention on unexpected events. In addition, these ideas are closely tied to teacher noticing (e.g., Sherin, et al., 2011)—what a teacher attends to (or fails to attend to) during a lesson. In the context of teaching, the teacher’s failure to observe student thinking may mean that the teacher is not paying attention to student thinking or does not notice a particular instance of student thinking, rather than that there is no observable evidence of student thinking. Thus, for the purposes of our work, *observable* refers to thinking that could be observed by someone (e.g., the teacher, other students, a researcher) who witnessed the instance, either by being present or by engaging with a record of the interactions.

### Mathematically Significant

In order to be a MOST, the mathematics in an instance must warrant use of limited instructional time; that is, it must be what we call *mathematically significant*. We use the term *mathematically significant* in the context of teachers engaging a particular group of students in the learning of mathematics. Thus, we see it as a subset of important mathematics, which can be determined apart from a specific classroom context. In the mathematical analysis of an instance, we consider mathematically significant in relationship to three key criteria: the importance of the mathematical idea of the instance, the appropriateness of the mathematics to the students in the classroom, and the extent to which the mathematics is connected to the mathematical goals for this group of students.

To determine whether the **important mathematics** criterion is met one must first determine whether the student thinking is mathematical in nature and, if so, what mathematics the student is expressing—what we call *the mathematics of the instance*. In order to determine the importance of the mathematics of the instance, one must be able to articulate an important mathematical idea that is closely related to the mathematics of the instance. Because this determination is purely mathematical, it can be made independent of a particular classroom context.

A second criterion for mathematically significant is that the mathematics of the instance be **appropriate** for the students in the classroom. That is, it must help students develop mathematically and move forward in their learning. Meeting this criterion

requires two things. First, the mathematics of the instance must be accessible to the students given their prior mathematical experiences; they must have adequate background knowledge to engage with the mathematical idea. Second, the students must not yet have mastered the mathematical idea related to the mathematics of the instance. If they had, pursuing that idea would not likely move them forward in their learning. Thus, the appropriate mathematics criterion requires that the mathematical idea be accessible to students with a particular level of mathematical experience while not being likely to have been already mastered.

A third criterion of mathematically significant is that there is a viable mathematical connection between the mathematical idea related to the instance and **mathematical goals** for student learning in that class. The mathematical goals for the classroom encompass both mathematical content and mathematical practices. They could be determined by the teacher or by an external source, such as curriculum documents, or they could be inferred by an observer who is knowledgeable in the field of mathematics education, such as another teacher, a researcher or a teacher educator. When analyzing the mathematical idea related to an instance in relation to the mathematical goals for student learning, it is important to consider a range of goals, from those for the lesson in which the instance occurs, to those for the unit of instruction in which the lesson occurs, for the course students are taking, or for their broader mathematical learning. In the case of lesson goals, the instance may focus on a particular mathematical idea or connections among ideas within a lesson. In the other cases, the instance might involve making connections to other areas of mathematics, revisiting ideas from prior courses, or previewing ideas from future courses. Developing mathematical ways of thinking could be goals at any of these levels.

### **Pedagogical Opening**

Conscientious teachers continuously seek evidence of their students' engagement with a wide variety of instructional goals. They take cues from actions big and small, making adjustments and pushing students to elaborate, explain and justify their thinking. Not all student actions, however, are "critical moments" (Walshaw & Anthony, 2008, p. 527) that create "potentially powerful learning opportunities" (Davis, 1997, p. 360). In the interest of differentiating student actions that meet this higher threshold, we define *pedagogical openings* as observable student actions that provide compelling opportunities to work toward an instructional goal. To determine whether an opening has been presented one must consider both the *positioning* and the *timing* of an observable student action.

Building on the notion from the discourse analysis literature in general and the work of Harré (e.g., Davies & Harré, 1990) in particular, we define *positioning* as the way in which an observable student action positions that student with respect to the content of an instructional goal. Students are positioned well with respect to an instructional goal when they engage "deeply" with the content of that goal as opposed to "at a surface level." Whereas good positioning is determined by a *particular* student's engagement with the content of an instructional goal, good timing is determined with respect to the

preparation of the class as a whole to engage with the idea being raised in ways that support, rather than supplant, overall instructional goals.

## **PUTTING THE THEORY INTO ACTION**

When determining whether a MOST has occurred, the focus of our analysis is an “instance”—an observable student action or small collection of connected actions (such as a verbal expression combined with a gesture). Typically an instance is one conversational turn or physical expression (such as writing a solution on the board), but it can involve multiple turns. For example, if the expression of an idea were interrupted by another speaker with a comment that merely encouraged the initial speaker (e.g., “yeah,” “okay,” or “um-hum”), the speakers’ initial idea and the continuation of it would be considered a single instance. Determining whether an instance qualifies as a MOST involves a systematic analysis of whether the instance embodies the three MOST characteristics (see Figure 1). This analysis begins with questioning whether the instance provides observable evidence of student thinking. If it does not, the analysis ends because the instance cannot be a MOST. Focusing first on this characteristic stems from the perspective that what students say or do during a lesson is critical and should inform the teacher’s actions. If observable evidence of student thinking is present, the mathematics of the instance is then analyzed to determine whether the instance is mathematically significant; that is, whether it satisfies the important mathematics, appropriate mathematics and mathematical goals criteria. This mathematical analysis takes place linearly; if any mathematics criterion is not met, the analysis ends. The instance is not mathematically significant and therefore not a MOST. This mathematical analysis of the instance distinguishes our work from more general work on classroom discourse or even “teachable moments” in that we focus on instances that are likely to advance students’ development of mathematical ideas. If the instance is determined to be mathematically significant, the instance is analyzed in terms of whether the positioning and timing are right to create a pedagogical opening. Again, if either criterion is not met, the analysis ends; if both are met, the instance has met the criteria for all three characteristics and is deemed to be a MOST. We have found that taking this flowchart approach to the analysis of an instance brings structure and simplicity to an often chaotic and complex task.

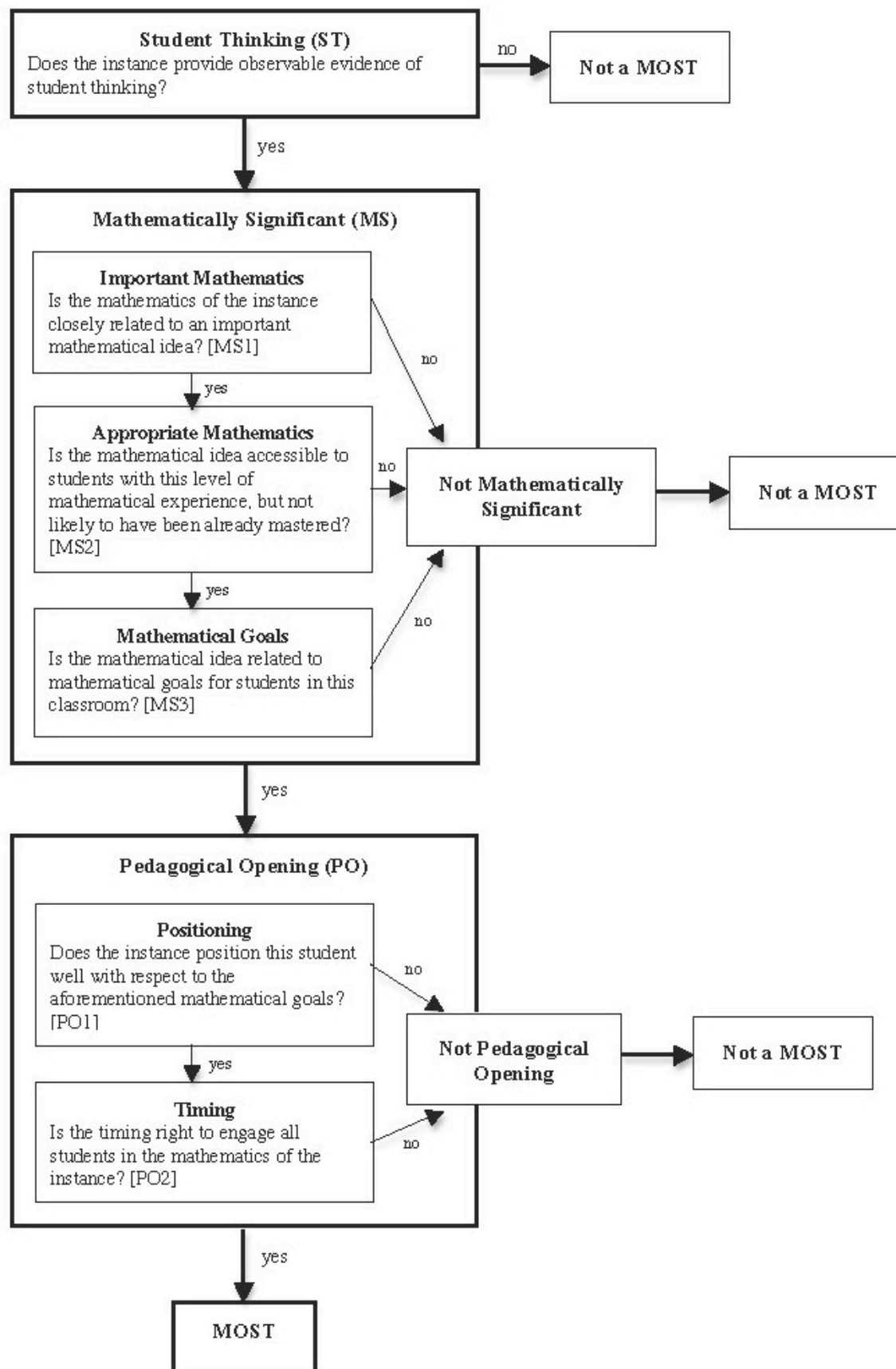


Figure 1. Analysis process for determining whether a classroom instance is a MOST.

## CONCLUSION

By clearly defining three critical characteristics that distinguish instances that provide high-leverage opportunities to advance students' mathematical understanding from those that do not, the MOST construct has the potential to become a tool to make sense of classroom interactions. In particular, the construct provides both a means for systematically analysing instances of classroom discourse and a vocabulary for discussing the mathematical and pedagogical importance of student thinking that arises within such discourse. Considering whether an instance embodies the three characteristics of a MOST requires identifying the mathematics in an instance of observable student thinking, as well as the larger mathematical idea to which it is related. Instances that are determined to be mathematical are then framed in terms of both mathematical significance and the pedagogical opening they provide. Engaging in this analysis provides a mechanism for teacher educators and researchers to frame teachers' practice in terms of their use of high-leverage instances of student mathematical thinking. This framing shifts the focus of the work from *whether* a teacher is using student thinking, to *what* student thinking a teacher is incorporating into a lesson and *why* that incorporation is valuable.

Although we acknowledge that mathematics teachers', teacher educators' and researchers' considerations are influenced by a wide range of beliefs about the nature of mathematics and about its teaching and learning, as well as by their own mathematical knowledge, we present the MOST framework as a mechanism for building mutual recognition and appreciation of high-leverage opportunities to build on students' mathematical thinking. Engaging in discussions of instances of student thinking using a common language and framework provides an opportunity to advance understanding of the productive use of student mathematical thinking, and consequently, enhance the teaching and learning of mathematics.

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# WHEN (AND HOW) DOES A POSED PROBLEM BECOME A PROBLEM?

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*We explore opinions, beliefs, and techniques on problem posing of some participants at a PME Research Forum. The study brings some evidence that problem posing is a complex activity that requires deep understanding and involvement, even from experts. Moreover, although expert-teachers do prefer real-world starting points in problem posing, producing rich learning tasks for students is a difficult process, and spontaneously rare.*

## INTRODUCTION

In this study, we explore opinions and beliefs concerning problem-posing (PP) activities of a group of researchers and teachers who participated at PME 35, in Ankara, Turkey. We confront these opinions with specific tasks of PP, the participants had to accomplish. We use this opportunity to address the following research question: *To what extent do mathematics education experts spontaneously generate meaningful mathematics problems and /or rich learning tasks for students?* – a question to which we provide a qualitative answer.

The specific literature shows a variety of definitions and views of PP: creating new problems from a situation and reformulation of problem data (Silver, 1994); making a succession of problems, starting from a situation (Leung, 1993); or, a process by which students construct personal interpretations of a situation and formulate meaningful mathematical problems (Stoyanova & Ellerton, 1996), or a resultant activity when the problem is inviting the generation of other problems (Mamona-Downs, 1993). In this paper, we adopt the definition given by Stoyanova and Ellerton.

The lack of a conceptual agreement on a definition for PP has propagated many controversies. For example, there are controversies about the role of PP in learning, and about the "actors" involved in the PP process as well. While some studies (e.g. Silver, 1994) identify PP as a critically important aspect of mathematics education, some others (e.g. Hirashima et al., 2007) are rather cautious arguing that students' posed problems may be wrong or may be too simple to be useful in learning.

Literature highlights various differences between experts and novices in PP situations. For example, Silver and Marshall (1990) show that experts tend to focus on the qualitative analysis of the posed problems, allocating more time for formulating and reformulating them, unlike novices who do not pay much attention to reformulations. In this study we continue this line of research by examining some differences between the PP process of experts and novices.



## METODOLOGY

### Participants

The sample used in this experiment consisted of 34 participants at a session of the Research Forum (RF) *Problem Posing in Mathematics Learning and Teaching: A Research Agenda*, where the authors of this paper had been coordinators or contributors. Due to various aspects (such as the specialized content of the conference, the selection of accepted papers, and the presentation process), the participants of the PME conferences can be considered experts (in the sense of Schneider et al., 1989).

The RF proceeded through two sessions, held on two consecutive days. On the first day, the participants were informed about the overall organization of the RF: namely, the first day they will be invited to complete two tasks whose preliminary analysis will be presented on the second day. None of the participants expressed dissent regarding the use of their answers. During the second session, a preliminary evaluation of the answers was presented and discussed. The present report is a detailed and more profound version of that preliminary assessment.

### Tasks and design of the study

During the first session of the Research Forum we engaged participants in two types of activities. First we asked them to answer some challenging questions presented in a questionnaire. Four types of quizzes with 6 questions each were distributed, and the participants had about 30 minutes to answer. In this paper, we focus on the analysis of responses to the following question:

*Question 1. In no more than three lines, outline what you understand by the term “problem posing”.*

The second activity consisted of a problem posing/problem modification task. More precisely, we asked the participants to work in small groups (3 to 5 members) and to pose a problem based on one of the images presented in Figure 1. The groups were heterogeneous in terms of the cultural context of participants' backgrounds, mathematical knowledge, and teaching experience.

After posing a problem, each group exchanged its work with another, and all were instructed to modify the problems received from their counterpart. Finally, each group received both problems (the problem originally posed and the modification offered from the other group) and they compared the two problems, identified similarities and differences, and wrote a comment. In total, there were 11 working groups and the activity lasted about 45 minutes.

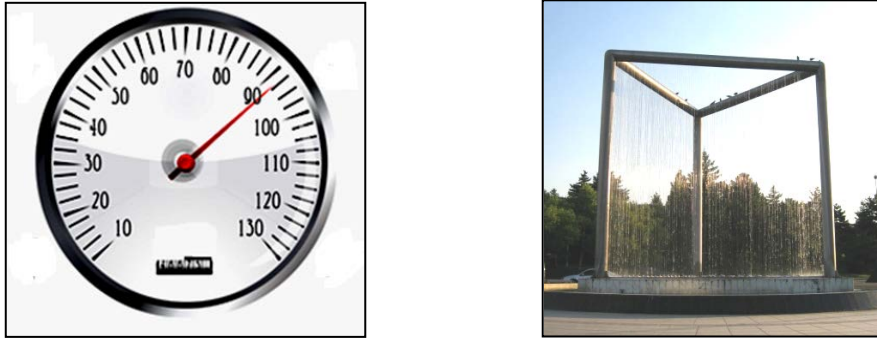


Fig. 1. The two images presented to the participants for activity 2.

## Data analysis

We analyzed, holistically, the two sets of problems generated during the workshop. In this analysis, we looked at the three components (posed problem, modified problem, and comment made by each group) as a unitary issue, to better understand the options and visions of each group. The participants' answers were grouped based on their perspective on PP concerning: the relationship between problem solving and problem posing; PP seen as a product or as a process; the teaching-learning role of PP; the problem formulation, etc.

## RESULTS AND DISCUSSION

### Questionnaire

The participants' answers to Question 1 demonstrated many different views on problem posing. This is not surprising; as shown above, different perspectives and definitions coexist in literature. In the next section we discuss and categorize these opinions.

Some participants regard PP as a useful step in problem solving; the excerpts below support this idea:

*Raising questions to understand a problem*

*To focus on one aspect in the problem posed*

*Making a better problem for a given problem*

*Construct knowledge and procedures for solving problems*

So, for some of the participants, PP is a collection of procedures useful in solving a problem, such as, for example, formulation/reformulation to help with problem solving. In other words, PP is seen as a sub-process of problem solving (PS). In the research literature we can find similar interpretations (e.g., Brown & Walter, 2005)

Other participants connect PP to novelty: they characterize PP as "authentic problems" or as the "formulation of conjectures". In these cases, PP is seen only through the final product, which must be "authentic", genuine, new. The word "conjecture" suggests a separation between PP and PS – the proposer is only meant to generate a (plausible) statement, but he/she does not have to hold a solution strategy for it. For example:

*What makes a problem “the problem” is that you don’t know the solution/ method of solution. That’s why it is a problem! It’s not necessary to know the solution. Posing is more related to curiosity.*

The two perspectives outlined above differ by looking at PP as a process – in the first case, or as a product – in the second case.

Other participants emphasized the usefulness of PP and, related, effective ways for classroom implementation of PP activities. For example:

*It is an opportunity to initiate mathematical thinking. An activity in which you have to pose a problem to reach a goal.*

*In my experience, I think investigation activities provide students to pose problems as a mathematics process. Projects are also an opportunity.*

*Given a context, triggering/ engaging students to come up with different possible questions and then teachers reformulate them in a meaningful way.*

*Students identifying lines of enquiry for rich starting points.*

The above excerpts reflect a teaching perspective: PP is seen as an important part of teachers' specialized content knowledge (as in Ball, Thames & Phelps, 2008).

It is important to highlight the differences, on a more general level, between these perspectives presented above. The first two positions do not restrict PP to mathematics or to teachers' practices. Rather, they represent a more general way of looking at PP: as a cognitive process or as a product. The view of PP through its potential role in teaching/learning or, as an activity arising from classroom interactions (teacher/students) or, as a student-centered activity posits PP in the context of schooling. In these situations, PP is seen as part of a teacher's tool to promote exploration, learning and understanding and, as counterpart, for students to do mathematics. It is expectable that these two different standpoints would manifest themselves differently when it comes to posing or modifying problems.

### **Problem posing and problem modification task**

As we have seen in the previous section, participants' perceptions on PP are quite different. The second activity analyzed in this paper was a group activity. This offered us the opportunity to see how the experts of our sample manifest and match different views, noting the contrasts between the posed problem, its modification, and the comments about the changes.

All the working groups had chosen the image on the right (see Fig. 1) as the support image for posing problems, although organizers have not made any reference to the choice of either picture. This option of the participants is not surprising, however: while the picture on the left is a schematic image, the one on the right (a photograph of an architectural structure in Ankara, near the conference venue) is contextually rich – there are various elements from daily life in the picture (cars, birds, water curtain, etc.). Everyday life context, more than a formal context, seems to be more relevant for our sample of experts. This option is consistent with most participants' perceptions about

the role of PP in the classroom, seen more as a way for "connecting real situations to mathematical ideas" or "to associate mathematical ideas with real world contexts".

In analyzing the proposals made by the participants, we compared the problem originally posed by a group with the changes made to it by another group. We tried to understand the impact of these changes on the proponents by a careful analysis of the comments. This analysis revealed several aspects, on which we include some significant examples in the following. To simplify, we denote  $Gx-p$  and  $Gx-m$  the problem  $p$  initially posed by the group  $x$ , and, respectively, the modification  $m$  proposed by another group for the same problem of the group  $x$ , and by  $Cx$  we note the comments made by the initial group  $x$  on the posed modification.

### Open-ended problems

A first interesting aspect of the problems posed by the participant experts is their tendency for open-ended problems – that is an exploration/investigation orientation. These problems are implicitly seen as projects for students. For example:

*G9-p: Notice the birds. How big do you think that the birds are? Use this to estimate the height of the prism.*

*G9-m: Estimate the volume of the prism. Estimate the area of the top (roof) of the structure.*

The two problems are focused on estimating sizes – height, area and volume. The solving of both problems requires the solver to identify possible tools for measurement, which means exploring the situation described by the image. There are however several differences between the problem initially posed and the proposed modifications. On the one hand,  $G9-p$  is more directive than the  $G9-m$ , as the first suggests the steps that can be used to solve. On the other hand, even if it requires the use of the same assessment tools,  $G9-m$  targets more advanced mathematical content.

Posing inquiry problems shows focusing on the process for both PP and PS. This option is consistent with the characterization of PP from G9 group perspective – as "construct knowledge and procedures for solving problems".

### From concrete to abstract

The image selected as input for the PP task belongs to a concrete everyday life context. We would expect that, in this case, the posed problems essentially keep this feature. As we saw in the previous example (G9), some of the posed problems valued the rich contextual starting point. Some others, however, just emphasize an abrupt passage from concrete to abstract. Let's analyze the following example:

*G7-p: How many faces are there in the picture? Is this shape categorized as platonic solid? Or an Archimedean solid?*

*G7-m: How many planes can you create from the vertices of this shape?*

*C7: We do not understand the intention of the problem posers.*

G7-*p* reduces the rich contextual image to a classical question: a solver has to observe the picture, to idealize the construction to a polyhedron, then to count the faces and to identify various properties.

G7-*m* makes a transcription in combinatorial thinking, because the solver must imagine and count all the planes determined by the vertices of the polyhedron. These two groups transferred the image representing a specific contextual situation in a purely theoretical approach in which the physical object is replaced by its geometric idealization.

We noticed in several cases this passage from real life to mathematical conceptualization, from concrete to abstract. It is possible that this shift to happen because "mathematized" problems are common in the teaching practice, so, more comfortable, for the experts as well.

### **Between ill-defined problems and open-ended situations**

Among the posed problems, some were incomplete. Consider the following example:

*G3-p: What is the area of the water walls?*

*G3-m: What is the volume of water that falls down the structure in one day?*

Solving the G3-*m* problem requires calculating/estimating the size of the construction, but also knowing the water flow of the water falling down along the "walls". When posed in the classroom, these problems may bring students to the need of analyzing the data: is there enough information, or some is missing, etc. However, given that the participants did not mention any approach of this kind, we characterized this problem as ill-defined (in the sense of Lynch et al., 2009), as some essential data is under-specified.

We encountered several situations of this type during the workshop. The occurrence of such ill-defined posed problems in our sample of experts is a phenomenon that deserves more attention. It is interesting to note that some previous studies (Singer, 2012; Singer, Pelczer & Voica, 2011; Singer & Voica, 2011, 2012; Voica & Singer, 2012) on students' PP (assimilated as novices in PS and PP), remarked that such ill-defined problems occur relatively rarely.

A possible explanation may be a tendency of students (obtained as a result of instruction) to generate problem "school like" with a well-defined structure. Another possible explanation is the methodology used in the cited studies: every time the students were asked to pose problems and to include a solution. Perhaps this focus, both on problem generation and solving, constrained the students to keep consistency of mathematical problems. This seems to confirm, from another perspective, an old conjecture: PP is deeply related to PS.

We continue to analyze this relationship by considering the following example:

*G4-p: Estimate the height of the structure in the right hand side picture.*

*G4-m: Estimate the height of the structure by using your own measurement tools.*

*C4: How is this different? They have not changed the problem!*

The G4 group members did not identify differences between their initial problem and the modified one. A closer analysis shows, however, that G4-m is more teaching-oriented. In fact, to solve the originally posed problem it is necessary, for example, that the solver compare the height of the birds at the top of the construction, or the length of the car in the background, to the height of the construction. The addition made by the second group for G4-m ("*use your own measurements tools*") is meant to guide the solver in approaching the problem.

Students, as well as teachers, have difficulties when it comes to posing an open situation. The validity of such proposal depends on the poser's purpose. An open ended situation can become a good source for project activities to enrich learning, a fact underlined by some of the groups.

## CONCLUSION

In an unfamiliar context of PP, experts behave like novices (a fact confirmed by other studies – see for example Fischer, Yan, & Stewart, 2003). However, an unexpected difference appeared between the tendency in our sample for problems in which some data are under-specified compared to the careful control of problem modification process shown by students in grade 6 (Singer & Voica, 2011). PP is a complex activity that requires deep understanding and involvement, even from experts.

Although expert-teachers do prefer real-world starting points in PP, producing rich learning tasks for students is still a difficult process, which rarely appears spontaneously. Of course, in a group, negotiation may interfere and some comments may emerge from the difficulty of a consensus. However, when mathematics education experts add commitment within a purpose oriented approach in open discussions, the results may become spectacularly good. Working group activities and inter-assessment contribute to raising the standards of the products among non-homogeneous teams.

This small study suggests some concerns and challenges for teacher-training programs. First, PP is very important because the tasks the students – prospective mathematics teachers get have to be relevant for effective learning. Second, teacher-training programs should explicitly prepare teachers for posing problems because this capacity does not necessarily derive from mathematical knowledge or simple pedagogical knowledge. Third, exposing (future) teachers to interactive PP sessions has the potential to be the more effective way to help them learn how to develop rich learning tasks for students.

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# MATHEMATICS FOR TEACHING AND PRE-SERVICE MATHEMATICS TEACHERS' ABILITY TO NOTICE

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*The study investigates the nature of pre-service teachers' ability to notice phenomena in a classroom, as directly related to the content of mathematics, mathematics teaching and learning. An analysis of 122 pre-service teachers' commentaries on short video clips with clearly framed episodes against relevant expert identified phenomena yielded results that confirm an overall low level of attention given to content-specific aspects of teaching. It also identifies some most significantly omitted issues and suggests a connection with pre-service teachers' mathematics knowledge for teaching.*

## INTRODUCTION

In our teaching pre-service mathematics teachers we noticed the difference between their ability to recognize general pedagogical phenomena and phenomena that are related directly to the teaching of mathematics when observing a mathematics lesson, be it in person, or mediated by a video recording. We realised that this *ability to notice* was possibly a manifestation of mathematical knowledge for teaching (MKT). In a previously conducted study (Vondrová, Žalská, 2012) we confirmed our initial notion about the lack of content-related attention (the term *mathematics-specific* was defined) when our students (pre-service mathematics teachers) commented on a mathematics lesson on video. In order to get a better view of the nature of this problem, we next decided to focus the pre-service teachers' attention by using a collection of selected class episodes, rather than a full class-length video recording, and by providing them with certain general guidelines for the observation assignment.

## THEORETICAL FRAMEWORK

### Ability to Notice and Mathematical Knowledge for Teaching

In the professional realm of teaching, it makes sense to speak about teachers' professional vision which Sherin (2007) describes as consisting of two distinct, but intertwined, subprocesses: *selective attention* and *knowledge based reasoning*. Something stands out for the teacher and he/she interprets it. The term *ability to notice* has been used by several authors (Sherin & van Es, 2005, Star & Strickland, 2008). According to Sherin and van Es (2005), *noticing* involves a) identifying what is important in a teaching situation, b) making connections between specific classroom interactions and the broader concepts and principles of teaching and learning that they represent, c) using what teachers know about their specific teaching context to reason about a given situation.

The nature, aspects and, subsequently, the measure of a mathematics teacher's knowledge necessary for teaching have been a subject under scrutiny throughout the longer history of mathematics education research. The particular endeavour to



recognize this essential part of professional knowledge specific to mathematics teachers gained momentum with Shulman's (1986) theory of teacher knowledge and pedagogical content knowledge, which was recently developed into Ball and others' model of content knowledge for teaching and, specifically, *mathematics knowledge for teaching* (MKT) (Hill et al, 2004). In this paper, we will use this model for a reference point, as it seems to be a suitable one for distinguishing the content-specific phenomena and knowledge. The main areas of knowledge involved in our specific task fall roughly in the domains of Specialized Content Knowledge, Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT), as described by Ball et al (2008). For the purpose of this study, we will also extend the notion of content of mathematical knowledge from "topics and procedures" (Ball et al., 2008, p. 395) to include also knowledge of ways of thinking; mainly problem solving, which in turn we will take to encompass proving, generalizing, and interpreting in mathematics (see, e.g., Harel, 2008).

### **Mathematics Specific Events, Facts and Phenomena**

For the purpose of the paper it is necessary to draw up a definition of relevant *mathematics specific* (MS) entities. We will not distinguish between *facts* and *events* (in accordance with Chevallard, 1988, and Goodwin, 1994). Furthermore, as the idea of noticing is based on specific, concrete, data observation, we will use the word *phenomenon* to refer to an observable fact. Mirroring the above description of the extended model of mathematical knowledge of teaching, here we will understand by *mathematics specific phenomena* (here MSPs) such that could be observed, explained, inferred or interpreted in relation to either mathematical or didactical issues pertaining to the teaching or learning of mathematics (as opposed to the teaching and learning of other subjects). Thus, noticing MSPs can be seen as part of professional vision of a teacher of mathematics as opposed to a teacher of other subjects. Further clarification of this concept will be provided in the methodology and data analysis section.

### **Studying the Ability to Notice in the Context of Mathematics Teacher Education**

There is an agreement in literature that the ability to notice can be studied (and also developed), among others, by letting (pre-service) teachers analyse video recordings of the teaching of others and/or their own (e.g., Alsawaie & Alghazo 2010; Hošpesová, Tichá & Macháčková, 2005; Krammer et al, 2006; Llinares & Valls, 2009; Santagata, Zannoni & Stigler, 2007; Sherin & van Es, 2005; Star & Strickland, 2008). Most of the studies confirm that (pre-service) teachers must learn *what* to notice. Santagata, Zannoni and Stigler (2007) found out that "more hours of observations per se [...] do not affect the quality of pre-service teachers' analyses" (p. 139). This implies that even if the ability is skill-based, it must be supported by some learning experience or reflection supported by knowledge.

With the exception of one study (Kersting et al, 2010), none of the various investigations undertaken to shed light on the ability to notice in an observed teaching situation focused primarily on MSPs. On the other hand, many of them used (written) observation analysis texts to assess the observer's cognitive and/or affective qualities.

In Sherin and van Es (2005) and Alsawaie and Alghazo (2010) the impact of particular video-based training programs on teacher learning was measured by the difference in the quality of pre-service teachers' analyses of an observed lesson, pre- and post-program. They found out that prior to the video based course, the students' analyses of a mathematics lesson tended to include chronological descriptions of most what happened in the lesson with no interpretation and no identification of noteworthy events. Both studies report a significant change after the course in the choice of noteworthy events and the way the participants saw them – they used fewer evaluative comments and more evidence-based comments.

Star and Strickland (2008) conducted a study with 28 student mathematics teachers. On the basis of an expert analysis of a mathematics lesson, they created a set of questions which the students were asked to answer from their memory after seeing a lesson on video. Star and Strickland conclude that the students “largely did not notice subtleties in the ways that the teacher helped students think about content” (p. 118) and that when content was noticed, the students “tended to comment only about whether the content was presented accurately and clearly and/or to provide a chronological description of what the teacher wrote on the board during the lesson” (p. 122).

Finally, Kersting et al (2010) use written video analysis to assess pre-service teachers' knowledge and relate it to their MKT test results. The video analysis was done on 13 video clips that showed classroom situations related to the area of fractions. The observer's analysis was coded and scored for Mathematical Content, Student Thinking, Suggestions for Improvement, and Depth of Interpretation. An overall correlation between the MKT test and the video analysis score was demonstrated, with the Mathematical Content code as the strongest predictor, explaining 37 % of the variance in MKT scores.

In our work, the last two studies (Star and Strickland's and Kersting et al's) play a crucial role: the latter gives tentative validity to using video analysis as indicative of MKT, the former suggests a successful operational use of expert analysis in assessing observers' comments, allowing for a more detailed picture of the exhibited MKT in regards to different possible contexts and mathematical content in the video clips.

## **METHODOLOGY AND ANALYSIS OF DATA**

The research question of our study is: *Which mathematics specific phenomena do student teachers attend to and which do they miss when given a specific focus area?*

The study is based on short video episodes from mathematics lessons which are meant to draw the viewer's attention to particular MS phenomena. The choice of suitable clips was made during a semester long PhD seminar in which ten video recordings of mathematics lessons from TIMSS 1999 Video Study were analysed. Finally, six clips were chosen which included teaching situations which made sense by themselves without the context of the whole lessons and which were not too long (between 2:20 and 8:10 minutes). All of them are from Grade 8.

An *expert analysis* of the clips was made prior to analysing students' responses by the authors and three PhD students from the department. This analysis defined what was considered to be Expert Mathematics Specific Phenomena (or Expert MSPs). This process was made easier by the fact that the clips were rather short and chosen because they included some clear MS aspects which the participants of the PhD seminar agreed on. Strictly adhering to MS definition, the expert analysis consisted of 19 Expert MSPs (two for clip 1, three for clip 2, five for clip 3, four for clip 4, three for clip 5 and two for clip 6).

To illustrate the nature of these MSPs, we will describe those pertaining to one of the clips. In clip 5, two problems involving the concept of similarity are posed by the teacher. The first one is set in the real-life context of finding the length between two places around the corner of a school building. The teacher's choice of the problem is relevant both to pupils' life experience and as an example of applying the knowledge of similar triangles in a problem-solving situation. The second problem concerns finding the distance between two places over the river. The implementation of the problems is, however, didactically lacking: the teacher herself finds a geometric model of the second problem context and she only makes an attempt to involve pupils in the problem-solving by asking a series of closed questions and making them perform a sequence of trivial calculations. Also, using a particular similarity ratio to solve the problem, she does not explain the arbitrariness of her choice of the ratio to her pupils. This analysis thus yielded 3 Expert MSPs: 1) the teacher's choice of a problem, 2) the implementation of problem-solving, and 3) a didactically lacking use of similarity ratio when solving the problem.

A questionnaire was used to collect data. It consisted of the links to the video clips, their short contexts and several prompts (unfinished sentences). With these prompts, we hoped to prevent students from writing superficial general comments. Note, however, that no prompt focused specifically on MSPs (e.g., In this clip, the pupils ...; In this clip, the teacher ...; This activity I would .... in my teaching because ...; In my opinion, this activity was ...).

The *participants* of the study are students, pre-service mathematics teachers of pupils aged 12 to 19, mainly in their 4th year of study. They were asked to fill out the questionnaire online, at the beginning of their mathematics education courses. Between February 2009 and October 2012 we received 122 responses.

The *analysis of data* consisted mainly in evaluating the content of the responses against the expert analysis. The filled-in questionnaires were uploaded to Atlas.ti software and coded for the Expert MSPs. Each student's quotation was assigned at most one Expert MSP code. From the very beginning, some restricting rules had to be set as it became clear that students commented on Expert MSPs without actually making any connection to mathematics content. A general comment such as "The teacher chose the problem to motivate the students." could not be included as mathematically specific without further elaboration involving content. Under such restrictions, the authors had to negotiate the grey area of border-line text content. In the end, and in consistence

with previous work, general comments about real-life connections, and motivational aspects of activities were excluded from the coding. Even so, it was difficult at times to decide whether a particular quotation had a MS dimension to be coded and the authors had to negotiate to get a reasonable degree of agreement.

## RESULTS

No of Observable MSP's Noticed	0	1	2	3	4	5	6	8	10
Number of students	22	16	23	21	19	12	6	2	1

Table 1: Pre-service teachers noticing Observable MSP's

Table 2 shows that the most frequently noted MSP were “Pupil offers a different solution” in one clip (note that only about a quarter of students commented on the way the teacher reacted to the pupil's suggestion), and “Teacher establishing arbitrary convention of right angle notation” in another. Just under a half of the students commented on each of these. The latter MSP is rather an exception in that the students did not notice other instances of the teacher's mathematical imprecisions.

The MSPs dealing with the teacher's choice of problems were among the least noted. However, it must be added that the comments here were quite frequent, not from the point of view of mathematics per se (e.g., the modelling of a real-life situation mathematically) but rather general comments about choosing real-life contexts and about motivating students.

In both clip 3 and 4 a MSP concerning the connection with the previously learned content was present. The topic in both clips was the same – an introduction of the same new content but done in different ways. However, while in clip 3 this MSP was noticed by 15 % of students, in clip 4 it was only 5 %. The difference might be explained by the fact that the teacher in clip 3 makes the connection explicit. Noticing this MSP in clip 4 thus demonstrates better ability to notice.

It is beyond the scope of this paper to give results in further detail. Instead, we will illustrate the overall situation on one of the clips: In this clip, the teacher introduces a ‘convention’ that for a right angle triangle  $ABC$ , the right angle will always be at  $C$ . We are not sure why the teacher introduces it, the fact is that he presents it as a valid rule even though he probably means to say that they will use this convention only in the following lessons when they practise Pythagoras' theorem. As this MSP was quite prominent for us, we were surprised that only 46 % of students noticed it, and only 29 % were able to give reasons why they think that the convention is not correct (e.g., “pupils will have problems when a triangle is labelled with different letters than  $ABC$ ”, “they will have problems when the right angled vertex is labelled  $A$ ”).

Observable MSP	Clip #	Fre- quency	% of students
Teacher establishing arbitrary convention of right angle notation	2	56	45.9
Pupil offers different solution	3	54	44.3
Teacher's admitted error and work with it	3	30	24.6
Teacher's work with pupil's solution	3	30	24.6
Problem selection: teacher typifies problems	2	5	4.1
Ineffective use of visuals in teacher's presentation	1	4	3.3
Mathematically misleading treatment of proof/demonstration	1	4	3.3
Teacher uses incorrect or imprecise terminology	4	4	3.3
Problem selection: mathematical modelling	6	4	3.3
Problem selection: absence of concrete values	6	3	2.5

Table 2: Most and least noticed Expert MSPs

The second MSP consists of the fact that the teacher stresses procedural problem solving habits – he stresses a model problem and clearly explains steps in its solution. He implies that the pupils should follow this model procedure. 18 % of students comment in any way on this fact.

Finally, the teacher typifies the problem to be solved in the clip as “a problem on a hypotenuse” (as opposed to “a problem on a leg”) thus, in fact, saying that the formula for Pythagoras' theorem should be used in the form of  $c^2=a^2+b^2$ . 4.1 % of students commented on this critically. The others did not mention it.

## CONCLUSIONS AND LIMITATIONS

In this study, we have investigated the possibility to assess pre-service mathematics teachers' content knowledge for teaching through their ability to notice mathematics specific phenomena. In a previous study, we found and confirmed that pre-service teachers pay little attention to mathematics specific issues when watching a video of a full lesson (Vondrová, Žalská, 2012). In this present investigation we asked whether and how pre-service teachers' ability to notice such issues would improve when the 'test' contains fewer distracting aspects, i.e., when the observed video material consists of short episodes with prominent mathematics content related phenomena. Our findings confirm our preliminary notion that the ability is not significantly improved by either such a choice of video material or by giving the observer more specific guidelines (but not pointing out the mathematics aspect).

Furthermore, we were able to get some insight into the main characteristics of the issue at hand. Most interestingly, pre-service teachers do not look at the choice of a mathematical activity (problem) through the lenses of teaching and learning a specific topic but rather through those of general motivational aspects. Rather discouraging, albeit consistent with our previous study, is the fact that mathematical imprecision and/or incorrectness deployed by the observed teachers overall was noticed in a small minority of cases. This can be attributed to either a trusting, non-critical attitude of the observers towards experienced colleagues or, more worryingly, by their own lack of necessary content knowledge.

Accepting the proposition that MKT is partly manifested through the ability to notice, or that at least – as tentatively confirmed by Kersting et al (2010) – there exists a correlation between the two, leads us to suggestions for further research: does an assessment of a lesson done by a practising teacher show different qualities (in terms of noticing content specific phenomena) to the ones of a pre-service teacher? What specific tasks do pre-service teachers need to be given in their observation assessment in order to show their content knowledge for teaching? The difference in noticing the links to previously learned content in two clips points to an interesting question of studying the ability to notice MSP which are less explicit in the video.

Finally, we are aware of the limitations of our study. Most notably, a one-to-one relationship between a noticed phenomenon and one that is chosen for a comment is difficult to establish; here we could only work with its conjectured existence. Also, using the tool of a written text analysis was weak in cases of vague comments that may have been written with a content specific issue in mind. Lastly, by focusing only on MS comments, we may have painted a rather distorted picture of our students' analyses, leaving them rather flat, although the opposite may be true in many a case. The students commented richly on other aspects, general pedagogical ones, psychological, of management etc., but due to our research question, these were not in our focal point.

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# A THEORETICAL LENS ON LESSON STUDY: PROFESSIONAL LEARNING ACROSS BOUNDARIES

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*Issues of developing lesson study practices as a model of professional learning across boundaries are explored in this paper from a Cultural and Historical Activity theoretic perspective. We consider the work of professional learning communities as teachers seek to expand the object of their classroom activity and in doing so identify systemic contradictions and personal conflicts that arise, leading to the potential for expansive learning. Further to this, we identify potential use for theory-in-action and directions for future research and development.*

## INTRODUCTION AND BACKGROUND

This paper explores, from a theoretical perspective, the professional learning of teachers who seek to develop their practice in relation to supporting students' problem solving in mathematics. Here we use the lens of third-generation Cultural Historical Activity Theory (CHAT) (see for example Engeström, 1987) to consider the development of lesson study groups as professional learning communities. This has been motivated by our work with several new lesson study groups in England aimed at motivating students' study of mathematics through inquiry-based teaching and learning, with a particular focus on problem-solving.

The focus of these lesson study groups on problem solving adds a complexity beyond the 'iconic' Japanese model of lesson study as practised and developed since the nineteenth century. In Japan lesson study is long-established, has high status and is central both to pre-service teacher education and to a teacher-led process of sustained professional development in an education system that 'discourages professional isolation' (Collinson & Ono 2001, p. 227). To this end, lesson study involves a community of teachers and 'knowledgeable other(s)' collaborating in a cyclical process that involves planning a 'research lesson', joint observation of the lesson and critical reflection in a detailed post-lesson discussion. This may lead to the collaborative development of a revised version of the lesson plan and progression once more around the cycle. An important member of the lesson study group is the *knowledgeable other*, a mathematics education 'expert' who aims to make a particularly significant contribution to the post-lesson discussion by providing insights informed by research and in-depth curriculum knowledge.

Lesson study based on the Japanese model has become increasingly widely known and adapted for use across geographical and cultural boundaries since the publication of the Stigler and Hiebert's book *The Teaching Gap* (1999). Perry and Lewis (2009), discussing implementation in the US, for example, describe how US teachers found it hard to elicit students' thinking and to keep that at the heart of post-lesson discussions. Likewise, Doig and Groves (2012), drawing on experiences in Australia, point out a



number of factors that militate against direct transfer of what works in Japan to other cultures. For example, they highlight the high status and stable communities of teachers, stability of educational policy and the less individualistic quality of Japanese culture, and the more flexible approach they take to scheduling research lessons and their subsequent debriefings.

## **MATHEMATICS LESSONS: WHAT MATHEMATICS?**

Central to professional learning through lesson study is understanding and development by the lesson study group of a shared vision of the nature of mathematics teaching. It appears from our survey of the literature in relation to lesson study, and comparative mathematics education more widely, that mathematics teaching is primarily considered as focused on ‘content’ as opposed to mathematical processes. This is noticeable in a number of international studies. For example, in seeking to uncover similarities and differences in patterns of teaching within and across national boundaries, analysis based on the TIMSS video studies (see for example, Givvin *et al.*, 2005) focused on purpose in terms of content, classroom social interactions and content activity (measuring the time that students worked on problems). The ‘Learners’ Perspective Study’ (Clarke, 2006), also gauged students’ performance in terms of mathematical content learned. Similarly, in the United States, research exploring knowledge for teaching (Ball *et al.*, 2008) broadens our thinking in this area beyond subject knowledge to include areas such as knowledge of curriculum, students and general and subject-specific pedagogies but omits reference to underlying issues of mathematical processes.

More recently the OECD PISA series of international comparative studies that quantify student performance on a range of tests in mathematics, and also science and literacy, have raised the profile of problem solving. The framework used by these studies to define the mathematics domain (OECD, 2003), in addition to content, identifies competencies and context and how these blend together in the mathematics tasks which we give to students, thus recognising the mathematical practices in which students then engage. In seeking to ensure our students become better mathematical problem solvers our lesson study communities are therefore attempting to focus on important mathematical processes. In the most recent formulation of the English national curriculum in mathematics with which our schools are working, these are organised using a problem-solving cycle and are termed ‘representing’, ‘analysing’, ‘interpreting’ and ‘evaluating’, with over-arching competencies identified as ‘communicating’ and ‘reflecting’. Our lesson study therefore focuses on research lessons in which students develop mathematical problem solving skills rather than build specific mathematical content knowledge.

## **THEORETICAL FRAMEWORK**

Cultural Historical Activity Theory (CHAT) considers how the activity of a community viewed as an Activity System is mediated by a range of different influences. It builds on the fundamental thinking of Vygotsky about how the action of an individual (subject) in pursuit of a goal-directed outcome is mediated by artefacts,

tools and ‘instruments’ (upper triangles in Figure 1). Luria, Leont’ev and followers (for a summary of their work see Engeström & Cole, 1997), in considering the unit of analysis to be extended to a collective of individuals, identify the additional mediating influences of the community with its division of labour and rules and norms (lower triangles in Figure 1). Central to lesson study, and of course teaching in general, is the Activity System of the mathematics classroom (left-hand triangles in Figure 1). Here teacher and pupils work as a community, with the learning of mathematics as object and, for many, pursuit of certification/qualification as the eventual outcome. There is a clear division of labour in the classroom, with teacher and pupils drawing on a range of tools in their individual actions. In many instances these tools are selected by the teacher, for example the questions they ask, the ordering of conceptual development they choose, texts, manipulatives, technology and so on, but individual pupils may additionally select from their own set of aids and resources. The *modus operandi* of the classroom community, as is the case in all Activity Systems, is culturally and historically situated and develop over time. As a result pupils, teachers and society more widely can be considered to operate within *le contrat didactique* (Brousseau, 1997) that encapsulates the current manifestation of culturally situated expectations of what constitutes a mathematics lesson.

Teachers in their professional lives are members of different communities in multiple Activity Systems determined by the structural organisation of their school and the educational system more widely. For example, mathematics teachers are frequently organised to work collectively in distinct departments in pursuit of the learning of mathematics, but in which individual teachers are involved in very different actions, from those that they carry out in classroom settings, such as developing schemes of work to organise their curriculum.

Lesson study brings into the shared experience of teachers and other educators a new Activity System with the object of professional learning (right-hand triangles, Figure 1). The joint activity of the lesson study group includes developing the lesson

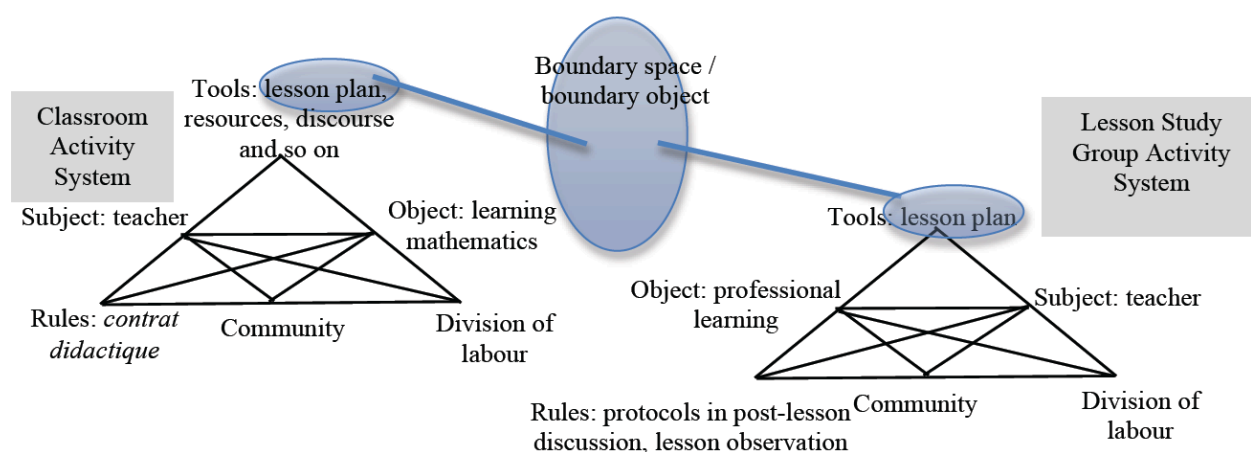


Figure 1: Interacting Activity Systems of classroom (student learning of mathematics) and lesson study group (professional learning)

plan, teaching and observing the research lesson, taking part in the post-lesson discussion and (optionally) refining the lesson plan. Here we are using the term lesson

plan in an inclusive sense to incorporate more than the written document that is developed as a communicative production, but also the shared understanding of teaching and learning intentions that its production-in-action facilitates amongst all participants in the process. We wish to be equally inclusive about professional learning, recognising that for teachers, as for other workers, it is an ongoing process that importantly includes reflection on action on a daily basis. Lesson study, like other forms of formalised professional development activities, however, attempts to make specific aspects of professional learning explicit and provides both time and space in which this can occur and be shared with colleagues, as opposed to day-to-day learning that is often individual, ad-hoc and tacit.

The focus of our reflections in the remainder of this paper is therefore on two Activity Systems of learning: one centred on the learning of pupils in classrooms and the other on teachers and educators and their professional learning in the lesson study community. Teachers are key actors in each of these systems and can be considered as boundary crossers. Here we draw on the notion of boundary as expressed by Akkerman and Bakker (2011, p. 133) “as a socio-cultural difference leading to a discontinuity in action or interaction”. In their review of research into boundary crossing and boundary objects Akkerman and Bakker (2011) argue that from a socio-cultural perspective all learning involves boundaries although as they point out researchers use the term in different ways. Here we emphasise, as do Akkerman and Bakker, the importance of discontinuity as being essential at the boundary, whether it be within or between Activity Systems. In lesson study we consider that it is in the boundary between classroom and lesson study group Activity Systems that teachers, in their reflections on their actions and interactions in each, experience professional learning. The lesson plan, understood in its widest sense, plays an important role as a boundary object. Star and Griesemer (1989), in defining boundary objects, point to their having different meanings in different Activity Systems while retaining a common essence. We see the lesson plan as being at the nexus of understanding of teaching and learning intentions. In the classroom it acts as a mediating instrument as a script by which the teacher organizes the research lesson, but it has other roles to play beyond this at different times in the activity of the lesson study group. For example, in initial planning the lesson plan provides documentation of, and encapsulates, their values, understandings, beliefs and intentions, whereas in the post-lesson discussion it again acts as a mediating instrument, this time facilitating discussion of these and their enactment as pedagogical practices in the classroom.

We contend that the teachers’ professional learning takes place at the boundary and is centred on the lesson plan that, as a boundary object, embodies the group’s shared and emerging perspectives on practice. In this sense it facilitates reflection on action and perspective making and taking (Boland and Tenkasi, 1995) on issues in relation to teaching and learning. Crucially, in this space identity development is supported which, with Wenger (1998), we see as fundamental to learning. Further, we consider members of the lesson study group as undergoing, in Beach’s terms (1999), a *consequential transition*, being in developmental change as they renegotiate their relationship with the social activities with which they are involved. We find ideas of

transition helpful in consideration of lesson study, which by design supports teachers in ‘the construction of knowledge, identities, and skills, or transformation, rather than the application of something that has been acquired elsewhere’ (Beach, 1999, p.119). We therefore view lesson study as supporting teachers in transition as they expand and enrich their mathematics educator identity.

At a system level, Engeström (2001) points to the central role of historically accumulating tensions within and between activity systems, which provide contradictions that are potential sources of change and development. These contradictions often give rise to conflict (Vasilyuk, 1988) for individual members of the community, who may experience destabilization of personal and interpersonal equilibrium. These conflicts often have a much shorter life-cycle than the contradictions (Sannino, 2008), that have their roots in systems that are much less open to change and quick resolution. As individuals reflect upon, question and adopt new actions, the community reconceptualizes the object and motive of the Activity System, giving rise to *expansive learning* that produces new patterns of activity. In relation to professional development, Engeström and Sannino (2010) advocate the term *expansion* to capture the key idea that “learners construct a new object and concept for their collective activity, and implement this new object and concept in practice”. We find these concepts useful in understanding professional learning as more than training or even reflective participation but rather as professional exploration of the nature of current and proposed objects of activity.

## **THEORY IN ACTION, NOW AND FOR THE FUTURE**

The theoretical ideas we have set out above provide a valuable lens which we have used to reflect on our work on professional learning using lesson study with networks of schools in England. In addition to our role as researchers, we also act as ‘knowledgeable others’ and our use of CHAT has provided us with some useful tools, including discourse, with which to consider the conflicts and contradictions that we have not only observed but also experienced.

Focusing lessons on students’ learning of key problem solving processes has proven particularly problematic: from lesson planning, even though drawing on classroom tasks specifically designed to support this, through enactment in the ‘research lesson’ to the post-lesson discussion. Our analysis points to a fundamental contradiction we have introduced in the classroom Activity System, where we wish to expand the object of activity so that pupil learning is understood to include the processes of problem solving whilst continuing to draw on and develop pupils’ facility with mathematical content. As suggested earlier this is not a commonly considered issue in mathematics classrooms and consequently should be central to the professional learning of teachers both as individuals and as collectives. It is apparent that in both the classroom and the lesson study group Activity Systems we are inadequately served in terms of ‘tools’ with which to support teaching and learning of problem solving; for example, well-developed and understood pedagogies and understanding of what constitutes learning, and progress in learning, in relation to the key processes. Questions that arise, for example, include, ‘What are more sophisticated models of mathematical

representations that we might expect from pupils?', 'Which representations provide for useful mathematical insight?', and so on.

Our introduction of lesson study to support professional learning in relation to this expanded object of classroom activity provides a challenge for members of the lesson study group in addition to developing the new activity system. This on its own requires a negotiation of a division of labour and sense of community, a development of a shared understanding of new rules together with an importation of existing, and development of new tools that will mediate the activity of the collective and actions of individuals in pursuit of a common goal and outcome. The establishment of the lesson study group provides many opportunities for disturbance of personal equilibrium across aspects of each individual's professional life ('conflict', in CHAT terms), and we found this to be conflated with the personal dissonance that the contradiction of the expanded object of classroom activity provides. Together these provided a considerable challenge for individuals and the collective in their ongoing day-to-day professional learning.

We advocate CHAT as a powerful theoretical lens through which to view the introduction of lesson study across cultural boundaries, whether geo-spatial or temporal. Our reflections as outlined here lead us to conclude that the theoretical tools that CHAT provides for analysis of lesson study in action will be of use to lesson study groups in facilitating better conceptualisation and understanding of the professional learning with which they are involved. They can be employed to facilitate the development of a common understanding of goals and outcomes of the group, and a discourse with which they can articulate and discuss these. (This parallels the way in which Engeström has used CHAT in his Helsinki change laboratory in his work with Health workers.) In this regard the crucial role of the lesson plan as a boundary object requires careful consideration. The 'new' status of the lesson plan as more than a script for the lesson needs to be recognised. Its use as a tool in mediating teacher activity in the classroom is enriched to encapsulate a practitioner research agenda: the plan embodies the group's research question(s) and methodology. Beyond their use in a single lesson, the lesson plans provide a means by which developing professional knowledge and perspective making of the individual and lesson study group can be used to communicate, both internally and externally, their new and emerging shared understanding (perspective taking).

We additionally value the insights that CHAT provides in relation to systemic contradiction and individual conflict, such as those illustrated here, as potentially useful to 'knowledgeable others' as they seek to negotiate their role within the lesson study group. Beyond this, as lesson study attempts to become established in new cultural settings and mathematics educators act as brokers to facilitate new modes of professional learning, viewing lesson study through the lens of CHAT has the potential to provide insight to inform new research agendas, for example, designing programmes that seek to explore potential systemic change that will introduce contradictions and professional conflict. We see careful design with attention to boundary conditions as having the ability to inform new directions in professional learning and future practice.

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# THE USE OF EXAMPLES TO PROVIDE REPRESENTATIONS IN PROVING

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*To contribute to debate about the need and nature of example generation in the proving process, we analysed videos of students working on different proving problems. Our analysis was based on integrating three frameworks: achieving conceptual insight and a technical handle while proving; manipulating, getting-a-sense, and articulating in mathematical work; and the use of syntactic and semantic modes of proving. A key aspect of successful use of examples in proving is to alight on a representation, or a coordination of representations, that provides both conceptual insight and a technical handle. We illustrate this finding in one problem.*

## INTRODUCTION

It is our personal experience that the exploratory creation of examples can help towards constructing proofs. This practice is supported by Pólya (1962) and many others. Nevertheless the notion that students could be usefully advised to do this has recently been challenged by Ianonne, Inglis, Mejia-Ramos, Simpson, and Weber (2011, p.13) whose experiments concluded that there was not enough evidence for example generation to be ‘a viable pedagogic recommendation’. This apparent contradiction led us to focus not on example generation as an imposed strategy, but on how students who had experienced the use of examples used it in their proofs.

In our previous work, we analysed video data from 27 university students working in groups on different proving problems, aiming to understand the contribution made by example generation when used naturally (i.e., as a tool that is available to the prover) in the proving process. In Sandefur, Mason, Stylianides and Watson (forthcoming) we identified and illustrated the following four aspects of situations in which example generation has a positive role to play in proving. These aspects conjoin qualities of students and of problems:

- (1) *Experience of utility of examples in proving.* Students have experience of constructing examples and are disposed to do so (e.g., they know how examples can expose structural relationships).
- (2) *Problem formulation.* The problem does not point directly to a productive direction for its solution (e.g. it might be phrased ‘prove or disprove’ or might require reformulation).
- (3) *Personal example spaces.* Students’ personal example spaces include appropriate familiar objects and methods, which can display underlying relationships.



(4) *Relational necessity*. The combination of the problem and the students' resources are such that it is necessary to attend to underlying relations, and cannot be completed by only manipulating symbols.

Our analysis also suggested that a key feature of successful use of examples in proving is the selection of representation, which we discuss in this paper.

## EXAMPLE CONSTRUCTION AND USE

There has been significant research into how students and mathematicians generate examples. Dahlberg and Housman (1997) noticed that students who exemplified widely and visually did gain a better understanding of a new concept than those whose actions were more limited. Iannone *et al.* (2011) extended this work with a large number of undergraduates from different universities. In one study they compared the success rates for proving statements of students who had been prepared in two different ways: one group through generating many examples of the central concept, the other by reading worked examples. Neither group knew in advance what would follow these activities. The authors found no significant differences between the two groups, neither of which seemed to benefit more than the other from the specific preparation in using examples. However, in these studies exemplification was imposed by the researchers. Methods of example generation and use may be different if exemplification is a tool available for use in order to achieve a mathematical purpose (Watson & Chick, 2011). Proving behaviour has been described as either syntactic or semantic. 'Syntactic' means the manipulation of symbols within the given representation system; 'semantic' is indicated by the introduction and use of other representations, which would include exemplification. Alcock and Weber (2010) suggested that semantic use of examples has four conditions for success: the prover can exemplify; examples are 'correct'; examples relate to formal definitions; and examples suggest inferences. The first three of these are subsumed in our third aspect above; the last we see in our fourth aspect, that is in interaction between the problem and the solver. The representation system in which the problem has been presented is also likely to have an influence on the approach to proof.

To summarise, there are few studies of how people spontaneously incorporate example use into purposeful mathematical work and those that exist suggest that it is not only the problem, and not only the disposition of the prover, that influence whether examples might be used or not.

## METHOD

We used a body of videos of students working in groups of two or three to produce proofs which arise in an 'Introduction to Proof' course. The second author has worked with others to create an online video-library for use as case studies, to be discussed in courses on proof (Birky, Campbell, Raman, Sandefur, & Somers 2009) (NSF grant #1020161). The students were either taking a basic 'Introduction to Proof' course or were advanced students who had taken this and some higher-level mathematics courses. They have all been introduced to a variety of proof techniques, including deduction and proofs by contraposition/contradiction/mathematical induction. They

have seen and experienced the need to engage in concepts rather than merely manipulate symbols in proving. The proofs produced in all the videos we analysed were correctly reasoned, but some were incomplete.

## ANALYTIC APPROACH

We selected and analysed 11 videos of students working on three problems with sufficiently different approaches to warrant closer attention. In this paper we use extracts relating to one problem to illustrate our analytic approach, but our final conclusion is based on the full set of videos. We viewed the videos several times separately and together, having extensive discussions about the ways in which students used examples. We compared their actions and words to three established frameworks using a cyclic process of analysis, refinement and re-analysis which tested the frameworks and data mutually against each other. The frameworks were:

*MGA (Manipulating; Getting-a-sense-of; Articulating)*: MGA integrates ideas of Bruner (1996) into a spiral of activity during mathematical thinking (Mason, Burton & Stacey, 1982). The manipulation of mathematical objects includes manipulation and inspection of examples to ‘get a sense of’ underlying structure and relationships. As that structure gradually becomes more coherent it can be articulated.

*S/S (Semantic/Syntactic)*: We intended to distinguish between working within the symbolic system in which the proof statement is made (syntactic), and stepping outside the symbolic system (semantic) such as considering examples (e.g. Alcock & Weber, 2010). From our experience we knew this distinction can be problematic. It is possible to work formally and correctly with symbols and not consider underlying concepts, or for the symbolic form to be used and understood as a conceptual embodiment. It is also possible to use the same representation procedurally or meaningfully at different stages of proof.

*CI & TH (Conceptual Insight & Technical Handle)*: Birky *et al.* (2009) (based on ideas of Raman (2003)) suggested that an important component in proving is recognition of the key idea in a problem. They observed that sometimes students gain CIs into the key idea but do not have access to THs with which to reason, and sometimes students have access to THs but have no CIs to direct their use. We extended this to fit our early observations of prover behaviour and identified a need for the prover to: (1) gain CI that indicates why the statement is likely to be true, and (2) find TH to convert CIs into acceptable proofs.

## PROBLEM: FUNCTION COMPOSITION

We now illustrate how we used these frameworks to describe the work of an advanced pair of students (referred to as students ‘Pip’ and ‘Sam’) on one problem.

Given that  $g$  and  $h$  map  $A$  to  $B$  and that  $fog = foh$ , prove or disprove the following: (1) If  $f$  is onto from  $B$  to  $C$ , then  $g = h$ . (2) If  $f$  is one-to-one from  $B$  to  $C$ , then  $g = h$ .

Part 1 is false and requires a counterexample while part 2 is true and requires a proof. A useful conceptual insight would be for the prover to understand the distinction

between ‘one-to-one’ and ‘onto’. Plausible technical handles would depend on what students had previously found useful in manipulating function composition.

### Students’ Work on Part 1

Pip and Sam were first given part 1. They wrote the problem down. Pip said: “My first instinct is to construct an example.... Maybe we should write down what onto means.” They then drew Figure 1 and stated that this cannot happen if  $f$  is 1-to-1.

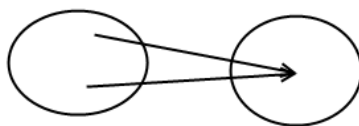


Figure 1: Understanding definition of one-to-one.

They wrote down the definition of ‘onto’ but did not give a specific example, unless we regard the diagram as an example of a way this could be represented. Then they wrote  $f(x)=x^2$  mapping  $\mathbb{R}$  to  $\mathbb{R}$ . They also wrote that  $g$  and  $h$  are from  $\mathbb{R}$  to  $\mathbb{R}$ .

Sam : We have to show they are the same if their composition is the same.

Pip: Our example is too complicated. ...Is your instinct that it works?

Sam : Yeah.

Pip: Yeah, my instinct too, I don’t know if it’s true, but here is what I’m thinking, let’s say that we have A (draws circle as in Figure 2), we’ve got B (draws and labels circle B) and we’ve got C (draws and labels C), okay, now, I’m thinking, (unintelligible) for a second, ... all we know is, we know, we know,  $f$  from B to C, (pauses with pen ready to draw from B to C) I think it’s false (pulls pen back), let me show you why (moves pen back toward figure) if, let’s say this is  $f$  (writes  $f$  between B and C) and it takes this point to C (plots point in B and draws line to C) then all that it’s saying, as long as  $g$ ...

He then drew  $g$  going from  $x_1$  and  $h$  going from  $x_2$  in A to point F in B, as in Figure 2. After a little mumbling by both, Pip said “but these are different  $x$ ’s.”

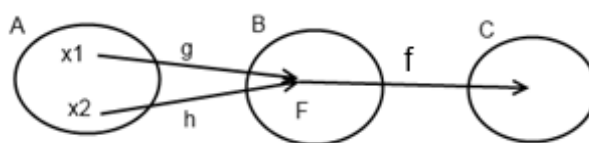


Figure 2: Initial attempt at understanding composition.

Sam then drew the arrow from  $x_2$  to a different point in B and from that to the same point in C. After a little more discussion, Pip said: “What you drew shows that it’s not true. We don’t know anything about  $g$  and  $h$ , so imagine, we can say that  $g$  and  $h$  aren’t 1-1, something like that, right, so imagine whatever crazy we can come up with, let’s say that  $g$  and  $h$ , they have to both act on the same  $x$  and let’s say  $g$  maps it to one point and  $h$  maps it to another point and  $f$  maps both points over here (points to one point in C) because  $f$ ’s not 1-1 so that’s a counterexample, see what I’m saying.”

Pip then drew the correct diagram, as seen in Figure 3. Pip said that this figure does not exclude  $f$  being onto. At one point Sam said: “ $g$  could be  $x+1$  and  $h$  could be  $x+2$ ” (at this point both  $B$  and  $C$  had been defined as given), and then they said in unison: “ $g$  takes 0 to 0 and  $h$  takes 0 to 1”. Sam pointed to  $h$  and said “that could just be a translation.” After some discussion and explanation for Sam, Sam then said “we could have  $f$  map everything onto the same point in  $C$ ”. Pip said that then  $f$  would not be onto, but Sam countered that  $C$  could consist of only one point. They proceeded to construct the counterexample where  $A=\{0\}$ ,  $B=\{0,1\}$ , and  $C=\{0\}$ ,  $g(0)=0$ ,  $h(0)=1$  and  $f(x)=0$ .

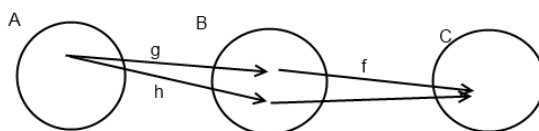


Figure 3: Correct interpretation of the problem.

### Commentary on Part 1

The students initially tried to construct an example of a function to get more insight into the problem while almost simultaneously trying to understand the definition of ‘onto’. They also used a diagrammatic representation from the start, so have made two moves which in the S/S distinction would be seen as semantic. The diagram is manipulable (M), whereas they were unable to manipulate the example they have chosen. They then tried to develop a ‘less complicated’ example, using the arrow representation as a technical handle (TH) to illustrate what might be possible (G), and they appeared to get some conceptual insight (CI) about the statement being false. They then developed a counter-example (A). The problem formulation of ‘prove or disprove’ seemed to have prompted a need for insight into whether the statements were true or false. They had drawn on personal example spaces (PES) of simple functions and of general representations to provide manipulable objects. The representation embedded a CI about possible routes between domains and images more obviously than did their algebraic examples. This suggests that it was helpful that students’ PES afforded movement from one representation to another until they gained insight into proving the statement. This could be described as finding an alignment between CI and TH through alighting on an appropriate representation. However, only one student seemed to have this facility. Pip wanted  $C$  to be the real numbers, but Sam talked him into using  $\{0\}$ . It is plausible that Pip’s past experience of functions which have the reals as the domain has led to  $R$  being treated as a prototypical function domain. We also note here the shift from a general class to finding a single counterexample, and Sam seemed to want formulae for  $g$  and  $h$  even though  $A$ ,  $B$  and  $C$  could only have 1 or 2 values. It is possible that Sam’s PES of functions consists of formulae. The two students appeared to want different levels of abstraction for the final articulation and also used numerical examples differently, possibly because of their different PES. The construction of a special, economical, numerical example in this case to counter a statement and to demonstrate the underlying structure seems to straddle the S/S distinction.

## Students' Work on Part 2

When given part 2 of the problem, the students wrote it down with a definition of 'one-to-one'. Pip drew Figure 4 and said: "It's obvious this time that if it is mapping to a point (from B to C) it's only coming from one because that's the one-to-one part." He drew arrow from B to C in the figure and continued "But here's the problem for me. Does the fact it is not 'onto' mean anything?" They convinced themselves that it does not matter. They then drew the two arrows from A to B in Figure 4. Sam then started talking about  $g(x)=x+1$  and  $h(x)=x+2$ , examples proffered but discarded in part 1, and drew Figure 5. Pip was at first convinced that this was a counterexample, but then realized that both  $g$  and  $h$  must start from the same point.

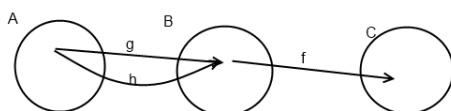


Figure 4: Initial understanding of part 2 of problem.

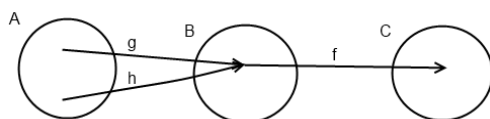


Figure 5: A misunderstanding of problem.

Pip said: "We are trying to show for every  $x$ ,  $f(g(x))=f(h(x))$ , but this shows there is some  $x_1$  and  $x_2$  such that  $f(g(x_1))=f(h(x_2))$  and these are different." Pip still gets confused and is not sure what this shows, so he says, "let's take your example,  $g(0)=2$  and  $h(1)=2$ .... don't we have to show that  $g(0)$  does not equal  $h(0)$ ?... and then you have to show  $f(g(0))=f(h(0))$  ... (he then started to draw Figure 6) and from your example  $h$  has to map here (draws  $h$  arrow from same point as  $g$  arrow in A to different point in B) but then  $f$  has to map back to the same point on C (draws arrow from B to C to same point as other arrow) NOT ONE-TO-ONE" and slams pen down. From here, they were able to write a proof of the statement using the diagram; they assumed  $g \neq h$  which means for some  $a$ ,  $g(a) \neq h(a)$ . Since  $f$  is one-to-one,  $f(g(a)) \neq f(h(a))$ , hence contradiction.)

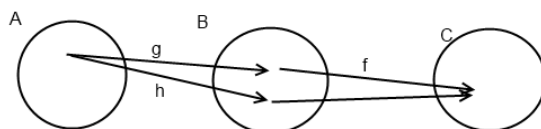


Figure 6: Basis for proof by contradiction.

## Commentary on Part 2

We note first that if our earlier conjectures about students' PES for functions and their domains are true, their PES did provide the appropriate tools for a proof. However, the diagrammatic approach that was successful for the students in part 1 did not appear to help the students in part 2 and so Pip resorted to numerical examples. This to-and-fro between numerical examples and diagrams provided the basis for a proof by contradiction. The example was not used to signify generality at first, but to explore

structure. In part 2 (similarly to part 1) the students tried to work deductively with definitions of one-to-one and onto, and resorted to the diagrams and examples to ‘get a sense of’ the definition. This is *relational necessity*: for these students in this situation examples were necessary for them to understand the mathematical relations. They could not work with the symbols until they understood the effect of ‘one-to-one’. This does not imply that this would be true for everyone tackling this problem, nor for these students for all problems. The formulation of this problem is such that part 1, which requires a counterexample, may have established exemplification as a useful approach for part 2. They had chosen a representation that allowed the definitions to be drawn and seen as mappings, rather than remain as abstract qualities. Alignment of the CI, that is the nature of ‘one-to-one’ functions, and the TH, that is the possible mappings from the same input, came about through their use of the particular representation involving arrows and ‘blobs’ for sets.

## CONCLUSION

This discussion gives a view of our analytical approach and also typifies the kind of behaviour we observed. This problem situation had the four required characteristics for the use of examples for proving that we listed in the introduction, i.e., suitable experience of utility of examples; problem formulation; suitable PES; and there was relational necessity. It also illustrates Alcock and Weber’s (2010) requirements, i.e., the ability to exemplify; that examples suggest inferences; that examples are ‘correct’; and that examples relate to formal definitions. We also need to state that our whole sample of videos included several pairs who began by trying to prove a statement and resorted to examples only if relational necessity arose, thus showing that the ‘Introduction to Proof’ course had not imposed exemplification as a necessary action in proving.

We are able to say more: that it is the alignment of CI and TH within a suitable choice of representation that appeared to lead to a method of proof in many of our videos as it did above, that is the finding of a technical handle that somehow models the students’ emerging grasp of the underlying concepts.

Figure 7 represents the integration of CI & TH with MGA that we have demonstrated above. We would not claim that a syntactic approach might provide a technical handle while a semantic approach provides CI, although it was tempting to assume this when we set out to do our analysis. Rather, in most of our videos either the TH or the CI could arise while translating between representations. Finding a representation that aligns TH and CI was a key step in all the proof processes. This can be seen in Figure 7. In only one of our cases was alignment found in the representation in which the problem was posed.

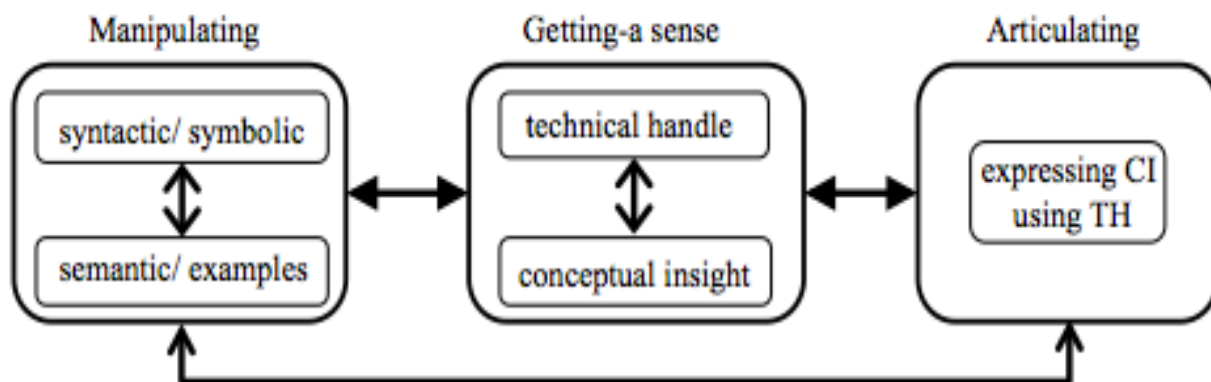


Figure 7: Integration of CI/TH with MGA.

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# WHICH KINDS OF TASKS DO MATHEMATICS TEACHERS SELECT FOR INSTRUCTION, AND WHY?

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*Instructional tasks are a central element of mathematics lessons. We report on an exploratory interview study with N=17 mathematics teachers, investigating their lesson planning activities. Our aim was to explore the role of aspects of instructional quality in teachers' reasoning when selecting instructional tasks for a particular lesson. We coded the potential of the selected tasks as well as the types and quality of reasons teachers give when selecting instructional tasks. The results suggest that teachers differ considerably in the type and quality of reasons they use to explain their task selection, in particular with respect to reasons that refer to creating cognitive learning opportunities. Based on theoretical explanations, we discuss hypotheses for future research on teachers' lesson planning and task implementation.*

## INTRODUCTION

Stein and Lane (1996) describe instructional tasks as activities teachers and learners engage in that are “oriented toward the development of a particular skill, concept, or idea”. The TIMSS study reports that students spent 80% of their time in mathematics instruction working on instructional tasks (Hiebert et al., 2003). Bromme (1981) found that mathematics teachers strongly concentrate on the task selection and on the anticipation of the enactment of these tasks when planning a lesson. Accordingly, selecting instructional tasks that have the potential to support students' learning, implementing these tasks in the classroom, and unfolding their potential may be considered central aspects of mathematics teachers' professional competence. Research on instructional quality yields first criteria for adequate task selection and implementation from a theoretical perspective. Nevertheless, even though there is considerable research on teachers' professional knowledge and competencies (c.f. Lindmeier, 2011), it is unclear to which extent teachers draw upon aspects of instructional quality or if their task selection is guided by other ideas.

## THEORETICAL BACKGROUND

### Quality of mathematics instruction

Over the past decades, educational research has identified three basic dimensions of instructional quality (c.f. Baumert et al., 2010): classroom management, student orientation and cognitive activation. Classroom Management describes the extent to which classroom time and activities are devoted to learning processes, as compared to organizational or disciplinary issues. Student orientation means to address students' individual needs and learning potentials (e.g. Buff et al., 2011). Indicators of student-oriented instruction include giving the students possibility to choose autonomously between alternative learning tasks or learning paths, providing



individualized student support, or a supportive classroom climate (e.g. an error tolerant classroom atmosphere). Cognitive activation describes the potential of a learning opportunity to stimulate learners' insightful cognitive learning activities. Cognitive activating instruction offers for example challenging tasks, activation of prior knowledge, and deals with errors in a discursive way (Baumert et al., 2010).

While cognitive activation and student orientation comprise subject-specific aspects (e.g. the cognitive demands of a task, or the nature of relevant subject-specific student support), classroom management may be regarded as a general, less subject-specific dimension of instructional quality.

### **Instructional tasks as indicators of potential for cognitive activation**

Given the theoretical consensus that insightful learning goes along with high-level cognitive processes like analysing, arguing, or reflecting (Seidel & Shavelson, 2007), instructional tasks that stimulate these higher-order activities are considered one effective indicator for the dimension cognitive activation and also insightful learning processes. Indeed, it has been shown that the didactical quality of tasks, especially their potential for cognitive activation, predicts student learning (Baumert et al., 2010; c.f. Hiebert & Wearne, 1993; Stein & Lane, 1996). Recent research has developed categorization schemes to describe the potential of a task for cognitive activation. A scheme by Jordan et al. (2008) describes the type of a mathematical task (three levels: purely technical, computational modeling, conceptual modeling), the level of mathematical argumentation required, the necessity to use mathematical representations or the level of modelling (inner and extra mathematical; all classified in four levels: not required, low level, intermediate level, high level). Baumert et al. (2010) could show a direct connection between quality of tasks selected by teachers and student learning within one school year. Nevertheless, studies report a low level of task quality in educational practice (Jordan et al., 2008). It remains an open issue, if teachers consider relevant criteria of task potential in their planning decisions.

From a theoretical perspective, high task potential for cognitive activation may be seen as necessary for cognitively activating instruction. Nevertheless, as Stein and Lane (1996) point out, high task potential alone is not sufficient for sustainable learning processes. Tasks must also be “orchestrated” adequately in classroom instruction to initiate learning.

### **Task selection as a part of mathematics teachers' professional competence**

Planning instruction is seen as a core teacher task (Blömeke et al., 2008; Lindmeier, 2011), and mathematics teachers strongly concentrate on the task selection when planning a lesson (Bromme, 1981). Thus, we consider the tasks selected by teachers as the “substrate” of their lesson planning. Nevertheless, aspects of instructional quality other than task potential have to be considered in their interaction with task selection: Cognitive activation includes, apart from challenging tasks, activation of prior knowledge, exploration of student thinking and, in general, a constructivist understanding of learning (Baumert et al., 2010). To address student orientation, one

should consider individualized instructional support, possibilities for choice between learning opportunities, and self-regulated learning (Buff et al., 2011).

There are at least two ways how teachers may realize these additional aspects of instructional quality. Student support and cognitive activation may to a certain extent be results of spontaneous reactions in the classroom, based on an experienced teacher's accumulated professional routines and implicit knowledge. Yet, adequate task implementation may depend as well on conscious decisions during lesson planning. Sullivan, Clarke, Clarke, and O'Shea (2009) report that a single task may be implemented totally differently, depending on teachers' goals and professional knowledge. Higher order thinking will only occur if good tasks are used in a challenging way (and not broken down to trivial sub-tasks; Stein & Lane, 1996), which is more likely if the teacher is aware of the tasks' specific potential. Thus, we may assume that considering aspects of instructional quality during lesson planning is a valid indicator of high quality instruction in the classroom.

Research shows a clear influence of teachers' planning processes on student learning, which is due to the cognitive potential of instructional tasks (Baumert et al., 2010). Yet, even though these aspects have a clear impact on student learning (Seidel & Shavelson, 2007), it is widely unknown if mathematics teachers consider aspects of cognitive activation and student orientation in lesson planning. Nevertheless, knowing this is important to conceptualize pre- and in-service teacher training.

## RESEARCH QUESTIONS

The aim of our study was to explore the role of aspects of instructional quality in teachers' reasoning when planning a particular lesson, with a specific focus on task selection. We included teachers of varying qualification to get a broad view of different planning processes. Our study was guided by the following questions:

- Do teachers refer to basic dimensions of instructional quality when giving reasons for selecting a particular task?
- Which other types of reasons do teachers give for selecting a particular task?
- Do teachers with additional qualifications in mathematics education give more or more elaborate reasons for their task selections than regular teachers with basic qualification in mathematics education?

## METHODOLOGY

We conducted semi-structured interviews with N=17 German secondary school teachers. Eight teachers were classified as having particular expertise in teaching mathematics and additional qualifications in mathematics education (AQ group, e.g. serving as subject-coordinator), nine teachers were classified as regular teachers with basic qualifications in mathematics education (RQ group). These RQ teachers had finished regular university studies and a 2-year in-service teacher preparation but had no additional responsibilities for mathematics education at their school. We asked each teacher to plan a unit on addition of fractions with unequal denominators for a fictitious class. We gave a rough characterization of the learning group (average, but

heterogeneous achievement, equal number of girls and boys) and detailed learning goals (establishing understanding of addition of fractions with unequal denominators). The teachers could use their usual text book, self-developed tasks or tasks from a pool of 18 tasks provided by the researchers. These 18 tasks from the literature were selected by the research team to offer a broad basis of tasks. Teachers were not required to solve the tasks. We asked the teachers to verbalize their thoughts during the planning process. To focus teachers' verbalizations on relevant aspects for this study, they were provided with a planning form to record the selected tasks, learning goals associated with each task, ideas about task implementation, and reasons for choosing this particular task at this position of the lesson.

The interviews were videotaped and analysed together with the planning forms. We analysed each of the  $N=143$  single task selections made by the 17 teachers separately. The research team coded the *potential for cognitive activation* for each task, covering the five dimensions of cognitive activation from Jordan et al.'s (2008) coding scheme (see above), as well as the following four additional dimensions: activation of prior knowledge, potential for cognitive conflicts, possibility for making estimations, and opportunities to find mathematical structures. These dimensions were coded on a four-level scale (0: not required/ 1: low level/ 2: intermediate level/ 3: high level).

A detailed coding protocol was developed in two steps to code teachers' reasons for task selection: First, reasons referring to basic dimensions of instructional quality were identified using a theory-based coding scheme. For each task selection, the *quality of reasons* referring to each of our nine dimensions of cognitive activation was coded on a three-point scale (0: not mentioned/ 1: only mentioned/ 2: mentioned and elaborated with respect to the potential of the specific task). Similarly, the reasoning with respect to the following dimensions of student orientation was coded on a two-point scale (0: not mentioned/ 1: mentioned): clarifying content relevance, providing choice, allowing self-regulation, and individualization. In a second step, an explorative coding approach was used to identify other types of reasons for task selection. These reason types were then coded by the first author and an independent coder on a two-point scale (0: not mentioned/ 1: mentioned).

## RESULTS

Teachers selected between six and twelve tasks (Mean=8.5; SD=1.7), with only small differences between the AQ and RQ group (Mean<sub>AQ</sub>=9.0; Mean<sub>RQ</sub>=7.9). The average task potential for cognitive activation was very low in general (Mean=0.68, SD=0.37, codes run from 0 to 3), replicating findings by Jordan et al. (2008). It did not differ significantly between tasks from the two teacher groups ( $t(141)=0.16$ ,  $p=.87$ ).

Figure 1 shows the ratio of task selections that were explained by different reason types in both teacher groups. On average, 50% of the task selections were supported by reasons that were connected to student orientation. Most of these reasons referred to allowing self-regulation and individualization. Task selections from both groups were explained by this aspect almost equally often ( $\chi^2(1)=.01$ ,  $p=.93$ ).

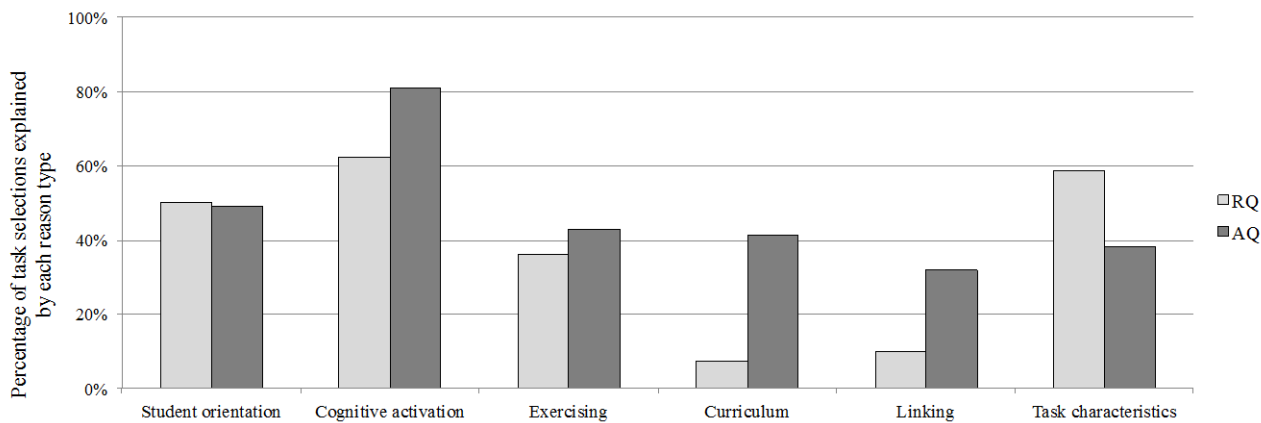


Figure 1: Reasons for task selection (individual ratio of task selections supported by each reason type, averaged over participants).

On average, 71% of teachers' task selections were explicitly explained by aspects of *cognitive activation*. While teachers rarely mentioned the potential to activate cognitive processes like mathematical modelling or mathematical argumentation when explaining a task selection, more reasons concerned activation of prior knowledge, provoking cognitive conflicts, or using mathematical representations. AQ teachers (81% of all task selections) mentioned this aspect significantly more often than RQ teachers when selecting tasks (63%;  $\chi^2(1)=5.79$ ,  $p < 0.05$ ).

In some cases, teachers referred to instructional concepts without explicating a connection to one of the a-priori categories analysed above. In an exploratory coding, further types of reasons for task selection were identified. These reasons could finally be grouped into the dimensions *exercising*, *curriculum*, *linking*, and other *task characteristics*. When explicitly stating *exercising* as a reason for task selection, teachers usually gave a goal for the exercise: automation of the procedure to add fractions (50% of these task selections), or elaborative exercises that should strengthen understanding for this routine (e.g. finding errors in examples, 35%). Application exercises played a minor role, which might be due to the specific aim of the lesson. Reasons referring to goals in standard or curriculum documents (i.e. a task that should help to implement a certain standard, 22%) were coded on the dimension *curriculum*. With about every fifth task, the teachers stated they wanted to link ideas within lessons, between lessons, or between mathematical content areas. These three types of *linking* (Shimizu, 1999) were mentioned almost equally often. The dimension *task characteristics* describes other arguments including, e.g., comments relating to the linguistic complexity of a task.

*Linking* was mentioned in task selections made by AQ teachers (32%) more often than in those made by RQ teachers (10%;  $\chi^2(1)=10.59$ ,  $p < .01$ ). AQ teachers also explained significantly more selections by *curriculum* reasons (41%) than RQ teachers (8%;  $\chi^2(1)=23.14$ ,  $p < .001$ ). There was no significant difference for *exercising* ( $\chi^2(1)=.65$ ,  $p=.42$ ). Contrary, other aspects from the category *task characteristics* were more often posed by RQ teachers (59%) than by AQ teachers when selecting tasks (38%;  $\chi^2(1)=6.02$ ,  $p < .05$ ).

For each of the nine sub-categories referring to cognitive activation, also the *quality of teachers' reasons* was coded on the three-point scale (0: not mentioned/ 1: only mentioned/ 2: mentioned and elaborated with respect to the specific task). For the following analysis, we averaged quality rating of teachers' reasons over the nine aspects of cognitive activation for each task selection. Similarly, the task potential ratings were averaged to obtain a measure of *task potential* for cognitive activation. *Reason quality* ratings and *task potential* were significantly correlated ( $r=.52$ ,  $p=.01$ ). As the scatter plot (figure 2) indicates, RQ and AQ teachers generally gave arguments of relatively low elaboration for tasks with low potential. Nevertheless, with growing task potential, the elaboration of AQ teachers' arguments increased more strongly than that of RQ teachers' arguments. This difference was also supported by a significant factor-covariate interaction in an ANCOVA for *reason quality* with factor *teacher group* and covariate *task potential* ( $F(1,139)=4.63$ ,  $p<.05$ ).

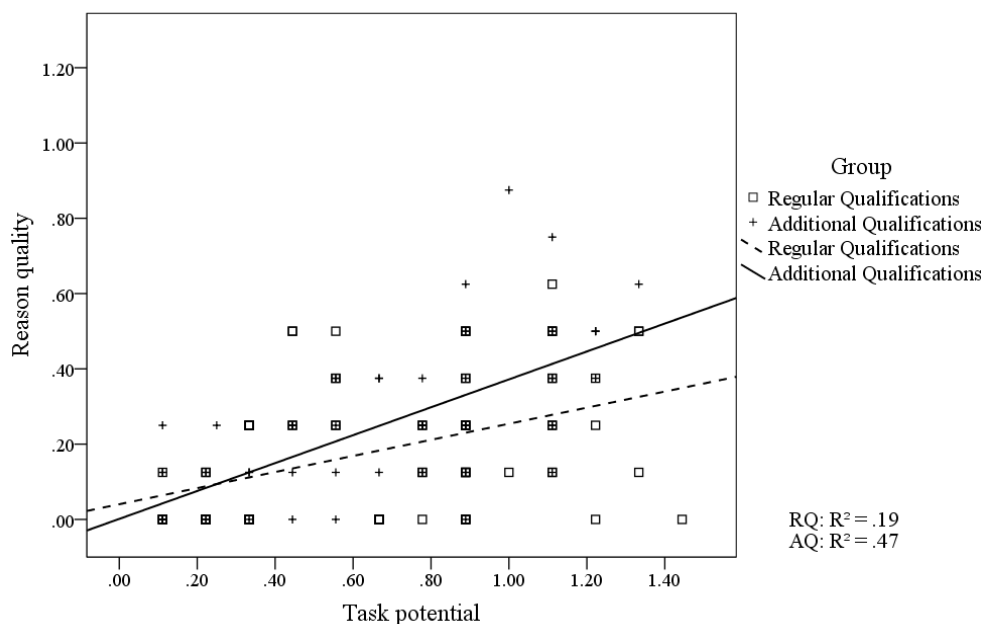


Figure 2: Reason quality and task potential for task selections from both teacher groups.

## DISCUSSION

The aim of our study was to explore the role of aspects of instructional quality in mathematics teachers' lesson planning. Of course our exploratory study with a small number of teachers, planning a very restricted lesson, cannot provide final evidence on this topic. Nevertheless, the results of our study lead to some hypotheses that will be subject to our further research. Firstly, teachers selected tasks that generally had low potential for cognitive activation from a normative point of view. This was expected (cf. Jordan et al., 2008). Nevertheless, the tasks selected by teachers with additional qualifications (AQ) did not show a higher task potential than those selected by the regular teachers (RQ). This is in particular interesting, since task potential was identified as a central predictor of student learning in Baumert et al. (2010).

Secondly, teachers did indeed explain task selections by reasons that address central aspects of high-quality mathematics instruction like cognitive activation and student orientation. Nevertheless, also other important aspects are taken into account like *linking* mathematical ideas (Shimizu, 1999), the learning goals from standard documents or specific classroom activities like *exercising*. Thus, also other aspects of instructional quality informed teachers' task selections.

Thirdly, teachers of different qualification showed specific differences in the types of reasons they explicated. Teachers with additional qualifications explained their task selections by aspects of *cognitive activation*, *linking* and goals from standard documents (*curriculum*) more frequently than teachers with regular qualifications. Almost equally often, both groups referred to *student orientation* and the aims of exercise tasks. *Exercising* refers more to general aims of instruction like "automation", "understanding" and corresponding classroom activities. These reasons focus less on aspects of activities that provide cognitive opportunities for sustainable learning. On the other hand, *cognitive activation*, *linking*, and clear learning goals (*curriculum*) – reasons reported more often by teachers with additional qualifications – may be regarded to be more proximal to aspects of instruction that support cognitive learning opportunities (Seidel & Shavelson, 2007).

Finally, how much teachers elaborate on the cognitive task potential when explaining their selection seems to depend on teachers' qualifications. It is plausible that for low-potential tasks like simple training problems, elaborate reasons are hard to find. Even though these tasks have an educational value as well, using them in the classroom may be straightforward to teachers. Implementing high-potential tasks, that involve for example cognitive conflicts or multiple solutions, requires at least some awareness of their specific potential. Thus, it would be problematic if teachers from the RQ group did not only fail to elaborate on these aspects, but were completely unaware of them, even though they selected them for their lessons. The relation between task selection, reflection on task potential, and task implementation (real or anticipated) should be subject to further research to identify relevant aspects of teachers' lesson planning.

The aim of our study was to explore the role of instructional quality in teachers' task selection during lesson planning. We found that there are clear traces of aspects of instructional quality in teachers' reasoning, but we also found first indications of differences in type and quality of reasons between teachers with varying qualifications. Our research cannot clarify if these differences are indeed indicators of different awareness of task potential and instructional quality. However, under the assumption that teachers' reasoning is at least a proximal indicator of professional task implementation (which might also be subject to further research), particular attention should be paid to aspects that are necessary to create cognitive learning opportunities, like cognitive activation, linking and goal clarity. This concerns research on teachers' competencies and, if the results can be substantiated, is also important to conceptualize pre- and in-service teacher education.

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# ASSOCIATIONS BETWEEN THE ONTOGENESIS OF CONFIDENCE AND INCLINATION TO EXPLORE UNFAMILIAR MATHEMATICAL PROBLEMS

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*This video-stimulated post-lesson interview study of students displaying confidence in mathematics examines the nature of confidence theoretically by linking it to Seligman's (1995) indicators of optimism. It also explores the activity of confident students empirically; examining their inclination to explore unfamiliar challenging mathematics problems. Findings include associations between student inclination to explore challenging mathematics problems, and the ontogenesis of their confidence. These findings have implications for the teaching of mathematics: 'a transmissive teaching approach' was associated with an absence of the inclination to explore.*

## INTRODUCTION

The Melbourne Declaration of Educational Goals for Young Australians [http://www.mceecdya.edu.au/mceecdya/melbourne\\_declaration,25979.html](http://www.mceecdya.edu.au/mceecdya/melbourne_declaration,25979.html) and the Australian Mathematics Curriculum (see <http://www.australiancurriculum.edu.au/Mathematics/Rationale>) require the development of creative, innovative and resourceful problem solvers. Confidence has previously been linked to mathematical performance (see for example, Fennema & Sherman, 1976). Such links have been stronger between students' performances on multiple-choice questions than performance on open-ended mathematical tasks (Pajares & Miller, 1997). It is time to take a closer look at confidence as a construct to find whether and how confidence is exhibited differently, and whether only some types of confidence are associated with an inclination to explore unfamiliar challenging problems. Williams (2012) identified confident students who do perform well on open-ended problem solving tasks, and confident students who do not. The confident student who was not inclined to explore performed well on tests requiring recall of rules and procedures. This paper examines the confidence and the problem solving activity of five high performing final year elementary students. The research questions are: Are there differences in the nature of the confidence these students exhibit? And if so, are their associations between the type of confidence exhibited and student inclination to explore? This focus is important because problem-solving activity can 'deepen' mathematical understandings (Cobb, Wood, Yackel, & McNeal, 1992; Williams, 2005).

## THEORETICALLY FRAMING THIS STUDY

'Optimism' is an orientation to 'failures' and 'successes' (Seligman, 1995). Optimistic children perceive failure as 'temporary' (able to be overcome), 'specific' (to the situation at hand), and 'external' (can be associated with factors beyond their control). They perceive successes as personal (achieved through their own effort), permanent



(able to be achieved again), and pervasive (internalized as characteristics of self: ‘I did this, I am good at this’).

With regard to optimistic activity specific to mathematical problem-solving successes (‘optimistic problem-solving activity’), ‘failure’ is taken to be ‘not knowing’, and ‘success’ as ‘finding out’. An optimistic problem solver perceives *not knowing* as temporary [Failure as Temporary] and able to be overcome through personal effort [Success as Personal] associated with looking into the situation of failure, and identifying what they can change [Failure as Specific] and what is outside their control [Failure as External], to help decide what to vary to increase their likelihood of success [Failure as Specific]. They perceive they can achieve such successes again [Success as ‘Permanent’] because they internalize their successes as a characteristic of self [Success as ‘Pervasive’].

‘Confidence’ has previously been defined as the “degree to which a person feels certain of her or his ability to learn and perform well in mathematics” (Hart, 1999, p. 243) [Success as Permanent], and related to a personal characteristic: “... one's ability to learn and to perform well on mathematical tasks” (Fennema & Sherman, 1976, p. 326) [Success as Pervasive]. When considered from the perspective of optimistic indicators, confidence is thus consistent with perceptions Success as Permanent and Success as Pervasive.

As optimistic students are inclined to explore unfamiliar mathematical ideas (Williams, 2005), and a confident student possesses the pair of optimistic indicators (Success as Permanent, Success as Pervasive), this study explores the combinations of other indicators of optimism possessed by confident students to see whether the combination of optimistic characteristics possessed provides insights into differences in the nature of confidence.

## RESEARCH DESIGN

The study is part of a broader study of the role of optimism in collaborative problem solving and whether building optimism leads to increased problem-solving capacity. Data selected for this smaller study relates to five students (Patrick, Eliza, Sam, Aisha, and Hank. Williams 2007 and 2008 include some of the data used here for Sam, Eliza, and Patrick but the data is used for a different purpose in the present study. Hank and Aisha have been included as two more confident students who were not inclined to explore. All five of these students were high achieving students on their usual mathematics tests in class, with Hank, Sam, and Aisha in general achieving higher performances on these tests than Eliza and Patrick. The students were in various Grade 6 classes in the same school. Their usual mathematics tests for these students, like the mathematics tests for students in many Australian schools, were predominantly tests about recalling rules and procedures rather than about using the mathematics they have learnt to undertake unfamiliar challenging problems. This following section describes the students selected, one of the problem solving task they undertook (‘How Many Boxes’), the pedagogical approach employed, and the data collection instruments utilized including rationale for why they were appropriate for collecting the data to

answer the research questions herein. The students selected were all confident but they differed in whether or not they were inclined to explore.

Students undertook three complex problem-solving tasks each year (six eighty-minute sessions) for one to three years within the broader study. Tasks were accessible through a variety of representations and levels of mathematical sophistication to give groups opportunities to idiosyncratically discover and explore complexities just beyond their present understandings. Priority was given to the selection of evidence within one task, the How Many Boxes Task for this study to limit the amount of space required for task description, and decrease the amount of information the reader needed to become familiar with to consider the data. A summary of the How Many Boxes Task is included in Figure 1.

Task Introduction: the features of rectangular prisms were discussed and these shapes were identified as the 'boxes' in this task.

Part 1: Groups were asked how many different solid boxes they could find that each contained 24 'little' cubes (cubic centimetres but the term was not provided at that stage). As they worked with this task, they were asked questions like: How many can you make? How do you know that you have got them all? Can you make a mathematical argument for how you know you have got them all?

Part 2: Groups were told they were to participate in a game where each group was trying to be first to find the dimensions of a box given the number of cubes within. Each group could ask a Yes/No question and all groups would have access to the questions and answers before beginning to find the dimensions of the box. Students were given five minutes to brainstorm the types of questions they might ask and during this time they had access to twenty-four little cubic centimetre blocks.

Figure 1. Summary of How Many Boxes Task

The pedagogical approach employed, 'Engaged to Learn', was developed by the researcher (author), informed by her research (Williams, 2005), and her teaching (Williams, 2002). As teacher, the RT and classroom teacher (T) team-taught with the RT as primary implementer of the task. Students worked in small groups composed by RT advised by T (3-4 students) (see Williams, 2008). Students gave brief reports to the class at 5-10 minute intervals. The order in which the groups reported was decided by RT and T. RT and T did not affirm pathways taken nor ideas presented but rather asked questions to stimulate further thinking. For more information about the teaching and learning approach, see Williams (2007).

## DATA COLLECTION TECHNIQUES

Four video cameras captured the activity of each group in class during their problem solving sessions. After each session, the worksheets and artefacts produced by each group were collected and used as additional stimuli during individual post-lesson video-stimulated student interviews undertaken individually with four students after each lesson. Video-stimulated interviews increase the validity of student

reconstructive reports through focus on memory traces related to specific activity that occurred (Ericsson & Simon, 1980). In their interviews, students had simultaneous access to video of their group and the reporting sessions. They identified and discussed parts of the lesson that were important to them and reconstructed their thinking and feelings during those parts of the lesson. These interviews informed the analysis of lesson video by helping to locate the parts in the lesson where new ideas were developed and the processes through which this occurred, and providing information about how students learnt mathematics, and how they perceived themselves as learners. Students also answered questions designed specifically to provide dialogue to identify optimistic or non-optimistic perceptions. For example, the questions: “How do you think you are going in maths, and how do you decide?” “How do you learn something like that [mathematics associated with the problem solving task]?” and “Does anyone help you with maths at home/outside school? tended to elicit information about whether students perceived Success as Personal or External. Students displayed indicators of Success as Personal where they perceived learning as predominantly associated with personal effort in reorganisation and synthesis of previously developed ideas to create new mathematical ideas. Indicators of Success as Personal were also displayed when students primarily evaluated their mathematical performance internally rather than through external sources like test results, or teacher or parent evaluations. In contrast, where students relied primarily on external judgments of their performance and described the way they learnt as occurring through ‘taking in’ and repeating of knowledge from external sources, they displayed indicators of Success as External. These questions also elicited information about how students perceived their performances in relation to future performances, and in terms of characteristics of self. For example: “I am really good at maths because I always get high marks on tests” indicates Success as Pervasive “I am really good” and Success as Permanent “I always get”. Questions like “Can you tell me what you were thinking about there? And, how did you work that out?” can elicit data about how a student altered variables to increase the likelihood of success [Failure as Specific]. Indicators of optimism or lack thereof were also displayed when students discussed how others might consider them when they ‘got something wrong’ (e.g., “they might all think I am an idiot”) [Failure as Pervasive], or “they would know that calculation is tricky” [Failure as Specific]. Optimistic or non-optimistic indicators are not always evident in responses to a particular question, and the probes following each question depend upon the student’s response. Thus indicators of optimism can be found in various parts of the interview transcript depending on the responses students give, and the probes the researcher is able to introduce.

## RESULTS

In Table 1, data drawn from interviews and lesson activity has been synthesised to develop a summary of how each student perceived learning occurred, and illustrate the nature of their responses to the ‘box’ task. A small selection of illustrations of the data is then presented to illustrate the type of analysis undertaken.

<b>Student</b>	<b>What is Learning</b>	<b>Excerpts of Activity During Box Task</b>
<b>Sam</b>	Listen to teacher, read text book, search on internet	Knew volume formula initially; generated many relevant sets of three factors; realised these were the box dimensions; at end of task, still did not know why multiplying these dimensions gave number of cubes in box.
<b>Hank</b>	No interview in Grade 6. T confirmed learnt from external sources.	Recognised factors of 24 pattern after two boxes were made. Generated as many sets of three factors of 24 as he could by considering only the numerical pattern. Disregarded other group members, and the RT when they asked 'why?' the pattern existed. By the end of the task, still had not worked out why that particular pattern helped give the number of little cubes in the box.
<b>Aisha</b>	Listen to a teacher, family member, and expert student, explaining bit by bit.	Generated two sets of two factors of 24 fast. When another member asked for explanation, she gave only the procedure. Listened intently to other groups linking their 3 factor solutions to the structure not just the dimensions of the box. Excitedly identified that there were an infinite number of possibilities using fractional and decimal triplets (with product 24) began to produce them but not link to box.
<b>Patrick</b>	Think about what others are still trying to work out, and about mistakes other groups make.	Reflected on why Sam had not recognised a member of his group was reporting on a 3x3x3 cubic box rather than a box containing 24 cubes. Wondered if they had missed the middle cube. Used ideas his group developed about layers of cubes in a box to solve a problem encountered by a group who had made a box with 24 cubes when they had intended to make one with 12 cubes within it.
<b>Eliza</b>	Think hard, puzzle it out, if 'stuck' ask parents to ask questions not give answer.	Relied heavily on building with cubes until she began to work out what was happening. Provided lateral contribution when group did not have sufficient cubes to make the box they wanted to explore: drew grids of layers of cubes they could use to finish making the block stack and work out how many were needed.

Table 1. Students, how they perceive learning, and responses to parts of Box Task

Table 1 shows that Sam, Hank, and Aisha perceived learning to involve the assistance of external resources or experts that provide information about relevant rules and procedures. These three students' performances with the 'box' task were consistent with this. They were unable to develop new ideas so remained within what they knew because there was not an 'expert' to rely on. Each used numerical procedures in ways

they had previously used them, and generated long lists of examples of the number pattern they identified. They focused on the numerical representations completely (Aisha, Sam) or also linked the numbers generated to the rule previously known to identify the dimensions of the box (Hank). In class Hank was sure he had finished and that the RT just did not understand what he knew:

RT: [to Hank] ... does the number pattern, which you beautifully explained yesterday, fit with (pause) those actual cubes in that box- ... I don't just mean length width and height (pause) why when you multiply those together (pause) do you get (pause) the total number [of little cubes] (pause) in that box [RT leaves group] ...

Hank [to group, fast and soft] factors are numbers that are multiple ... you can multiply factors to get the number

The RT's comment (in transcript above) is intended to elicit thinking from Hank about the actual structure of the cubes in the boxes. Hank's response is a cyclical argument that draws on the definition of 'factor'. He is communicating that the dimensions must be multiplied to get 24 because they are factors of 24. Even though asked as part of the class, and in his group, on many occasions, Hank gave no thought to why it was that product of the dimensions gave the number of cubes in the box. Neither Aisha, or Sam, or Hank was able to link the numerical work they generated to the structure of the cubes in the boxes. This is illustrated with Hank's response to the RT's hint about the cross-section of the box whose dimensions they were finding Part 2 of the task:

1. RT ... the cross section to it [the hidden box] has nine little squares in it
2. Hank [in his group writes/draws on page] ... *hu ... that's weird!*

The term 'cross section' had been explained earlier, and some groups were interpreting and using this clue. Hank's exclamations and his subsequent non-activity showed he was unable to use the information given. He was unaware of the structure of the cross section.

Patrick and Eliza on the other hand each described learning as an active process where they were making sense of ideas that became evident to them during their work with the task. In doing so they continually learnt more about the structure of the little cubes in the 'box'. The following comment made by Eliza in her interview captures some of her activity: "When I try to do things *in my mind* it is hard for me to figure it out 'til I really know how so the blocks help me to learn how to figure it out in my mind". Eliza illustrates the active nature of her engagement with the task and that she perceives *not knowing* as temporary and *finding out* more as something she can work out how to do. Later, before responding, she pauses and reflects on how she decides how she is going in maths:

I know it's *not* because (pause) I get things right ... I think it's because I ... contribut- [hesitant] (pause) ... rather than (pause) just agree and disagree ... I actually (pause) say what I think (pause) ...

Eliza further demonstrates her faith in her ability to think, and her ability to assess how well she is going with her mathematics internal interrogation (contributions she makes; that she is thinking deeply about ideas discussed) [Success as Personal]. Eliza contributed in several ways to the group's construction of new knowledge. It was her idea to use drawings on paper to represent more cubes. She also changed the orientation of the box (base 2x4) to produce four-layers-of-eight blocks to construct the box quickly. Eliza developed an understanding of the structure of the cubes in the 'box' to the extent of 'seeing' layers but not yet 'seeing' the base as an array.

## DISCUSSION AND CONCLUSIONS

These cases illustrate associations between the ways students developed confidence and their inclination to explore new (to the student) mathematical ideas. Although Sam, Aisha, and Hank were confident of their ability to do mathematics, this confidence developed as a result of their demonstrated ability to reproduce rules and procedures they had been taught, and praise by others in relation to this ability. Although these three students possessed confidence [Success as Permanent, Success as Pervasive] they did not possess persistence [Failure as Temporary, Success from Personal Effort]. They possessed 'disabling confidence' in relation to mathematical problem solving.

Patrick and Eliza on the other hand possessed 'enabling confidence'—confidence developed through successes with overcoming mathematical challenges through puzzling about ideas and constructing new knowledge during the process and knowing they had the ability to work out more through such processes in the future [Success as Permanent, Pervasive, and Personal; Failure as Temporary]. They also demonstrated the optimistic characteristic Failure as Specific through their looking in to situations and varying what they did to increase their chances of success. This raises questions about whether possession of confidence and persistence is always accompanied by the optimistic characteristic Failure as Specific.

The ontogenesis of confidence thus differed for those who were and were not inclined to explore. This fits with Pajares and Miller's (1997) finding of the stronger associations found between high mathematical performances and confidence when multiple-choice items (rather than open ended tasks) were used to assess performance. This stronger association between multiple-choice test items and confidence is consistent with the predominance of pedagogies likely to develop disabling confidence in our schools. To conform to the expectations of curriculum documents requiring creative mathematical thinking be developed, we need to decrease the use of 'transmission' pedagogies in our schools.

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# YOUNG CHILDREN'S COGNITIVE REPRESENTATIONS OF NUMBER AND THEIR NUMBER LINE ESTIMATIONS

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*This article reports findings from an exploratory multiple case-study investigating children's cognitive representations of number. The article considers the case of one child at the end of Year One (6 years old) and the findings from her participation in the pilot stage of a larger study following fifteen children over one school year. Multimodal analysis was carried out on video recordings of children's participation in individual task-based interviews. Clearly distinguishable strategies, in which many structural aspects of natural number were represented, were identified during interactions with number line estimation tasks. Structural similarities were identified between the imagistic representations of number demonstrated in the interview, and representations of number during number line estimation tasks.*

## Background

The foundational importance of cognitive representations of number to mathematics is clear, and research links immature number representations with not only lower mathematics performance but also with hindered learning of new mathematics (e.g. Booth & Siegler, 2008). The study reported here aims to provide depth of understanding of children's representations during the first year of formal schooling. Key research has focused on children's imagistic representations of number, and the structural development of representations more generally, and found them to progress through five clear stages of structural development (Mulligan, Mitchelmore, & Prescott, 2005). In addition to this is a substantial body of work within cognitive science, which has typically focused upon automaticized representations of numerical magnitude and their spatial aspects. This work has repeatedly documented an apparent 'shift' with age from a logarithmically to linearly calibrated mental number line, but despite a large number of studies in the field, there remain disagreements over key ideas and the interpretation of existing data (e.g. Thompson & Opfer, 2010).

This study addresses three gaps evident in the literature. First, little research has addressed children's interactions with number line estimation tasks - the primary research task of the field - since whilst the results of estimation processes are easy to measure, the processes themselves are difficult to reach. Petitto (1990) identified two strategies, 'counting on' and those involving midpoints, but was unable to provide further detail based solely on real-time observations. More recent work has made progress using eye-tracking (e.g. Schneider et al., 2008) and fine-grained statistical analysis (e.g. White & Szucs, 2012). However, both approaches have proven less successful with younger children, and these researchers have explicitly noted the need now for trial-by-trial analysis and the support of qualitative data.



The second gap addressed is the relationship between the structure of children's imagistic representations and their number line estimations. Empirical evidence points to strong and not yet fully understood connections between representations of number. These include the grounding of the mature concept of number in the numerosity representation systems present in infants (Carey, 2004); the spatial similarities between participants' automaticized and imagistic representations (Fias & Fischer, 2005); and the susceptibility of children's magnitude representations to alteration through carefully designed educational activity (Thompson & Opfer, 2010).

Thirdly and most importantly, the full multiple case study also addresses the current reliance on cross-sectional studies by following individuals through one school year.

### Theoretical framework

The research adopts the theoretical approach to cognitive representations described by Duval (1999). This framing acknowledges both intentional (semiotic) and automaticized (including perceptual) representations, and relations between them, an inclusive overall framework that is necessary given the connections noted above.

Duval argues that the customary distinction between mental and external representations is a "misleading division" (1999, p. 5), since this distinction addresses only the "mode of production" rather than the "nature" or "form" of representations. It is for this reason – the de-emphasis of the *mode* of representation – that this study utilises a multimodal approach (see Research Design).

The relation of cognitive representations to mathematics is that mathematical processes consist of transformations of representations, of which there are two kinds: processing (within registers) and translation (between registers) (Duval, 1999). Number line estimation tasks require transformation *between* registers: the translation between symbolic and verbal representations of number and spatial representations.

A key assumption from Piaget & Inhelder (1971) and expressed here by Presmeg (2006, p. 206), is that "when a person creates a spatial arrangement (including a mathematical inscription) there is a visual image in the person's mind, guiding this creation". Duval's classification of representations emphasises that there exist "two heterogeneous kinds of 'mental images'", firstly the "internalized semiotic visualizations", and secondly "'quasi-percepts' which are an extension of perception" (Duval, 1999, p. 6). Importantly, transformations (i.e. processing and translation) can be carried out on both kinds of mental image.

Theories of number concept development (see Nunes & Bryant, 2009) hold that children's understanding of structural aspects of number increases significantly during the early years of schooling, and this has been hypothesised as a cause of changes in children's interactions with estimation tasks (e.g. White & Szucs, 2012). The features expected to be potentially included in children's cognitive representations of number at this age are the sequential structure of the natural numbers, proportion between numbers, half/double relationships, and the base ten system (Thomas, 2004).

## RESEARCH DESIGN

The research design of the wider study is an exploratory multiple case study, in which children participate in video-recorded individual task-based interviews. This article presents analysis of one task-based interview with Imogen (age 6), from a south of England primary school, interviewed at the end of Year One as part of the pilot study. Imogen was assessed by her teacher to be of mid- to high- attainment in maths.

### Task-based interviews

Four tasks were completed, designed to stimulate and require translation of cognitive representations of number. The first task (T1) required children to close their eyes and imagine the numbers 1 to 100, then to draw and describe the picture in their mind (adapted from Thomas et al. (2002)). Following this, the children completed an estimation task (T2) in which they were asked to position number rocket stickers onto blank number lines (adapted from Thompson & Opfer (2010)). A third task (T3, not analysed here) asked children to estimate the quantity of sweets in clear plastic boxes. Finally, children were asked to estimate the number represented by already-positioned rockets on blank number lines (T4, adapted from Petitto (1990)). In both number line estimation tasks, children were presented with randomised target numbers across different ranges, to be placed on blank number lines with only the endpoints labelled. The ranges tested, and hence the endpoint labels, were 0-10, 0-20, 5-15, and 0-100 for T2, and 0-20 and 0-100 for T4. Before each number line estimation task, children completed a practice trial with the researcher, in which the target number consisted of an endpoint. No corrective feedback was provided during trials, only encouragement.

### Data analysis

Video data was transcribed separately for speech, gaze, and gesture (encompassing gesticulation, language-like gestures, pantomimes, and emblems, as defined by McNeill's strict classification (1992, p. 37)). These were then analysed alongside the paper-based representations created by children in T1 (drawings) and T2 and T4 (number line estimations). Imagistic representations evident in any task were first coded according to the type of component sign (pictorial, iconic, or notational) following classifications adapted from Presmeg by Thomas et al. (2002). Representations were then examined for structural features, as previously described.

In line with previous research findings, children's number line estimates were quantitatively analysed for their degree of linearity. These results were then compared to the representations inferred from video data to have been implicated in the estimation process. In the number-to-position task (T2), analysis first calculated the number indicated by the child's estimate as follows:

$$\frac{\text{Distance from left endpoint to estimate (mm)}}{\text{Total length of line (mm)}} \times \text{scale of number line}$$

In order to compare the linear accuracy of estimates in both T2 and T4, the absolute percentage error of each estimate was also calculated, using the following:

$$\frac{|\text{Estimate} - \text{Target Number}|}{\text{Scale of number line}} \times 100$$

Linear and logarithmic models were fitted to the target number estimates, for each range individually. For each model, the coefficient of determination  $R^2$  was calculated in order to compare model fit.

## FINDINGS

### Task 1 (T1)

Imogen's initial response to T1 was to ask for clarification. During the exchange that followed, she gesticulated in reference to the number sequence:

Interviewer: I'd like you to try to draw ... the picture you see in your imagination of all those counting numbers.

Imogen: So like one two ... and three four [*right hand hovers over paper and traces stair-shaped path from left of page: right-down-right, twirling pencil*]

This same stair-shape appears in the drawing then produced by Imogen (Figure 1). The drawing is composed of notational signs with an idiosyncratic and pictorial aspect, evidenced in both the drawing and Imogen's unprompted explanations: "When I saw it all it was bubble writing." Though the interviewer did not comment or enquire, Imogen explained that the particular form was also the reason for drawing a limited range: "I'll just do it up to ten ... Cos I don't want to waste all my time counting up to a hundred in bubble writing." The interviewer then pursued this:

Interviewer: If you did have time, where would one hundred go on that page?

Imogen: [*silently mouths the numbers one to ten, as right index finger jumps one by one along the number sequence already drawn*] Twelve [*finger jumps onto empty space to right of "11"*] ... thirteen fourteen fifteen ... [*finger jumps three steps to lower right, see Figure 2*] ... I have no idea!

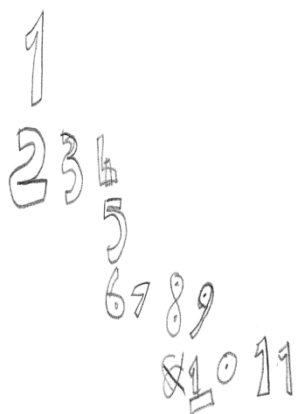


Figure 1: Imogen (T1)

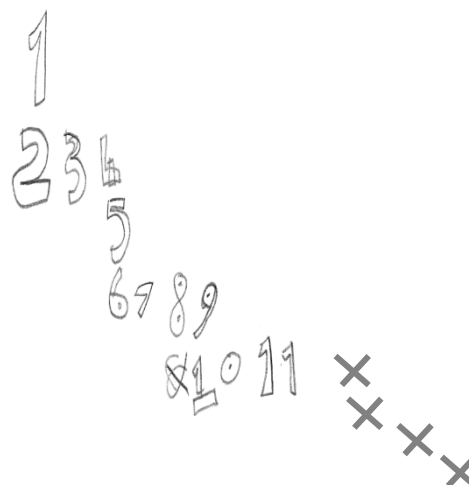


Figure 2: Imogen (T1) with gesture

The sequence structure of number is clearly represented in all modes, and the numbers as far as shown are evenly spaced. Numbers are also grouped, relatively consistently though unconventionally. Imogen's comments and re-drawing indicate clearly that the

“9” was originally intended to be positioned beneath the 8, consistent with her earlier grouping (changing direction on each multiple of two).

### Task 2 (T2) and Task 4 (T4)

Representation of structural elements of number occurred in all modes examined during these tasks: speech, gaze, and gesture. Representations occurred during trials both before and after giving an initial solution, and also in spontaneous comments and justifications. Clearly distinguishable strategies were identified in Imogen’s interactions with T2 and T4. The strategies, and the target numbers of the trials in which they were observed, were as follows:

Strategy	Task 2				Task 4	
	0-10	5-15	0-20	0-100	0-20	0-100
Counting on from L endpoint in 1’s	4, 6		2		2	3
Counting on from L endpoint in 5’s		9				
Counting on from midpoint in 1’s		11, 13	6			
Counting on from other point		13				25
Counting back from midpoint in 1’s			6			
Counting back from other point		11				
Referral to a previous trial		7, 14	7	48	15	18
Ref. to L endpoint	8, 2, 7, 3	6	4, 13, 18	2, 48, 4, 67, 25, 6	16, 4, 7	71, 6, 86
Ref. to R endpoint	9, 8, 7, 3	13	16, 18	2, 48, 4	7	71, 86, 48
Ref. to midpoint				71, 67, 18, 3, 25		
Ref. to other point						
Ambiguous		8	15	86	6, 18, 13	67, 5, 3

Table 1: Strategies identified in use in Task 2 and Task 4.

In the following example from T2 (range 5-15, target number 13), the strategies identified were “Ref. to R endpoint”, “Counting on from midpoint in 1’s”, and “Counting on from other point”:

Interviewer: It's thirteen. Where do you think thirteen belongs? [*Imogen's gaze goes quickly to right endpoint (15) then to interviewer proffering rocket sticker*]

Imogen: [*takes rocket with right hand, transfers to left hand, pauses*]  
 'Cause ten is here [*right hand points onto midpoint and holds*]  
 [*right hand 'hops' to right; both hands stick rocket to right of the 'hop'*]  
Fourteen fifteen [*right hand thumps line between rocket and right endpoint, then thumps right endpoint itself*]

Representation of aspects of number structure is apparent in these strategies. In the example above, the number sequence is represented in the two counting on strategies, and with a confidence that allows Imogen to start counting midway through the sequence. The units represented by gesture during the counting on represent a further aspect of structure: the spatial extent of each unit is approximately equally sized, and scaled so that Imogen's sequence from ten to fifteen covers the spatial extent from indicated midpoint to endpoint. Structure of number is also apparent in Imogen's use of the right endpoint (fifteen) as an appropriate 'landmark' for the target number thirteen. The midpoint structure of ten within the range 5-15 is clearly represented in speech and gesture.

Throughout T2 and T4, the sequence structure of number, with left to right orientation, was most frequently represented. Representations of number that Imogen spontaneously demonstrated on the ranges 5-15 and 0-20 encompassed further structure in the form of evenly spaced multiples of five. An example of this was Imogen's exclamation on seeing the first page of 5-15 trials: "Shouldn't it be five TEN ...?" [*Right hand points onto midpoint of line and holds*].

In agreement with the findings of previous research, the linear accuracy of Imogen's estimates decreased on larger ranges. In T2, the mean absolute percentage error was low for both 0-10 and 5-15 (7.6% and 4.9% respectively), and rose to 25.5% on the range 0-100. Interestingly, linear accuracy in T4 was higher than in T2 on both ranges tested; a paired samples t-test was conducted to compare the absolute percentage error in T4 and in T2 and found a significant difference between the error in T4 (mean=11.94, SD=10.00) and T2 (mean=21.59, SD=8.78);  $t(16)=2.89$ ,  $p=0.011$ .

On the range 0-10, Imogen's T2 estimates were best described by a linear model ( $R^2=.94$ , compared to .84 for logarithmic model). On the range 5-15, the comparison was inconclusive ( $R^2=.97$  for both models). On the ranges 0-20 and 0-100, Imogen's estimates were more consistent with a logarithmic model ( $R^2=.71$  and  $R^2=.57$  respectively, compared to .64 and .39 respectively for linear models). Imogen's T4 estimates were, in contrast, better fit by linear models for both ranges 0-20 ( $R^2=.97$  compared to .87 for logarithmic model) and 0-100 ( $R^2=.91$  compared to .77 for logarithmic model).

The linear accuracy of Imogen's T2 estimates is in line with previous research, which expects that by the end of Year One, estimates on the range 0-10 will demonstrate a good level of linearity, whilst those on larger number ranges do not. Overall, Imogen's linear accuracy was highest on the T2 trials on the range 5-15. Almost every strategy

identified was in evidence during these trials; and, furthermore, during this part of the interview cognitive representations with more structural detail and greater accuracy were inferred from Imogen's spontaneous behaviour, for example with regard to visualising the subdivision of the 5-15 line into equally sized fives.

## CONCLUSIONS AND FURTHER DIRECTIONS

The findings give good reason to infer that Imogen cognitively represents number in ways which encode significant structural elements, many of which are evident in her interactions with number line estimations. Particularly of note is that she successfully applied counting strategies, commonly regarded as a less sophisticated approach (e.g. White & Szucs, 2012). What is clear from this case study is that the detail of children's interactions must be attended to: whilst Imogen indicated appropriately sized unit jumps, consistent with the evenly spaced numbers in her imagistic representations and her adjustment of unit size depending on scale, this may not be the case among other children. Aspects that may vary are the size of 'jump', whether the child attempts to scale the 'jump', and whether the size of 'jump' is consistent within a trial. Conversely, the inclusion of apparently more sophisticated structure (for example midpoints) may still result in estimation with low linear accuracy, depending on the sophistication and accurate execution of other parts of the strategy.

In terms of the relationship between imagistic representations and number line estimations, structures seen in Imogen's imagistic representation task were clearly demonstrated in the processes of translating representations during estimation tasks. Structures in common were the sequencing of natural numbers, regularity of number spacing, and the grouping of number based on multiplicative relations.

Particular findings from this case will be interesting to follow up in the longitudinal study. An example is the difference between strategies and estimation results in Task 4 (position to number) compared to Task 2 (number to position) that was evident in this case. The full study will also indicate the extent to which other children with comparatively well-developed imagistic representations of number incorporate structures of the number system into their estimations with the frequency that Imogen demonstrates.

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# LEARNING TRAJECTORIES AND STUDENT-CENTERED TEACHING PRACTICES

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*We examine classroom observations of 19 elementary grades teachers participating in 60 hours of professional development designed to support learning of one learning trajectory. Findings describe the ways these teachers used their knowledge of the LT to enact practices that elicit and use students' mathematical thinking in instruction.*

Student-centered teaching is fundamental in promoting learning with understanding. In mathematics, teachers' knowledge of students has long been identified as a critical factor in developing mathematical proficiency, and mathematics education research results have long demonstrated the benefits of teachers' understanding of students' mathematical thinking (e.g., Fennema et al., 1996). Here, we consider two recent developments in mathematics education to examine the relation between teachers' knowledge of students and student-centered teaching: learning trajectories and practices that promote student-centered learning environments. Learning trajectories (LTs) are empirically developed descriptions of the ways in which students progress from less to more sophisticated understanding of specific mathematical ideas (Confrey et al., 2009; Clements & Sarama, 2004). A set of five, teacher-lead classroom practices to help orchestrate productive mathematics discussions (Smith & Stein, 2011) represent key practices that promote student-centered learning environments. Our research examines the ways in which elementary teachers' learning of one LT supported their use of these five practices in their classrooms with the research question: *in what ways does a LT support teachers in organizing instruction around practices that facilitate attention to student thinking?*

## LEARNING TRAJECTORIES AND MATHEMATICS TEACHING

Growing attention to students' LTs in the US renews the opportunity to explore the ways that teachers may use students' mathematical thinking in student-centered teaching. Current interest in LTs can be traced to Simon's (1995) *hypothetical learning trajectory*. In articulating mathematics pedagogy from a constructivist perspective, he proposed the term to describe the predictions teachers make about the ways mathematics learning may unfold, noting that this path is "hypothetical because the actual learning trajectory is not knowable in advance" (p. 135). Since that time, researchers have made significant progress in understanding patterns in the ways children formulate and refine mathematical ideas. *Taking Science to School* (NRC, 2007) highlighted this progress, noting that research on learning is beginning to map the successively more sophisticated ways of thinking about a topic that can build on one another as children learn a topic over a broad span of time. Though researchers define LTs differently, our group uses Confrey et al.'s (2009) definition that states a



LT is, “a researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction ... in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time” (p. 347).

Several studies outline positive outcomes of teachers’ learning and using LTs in their practice. For instance, the Cognitively Guided Instruction work showed that teachers’ learning of a research-based model for children’s mathematical thinking changed instruction toward being more cognitively guided and that these changes in instruction were positively related to student achievement gains (Fennema et al., 1996). In investigating 33 elementary grades teachers’ learning of a LT, Wilson (2009) concluded that LTs supported teachers’ interactions with students during instruction by sensitizing them to students’ strategies, providing a framework for organizing students’ work, and assisting in relating mathematical ideas across students’ approaches. Similarly, Edgington (2012) investigated five second grade teachers’ uses of a LT using a multi-case study and found that the LT supported them in specifying learning goals, anticipating levels of sophistication among expected strategies, attending to the processes their students engaged in, and recognizing important mathematical ideas that surfaced during instruction. Together, these studies are beginning to show the potential of LTs in supporting student-centered teaching.

## INITIAL CONJECTURES

In our work, we have developed a theoretical model for Learning Trajectories Based Instruction (LTBI) (Sztajn et al., 2012), which unifies various frameworks for mathematics teaching by considering students’ LTs as the center for instructional decisions. Using this model, we have engaged in design-based research to empirically test the various components of the framework, seeking to understand both teacher learning of LTs and the ways teachers use LTs in their teaching. We have considered our model as an initial conjecture about the role of LTs in teaching and have worked to examine whether its components can be confirmed. This report presents evidence from an examination of the LTBI model related to practices that promote student-centered learning environments.

Smith and Stein’s (2011) *Five Practices for Orchestrating Productive Mathematics Discussions* constituted the initial conceptual framework. These authors identified five practices to assist teachers in conducting lessons that use students’ thinking in whole class discussions to advance the mathematical goals of the lesson. *Anticipating* is the practice of predicting the ways students will mathematically approach a particular instructional task. *Monitoring* is closely attending to students’ engagement and progress with the task. *Selecting* is choosing particular students’ ideas to share in class discussion. *Sequencing* is making decisions about how to order the students’ ideas to maximize the chances of achieving the mathematical learning goals. *Connecting* is helping students develop relations among the ideas shared and making sure students’ contributions build on each other to develop powerful mathematical ideas. Although not included as one of their practices, Smith & Stein (2011) noted that learning goals

and task selection were a “critical starting point” for mathematics teaching that used students’ ideas (p. 13). In our work, we included *Setting Goals and Selecting Tasks* as a practice of interest. We consider that Smith and Stein’s definition of these practices has significantly advanced the discussion of what student-centered instruction entails but contend that the referent for these practices is underspecified. In formulating our LTBI theory, we interpreted what these practices might look like when taking a LT as this referent.

In this study, we empirically examined this portion of our initial LTBI framework by taking it as an initial conjecture for design-based research. In LTBI, instructional tasks surface and use students’ current understandings of mathematics from previous instructional and informal experiences to move learning forward. These tasks span several levels of a LT, permitting all students to engage with the mathematics of the task. Likewise, the LT levels may inform the anticipations teachers make by organizing the various strategies students will likely use, relating these strategies to underlying cognition, and marking common misconceptions. When monitoring, knowledge of the LT may attune teachers to the various strategies (and the mathematical thinking underlying these approaches) that students take as well as alert them to misconceptions. When selecting and sequencing students’ ideas, a LT may direct teachers toward strategies indicating a variety of mathematical thinking and support their ordering in ways that build toward more sophisticated understandings. When supporting students in making mathematical connections, a LT may inform teachers of the mathematical relationships among the various strategies shared.

## **METHODOLOGY**

This investigation is part of a larger design experiment to investigate teacher learning of LTs and LTBI. Design experiments seek to understand both the processes of learning and the environments that support such processes (Cobb et al., 2003). They are driven by conjectures (Confrey & Lachance, 2000) and seek not only to “engineer” a product but also to generate theoretical claims about the nature of the learning bounded by the design (Design-based Research Consortium, 2003). In our case, we sought to design PD to facilitate teachers’ learning of one LT and support teachers in using this learning in their instruction; this PD was the setting for our design experiment. This report focuses on one of the classroom-based activities from the PD, which we utilized to understand the ways teachers’ used their knowledge of the LT in their mathematics teaching. Using qualitative analysis of classroom video, questionnaires and interviews to test our interpretation of Smith & Stein’s (2011) practices, we sought to examine our initial conjectures about the ways teachers’ LT knowledge would manifest in practices that promote student-centered learning.

### **Intervention and Participants**

Our 60-hour, school-based PD focused on supporting teachers in learning one LT through targeted professional learning tasks. The PD started with a 30-hour Summer Institute, spread over six days immediately prior to the school year, with a primary emphasis on teacher learning of the LT itself and with a secondary focus on assisting

teachers in relating this learning to various instructional practices. A second portion of the PD, which occurred during monthly meetings after school (18 hours) during the school year, was organized around classroom-based activities designed to support teachers in exploring the LT in their classrooms. The final 12 hours of the PD occurred as a two-day meeting immediately following the school year and served mostly as evaluation of the PD conducted during the year.

Although various learning tasks in the PD made mention of Smith and Stein's practices, and we explicitly talked with teachers about the importance of using more open tasks in creating spaces to observe and elicit students' mathematical thinking, the learning of these practices was not an explicit goal. Rather, the practices served as a way to present what teachers might do when they wanted to focus on student thinking in their classroom. They were introduced to teachers without explicit mentioning of the LT in relation to these practices. Further, there was no clear message about how one might teach mathematics using a LT. We indicated to teachers that we were working with them to understand what LTBI might entail; therefore, we did not offer an a priori definition of LTBI to participating teachers.

Our PD focused on the *Equipartitioning Learning Trajectory* (EPLT) (Confrey, 2012). Confrey (2009) defines equipartitioning as the set of cognitive behaviors that has the goal of producing equal-sized groups, parts, or combinations of wholes and parts such as typically encountered by children initially in constructing "fair shares." The EPLT is organized as a two dimensional matrix. One dimension represents levels of cognitive proficiency through which students pass as they engage with equipartitioning tasks. These levels are organized by increasing sophistication, with the lower levels describe the various strategies children use to complete equipartitioning tasks as well as outline common misconceptions. Intermediate levels of the EPLT outline the mathematical practices students use to justify and communicate their work. The upper levels describe various relationships and generalizations related to equipartitioning. The second dimension is a set of task parameters, organized with increasing difficulty, that interact with the proficiency levels to affect the difficulty of a particular task. They begin with discrete collections, move to single wholes through a sequence of increasingly difficult numbers of parts, and conclude with multiple wholes.

Twenty-two elementary grades teachers from one mid-sized suburban elementary school in the US participated in the PD. Of these, eighteen were classroom teachers and the remaining four served as support to the classroom teachers. Their years of teaching experience varied between 2 to 26 years. All teachers volunteered and received a stipend to participate.

## **Data and Analysis**

In our PD, we designed one classroom-based professional learning task that asked teachers to create an equipartitioning lesson to teach in their classrooms. Teachers were asked to select an instructional task and plan a lesson to teach their students. Teachers completed a pre-observation questionnaire where they recorded their learning goals, a description of the task they would use, a plan for how they might

organize the lesson with students, and their anticipations of the ways students might engage in the lesson. The final question asked the teachers about the ways that they used the EPLT to prepare. A researcher video recorded the lesson of each participating teacher and, following the lesson, met with the teacher for a post-lesson interview focusing on teachers' general impression of the lesson, the difficulty of the tasks for the students, what teachers were looking for during instruction, their rationale for sharing different students' ideas, the connections they wanted students to make, and the ideas from the EPLT that teachers used. Nineteen of the teachers completed this classroom-based activity and served as participants for the present study. The data set for each teacher consisted of the pre-lesson form, the video recording, and the transcripts of the post-observation interview.

Beginning with theory-driven codes based on the *Five Practices* (Smith & Stein, 2011), we followed Decuir-Gunby, Marshall, and McCulloch's (2010) guidelines for the development of a codebook by operationally defining the following practices: task and learning goal, anticipating, monitoring, selecting and sequencing, and connecting. These codes did not focus on LT but rather explained what a coder should look for to recognize a practice when examining both written and video data. Our procedure specified that codes be applied to idea units, defined as segments of the transcript or video recording that, when taken in isolation, keep the whole idea or action of the participant together and represents the idea the participant was trying to convey. Idea units include interactions between the teacher and interviewer or between the teacher and students. We initially analyzed two of the nineteen teachers' data sets to clarify our definitions, idea units, and provide examples for the codebook. Through iterations of defining the codes and identifying idea units, we reached agreement on the definitions, procedures, and examples.

Two independent researchers were trained using the two data sets the research team used in the codebook development. Inter-coder reliability of 84% was established among these researchers and our team. Data for the remaining 16 teachers was divided between the two of them, and they identified and coded 1102 idea units. Two members of the research team open coded these idea units to identify instances related to the EPLT. Using the constant comparison method (Strauss & Corbin, 1998), they looked across these idea units to discern patterns related to the EPLT.

## RESULTS

Our analysis examined the idea units related to the EPLT (see table 1). From this, we conclude that the EPLT supported teachers in utilizing each of the practices that support student-centered learning environments. Across the observations, we found evidence that the majority of the teachers made use of the EPLT in: selecting a task and articulating learning goals; anticipating the ways that their students might engage with the task; monitoring students' work on the task; selecting and sequencing students' ideas to share in discussion; and connecting students' ideas in ways that attempted to advance the goals of the lesson.

Practice	No. of teachers	Practice	No. of teachers
Task & Learning Goal	19	Selecting & Sequencing	13
Anticipating	18	Connecting	18
Monitoring	14		

Table 1. Number of teachers using the EPLT for each practice (N = 19).

All 19 teachers used the EPLT when selecting an instructional task and when determining the learning goals for their lesson. By far, the aspects of the EPLT that the teachers drew upon most frequently to select a task and clarify learning goals were the proficiency levels and/or the task parameters. Some teachers chose to focus their goals on a single or set of levels while others focused on varying task parameters while holding the proficiency levels constant. Several teachers drew upon both the levels and task parameters in a coordinated way when setting their goals and planning for their tasks. The EPLT also supported teachers in the selection of instructional tasks. Seven of the teachers described using the EPLT to select or design instructional tasks to address important ideas about equipartitioning. Some used the EPLT to identify tasks that might elicit students' current understandings while others used the EPLT to create contexts for problems that might be meaningful for students.

Eighteen teachers used the EPLT to anticipate the ways their students might mathematically engage with the task. The most frequent when Anticipating was to identify specific strategies described in the levels. Eleven teachers specifically described ways students might approach the task. Another popular use was Anticipating in relation to the level of sophistication and difficulty expressed by the proficiency levels and task parameters. Ten teachers reported drawing upon the EPLT's structure to anticipate students' work on the instructional task. Some used the proficiency levels while others used the task parameters. Eight teachers anticipated various misconceptions that their students might exhibit during their lesson.

Fourteen teachers drew upon their knowledge of the EPLT when monitoring students' work during the lesson. The majority of teachers using the EPLT when monitoring did so by focusing on the strategies students used to engage with the task. Rather than making quick evaluations of students' work, these ten teachers focused on the processes that students were using to complete the task when monitoring students' exploration. Surprisingly, the evidence suggested that only one teacher purposefully attended to the misconceptions marked in the EPLT during monitoring.

Thirteen teachers used the EPLT when selecting and sequencing students' solutions for whole class discussion. Seven of them reported that they selected students' responses for whole class discussion based on a desire for their students to see "different" approaches. Two teachers made explicit reference to using the proficiency levels of the EPLT to attend to various levels of sophistication among the strategies present in the class for sharing in whole group discussion.

Eighteen teachers used the EPLT when supporting students in making connections. Twelve provided opportunities for students to make explicit connections to key ideas from the LT levels, including common misconceptions. Seven teachers provided

opportunities for students to make explicit connections among the different strategies that were shared in the discussion. These teachers often asked questions that provided opportunities for students to consider the various strategies presented.

## DISCUSSION

Our study provides supporting evidence for our initial conjectures. The LT structure assisted teachers in determining and specifying their learning goals and in selecting open tasks that elicit and build upon students' thinking. Teachers drew upon strategies and misconceptions described by the LT to anticipate how students might engage with the selected task. Thus, one clear way that a LT supports teachers in organizing instruction around student thinking is through informing their planning. While teaching, the strategies and misconceptions assisted teachers in looking beyond "answers" to the approaches taken by students. This knowledge informed the ideas that teachers selected to be shared with the class, and the LT levels suggested possible ways in which these ideas might unfold to move all students toward more sophisticated understandings. The levels and strategies allowed teachers to provide opportunities for students to draw connections across the ideas shared in class to advance the goals of the lesson. Thus, we conclude that LTs support teachers in organizing instruction around student thinking by: bringing specificity to teachers' learning goals; informing the selection of instructional tasks and anticipations of student approaches; framing the monitoring, selecting, and sequencing of student work; supporting the connecting students' approaches in whole group discussion; and bringing coherence to various student-centered practices.

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# USING MULTI-STAGE TASKS IN MATHEMATICS EDUCATION: RAISING AWARENESS, REVEALING INTENDED PRACTICE

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*In this paper we study task design and use for mathematics teacher education and research. We develop a former task design and enrich it into a more complex and multi-staged format. Our aim with this type of task is to explore: (i) teachers' mathematical knowledge, (ii) pedagogical perceptions and intentions and (iii) epistemological beliefs. We deployed one such task to engage 23 mathematics graduates, many already in-service teachers. In this paper we demonstrate the capacity of the task to allow insights into (i) – (iii), as well as reveal some discrepancies between the teachers' stated beliefs and intended practice.*

## INTRODUCTION

The design and use of tasks for pedagogic purposes is at the core of mathematics education and mathematics education research (Artigue & Perrin-Glorian, 1991). Especially, in the field of mathematics teacher education, significant attention has been paid to the nature, role and use of tasks. As Sullivan (1999) indicates, there is a need to educate new teachers in the use of complex tasks. Recent work (Tirosh & Wood, 2009) has focused on integrating tasks into the processes of teacher education. Also, a special issue of Journal of mathematics Teacher Education (Mason, Watson, & Zaslavsky, 2007) and the book edited by Zaslavsky and Sullivan (2011) signal this interest. Within teacher education a task can be used to explore teachers' mathematical knowledge for teaching and their pedagogical, didactical and epistemological perceptions and beliefs. An appropriately designed task, which addresses complex purposes, affords opportunity to engage with aspects of mathematics, didactical strategies, pedagogical theory and epistemological beliefs.

In this paper we study a type of task for teacher education that is a development of a former type of task, which was studied in (Biza, Nardi & Zachariades 2007; 2009) and (Nardi, Biza & Zachariades, 2012). The research team members, and three authors, are the task designers and users and bring expertise from mathematics, mathematics education and teacher education, both in terms of research and teaching. We use tasks of both the former and current types in research and under- / post-graduate programmes in mathematics education, particularly in courses run by the first author.

## TASK DESIGN

In (Biza et al, 2007) we studied teacher knowledge in situation specific contexts, using a type of task based on a hypothetical classroom scenario (Figure 1). This task was grounded on learning and teaching issues that previous research and experience had highlighted as seminal. With this first type of task we engaged mathematics teachers in a mathematically and pedagogically specific situation that had the following structure:



Solving (and reflecting upon the learning objectives within) a mathematical problem	Examining a flawed (fictional) student solution	Describing, in writing, feedback to the student
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Figure 1. First-type task structure.

From the teachers' responses to this type of task we aimed to explore teachers' subject matter knowledge and their gravitation towards certain types of pedagogy and didactical practices (Biza et al, 2007). We have used tasks of this type to explore the above in (Biza et al, 2007; 2009) and (Nardi et al, 2012).

In this paper we study a second type of task, which enriches and develops the previous one. A task of this type is a multi-stage task, which addresses complex purposes in teacher education. Our aim using this second type is, in addition to the aims of the first-type task, to invite the teacher to evaluate the didactical approach followed by another (fictional) teacher. The teacher fictional response to the student is grounded in issues identified as seminal in previous research into teacher knowledge and beliefs. Through engagement with the tasks we aim to explore various aspects of teachers' knowledge (mathematical, pedagogical, didactical, epistemological) and their ability to support their views and choices, especially when juxtaposed to those of another teacher. These tasks can be used in teacher education to explore, assess and develop teachers' mathematical knowledge for teaching.

In this respect in designing these tasks we bear in mind the following:

- The mathematical content of the task concerns a topic or an issue that is known for its subtlety or for causing difficulty to students (from literature and/or previous experience).
- The fictional student response reflects this subtlety (or lack of) or difficulty and provides an opportunity for the teacher to reflect on and demonstrate the ways in which s/he would help the student achieve subtlety or overcome difficulty.
- The fictional teacher's didactical approach concerns mathematical, pedagogical, didactical and epistemological issues that are known for their subtlety, or for being challenging to teachers.
- Mathematical content and fictional student/teacher responses provide a context in which teachers' knowledge, perceptions and choices (mathematical, pedagogical, didactical and epistemological) are allowed to surface.

The enriched version of the mathematically / pedagogically / epistemologically specific situations in the tasks that we invite teachers to engage with has the following structure (Figure 2):

Solving a mathematical problem	Examining a (fictional) student solution and a (fictional) teacher pedagogical approach	Offering, in writing, an evaluation of the fictional teacher's approach across mathematical, pedagogical and epistemological perspectives	Describing, in writing, the pedagogical approach which they would follow
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Figure 2. Second-type task structure

In this paper we focus on responses to a task of the second enriched type (we call this Task 2). Respondents were mathematics teachers who attended a post-graduate course in the teaching of Calculus, as part of their studies towards a Masters in Mathematics Education. The 23 participants, all mathematics graduates, had engaged with tasks of the above two types during the course. They engaged with the task during the course examination and, in a note attached to the exam paper, all agreed with the use of their responses for research purposes.

The task (Task 2, Figure 3) was designed by the three authors of this paper, all mathematicians and mathematics education researchers. The first author is a research mathematician and also mathematics teacher educator and the third author also has substantial experience in secondary mathematics teaching.

For the design of this task, results of previous studies based on a task of the former type were taken into account. The former task (called Task 1 in our earlier work) was used to study teachers' views on the role of visualization and their perceptions about the tangent at an inflection point of a curve (e.g. Biza et al, 2009) as well as to analyse their argumentations for or against certain pedagogical approaches (Nardi et al, 2012). The hypothetical scenario of Task 1 concerned two different solutions of two students in a problem that involved the tangent line at an inflection point, and the participants were asked to examine these answers and describe their feedback to those students.

In the problem, the algebraic expressions for a line and a tangent were given and the students were asked to explore if the line is a tangent of the curve. The solution of the first student was algebraic and that of the second student was geometrical.

Using Task 1 our aim was to explore the participants' mathematical knowledge in two issues that previous research (e.g. Biza, Christou & Zachariades, 2008) has identified as critical:

- students often believe that having one common point is a necessary and sufficient condition for tangency; and,
- students often see a tangent as a line that keeps the entire curve in the same semi-plane.

Also our aim was to explore the relationships between teachers' beliefs about the sufficiency of a visual argument, their views on the persuasiveness of a visual argument and their personal mathematical images about tangent line. Whether a visual representation can be used not only as evidence and means of insight from a

mathematical statement, but also as a part of its justification (Hanna & Sidoli, 2007), was also a central issue.

In Task 2, we kept the mathematical problem as it was; we also kept the solution of the first of the students, and we enriched the scenario with a fictional teacher's pedagogical approach. This approach was inspired by teachers' perceptions and views about the tangent at an inflection point and the role of visualization as they emerged from a study in which we used Task 1 (see Biza et al, 2009).

Using Task 2, our aim was to explore the participants' knowledge and perceptions, not only through the way in which they would tackle the situation in Task 1, but also in relation to the arguments they use in evaluating another teacher's approach. Also, comparing the participants' responses to the two questions of Task 2 (Q1: comment on T's response; Q2: describe how you would tackle the situation), we can explore and identify possible discrepancies between their stated beliefs, as evident, for example, in the critique of the teacher's approach in the first question of the task; and, their intended practice, as evident in the response to the second question. We recognize that, in the course of their engagement with the task, the participants were not in the classroom and they had some time to think about their reaction. However, we consider that, even under these circumstances, their responses can still be representative of their intentions and can also be particularly reflective.

### Scenario

In a Year 12 class of students specializing in mathematics, the teacher gave the following problem:

"Examine whether the line with equation  $y=2$  is tangent to the graph of function  $f$ , where  $f(x)=3x^3 + 2$ ."

A student responded as follows:

"I will find the common points between the line and the graph solving the system:

$$\begin{cases} y = 3x^3 + 2 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} 3x^3 + 2 = 2 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} 3x^3 = 0 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 2 \end{cases}$$

The common point is A(0, 2).

The line is tangent of the graph at the point A because they have only one common point (which is A)."

The following dialogue then took place in the classroom:

T (Teacher): The parabola  $y=x^2$  and the line  $y=2$  have only one common point, the point (0, 0). Is the line  $x=0$  tangent of the parabola at this point?

The student sketches the parabola and the line on the board and answers:

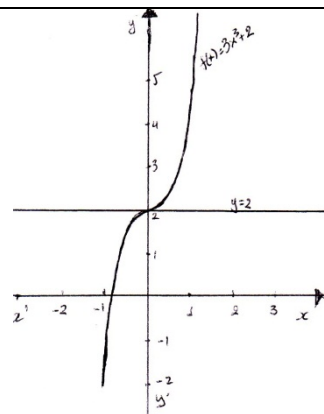
S (Student): No, it isn't, because the line cuts the parabola at this point.

T: OK. In our case [the teacher shows the problem in question] what is happening?

The student sketches the following graph and answers:

S: As we see from the graph, the line  $y=2$  cuts the curve  $y=3x^3 + 2$  at the point  $(0,2)$ . So, the line is not a tangent of this curve.

T: This is correct but you also need to justify it algebraically. Even if a graphical understanding of functions is particularly useful, you should not forget that it is not always possible to use graphical representations and that you should learn to solve problems also algebraically.



### Questions

Q1. How do you evaluate the teacher's management from:

- a mathematical perspective?
- a pedagogical perspective?
- an epistemological perspective, especially regarding the teacher's beliefs about the role of visualization in mathematics?

Q2. If you were the teacher, how would you manage the situation following the student's answer to the problem?

Justify your answers.

Figure 3. Task 2

### TASK USE: SAMPLE OF DATA AND FINDINGS

From the analysis of the participants' written responses evidence emerged about their knowledge concerning the tangent, their pedagogical intentions in the teaching of this concept and their epistemological perceptions about the role of visualization. Also, through their evaluation of the fictional teacher's approach in the task, some of their general pedagogical beliefs emerged. These beliefs concern the role of interactive teaching, the role of examples in teaching, the role of obstacles etc.. This task allows evidence to surface on how teachers perceive and weave together mathematical, pedagogical and epistemological issues. To give a flavour of the surfacing of this evidence, and in awareness of space limitations, in this paper we focus on:

- evidence of teacher beliefs about the role of visualization in mathematics as it emerged from their evaluation of the fictional teacher's practice in Q1c; and,
- the use of visualization in their intended practice as it emerged from their answers in Q2.

We note that, during the course that these teachers attended as post-graduate students, the role of visualization in mathematics, and especially in teaching Calculus, had been discussed extensively. The common perception of the participants, as it emerged from their written answers in Q1, was that visualization plays a very important role in the teaching of Calculus, because it helps students comprehend a concept. In consistence with their answer in Q1, most participants wrote in their answer in Q2 that they would use graphs to help the student construct a good concept image of tangent line and many of them sketched some graphs in their paper. Some scripts, however, were not as

internally consistent. For example, we identified two potential discrepancies in the scripts of T18 and T1.

T18, in his answer in Q1c, writes

Visualization is a very important part of mathematics because, through this, intuitions, conjectures and concept images form and give to the student the possibility to understand concepts better.

However, in his answer in Q2, he does not use or refer to any visualization. He uses only formal mathematics and completes the algebraic solution of the students. So, in the above excerpt, the teacher expresses a view about visualization that echoes that which was discussed, and perhaps prevailed, during the course. But his intended practice is completely different.

We identify a similar potential discrepancy in participant T1's answer. This participant criticized the fictional teacher's beliefs about the role of visualization in mathematics and she writes in her answer in Q1b:

I consider it to be wrong, the teacher's claim that he gives a secondary role for graphs and that he asks from the student to solve problems algebraically.

Later, in her answer in Q1c, she writes

From an epistemological point of view, the teacher seems to downgrade the role of visualization and the possibility of a proof based on geometry and the properties of curves.

From the above it seems that this teacher's perception is that a graph-based proof in Calculus is acceptable. However, in her answer in Q2, she writes:

Graphical representations of examples do not constitute mathematical claims. We accept only what they have proved completely algebraically [...] Proof is one thing, and conjecture is another.

From the last excerpt a perception that differs significantly from the previous one seems to emerge.

Participant T18 seems to agree theoretically with the perceptions about the role of visualization that prevailed in the discussions during the course. But he appears less willing to mirror these perceptions in his practice which is characterised by absolute adherence to a formal approach. Not dissimilarly, participant T1 critiques the perceived 'downgrading' of visualisation by the task's fictional teacher, and seems to suggest that a proof in Calculus based on graphs can be acceptable. But, in her statement of intended practice, she seems to believe that we can only obtain conjectures, not proofs, from a graph. It is to the credit of Task 2, its multi-stage structure and content, that such discrepancies can be revealed.

## **DISCUSSION AND CONCLUSION**

This type of task develops a former type. The new structure is complex and multi-stage, reflecting the complexity and multi-layeredness of teaching situations. The scenario of a task of this type (Task 2) is composed by one or more solutions to a

problem given by students, a teaching episode that can occur in a secondary mathematics classroom, and questions based on this episode. The teaching episode describes a teacher's management of a student's response. The questions ask participants to evaluate this management and propose their own preferred one, especially in the light of this evaluation. For example, if their views were at some distance from the fictional teacher, the participants were expected to propose an alternative management of the situation. This type of task can address complex purposes as discussed, for example, by Kilpatrick, Swafford and Findell (2011). With such tasks we can explore conceptual mathematical understanding, pedagogical strategies and epistemological perceptions and beliefs of pre-service and in-service mathematics teachers.

The enriched task we discuss in this paper concerns the teaching of tangent line of a curve and its design is supported by research results generated after the application of a related task of an earlier type. We changed the structure of the former task and enriched it using pedagogical strategies and epistemological beliefs expressed by participants in previous studies in which we used that former version of the task. By using the new version of the task we explore the participating teachers' mathematical knowledge concerning the tangent line of a curve, their pedagogical intentions in the teaching of this concept and their epistemological perceptions about the role of visualization. Further – as evident, for example, in the excerpts in this paper – we observe how the teachers weave together their views on mathematical, pedagogical and epistemological issues; and, we identify some discrepancies between stated beliefs and intended practice.

The insights that the use of this type of task has allowed are many and in this paper we illustrated and elaborated upon a fraction of these. The task design and use we outline here bring together expertise that cuts across the intertwined communities of mathematics and mathematics education, as well as of mathematics education research and mathematics teacher education. We aim that this cross-community task design and use continues to enrich research outcomes as well as improve teacher education provision.

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# **PRESERVICE PRIMARY TEACHERS'S NOTICING OF STUDENTS' GENERALIZATION PROCESS**

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*The objective of this research is to characterize levels of development of teaching competence in noticing students' mathematical thinking in the specific area of the generalization process. The findings provide descriptors of development levels of this teaching competence characterized by the way prospective primary teachers identified the relevant items in generalization processes from the answers given by primary students to generalization problems. The findings provide information for designing interventions in teacher training aimed at developing teaching competence in noticing students' mathematical thinking.*

## **INTRODUCTION**

Research on the professional development of mathematics teachers has highlighted the importance of teaching competence in noticing mathematical teaching-learning (Fernández, Llinares, & Valls, 2012; Mason, 2002; Jacobs, Lamb, & Philipp, 2010). Developing this teaching competence is one of the goals of teacher training programs and a relevant subject of study in research on mathematical education in recent years (van Es, & Sherin, 2002; Fernández, Llinares, & Valls, 2011).

Jacobs, Lamb and Philipp (2010) characterize this teaching competence using three skills mathematics teachers must develop: (a) identifying strategies used by the students; (b) interpreting student understanding; and (c) deciding what actions to undertake in the classroom. The research reported on here aims at providing information on how prospective primary teachers identify and interpret the mathematical thinking of primary school students in generalization processes. Generalization processes in this context are understood as linked to tasks in which the first terms of a succession are given in graphic form and students are asked: (a) to continue the succession; (b) to provide the number of elements making up the figures for distant terms; (c) to identify the rule; (d) to identify the position of a figure given the number of elements.

Research into how tasks of this sort are resolved by primary school students has shed light on the important role of the following components in the development of generalization processes:

*Coordination between spatial and numerical structure:* To extend a figural sequence, the students need to grasp a regularity that involves the linkage of two different structures: one spatial and the other numerical. From the spatial structure there emerges a sense of the elements' spatial position, whereas their numerosity emerges from a numerical structure (Radford, 2011; Rivera, 2010).



*Functional relationships:* In order to identify a distant (or non-specified) term it is necessary to establish the relationship between the position of a figure and the number of elements that make it up.

*Inverse process:* To identify the position of a known figure it is necessary to establish a functional relationship that is the inverse of the above. Although many students are capable of establishing the relationship between the position of a figure and the number of elements making it up, they find it difficult to reverse the thinking (identifying the position of a figure when given the number of elements in it) (Warren, 2005).

In cases in which the functional relationship is a affine function,  $f(n)=an+b$ ,  $b\neq 0$ , we must consider the *independent term* that appears as a constant in the function's expression. These elements play an important role in the development of the generalization process (Figure A).

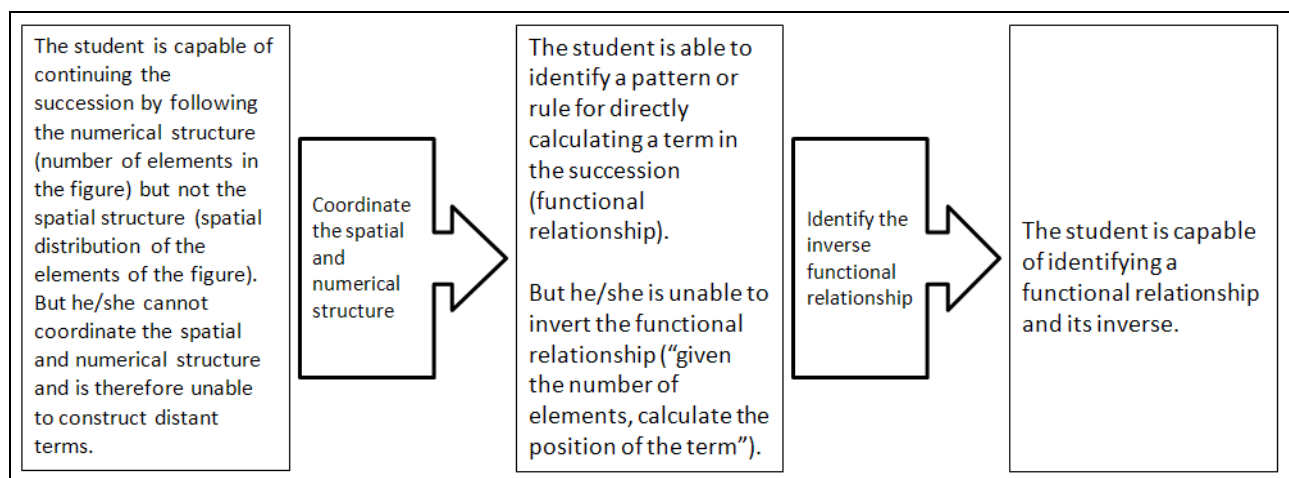


Figure A: Levels of development of the generalization process

The objective of this study is to characterize the development levels of the teaching competence of prospective primary teachers in noticing primary students' mathematical thinking in generalization process.

## METHOD

### Participants

The participants were 40 prospective primary teachers (PPTs) in the second semester of their academic program studying subjects focused on primary students' development of a numerical sense.

### Instrument

Based on prior research on the development of the generalization process in primary students (Radford, 2010; Carraher, Martínez, & Schliemann, 2007) we devised a questionnaire made up of the responses of three students to three problems displaying a succession of figures that follow a pattern of additive growth (Figure B).






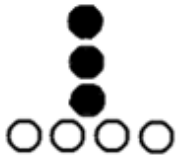

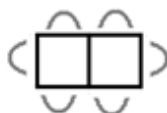
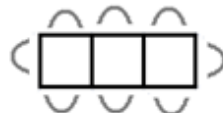
<p><b>Problem 1</b> Observe the following figures:</p> <div style="display: flex; justify-content: space-around; align-items: center;"> <div style="text-align: center;">  <p>Figure 1</p> </div> <div style="text-align: center;">  <p>Figure 2</p> </div> <div style="text-align: center;">  <p>Figure 3</p> </div> </div> <ol style="list-style-type: none"> <li>1. Continue the succession and draw figures 4 and 5.</li> <li>2. Without drawing figure 25, can you tell how many squares it would have? Explain how you figured this out.</li> <li>3. How would you calculate the total number of squares for a given figure?</li> </ol>	<p><b>Problem 2</b> Observe the following figures:</p> <div style="display: flex; justify-content: space-around; align-items: center;"> <div style="text-align: center;">  <p>Figure 1</p> </div> <div style="text-align: center;">  <p>Figure 2</p> </div> <div style="text-align: center;">  <p>Figure 3</p> </div> </div> <ol style="list-style-type: none"> <li>1. Continue the succession and draw figures 4 and 5.</li> <li>2. Without drawing figure 30, can you tell how many circles it would have? Explain how you figured this out.</li> <li>3. How would you calculate the total number of circles for a given figure?</li> </ol>
<p><b>Problem 3</b> Observe the following figures representing tables and chairs:</p> <div style="display: flex; justify-content: space-around; align-items: center; text-align: center;"> <div>  <p>1 table 4 chairs</p> </div> <div>  <p>2 tables 6 chairs</p> </div> <div>  <p>3 tables 8 chairs</p> </div> </div> <p>As you can see, we have put 4 chairs around one table, 6 chairs around two tables, and 8 chairs around three tables.</p> <ol style="list-style-type: none"> <li>1. Can you draw 4 tables and the number of chairs it should have?</li> <li>2. How many chairs can we put around 5 tables in this way? And around 6 tables?</li> <li>3. For a party we put 18 tables together along with the appropriate number of chairs. How many guests will be able to sit? Explain how you found your answer.</li> <li>4. If there are 42 children invited to a birthday party, how many tables will we need to put together in a row? Explain how you found your answer.</li> <li>5. Explain in your own words a rule connecting the number of tables and the number of chairs.</li> </ol>	

Figure B: Problems solved by primary students

In problems 1 and 2 the rule is  $2n+1$ ,  $n$  being the number of the figure. In the first problem the independent term corresponds to the only black square in each figure, and in the second to the difference between the number of white and black circles. The rule for the third problem is  $2n+2$ .

The answers of the three students to the three problems were chosen based on different levels of development of the generalization process (Figure A) (Radford, 2011; Warren, 2005):

*Student A's* answers to the three problems show a generalization process development that allows him to continue the succession for close terms, obeying the quantitative growth pattern but not the figures' spatial structures. However, the student is unable to construct the distant terms, as he/she cannot coordinate the spatial and numerical structure of the figures and ignores the independent term.

*Student B's* answers to the three problems point to a generalization process development that allows him/her to coordinate the spatial and numerical patterns, recognize functional relationships in specific cases, and express the rule with their own

words as a functional relationship. However, the student is not able to invert the functional relationship in specific cases (without the inverse process).

*Student C's* answers add to the above skills the ability to invert the functional relationship in specific cases (with the inverse process).

The PPTs were asked to respond to the following three questions:

1. *What aspects of student E's answers with respect to each of the problems would you stress, indicating to what problem you are referring.*
2. *Based on the aspects you have pointed out, identify characteristics of the generalization process of student E for the three problems.*
3. *Given the characteristics of the generalization process you listed in the above point, if you were a teacher, what would you do to improve this process?*

In this article we discuss the findings of the analysis of the first two questions.

## **Analysis**

The analysis was carried out in two phases. The first analyzed the responses of each PPT to the first two questions from the questionnaire. The objective of analyzing the first question was to see to what extent the PPTs identified the significant mathematical elements used by the students in solving generalization problems. In analyzing the second question, we examined to what extent the PPTs identified characteristics of the development of the generalization process of each of the primary students.

The goal of the second phase of the analysis was to generate descriptors of different degrees of development of teaching competence in noticing students' mathematical thinking in the realm of generalization process development. To do so, we jointly examined the mathematical elements and the characteristics of development of the generalization process that each PPT had identified. We used an inductive procedure for generating categories in which the results of the various steps were contrasted independently by three researchers, who discussed any initial discrepancies. Based on a preliminary analysis of a sample of answers we generated an initial system of categories to bring to light aspects that could be considered relevant to teaching competence in noticing students' mathematical thinking in the generalization process. As a result of this process we generated descriptors on three levels of the development of noticing:

*Level 1.* PPTs identify that the student continues the succession for close terms, obeying the pattern of quantitative growth but not the figures' spatial structure.

*Level 2.* PPTs identify that the student coordinates the spatial and numerical scheme, that he/she recognizes the functional relationship in specific cases, and that he/she is able to express the rule as a functional relationship.

*Level 3.* PPTs identify that the student coordinates the spatial and numerical scheme, that he/she recognizes the functional relationship in specific cases and is able to express the rule as a functional relationship, and that he/she can invert the functional relationship in specific cases.

## FINDINGS

Our analysis has enabled us to classify 35 of the 40 PPTs (87.7%). The responses to the questionnaire by 5 of the PPTs could not be classified into any of these levels as they did not identify the generalization level of any student (2 PPTs) or they did not identify the most elementary level (3 PPTs identified only the two students who expressed the rule).

	Level 1	Level 2	Level 3
PPTs	10	12	13

Table 1: Number of PPTs on each of the levels identified

*Level 1.* PPTs identify that the student continues the succession for close terms, obeying the pattern of quantitative growth but not the figures' spatial structure (10 PPTs).

The PPTs on this level identified only the case of the student that continues the succession for close terms and has difficulty in calculating the number of elements of a distant term because he/she cannot coordinate the spatial and numerical pattern of the succession and omits the independent term. For example, one PPT, when referring to solving problem 1, mentioned the inability to coordinate spatial and numerical structures and that the student ignored the independent term:

The exercises are wrong, since he/she did not follow either the numerical or spatial order and the result is not correct. The strategy the student used of multiplying the two rows would not be wrong if he/she added one to the multiplication (...). The difficulty the student had was because he/she didn't realize that you have to add the black square.

The PPTs at this level of development of teaching competence did not recognize the characteristics of the generalization process in the responses of the other students. Thus, they did not mention the functional relationship when interpreting the answers of the two students who successfully expressed the rule (B and C).

*Level 2.* PPTs identify that the student is able to coordinate the spatial and numerical structure, that he/she recognizes the functional relationship in specific cases, and that he/she can express the rule as a functional relationship (12 PPTs).

The PPTs at this level were capable of identifying and differentiating the students who were able to express the rule. Nevertheless, they did not find it relevant whether the student was able to carry out the inverse process. Therefore, they did not recognize the difference in the generalization process involving the ability to invert the functional relationship in specific cases. For instance, one PPT, when referring to the answers of the student who was unable to carry out the inverse process (B) omitted this aspect:

Student B adequately resolves each step. One could say that he/she follows and maintains the spatial and numerical structure. He/she describes verbally what guidelines or pattern should be followed.

*Level 3.* PPTs identify that the student coordinates the spatial and numerical pattern, that he/she recognizes the functional relationship in specific cases and expresses the rule as a functional relationship, and is able to invert the functional relationship in specific cases (13 PPTs).

The PPTs at this level were able to identify the different characteristics of the generalization process in the answers of the three primary students.

One PPT, for example, interpreted the solution of problem 1 by the student who was limited to continuing the succession for close terms (A), mentioning the lack of coordination between the spatial and numerical structures and considering that this was the reason that he/she ignored the independent term:

The answer to this problem was incorrect. In carrying out this activity, the student did not take into account the figures' spatial distribution, but did include the numerical distribution (...) It is possible that when doing the second step he/she did not see the black square and therefore did not count it.

The PPT described the characteristics of student B's generalization process indicating that he/she was able to express the rule but cannot carry out the inverse process:

In the first and second exercises he/she is able to make a generalization globally, since the student has no problem in resolving all the steps. In the third exercise he/she also successfully finds the general pattern of the problem but does not manage to apply it the other way around.

And the PPT identified the realization of the inverse process as a differentiating trait of comprehension of the generalization process between students B and C, since both coordinate spatial and numerical structures and identify the global pattern, but only student C is capable of applying the inverse process in specific cases:

He/she correctly carries out the generalization of step (1), following the pattern of the question's figures and obeying their spatial and numerical distribution. The student is also able to make a more far generalization since by using a strategy of spatial counting he is able to solve problems with higher numbers without difficulty. Finally, this student is capable of making an global generalization, since he/she is able to see the pattern followed by all the figures.

## **DISCUSSION**

The aim of this research is to characterize levels of development of teaching competence in noticing students' mathematical thinking in the generalization process. The results have given us descriptors of the development of this teaching competence characterized according to how prospective primary teachers identified the elements relevant to the generalization processes from the answers given by primary students to generalization problems.

One finding of this study is that PPTs have trouble interpreting the mathematical thinking of students in the development of the generalization process. The fact that some PPTs identified only the characteristics of the generalization in specific cases and were unable to recognize other characteristics of the generalization process displayed in the primary students' answers could be due to a deficiency in their mathematical knowledge. This fact showed through because the PPTs at level 1 of noticing development produce a generic discourse that does not reflect the relevant components that are integral to the generalization process. By contrast, the discourse generated by

the PPTs who were capable of recognizing when the primary students had expressed a rule was more specific.

Another finding from this study is the characterization of three levels of development of this teaching competence. Moving up from level 1 to 2 occurs when the PPTs are capable of identifying as relevant the functional relationship when interpreting the responses of primary students. The step from level 2 to 3 takes place when the PPTs are able to recognize as relevant to the generalization process the realization of the inverse process based on identifying the functional relationship.

These findings provide information for designing materials for teacher training that take into account the characteristics of the PPTs learning. In this respect, the instrument designed can be a springboard for devising teaching materials in teacher training programs whose objective is the development of skills in identifying the mathematical components that are relevant in solving these types of tasks and interpreting the students' answers.

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# EXPLORING MATHEMATICS VIA IMAGINED ROLE-PLAYING

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*Role-playing is considered a valuable pedagogical strategy in many contexts. We adopted this strategy in the context of a mathematics course for prospective teachers. Participants were presented with opposing viewpoints with respect to a mathematical claim and were asked to write a dialogue in which the characters attempted to convince each other of their point of view. In the process of elucidating both roles they had to imagine and articulate fictional students' reasoning as well as design a potential pedagogical intervention. We outline what imagined role-playing revealed about the participants' understanding of the structure of natural numbers and of mathematical argumentation.*

## ROLE-PLAYING

Role-playing activities are traditionally used in social studies classrooms for exploring ethical issues and the complexities involved in social situations. The effectiveness of these activities was shown in examination of cases of prejudice, such as racism, ageism or homophobia (e.g., McGregor, 1993, Plous, 2000). The use of role-playing in mathematics classrooms is infrequent; Pimm and Johnston-Wilder (2001) consider role-playing an “underused resource in mathematics teaching” (p. 72). Though it is occasionally found in mathematics at an elementary school level (Jarett, 1997, Tucker, 2010), where, for example, children assume roles of a buyer and seller in order to improve their computational skills in a “realistic” situation. Role-playing also happens in video games, where in order to progress students' avatars have to solve several mathematics tasks.

Another use of role-playing, referred to as “participatory simulation”, intends to explore how complex dynamic systems evolve over time (Resnick & Wilensky, 1998; Wilensky & Stroup, 1999). In participatory simulations students act out the roles of individual system elements (e.g., points in a system of coordinates) in order to explore how the system as a whole (e.g., function) can emerge over time.

Increased interest and involvement, empathy and understanding various perspectives, as well as deeper engagement with content and increased retention have been mentioned as advantages of role-playing.

## SCRIPT WRITING, OR IMAGINED ROLE-PLAYING

Despite the recognized advantages, time and participation logistics are a significant limitation of role-playing. If we intend to engage our students in role-playing during class time, only a few will be active players and the remainder will serve as an audience. To give all students the opportunity to participate in the role-playing scenario we turned to imagined role-playing, that is, writing a script for a dialogue between characters.



The use of script writing as an instructional tool has been implemented in prior research. For example, Gholamazad (2007) developed the “proof as dialogue” method. The participants in her study were asked to clarify the statements of a given proof by creating a dialogue, where one character had difficulty understanding the proof and another attempted to explain each claim. Additionally, the “lesson play” method was developed and used in teacher education in which participants were asked to write a script for an imaginary interaction between a teacher-character and student-character(s) (Zazkis, Sinclair, & Liljedahl, 2013).

Here we extend the script writing method by considering a disagreement between characters with respect to a number property. We consider this to be imagined (rather than enacted) role-playing.

## THE STUDY

### Participants and Data

The prospective elementary school teachers that participated in this study were enrolled in a mathematics for elementary school teachers course. In this course they completed a unit on number theory in which the topics included divisibility, prime and composite numbers, prime factorization, and the Fundamental Theorem of Arithmetic. In particular, they learned how to calculate the number of factors of a given natural number, based on its prime decomposition and the fundamental principle of counting. That is to say, if  $N$  is a natural number and the prime decomposition of  $N$  is  $p_1^{a_1} \times p_2^{a_2} \times \dots \times p_k^{a_k}$ , then every factor of  $N$  can be written as  $p_1^{s_1} \times p_2^{s_2} \times \dots \times p_k^{s_k}$ , where  $0 \leq s_i \leq a_i$ . Therefore the number of factors of  $N$  is equal to  $(a_1+1)(a_2+1) \dots (a_k+1)$ .

Having completed the number theory unit, the participants were asked to provide a written response to the Task as one of their elective assignments in the course. 24 responses were collected and analysed.

### The Task

The participants were presented with a scenario in which two characters have opposing points of view with respect to a number property. They were assigned to create a script for a conversation in which each character is attempting to convince the other of the validity of their viewpoint. The participants were further advised that the characters in their scripts may expose incorrect justification or inappropriate mathematical expressions, but this should be acknowledged in the commentary. That is, the commentary was meant to distinguish between the language and argumentation that the script-writers attribute to their characters and those that they themselves find appropriate.

The scenario and specific instructions for completing the Task are presented below.

*Bonnie and Clyde are discussing numbers and their factors. Bonnie claims that the larger a number gets, the more factors it will have. Clyde disagrees.*

Write a script for a conversation between these two characters that includes their exchange of arguments as both sides are convinced they are right. Consider what examples and what experiences could have led Bonnie to this conclusion. Consider why Clyde would disagree. Consider what arguments and what examples they both use to convince each other and what each one of them finds convincing.

Annotate your script analysing the arguments of your characters and their examples.

The scenario presented in the Task was adapted from Zazkis (1999). In that study the erroneous claim of students – that larger numbers have more factors – was analyzed in terms of the “intuitive rules” framework suggested by Stavy and Tirosh (1996). This claim was seen as an example of the “more of A – more of B” intuitive rule.

Indeed, if one were to choose a random  $n$ -digit number the expected number of factors for that number would be higher than in a randomly chosen  $(n+1)$ -digit number. For example, the expected number of factors for a 2-digit number is about 5 (450/89) and the expected number of factors for a 3-digit number is about 7.3 (6580/899). However, there are infinitely many examples of large numbers with a relatively small number of factors. In fact, for each natural number  $n > 1$ , there are an infinite number of natural numbers that have exactly  $n$  factors.

### Research questions

In our data analysis we aimed at addressing the following questions:

What arguments were used by the characters, and what arguments did the characters find convincing? What is revealed by the imagined role-playing method about the participants’ understanding of mathematics?

### THEORETICAL PERSPECTIVE

In mathematics one counterexample is sufficient to conclude that a statement is not true in general. However, this foundational norm in logic and in mathematics is not the one practiced in everyday reasoning. Researchers have observed students’ readiness to accept conjectures following several confirming examples and students’ reluctance to abandon conjectures when having disconfirming evidence (e.g., Harel & Sowders, 1998).

In non-mathematical situations one often weights evidence in order to confirm or support an assertion. Fuzzy logic – as a response to the limitations of Aristotelian ‘crisp’ logic – was developed to capture decision making in situations of uncertainty and to acknowledge a ‘grey area’ in which the truth-value of a statement is represented by a number between 0 and 1. However, it was observed that students may have a tendency to apply reasoning consistent with fussy logic to mathematical situations. For

example, the statement “even numbers are divisible by 4” was considered 50% true, as it is true for every second even number (Zazkis, 1995).

In experiments with science students, Chinn and Brewer (1993) identified seven responses to what they called “anomalous data.” The non-normative responses were: ignoring, rejecting, or reinterpreting the data, excluding the data from the current theory, and holding it in abeyance (not rejecting it, but not using it to modify the theory, either). Only two responses followed scientific or mathematical norms: the first, making peripheral changes to the currently-held theory, and the second, making substantial changes. Our analysis focuses on participants’ ways of dealing with confirming and disconfirming evidence.

## RESULTS AND ANALYSIS

In examining the scripts and their annotations written by the participants we focused on several reappearing themes. Though the students were exposed to the algorithm for calculating the number of factors of any given number, and were mostly successful in applying this knowledge on a test, only 3 out of 24 scripts alerted to it. Most scripts restated the definition of a prime number and rehearsed the method for determining whether a given number is a prime.

### Who is right?

While no participant agreed with Bonnie (that larger numbers have more factors), there were various degrees of disagreement. Only a third of the scripts had a clear rejection of Bonnie’s claim, which is in accord with mathematical convention.

Clyde:           The answer to “True or False”: As a number gets bigger the more factors it will have” is False. It may sometimes have more factors, but to say that it always does would be incorrect.

However, the verdict of “false” to a statement that is “sometimes true”, or true in a large number of cases, is inconsistent with everyday reasoning. As such, even when a mathematically correct conclusion was drawn, some participants attempted to amend the theory referring to a limited scope of applicability. Theory amendment is in accord with a mathematical/scientific norm (Chinn & Brewer, 1993), however, the amendment itself was usually incorrect. For example, the following was included in the commentary on one of the scripts: “Large numbers do not always have more factors. [...] Her statement could be true for even numbers that are increasing but it is not true for all numbers as a collective”.

A common tendency was to consider that Bonnie was “not totally wrong”, “not completely right”, or “only partly correct”. This is consistent with Chinn and Brewer’s (1993) category of “excluding data from the currently held theory” as well as with intuitive application of fuzzy logic to a mathematical situation. Prime numbers were the most notable exceptions.

## Prime numbers as exceptions

Prime numbers immediately falsify Bonnie's initial claim. All script-writers attended to prime numbers, but this attention had different forms. Initial examples of 'small primes' – such as 5 has fewer factors than 4, or that 7 has fewer factors than 6 – were initially treated by Bonnie in many of the scripts as an anomaly. Consider the following reaction to disconfirming evidence:

Clyde: Exactly! Now haven't we just shown that larger numbers don't always have more factors?

Bonnie: Damn you and your tricks! No, I refuse to give in, maybe you have just selected the only two numbers that this general rule does not apply to. Maybe you chose an anomaly, the only exception to the rule; it's going to take more than just one counter example to persuade me!

Providing evidence that supports the claim was the usual reaction to the initial disconfirming evidence. However, as KJ wrote in his commentary, "*Bonnie is selecting only composite numbers, and that is her mistake, she seems to be forgetting that there are more than just composite numbers.*" This comment implicitly suggests that the statement is correct for composite numbers, that is, leaving out the primes. Other script-writers attributed this perception to their characters explicitly.

After considering several examples, the following conversation concludes the script:

Bonnie: The larger the number, the more factors it has.

Clyde: True, unless it's a PRIME NUMBER.

Bonnie: Why didn't you tell me this rule before, it could have helped save so much time!

Clyde: I wasn't sure myself either, I just didn't want you to think you were right so I denied it.

In a different script a similar idea is explicitly reiterated, after revisiting the algorithm for determining the number of factors of a composite number:

Bonnie: This proves that I am right! That the larger the number, the more factors it will have.

Clyde: No! Actually this proves that these methods will work for composite numbers (large or small) and not prime numbers.

Students' tendency to reject evidence that is not in accord with their held beliefs was noted in several studies (e.g., Edwards, 1997). Given that script-writers are prospective teachers, this tendency of their characters exemplifies their awareness of such behavior among students. However, when erroneous claims of characters are not acknowledged in a commentary, we conclude that they are in accord with the writers' personal views.

## **Powers of primes as exceptions**

While prime numbers were the most frequently acknowledged exceptions, they were not the only ‘exceptions’ to the rule. The following exchange takes place after considering several examples of prime numbers.

- Bonnie: Prime numbers are the exception to the rule. They do not behave like other numbers. [...] The numbers that I am talking about when I say that the factors increase as the value of the numbers increase, are any number other than a prime.
- Clyde: Okay, so what about the squares of prime numbers. For example, the square of 7 is 49, so its only factors are 1, 7 and 49. That means that a smaller number, like 12, actually has more factors than the larger number which is 49. I am confused.
- Bonnie: Again, Clyde, we are looking at prime numbers in this situation. Any square of a prime will only have three factors just like you said. The same thing happens when you find the number of factors in the cube of a prime. But remember what I said before: prime numbers are the exception to the rule.
- Clyde: So what you are saying is that the number of factors increases with the value of a number, unless the number you are looking at can be factored into the base of a prime number. For example,  $81 = 3^4$  so it does not follow the trend that you are describing.
- Bonnie: Yes! Clyde, I am really glad that you challenged me when I first suggested that the larger a number is, the more factors it will have. I have also realized that this is not always true. However, when leaving out numbers that can be factored into the base of a prime number, like you said, the rule does hold true.

Here Bonnie acknowledges prime numbers as exceptions, but later she is invited to look at squares of primes. As a result, the ‘exceptions’ to the rule are extended to include powers of primes. Though the expression “numbers that can be factored into the base of a prime number” used by both characters is inappropriate, it is clear from the examples that this refers to numbers whose prime factorization is a power of a single prime. As the script-writer does not comment on Bonnie’s conclusion – that “the rule does hold true” once some numbers are excluded – we conclude that the participant shared this belief. Other possible clusters of ‘exceptions’, such as the product of two large primes, were not discussed in any of the scripts.

## **On the power of large numbers**

In all the scripts one counterexample was insufficient in convincing Bonnie to abandon her claim. This shows the awareness of script-writers to possible robust beliefs of their potential students, beliefs that they themselves may also have possessed. The following commentary, which is a clear demonstration of empirical proof scheme (Harel & Sowder, 1998), summarizes this phenomenon:

*“Bonnie insisted she was right until Clyde did more examples to prove her wrong. In order to thoroughly prove that this theory is a reliable one (without just taking someone’s word for it), one must test the theory multiple times with various numbers. After picking a few strategic numbers, only a few examples are required before the trend can be seen that the size of the number does not influence the number of factors.”*

We mentioned the tendency to treat counterexamples as exceptions above, as in the case of ‘large primes’. However, counterexamples that included ‘large’ composite numbers that were close to each other had more convincing power than others. For example, comparing the number of factors of 512 (having 10 factors) and 513 (having 8 factors), or, in a different script, comparing the factors of 3800 and 3600 helped Bonnie reconsider her position. The script-writers demonstrated not only that several examples are essential, but also that examples with large numbers are more ‘exemplary’, that is, are more likely to serve the intended purpose.

## DISCUSSION

Role-playing activities are rare in mathematics classrooms. In social studies a major goal of these activities is to help students adopt the perspective of another person. However, in dealing with mathematical content there is a need for detached analysis, not participatory perspective taking. When content-based role-playing is implemented in mathematics teacher education, the goal is not to adopt a perspective of another person, in particular, of a mistaken student, but explore it further. The goal is to understand the origins of erroneous claims and explore ways of helping students abandon them.

By playing the roles of Bonnie and Clyde the participants revealed their perceptions of the sources of the mistaken claim and their ideas of what the characters may find convincing. While erroneous conclusions may have been attributed to the characters, the included commentary pointed to participants’ personal understandings and biases. Although the given task is based on a falsifiable mathematical statement, which can be disproved with a single counterexample, the characters created by the participants treated the task as if it involved coordinating a general tendency and the exceptions to that tendency. Very few included commentary that clarified this point.

Primes are perhaps the most accessible counterexample since they have a formal name and are defined in terms of their factorizations. Actually, most participants created characters who viewed primes as the only exceptions to the rule. Several script-writers expanded the set of exceptions to include powers of primes, that is, numbers of the form  $p^k$ . Only two participants were able to differentiate between, what they called the ‘potential to have many factors’ and the actual general case.

The fact that participants used prime numbers as the first and most frequent counterexample is in accord with prior research (Zazkis, 1999). However, the method of script-writing, or imagined role-playing, has revealed that primes are often mistakenly treated by the participants as ‘removable’ counterexamples. That is, the statement is treated as if it holds true if its scope of applicability is reduced.

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# SECONDARY MATHEMATICS TEACHERS' UNDERSTANDING OF THE INFINITE DECIMAL EXPANSION OF RATIONAL NUMBERS

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*The study examines practising secondary mathematics teachers' understanding of the decimal periodic expansion of a rational number. We adopt the theoretical perspective framing the notion of 'procept' in order to explore the participants' relevant conceptions as expressed in responding to the questions of a teaching scenario and of an interview, both serving the purposes of the study. The results reveal interesting shifts on the way to understanding the nature of the decimal expansion of a rational number of period 9.*

## INTRODUCTION

The decimal expansion of a rational number is either finite (non zero) or infinite and periodic. However, a rational number with finite decimal representation has also an alternative decimal periodic representation of period 9. Significant problems in students' understanding of the relevant representations are identified in the relevant literature.

For example, Tall and Schwarzenberger (1978) found that first-year university students have difficulties in comparing  $0.999\dots$  to 1 likely to be due to a) a lack of understanding of the concept of limit, b) misconceptions related to the representation ' $0.999\dots$ ' seen as long but finite sequence of 9, c) students thinking in terms of infinitesimal and d) that a rational number has two different decimal representations. Artigue's (2000) findings that students who recognize the density of real numbers relate it with the existence of numbers just before or just after a specific real number might be seen to shed some light on students' difficulty in understanding the nature of  $0.999\dots$ . Giannakoulis, Sougioul and Zachariades (2007) exploring graduate students' difficulties in understanding real numbers, found that about half of them fell into contradiction, considering that  $2.999\dots$  is less than 3 and there is no number between them. These students, answering another question, considered that there is no pair of numbers having no number between them. Also, the historical arguments related to the potential and actual infinity called upon by Dubinsky, et al (2006a, b) to justify this difficulty provide another reading of it.

Working on similar issues, Edwards & Ward (2004) argued that students accept more easily the equality  $1/3 = 0.333\dots$  than the equality  $1 = 0.999\dots$ , as  $0.333\dots$  is the quotient of the division  $1:3$ , while  $0.999\dots$  is not the quotient of a division. Moreover, Mamona (1987), exploring students' wrong assumptions related to decimal notation, found that decimal numbers with infinite digits are not always seen as specific numbers, periodic decimal numbers are understood to be irrational numbers, while the equality  $0.999\dots=1$  is viewed as incorrect.



The research highlighted above sought to interpret students' difficulties with periodic rational numbers of period 9 mainly in relation to the corresponding fractions or integers, taking for granted, though, that '0.999...' was considered to be a number.

In this paper, we assumed that a representation such as '0.3999...' is not necessarily seen as a number. In particular, we look at the meaning attached to this representation by practicing secondary mathematics teachers. The number '0.3999...' was chosen instead of '0.999...' in order to avoid additional difficulty likely to be caused by the fact that '0.999...' represents an integer number.

## THEORETICAL FRAMEWORK

The preceding brief report of studies examining high school and university students' understanding of the periodic decimal expansion of a rational number reveals two prominent types, one that comprehends this expansion as a process and another that conceives it as a concept. This twofold pathway to making sense of the notion at hand has been addressed in Mathematics Education research.

Sfard (1989) claims that the ability to conceive mathematical notions as processes and objects at the same time, although ostensibly incompatible, is in fact complementary. Much along the same line, Gray & Tall (1994) argue that "[professional mathematicians and all those who are successful in mathematics] employ the simple device of using the same notation to represent both process and the product of the process [...]". Furthermore, they advocate that "the notation  $\lim_{x \rightarrow a} f(x)$  represents both the process tending to a limit and the concept of the value of the limit, as does  $\lim_{n \rightarrow \infty} s_n$ ,  $\lim_{n \rightarrow \infty} \sum_{k=1}^n a_k \dots$ ". They proposed "the portmanteau word 'procept' to refer to this amalgam of concept and process represented by the same symbol". An *elementary procept* is the amalgam of three components: a *process* which produces a mathematical *object*, and a *symbol* which is used to represent either process or object" (Gray & Tall 1994, p. 118 - 120). It should be noted at this point that Gray and Tall distinguish the terms *procedure* and *process*, arguing that "the word *procedure* is used to mean a specific sequence of steps carried out a step at a time. The term *process* is used in a more general sense to include any number of procedures which essentially 'have the same effect'" (Tall et al, 2001, p.87).

Gray, et al (1999) suggested a model for the performance spectrum of different individuals in different contexts, when using mathematics procedures, processes and procepts. Gray and Tall (2001) expanded this model, providing four possible outcomes of progressively higher level of sophistication: pre-procedure (no solution or partial solution), procedure (step-by-step solution for a routine problem), process (flexible solution with conceptual alternatives) and procept (ability to think about mathematics symbolically). In a latter work Tall (2007) slightly modified the model advocating that it "models the way in which a procedure which is a thinkable sequence of steps *to do* in time is steadily enriched to give the efficiency of choosing the most suitable procedure to perform the task in a particular concept, condensed into a process and compressed as a procept *to think about* and to manipulate mentally in a flexible way (Tall 2007, p. 25).

The infinite decimal expansion of a rational number is the limit of a sequence and, according to Gray and Tall (1994), symbolizes both the process tending to the limit and the concept of the limit (its value). We employed this notion to investigate the meaning attached to the relevant symbol by practicing mathematics teachers.

## METHODOLOGY

Based on the literature and the assumption articulated in the last paragraph of the introductory section, we developed the following hypothetical classroom/ teaching scenario for research purposes. The questions raised in the scenario were answered by 106 practicing secondary school mathematics teachers in writing (of a wide range of years of teaching experience). These answers were provided in the context of an examination paper for a Master's degree in Didactics and Methodology of Mathematics. The scenario had as follows:

A final year secondary school teacher gave the following question to his students: 'Which is the meaning of the representation  $0.3999\dots$  (infinite number of 9)'?

Four students gave the following answers: (i) *Student A*: The representation  $0.3999\dots$  means a process that tends to 0.4, (ii) *Student B*:  $0.3999\dots$  is a number that tends to 0.4, (iii) *Student C*:  $0.3999\dots$  is the number just before the 0.4 and (iv) *Student D*: The representation  $0.3999\dots$  is the sum of  $0.3+0.09+0.009+\dots$  but, as it continuously increases, it cannot be equal to a number. (a) What could be the teacher's goal in asking this question? (b) Comment each student's answer according to what was the student's thought process, which are the positive and the negative points in his/her view and which are his possible misconceptions (if there are any) and (c) If you were a teacher in this class, how would you help these students to overcome the misconceptions, you identified?

Teachers' responses to a scenario such as the above can serve as a means a) to study their understandings related to subtle differences of meaning such as expressed via the scenario students' replies and b) to explore teachers' practices of helping students understand this subtlety and overcome their possible misconceptions (Biza, Nardi & Zachariades, 2007). In the present study we focus only on the first issue, arguing that the hypothetical teaching scenario facilitates teachers to express their own understanding of the notion at hand through commenting on the hypothetical students' relevant understanding. Furthermore, it allows them to expose an account of this understanding unconstrained by teaching limitations. It is important to note that teachers' understanding is not dealt with as deficient but as part of a system of meanings of interest to access.

The written answers to the scenario questions were classified according to whether the employed arguments promoted a view that the notation ' $0.3999\dots$ ' represents a process, a concept or an amalgam of the two. Following the results of this analysis, 10 participants were selected based on the degree of incompatibility of their answer to the formal one. These teachers were asked to respond to the questions of a semi-structured interview which sought to deepen into their personal understanding of the representation ' $0.3999\dots$ ' directly (e.g., "what does this presentation express for you?") and indirectly, through asking clarifications on aspects of their answer to the teaching scenario (e.g., "What do you mean writing that it is not only a number?"). The

responses to the teaching scenario and the interview were used to substantiate and exemplify teachers' understanding of this particular notation.

## RESULTS

The analysis of the responses gave rise to five categories of teachers, which are presented in table 1 together with the description and the criteria used for each as well as their frequencies.

Categories	N	Descriptions and Criteria
No response/irrelevant	10	
Process	30	0.3999... is dominantly seen as a process
Process & Concept(number), at least one of the two false	31	0.3999... is viewed as a process and the result of a process (concept/number), but there is somewhere a mistake
Process & Concept (number), wavering between the two	12	Coexistence of a correct view of the notation 0.3999... (procept) with incorrect ones
Procept	23	Awareness of a correct view of the notation 0.3999...

Table 1: Categories of responses to the teaching scenario questions and frequencies.

In the following, each category is first commented with respect to its content and frequency and then some characteristic examples are cited, which exemplify the category.

According to the information included in Table 1, about 10% of the teachers gave no or an irrelevant answer. For example, teacher T60 wrote:

The teacher's goal is to introduce the concept of limit, but it could also be the concept of continuity of a function [...]. Of course, it could be an introduction to the limit of a function, where  $x \rightarrow \pm\infty$ , because 0.3999... has infinite 9s (I prefer the last example).

A noticeable percentage of the teachers, about 28%, view explicitly or implicitly the representation 0.3999... as a process. Here are classified responses in agreement with A, B or D. Thus, some attribute properties of a sequence to a number, e.g., 'tends to a number' (in agreement with student B), while others argue that 0.3999... expresses something variable continuously increasing (in agreement with student D). One of the teachers of this group, T23, in his written answer, implicitly approached 0.3999... as a process. In particular, he wrote: "This number may continue to increase but this is not the reason why it doesn't become 0.4". His view is that 0.3999... represents an increasing variable number. During the interview, after a question of the researcher, he explicitly expressed the view that 0.3999... is a process and not a number. He actually argued:

I might have said 0.333... is  $1/3$  but I do not consider it as a number right now. Logically, I may use it but in reality I do not. I believe that this operates like a sequence, like a limit, coming closer and closer. The straight line and the arrow which ... shows us that it tends ...in the neighbourhood ... The drawing we make when we want to show the sequence which tends to a neighbourhood.

Another teacher of this group, T82, expressed the same view in his written answer:

I would say to my students that  $0.3999\dots$  is a representation and not a number. It is a sequence  $(a_n)$  and we need to calculate the limit such as  $n \rightarrow \pm\infty$  where  $n$  is the number of 9s.

During the interview T82 clarified his view:

If I was the teacher who raised this question, I would first clarify to the students that  $0.3999\dots$  is an expression/ representation and not a number. Then, I would ask them to think the answer keeping in mind that it's about a sequence  $(a_n)$  of which they would have to calculate the limit, since  $n \rightarrow \pm\infty$  (where  $n \in \mathbb{R}$  is the number of digits 9).

A similar percentage of teachers (29%) envisage the representation  $0.3999\dots$  not only as a process but also as the product of a process, without all necessarily considering that  $0.3999\dots = 0.4$ . Also, some come up with their own 'small theory' in trying to interpret this representation. For example, T92 expressed the view that  $0.3999\dots$  is a constant number unequal to 0.4 but at the same time, he considered that it changes. He wrote:

Student B connected number  $0.3999\dots$  with number 0.4 as its final value [...]. Number  $0.3999\dots$  is in between 0.4 and 0.38. We could say that this number expresses a distance continuously changing.

Also, teacher T46, although he considered that the symbol  $0.3999\dots$  represents a process and a concept, he wrote about "the limiting approach of a number":

The expression  $0.3999\dots$  does not constitute *only* a number [...] Behind  $0.3999\dots$  there is hidden an approach of the values of a function [...]. Student C has not understood the concept of the *limiting approach* (sic) of a number.

This teacher didn't include in his answer something indicating that he considers  $0.3999\dots = 0.4$ .

A notably interesting category of answers concerns about 11% of the teachers who, despite knowing the equality  $0.3999\dots = 0.4$ , they contradict themselves elsewhere in their answer either agreeing with one or more of the students or arguing that  $0.3999\dots$  is an irrational number. For example, teacher T32 proved that  $0.3999\dots$  is equal to 0.4 but he also wrote:

The representation  $0.3999\dots$  is an infinite process that tends to 0.4. There are not misconceptions in student's A response.

Another teacher of this group, T25, although he provided two proofs of the equality  $0.3999\dots = 0.4$  appears to be unsure about this. He wrote:

Therefore  $0.3999\dots$  is a number that tends to 0.4 - if not 0.4 itself'. Also, he gives two proofs of the equality  $0.3999\dots = 0.4$ .

Some teachers of this group proposed some 'peculiar' symbols. For example, teacher T33 quoted two correct proofs for the equality  $0.3999\dots = 0.4$ , but he also wrote:

The positive point in student A is that he corresponded the representation [ $0.3999\dots$ ] by a continuous function and this thought is correct.[...] Then  $\lim 0.3999\dots = 0.4$ .

Only 22% of teachers appear to understand the representation  $0.3999\dots$  as equal to 0.4 with no accompanying contradictions or falsifications. For instance, teacher T49, evaluating students' C and D answers wrote:

Student C recognizes  $0.3999\dots$  as a number but he is carried away by the order in the set of natural numbers and he conceives it as the number immediately before 0.4 [...] (In student's D view) the sum  $0.3+0.09+0.009+\dots$  is the infinite sum of the terms of a geometric progression of rate  $1/10$ , which exists.

She proved also the equality  $0.3999\dots=0.4$ .

Some of the teachers of this group, although wrote explicitly that  $0.3999\dots = 0.4$ , they didn't provide any proof or further explanations.

## **IN CONCLUDING**

The results show that, overall, the teachers of the sample are divided into three groups with respect to the way they make sense of the representation ' $0.3999\dots$ '. Almost 3 to 10 of them come to view it only as a process, about 4 to 10 argue for a combination of process and concept (number) with deficiency (ies) somewhere, while nearly 2 to 10 consider the given representation correctly both as a process and as a concept (procept). This suggests that, despite the strong mathematical background of the subjects, all of them had a mathematics degree, notably few appear to hold an accurate understanding of this particular infinite decimal representation of a rational number, whereas a significant number see it only as a process. These findings draw a picture that does not differ substantially from the one emerging for students according to the relevant literature and could be attributed both to the difficulty of the idea itself and to its weak approach in education. We believe that the teaching scenario provided for the collection of the data, methodologically facilitate the above conceptions spectrum to emerge mainly by not promoting ' $0.3999\dots$ ' being seen only as a number.

From a semiotic point of view, for many mathematics teachers, the symbol of the decimal expansion of a rational number with period 9 appears to refer to mainly a process and not to a number. Furthermore, because the possibility of a rational number to be expressed with a second alternative decimal expansion is not clear or known to several teachers, even when they understand this notation as a number, many do not have in mind the right number. Their rigorous mathematical knowledge with respect to the infinite decimal expansion of a real number and the properties of this expansion in the case of rational numbers (Spivak, 1994) appears to be rather weak and causing the misconceptions/problems discussed above. This particular knowledge can be seen to belong to the specialized content knowledge (Ball, Thames & Phelps, 2008) which is crucial for teaching mathematics. The results of this paper highlight some significant aspects of this knowledge and could be fruitfully used in designing and implementing professional development programs of study for mathematics teachers.

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