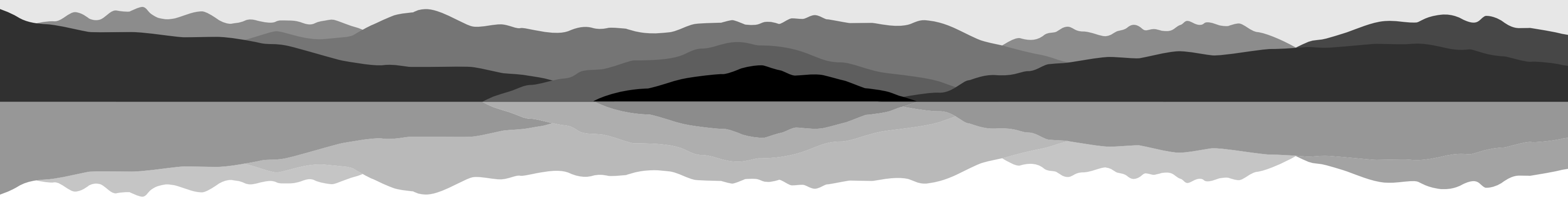
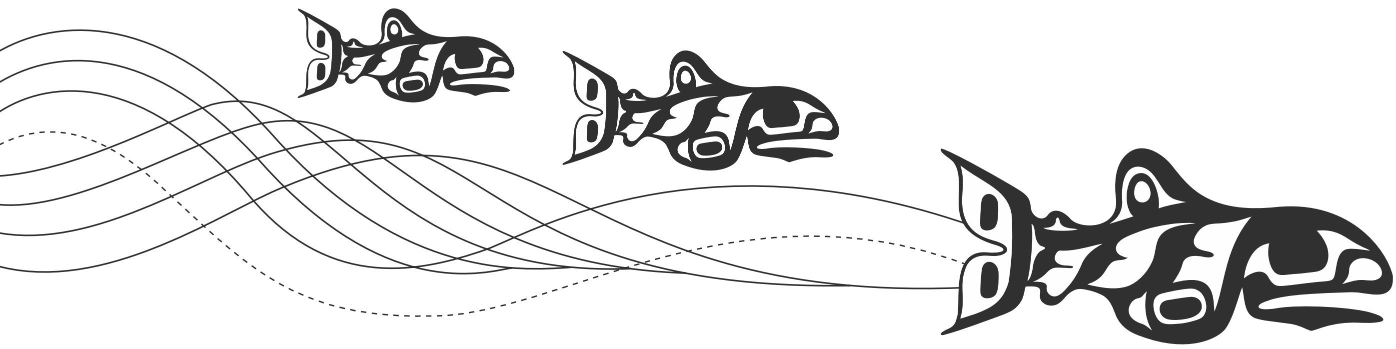


PME 38 / PME-NA 36

Proceedings

Vancouver, Canada | July 15-20, 2014

Of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education



Volume 4

Editors | Peter Liljedahl, Susan Oesterle, Cynthia Nicol, Darien Allan

PME 38 / PME-NA 36

Vancouver, Canada | July 15-20, 2014

Cite as:

Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.). (2014). Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education (Vol. 4). Vancouver, Canada: PME.

or

Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.). (2014). Proceedings of the Joint Meeting of PME 38 and PME-NA 36 (Vol. 4). Vancouver, Canada: PME.

Website: http://www.pme38.com/

The proceedings are also available via http://www.igpme.org

ISSN 0771-100X

Vol. 4 ISBN 978-0-86491-364-7

Set ISBN: 978-0-86491-360-9

Cover Design: Kirsty Robbins, University of British Columbia

Salmon Design: Haida artist, William (Billy) NC Yovanovich Jr. Ts'aahl

Formatting Assistance: Robert Browne, Jacky McGuire

Production: Lebonfon, Québec

TABLE OF CONTENTS
VOLUME 4

RESEARCH REPORTS

KNO – PI

A MODEL FOR SYSTEMIC CHANGE IN RURAL SCHOOLS 1
Libby Knott, Jo Clay Olson, Ben Rapone, Anne Adams, Rob Ely

PRIOR TO SCHOOL MATHEMATICAL SKILLS AND KNOWLEDGE OF CHILDREN LOW-ACHIEVING AT THE END OF GRADE 1 9
Sebastian Kollhoff, Andrea Peter-Koop

INVENTION OF NEW STATEMENTS FOR COUNTEREXAMPLES 17
Kotaro Komatsu, Aruta Sakamaki

THE INFLUENCE OF 3D REPRESENTATIONS ON STUDENTS' LEVEL OF 3D GEOMETRICAL THINKING 25
Yutaka Kondo, Taro Fujita, Susumu Kunimune, Keith Jones, Hiroyuki Kumakura

FLEXIBLE USE AND UNDERSTANDING OF PLACE VALUE VIA TRADITIONAL AND DIGITAL TOOLS 33
Ulrich Kortenkamp, Silke Ladel

PERCEPTIONS AND REALITY: ONE TEACHER'S USE OF PROMPTS IN MATHEMATICAL DISCUSSIONS 41
Karl W. Kosko, Yang Gao

LOOKING FOR GOLDIN: CAN ADOPTING STUDENT *ENGAGEMENT STRUCTURES* REVEAL ENGAGEMENT STRUCTURES FOR TEACHERS? THE CASE OF ADAM 49
Elizabeth Lake, Elena Nardi

COOPERATION TYPES SPECIFIC FOR BARRIERS 57
Diemut Lange

EXPLAINING MATHEMATICAL MEANING IN "PRACTICAL TERMS" AND ACCESS TO ADVANCED MATHEMATICS 65
Kate le Roux

TEACHERS' PERCEPTIONS OF PRODUCTIVE USE OF STUDENT MATHEMATICAL THINKING 73
Keith R. Leatham, Laura R. Van Zoest, Shari L. Stockero, Blake E. Peterson

A COMPARATIVE STUDY ON BRAIN ACTIVITY ASSOCIATED WITH SOLVING SHORT PROBLEMS IN ALGEBRA AND GEOMETRY 81
Mark Leikin, Ilana Waisman, Shelley Shaul, Roza Leikin

PRINCIPLES OF ACQUIRING INVARIANT IN MATHEMATICS TASK DESIGN: A DYNAMIC GEOMETRY EXAMPLE 89
Allen Leung

HOW THE KNOWLEDGE OF ALGEBRAIC OPERATION RELATES TO PROSPECTIVE TEACHERS' TEACHING COMPETENCY: AN EXAMPLE OF TEACHING THE TOPIC OF SQUARE ROOT 97
Issic Kui Chiu Leung, Lin Ding , Allen Yuk Lun Leung, Ngai Ying Wong

EXPLORING THE RELATIONSHIP BETWEEN EXPLANATIONS AND EXAMPLES: PARITY AND EQUIVALENT FRACTIONS 105
Esther S. Levenson

HOW A PROFESSOR USES DIAGRAMS IN A MATHEMATICS LECTURE AND HOW STUDENTS UNDERSTAND THEM 113
Kristen Lew, Tim Fukawa-Connelly, Pablo Mejia-Ramos, Keith Weber

THE INCIDENCE OF DISAFFECTION WITH SCHOOL MATHEMATICS 121
Gareth Lewis

MATHEMATICIANS' EXAMPLE-RELATED ACTIVITY IN FORMULATING CONJECTURES 129
Elise Lockwood, Alison G. Lynch, Amy B. Ellis, Eric Knuth

INFLUENCE OF EARLY REPEATING PATTERNING ABILITY ON SCHOOL MATHEMATICS LEARNING 137
Miriam M. Lüken, Andrea Peter-Koop, Sebastian Kollhoff

THE EFFECT OF AN INTELLIGENT TUTOR ON MATH PROBLEM-SOLVING OF STUDENTS WITH LEARNING DISABILITIES 145
Xiaojun Ma,Yan Ping Xin, Ron Tzur, Luo Si, Xuan Yang, Joo Y. Park, Jia Liu, Rui Ding

CHILDREN'S CONCEPTUAL KNOWLEDGE OF TRIANGLES MANIFESTED IN THEIR DRAWINGS 153
Andrea Simone Maier, Christiane Benz

CHARACTERISTICS OF UNIVERSITY MATHEMATICS TEACHING: USE OF GENERIC EXAMPLES IN TUTORING 161
Angeliki Mali, Irene Biza, Barbara Jaworski

CARDINALITY AND CARDINAL NUMBER OF AN INFINITE SET: A NUANCED RELATIONSHIP 169
Ami Mamolo

MATHEMATICAL CERTAINTIES IN HISTORY AND DISTANCE EDUCATION 177
Benjamin Martínez Navarro, Mirela Rigo Lemini

HIGH SCHOOL STUDENTS' EMOTIONAL EXPERIENCES IN MATHEMATICS CLASSES 185
Gustavo Martínez-Sierra, María del Socorro García González

YOUNG LEARNERS' UNDERSTANDINGS ABOUT MASS MEASUREMENT: INSIGHTS FROM AN OPEN-ENDED TASK 193
Andrea McDonough, Jill Cheeseman

MAKING SENSE OF WORD PROBLEMS: THE EFFECT OF REWORDING AND DYADIC INTERACTION 201
Maria Mellone, Lieven Verschaffel, Wim Van Dooren

STUDENTS' MANIPULATION OF ALGEBRAIC EXPRESSIONS AS 'RECOGNIZING BASIC STRUCTURES' AND 'GIVING RELEVANCE' 209
Alexander Meyer

AN INFERENTIAL VIEW ON CONCEPT FORMATION 217
Michael Meyer

FUNCTIONS OF OPEN FLOW-CHART PROVING IN INTRODUCTORY LESSONS OF FORMAL PROVING 225
Mikio Miyazaki, Taro Fujita, Keith Jones

TEACHERS KNOWLEDGE OF INFINITY, AND ITS ROLE IN CLASSROOM PRACTICE 233
Miguel Montes, José Carrillo, C. Miguel Ribeiro

PROBABILITY, UNCERTAINTY AND THE TONGAN WAY 241
Noah Morris

FUNCTIONS OF EXPLANATIONS AND DIMENSIONS OF RATIONALITY: COMBINING FRAMEWORKS 249
Francesca Morselli, Esther Levenson

TEACHERS' ABILITY TO EXPLAIN STUDENT REASONING IN PATTERN GENERALIZATION TASKS 257
Rabih El Mouhayar

USING MODELING-BASED LEARNING AS A FACILITATOR OF PARENTEL ENGAGEMENT IN MATHEMATICS: THE ROLE OF PARENTS' BELIEFS 265
Nicholas G. Mousoulides

DEVELOPING CONCEPTUAL UNDERSTANDING OF PLACE VALUE: ONE PRESERVICE TEACHER'S JOURNEY 273
Jaclyn M. Murawska

FUNCTION NOTATION AS AN IDIOM 281
Stacy Musgrave, Patrick W. Thompson

THE INTERPLAY BETWEEN LANGUAGE, GESTURES, DRAGGING AND DIAGRAMS IN BILINGUAL LEARNERS' MATHEMATICAL COMMUNCIATIONS 289
Oi-Lam Ng

INVESTIGATING STUDENT PARTICIPATION TRAJECTORIES IN A MATHEMATICAL DISCOURSE COMMUNITY 297
Siún NicMhuirí

THE INFLUENCE OF SYMMETRIC OBJECTS ON SPATIAL PERSPECTIVE-TAKING – AN INTERVIEW-STUDY WITH YOUNG ELEMENTARY SCHOOL CHILDREN 305
Inga Niedermeyer, Silke Ruwisch

SURVIVAL OF THE FIT: A BOURDIEUIAN AND GRAPH THEORY NETWORK ANALOGY FOR MATHEMATICS TEACHER EDUCATION 313
Kathleen Nolan

AN INFERENTIALIST ALTERNATIVE TO CONSTRUCTIVISM IN MATHEMATICS EDUCATION 321
Ruben Noorloos, Sam Taylor, Arthur Bakker, Jan Derry

TECHNOLOGY-ACTIVE STUDENT ENGAGEMENT IN AN UNDERGRADUATE MATHEMATICS COURSE.. 329
Greg Oates, Louise Sheryn, Mike Thomas

EXPERT MATHEMATICIANS' STRATEGIES FOR COMPARING THE NUMERICAL VALUES OF FRACTIONS – EVIDENCE FROM EYE MOVEMENTS 337
Andreas Obersteiner, Gabriele Moll, Jana T. Beitlich, Chen Cui, Maria Schmidt, Tetiana Khmelivska, Kristina Reiss

THE DEVELOPMENT OF SOCIOPOLITICAL CONSCIOUSNESS BY MATHEMATICS: A CASE STUDY ON CRITICAL MATHEMATICS EDUCATION IN SOUTH KOREA 345
Kukhwan Oh, Oh Nam Kwon

EXAMINING THE COHERENCE OF MATHEMATICS LESSONS FROM A NARRATIVE PLOT PERSPECTIVE 353
Masakazu Okazaki, Keiko Kimura, Keiko Watanabe

TEACHER METAPHORS – DIFFERENCES BETWEEN FINNISH IN-SERVICE AND PRE-SERVICE MATHEMATICS TEACHERS 361
Susanna Oksanen, Päivi Portaankorva-Koivisto, Markku S. Hannula

PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' CONSTRUCTION OF BOX PLOTS AND DISTRIBUTIONAL REASONING WITH THREE CONSTRUCTION TOOLS 369
Samet Okumuş, Emily Thrasher

RECONSTRUCTION OF ONE MATHEMATICAL INVENTION: FOCUS ON STRUCTURES OF ATTENTION 377
Alik Palatnik, Boris Koichu

REVISITING MATHEMATICAL ATTITUDES OF STUDENTS IN SECONDARY EDUCATION 385
Maria Pampaka, Lawrence Wo

'VALUE CREATION' THROUGH MATHEMATICAL MODELING: STUDENTS' DISPOSITION AND IDENTITY DEVELOPED IN A LEARNING COMMUNITY 393
Joo young Park

IMPROVING PROBLEM POSING CAPACITIES THROUGH INSERVICE TEACHER TRAINING PROGRAMS: CHALLENGES AND LIMITS 401
Ildikó Pelczer, Florence Mihaela Singer, Cristian Voica

TEACHER SEMIOTIC MEDIATION AND STUDENT MEANING-MAKING: A PEIRCEAN PERSPECTIVE 409
Patricia Perry, Leonor Camargo, Carmen Samper, Adalira Sáenz-Ludlow, Óscar Molina

ANALYZING STUDENTS' EMOTIONAL STATES DURING PROBLEM SOLVING USING AUTOMATIC EMOTION RECOGNITION SOFTWARE AND SCREEN RECORDINGS 417
Joonas A. Pesonen, Markku S. Hannula

DIAGNOSTIC COMPETENCES OF MATHEMATICS TEACHERS – PROCESSES AND RESOURCES 425
Kathleen Philipp, Timo Leuders

THE PREDICTIVE NATURE OF ALGEBRAIC ARITHMETIC FOR YOUNG LEARNERS ... 433
Marios Pittalis, Demetra Pitta-Pantazi, Constantinos Christou

INDEX OF AUTHORS AND COAUTHORS VOLUME 4 443

RESEARCH REPORTS

KNO – PI

A MODEL FOR SYSTEMIC CHANGE IN RURAL SCHOOLS

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This paper describes a theoretical model for systemic change as it concerns the learning and teaching of mathematics in K – 12 schools, with particular attention being paid to the rural context. Systemic change is the active process of establishing change in the community through lasting, long-term relationships, practices, and procedures (Adelman & Taylor, 2003). Our purpose is to describe the mechanics of such change provided by the strategic, continuous, and monitored support of all three of the constituents: Teachers, administrators and community, and externally supported by a temporary catalyst. Systemic change is achieved when the removal of the external catalyst does not affect the rest of the model. Evidence to support this claim has been derived from our case studies.

INTRODUCTION

This paper describes a theoretical model for systemic change as it concerns the learning and teaching of mathematics in K – 12 schools. The motivation for this study comes from an increasing demand for sustainable change in educational systems that seek to improve the performance of American students in mathematics (Stigler & Hiebert, 2009). With the implementation of the Common Core State Standards in Mathematics, there is a shift in emphasis on educational goals that may necessitate sustained school-wide change. Systemic change is an active process of establishing change in a community through relationships, practices, and procedures that become a lasting part of the community and is promoted by school leaders to institutionalize instructional strategies that increase student learning (Adelman & Taylor, 2003). Many recipients of major grants designed to increase students' achievement in mathematics that are intended to produce lasting change in schools struggle with ways to make the change systemic.

Despite the attention this topic has drawn from funding agencies, there is surprisingly little research that describes in detail successful implementation of sustainable system-wide change in mathematics instruction, particularly in rural schools. The proposed model emerged from our work with 34 rural schools in the Pacific Northwest through a NSF funded grant, Making Mathematical Reasoning Explicit (MMRE). The model attempts to describe the interconnections and interactions among teachers, administrators, and the community. We use the term community broadly to capture individuals or groups connected with the school, such as parents, and civic organizations with an interest in education. It is our intent to describe (a) the nature of the strategic, continuous and monitored support provided by these three constituents and their interactions with each other and (b) describe a fourth temporary, external catalyst (in our case a federally funded grant). We illustrate this process using our case study data.

LITERATURE REVIEW

Three major constituents appear to influence the degree to which systemic change becomes sustainable: Teachers, administrators, and communities (Loucks-Horsley et al, 2009). Stigler and Hiebert (2009) make a strong case for the position that if we want different outcomes in student learning, then teachers must change what they do in the classroom. The process of teacher change is critical to the systemic change (Silvia, Gimbert, & Nolan, 2000). However, we limit our literature to the interactions and influences between, rather than within, the three groups: teachers, administrators, and community. Thus, our brief literature is focused on the influences of teachers, administrators, and community on systemic change.

Teachers as supporters of systemic change

Literature on the influence of teachers on systemic change is focused on their implementation of high quality professional development and the details of change process itself (e.g., Pegg & Krainer, 2008). Pegg and Kainer describe four large scale initiatives in Austria, United States, Australia, and South Korea. These projects focus on supporting individual teachers through collaboration, communication, and partnerships. Teachers collaborated with each other and university staff members as valued members of the community with specific expertise. The inclusion of outside experts leads to in-depth discussions that facilitate the development of new instructional practices. Partnerships and communication were between the national funding agency, teachers, and university staff members. Teachers' professional development is critical to systemic change. Developing teacher leaders have been seen as one component that can support systemic change. Unfortunately, literature on teacher leadership describes their development, roles, and interactions with colleagues (e.g., Christensen, 2012) and does not examine the interactions between the school administrators and the community.

Administrators as supporters of systemic change

Guskey & Sparks (2002) describe the types of support that administrators may provide to teachers: supervision, professional opportunities, coaching, and evaluation, as well as their leadership and its influence on the school community and culture. Specifically, principals who support their teachers by individually participating in and allotting time for professional development have reported increases in teacher effectiveness and organization (Darling-Hammond & Bransford, 2007). Pegg and Krainer (2008) summarize the influence of principals on teacher change in a large-scale Austrian initiative. Teachers who had support from their principal and colleagues were more motivated to use new instructional practices and their students were more enthusiastic. In contrast, teachers with little support or who felt pressure from the administration had little intrinsic motivation to use new instructional practices.

Community support for systemic change

To support systemic change, it is important for all stakeholders to articulate the vision of this change (Adelman & Taylor, 2003). Stakeholders must anticipate barriers to change, create structures within the school system, and appropriately allocate resources to confront and remove these barriers. The likelihood of long-term systemic change is greatly enhanced when parents and the community support the innovation (Joseph & Reigeluth, 2010). Including parents into the change process is a "step toward helping parents not only to get *involved*, but also to take *ownership* of the change process." (p. 10).

SYSTEMIC CHANGE MODEL

Anderson (2003) suggests that research is needed to investigate the reciprocal interactions among the various components of a school system. To investigate these reciprocal interactions, we needed a model to help us analyse the interactions among school personnel and the community. Our 3-D model (Fig. 1) consists of a double tetrahedron that represents the multidimensional aspects of these influences, with the central plane representing the playing field where interactions among teachers, administrators and community constituents occur.

The playing field is supported by the external funding agency, and in turn supports student learning. The edges represent the interactions among these constituents. The goal of systemic change is to accomplish the shrinking of the lower tetrahedron until the external funding agency support is no longer necessary, while still maintaining the integrity of the upper tetrahedron. We next describe the external support, followed by a discussion of the interactions between community and school personnel.

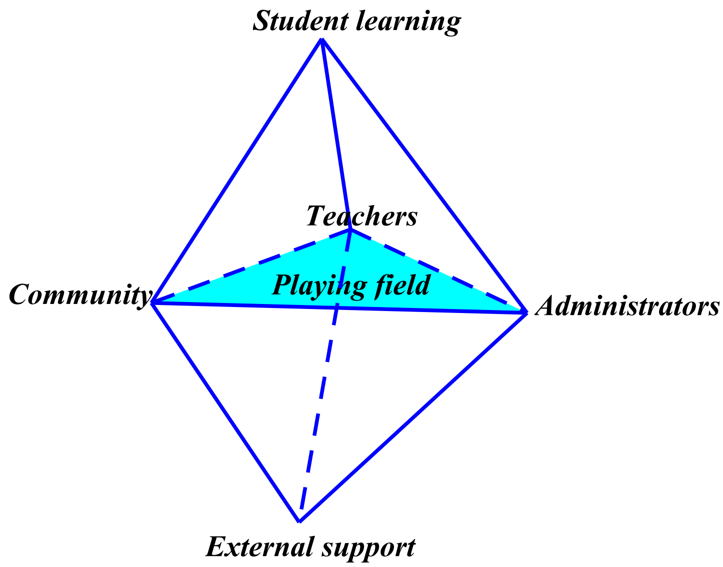

Figure 1: The goal of the systemic change model is to describe the school and community-based interactions that can provide continuous support to improve instruction and increase student learning.

External support

External support most often occurs as the result of local, state or federal grants. Valley School District, the focus district of our case study, is supported by MMRE, an ongoing five-year National Science Foundation Mathematics and Science Partnership grant project. The goals of MMRE are to (a) develop teachers' understanding of generalization and justification so that they can create opportunities for students to engage in these actions, (b) build mathematics teacher leaders, (c) support school districts create structures that increase students' intellectual engagement in mathematics, and (d) boost student achievement. MMRE Teacher leaders are expected to mentor their colleagues during their second and third year of participation and then to continue working with colleagues for two additional years.

Each year of MMRE teachers' participation includes: attendance at a 2½ week Summer Institute, four to six half-day regional meetings during the school year, and three classroom observations by MMRE staff. During the summer, we engage teachers in mathematical reasoning through the content areas of algebra, geometry and proportional reasoning. We hold daily sessions on leadership, designed to equip the teacher leaders with the skills necessary for leading professional development with their colleagues. Administrators from the participating school districts join their teacher leaders for three days to work on a three-year school district plan to support other teachers in the district implement instructional practices that support students to reason mathematically. In addition, administrators attend sessions to help them recognize mathematical reasoning as it occurs in mathematics classrooms and instructional strategies that promote it.

Influences and interactions between school administrators and teachers

School administrators include the superintendent, principals, curriculum coordinators, mathematics coaches, specialists, and their assistants. These individuals set the budget, define school district goals, policy and vision, set the schedule of classes, oversee curriculum, hire staff, and provide supervision and evaluation of them. Administrative support for change ensues from (a) allotting time for professional development, (b) allocating money for substitute teachers or supplies, and (c) revising policy to create a safe environment for teachers to use new practices. Through these actions, administrators influence teachers and their practice by providing opportunities for teachers to learn new instructional practices, collaborate, plan and enact instruction using them, and reflect on the impact of these new practices on student learning.

Teachers influence administrators by their enthusiasm and willingness to embrace a change. They may discuss with staff members and parents the importance of changing instructional practices and their impact on students' learning. They provide specific anecdotes to administrators, illustrating the positive and negative impacts of the new practices. Teachers discuss the importance of changing instructional practices and their impact on students' learning and request resources to support them. Administrators often find these professional conversations inspiring and energizing.

The superintendent and the elementary principal in our case study school district, Valley School District, were enthusiastic about participating in MMRE from the outset. During an informational meeting, four teachers expressed interest in participating in the project. Funding to support two additional teachers was requested and provided by the school board, setting a tone of support which was in place from the advent of the project and established their continued investment in the goals of MMRE.

The four teacher leaders and principal formed the MMRE Valley team. During the first year, they created a plan for implementing, sustaining and spreading MMRE instructional practices over a three-year time period. The focus of the first year was on developing teachers' own practices. The principal observed the teachers and gave supportive feedback to them. The principal reflected on his early observations and noted,

> I could see impact. The difference was in instruction. I saw entry points for students across the spectrum. Students got connected to the problems and were engaged. It was this student engagement that sold me on MMRE [during the first year].

From this reflection, it is clear the teachers and their students influenced the principal, leading to further support from the principal. This additional support came in the form of a reassignment of committee work for the three elementary teachers to focus solely on MMRE and he attended these meetings. This was a significant contribution as small schools have many needs to fill with very few staff members to contribute.

During the second year, the MMRE teachers each selected a teacher to mentor, planned presentations for the school board, and provided professional development for staff members. The teachers continued to influence the principal and the administration during the school year. Their enthusiasm and students' excitement about learning math were contagious. One teacher wrote,

> The students were engaged in math conversations. They found patterns, made conjectures based on their observations, and were able to defend or explain why things happened the way that they did. I was excited to see the kids all use exhaustion as their first strategy, but very few of them use that as a prevailing strategy as the problems became more difficult… Kids were excited about math and enjoyed working in the group setting. (Teacher reflection, September 2012).

The students' intellectual engagement in solving problems further encouraged the teacher to continue to implement these types of problems to teach mathematics. All of the MMRE teachers in Valley Schools shared these insights and commented on how students were able to think more deeply than they expected. A different teacher noted that, "I don't have to teach them anything. I just give them the opportunity to explore and they figure out what I want them to learn."

A key element of Valley Schools' success was the weekly meetings between teachers and the principal. The anecdotal stories that teachers shared increased their commitment and helped them gain confidence. The principal commented on the transition of one elementary teacher who went from someone who was "apprehensive about teaching math to a teacher who wanted to go to a two-day state conference [on math instruction] to become a better resource for teachers."

Influences and interactions between teachers and community

Teachers have opportunities to share new instructional practices directly with parents during informal and formal meetings. During parent-teacher conferences, teachers may provide work samples showing how the new instructional practices are directly impacting individual students. Parents share personal observations of their students at home and ask questions about their students' learning during conferences, through email communication, and casual exchanges in or outside school. These communications build support from the community. Although students attend a school, they can also be considered as part of the community and are one of the strongest supporters for teachers. When students are engaged in mathematical thinking with carefully scaffolded activities, they often show enthusiasm for mathematics. Their positive attitude and statements like, "Now I get it," encourage teachers to persevere in using the new practice.

All of these interactions occurred in our case study school. The teachers were proactive in communicating with both parents and the broader community. Teachers in Valley Schools were responsible for communicating instructional changes with parents and the community through three venues. They met with parents during Back-to-School night when they could help parents "understand what the kind of work that students would be bringing home." Students from the MMRE classrooms "talked about something very different [in mathematics instruction]." Second, parents and teachers discussed students' learning during conference. The teachers communicated the new depth of understanding using classroom examples. Parents were pleased about their students' enjoyment of mathematic and new abilities. Third, teachers made yearly presentations to the school board about MMRE, its impact on their instruction, and anecdotal stories about students' learning. The school board responded by acknowledging the teachers' efforts and continuing their financial support of the project.

Influences and interactions between school administrators and community

Administrators in the United States typically meet monthly with their school board, (community representatives elected to provide oversight of the school district) approve policy and budget, and support the education of students. Administrators gain the support of the board by providing updates of educational programs, discussing new research-based instructional practices, and describing how these practices will enhance students' learning. Administrators also create relationships with various community and civic groups. The support of the school board and other groups can be a critical factor in the success of systemic change.

The administrators also interact with the community through the Parent Teacher Organization. Four meetings are held each year in which the principal provides a school update, including a summary of MMRE activities along with progress on the school improvement plan. Another venue for communication among the administration and parents is at fall registration, and in newsletters that are sent home with students and posted on the school webpage.

The Rotary Club is an example of a civic organization that supports the schools, promotes the community, and helps to develop its economic vitality. In Valley, the Rotary Club holds one meeting each year in the school. The principal arranged for students from the MMRE teachers' classrooms to lead a mathematics problem, much to the surprise and delight of members! They commented that the math was more interesting than what they had experienced in school and thanked the students for the opportunity to work with them. These interactions garnered community support for MMRE. It is important to note the members of the Rotary Club tend to be the most influential individuals in a small rural community.

SUMMARY

When we began MMRE, we knew that administrative support was important. As we worked with schools, we began to notice differences in how school districts supported teacher leaders. We needed a theoretical model to provide a lens to describe the school district's playing field and identify supporting interventions. From this model we were able to analyse the interactions between the three players, teachers, administrators, and community. The case study serves as an illustrative example of a school district that created a strong base. However, not all of the school districts create a strong base like our example.

The model suggests interventions that we can use to shore up the base of school districts that may rely on only one or two players. Some of our school districts support the MMRE teachers in very superficial ways. For example, they provide substitute teachers to attend school-year meetings but do not provide time or resources for collaboration or professional development for teachers in the school district. The model suggests that the MMRE leadership team needs to work with the school district administration to help them understand their role in providing support to the project if long term gains are to be systemically induced and maintained. We also noticed that many school districts do not provide information to the community about the project. The model suggests that the community is an important constituent in reaching sustainability.

The model helped us identify ways that the school administrators and the community can help a project become sustainable and suggest interventions that can support our goals of increasing students' mathematical achievement by engaging students in making generalizations and justifications. Additional research is needed to describe the usefulness of the model in our understanding of the playing field and the specific interventions that build school districts' bases so that systemic change can be realized in school districts receiving temporary support from an external source.

Acknowledgement

Made possible through funding on NSF grant DUE #1050397

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PRIOR TO SCHOOL MATHEMATICAL SKILLS AND KNOWLEDGE OF CHILDREN LOW-ACHIEVING AT THE END OF GRADE 1

Sebastian Kollhoff, Andrea Peter-Koop

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Recent psychological studies as well as research findings in mathematics education highlight the significance of early number skills for the child's achievement in mathematics at the end of primary school. In this context, first results from an ongoing four-year longitudinal study are reported. The study investigates the development of early numeracy understanding of 408 children from one year to school entry until the end of grade 2. The study seeks to identify children that struggle with respect to their mathematics learning after the first year of school mathematics and compare their performance with their number concept development one year prior to school as well as immediately prior to school entry.

INTRODUCTION

In their play, their everyday experiences at home, and in child care centres, children start developing mathematical knowledge and abilities a long time before entering formal education (e.g. Anderson, Anderson & Thauberger, 2008). However, the range of mathematical competencies children develop prior to school varies quite substantially. While most pre-schoolers manage to develop a wide range of informal knowledge and skills, there is a small number of children who tend to struggle with the acquisition of basic number-skills (e.g. Peter-Koop & Grüßing, 2014). Clinical psychological studies suggest that these children potentially at risk in learning school mathematics can already be identified one year prior to school entry by assessing their number concept development (e.g. Krajewski, 2005). These children benefit from interventions prior to school helping them to develop a foundation of knowledge and skills (e.g. Peter-Koop & Grüßing, 2014). This seems to be of crucial importance as findings from the SCHOLASTIK project (Weinert & Helmke, 1997) indicate that students who are low performing in mathematics from the beginning of primary school tend to stay in this position.

THEORIES ON NUMBER CONCEPT DEVELOPMENT

Research and curricula increasingly stress the importance of students' early engagement with sets, numbers and counting activities for their number concept development. Clements (1984) classified alternative models for number concept development that deliberately include early counting skills as skill integrations models. While Piaget emphasized that the understanding of number depends on operational competencies and that counting exercises do not have operational value and in this respect no conductive effect on conceptual number competence, research findings

2014. In Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 4, pp. 9-16. Vancouver, Canada: PME.

suggests the development of number skills and concepts result from the integration of number skills, such as counting, subitizing and comparing (Fusion, Secada, & Hall, 1983; Clements, 1984).

Krajewski & Schneider (2009) provide a theoretical model that is based on the assumption that the linkage of imprecise nonverbal quantity concepts with the ability to count forms the foundation for understanding several major principles of the number system. Their model depicts the acquisition of early numerical competencies via three developmental levels. On the first level (*basic numerical skills*) number words and number word sequence are isolated from quantities. Children compare quantities by using comparatives like “less”, “more” or “the same amount”. At the age of three to four years most children start to link number words to quantities and hence enter the second level (*quantity number concept*). The understanding of the linkage between quantities and number words is acquired in two phases: (a) an *imprecise quantity to number-word linkage* (e.g. 3 is “a bit” while 8 or 20 is “much” and 100 is “very much”), and (b) the *precise quantity to number-word linkage*, where quantity discrimination is based on counting. At this level children gain experiences with non-numerical relations between quantities as they increasingly understand part-whole and increase/decrease schemata (Resnick, 1989). At the third level (*linking quantity relations with number words*) children understand that the relationship between quantities also takes on a number-word reference. They realise that numerically indeterminate quantities can be divided into smaller amounts, and understand that this can also be represented with precise numbers. Furthermore they discover that two numerical quantities differ by a third numerical quantity. However, Krajewski and Schneider (2009) stress that children are not necessarily at the same developmental stage with respect to number words and number symbols and that the use of manipulatives also effects the children’s performances on the different levels. Hence, with respect to their numerical development, it is very difficult to assign children exactly to one level.

In summary, Krajewski (2008) states that the quantity-number-competencies that children develop up to school entry build the foundations for their later understanding of school mathematics. While competencies on the third level reflect first computation skills and in this respect initial arithmetic understanding, the first to levels can be accounted as “preparatory mathematical skills” (ibid).

EARLY NUMBER-QUANTITY COMPETENCIES AND THEIR INFLUENCE ON LATER SCHOOL MATHEMATICS LEARNING

In their longitudinal study Krajewski & Schneider (2009) investigated the predictive validity of the quantity-number competencies of these developmental levels for mathematical school achievement. Their results indicate that quantity-number skills related to the second level measured in kindergarten predict about 25% of the variance in mathematical school achievement at the end of grade 4. Moreover, a subgroup analysis indicated that low-performing fourth-graders had already shown large deficits in their early quantity-number competencies. It can be concluded that these early quantity-number competencies constitute an important prerequisite for the understanding of school mathematics.

An intervention study by Peter-Koop & Grüßing (2014) with a pre-/post-test design (one year prior and immediately before school entry) and follow-up tests at the end of grades 1 and 2 suggests that an eight months intervention had a long-term effect lasting until the end of grade 1. Children in the treatment group demonstrated increased skills in the areas addressed in the intervention, i.e. knowledge about numbers and sets as well as counting abilities, ordinal numbers, and part-whole-relationships. A total number of 854 children performed on a standardised test as well as an individual interview one year prior to entering grade 1 and the analysis of their results lead to the identification of 73 children potentially at risk learning school mathematics that took part in the intervention. Children with a migration background who speak at least one other language than German at home were overrepresented in the group of pre-schoolers potentially at risk learning school mathematics. This group, however, demonstrated the highest increases in their performance within the treatment group.

Since the study lacks a control group (due to missing parental consent with respect to their children not being given the opportunity to take part in the intervention group) it could not be investigated how many of the children identified to be potentially at risk learning school mathematics based on their number concept development one year prior to school would have shown at least an average performance at the end of grade 1 without participating in the intervention. Hence, the number concept development of 5- to 8-year old children in the transition from kindergarten to school is addressed in the ongoing longitudinal study (2011 – 2014) that is reported in this paper. In contrast to the previous intervention study, this study is recursive in nature, i.e. it seeks to identify the low-performing students at the end of grade 1. The longitudinal data from standardised tests and one-on-one early numeracy interviews one year prior to school and immediately before school entry is (and will further be) analysed to investigate whether these children already showed lower performance with respect to sets, numbers, quantities and counting than their better achieving peers in grade 1. It will further be analysed which areas these children – in contrast to their peers – did struggle with prior to school. The main questions addressed in the study are:

- Which children perform clearly below average at the end of grade 1?
- Which content areas do they struggle with the most?
- How did they perform one year prior and immediately before school entry?
- Which content areas did they struggle with the most prior to school?

METHODOLOGY

The data collection involves four measuring points (MP1 – MP4), i.e. one year prior to school, immediately before school entry, end of grade 1 and grade 2 (which will be conducted in June 2014). At each measuring point the children performed on both a standardised test on number concept development that is suitable for their respective age (OTZ, DEMAT 1+ / 2+) as well as on a not standardised task-based one-to-one interview (EMBI-KiGa, EMBI) that focuses on the strategies that children apply on mathematical tasks or problems. Table 1 provides an overview of the study design.

Measuring points		Instruments	Participants
June 2011	MP 1	OTZ	children participating in the study (n = 538)
		EMBI-KiGa	children participating in the study (n = 538)
June 2012	MP 2	OTZ	children participating in the study (n = 495)
		EMBI-Kiga	children participating in the study (n = 495)
June 2013	MP 3	DEMAT 1+	all grade 1 classes with children participating in the study (n = 2250)
		EMBI	children participating in the study (n = 408)
June 2014	MP 4	DEMAT 2+	all grade 2 classes with children participating in the study
(to be conducted)		EMBI	children participating in the study

Table 1: Measuring points, instruments and number of participants in the study (for a detailed description of the instruments see Peter-Koop & Grüßing, 2014)

At MP3 and MP4 the whole learning group of children in the study is tested in order to compare the children's performance to their peers' and to diminish intra- and inter-group effects. Since the data collection is still in progress the analyses in this paper are only based on data from MP1 to MP3 while the whole learning group data has not been analysed yet. More detailed and complex analyses will be conducted after the completion of the data collection. For a total of 408 children (206 male, 202 female) complete data sets from the first three measuring points are available and provide the basis for the following initial analyses. Whereas 215 children (52.7%) in the sample only speak German at home, 193 children with migration background (47.3%) speak at least one language other than German at home.

In order to analyse the differences between the performances of the low-performing group of first-graders and their peers mean value comparisons have been computed using t-tests for independent samples.

RESULTS

Identification of low-performing children in the sample

In order to identify the children in the sample who are low performing at the end of grade 1 a cross mapping of the results in the standardised DEMAT 1+ and the EMBI-Interview was used to eliminate the children with low performance in only one of both tests. In this respect the DEMAT 1+ values provided a pre-selection of the lowest 20%, which was further validated with the children's performance on the EMBI. As a result 49 children (12% of the overall sample n=408) performed low in both, the standardised test and the interview (lowest 16% in EMBI scores) as well. This group of 49 children provides the basis for all further analyses.

In comparison with the complete sample children with non-German-speaking background are significantly ($p<0.001$) overrepresented in the group of low-performers (35 of 49 children, i.e. 71.4%), while there is no major difference in the sex distribution (21 male, 28 female) to the overall sample.

Performance on the DEMAT 1+ subtests and EMBI interview parts

With respect to the standardised DEMAT 1+ the group of low-performing first-graders scored significantly (p<0.001) lower in all nine DEMAT 1+ subtests. This also holds true for their results on all four interview parts of the EMBI. On average the low-performers scored one to two points less in each of the four interview domains (see Figure 1). Apart from domain A (*counting*) the low- performing first-graders only get assigned the first point in each domain. The biggest difference between both groups is found in domain C (*strategies for addition and subtraction*), in which the mean difference accounts for more than two points.

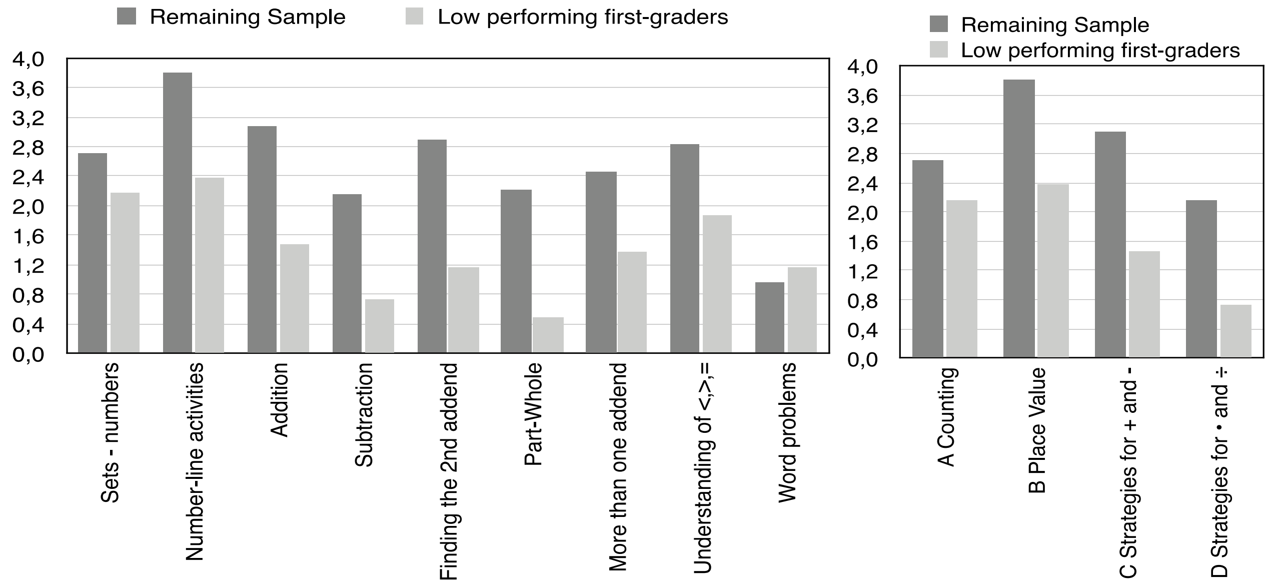

Figure 1: DEMAT 1+ subscales (left) and EMBI mean scores at MP 3

Performance prior to school (MP 1 and MP2)

The analysis of the data from MP1 and MP2 showed that the group of low- performing first-graders already performed lower prior to school entry. Their total scores on the OTZ (MP1: Low-performing first-graders: Mean: 12.96, SD: 5.156 – Remaining sample: Mean: 21.04, SD: 6.889; MP2: Low-performing first-graders: Mean: 21.92, SD: 5.235 – Remaining sample: Mean: 29.67, SD: 5.473) and their total scores on the EMBI-KiGa (MP 1: Low-performing first-graders: Mean: 3.159, SD: 1.775 – Remaining sample: Mean: 6.632, SD: 2.230; MP 2: Low-performing first-graders: Mean: 6.693, SD: 1.978 – Remaining sample: Mean: 8.972, SD: 1.337) show significant (p < 0.001) differences. While the overall scores at MP2 have improved for both groups as it was expected, the significant difference between the mean scores of both groups remains at an average difference of 2 points.

Analysis of the performance with respect to the different content-specific items in the EMBI-KiGa

The results on the EMBI-KiGa show that the group of low-performing first-graders performs significantly (p<0.001) worse in all content specific items apart from *one-to-one correspondence* (p>0.1) one year prior to school.

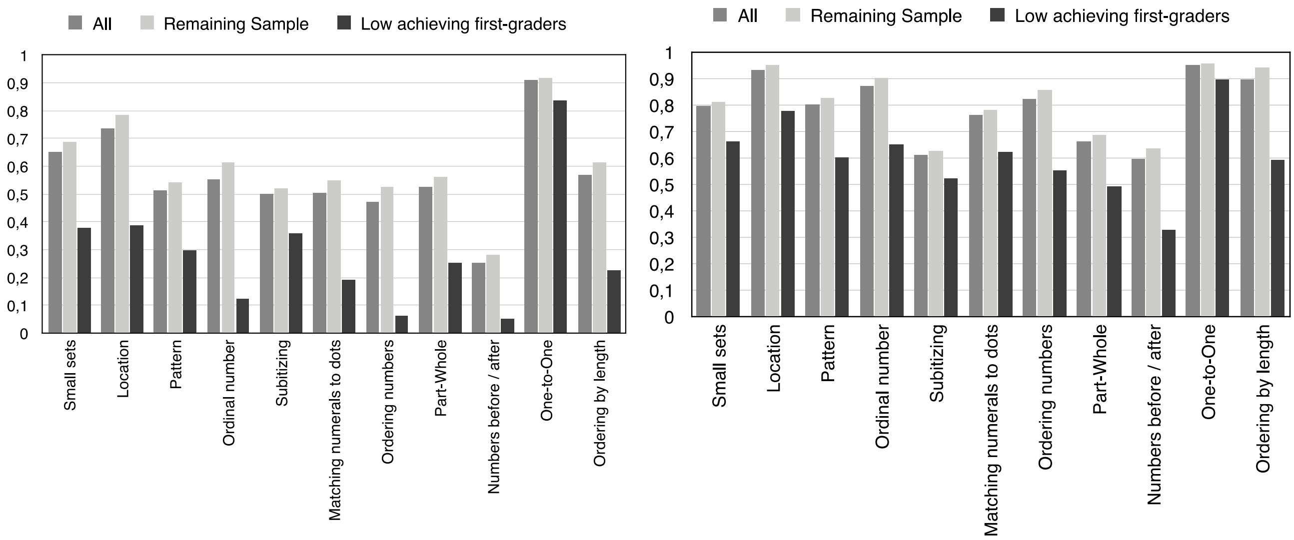

Figure 2: EMBI-KiGa subcategory mean scores MP1 (left) and MP2 (right)

They severely struggle with naming *numbers before and after* (mean=.051), *ordering numbers 0 to 9* (mean=.063) and *ordinal number* (mean=.122). While the group of low-performing first-graders showed overall improvements in all categories of the EMBI-KiGa from MP1 to MP2, they still score significantly (p<0.001) lower than their peers and there is still a major difference on their performance in the areas *numbers before/after* (mean=.326), *part-whole* (mean=.489), *ordering numbers* (mean=.551) *ordering by length* (mean=.591) and *ordinal number* (mean=.653).

DISCUSSION AND IMPLICATIONS

The first analyses of the data collected in MP1 to MP3 suggest that low-performing first-graders already demonstrate a significantly lower understanding of sets and numbers as well as less elaborate counting skills than their peers at both measuring points prior to school. Again, children with a migration background are overrepresented in this group (see Peter-Koop & Grüßing, 2014). At the end of grade 1 their performance is significantly lower in all subtests (DEMAT 1+) and in all content domains (EMBI). With respect to the DEMAT 1+ they particularly struggle with items on *subtraction, part-whole relationships, addition with more than one addend*, and *finding the second addend*. The subtests on *part-whole relationships, subtraction, addition with more than one addend, and word problems* proved to be the most difficult items for their better performing peers.

In contrast to the DEMAT 1+ that focuses on correct results, the EMBI seeks to identify strategies that children apply on mathematical tasks and problems. In this perspective the identified group of low-performing first-graders demonstrates less elaborate strategies for addition and subtraction. This is in line with their understanding of number and their number skills prior to school. In order to solve problems such as 8+6 other than counting, an understanding of part-whole schema is required in order to add up to 10 and then on (8+2+4). While they still struggle with part-whole relationships in grade 1, they already demonstrated less insight into this concept than their peers prior to school. In addition, the low-performing first-graders demonstrate less insight in counting procedures and place value. This implies that their better achieving peers show significantly more elaborate knowledge and skills with respect to high numbers. In how far this can be compensated at the end of grade 2 so far remains unclear.

It is important to note that the group of low-performing first-graders experience special difficulties with respect to items that require elaborate language skills, i.e. *language of location, numbers before/after*, and ordinal numbers. This might explain the overrepresentation of children from a non-German-speaking background in this group. However, since the assessment of German language competencies has not been included in the study design, this concern needs further investigation.

Prior to school the lower-performing first-graders demonstrated significantly less knowledge and understanding of number symbols, which suggests that their command of the German language might only be one factor among others that would explain why they tend to struggle with the development of number skills and counting a lot more than their peers.

However, as the comparison of the results on the EMBI-KiGa suggests this group of children does improve from MP1 to MP2. Immediately before school entry they show about the same average scores as their peers did one year before school entry. This complies with findings from Aunola et al. (2004), who describe cumulation effects of number-related knowledge and skills deficits prior to school, i.e. pre-schoolers who only demonstrated weak competences in dealing with numbers and sets showed a slower development of their mathematical competencies in primary school with an increasing gap towards their peers who started school with higher number skills and knowledge.

In summary the initial results of the study in progress that are reported in this paper confirm previous findings that understanding and skills with respect to number and counting are important precursors for later achievement at school. The children that were identified as low-performers at the end of grade 1, prior to school demonstrated significantly lower knowledge and skills than their better achieving peers. However, these results provide only first insights into the development of number and counting skills. Further in-depth analyses of the individual development of the children will help to better understand and describe the factors that explain the difference in achievement in the transition from kindergarten to school.

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INVENTION OF NEW STATEMENTS FOR COUNTEREXAMPLES

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From a fallibilist perspective, mathematics gradually develops with problems, conjectures, proofs, and refutations. To attain such authentic mathematical learning, it is important to intentionally treat refutation in mathematics classrooms, such as facing or proposing counterexamples and coping with them. In particular, analysing students' behaviour in response to counterexamples can lead to a design of teaching materials and instruction based on students' existing knowledge and strategies. In this paper, we construct a framework for capturing students' actions of inventing a new statement that holds for counterexamples to an original statement. We then illustrate a specific aspect of this framework with an episode that took place in an eighth grade classroom, and discuss two approaches to deductively generating a new statement.

INTRODUCTION

According to Lakatos (1976), mathematics progresses through the consideration of conjectures, proofs, and refutations, not just by monotonously increasing the number of indubitably established theorems. To introduce this authentic process in mathematics classrooms (Lampert, 1990), it is essential to deal with not only proving that a statement is true, but also refuting a conjecture by counterexamples, restricting the domain of the conjecture to exclude the counterexamples, and inventing a new statement to account for the counterexamples. In particular, it is fundamental to construct frameworks of analysis for students' behaviour in response to counterexamples, because these frameworks will enable mathematics teachers and educators to deepen their understanding of students' thought processes; such understanding may provide insights into a more effective design of teaching materials and instruction based on students' existing knowledge and strategies.

There are at least two research strands on students' behaviour related to counterexamples. The first centres on the production of counterexamples; researchers have investigated whether students and teachers can produce a proper counterexample to show that a statement is false, how they generate counterexamples, and what types of counterexamples they create (e.g. Hoyles & Küchemann, 2002; Peled & Zaslavsky, 1997; Weber, 2009). The second strand of research focuses on the recipients of counterexamples (Zazkis & Chernoff, 2008). In particular, some researchers utilise the mathematical actions shown in *Proofs and Refutations* (Lakatos, 1976) to analyse how students respond to counterexamples (Balacheff, 1991; Reid, 2002; Yim, Song & Kim, 2008). For instance, Larsen and Zandieh (2008) construct a framework that consists of "monster barring", "exception barring", and "proof analysis" (lemma incorporation), and they describe an undergraduate classroom episode to argue that this framework can serve as a description and explanation of students' mathematical activity. This paper intends to contribute to this second strand of research.

However, most researchers of the latter strand have focused on students' behaviour to exclude counterexamples, and they have not dealt with the invention of a new statement that holds for the counterexamples. In fact, monster barring, exception barring, and lemma incorporation were formulated as methods for excluding counterexamples in Lakatos (1976). Although Balacheff (1991) shows that some students created new conjectures to account for counterexamples to their initial conjectures, he summarises various student responses as "modification of conjectures" and does not examine in detail how the students modified the conjectures or what relationships the modified conjectures had with the original ones. It is valuable to focus on the invention of a new statement for counterexamples because this invention can be regarded as a brave attempt to explain the counterexamples rather than disregard them.

Consequently, this paper has two research purposes. First, we construct a framework for capturing students' action to invent a new statement that holds for the counterexamples to an original statement. Second, we illustrate a specific aspect of this framework by describing an episode that took place in an eighth grade classroom, and discuss two approaches to deductively generating a new statement.

THEORETICAL FRAMEWORK

Lakatos (1976) referred to the invention of new conjectures to account for the counterexamples to a primitive conjecture, though the description has not been sufficiently considered in mathematics education research. It was mentioned as "increasing content by deductive guessing", which means the deductive invention of a more general conjecture that holds even for the previous counterexamples (Lakatos, 1976, p. 76). Komatsu (2011) demonstrates that Lakatos's notion of increasing content by deductive guessing is useful for describing certain behaviour by ninth grade students.

However, it may not be appropriate to directly introduce this notion for describing students' behaviour in general because Lakatos's main interest lay in describing a process of growth in the discipline of mathematics, and there are differences between mathematicians' and students' behaviour. In addition, Lakatos seemed to think that his heuristic rules, which included increasing content by deductive guessing, were not universal or obligatory (Kiss, 2006). Therefore, in the following, we examine alternatives to increasing content by deductive guessing to construct a framework for capturing students' invention of a new statement that holds for previously given counterexamples.

There are two characteristics of increasing content by deductive guessing. The first is related to 'increasing content', that is, the product of invention. As mentioned earlier, increasing content by deductive guessing refers to inventing a general conjecture that holds even for the counterexamples to the previous conjecture. Therefore, the new generated conjecture is more general than the previous one in that it includes the counterexamples to the previous conjecture as its examples. However, there may be another case in which even if students can produce a statement for counterexamples to an original statement, the original statement and the produced statement do not always have such a particular-general relationship. In other words, the students may generate a statement separated from the original one, and these two statements may be regarded as just case analysis (see the following sections for an example).

The second characteristic is related to 'by deductive guessing', that is, the approach to creating a new conjecture. When Lakatos mentioned increasing content by deductive guessing, he seemed to consider the deductive invention of conjectures that were difficult to find through empirical or perceptual approaches (Lakatos, 1976, p. 82). However, there are types of mathematical reasoning other than deduction, such as induction and analogy. Therefore, it is expected that students may generate a new statement for previous counterexamples in non-deductive ways, such as through inductive, perceptual, analogical, and ad-hoc methods.

From the above, it is possible to construct a framework as shown in Table 1 for capturing students' actions to invent a new statement that holds for counterexamples to an original statement. Regarding the horizontal structure of this framework, a particular-general relationship is more desirable than a case-analysis relationship because the former can unify an original statement and its counterexamples under a new statement, without separating them (Nakajima, 1982). Although the vertical direction does not have this desirable structure, a deductive approach may be more difficult for students than a non-deductive approach. In addition, the vertical structure of this framework is relevant to the functions of proof (De Villiers, 1990). A deductive approach involves the discovery function of proof, especially if students use the proof of an original statement to generate a new statement. On the other hand, the verification and explanatory functions of proof are relevant to a non-deductive approach if students produce a statement in a non-deductive way and then prove it.

Invention approach	Relationship between original and new statements	
	Case analysis	Particular-general
Non-deductive	Type I	Type II
Deductive	Type III	Type IV

Table 1: A framework for invention of new statements to account for counterexamples

Lakatos's notion of increasing content by deductive guessing corresponds to Type IV in this framework, and this framework implies three possibilities of students' behaviour other than increasing content by deductive guessing. Nevertheless, this framework is derived from purely theoretical considerations, and it therefore needs empirical support, which we describe in the following sections.

METHODS

The classroom episode examined in this paper is taken from our larger study that aims to develop, through design experiments, a set of tasks and associated teachers' guidance to foster student engagement in proofs and refutations (Komatsu & Tsujiyama, 2013). We selected this episode because it is suitable for one of the purposes of this paper, that is, illustration of the framework in Table 1.

The second author, who has over 10 years of experience teaching in secondary schools, carried out a teaching experiment that consisted of two lessons (50 minutes per lesson) with 36 Japanese eighth graders (13–14 years old). He was not familiar with the above framework, but he took an active role in the lessons, encouraging the students to think of counterexamples and challenging the students' thinking. Both authors were involved in the lesson design, and the first author observed all the lessons.

On average, the students' mathematical abilities were above standard. They could prove geometric statements related to various properties of triangles and quadrilaterals, using conditions for congruent triangles, and had learnt counterexamples as well.

All the lessons were recorded and transcribed. The data for analysis included these transcripts, the students' worksheets, and field notes taken during the lessons. We analysed the data with a focus on the students' behaviour after proof construction, in particular, how they invented new statements to account for counterexamples to the original statement. We translated the problem sentences, the students' words and proofs from Japanese to English. All the students' names used here are pseudonyms.

RESULTS

Original statement and its proof

We used the problem shown in Figure 1 in our teaching experiment because it enables students to find counterexamples to the statement PQ = DQ – BP, as described later.

As shown in the right figure, we draw line l that passes point A of square ABCD, and perpendicular lines BP and DQ to line l from points B and D, respectively. Prove that PQ = DQ – BP.

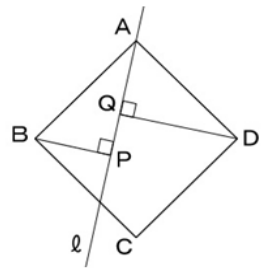

Figure 1: The problem in the lesson

The teacher presented this problem at the start of the first lesson. We describe only briefly how the students proved the statement, because the focus of this paper is on their processes after proof construction. After discussing a plan for solving the problem, the students worked individually. Next, the teacher had a student, Emi, write her proof on the blackboard. Her proof was examined in a classroom discussion, which revealed that the part which showed the congruence of angles ABP and DAQ was complicated for the other students. The teacher therefore had Mai give a complementary explanation with a different expression (Figure 2).

Proof by Emi and Mai:
In ΔABP and ΔDAQ,
From the supposition, $\angle APB = \angle DQA = 90°$
Since quadrilateral ABCD is a square, $AB = DA$
Let $\angle BAP = a$
Since the sum of the interior angles of triangle ABP is 180 degrees,
$\angle ABP = 180° - \angle APB - \angle BAP = 90° - a$
Since an interior angle of a square is 90 degrees, $\angle DAQ = 90° - \angle BAP = 90° - a$
Therefore, $\angle ABP = \angle DAQ$
Since the hypotenuses and a pair of corresponding angles in right triangles are equal, $\Delta ABP \equiv \Delta DAQ$
Since the corresponding sides of congruent figures are equal, $AP = DQ$ and $BP = AQ$
Therefore, $PQ = AP - AQ = DQ - BP$

Figure 2: The proof constructed by the students

Counterexamples and new statements

After this proof, the teacher asked, “Now, we drew line *l* which passed point A like this [Figure 1], but when the place of this line *l* is different from here [Figure 1], is it possible to say that this [PQ = DQ – BP] is true?” A few students responded “maybe impossible”. Then, the teacher told the students, “Draw various lines, *l*, which pass point A and investigate by drawing your own diagrams”. The first lesson finished with the students individually drawing diagrams on their worksheets.

Analysing their worksheets after the lesson, we found that many students drew diagrams similar to those shown in Figure 3 (these figures are examples of the students’ actual drawings). In the case of Figure 3-a, the students wrote, “Segment BP becomes longer than segment DQ” or “DQ – BP becomes negative”. For Figure 3-b, they wrote, “Segment PQ is longer than segments DQ and BP” or “[DQ – BP] becomes negative as well”. Their worksheets evidenced that they grasped these cases as counterexamples refuting the statement in the original problem, PQ = DQ – BP.

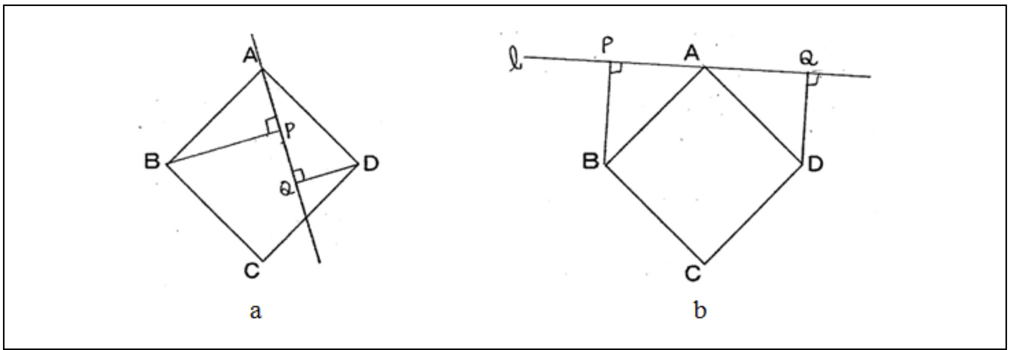

Figure 3: Counterexamples drawn by the students

In the second lesson, the students investigated what relationships among PQ, DQ, and BP held in the cases in Figure 3. At this point, the teacher told them they were allowed to utilise the previous proof by Emi and Mai (Figure 2).

After the students engaged in this investigation individually, the teacher had Manabu and Ken write their ideas on the blackboard. Regarding the case represented by Figure 3-b, Manabu wrote, "I prove the congruence of triangles ABP and DAQ as we did in the last lesson, and from PQ = AQ + PA, it should be true that PQ = BP + QD". Thus, Manabu deductively invented a new statement, PQ = BP + QD, for this case that had been a counterexample to the original statement, by utilising the congruence of triangles ABP and DAQ as a reason which, he thought, could be shown by the same proof as the previous one. Ken thought similarly to Manabu, writing his idea for Figure 3-a as follows: "[From the previous proof, I found DQ = AP and AQ = BP.] Since PQ = AQ – AP is true, the relationship among PQ, DQ, and BP is PQ = BP – DQ" (he wrote the square brackets on his worksheet, but not on the blackboard).

Next, the students examined whether the congruence of triangles ABP and DAQ could actually be shown by the same proof as Emi and Mai's one. For example, the teacher asked the students whether Emi and Mai's proof was directly applicable to the case shown in Figure 3-b. Daisuke answered that it was possible to apply this proof up to its part deducing AP = DQ and BP = AQ, and many students seemed to agree. Then, the teacher urged the students to inspect this applicability in more detail, and some students had doubts as to the part stating that since an interior angle of a square is 90 degrees, the degrees of angle DAQ are 90 – *a* (Figure 2). More concretely, Satoshi stated, "Because both angles DAQ and BAP are not inside it [angle BAD], I think it is not true". After that, other students added that it was enough to use the degrees of angle PAQ (180 degrees) to prove that the degrees of angle DAQ are 90 – *a*.

DISCUSSION

In this episode, the students proved the original statement (Figures 1 and 2) and then faced counterexamples that refuted it (Figure 3). In response, they produced new statements, that is, PQ = BP – DQ for the case as Figure 3-a, and PQ = BP + QD for the case as Figure 3-b. The original statement and these new statements written for the counterexamples do not have a particular-general relationship, and they are regarded as case analysis according to the position of line *l*. In theory, it is possible to generate a general statement that holds for all cases if we represent PQ as the absolute value of the sum of vectors BP and DQ (Shimizu, 1981). However, the students in this episode had not learnt vector, and it was impossible for them to consider such a generalisation.

The students invented the new statements for the counterexamples (Figure 3) in deductive ways, such as utilising a part of the previous proof as a reason for their thinking or constructing new deductive arguments. In addition to Manabu and Ken, Yuko wrote on her worksheet that "I had thought PQ = DQ – BP [in the case of Figure 3-a], similar [to the case shown in Figure 1], because the right and left were only reversed, but *I found [PQ = DQ – BP was] not true through copying [the previous] proof*". Toru also wrote that "In *the process of making proofs, I gradually understood* that I could represent the relationships between PQ, DQ, and BP by using + and –" (our emphases). In summary, these students' behaviour corresponded to Type III in the framework shown in Table 1.

This episode implies a possibility of dividing a deductive approach to invention of a new statement into at least two categories. The students in this episode thought that the congruence of triangles ABP and DAQ, which had been shown by Emi and Mai for the original statement, held for the counterexample shown in Figure 3-b as well, and they used this congruence as a reason to produce a new statement, PQ = BP + QD. At that point, they did not examine this congruence in detail, such as by considering whether the previous proof by Emi and Mai was directly applicable to the case as Figure 3-b. These students' behaviour can be regarded as *modularly deductive* in the sense that they thought a certain encapsulated part was true and invented the new statement by utilising this part as a reason for their thinking.

Considering an alternative to a modularly deductive approach, it is possible to think up a *sequentially deductive* approach that refers to confirming, from the beginning, that each detailed point is true and piling these points step by step to invent a new statement. This approach is only a research hypothesis because we could not directly capture the relevant students' behaviour in the episode reported in this paper. However, the relevant process can be seen in Lakatos (1976), which dealt with the Descartes-Euler conjecture on polyhedra, expressed as $V - E + F = 2$, where V, E, and F are the numbers of vertices, edges, and faces of polyhedra, respectively. In this literature, an imaginary teacher and students sequentially constructed polygons and polyhedra by marking points, connecting them, and pasting polyhedra whose values of $V - E + F$ were already known. They then examined each increase and decrease in the numbers of vertices, edges, and faces to invent a more general conjecture than the above conjecture. In the future, it will be necessary to investigate whether a sequentially deductive approach can be observed in actual students' activity.

Another future task should explore the characteristics of a modularly or sequentially deductive approach. In the episode reported here, the students who took a modularly deductive approach first believed that part of the previous proof by Emi and Mai, up to deducing AP = DQ and BP = AQ, was directly applicable to the case shown in Figure 3-b. After that, when the teacher urged the students to inspect this applicability in more detail, they could realise the necessity of modifying the part showing that the degrees of angle DAQ are 90 – a. In addition to such a pitfall, it will be valuable to investigate the advantages of each approach.

Acknowledgements

We are grateful to Taro Fujita (University of Exeter, UK) for his valuable comments on earlier drafts of this paper. This study is supported by Grants-in-Aid for Scientific Research (No. 23330255, 24243077, and 24730730).

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THE INFLUENCE OF 3D REPRESENTATIONS ON STUDENTS' LEVEL OF 3D GEOMETRICAL THINKING

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While representations of 3D shapes are used in the teaching of geometry in lower secondary school, it is known that such representations can provide difficulties for students. In this paper, we report findings from a classroom experiment in which Grade 7 students (aged 12-13) tackled a problem in 3D geometry that was, for them, quite challenging. To analyse students' reasoning about 3D shapes, we constructed a framework of levels of 3D geometrical thinking. We found that students at a lower level of 3D thinking could not manipulate representations effectively, while students operating at a higher level of 3D thinking controlled representations well and could reason correctly.

INTRODUCTION

In geometry teaching, despite the study of 2D figures often taking precedence over the study of 3D figures, most school curricula aim to develop learners' understanding of 3D figures. As such, an issue for research is to seek ways to develop learners' spatial thinking and reasoning in 3D geometry (Gutiérrez et al., 2004). In reporting an earlier study of students' reasoning in 3D geometry (see Jones, Fujita, and Kunimune, 2012), we focused on how particular types of 3D representation can influence lower secondary school students' reasoning about 3D shapes. The purpose of this paper is to propose a theoretical framework to capture students' levels of thinking with 3D shapes and their representations. In particular, we address the following research questions:

- What framework can be constructed to capture students' spatial thinking in 3D geometry?
- What characteristics of thinking can be identified when students tackle challenging problems in 3D geometry?

In what follows, we take as our starting point the levels of 3D geometrical thinking proposed by Gutiérrez (1992) and our previous study (Jones, Fujita and Kunimune, 2012). We then construct a framework to capture students' levels of thinking with 3D shapes. To construct our framework, we take a bottom up approach, i.e. our framework is mainly derived from data from 570 G7-9 students. We then evaluate our framework further by analysing classroom episodes taken from a sequence of two lessons from a teaching experiment with Grade 7 students. In considering how our framework is useful to capture students' levels of thinking effectively, insights from our findings are discussed in terms of how we might improve students' thinking with 3D shapes – a form of mathematical thinking which is challenging for many students.

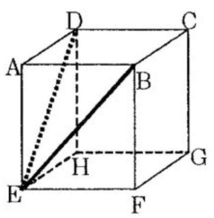
STUDENTS' REASONING IN 3D GEOMETRY: SURVEY RESULTS

By considering existing studies, and in order to obtain detailed analyses of students' thinking and reasoning with 3D shapes in relation to their interpretation of graphical information (Bishop, 1983) and decoding 3D figures (Pittalis & Christou, 2013), the following three theoretical components 'Reasoning with 3D shapes', 'Manipulation of 3D representations', and 'Levels of thinking of 3D shapes' are important.

Research on students' reasoning and representations of 3D shapes

While physical models of 3D shapes can be used in the teaching of geometry, representations of 3D shapes (on the 2D board or in textbooks or other materials) are the main mediational means. Existing research evidence indicates that representations of 3D shapes can have various impacts on learners' reasoning processes. Parzysz (1988; 1991), for example, reported that not only do learners prefer the parallel perspective (in which parallels are drawn as parallels), but, in particular, they prefer the oblique parallel perspective in which the cube is drawn with one face as a square. Such external representations can lead to some ambiguities for students with the result that particular geometrical relationships might appear as 'evident' in a way that can prevent geometrical reasoning from developing in the most appropriate way. In line with this, Ryu et al. (2007) reported that while some of the mathematically-gifted students they studied could, for example, imagine the rotation of a represented 3D object, other such students had difficulty in imagining a 3D object from its 2D representation.

Informed by research by Hershkowitz (1990) and Mesquita (1998), in research reported in Jones, Fujita and Kunimune (2012) we found that some students can take the cube as an abstract geometrical object and reason about it beyond reference to the representation, while others were influenced by the visual appearances of 3D representations and could not reason correctly. The 570 G7-9 students' answers for the question illustrated in Figure 1 (which was one of the survey questions) were classified into the following five categories: (A) global judgment; e.g. 90^o, no reason (19.3%); (B1) incorrect answer influenced by visual information; e.g. half of $\angle$AEF = $90^o/2$ = 45^o (44%); (B2) incorrect answer with some manipulations of a cube but influenced by visual information; e.g. drawing a net, and then $45^o + 45^o = 90^o$ (10.3%); (C) incorrect answer by using sections of cube but influenced by visual information; e.g. in triangle BDE, $\angle$B = $\angle$D = 45^o, therefore $\angle$AEF = 90^o (5%) ;(D) correct answer with correct reasoning; e.g. in triangle BDE, EB=BD=DE and therefore $\angle$BED = 60^o (6.3%) ;(E) no answer (15.3%). The result implies that it is difficult for many students to reason correctly with given representation.



What is the size of the angle BED?

State your reason why.

Figure 1: Angle in a cube problem (survey problem version)

3D geometry thinking levels

Based on the van Hiele model of thinking in geometry, something widely used to describe and analyse learners' thinking in 2D geometry, Gutiérrez (1992) proposed levels of 3D thinking. This was used by Gutiérrez et al. (2004) to investigate students' levels of thinking and their proof capabilities with problems involving prisms. The result was a characterization of students' levels of 3D spatial thinking, with the lower levels characterised as relying on simple descriptions based on drawings, while at higher level students begin using reasoning more analytically (pp. 512-513). Extending this, we note that Pittalis and Christou (2013) argue that interpreting representations of 3D figures utilises two capabilities: a) recognising the properties of 3D shapes and comparing 3D objects, and b) manipulating different representational models of 3D objects. From these points of view, we first undertook an initial analysis of the survey data mentioned above of 570 G7-9 students' answers. We found that students' incorrect responses were the result either of inappropriate reasoning with 3D shapes' properties or inappropriate manipulating of shapes, or both. In this paper, we utilise this information to refine Gutiérrez 's work, and propose the framework set out in Table 1 to capture students' geometrical thinking with 3D shapes.

Level	Reasoning with 3D properties	Manipulating	Features of students' 3D thinking
1	No	No	Students' thinking is influenced by 2D representation.
2a	Yes(not appropriate)	No	Students start utilising 3D properties of shapes but without effective manipulations.
2b	Yes(not appropriate)	Yes(not appropriate)	Students utilise 3D properties & manipulate the figure but it is not appropriate.
2c	Yes(not appropriate)	Yes	Students utilise 3D properties & manipulate the figure appropriately but with an incorrect answer.
3	Yes	Yes	Students utilise 3D properties & manipulate the figure appropriately, and obtain the correct answer.

Table 1: Levels of thinking in 3D geometry

STUDY CONTEXT AND METHODOLOGY

Building on our earlier classroom-based research in 3D geometry (e.g. Jones, Fujita, & Kunimune, 2012), in this paper, to address our research questions, we analyse episodes taken from two lessons in which students tackled the problem in Figure 1 by using our framework presented in Table 1 (note in these lessons the size of angle BGD is asked instead of the angle BED). We refer to the data from classroom episodes because the data from the survey is rather superficial to reveal students' thinking and further qualitative analysis of students' reasoning with this problem is worthwhile. In particular, the cube in the problem uses the oblique parallel projection and the angle to be found is changed from the original survey problem. All this means that it is not straightforward to know the size of the angle BGD because of this representation.

The main data are taken from a class of 28 Grade 7 students (aged 12-13) from a public school in Japan. The class teacher, Mrs M, has more than 20 years teaching experience, and is particularly interested in students' geometrical reasoning processes. Given that teachers' interactions with students are crucial to encourage students' reasoning (e.g. Jones & Herbst, 2012), in general her roles in the lessons were to facilitate students' discussions by suggesting where to direct their attention in the problem, which properties might be used, and so on. Through following the Japanese geometry curriculum, the students had already studied selected properties of solid figures such as nets, sections of a cube, surface areas and volume (note that the measure of the angle between two lines in 3D space is not formally studied within the prescribed curriculum). We video-recorded two lessons (each 50 minutes) in which the students worked with the problem. Field notes were kept and the audio was transcribed. In addition, student worksheets from both lessons were collected to obtain information on how the students' reasoning changed across the two lessons. All data were analysed qualitatively in terms of the theoretical framework presented above. We particularly analysed students' interactions with the teacher during the lessons and their answers and explanations in their worksheets. Through this we determined the levels of students' thinking in terms of the characteristics of their reasoning and manipulations of representations.

FINDINGS AND ANALYSIS

Lesson progression

A key to correctly answering the problem is to deduce that triangle BGD is an equilateral triangle. During the two lessons, the 25 students (3 students were absent during the first lesson) attempted enthusiastically to solve the problem and Mrs M led the class well. In the first lesson, after the problem was posed, the students began by tacking the problem individually. Their initial answers from their worksheets are shown in Table 2.

As can be seen from Table 2, only five of the students considered that the angle was 60°. This indicates that their reasoning is likely to be influenced by the external representation of the problem. After this initial stage, the teacher asked the students to share their ideas and answers, and the six answers in Figure 2 were presented. Mrs M then asked the students to comment on these, but no students stated their opinions. This was the end of the first lesson.

Answer	90°	60°	22.5°	30°	45°	35°	90° or 60°
Number of students	8	5	4	2	2	1	2

In total 24 answers; one student did not write any answer.

Table 2: Students' initial answers to the problem

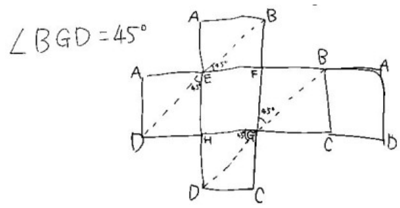
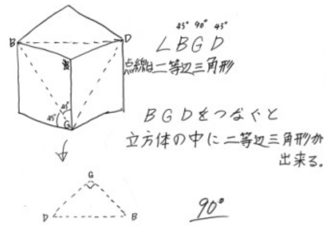
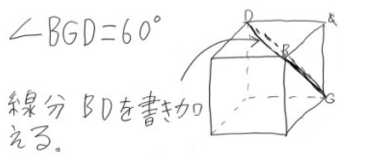
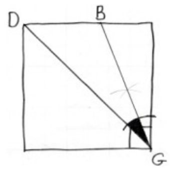
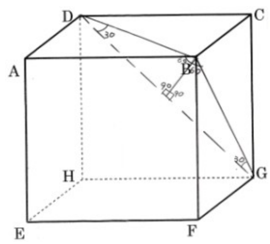
(1) 35° (Student F1) I thought it is 35° because I used a set square (to measure the angle in the representation), and it seems 10° smaller.	(2) 45° (student Y) I used a net and if I cut it from B to D, then it is 45°. So ∠ BGD＝45° 	(3) 90° (Student F2) I rotated the cube and I can make an isosceles triangle BGD. And ∠ BGD＝90° 
(4) 60° (Student IM) In a cube all diagonals of each face are the same. I added a line BD and we have a triangle BGD which is an equilateral triangle. And ∠ BGD＝60° 	(5) 22.5° (Student K) A line DG halves a square, and another line BG further halves it. So 90×1/2＝45, and 45×1/2 ＝22.5. 	(6) 30° (Student H)  No explanation and angles are measured as 2D angles.

Figure 2: Presented students' answers

At the start of the second lesson the 24 students (four were absent during the second lesson) continued to exchange their ideas and reasoning. First, Mrs M asked the students whether they changed their answers or not, based on the presentations at the end of the first lesson. The students' revised answers are shown in Table 3. While the number of students giving the answer 60° had risen to 16, during the second lesson some students still argued why they could not see the angle as 60°.Mrs M asked the students to consider an explanation which would help everyone in the class to consider whether the answer was 60°. One student explained BGD is an equilateral triangle. After another student (student IM) presented his proof to refute 90°, Mrs M then used a physical model of a cube to demonstrate the reasoning. That completed the second lesson.

Answer	90°	60°	22.5°	30°	45°	35°	22.5° or 60°
Number of students	6	16	0	1	0	0	1

Table 3: Students' revised answers to the problem

Students' level of thinking

As might be expected, students in the class illustrated various levels of 3D thinking. In the first lesson, some students determined the size of the angle neither by referring to the properties of 3D shapes nor by manipulating the presented figure. In our case, students F1 (the Figure 2-(1)) or H (in Figure 2-(6)) for example, used measurement from the given representation, and did not have any idea why this would not be correct. These students can be considered as Level 1 in our framework.

Meanwhile, some students started using the properties of a cube and simple (but ineffective) manipulations. For example, like student K in Figure 2-(5), student J (Figure 3 left) did not add anything on the given figure but used properties of angles of a square (90°) and concluded 22.5°. This is Level 2a. Student C utilised a net to consider the size of the angle BGD to deduce the angle is 90°, as illustrated in Figure 3 (right). As evident, this approach does not work because the angles in the net and the angle required in the problem are different. Yet this student cannot see this by using only the net representation, i.e. they cannot utilise properties of cubes independently from the used representations (a similar case is student Y, Figure 2-(2)). Such thinking can be considered as level 2b, i.e. utilised properties of shapes and started to manipulate the given representation, but neither of them were effective and appropriate.

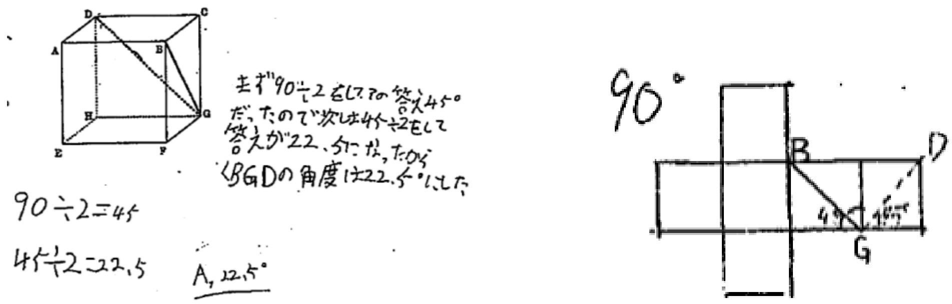


Figure 3: Answers by student J (left) and C (right)

Other students used more manipulations of shapes and also started using properties of shapes to construct some simple deductions. In our study an example of this was when student F2 joined B and D to form triangle BDG and started examining what the triangle BDG would be to deduce the size of the angle. However, the following exchange shows that this student could not recognise the triangle BGD as an equilateral because of how the representation of the cube looked:

Student F2	I joined B and D.
Mrs M	Join B and D, and then?
Student F2	Then I see a right-angled isosceles triangle (Figure 2-(3)).
Mrs M	OK, you thought the triangle is a right-angled isosceles triangle…
Student F2	So, G should be 90°?

It is notable that student F2 changed the oblique parallel projection to an orthogonal projection (see Figure 2-(3)) but still deduced an incorrect answer. This student's manipulation was appropriate (i.e. can lead to the correct answer), but their reasoning was influenced by the visual appearance of the triangle in the figure, and they were not able to utilise properties of shapes effectively.

Nevertheless, it was the case that even in the first lesson some students started manipulating the representations effectively to solve the problem. In such cases, their reasoning was not overly influenced by the external representation of the cube but was more controlled by logical thinking, which can be considered as level 3 thinking. For example, student IM explained his reasoning very clearly as follows:

Student IM Because in a cube all diagonals should be the same length, this triangle is an equilateral.

Mrs M OK, you thought it will be an equilateral because of the length of the diagonals.

Student IM [nods] Each angle of an equilateral triangle is 60, ° so ∠BDG is 60°.

Student IM also showed his clear and advanced thinking and explained how he could refute 90 °as an answer as follows:

Student IM (writing the following answer) *The sum of the inner angles of a triangle is 180°so 90 does not work. D=90, B=90, G=90. D+B+G=90+90+90=270.*

Mrs M Can you explain this?

Student IM If the line BD, DG and BG are all the same, then ⋯ no, sorry. If you add the angles of a triangle, then 180°, and if the ∠BGD is 90°, then it is an equilateral, so all angles should be the same and the other two angles are also 90°, and add them together it will be 270°. This does not work.

DISCUSSION

For our first research question 'What framework can be constructed to capture students' spatial thinking in 3D geometry?, we developed the framework of the levels of geometrical thinking with the two aspects 'reasoning with 3D properties' and 'manipulating 3D shapes representations' derived from our large empirical data set. In order to evaluate our framework, in this paper we used data from the classroom episodes in the form of students' explanations and tested how our framework can capture students' characteristics of thinking. As we have seen, the students presented a wide variety of their answers and reasoning, and our framework can provide a comprehensive classification of these answers and reasoning.

For our second research question 'What characteristics of thinking can be identified when students tackle challenging problems in 3D geometry?', from what we have observed in the two lessons and students' worksheets, our framework can successfully characterise students' thinking. At the first level, there are students who are strongly influenced by visual appearances of external representations. These students should be encouraged to explore their reasoning without relying on their naïve visual thinking.

Students at the second level cannot utilise manipulations of representations or properties of shapes. For example, although some students used effective manipulations such as drawing a line BD, some of these students could still not reach the correct answer. For these students, it is necessary to make them reflect on their reasoning or manipulations. Indeed, in our classroom episodes, it was useful when students in the class shared their ideas of various manipulations of shapes, reasoning and so on. In particular, student IM's refutation which was appropriate use of reasoning with properties worked well in leading many students to the correct answer. In future research, we plan to examine these modified level descriptions further using a larger data set. While we only have space in this paper to present one problem, we have additional analyses of other problems that reveal students' levels of thinking with 3D shapes in general.

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FLEXIBLE USE AND UNDERSTANDING OF PLACE VALUE VIA TRADITIONAL AND DIGITAL TOOLS

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Germany

Place value is a key concept for numbers that is introduced in early mathematics. It is necessary to have a flexible understanding of place value for efficient arithmetic strategies and success in written algorithmic arithmetic. In our research we explore typical mistakes and misconceptions that occur with 3rd graders in German elementary school and investigate the various underlying actions when manipulating numbers in a place value chart. We also report on a follow-up quantitative study that compares real and virtual manipulatives for place value and their effect on learning of place value.

INTRODUCTION

As demonstrated in the PME 36 plenary by Mariotti (2012), it is helpful to analyse artefacts from a semiotic perspective for their semiotic potential. The Theory of Semiotic Mediation (TSM) can be used to analyze teaching experiments at all school levels and with artefacts of any type. This latter is particularly important when we want to find out whether and how new artefacts, in particular digital ones, can improve the teaching and learning process. In our research about place value and flexible interpretation of place value charts we realized that instead of focusing on sign productions it was more helpful to look at the individual *actions* that can be observed when children work with a place value chart. Using Artefact-Centric-Activity-Theory (ACAT, Ladel & Kortenkamp 2013) we designed a digital artefact as an alternative to traditional paper-and-pencil place value charts. This paper will report about the necessity of teaching place value, will analyze typical tasks found in textbooks, and show how these tasks correspond (or not) to certain actions with either artefact, the traditional and the electronic one. Finally, it will elaborate on a pre-study that helped to design an experiment that is currently conducted with 3rd graders (N > 300) in a quantitative study that shall contribute to the research for designing learning environments for place value that has been started by Hiebert and Wearne (1992), who conclude:

> The data reported here suggest that understanding, as measured by the place-value tasks, does not translate directly into procedures but that it does interact with procedures to yield increased flexibility and power. However, this interaction is influenced by the in-structional environment and, in this case, flourished more when instruction attempted to facilitate students' understanding rather than procedural proficiency. (p. 121)

PLACE VALUE AND NUMBER SYSTEMS

The decimal number system (in fact, any positional system) is a powerful tool for writing mathematics and doing arithmetic. Any number[1] can be written using only 10 different digits *in a unique way using a finite number of places*. It is easily possible to compare, add, subtract and –a bit harder– multiply or divide two numbers when we are given the numbers of bundles in the *maximal bundling*.

The underlying idea of the number system carries the proof for this non-obvious fact: By *creating bundles* of ten objects, bundles, bundles of bundles, etc. *repeatedly* until there are less than 10 objects of any same bundle size available we end up with a unique maximal bundling of objects. Creating bundles of objects is a basic task in many exercises in early mathematics learning, as it constitutes the basic operation for working with larger numbers.

Another key concept besides bundling is the *part-whole-concept* that concerns the fact that each number can be partitioned into smaller parts that add up to whole. While this is trivial for most of us, it is still an important fact that is not obvious to all children. It is used in almost all further arithmetic work. Creating bundles and the part-whole-concept are connected: By replacing 10 single objects with a bundle of ten objects we do not change the whole quantity, as we replace a part with another part of same size.

While any partition of a quantity into parts is feasible, some are more useful than others. If we are working with bundles, any partition that is created by (repeated) bundling, counting all bundles of the same size in one part, is called a *decimal partition*. More formally, if $n = a_0 \cdot 10^0 + a_1 \cdot 10^1 + \ldots + a_k 10^k$ then the summands form a decimal partition of n. The unique representation of a number results in a decimal partition created by less than 10 bundles of each bundle size and is called the *standard (decimal) partition* of n. In German schools, a different notation for decimal partitions is used: Instead of powers of ten a single or double letter is used. E (Einer/Ones) is used for 10^0, Z (Zehner/Tens) for 10^1, H (Hunderter/Hundreds) for 10^2, T (Tausender/Thousands) for 10^3, ZT (Zehntausender/Ten thousands) for 10^4, HT, M, ZM, …. A typical notation used for the number 324 while introducing place value is 3H 2Z 4E (standard partition), but also 32Z 4E, 24E 3H, or even 324E (nonstandard partitions). We follow the naming convention of Ross (1989), who also claims that "Understanding place value requires an elaboration of the student's emerging understanding of a part-whole concept." (p. 47)

Due to the structure of *decimal partitions* we can easily find a correspondence between tokens in a place value chart and such a decimal partition. Any placement of tokens in a place value chart corresponds to the decimal partition that has exactly as many bundles of a given size as there are tokens in the table cell for that size.

[1] We consider positive integers, though most of the following is applicable to decimal fractions, too.

Nonstandard partitions are important for flexible arithmetic. Dividing 320164 by 4 is much easier if the dividend is interpreted as 32 Tenthousands + 16 Tens + 4 Ones, just to give one example. They become even more important in written arithmetic, as we start with single digits and carryovers that directly lead to nonstandard partitions.

Abstraction levels

Sayers & Barber (2014) discuss teaching of place value to 5-6 year with a particular emphasis on the teacher and the manipulatives used in the classroom, and they conclude "In sum, place value is difficult to understand and to teach." (ibid., p. 34)

This difficulty is documented in the literature. Gerster & Walter (1973) describe eleven levels of abstraction from bundling up to standard notation of numbers. Levels 1–6 are just for creating bundles and exchanging between them. Only levels 7–11 are concerned with place value: (7) sort objects into a place value chart, (8) replace objects by iconic representations, for example coloured tokens, (9) replace icons by undistinguishable tokens, (10) replace tokens by a digit representing their number, (11) omit the chart and write the number as a sequence of digits.

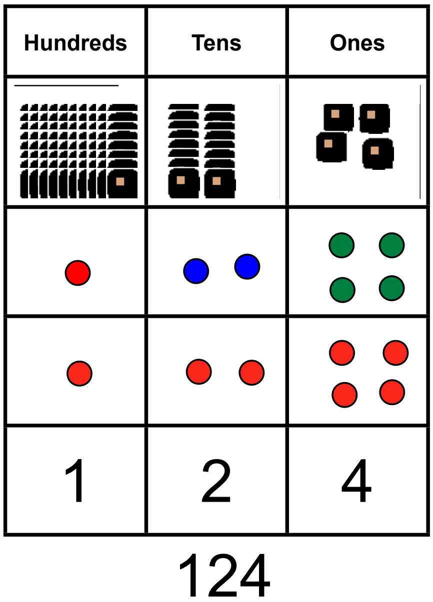

Figure 1: Abstraction 7–11

While it is neither clear that all these abstractions have to be followed one by one, nor at all, they influence teaching in primary school. According to Grevsmühl (1995) it is important to exchange and replace not only objects that are clearly distinguishable by their volume, shape or colour, but special care should be taken to exchange tokens of the same kind differing only by their position in a place value chart.

Typical activities when teaching place value

An analysis of seven German textbook series for primary schools showed that we can categorize place value related demonstrations and exercises into

- Bundling activities as a preparatory task,
- Grouping and sorting of bundles and placing them in charts,
- Exercises based on the transfer between different representations of numbers (tokens or digits in a chart, the standard number representation, or spoken number words),
- Exercises that use a place value chart as a tool, and
- Exercises where number representations (either in a chart or in standard notation) are changed.

Our main focus lies on the last two categories, as they involve nonstandard partitions as well as standard partitions. The third category, representation transfer, relies on students being able to create standard partitions from nonstandard partitions in a place value chart using repeated bundling to find a maximal bundling. This includes questions for the minimum number of tokens needed to represent a number.

The tool use of place value charts occurred mainly when the algorithms for written addition and subtraction were introduced. For addition, the carryover is easily explained using the bundling process in a chart, and for subtraction, depending on the strategy used, we can demonstrate how to unbundle for borrowing.

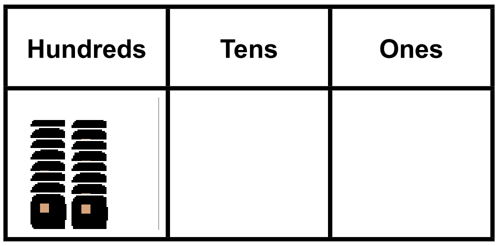

Figure 2: What number is this?

Some questions ask for the consequence of the movement of one or several tokens from one place to the other. It is important to note that the place value chart as a tool is now used differently: When we are placing objects that still carry their bundle size as an attribute, for example by being a base block of 100 or 10, by having a different colour, or by being an iconic representation of a bundle, we must not place these in the wrong cell of the chart, because this violates the rules and leads to a conflicting situation. A 10-stick place in the tens' place represents *one ten*, while the same stick in the hundreds' place could represent either *one hundred, ten hundreds*, or again *one ten* – or, even better: none of these, as it is illegal to place 10-sticks there (Fig. 2). On abstraction level 9 –having the same tokens for counting any bundle size– it is no longer visible that this is illegal, and it leads to exercises like the following: "Paul places 324 with 9 tokens in a place value chart. He moves one token from the tens' place to the hundreds' place – what number does he have now?" While this exercise makes sense for children that already understand place value, it can be confusing for children who just learned (or are about to learn) that moving a token between different places changes their value and is illegal unless the token is replaced by its corresponding unbundled bundle.

We conclude with the statement that there are both *value-preserving* and *value-changing* operations taking place in typical exercises. For us, it is important to support *value-preserving* operations, as these are the underlying mechanism for creating standard partitions from nonstandard partitions and also to create many new decimal partitions from one representation of a number in the place value chart.

THEORETICAL FRAMEWORK: ACAT

Our theoretical framework has been introduced in detail in Ladel & Kortenkamp (2013) and we recall it here only briefly and for better understanding of our study. In ACAT (Artefact-Centric Activity Theory), based on the work of Engeström (1987) and Leont'ev (1978) special attention is given to a mediating artefact between a subject (here: the student) and an object (here: the notion of place value), and the internalization and externalization processes occurring along this line of interaction. ACAT allows for the derivation of rules for the design of the artefact.

For our research, we used an interactive place value chart[2] that works as an App on iOS devices. Its major design decision has been that moving a token with the finger should be *value-preserving*. In order to achieve that effect, the app is automatically unbundling tokens of a higher value when moving them to a lower place. When the student is moving tokens to a higher place this is only possible when there are enough further tokens of the same bundling size that can be bundled with the moving token. If so, tokens are automatically bundled and replaced with a single token. The standard configuration of the App does not colour the tokens in order to support the abstraction process described earlier, but it is possible to switch on an automatic recolouring as well ("Montessori mode").

RESULTS

The App and the theoretical framework leading to its design have been demonstrated at PME 2013. We report on experimentation with it that has been carried out by us in laboratory situations and the classroom.

Interview Pre-study

In our pre-study we combined several exercises into a guided interview. The children were interviewed by one of the authors following the various questions and tasks that had to be carried out. The interactive place value chart App was introduced during the interview by the interviewer (I). We give some exemplary results.

In the first set of questions children had to compare nonstandard decimal partitions in the typical notation (e.g., 32Z 4E, see above). After deciding which one is larger, or whether they are equal, they were asked for a justification of their answer. We could identify the following types of mistakes: (a) Numbers were created just by omitting the letters denoting the bundle size – 14E 2Z becomes 142; (b) Only the largest bundle is considered, such that 5Z 3E (=53) is considered to be larger than 4Z 15E (=55); (c) The bundle size letters are ignored completely (5Z 3E becomes 5 and 3); (d) Only the largest number of a certain bundle type is used to decide which number is larger. The answers showed that not all children have understood the notation that is used on a daily basis in schools for decimal partitions.

The next item was to ask children how many tokens they need to represent 35 in a (two-column) place value chart. Next they were asked to show one representation. The children had red and blue tokens and strips of 10 blue tokens. One of the interviews highlighted the problem of mixing abstraction levels mentioned before. Here is part of the transcript after a student (S) placed three red tokens in the tens' place and five blue tokens in the ones' place:

I: Is it necessary to use blue tokens there (*points to the ones' place*) and red tokens there (*points to the tens' place*)?

S: No, not really. I just did it that way. […]

[2] Available from https://itunes.apple.com/app/id568750442.

I: Is there another way to put the tokens?

S: Yes, 30 tokens here (*points to the tens' place*) and 5 tokens there, as there are.

I: (*points to the tens' place*) Does it matter whether there are 3 or 30 there?

S: It does not, not really.

I: Can you explain that?

S: Actually it is the same. The 3 is there *(points to the tens' place*), the 5 is there (*points to the ones' place*) and when I put 30 there then there is a 0 instead of a 5 there.

The student shows that he is able to abstract from the actual colour of the tokens, but he uses an interpretation of nonstandard partitions that will interfere with activities that require bundling and unbundling. 30 tokens in the tens' place are not the same as 3 tokens in the tens' place.

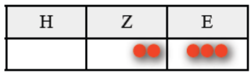

Figure 3

The last item was concerned with the action of moving a token from one place to the other. Children were asked what number is shown in the place value chart of Fig. 3. After answering the question they had to tell what they think will happen when one token is moved from the tens' place (Z) to the ones' place (E).

Of course, there are two possible answers to that question, depending on whether the token is unbundled or not. Our textbook analysis showed that both behaviours are used currently: In a "what if one of the Z-tokens is used as a E-token" scenario students should answer that the value of the number is decreased by 9. Questions of that type are also contained in the national comparison VERA-3 (see Stanat et al. 2012). On the other hand, using the place value chart as a tool for the introduction and explanation of written arithmetic the behaviour must be value-preserving, such that we end up with one Z-token and thirteen E-tokens.

Unsurprisingly, the students preferred the behaviour of traditional place value charts in their answers. Immediately after that they were given the interactive place value chart and were asked to move a token. Here is a transcript excerpt of another student.

I: Let's try it here. Here is the 23. Move a token from the tens to the ones!

S: (*moves a token*). Ooooh. Ey Caramba.

I: Ey Caramba. What is happening?

S: They become many.

I: Yes. Look here, the numbers are shown above. Can you try again and see? […]

S: I move (S. *moves a ten token to the ones*)

I: What number is it now?

S: 13 Ones.

I: Yes, and there is a ten (*points to the ten token*)

S: Strange.

I: What happened?

S: (*thinks*) These are 10 single ones! Now, these (points to tokens).

I: Why?

S: Because that wouldn't work. See, if you did on token, and then, if you do not have ten single ones, then it is only one. If you have ten single ones, then it is a ten. (S *moves one token from the ones to the tens. Automatically, nine other tokens are bundled into it and the ten tokens are replaced by one token.*)

S: Oops, what happened there?

I: Yes, what happened there?

S: The others moved over. Because the one, you know, if it moves to the tens, it would be eleven only and then nine come as well and 9 plus 1 is ten, so its two tens again, twenty, so its 23 again.

After being surprised by the magic behaviour of the App the student is able to explain this behaviour in detail. It seems to support the necessary flexible representation of numbers in the place value chart.

Quantitative Study

Based on the above we designed a study which shall reveal whether our digital artefact that interprets the action of moving a token from one place to the other differently, that is, preserving the value of the number, can improve the fluency of the students when creating nonstandard partitions. Also, we measure how the fluency –either acquired using the digital or virtual artefact, or already existing before– in creating nonstandard partitions in a place value chart influences the ability to transfer between written nonstandard partitions and the numbers represented by these.

Currently, we run the experiment with over 300 students in grade 3. The students are assigned by random to one of two groups; in group A each student has access to an iPad with our place value chart software, in group B students work with paper and pencil only. Each student has to work without further support on a test with three parts. Part I and III contain questions of the type "Which is larger – 22E 5Z or 22Z 5E?" or "Which is larger – 1H 12Z 5E or 1H 3Z 5E?" where two decimal partitions have to be compared, and questions where a (nonstandard or standard) decimal partition should be written in standard notation. The questions are designed such that students without a proper understanding of decimal place value are likely to fail.

Part II of the test consists of two activities. The students are asked to interpret numbers given by tokens in a place value chart as a number and to write them down. Next, students are asked to place tokens in a chart in order to represent a given number. For each number they are encouraged to find distinct representations.

The data will be interpreted using statistical implicative analysis (SIA, Gras et al. 2008) with Boolean variables that further differentiate between the ability to interpret non-standard representations and only standard representations. Our first data set (N=37) supports our hypotheses that the use of the iPad lets students create more nonstandard partitions (1.58 vs. 1.06 on average) than in the non-technology group. Also, the iPad group performed about 7.9% better on average in part III than in part I, while the other group performed 7.6% *worse*. However, without further data, to be collected in late January 2014, it is impossible to discuss this further, but we hope to answer the question whether an activity-theoretic driven design of an electronic place value chart can support a flexible understanding of place value and whether this leads to better performance in place value tasks.

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PERCEPTIONS AND REALITY: ONE TEACHER'S USE OF PROMPTS IN MATHEMATICAL DISCUSSIONS

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We examined one primary teacher's knowledge for facilitating mathematical discussion (MKT-Disc) via approximations of practice and compared her use of certain questioning prompts in these vignettes with her facilitation of discussions in her actual mathematics teaching. Findings showed differences in what the teacher reported she knows and what she actually did in practice. Evidence suggests the teacher's institutional obligations to a mandated curriculum, as well as the nature of her MKT-Disc, were the primary reasons for the mismatch between approximations and actual practice.

BACKGROUND AND OBJECTIVES

Teachers' use of questioning to facilitate mathematical discussions has been a topic of interest in much of the literature (Boaler & Brodie, 2004; Hiebert & Wearne, 1993). Much of this research has identified various ways of questioning that are more effective in promoting students' mathematical understandings and achievement. However, certain issues have problematized applying what research has uncovered as more effective mathematics pedagogy, and what teachers do in the classroom. A growing body of research suggests discrepancies between what researchers describe as effective questioning and classroom teachers' interpretations of such descriptions (Hill, 2005; Kosko et al., in press). Yet another issue lies in the potential discrepancy between what a teacher *knows* and what they actually *do* in their teaching. Specifically, teachers' questioning to promote mathematical discussion can be considered as a form of pedagogical content knowledge, and such knowledge is only pragmatically useful if it can be applied to pedagogical practice. Hill et al. (2008) observed relationships between teachers' mathematical knowledge for teaching (MKT) and their quality of instruction, and Kersting (2008) has observed similar connections in her use of vignettes to measure MKT. So, it appears that teacher knowledge does inform practice. Yet, teachers are influenced by a myriad of factors, stakeholders and context-specific conditions (Herbst & Chazan, 2012). Such influences may impose restrictions on how able a teacher is to apply what they *know* to be effective pedagogy in their classroom. The present study describes the case of a primary grades teacher with higher than average MKT and a disposition towards using dialogic mathematics discourse. The purpose of this study is to explore whether the teacher's conceptions of appropriate mathematical discussion aligns with her facilitation of such discussions in her classroom.

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TEACHER QUESTIONING

One of the primary means for teachers to facilitate mathematical discussion is through the purposeful use of effective questioning practices. Kazemi and Stipek's (2001) observations of four upper primary teachers in the U.S. revealed a connection between a press for meaning via questions that solicited explanation and justification with deepening students' mathematical understandings. Supporting these observations, Hiebert and Wearne (1993) observed six early primary grades teachers and found that students whose teachers elicited more explanation and justification via questioning had higher mathematics achievement. Boaler and Brodie (2004) refer to such questions that solicit explanation and justification as *probing* questions. While probing questions are generally encouraged by findings from observational and empirical studies, another form of questioning is much more prevalent. Referred to as *gathering informatio*n questions by Boaler and Brodie (2004), such prompts solicit factual/answer-only responses, recalled/memorized procedures, and similarly simplistic mathematical statements. Gathering information prompts are the dominant form of questioning by teachers in math lessons in the U.S., and while such prompts may provide opportunities to stimulate discussion in certain contexts, probing questions are more consistent in eliciting deeper descriptions of mathematics (Temple & Doerr, 2012).

While the literature generally supports the more prevalent use of probing questions, recent research has suggested inconsistencies between researchers' descriptions of appropriate questioning and some teachers' interpretations of such descriptions (Hill, 2005; Kosko et al., in press). This has led some researchers to argue for a more explicit approach to teacher education in facilitating mathematical discussions, particularly via questioning approaches (Boerst et al., 2011; Kosko et al., in press). One such specification applied here is the conceptualization of such questioning as a particular domain of teacher knowledge composed primarily of what Ball et al. (2008) would refer to as *knowledge of content and students* (KCS) and *knowledge of content and teaching* (KCT). Thus, using questioning to facilitate mathematical discussion is one portion of a subdomain of MKT which we refer to as MKT-Disc (although, MKT-Disc may also include revoicing, task selection, etc.). Relationships between MKT and more effective use of questioning have been observed by qualitative studies (e.g., Hill et al., 2008) and more recently by quantitative analysis (Kosko, in review). Yet, as argued by Kosko (in review), such relationships may be more pragmatic in defining teacher questioning as a domain of knowledge. As such, MKT-Disc requires the teacher to utilize, often simultaneously, KCS in listening and decoding student responses in discussions and KCT in framing an appropriate prompt to both attend to the student's thinking and facilitate the general instruction of mathematics at hand for the whole class. The appropriate use of probing questions is only one element of MKT-Disc, and its complexities extend our brief review here. Yet, questioning is widely researched and, therefore, an appropriate gateway to examining MKT-Disc. For this reason, it is the central focus of the present study.

TEACHER PERCEPTIONS VERSUS ACTUAL PRACTICE

Research on teacher perceptions (beliefs and knowledge) has shown that in certain cases, teachers' perceptions align with their practice; while in other cases there is inconsistency (e.g., Kuhs & Ball, 1986; Stipek et al., 2001). For example, Stipek et al. (2001) found primary teachers beliefs about mathematics teaching correlated with their students' perceptions about mathematics. Following a review of literature, Kuhs and Ball (1986) argue that inconsistencies between teachers' beliefs and practice can be explained by their level of mathematical knowledge. Pajares (1992) suggests that internal (e.g., beliefs) and external (e.g., school/administrative) factors mediate beliefs and practice, thus causing some dissonance between teacher beliefs and practice. Given the potential both for alignment and misalignment between perceptions and practice, the current study examines the case of one teacher to ask:

Do conceptions of using probing questions align with actual practice?

METHODS

Data was collected from a grade 3 teacher in the Midwestern U.S. whom we refer to as Mary. Mary was an early career teacher (2 years experience) with a Bachelors and Masters degree in education. She taught in a school district using Saxon Math curriculum. In the year data was collected, Mary indicated that district expectations were to adhere to the curriculum materials stringently. Mary participated in a two-phase study in which a larger sample completed a survey packet with open and closed response items regarding dispositions and knowledge for facilitating mathematical discussion. After completing the survey packet, Mary was randomly selected for the second phase of data collection which involved observing 10 of her mathematics lessons. The goal of the second phase of study was to compare teachers' actual practice with findings from the survey packet. For the present study, we report on data from both phase one and two.

Phase One: Data from Survey Packet

The survey packet included items assessing knowledge and dispositions regarding facilitation of mathematical discussion, as well as some background information. We briefly describe some measures here, along with descriptive indicators for Mary. One measure was Truxaw et al.'s (2010) assessment of teachers' dispositions for dialogic discourse. Mary's score of 3.67 (*Range* = 1.00 to 4.00) indicated a strong disposition towards supporting dialogic discourse. Mary also had a higher than average score for MKT as assessed by a 2006 version of Hill et al.'s (2004) assessment (*IRT Score* = 1.72 where a score of 'average ability' is 0.00). In addition to these measures, Mary completed one open-response question asking "What are the essential things a teacher must do to facilitate mathematical classroom discussions so they are effective in helping students' understanding of the content?" This open-response prompt was followed by three incomplete vignettes which Mary and other participants were asked to complete so they were 'model' vignettes of teachers facilitating mathematical discussions. While we provide excerpts of Mary's responses in the analysis and results section, the general nature of her descriptions were supportive of using probing questions to solicit explanation and justification from students. Lastly, we presented Mary and other participants with three cartoon-based vignettes in which they were asked to select one of four provided prompts that would best aid students' understanding in the discussion. In two of the three vignettes, Mary selected a probing question (see Kosko, in review for analysis of all Phase One participants). Responses Mary provided in the survey packet indicated a disposition towards facilitating mathematical discussions, aligned with a higher than average level of MKT.

Phase Two: Observations of Whole Class Mathematical Discussions

Observational data was collected from 10 class sessions over three weeks in Spring 2013 via video and audio recording and were transcribed. Segments of recordings were selected that included whole class discussions generally about 25 minutes in length. Both authors coded Mary's questioning following Boaler and Brodie's (2004) rubric. Coding reliability was sufficient (Kappa = .61) and differences were reconciled before analysis. For purposes of space and simplicity, we limit discussion of all coding for Mary's questioning to gathering information (n = 323; 89.5% *of all prompts*) and probing questions (n = 19; 5.3% *of all prompts*) across the 10 observations. However, descriptive statistics for all question types is reported in Table 1.

Gather Info	Insert terms	Explore math meaning	Probing	Generate Disc.	Link & apply	Extend think	Orient & focus	Establish context
323	3	9	19	5	1	0	1	0
89.5%	0.8%	2.5%	5.3%	1.4%	0.3%	0.0%	0.3%	0.0%

Table 1: Descriptive statistics for Mary's questioning across 10 observations.

ANALYSIS AND RESULTS

We examined Mary's responses from the survey packet and her use of questioning as coded in the transcripts from observational data. Our intent was to examine areas where there was overlap in Mary's perceptions via beliefs and knowledge with her actual pedagogy. To do this, we first attended to Mary's responses to the open and closed response items that included approximations of practice (i.e., vignettes). The open response items included one general question asking teachers to describe the essential things a teacher must do to facilitate mathematical discussions. Among other features, Mary's response included the following statement related to teacher questioning: "teachers must encourage students to explain their thinking so that teachers can correct misconceptions." Mary also provided responses to three incomplete vignettes, of which an example is shown in Figure 1. Mary's response is also shown, with her use of prompts in bold.

Incomplete Vignette:

Mr. William's 2nd grade class is discussing the problem *80 + 15 + 5 = ☐ + 60*. Jared blurts out "twenty plus fifteen plus five makes forty!" Keisha replies, "but you've got to do it in order!"

Mary's Response:

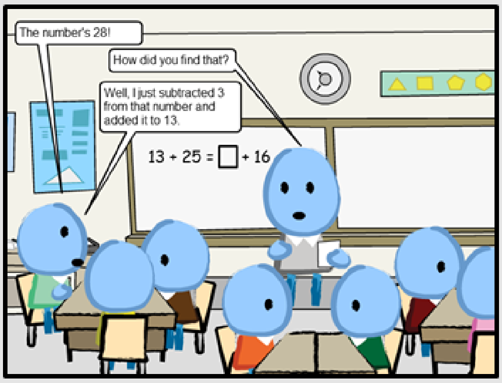
Jared, please explain your thinking. I see 15 and 5 in the problem, but not 20 - why did you think to add 20? Let Jared explain + correct any misconception. **"Keisha, what did you mean when you said 'you've got to do it in order'?"** Let Keisha explain her thinking and be sure to clarify, "The commutative property says that it does not matter in what order we add numbers."

Figure 1: Mary's response to the first incomplete vignette.

Mary's response in Figure 1 includes a probing question for both hypothetical students, Jared and Keisha. This is followed by a description that infers a check for understanding. Mary provided a similar description for another vignette. Taken together with her response to the general prompt, Mary's descriptions suggest a conception of using probing questions primarily as a means to assess students' mathematical thinking. Her response to the third incomplete vignette, however, demonstrated a dominant use of gathering information prompts, but no use of probing questions. One potential reason for this is the nature of the vignette stem. Specifically, the third incomplete vignette includes a student's description of mathematics where in the other vignettes, students provided only answers initially. Thus, Mary appears to be using probing questions to solicit student's thinking, but when such thinking is present she is comfortable with the use of gathering information questions.

Mary's answers to the closed response vignette items provide support for this assessment. Figure 2 provides an example of one such item which, like the third incomplete vignette, includes a student's description in the vignette. Mary's response to this scenario was to select option 4, a gathering information prompt. Interestingly, the other two closed response items did not include initial student descriptions of their mathematics, and Mary selected options that were probing questions. Thus, from Mary's responses to these six vignettes, it appears Mary uses probing questions as a means to assess students' mathematical thinking, and uses gathering information questions when such thinking is apparent. Additionally, Mary provided responses to several survey items asking how frequently she used probing questions in her class discussions. Responses were along a Likert-scale (*1=Never/Hardly Ever; 2=Some Discussions; 3=About Half of Discussions; 4=Every/Almost Every Discussion*). Her response to each item was *Every/Almost Every Discussion*. Taken together, we can infer that Mary knows to use probing questions to elicit student thinking, and believes she does so in almost every mathematical discussion in her classroom. What is less apparent in examining her responses to vignettes is whether she knows to use probing *sequences* in certain contexts. As such, she has demonstrated both in the open and close response items that probing questions are only used to solicit student thinking, but not to press students for mathematical meaning.

Q30. In this scenario, Mrs. Patel is playing "guess my number" with her second-grade class when Katie tells her guess:



Mrs. Patel wants her students to have a solid understanding of the math involved.
Of the options below, which would be the best question for Mrs. Patel to ask?

① "Does anyone have an opinion about what Katie did here?"

② "So Katie subtracted 3 from 25 and then added it to 13. Why do you think she subtracts and then adds the same number over here?"

③ "Katie, why did you subtract 3 from that number?"

④ "Katie, what number did you subtract 3 from?"

Figure 2: Example closed response item assessing teacher questioning.

We next compared Mary's survey response data via approximations of practice (i.e., vignettes) with her questioning in her whole class discussions (i.e., actual practice). The discrepancy in actual use of probing questions and what Mary 'knew' or perceived to be appropriate use of probing questions was stark. To assess difference in frequencies in vignettes versus actual practice we used a Chi-Square analysis, which accounts for the different count data in each category and examines the distribution across the contingency table in Table 2. We found that the differences were independent from chance ($\chi^2(df=1)=25.78$, $p<.001$). As we alluded to earlier, Mary's school district had strict curriculum requirements. Mary expressed the belief that these demands influenced her teaching and the nature of her class discussions. Specifically, she was concerned that the quality of her class mathematical discussions would not be as rich because she needed to keep pace with the curriculum guide. Findings from the Chi-Square analysis suggest her concerns may have been justified.

	Approximations of Practice	Actual Practice
Gather Information	45.5% (n=5)	92.9% (n=404)
Probing	54.5% (n=6)	7.1% (n=31)

Table 2: Comparison for use of gather information and probing questions.

We also compared Mary's self-reported frequencies of using probing questions via survey items with explicit statements with the number of class discussions observed (n=10) where she used at least one probing question. Mary used probing questions in 40% of observed mathematical discussions. To assess whether this was a significant variation from her self-reported survey data, we used a binomial test and assumed that usage of probing questions *Every/Almost Every Day* would account for a minimum of 80% usage. Results indicated a statistically significant difference in perceived versus actual use of probing questions in class discussions (p=.006). However, if Mary had selected the next frequent response available, *About Half of Discussions*, the binomial test would have found no statistical difference (p=.75). The difference in perceived frequency of probing questions versus actual frequency may be an over-estimation on the part of the teacher, not uncommon among those with reform-oriented beliefs. Yet, it may also represent a misalignment of Mary's conceptions of appropriate questioning practices in mathematical discussions with those advocated by researchers.

DISCUSSION AND CONCLUSIONS

Mary's responses to the open and closed response vignettes indicated both a disposition and knowledge for using probing questions to solicit students' mathematical thinking. However, observations of her facilitation of actual class mathematical discussions showed starkly different patterns in her use of probing questions and gathering information questions. The most obvious rationale for this difference between knowledge (via approximation of practice) and practice (via observation of practice) stems from descriptions Mary provided about curriculum demands. Namely, she was expected to follow a curriculum guide at a set pace as mandated by her school district. There is evidence to suggest that curriculum demands influence teachers' questioning (e.g., Boaler & Brodie, 2004). Herbst and Chazan (2012) classify such influences on teachers' decision making as *institutional obligations*, or obligations to institutional demands that teachers must adhere.

While it is tempting to identify the institutional obligation as the main reason for Mary's infrequent use of probing questions, data suggests additional influences. Mary's dialogic disposition and MKT scores were relatively high, but her responses to the vignette items indicated a specific understanding of questioning in mathematics. We surmise from her responses that Mary knows probing questions are useful for soliciting student thinking. However, when students have provided explanations, she did not elect to press for meaning either via vignette items or actual practice. Therefore, Mary's MKT-Disc may have included a gap in knowledge which could further explain her infrequent use of probing questions in actual practice.

The findings presented here are useful in continuing the work of understanding teachers' questioning strategies to facilitate mathematical discussion, and thus further conceptualize MKT-Disc. Further research comparing teachers of similar levels of MKT-Disc, but under different sets of institutional obligations, would help illuminate how such institutional demands interact with MKT-Disc. By further investigating this interaction, we believe the field will be better informed to improve teacher education and professional development efforts related to facilitating MKT-Disc.

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LOOKING FOR GOLDIN: CAN ADOPTING STUDENT *ENGAGEMENT STRUCTURES* REVEAL ENGAGEMENT STRUCTURES FOR TEACHERS? THE CASE OF ADAM

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Goldin et al. (2011) suggest nine 'engagement structures' for describing complex, 'in-the-moment' affective and social interactions as well as student beliefs. The study we report here examines the conjecture whether the 'engagement structures' construct can be appropriately adapted to allow such descriptions for secondary mathematics teachers. If this can be the case then linking teacher and student engagement structures could support detailed examination of classroom interactions. The aim of this paper is to consider one such adaptation and demonstrate some of its parts through the case of one teacher. We draw on this case study to indicate that such an approach has value, in particular in the ways in which it reveals relationships between engagement structures and norms in classroom interactions.

INTRODUCTION

There is a growing body of literature exploring affect in mathematics education (McLeod, 1992) and on the beliefs of mathematics teachers (Holm & Kajander, 2012). However there is less research on the complexity of teacher emotions as they engage in teaching mathematics (Hargreaves, 2000).

Goldin et al.'s *engagement structures* (2011) are designed as a tool for framing analysis of the complex nature of affect, and particularly the interaction between individual and social aspects of students' problem-solving experiences in mathematics. Goldin sees *engagement structures* as a useful, idealised multileveled hypothetical construct, one that covers a broader part of affect and more than emotions. He suggests that the construct of *engagement structures* can be used to describe complex "in-the-moment" (2011 p548) affective and social interactions for students by identifying positions that students can adopt when learning mathematics; and also locating the patterns which characterise individual behaviour, but are evoked in social situations.

In this paper, we draw on an ongoing study to propose that the construct of *engagement structures* can be adapted to apply also to teachers of mathematics. To this purpose, we first introduce *engagement structures* and then exemplify their use in a sample of our data. We conclude with an outline of where the larger study is currently heading.

ENGAGEMENT STRUCTURES AND THE AIMS OF THE STUDY

This paper aims to provide evidence that supports the existence of *engagement structures* (in the sense of Goldin et al.) for teachers of mathematics that are similar to those of students. This is potentially interesting as a position of power in the classroom means a teacher can manipulate the social situation and have a profound impact on the social dynamics of the classroom in a multitude of ways. For example, the teacher can, by condoning and modelling selected practices, through language and engaging in emotional interaction, act as gatekeeper to the community of practice of mathematics, establishing both 'norms' and 'endorsed narratives'(Sfard & Prusak, 2005).

Engagement structures are by no means fixed, but do emerge from common observable characteristics. Students can dip in and out of the positions suggested by these structures, sometimes showing characteristics of more than one structure, although at any one moment there will be a dominant structure, which directs their emotional reactions and hence their learning. Since Goldin et al. suggest that 'different motivating desires may result in similar behaviours' (2011 p550), this similarity implies there are a limited number of affective structures that encode current possibilities for the individual *engagement structures* for mathematics students. It also implies that, despite differing motivations for a mathematics teacher, the result may be similar structures. The nine original *engagement structures* that Goldin et al. (ibid.) suggest are: 'Get the job done'; 'Look how smart I am'; 'Check this out'; 'I'm really into this'; 'Don't disrespect me'; 'Stay out of trouble'; 'It's not fair'; 'Let me teach you'; and, 'Pseudo engagement' (p.553-557).

To illustrate one of these *engagement structures*, 'I'm really into this' is in evidence when a student's self concept appears to be that of a serious, involved thinker who values mathematical problem solving for its own sake, and is driven by an underlying mastery goal. This contrasts strongly with *engagement structures* such as 'Stay out of trouble' or 'It's not fair', both representing lower levels of engagement. We illustrate more *engagement structures* later in the paper, when we consider the case of Adam.

Goldin et al. (2011) mean to show patterns that are repeated or occur commonly; that are present in many different people and are therefore transferable. It seems reasonable to suggest that some recognisable patterns will also appear for mathematics teachers to form archetypal *engagement structures*. Here we examine this suggestion in the context of a study that involves secondary mathematics teachers in the UK.

If evidence of such structures emerges likewise for teachers, then we may have a unified language to examine complex classroom interactions, especially emotional interactions. This would allow a closer examination of how the teacher functions in guiding and supporting shifts in *engagement structures* for students, particularly in ways that support their learning. This may also mean that we can begin to examine how a teacher limits or encourages certain *engagement structures* in students, both through which *engagement structures* they themselves adopt, and through setting norms in a classroom context. We may also then be able to examine the place of beliefs within 'in-the-moment' interactions. Our experiences as teachers – and conversations with other teachers – suggest the viability of this plan and indicate a high degree of resonance and recognisability in these structures.

RESEARCH QUESTIONS, PARTICIPANTS AND DATA COLLECTION

This paper draws on a larger study which enquires into how mathematics teachers perceive and feel about their subject, and how they share their emotional engagement with mathematics, especially enjoyment, with their students.

The data collected for the full study comprises of three data sets for each participating teacher: data on their life history; lesson observations captured in video; and, post-observation discussions of video extracts where the teacher is asked to recall and articulate their emotions and thoughts during the incident presented in video extract. The selection of these extracts is guided by data collected through a galvanic skin sensor, worn by the teacher during the lessons, which records moisture changes in the teacher's skin. These changes are taken (van Dooren, de Vries, & Janssen, 2012) as indicators of emotional shifts and, in our study, as potential indicators of shifts in the intensity – or otherwise – of the teacher engagement at given points in the lesson. The sensor generates a timed graph of aforementioned shifts.

Participants to the study are UK secondary mathematics teachers who teach the age range 11 to 16 and are at various stages of their career, but not newly qualified. We have representatives from both urban and rural schools, and by gender and age. We are currently sampling across the school year, for example in early autumn, when norms are set with new classes. We expect to visit our teachers more than once. At the time of writing, data was being collected from twelve teachers.

In this paper, we exemplify the proposed use of Goldin's *engagement structures* in a small sample of our data, from one mathematics teacher, Adam. To this purpose we offer a snapshot of Adam's practice in a rural UK secondary school and of his talking about mathematics and his teaching. We heard the teacher relating his life history and talking about an observed lesson whilst watching a selected part of the video recording of the lesson. The transcriptions from the three phases of the data collection (life history, recorded lesson, post-lesson interview) is a rich source from which to construct a profile of this teacher's *engagement structures*, and to explore the place of his affect, as exemplified by the data, within his mathematical identity.

ADAM'S AFFECT: AN OVERVIEW

Adam, as evidenced from both interviews and observation of practice, values helping others, as he sees himself as being able to do mathematics when others cannot. He enjoyed his school work-experience helping primary children in mathematics:

> ...I used to help students with their homework in the mornings, on the bus, in payment, [laughs] give me like a can of coke or a chocolate bar and I'd help them with their maths homework...

Whilst training as a teacher, "...just having that opportunity to work with students and show them bits and pieces..." gave him a renewed enthusiasm for mathematics.

Adam may then experience discomfort if he feels he has not helped enough, for example if students were leaving without full understanding,

> ...and this is where...possible...I was thinking [groans] they didn't get this... so we thought, give them a bit to emphasise […]. So I don't like it when students don't get something.

He evidently finds pleasure and satisfaction in his interaction with students, especially students who are willing to engage in effortful learning of mathematics,

> and um...teaching at that level, at that kind of GCSE/A-level pitch [UK age equivalent: 15-19], I just get such a buzz […] when it starts to get a bit more um...like algebraic um....a bit more 'mathsy' and bits, when they get it, when they like it and love it like I do, it's brilliant, I love it...hmm [sound interpreted as strong satisfaction and contentment].

Yet particularly in the short video extract of Adam's teaching discussed during the interview, his motivation is primarily time, to cover the syllabus content quickly. He seems to value rapid pace, as in the utterance "I'll show you something quickly to help tie this together". The pace was clearly troubling him as he returned to this theme often, in the interview. For example: "I was talking quite a lot and we weren't getting through the content as quick as we should have done."

He possibly experienced some discomfort or perhaps frustration in that the students did not have enough consolidation time and there would not be enough time to round off the lesson properly:

> ...in the normal way...I think that I was also aware that again I hadn't […] kind of switch off and just sit and let them do something for a longer period of time...

There is evidence of competitiveness, which may be rooted in his stories about his early mathematical experiences. At about age 5, he says, in comparison to other children: "I was just able to do it...I just got any kind of numbers or anything."

Achievement was a repeating theme. At age 8, he was rewarded with early peer esteem for being good at mathematics. Adam's story is about who was top in a test, and his empowerment when he got recognition from peers.

However, vulnerability appears when Adam found university mathematics challenging. When other people were better at it, he "...lost the love a bit for mathematics."

A further thread in Adam's stories is the place of significant others, in his case a high school mathematics teacher. Adam experienced successful learning in a 'traditional' practice orientated way (and the observation showed Adam also teaching 'traditionally'). Yet we also find that he kept an open mind about not being concerned about any mistakes in his board work, modelling accepting error as normal, "Yeh...I'm not fussed with that. It happens quite a lot. I always say to the students...I'll make mistakes, and they'll make mistakes...and there it goes."

Although all of the above is merely a snapshot of our data, we suggest that it reveals much about Adam and his potential *engagement structures*. We illustrate some of these next.

GOLDIN'S *ENGAGEMENT STRUCTURES* IN THE CASE OF ADAM

Goldin et al. (2011) identifies for students 'Get the job done' through characteristics such as deference to establishment and following of the rules. In Adam's case, these are also often the expected behaviours for a mathematics teacher: a need satisfied by achievement of the perceived obligations and through task completion regardless of whether, or what type of, learning is achieved. Such a position sees school mathematics as procedural. A story Adam tells about when his school was short of mathematics specialists at one point illustrates his unease with this position, "Um...so you kind of lose some of the nice bits of the job, all the perks, all the nice feeling, you are just trying to get the job done."

One of the conventional 'expected' behaviour and social interaction rules for success in mathematics is quickness (Black, Mendick, & Solomon, 2009). This is illustrated here by how Adam fulfils an identified desire for timely completion. The lesson observation data suggest that he inhibits comments or questions from the students in order to complete a mathematical task quickly and promptly. Yet he is not entirely comfortable with this, since he simultaneously engages in 'in-the-moment' behaviour, acknowledging by eye contact, use of 'we', and facial expression some student contributions, thus maintaining his approachable style. So, although we have examples of engagement within the 'Get the job done' structure within the data, we simultaneously have evidence that this is not entirely satisfying for Adam.

A second *engagement structure* we have evidence for is 'Look how smart I am'. A teacher adopting such a structure, as in the case of a student, would try to impress with ability or knowledge, both highly valued, and would give value to where self-regard has been increased. They would respond to an admiring audience and may have a performance goal orientation that includes competitiveness. Adam's emphasis on pace and identified examples of competitiveness both in mathematics and in administration tasks places Adam within this structure at times. His losing some of his faith in mathematics, exactly when he was challenged at university and could not perform highly enough to meet the demands of this competitiveness, also reveals a perception of a need for affirmation that was unfulfilled, and the subsequent seeking of a new, more satisfying path.

Yet Adam also exhibits elements of another *engagement structure*, 'Check this out', where value is given to utility yet also to mathematics solely as an enjoyable experience motivated by intrinsic or extrinsic reward, and includes both conscientiousness and consideration for what benefits there are in the activity. He seems to feel the need for completion of the activity, even if it means de-prioritising other aspects of learning, thus perhaps valuing utility. He also appears to find personal satisfaction in his own successes, both intrinsic and extrinsic.

To a lesser degree there is some evidence of a further *engagement structure*, 'I'm really into this', in Adam's data. In the video he appeared able to focus single-mindedly on a task, exhibiting a desire to experience flow – complete absorption in what one does and for tuning out of the rest of the world (Csikszentmihalyi, 1990). Goldin suggests that the underlying need within this *engagement structure* is for understanding i.e. a mastery/goal orientation. We would also suggest that Adam finds satisfaction in the experience of teaching itself, and in finding solutions to the challenges within his role. Both are strongly associated with this *engagement structure*. The satisfaction of mastery of teaching skills is perhaps illustrated through his use of idiosyncratic, observed yet subtle gesture and interjections, used to modify behaviour. These gestures were quick and clearly 'norms' for the group. For example, he used a rapid and directed 'Shh' for seeking the attention of the class, and he used the word 'travellate', which had meaning for this class (they were expected to assess their learning in the session) and the students immediately responded as expected.

Yet, of all the *engagement structures*, we would suggest that the strongest match, (unsurprisingly) is with 'let me teach you', the strong evidence of a desire to help others understand and adopting a position of nurturance. Adam shows that he finds satisfaction in fulfilling this desire, and that this belief that he will find gratification in a positive response or appreciation is well established. This well established belief is evidenced when Adam talks about his own achievements, in particular the frequent use of a contented 'hmm' when he is proud of a remembered experience. Other examples include his statement that the students liking mathematics because they also like him is rewarding: "I think um...students I teach get that enthusiasm from me, and they like the subject."

Adam also used vocal tone and emphasis to stress mathematical points, and his speaking pattern was different for this purpose than for other parts of the lesson: the pace in these parts of the recording became slower and more repetitive. His voice had contrasting volume, and became louder for significant junctures in mathematical explanation. We would suggest that students, exposed to this pattern regularly, would soon 'tune in' to what Adam intended to highlight as important.

TEACHER *ENGAGEMENT STRUCTURES*: AN EMERGING PERSPECTIVE

The preceding analysis sample, based on data from observation, life history interview and post-lesson interview with one participant in our study, reveals more about Adam's stable beliefs, as opposed to 'in-the-moment' emotional structures. This may be due to the broadness of the structures, especially what comprises 'Let me teach you', an issue which may later prove to limit the value of *engagement structures* for analysing interactions between teacher and student. Hannula (2012) also questions the stability issue, in that emotions are stable if the emotion patterns are similar in similar situations, becoming similar to beliefs which appear with particular triggers and this is what this analysis is revealing, and less about 'in-the-moment'. This needs further investigation.

Adam, experienced and comfortable in his role, very openly shows his shifting emotions in a classroom context. Therefore, his affective pathways (Gerald A. Goldin,

2000) are orientated into his beliefs and identity as a mathematics teacher. According to Hochschild (2003), emotions are generally managed according to organisational expectation rules, such as for display, framing and feeling. The role then becomes a baseline for appropriate emotional display, which we see as very much the case for a teacher of mathematics. Suppression may be evidence that the teacher is self-regulating his affect. We are not sure whether the teacher can, given role expectations, experience the meanings of a mathematics classroom as either emotionally engaged or disaffected in the same way as a student. We would, for example, suggest Adam appreciates and articulates times when he experiences class as pleasurable, yet is not so likely to reveal feeling bored, nervous, mean, mad or frustrated in class, as he may think that this would imply some valuing for unacceptable negative emotions.

Nevertheless, a key part of *engagement structures*, meta-affect, a strand which G.A. Goldin (2002) suggests is ‘affect about affect’, provides stronger evidence to establish any *engagement structures* in the case of a teacher. The teacher is more likely than a student to reflect emotionally on experienced emotions, including self-monitoring of their emotions (DeBellis & Goldin, 1997; DeBellis & Goldin, 2006). For example when, as discussed earlier, Adam adds a contented ‘hmm’, he seems to be assigning positive attributes to the described emotion. One could interpret this unconscious purr as the very act of internally experiencing affect as a transformative tool for converting the experienced affective pathway into a positive experience or a more permanent belief. If this interpretation is valid, then such a response indicates the presence of an *engagement structure* since, according to Gerald A. Goldin et al. (2011), it is the structure that evokes meta-affective responses.

So, to address our intended use of the model as a tool for analysis, there may also be new positions emerging as the research progresses and the model is applied to other teachers. However, we think at this stage that similar structures apply to teachers as well as to students, but with important provisos. *Engagement structures* for teachers cannot be divorced from the differentiated power relationship between teachers and students, and norms play a significant role as regulators of classroom behaviour management. For Adam, there seems to be a high level of norm setting in the relationship with students, which appears to facilitate the opportunities for learning in his classroom. Norm setting may therefore act in combination with Adam’s beliefs, acting as both promoter and limiter of ‘in-the moment’ interactions. Approaching the data in this way has revealed this strong association.

To conclude, we would tentatively concur with Goldin’s (2011) suggestion that the value of *engagement structures* lies in enabling practical access to the complexity of teacher emotions as they engage in teaching mathematics and that developing a deeper understanding of these structures could provide a unified tool towards deeper understanding of the interplay between teacher and student emotions.

Acknowledgements

We thank the teachers and the schools for their time and willingness to fully participate in this research. This work is part of a doctoral study funded by a studentship at the authors' institution, carried out by the first author and supervised by the second author.

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COOPERATION TYPES SPECIFIC FOR BARRIERS

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We present the results of a study analysing the cooperative behaviour of fifth-grade student pairs when they have to overcome a difficulty (a barrier) in a mathematical problem task. As analysing method we used content analysis as well as frequency-tests (χ^2-test, CFA, Freeman-Halton-test). Our results help to confirm and elaborate the vague assumptions about cooperation types occuring at barriers based on the results of existing studies. In addition, our results suggest that 'presenting an idea of how or why to do sth.' could be relevant for overcoming the barrier in a pair.

It could be useful to solve a mathematical problem in a group: "The reasons given for the use of group work in problem solving include the opportunity for pooling of ideas, the natural need that arises to explain and express ideas clearly, and the reduction in anxiety for tackling something hard" (Stacey 1992, p. 261). If students reach a difficult point, they can stimulate and encourage each other to elaborate and question ideas. But surprisingly Stacey's study (Stacey 1992) as well as other studies suggest that there doesn't exist such an easy connection like 'groups solve math problems better than individuals'. This study will provide a more nuanced view to this phenomenon by describing the cooperation types specific to problem solving.

THEORETICAL FRAMEWORK

In mathematics as well as in psychology, a *problem* is understood as a task in which the problem solver has to overcome at least one difficulty or barrier (e.g. Dörner 1979). In contrast to a problem, in our study a *routine task* is taken to be a task without such a barrier for the problem solver. Thus, if the task is a problem or a routine task *depends on the problem solver* who tries to solve the task (e.g. Schoenfeld 1985). The term barrier is circumscribed vaguely in the literature (e.g. Schoenfeld p. 74: "intellectual impasse"). In our study a *barrier* is defined as a passage in a solving process where a solver does not perform something self-evidently and cannot remember essentials for solving. For example a person hesitates what to do next or questions the last ideas. For solving the problem the solver can work heuristically.

In our study we are interested in describing the kind of cooperation at barriers (for our understanding of cooperation see below). Only a few studies differentiate between separate cooperation types (e.g. asking, checking, explaining) and connect them with a difficulty in a problem. The research results suggest that there might be cooperation types specific for barriers and hints at these cooperation types.

Gooding and Stacey (1993) analysed cooperation-processes when students were working on difficult tasks in mathematics. For that, they modified a coding system which Sharan and Shachar used for tasks in geography and history. Gooding and

Stacey coded the categorie *thinking aloud* more frequently than Sharan and Shachar. The researchers explained this difference with the distinctions in the nature and the difficulty of the tasks: In order to solve these tasks students have to spend more time in understanding difficult ideas and in stating a problem over and over.

Apart from this cooperation type, students possibly cooperate at barriers more frequently in types which focus on the 'how' (e.g. explanations, elaborations and demonstrations) rather than in types which focus on the 'what' (informative) or on the 'why' (evaluative). So, Hertz-Lazarowitz (1989) compared the cooperation types when students were working on a low-cooperative task with the types when students were working on a high-cooperative task (working on the task-process together). If the task was interpreted as high-cooperative, the students discussed most frequently on the how-level.

In addition, there might be a difference in cooperation types if only one student has a barrier or if both students have the barrier. So, Goos et al. (1996) observed that if the relative expertise of the students is unequal, the dominant cooperation type is *peer tutoring* – in the other case *collaboration* (if there is a degree of challenge for the students inherent in the task).

In sum, we hypothesize that there might be cooperation types specific for barriers. So the research results give hints for these cooperation types (*thinking aloud*, *peer tutoring* or *collaboration*). We also assume that students who have to overcome a barrier interpret the task as high-cooperative. So students are expected to cooperate at barriers most frequently on the *how-level*. In addition, the distinction 'barrier only for one student' and 'barrier for both students' might be suitable for our research purpose. But the assumptions based on these research results remain superficial, since the researchers use different definitions of the terms barrier and cooperation.

Consequently, the research questions explores the characteristics of cooperation types specific for barriers. In order to get precise results, we will differentiate between cooperation types that (a) occur only at barriers and not in the rest-process, that (b) can be found more frequently at barriers than in the rest-process, and that (c) appear both at barriers and in the rest-process but bear special characteristics when employed at a barrier.

As a system for cooperation types we don't use one of the systems in these three studies because the differentiated types are either too rough for describing the cooperation behaviour at barriers (Hertz-Lazarowitz and Goos et al. differentiate only between three different cooperation types) or not suitable to model the whole cooperative behaviour in the own data (Gooding & Stacey). So, we build upon Naujok's definition of cooperation (Naujok, 2000, p.12: "every kind of task-related interaction") and her task-related cooperation types *explaining, asking, comparing, prompting* and *copying*: Her understanding of *cooperation* as an "empirical phenomenon" (p. 12) is more useful for our study than the definition as an ideal way to work together (Slavin, 1983), since first of all we aim at describing the cooperation behaviour without a normative direction and since we expect various kinds of cooperation when solving mathematical problems. In a former study (Lange, 2012) we modeled the variety of cooperation acts when solving mathematical problems. We adapted and completed the list of cooperation types (see below).

DESIGN AND METHOD

Study: Between November 2008 and June 2010 we organized a math club at the University of Hanover (MALU), an enrichment project for fifth-grade students (age 10-12). In this math club the students had to solve one or two mathematical problem tasks in pairs one afternoon a week. After working in pairs, we discussed possible ways to solve these tasks with the whole math club group. Based on the results of two tests (a general giftedness test CFT-20R, a mathematical giftedness test) we selected a group of 9 to 14 fifth-graders with different results for each of the four MALU semesters (for more details concerning the tests see Gawlick & Lange 2011). Since the kind of cooperation may depend on the composition of a pair, we arranged homogeneously as well as heterogeneously composed pairs (criterium: test results), pairs which were held constant during the semester as well as pairs with changing partners.

The problem solving processes were videotaped and the students' notes were collected. In addition, a log was kept of the children's main thoughts and the observers' subjective impressions.

Tasks: Cooperation and the kind of barrier possible for fifth-graders in the tasks may vary with different problem tasks, so we looked at task-collections (e.g. competition tasks) and problem solving books, analysed the tasks and chose tasks with different features. Fifth-graders should be able to solve the chosen problem solving tasks with their mathematical knowledge. Two of the tasks we presented are the following:

The seven gates (Bruder 2003, p. 12)
A man picks up apples. On his way into town he has to go through seven gates. There is a guardian at each gate who claims half of his apples and one apple extra. In the end the man has just one apple left. How many apples did he have first?

Oh yes, the chessboard (idea: Mason et al. 2010)
Peter loves playing chess. He likes playing chess so much, that he keeps thinking about it even when he isn't playing. Recently he asked himself how many squares there are on a chessboard. Try to answer Peter's question!

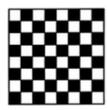

Figure 1: Two problem tasks

Evaluation Method: Because of the large amount of data and in order to receive a maximum diverse sample of cooperation types and barriers, we selected a cross-section from the data (different pairs, different semesters, different tasks) – altogether 23 processes (the analysing method is very time-consuming). The videographies of the processes were transcribed, and the transcripts were revised before the coding process started. As coding method for identifying cooperation acts and barriers we used qualitative content analysis (Mayring 2008).

In reference to cooperation, we were able to build upon Naujok's cooperation acts and upon her descriptions of the cooperation phenomenon (see above). Since the setting in Naujok's investigation (routine tasks, tasks from different subjects in school, younger students, school setting) differs in some kinds from our setting, the cooperation acts were collected on the one hand deductively from her study and on the other hand empirically from the MALU-transcripts (for more details in the development of the coding system see Lange 2012). The following table represents an excerpt from the original coding system for cooperation types (some cooperation types are explained together). An important point is the differentiation between the cooperation on the *what-, how-* and *why-level*. The *why-level*-cooperation in our study involves giving reasons and evaluating something. If students discuss products like results / answers or part of results / answers, they cooperate on the *what*-level – if they are interested in possibilities to understand or solve the task, they cooperate on the *how-level*. In our transcripts students use the terms *how* and *why* not selectively, partly even synonymously, that means they ask for example for reasons (*why*), but get an answer on the how-level ("first I did ..., then I did ..."). Therefore we cannot differentiate between these two levels (how- and why-level). If the interaction takes place visually, we defined this as *non-verbal*, otherwise as *verbal*.

presenting an idea	One person mentions an idea about a possible solution or about possible parts of the solution path (e.g. next steps, task-features, argumentation) before this person has solved the task or the parts of the solution path, which the person presents.
informing about sth.	One person says her/his (part) solution or says how she/he solved the task or parts of the task.
helping	One person has an advance in knowledge belonging to a part of the task and shares this knowledge with a partner within this cooperation type. The person can write it down (*passing sth. non-verbally*) or *prompt* or *explain* it. The partner can also be active by copying information from the person's notations.
comparing sth.	At the moment of comparing the comparing persons have solved the part of the task, so they (or only one person) inform themselves about this part of the task ((part-)result, (part-)answer, process-step, whole process) of the other person.
evaluating, checking, pointing out a mistake	If persons check something, they go through the solution path and reflect the steps. Instead one person can *point out a mistake* or can assess the correctness or usefulness of something (*evaluating*). These three cooperation acts occur in the MALU-processes as *saying-what* as well as as saying-how. In the case of these acts the person could either have the solving step already done or haven't yet.
commenting on the task	The persons say something about the task (e.g. familiar, funny) or comment on the difficulty of the task (e.g. easy, difficult).
asking	One person asks sth., the partner responds. In order to reduce an overlap, this category is only coded, if the passage cannot be subsumed under another category.

Table 1: Excerpt from the coding system for cooperation types

The transcripts were coded in two steps: First, we marked the points if something changes (cooperation theme, cooperation type, cooperation into non-cooperation or reverse). Second, we dedicated a cooperation type-label to the marked out transcript-passages.

Belonging to barriers in the process, we started with the formulated definitions (see above) and coded first all barriers in the process. In a second step we decided, if only one person face the barrier or if both students experience the barrier. As signs for a barrier we observed among others the following aspects: A person says that she doesn't know or that she's unable to do something / A person does something not self-evidently (e.g. she/he hesitates or questions something). / A person changes the perspective or has an illumination.

We trained three students in coding with both category systems. For the first (marking the begins of new cooperation-phases) and the third decision (decision for a barrier) the pairwise interrater-agreement varies between 60% and 69%. For the second decision (labeling the marked cooperation-phases) Cohen's Kappa varies between 0.64 and 0.68. Also the percentual-agreement-values can be termed as good because we only considered those transcript-passages where at least one person coded a cooperation-change or a barrier. After coding independently, we discussed the decisions where we disagreed, and coded jointly.

For answering the second part of our research question (b) (if certain cooperation types occur more frequently at barriers than in the rest-process), we used the kx2-χ^2-test with the CFA (Configural Frequency Analysis) as a posthoc-test. Since the codings are independent of each other and since every passage can be coded clearly, two of the three test-assumptions are fulfilled. As the third test-assumption (at least 80% of the expected frequencies should be greater than or equal 5) is not fulfilled in our study (although it is disputed), we also did the Freeman-Halton-test as an exact test.

RESULTS

Among the 23 chosen processes for analysis we coded 38 barriers in 16 problem solving processes. At about 82% of the barriers, the students cooperated. In figure 2 the cooperation types occuring at barriers are marked in bold. The non-marked cooperation types occurred only in the rest-process. The superficial assumption that none of Naujok's cooperation acts (blue-grey background) but only the added cooperation types (white) could be observed at barriers proved incorrect. Comparing-types as well as some of the evaluating-types first of all help to uncover a mistake, so that these types do not occur in the context of discussing a barrier. The helping-types *copying, prompting* and *passing sth. non-verbally* are used to get information – often without questioning this information. However, the question remains whether the cooperation *type informing about how or why have done sth*. occur at barriers. Perhaps students only seldomly question these contents (see Goos 2002 for reasons for metacognitive failure).

Regarding the three above formulated criteria, we found the following results: None of the cooperation types were observed only at barriers (a). For analysing how far the second criterion (b) is fulfilled, we first summarized the cooperation types each belonging to *helping, presenting, informing, comparing, evaluating* and other types. This approach seems to be useful for testing purposes (otherwise: many cells with an expected frequencies smaller than 5) and also possible because of the contentual relationship of the types belonging to the same cooperation intention. For this analysis we take all transcripts from two tasks (see fig. 1) into account, in order not to distort the quantitative results through sporadic chosen problem solving processes. See table 2 for results (expected frequencies in brackets).

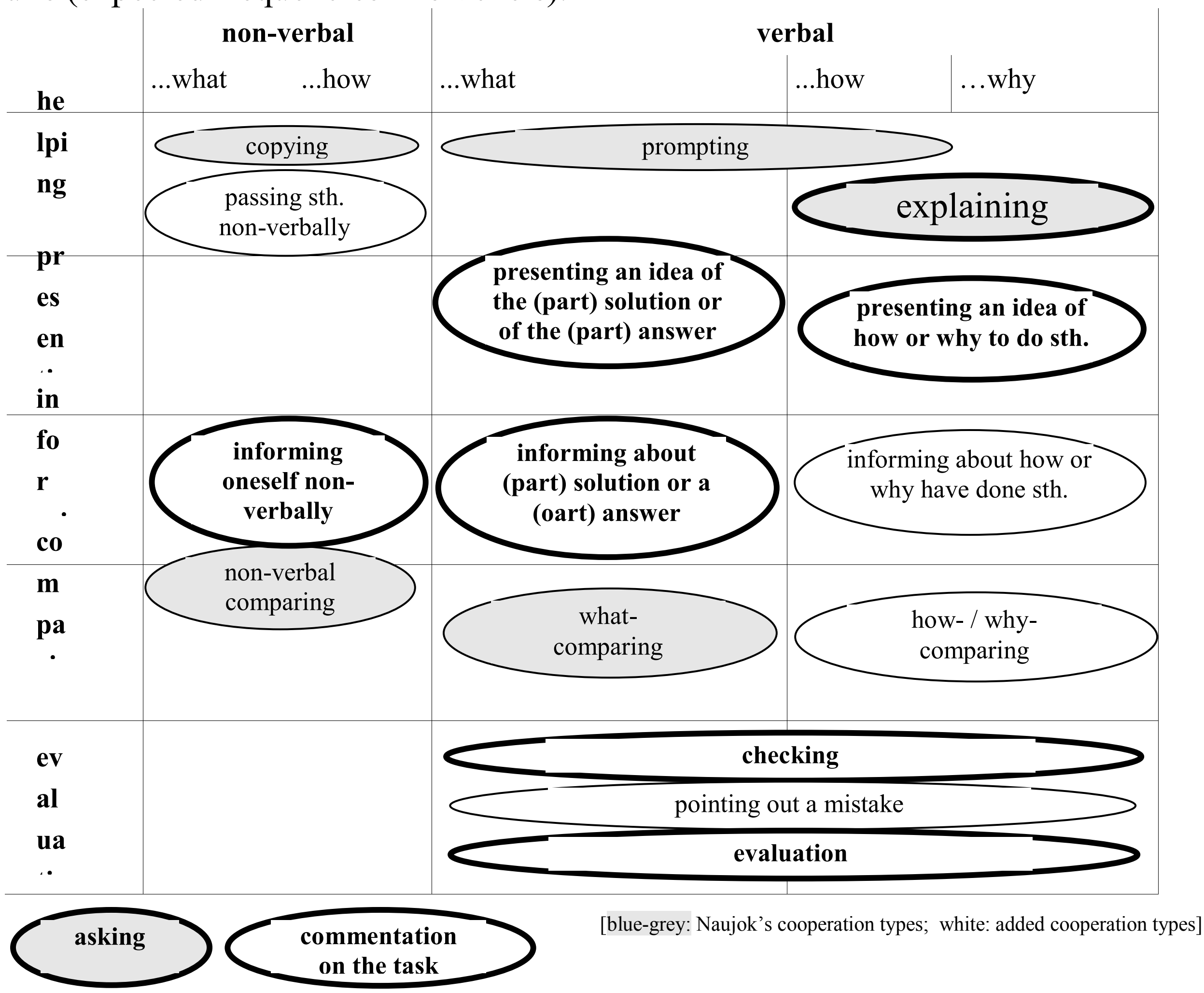

Figure 2: Cooperation types occuring at barriers (marked boldly)

cooperation intention	**barrier**	**rest-process**	**Σ**
helping	2 (2.81)	20 (19.19)	22
presenting	**14 (6.77)**	39 (46.23)	53
informing	0 (3.45)	27 (23.55)	27
comparing	0 (2.68)	21 (18.32)	21
evaluating	4 (4.60)	32 (31.40)	36
other types	3 (2.68)	18 (18.32)	21
Σ	23	157	180

Table 2: Contingency table – cooperation types at barriers vs. in the rest-process

The 6x2-χ^2-test as well as the Freeman-Halton-test gave a significant result: both-sided test, χ^2=16.28>11.1= $\chi^2_{\alpha,df}$ df=5 (the significance level α=0.05 is appropriate, because the study is an explorative one). The effect is medium-sized ($w \approx 0.3$). The configuration *presenting / barrier* can be accepted as statistical ascertained (CFA-results: $\gamma \approx 0.0085 \leq 0.05 = \alpha = \alpha^*$). That means the cooperation intention *presenting* occurs significantly more frequently at barriers (14) than expected (6.77) (shaded in grey in tab. 2). By calculating the χ^2-values for both presenting-types separately, we found that the above mentioned statement is only true for the cooperation type *presenting an idea of how or why to do sth*. ($\chi_{ij}^2 == 14.46 > 3.84 = \chi_{0.05,1}^2$). So, only this cooperation type occurs significantly more frequently at barriers than expected.

With regard to the cooperation type *presenting an idea of how or why to do sth*. we had enough transcript passages in order to compare the appearance at barriers with the appearance in the rest-process (c). In the rest-process this type was coded when students proposed a procedure for solving the next steps or when students considered or implemented an idea. On the contrary at barriers this type was coded when students didn't know how to proceed next or how far the previous ideas were appropriate for solving the problem. In sum, based on this cooperation type we found manifestations occuring only at barriers and others we found only in the rest-process.

DISCUSSION AND CONCLUSIONS

Our results are compatible with and elaborate the results found in the three studies referenced in the theoretical section. Gooding and Stacey's cooperation type *thinking aloud* contains facets from both presenting-cooperation acts. Both types occur at barriers (fig. 2), but as elaboration of their result only one type, namely *presenting an idea of how or why to do sth.*, appeared more frequently at barriers than expected.

The assumption that barriers occur more frequently in high-cooperative than in low-cooperative tasks is correct for our data: For that we recoded the MALU-transcripts in terms of low- and high-cooperative-processes. In accordance with Hertz-Lazarowitz' results, at barriers students cooperated most frequently on the how-level. Our results offer two specifications: Only the type *presenting an idea of how or why to do sth.* represents the type the students cooperated in most frequently. In addition, students cooperated at barriers not only on this level, but rather on the other two levels.

Also with regard to the results of the third study, our results can help to differentiate the statements: So, we observed not only helping-types if only one person has the barrier, but also for example presenting-types. Further research is needed concerning the differentiation between a barrier only for one and a barrier for both students: Our results couldn't confirm the results found by Goos et al. (1996). One reason might be a (possibly too) small data-base for this differentiation in our study.

All together, our results confirm the vague studies' results. Beyond this, our system allows for more specific predictions because our types are finer with more facets. In addition, we now can make a statement about the types at barriers as well as about quantitative connections between cooperation acts and the occurrence of barriers.

Our results suggest two additional conclusions: Firstly, because of our results to the first criterion (a), one cannot decide if students contemporarily have to overcome a barrier only based on the cooperation types. But the results to the third criterion (c) let us assume that this is possible when we distinguish between different manifestations related to each cooperation type. Secondly, our results emphasize the relevance of the cooperation type *presenting an idea of how or why to do sth*. for problem solving in pairs, because amongst all cooperation types at barriers this cooperation act appeared at barriers most frequently and furthermore significantly more frequently at barriers than in the rest-process. With a view to learning theories, cooperation in this type might help to overcome the barrier in pairs.

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EXPLAINING MATHEMATICAL MEANING IN "PRACTICAL TERMS" AND ACCESS TO ADVANCED MATHEMATICS

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This paper uses a socio-political practice perspective of mathematics to investigate the action of first year university students as they work in small groups to explain the meaning of mathematical objects in "practical terms". Written transcripts representing video recordings of the action were analysed using critical discourse analysis and focal analysis. I present a description of the micro-level action and locate this action relative to the mathematical discourses in the university space. This analysis raises questions about the relationship between problems requiring the use of the practical terms genre and access to advanced mathematics at university.

INTRODUCTION

The relationship between practical problems[1] and access to school mathematics has been in view in mathematics education research for the past two decades. The sociological work of Basil Bernstein and Paul Dowling has been used extensively in this research, with this and related research showing that solving practical problems involves recognising the practical/mathematical boundary (e.g. Straehler-Pohl, 2010) and the esoteric domain knowledge driving the recontextualization of the practical (e.g. Gellert & Jablonka, 2009). It has been argued that possession of these recognition rules is not equally distributed across social class (e.g. Cooper & Dunne, 2000), with Nyabanyaba (2002) arguing that student agency in choosing not to answer certain practical problems in high-stakes examinations is enabled by socio-economic status.

In this paper the relationship between practical problems and access to advanced mathematics[2] at university is in view. I focus on problems requiring students to explain the meaning of mathematical objects in "practical terms" (see Figure 1). The design of this problem draws on calculus reform texts of the 1990s. These texts argue that "formal definitions and procedures evolve from the investigation of practical problems" (Hughes-Hallet et al., 1994, p.vii) and that explanations in "practical terms" (p.vii) strengthen the meaning students attach to mathematical objects. The transition from intuitive to advanced calculus has long been a concern in mathematics education research. However, the role of practical problems in this transition has traditionally not been in view in the psychological research on advanced mathematics (e.g. Tall, 1991)

[1] My use of this term is consistent with its use in calculus reform texts (e.g. Hughes-Hallet et al., 1994). The literature variously uses realistic problems, real-world problems, and word problems.

[2] I use this term for the formal abstract mathematical practice at university (called advanced mathematical thinking in much of the literature, e.g. Tall, 1991). At South African universities, first year mathematics is preparation for advanced mathematics in the second year of undergraduate study.

and the untheorized evaluation studies of calculus reform curricula (e.g. Garner & Garner, 2001). A recent trend sees the uptake of sociological and systemic functional linguistic perspectives in research on undergraduate mathematics (e.g. Jablonka, Ashjari & Bergstrom, 2012), and this paper is part of this move.

A flu virus has hit a community of 10 000 people. Once a person has had the flu he or she becomes immune to the disease and does not get it again. Sooner or later everybody in the community catches the flu. Let $P(t)$ denote the number of people who have, or have had, the disease t days after the first case of flu was recorded.

c) What does $P(4) = 1\,200$ mean in practical terms? (Your explanation should make sense to somebody who does not know any mathematics.)

d) What does $\frac{P(7)-P(4)}{7-4} = 350$ mean in practical terms? Give the correct units.

e) What does $P'(4) = 400$ mean in practical terms?

Course Answers:

c) 4 days after the first recorded person got flu, 1200 people had the flu.

d) From the 4th to the 7th day after the first recorded person got flu, the number of people on average who had the flu was increasing by 350 people per day.

e) 4 days after the start of the epidemic, the number of people who had the flu was increasing by 400 people per day.

Figure 1: The Flu Problem, questions (c) to (e), with course answers in italics.

This paper investigates the action of students on the questions in Figure 1. The students are enrolled in a first year university mathematics course (called *foundation mathematics*) at a South African university. The course aims to provide epistemological access to advanced mathematics for students considered educationally disadvantaged on the basis of their race, socio-economic status and language. A pass in this course provides formal access to advanced mathematics courses. I use a socio-political practice perspective of mathematics (a) to describe the student action, and (b) to ask whether this action reproduces or diverges from the university mathematical discourses. This analysis problematizes, at the level of the individual student, the relationship between using practical terms and advanced mathematics.

THEORETICAL FRAMEWORK

The socio-political practice perspective of mathematics used in this paper is based on Fairclough's (2001, 2003, 2006) critical linguistics, work that draws on Bernstein's sociology and the systemic functional linguistics of Halliday. For the description of *mathematical discourse* I use Morgan (1998), Moschkovich (2004) and Sfard (2008). In this perspective, the student action as text is located in the socio-political practice of *foundation mathematics*. Language (or *mathematical discourse*) is used in the practice to represent mathematics, with the *discourse type* of foundation mathematics involving particular ways of talking about and looking at mathematical objects, of operating on objects, and of making and evaluating arguments. Language is used to interact communicatively (the *genre*), making discursive links between texts and practices, and between students. Lastly, language identifies the students as particular types of people (the *style*).The term *practical* in this paper recognises the "relationship of recontextualization" (Fairclough, 2006, p.34) between foundation practice and the virus in epidemiology, with the discourse types, genres and styles in the latter practice "filtered" (Fairclough, 2003, p.139) by the recontextualising mathematical practice.

In this perspective, using practical terms to describe mathematical objects is a *genre*. Using the genre involves, for example, looking at the spread of the flu mathematically as an increasing function and looking operationally at the subtraction $P(7)–P(4)$ as the change in the number of people who have or have had the flu (Le Roux & Adler, 2012). The absence of mathematical words like "rate" is a key evaluation criterion in the genre. Practical terms are sourced via links to the problem text and the course lecture text where the genre was demonstrated prior to answering the Flu Problem. Students interact socio-politically as they talk in small groups. At the university at which this study was conducted this genre is restricted to foundation mathematics, its use thus identifying the students in the style of foundation students.

The student action as text that is the focus of this paper is, according to Fairclough (2001), shaped by / a repetition of the mathematical discourse of the foundation practice. In fact, the students are enabled to act, provided this is within the constraints of the discourse. Yet the text also creates meaning and how it represents mathematics, interacts communicatively, and identifies students may diverge from the discourse type, genre and style respectively of the foundation practice. This is a result of the text cutting across practices and the agency of students. Fairclough (2001) points to likely asymmetries in the extent to which students resist the constraints of the discourse.

Thus from this perspective, answering the research questions in this paper involves, firstly, describing the mathematical, discursive and socio-political action of the students. Second, I consider whether this meaning reproduces or diverges from the discourse types, genres and styles of the university mathematical discourses.

METHODOLOGY

The texts used in this paper are from a wider study of the use of practical problems and a learner-centred pedagogy in the foundation course (Le Roux, 2011).[3] Video recordings of students working in small groups were transcribed to represent the non-verbal action (shown in bracketed *italics*) and the verbal action. For the latter, pauses are represented using three points..., emphasis is marked by underlining, rising intonation shown by the up arrow ↑, and square brackets [ ] enclose overlapping talk.

[3] I thank Jill Adler for her supervision of the wider study. I also acknowledge the research funding provided by the National Research Foundation (Grant No. TTK2006040500009). The opinions, findings and conclusions are those of the author and the NRF does not accept any liability in regard thereto.

For the analysis, the theoretical view of text as both repetition and creation was operationalized using Fairclough's three-stage method of critical discourse analysis. The specific action on mathematical objects was brought into view using an adapted version of focal analysis (Sfard, 2000). First, the descriptive stage involved working line by line through the transcript to identify what students are looking at (the *attended focus*) and saying or writing (the *pronounced focus*). The latter was analysed in detail to identify textual features such as the choice of words, the tense and the mood. The interpretation stage involved working across longer pieces of text to identifying what meaning these features give to the mathematical discourse. For example, specialist terms may represent the text as mathematical, and rising intonation at the end of a statement may identify the student's claim as tentative. Finally, I consider the mathematical discourse of the students in relation to the discourse-types, genres and styles of the foundation and advanced mathematics practices.

In this paper I present the texts of two groups of five students each, with pseudonyms Bongani, Lungiswa, Mpumelelo, Siyabulela and Vuyani used in Group A and Hanah, Jane, Jeff, Lulama and Shae for Group B. The similarities and differences between the groups as they work with the function and both the average and instantaneous rate of change of this function enable a rich description of the student action. I note here that these groups are mixed with respect to gender, home language, socio-economic status, race and schooling, a point I return to at the end of this paper.

DESCRIPTION OF THE STUDENT ACTION

Describing the function value P(4) = 1 200 in practical terms

The students write practical descriptions of $P(4) = 1\ 200$ relatively quickly. They evaluate one another's verbal pronouncements about the two variables, for example, "After 4 days" for the independent variable. Shae looks at the problem text to identify that "t is in days", but the students do not repeat the description of this variable in that text, where t represents the number of days "after the first case of flu was recorded". The use of the preposition "after" for the time when four days have passed reproduces the course lecture text.

To describe the dependent variable, some students repeat the problem text ("1 200 people will have it or have had it", Jane). It is not possible to tell whether they are simply copying the problem text or actually looking mathematically at $P(t)$ as an increasing function, a gaze needed in some parts of the Flu Problem. Other students reword the problem text to people who "have been infected" (Hanah) or "are infected" (Lungiswa). The students' lack of attention to variations in tense may suggest that they are acting in the style of students solving word problems; Gerofsky (2004) argues that, since these problems are only a pretence of the real world, inconsistencies in tense are not considered a problem by students familiar with the word problem genre.

While co-constructing answers in practical terms, Siyabulela and Lungiswa in Group A express surprise at the large number of people infected; Lungiswa laughs as she says

"That's too much". Referring to the nameless "somebody" in question (c), Shae in Group B asks Jeff, "If he doesn't know any mathematics, don't we have to teach him numbers and stuff?" They link this action to the "yellow books" called "Maths for Dummies". Thus these students give significance to mathematical action, with Siyabulela and Lungiswa only pretending that the flu virus exists (Gerofsky, 2004) and Shae and Jeff representing the practical terms genre as targeting "dummies".

Describing the average rate of change in practical terms

Characteristic of the action in Group A, in lines 375 to 381 Siyabulela and Lungiswa co-construct the first verbal pronouncements, seeking feedback (the rising intonation ↑) and giving feedback ("ja"). The words "from four to seven days" suggest that they are looking operationally at the denominator of the fraction $\frac{P(7)-P(4)}{7-4}$, with the subtraction representing the change in time. Their choice of prepositions "from… to…" reproduces their lecture text, although they do try out "between". Siyabulela's use of the unit "people" rather than "people per day", and his evaluation of his peers' pronouncements as described later in this section, suggests that his operational view does not extend to the numerator as the change in the number of people infected and the division as the change in the number of people infected relative to time.

375 Lungiswa:	What does mean in practical terms↑° ((Reads text))... From four to seven days
376 Siyabulela	Oh... ai that one is bet... that one is between seven uh
377 Lungiswa:	Four to seven days
378 Siyabulela:	Oh ja… four to seven days… the number of people infected
379 Lungiswa:	Uh huh↑ Between ↑··· ja
380 Siyabulela:	Ja [from four to seven days]··· 350 people were infected↑
381 Lungiswa:	[from four to seven days]

Vuyani enters the discussion with a tentative suggestion (using the imperative mood and the negation "aren't"); "Aren't we supposed to include the word... average?" He is looking structurally at the fraction as an object and linking to the lecture text where "the… word average" is used for such objects. This prompts other students to look structurally at the fraction ("this one" and "<u>this</u>") as an average rate of change; "It's the average this one", and "<u>This</u> is a rate of change". However, Lungiswa reminds her peers about using "practical terms", and the students proceed to verbally insert "the word... average" into Siyabulela's first attempt in line 380, as in "the average in the number of people were… " (unidentified), and "… between that period of 4 and 7... there were like … how many people infected? Ja like 350 people ... on average average" (Bongani). However, trying to produce what sounds right in the lecture text ends in frustration, suggested by Bongani's loud, "Aargh".

As usual, Siyabulela acts as an authority in the foundation practice by, firstly, critiquing Mpumelelo's use of "a mathematical term" in "rate of infection". Yet his laughter mocks the practical terms genre. Secondly, he dismisses as "the derivative"

Bongani's use of "increasing" in "the number of people were increasing that were infected by 350". Since Bongani may be looking operationally at the subtraction $P(7)–P(4)$ as required in the genre, Siyabulela diverges from his usual role as the student who enables the action in Group A. Not knowing how to proceed, the students discuss the prepositions and whether their choices are "bad English" (Siyabulela).

The Group B students initially view the fraction $\frac{P(7)-P(4)}{7-4}$ as an object linked to the words "average" and "rate of change". They also insert the word "average" into talk about the function ("the average... people who will be infected is 350... from", Jane). Two actions enable a shift from testing what sounds right relative to the course. Firstly, Hanah's attention to the units ("the average amount... per day") suggests she is looking operationally at the division. Secondly, Lulama links the words "average" and "rate of change"; "is this an average or an average rate of change?" Attention to the voices of Lulama and Hanah is rare in this group, yet progress is made in the practical terms genre, for example, "From 4 to 7 days the average number of people infected per day are 350 people". Shae is the exception, using the "rate of change" in his answer.

Describing the instantaneous rate of change in practical terms

In Group A Siyabulela and Bongani link the symbols $P'(4)$ to their earlier talk about "the derivative". Bongani avoids mathematical words by naming it "my thing thing". As usual, Lungiswa and Siyabulela begin to co-construct verbal answers in practical terms. Lungiswa's initial description of the independent variable ("After each four days") reproduces the preposition in the lecture text, with Siyabulela refining this to "After four days". Siyabulela then adds the meaning of the dependent variable, using Bongani's word "increasing"; "the number of people… who were infected were increasing by… 400 <u>per</u> day". The other students join in, and considerable effort is made to produce practical wording that sounds right (irrespective of the tense).

In contrast, the Group B students complete question (e) at different times and use mathematical words. As usual, Jeff and Shae begin and, after stating words like "rate of change" and "instantaneous" settle on "infection rate" in their writing. Answering question (e) later, Jane and Hanah identify Jeff as the authority in the practice, relying on him to confirm or rework their words. In one such version Jeff uses practical terms only ("<u>at</u> day 4, 400 people <u>per day</u> are being infected), although he gives significance to mathematical terms in his writing. For the time variable the students switch between the prepositions "after" of the lecture text and their own "at" for "instantaneous". Lulama does not talk with his peers and writes "average rate of change".

DISCUSSION

Using practical terms involves students verbally stating the meaning of the variables and reworking and adding to one another's talk, action that is initiated by the students who identify and are identified as authorities in the foundation practice. Words from the problem and lecture texts are combined to sound right. Progress is enabled or constrained by the group interaction, as students invest in or resist the style of students who collaborate in groups and as the authorities in the practice control who has a voice.

Viewing this action relative to the discourse-types, genres and styles of the university mathematical discourses raises concerns about the relationship between the practical terms genre and advanced mathematics. Firstly, it is possible that, by repeating words in the problem and lecture texts and ignoring inconsistent tenses, students use the practical terms genre without recognising the esoteric domain knowledge of calculus as intended (Gellert & Jablonka, 2009). Secondly, the students' jokes about the genre suggest that they recognise the practical/mathematical boundary (Straehler-Pohl, 2010). Yet they still invest considerable time using the genre, sometimes unsuccessfully. It may be that not repeating the genre involves, in Lulama's words, being "wrong" in the foundation practice, a practice that provides formal access to advanced mathematics. Paradoxically, using this genre requires producing meaning in the public domain which does not provide access to the vertical mathematics practice (Dowling, 1998) and, in question (d), looking operationally at an object rather than structurally as valued in the vertical practice.

Finally, some students (e.g. Shae and Jeff) appear to resist using the practical terms genre more easily than others (e.g. Lulama and Lungiswa). It may be that, like the students in Nyabanyaba's (2002) study, Shae and Jeff choose not to reproduce foundation questions targeting "dummies" in the knowledge that they can still pass the course. I note at this point that Shae and Jeff would not be classified as educationally disadvantaged in the South African context and only joined the foundation class after performing poorly in the first assessments in the mainstream mathematics course. While the conceptualization of this study does not allow claims in this regard, there is scope for an investigation of whether the foundation practice acts in reproductive ways with respect to race, socio-economic status and language.

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TEACHERS' PERCEPTIONS OF PRODUCTIVE USE OF STUDENT MATHEMATICAL THINKING

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We argue that the teaching practice of productively using student mathematical thinking [PUMT] needs to be better conceptualized for the construct to gain greater traction in the classroom and in research. We report the results of a study wherein we explored teachers' perceptions of PUMT. We interviewed mathematics teachers and analysed these interviews using and refining initial conjectures about the process teachers might go through in learning PUMT. We found that teachers' perceptions of PUMT ranged from valuing student participation, to valuing student mathematical thinking, to using that thinking in a variety of ways related to eliciting, interpreting and building on that thinking.

INTRODUCTION

Instruction that meaningfully incorporates students' mathematical thinking is widely valued within the mathematics education community (e.g., NCTM, 2000, 2007). Past research has suggested both the benefits of instruction that incorporates student mathematical thinking to develop mathematical ideas (e.g., Fennema, et al., 1996; Stein & Lane, 1996), and the challenges of learning about and enacting such instruction (e.g., Ball & Cohen, 1999; Sherin, 2002). One reason for these challenges may be the under conceptualization of *the teaching practice of productively using student mathematical thinking* [PUMT].

The literature uses multiple terms, and the same terms in multiple ways, to describe PUMT. For example, some (e.g., Franke & Kazemi, 2001; Peterson & Leatham, 2009) talk of teachers *using* student mathematical thinking. Others (e.g., Hill, Ball, & Schilling, 2008; Leatham, Peterson, Stockero, & Van Zoest, 2014) discuss teachers *building* on student mathematical thinking, and still others (e.g., Feiman-Nemser & Remillard, 1996; Lampert, et al., 2013) refer to students *attending to* the mathematical thinking of others. Thus, although many advocate teachers *being "responsive to* students and… their understanding" (Remillard, 1999, p. 331), the nature of such responses is ill defined.

This imprecision in language causes challenges when supporting teachers in developing PUMT, leaving them with multiple, and sometimes unhelpful, interpretations of the practice. This imprecision also hinders productive discourse within the research community and inhibits researchers from building on each other's work. Our broader work on PUMT is designed to support teachers in developing this critical practice; thus we chose as participants practicing teachers so that we could use their thinking to begin to address these imprecision-related challenges. Our goal is to better understand the multiple interpretations of PUMT that teachers have developed, and to initiate a discussion about what the mathematics teacher education field means by PUMT. Specifically, we investigated the question, “What are teachers’ perceptions of productive use of student mathematical thinking during whole class discussion?”

THEORETICAL PERSPECTIVES

For us productive use of student mathematical thinking requires first that one honor students as legitimate creators of mathematics. In addition, productive use in a mathematics classroom must be in the service of facilitating the learning of significant mathematics. Finally, we use “use” in the immediate sense of a teacher orchestrating student learning during a lesson. Productive use of student mathematical thinking “engages students in making sense of mathematical ideas that have originated with students—that is, it builds on student mathematical thinking by making it the object of rich mathematical discussion” (Leatham et al., 2014, p. 5). For example, suppose students in a pre-algebra class are discussing how to solve the equation $m - 12 = 5$ and someone in the class suggests subtracting 12 from both sides. A teacher could productively use this student mathematical thinking by pursuing it with the class and making sense of the outcome, all in the service of facilitating better understanding of the use of inverse operations to isolate variables when solving linear equations. (See Leatham et al., 2014 for further elucidation of this and other such examples.)

As we have already argued, enacting practices related to productively using student mathematical thinking is complex. As we have studied novice and expert teachers’ attempts to enact this practice (e.g., Peterson & Leatham, 2009; Van Zoest, Stockero & Kratky, 2010) we have developed conjectures about a *hypothetical learning process* [HLP] (Simon, 1995) related to PUMT. That is, it seems as though there are critical stages that build somewhat linearly on one another as a teacher develops PUMT (see Table 1). In professional development work, the HLP would combine with the goal of developing PUMT and with learning activities to form a *hypothetical learning trajectory* [HLT] (Simon, 1995).

Although this study contributes to research on teachers’ beliefs, we use the somewhat weaker term “perceptions” here because of the nature of the data collection and analysis. We use the term “perception” to mean, in essence, “initial reaction,” and recognize that perceptions are part of complex sensible belief systems (Leatham, 2006). Thus we expect that teachers may have more to say about these issues if they were explored in greater depth, and we make no claim to have sufficient data to infer deeper held beliefs. Initial reactions are very interesting, however, when looked at across a group of individuals because these commonalities can be construed, to some degree, as a “common wisdom” or “common viewpoint” (Leatham, 2009). Thus studying teachers’ perceptions will provide initial insights into the ways they conceptualize productive use of student mathematical thinking.

Hypothetical Learning Process for PUMT
Reject Active Student Participation – Teachers do not see the value of students being actively engaged during instruction.
Value Student Participation – Teachers want students to be actively engaged during instruction.
Value Student Mathematical Thinking – Teachers view students as capable of diverse legitimate ways of viewing and doing mathematics.
Elicit Student Mathematical Thinking – Teachers actively provide opportunities for students to share their mathematical thinking publicly.
Interpret Student Mathematical Thinking – Teachers conscientiously attend to and make sense of the mathematical thinking that is being shared.
Build on Student Mathematical Thinking – Teachers make student mathematical thinking the object of consideration in order to engage students in making sense of that thinking to better understand an important mathematical idea. (Teachers refine this practice first with individuals, then with small groups, and eventually in whole-class settings.)

Table 1: Hypothetical learning process for developing *the teaching practice of productively using student mathematical thinking* [PUMT].

METHODS

Our participants were 14 mathematics teachers (6 female and 8 male) with 1 to over 20 years of experience teaching a variety of mathematics courses in grades 6-12. In order to explore teachers' perceptions of productive use of student mathematical thinking we developed an interview protocol wherein we asked each teacher to sort a collection of cards describing teacher moves one might associate with classroom discourse (e.g., "get students' ideas out there for the class to consider and discuss," "juxtapose two student ideas that differ in an important mathematical way," "repeat an important student comment"). We compiled these teacher moves from the literature, from our own experience, and from an informal survey of mathematics education colleagues that asked them to describe what it meant to build on student thinking. We asked the participants to sort the moves along a continuum, from least to most productive use of student thinking during whole-class discussion, thinking aloud as they did so. We further prompted them to explain their reasoning or describe the criteria they seemed to be applying in making their decisions as they sorted the cards. We ended the interview by asking the participants what characteristics they saw as encapsulating the moves they placed at the top (as well as the bottom) of the continuum. Prior to conducting the 14 interviews we conducted two pilot interviews and made minor revisions to the protocol. All interviews were videotaped, with the video focused on the interviewees' sorting of the cards.

Initial analysis consisted of watching and writing brief summaries for each interview, in which we attempted to capture the essence of each teacher's overall perception of productive use of student thinking. Based on these summaries and on our initial learning trajectory (see Table 1) we developed a coding framework of potential perceptions and types of uses of student thinking and returned to the data to systematically code the interviews for evidence of these perceptions and uses (or for the emergence of others). We applied this framework to the interviews (refining and reapplying as appropriate) from six teachers who were selected to be representative of the range of perceptions based on analysis of the initial summaries. We then asked the following questions of the data: What are teachers' perceptions of productive use of student thinking? To what extent do those perceptions align with the PUMT HLP? Our answers to these questions make up the results section of the paper.

RESULTS

Initial analysis of the interviews revealed a variety of ways that teachers thought about PUMT, including different uses of student thinking during instruction. Further analysis revealed that types of use seemed to align in interesting ways with our conjectures about stages of the PUMT HLP (see Table 2). We thus organize this results section around these stages. As we discuss the stages we provide examples from the data to illustrate the participants' associated perceptions.

PUMT HLP	Type of Use
Reject Active Student Participation	
Value Student Participation	
Value Student Mathematical Thinking	
Elicit Student Mathematical Thinking	Engagement Replacement Validation
Interpret Student Mathematical Thinking	Assess Clarify Launch
Build on Student Mathematical Thinking	Pondering Establishing Extracting

Table 2: Conjectured relationship between the PUMT HLP and various types of use.

Before beginning our discussion of the stages on the HLP, it is important to note that an individual teacher may be functioning in several stages simultaneously. This multiplicity can be a reflection of a transition or a result of contextual factors. For example, some teachers' perceptions about productive use of student mathematical thinking were tied to the level of student (advanced vs. remedial, middle school vs. high school) or to school factors (pressure to prepare for high-stakes tests vs. freedom to vary the curriculum). These nuances are not our focus here, but deserve attention in future research.

Non-Use Stages

The first three stages of the PUMT HLP do not involve incorporating students' mathematical thinking into instruction. At the first stage, *Reject Active Student Participation*, teachers do not see the value of students being actively engaged during instruction. Instead, they consider the students as receivers of knowledge that the teacher presents to them. Teachers at the second stage, *Value Student Participation*, place a high regard on student participation, but in a way that seems to have little to do with the mathematical content of that participation. For example, one teacher wanted his students to understand that, "realistically, you might not use… any of these formulas in what you are going to do in life, but if you can learn to be a thinker… then that's going to be of great benefit." For this teacher participation through thinking yielded an important outcome regardless of the content of that thinking. At the third stage, *Value Student Mathematical Thinking*, teachers view students as capable of diverse legitimate ways of viewing and doing mathematics, but do not purposefully incorporate that thinking into instruction.

Elicit Student Mathematical Thinking

Teachers at the *Elicit* stage actively provide opportunities for students to share their mathematical thinking publicly. We have identified three types of use at this stage (not related hierarchically): (a) *Engagement*—The teacher elicits student mathematical thinking so that students will feel that they are an important part of the lesson and so that, by seeing others so engage, they will want to similarly participate. For example, one teacher indicated that any move that could elicit student mathematical thinking provided evidence that students were engaged and "trying to get the student involved is the most important thing. Everything else is secondary." (b) *Validation*—The teacher elicits student mathematical thinking to create an opportunity to provide positive feedback for students so they feel good about themselves. One teacher explained that "acknowledging that you are thinking is important because that gives you positive reinforcement." (c) *Replacement*—The teacher elicits student mathematical thinking in such a way that students say what the teacher wanted said. For example, teachers might share a student solution to a problem rather than working an example themself. Or, instead of making a statement teachers might ask a question (simple or fill-in-the-blank) so that student responses say what they would have said.

Interpret Student Mathematical Thinking

Teachers at the *Interpret* stage conscientiously attend to and make sense of the thinking that is being shared during their instruction. Three types of use (again not related hierarchically) were identified at this stage: (a) *Assess*—The teacher makes sense of the student mathematical thinking to determine whether given ideas are sufficiently understood to inform subsequent instruction. They may share this assessment with students, thus informing students about the correctness of their thinking. One teacher explained, "if they can verbalize how they are thinking about it then I actually get a better idea that they actually do know what is going on." (b) *Clarify*—The teacher makes sense of the student mathematical thinking and shares their own interpretation with the class with the intent to clarify the content of that thinking for the class. Some ways a teacher might clarify include adding mathematical language to a student comment, making a connection between the student thinking and a mathematical idea, and highlighting the importance of the thinking. (c) *Launch*—The teacher makes sufficient sense of the student mathematical thinking to see a connection to something they want to come out in the lesson. They then make the connection as a segue to making their point. As one teacher indicated, it is valuable to "give them suggestions about how they could advance their thinking about the mathematics, rather than just acknowledge *that* they are thinking."

Build on Student Mathematical Thinking

Teachers at the *Build* stage make student thinking the object of consideration in order to engage students in making sense of that thinking to better understand an important mathematical idea. There are three types of use connected to this stage: (a) *Pondering*—The teacher invites the class to think about the student mathematical thinking. For example, the teacher could give students a few moments to digest an idea before moving on. One teacher indicated that a major goal in having students share their ideas is to "have the class think about them." (b) *Establishing*—The teacher creates the space for the class to make sense of the student mathematical thinking and come to a mutual understanding of what was said or meant. For example, one teacher described how they "could have the student actually write what they just said and see if… the rest of the class could apply what the other student just said to the current problem they are working on." Another teacher spoke of the value of having students "convince the other person what you're thinking or try to understand the other idea." (c) *Extracting*—The teacher orchestrates a discussion that leads to a mutual understanding of the student mathematical thinking and helps the class to see the underlying mathematics that the student thinking embodies. For example, one teacher felt that it was extremely productive to elicit a variety of student ideas and "ask them to compare and contrast them, to try to work out how they might be related." It is this "work[ing] out how they might be related" that reflects the essence of extracting.

Different from the earlier stages that involve use, the three types of use in this final stage appear to be hierarchical. That is, we anticipate teachers first developing skill at supporting students in thinking about their peers' ideas, followed by increasing their abilities to create space for students to establish meaning from their peers' thinking, before finally being able to help students to see the underlying mathematics that the student thinking embodies. It is this final use that fully capitalizes on the potential of student thinking to improve the learning of mathematics.

DISCUSSION AND CONCLUSION

The perceptions and their accompanying uses represent a continuum of less to more productive ways of incorporating student mathematical thinking into instruction. Valuing student participation and student mathematical thinking is important, but on their own they do not make student mathematical thinking available for use in instruction. Likewise eliciting student mathematical thinking is a critical component of PUMT, but when it is thought of as an end in itself—rather than as a means toward building mathematical understanding—it fails to take full advantage of the possibilities student thinking offers. Interpreting student mathematical thinking allows for a broader range of productive use, but in these uses the teacher takes on the mathematical work, thus limiting students' opportunities to engage with the mathematics at a deep level. Building incorporates valuing, eliciting, and interpreting, but uses the information gained from interpreting the student mathematical thinking to turn that thinking back to the students. The productivity of uses categorized as building increases as one moves beyond asking students to ponder their peers' mathematical thinking, to engaging them in mutual sense making of that thinking in order to establish a mutual understanding, to collectively extracting important underlying mathematical ideas as a result of making the student thinking the object of discussion.

The PUMT HLP provides a starting place for conceptualizing PUMT and demonstrates that such a conceptualization is possible and worthy of additional investigation. The HLP could be further refined through using it to analyse more interviews as well as other sources of data, such as videotapes of classroom practice. The HLP could also prove useful as a means of analysing teachers' instruction to gauge proficiency with respect to this particular practice. We envision this work leading to the development of a HLT that could be used to support teachers in developing PUMT. As a result, this critical practice would gain greater traction both in research and in classrooms.

Acknowledgement

This research report is based on work supported by the U.S. National Science Foundation (NSF) under Grant Nos. 1220141, 1220357 and 1220148. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.

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A COMPARATIVE STUDY ON BRAIN ACTIVITY ASSOCIATED WITH SOLVING SHORT PROBLEMS IN ALGEBRA AND GEOMETRY

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This paper presents study that investigates brain activity (using ERP methodology) of male adolescents when solving short problems in algebra and geometry. The study design links mathematics education research with neuro-cognitive studies. We performed a comparative analysis of brain activity associated with the translation from visual to symbolic representations of mathematical objects in algebra and geometry. The findings demonstrate that electrical activity associated with the performance of geometrical tasks is stronger than that associated with solving algebraic tasks. Additionally, we found different scalp topography of the brain activity associated with algebraic and geometric tasks. Based on these results, we argue that problem solving in algebra and geometry are related to different patterns of brain activity.

INTRODUCTION

This paper presents a small segment of a large scale project that analyses components of mathematical abilities in three dimensions: basic cognitive traits, brain activity associated with solving mathematical problems and mathematical creativity (Leikin, Leikin, Lev, Paz, & Waisman, 2014). In this paper we choose information about neuro-cognitive activity related to solving short problems in algebra and geometry, and perform a comparative analysis of brain activity associated with translation from visual to symbolic representations of mathematical objects in algebra and geometry. Algebraic tasks required translation from graphical to symbolic representation of a function, whereas tasks in geometry required translation from a drawing of a geometric figure to a symbolic representation of its property. In this paper we do not analyse relationship between students' mathematical performance and their mathematical abilities since no interaction between students' abilities and test effect was identified.

BACKGROUND

Neuro-cognitive research in mathematics education

Neural basis of the use of mathematical cognition has been investigated in several directions mostly focusing on brain location of cognitive functions related to mathematical processing. Here we provide several examples. Research on number processing and simple arithmetic (Dehaene, Piazza, Pinel, & Cohen, 2003) emphasized the role of the parietal cortex to number processing and arithmetic calculations. The horizontal intraparietal sulcus has been found to be involved in calculations; the posterior superior parietal lobule has been linked with the visuo-spatial and attention aspects of number processing while the angular gyrus has been found to be associated with the verbal processing of numbers and involved in fact retrieval (Grabner, Ansari, Koschutnig, Reishofer, Ebner, & Neuper, 2009). The parietal cortex has been found to be involved, too, in more complex mathematical processing such as word problem solving (Newman, Willoughby & Pruce, 2011). The posterior superior parietal cortex has been found to be involved in visuo-spatial processing including the mental representations of objects and mental rotations (Zacks, 2008). Some studies demonstrated that when complexity of the problems rises, more brain areas simultaneously support the solving process (Zamarian, Ischebeck, Delazer, 2009). Note, however, that the neural mechanisms involved in complex mathematics have not been studied sufficiently, and our study enters this lacuna.

Studying functions and geometry in high school

Function is one of the fundamental concepts in mathematics in general and in school algebra and calculus in particular (Da Ponte, 1992). Kaput (1989) argued that the sources of mathematical meaning-building are found in translations between representation systems. The ability to translate from one representation of the concept of function to another highly correlates with success in problem solving (Yerushalmy, 2006) while flexible use of representations is part of cognitive variability, which enables individuals to solve problems quickly and accurately (Heinze, Star, & Veschaffel, 2009).

Geometry in school mathematics is considered an important source for development of students' reasoning and justification skills (Lehrer & Chazan, 1998; Hanna, 2007; Herbst & Brach, 2006). Learning geometry in high school involves analyzing geometric objects, their properties and the relationships between them. Mental images of geometrical figures represent mental constructs possessing simultaneously conceptual and figural properties (Fischbein, 1993). Geometrical reasoning combines visual and logical components which are mutually related (Mariotti, 1995), while perceptual recognition of geometrical properties must remain under the control of theoretical statements and definitions (Duval, 1995).

This paper focuses on brain activity associated with solving short geometry problems that require translation between visual representation of a geometric object and the symbolic representation of its property.

METHODS

The study goals

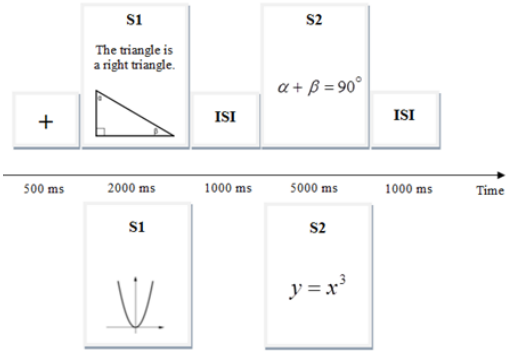
This study examined behavioural measures, i.e., Accuracy of responses (Acc) and Reaction time for correct responses (RTc), and electrophysiological measure, i.e., amplitudes, latencies, and scalp topographies, related to solving short problems that require translation between symbolic and graphical representations in algebra and geometry. We asked: How do the examined measures revealed when solving algebraic problems differ from those revealed when solving geometry problems?

Participants

We report herein our findings on 71 right-handed male adolescents. All participants were paid volunteers, native speakers of Hebrew, right-handed, with no history of learning disabilities and/or neurological disorders. All participants and their parents signed an informed consent form. The study received the approval of the Helsinki Committee, the Israel Ministry of Education, and the Ethics Committee of the University of Haifa.

Materials and procedure

A computerized test that required students to perform a translation between symbolic and graphical representations of function was designed with 60 tasks (trials) using E-Prime software (Schneider, Eschman, & Zuccolotto, 2002).



S1 – Introducing a situation; S2 – Question presentation;
+ – Fixation cross; ISI – Inter Stimulus Interval

Figure 1: The sequence of events and task examples

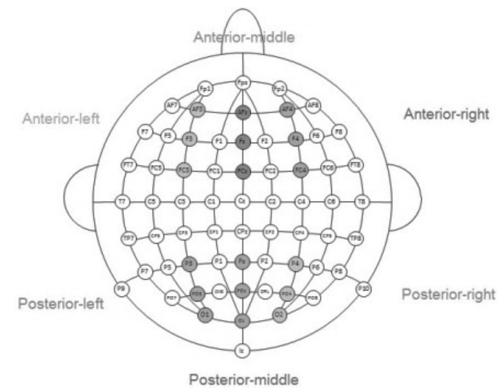


Figure 2: Location of the electrodes and selected electrode sites

Each task on each test was presented in two windows with different stimuli (S1 – Task condition; and S2 – Suggested answer) that appeared consecutively. The sequence of events and examples of the tasks are presented in Figure 1. At S2 each subject had to decide whether the suggested answer was correct or not by pressing an appropriate button on the keyboard. Alpha-Chronbach was determined by accuracy criteria and found to be sufficiently high (α_C = .859 and α_C = .760 for algebraic and geometric tasks, respectively).

Scalp voltages were continuously recorded using a 64-channel BioSemi ActiveTwo system (BioSemi, Amsterdam, The Netherlands) and ActiveView recording software. Two flat electrodes are placed on the sides of the eyes in order to monitor horizontal eye movement. A third flat electrode is placed underneath the left eye to monitor vertical eye movement and blinks. During the session electrode offset is kept below 50 µV. Figure 2 depicts location of the electrodes and the selected electrode sites.

Data analysis and statistics

Trials with correct responses were used for both ERP and behavioural analysis. Behavioural data of trials excluded from electrophysiological analysis by artifact rejection were excluded from the behavioural analysis.

Behavioural measures: We examined Accuracy (Acc) and Reaction time for correct responses (RTc) for each participant. Acc was determined by the mean percentage of correct responses to 60 tasks on the test. Reaction time for correct responses (RTc) was calculated as the mean time spent for verification of an answer (stage S2) and for correct responses only. We performed between the tests comparisons using repeated measures MANOVA.

Electrophysiological measures: Event related potentials (ERPs) were analysed offline using the Brain Vision Analyzer software (Brain-products). Ocular artifacts were corrected using the Gratton, Coles and Donchin (1983) method. The ERP waveforms were time-locked to the onset of S1 and to the onset of S2. Due to the space contrarians we do not report analysis of ERP early components.

Following visual inspection of grand average waveforms and appropriate scalp topographies we divided the late potential wave into three time frames: 300-500, 500-700 and 700-900 ms. We used repeated measures ANOVA tests on the ERP mean amplitude, to examine effects of Tests, Laterality (Left, Mid-line and Right) and Caudality (Anterior and posterior). Analysis was done for each of the two stages of a task (S1 and S2). Table 1 depicts Electrophysiological data analysis performed for Late Potentials.

ERP component	Stage	Time frame (ms)	Factors	Measures
Late potentials	S1 S2	300-500 500-700 700-900	*Laterality:* 3 levels: Left, Middle and Right *Caudality:* 2 levels: Anterior, Posterior *Test:* 2 levels: Algebra, Geometry	Mean amplitude

Table 1: Electrophysiological data analysis of Late Potential

RESULTS

We report on the significant effects and interactions only. If a particular effect (or interaction) is not reported, this indicates that it was not significant.

Differences in Acc and RTc

Measure	Algebra Mean (SD)	Geometry Mean (SD)	F (1, 67)	η_p^2
Acc	78.2 (9.7)	83.2 (7.5)	18.877^{***}	.220
RTc	1686.4 (398.6)	1593.7 (382.5)	5.513^{*}	.076

$p<.05$, $^{**}p<.01$, $^{***}p<.001$; Acc – Accuracy, RTc – Reaction time for correct responses

Table 2: Acc and RTc in the different groups of participants

Repeated measures MANOVA demonstrated significant effects of the Test factor [F (2, 66) = 10.082^{***}, *Wilks* ∧ = .766] both on the Acc and on the RTc (Table 2). On algebraic test we found significantly lower Acc along with significantly higher RTc as compared to Acc and RTc on the geometry test (Table 2).

Electrophysiological findings reflected in late potential

Figure 3 depicts examples of the grand average waveforms and topographical maps of the late potentials for Algebra and Geometry tests.

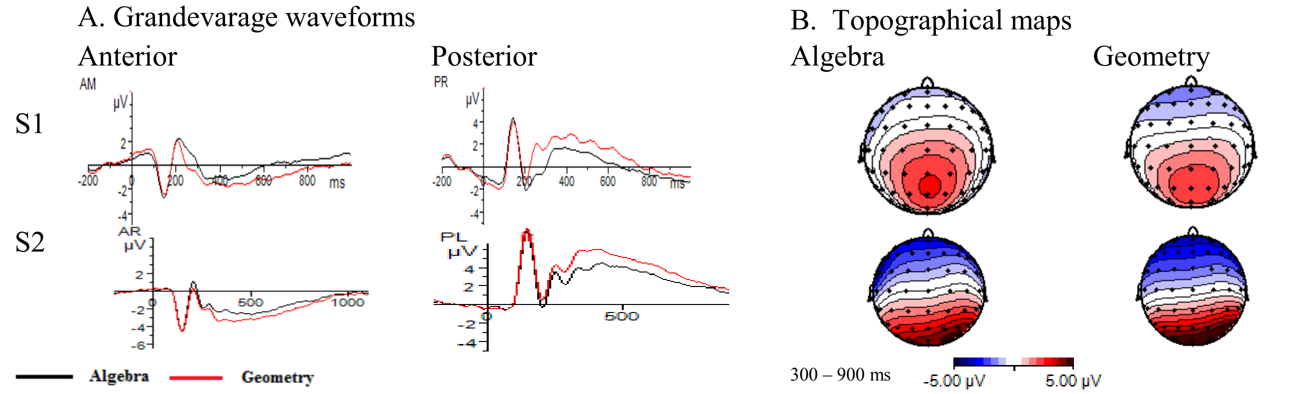


Figure 3: (A)Examples of the grandevarage waveforms associated with Algebra and Geometry tests in the selected electrode sites.; (B) Topographical maps of voltage amplitudes for Algebra and Geometry tests

	Significant factors and interactions	Time	F	η_p^2
S1	**Test** Amp(Algebra) < Amp(Geometry)	300-500 ms	7.687^{**}	.103
	Test × Caudality	300-500 ms 500-700 ms 700-900 ms	6.763^{*} 21.417^{***} 13.211^{***}	.092 .242 .165
	Test × Laterality Amp (Algebra) < Amp(Geometry) in **RH** Amp(Algebra) > Amp(Geometry) in **ML**	300-500 ms 500-700 ms	$F(1.657, 111.033) = 10.480^{***}$ $F(1.749, 117.165) = 8.561^{***}$	.135 .113
	Test × Laterality × Caudality	300-500 ms	$F(1.692, 113.371) = 7.709^{***}$	.103
S2	**Test** Amp(Algebra) < Amp(Geometry)	300-500 ms 500-700 ms 700-900 ms	46.560^{***} 25.155^{***} 5.615^{*}	.410 .273 .077
	Test × Caudality	300-500 ms 500-700 ms	41.390^{***} 10.248^{**}	.382 .133
	Test × Laterality Amp (Algebra) < Amp(Geometry) in **LH** and **ML** Amp(Algebra) ~ Amp(Geometry) in **RH**	500-700 ms	3.650^{*}	.052

$^{*}p \le .05$, $^{**}p \le .01$, $^{***}p \le .001$, when not mentioned *d.f.* (1, 67); RH –Right hemisphere; ML – Mid-line; LH – Left hemisphere

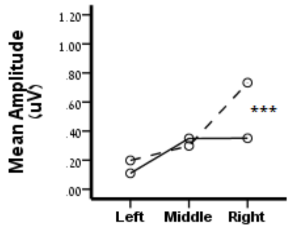
Table 3: Significant results in mean amplitude in the selected electrode sites associated with the Algebra and Geometry tests

Significant differences associated with the Test were found at both S1 and S2 stages. Table 3 demonstrates time frames at which Test, Laterality and Caudality factors had significant effects and interactions. We found that the *mean amplitude* was significantly higher for Geometry test than for Algebra test. Significant *interaction between Test and Caudality* revealed at posterior and anterior regions while absolute values of voltages for the Geometry test were larger than for Algebra test with negative voltages in anterior regions and positive voltages in posterior regions.

A significant *interaction of Test with Laterality* was found at S1 and at S2: At S1, the mean amplitude reveal for the Geometry test was larger than for the Algebra test in the right and left hemisphere, whereas in the mid-line the mean amplitude for Algebra test was higher as compared to mean amplitude for the Geometry test (Figure 4). At S2, the mean amplitude for the Geometry test was larger in the left hemisphere and mid-line as compared to that for the Algebra test, while the difference between the tests in the left hemisphere and the mid-line was significant. A significant *interaction between Test, Caudality and Laterality* was found at S1 (Figure 4).

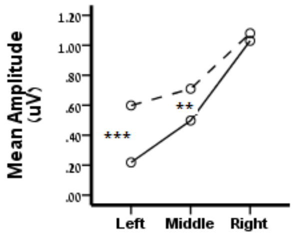
Test × Laterality

S1: 300-500 ms



**right:*
95% *CI* [.243, .521]

S2: 500-700 ms

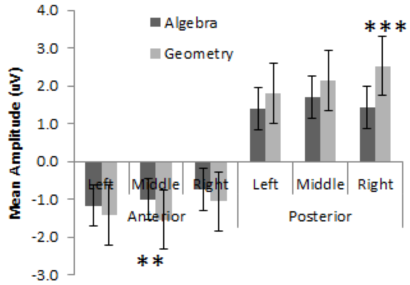


****Left:* 95% *CI* [.215, .546]
***Middle:* 95% *CI* [.072, .351]

— Algebra
- - Geometry

Test × Laterality × Caudality

S1: 300-500 ms



*** *posterior right*: 95% *CI* [.597, 1.583]
** *anterior middle*: p = .007, 95 % *CI* [.158, .936]

$^{*}p \le .05$, $^{**}p \le .01$, $^{***}p \le .001$

Figure 4: Mean amplitude for Algebra as compared to Geometry test in the Left, Middle and Right electrode sites in the 300-500 ms at S1 and in the 500-700 ms at S2.

DISCUSSION

Strength of electrical potentials: The first major finding of our study shows that solving geometry tasks emerged higher electrical potentials that solving algebraic tasks both at S1 and S2 stages at both the anterior and posterior parts of the scalp. Following previous research that connects cognitive functions of different types with activation of different brain regions, we hypothesise that the greater electrical activation in the anterior parts of the scalp demonstrated that geometry test requires greater cognitive control and activation working memory (Arsalidou & Taylor, 2011; Newman et al., 2011). The enhanced voltages in the posterior (especially right) parts of the scalp during geometry test may be connected to the greater demands in visuo-spatial processing, including manipulation of internal representations (Zacks, 2008) in geometry problem solving.

Brain topography: The second major result of our study revealed a significant interaction of hemispheric laterality with Test. At S1, the mean amplitudes associated with the geometry test were higher as compared to the algebra test in the right hemisphere, whereas the mean amplitudes associated with the algebra tests were higher in the mid-line. Moreover, in the right hemisphere the difference between the amplitudes elicited by the two tests was significant. In contrast, at S2 the mean amplitudes associated with the geometry test were significantly higher in the left hemisphere and the mid-line. Previous studies demonstrated that the left hemisphere is thought to be more involved in processing verbal and symbolic information and is also shown to be more analytic from the processing viewpoint, while the right hemisphere seems to deal more with the processing of visuo-spatial information (e.g., Dien, 2008). Thus we hypothesise that differences in activation patterns between the Algebraic and Geometry tests in our study may be explained by the differences between the processing strategies used by the participants in algebra and geometry. We also suggest that when solving geometry tests, at the visual and symbolic stages, participants activate different hemispheres.

Students' problem solving performance on geometry task as compared to their performance on algebra tasks revealed higher electrical potentials along with higher accuracy and shorter reaction times for correct responses. In contrast to the previous findings that task complexity lead to higher electrical potentials, by combining our research findings related to Acc and RTc with findings related to ERP measures, we speculate here that strength of electrical potentials when solving mathematical tasks are not necessarily connected to the task difficulty but subject-matter-dependent.

This study emphasizes the contribution of neuropsychological research, which adds important information to previous findings of cognitive studies in the field of the psychology of mathematics education. Based on our findings we argue that problem solving in algebra and geometry is associated with different patterns of brain activity and, thus, we hypothesize that teaching algebra and geometry may require different didactical approaches. We also assume that our findings on the differences in scalp topology associated with solving algebra and geometry tasks may explain why different students are not equally good in geometry and algebra.

Acknowledgements: This project was made possible through the support of a grant from the John Templeton Foundation and the generous support of the University of Haifa. The opinions expressed in this publication are those of the author(s) and do not necessarily reflect the views of the John Templeton Foundation.

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PRINCIPLES OF ACQUIRING INVARIANT IN MATHEMATICS TASK DESIGN: A DYNAMIC GEOMETRY EXAMPLE

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This paper is a theoretical discussion on a pedagogic task design model based on using variation as an epistemic tool. A set of four Principles of Acquiring Invariant is put forward that is complementary to the patterns of variation in Marton's Theory of Variation. These principles are then used as adhesive to tie together the epistemic modes in a model of task design in dynamic geometry environment that the author proposed in earlier research literature. A dynamic geometry task sequence is used to illustrate how the Principles of Acquiring Invariant can be used in mathematics task design.

INTRODUCTION

Marton's Theory of Variation is a theory of learning and awareness that asks the question: what are powerful ways to discern and to learn? In recent years, the theory has been applied in different pedagogical contexts (see for example, Lo, 2012; Lo & Marton, 2012). The Theory of Variation starts with a taken-for-granted observation: nothing is one thing only, and each thing has many features. In this theory, discernment is about how to go from a holistic experience of a phenomenon (e.g. seeing a forest) to separating out different features (e.g. seeing a tree) in the phenomenon (cf. Marton and Booth, 1997). It concerns with how to pick up meaningful experiences through our senses, and how meaning comes about from relationship between similarity and difference derived under simultaneous attention. In particular, there is a discernment ordering from difference to similarity. That is, learning and awareness begins with noticing difference before observing similarity. Suppose I can only perceive "grey" in certain situation, then "grey" has no meaning for me even if you show me a grey chair, a grey car, or a grey whatever. "Greyness" becomes meaningful to me only if I can perceive something else other than "grey". Thus, contrast (finding counter-examples focusing on difference) should come before generalization (which can be regarded as an inductive process focusing on similarity) in discernment. In this connection, a fundamental idea in the Theory of Variation is simultaneity. When we are simultaneously aware of (intentional focusing our attention on) different aspects of a phenomenon, we notice differences and similarities. By strategically observing variations of differences, similarities and their relationships, critical features of the phenomenon may be brought out. Morton proposed four patterns of variation as such strategic means: contrast, separation, generalization and fusion (Marton et al., 2004). A major undertaking of the Theory of Variation is to study how to organize and interpret a pedagogical event in powerful ways in terms of these patterns of variation (Lo & Marton, 2012).

THE THEORY OF VARIATION IN MATHEMATICS CONCEPT DEVELOPMENT

In PME 27, I presented the first application of the Theory of Variation to mathematics pedagogy in dynamic geometry dragging exploration (Leung, 2003). There the four patterns of variation were used to interpret dragging modalities in a dynamic geometry construction problem to explore the gap between experimental reasoning and theoretical reasoning. This began a long programme of study where in my subsequent work; the values of the four patterns were gradually changing from originally as means to categorize possible powerful ways to discern into epistemic functions that can be used to bring about mathematical concept development (see for example, Leung, 2008; Leung, 2012; Leung et al., 2013). An epistemic activity in doing mathematics is to discern critical features (or patterns) in a mathematical situation. When these critical features are given interpretations, they may become invariants that can be used to conceptualize the mathematical situation. In Leung (2012), I used classification of plane figures as an example to develop a variation pedagogic model. The model consists of a sequence of discernment units in which different variation strategies are used to unveil different feature types of plane figure: intuitive visual type, geometrical property type, and equivalent geometrical properties type. Each discernment unit contains a process of mathematical concept development that is fused together by contrast and generalization driven by separation. The sequence represents a continuous process of refinement of mathematical concept, from primitive to progressively formal and mathematical. Mhlolo (2013) later used this model as an analytical framework to interpret a sequence of richly designed mathematics lessons teaching the conceptual development of number sequence. The upshot is, in variation perspective, mathematical concepts can be developed by strategic observation and variation interaction in terms of contrasting and comparing, separating out critical features, shifting focus of attention (cf. Mason, 1989) and varying features together to seek emergence of invariant patterns. A variation interaction is "a strategic use of variation to interact with a mathematics learning environment in order to bring about discernment of mathematical structure" (Leung, 2012). It is also a strategic way to observe a phenomenon focusing on variation and simultaneity. I interpret "interaction" in the sense that the acts of observing may involve direct or indirect manipulation of the mathematical object under studied.

PRINCIPLES OF ACQUIRING INVARIANT

Simultaneity is the epistemic crux of variation. The four patterns in Marton's Theory of Variation are different types of simultaneous focus used to perceive differences and similarities which lead to unveiling of critical features of what is being observed. Looking for invariant in variation and using invariant to cope with variation are essences of mathematical concept development. A mathematical concept is in fact an invariant. For example, the basic concept of the number "three" is an invariant cognized out of myriad representation of "three-ness". Thus in acquiring mathematical knowledge, to perceive and to understand invariant amidst variation are central epistemic goals. Putting these together, I put forward a set of four *Principles of Acquiring Invariant* that are complementary to Marton's four patterns of variation in the context of mathematics concept development (the italic words are the four patterns of variation):

Difference and Similarity Principle (DS) *Contrasting* and comparing in order to perceive or *generalize* possible invariant features

Sieving Principle (SI) *Separating* under prescribed constraints or conditions in order to reveal ("make visual") critical invariant features or relationships

Shifting Principle (SHI) Focusing and paying attention to different or similar features of a phenomenon at different time or situations in order to discern *generalized* invariant

Co-variation Principle (CO) Co-varying or *fusing* together multiple features at the same time in order to perceive possible emergent pattern or invariant relationship between the features

These four principles work with the four patterns of variation in a concerted way. All four principles, just like the four patterns, are different aspects of simultaneity and contrast. They are cognitive activities to look for mathematical invariants leading to development of mathematical concept. In particular, they have the following predominant functions. DS is about contrast and generalization leading to awareness of perceptual invariant feature. SI is about awareness of hidden invariant feature that is being separated out under variation when only selected aspects of the phenomenon are allowed to vary. SHI is about diachronic (across time) simultaneity leading to possible generalization in the conjecture making process. CO is about synchronic (same time) simultaneity leading to fusing together of critical features in the mathematical concept formation process. These four principles are learner driven which can be cognitively mingled and nested together. During a variation interaction, a learner can apply these principles with different weight and transparency. In the next section, I will illustrate a pedagogical example of these principles in designing a sequence of dynamic geometry tasks.

MATHEMATICS TASK DESIGN: A DYNAMIC GEOMETRY EXAMPLE

In Leung (2011), I proposed an epistemic model of task design in dynamic geometry environment (DGE). It consists of a sequence of three nested epistemic mode of cognitive activities:

Practices Mode (PM) Construct DGE objects or manipulate pre-designed DGE objects. Interact with DGE feedbacks to develop (a) skill-based routines; (b) modalities of behaviour; (c) modes of situated dialogue.

Critical Discernment Mode (CDM) Observe, record, recognize and re-present (re-construct) DGE invariant.

Situated Discourse Mode (SDM) Develop reasoning that lead to making generalized DGE conjecture. Develop DGE discourses and modes of reasoning to explain and prove.

These task design modes are nested in the sense that CDM is a cognitive extension of PM and SDM is a cognitive extension of CDM. An exploration space is opening up for learners as the task sequence progresses to construct first-hand understanding of the mathematical concepts carried by the task. It is a nested expanding space where practices evolve into discernment followed by discernment evolves into discourses. Within each mode, cognitive activities can be organized by variation tasks. Thus, the four Principles of Acquiring Invariant can be used as a skeleton to frame this epistemic model of task design. A DGE task sequence can be designed combining the epistemic modes and the Principles of Acquiring Invariant to constitute an evolving process (not necessarily linear) that merges gradually from dominate perceptive experiential "thinking" to dominate conceptual theoretical "thinking". The following is an example of such a task sequence. It is conceptualized and designed by using a student DGE exploration studied in Leung, Baccaglini-Frank and Mariotti (2013).

TASK 1: Construction

PM: DGE Construction

Construct three points A, B, and C on the screen, the line through A and B, and the line through A and C. Construct a line *l* parallel to AC through B, and a line perpendicular to *l* through C. Label the point of intersection of these two lines D. Consider the quadrilateral ABCD (see Figure 1).

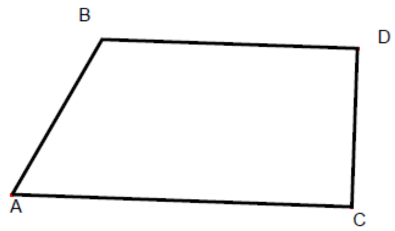

Figure 1

TASK 2: Contrast and Comparison

PM / DS: Variation tasks are used to bring about awareness of different and similar aspects/features in a DGE phenomenon that leads to observable invariants

2.1 Drag A, B, C to different positions to make different quadrilaterals

2.2 How many different or similar types of quadrilateral ABCD can you make?

2.3 Describe how you drag a point to make it changes into different types of quadrilateral

2.1 and 2.2 ask the learner to contrast and compare different positions of A, B and C as these vertices are being dragged to observe how many different types of quadrilateral can be formed. 2.3 ask the learner to think about the dragging strategies used to obtain different types of quadrilateral, thus motivating the learner to develop dragging skills and strategies, to relate feedback and dragging action, and to begin a DGE-based reasoning about perceiving DGE invariant. Figure 2 are two snapshots for different positions of A where B and C are fixed. There are only two types of possible quadrilaterals: right-angled trapezium and rectangle. This is making use of the Difference and Similarity Principle.

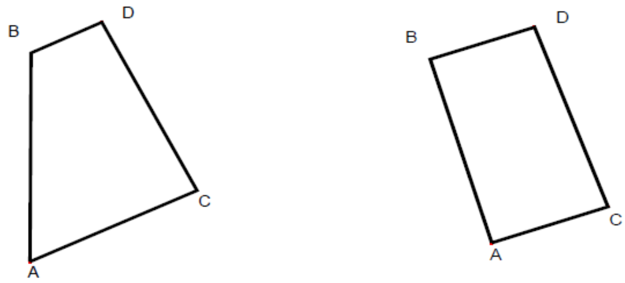

Figure 2

TASK 3: Separation of Critical Features

CDM / SI and SHI: Variation tasks are used to bring about awareness of critical (causal) relationship among the observed invariants

3.1 Activate the Trace function for point A. Drag A while keeping B and C fixed to maintain ABCD to look like a rectangle.

3.2 Describe your experience and what you observe

3.3 Make a guess on the geometrical shape of the path that A follows while maintaining ABCD to look like a rectangle. How do you make this guess? Call this guess a maintained-path (cf. Leung et al., 2013)

3.1 asks the learner to use a special function in DGE to record the trace of point A as it is being dragged to keep ABCD looks like a rectangle. Using rectangle as a perceptual invariant to constrain the dragging control makes visible the emergence of another perceptual invariant: the trace-mark of A which appears to take a geometrical shape (see Figure 3). Guessing and naming the trace motivates the learner to engage into a DGE discourse. This is making use of the Sieving Principle.

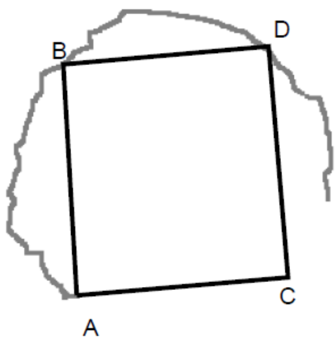

Figure 3

In 3.2 and 3.3, by asking the learner to describe his/her dragging experience and to make a guess on the geometrical shape of the traced path, the learner's cognitive mode is transiting from observation of DGE phenomena to discernment of critical features which could lead to concept formation. In particular, while the learner shifts his/her attention to the two perceptual invariants (the rectangular-like ABCD and the maintained-path) during dragging, attention to discern possible causal relationship between the two invariants may come about. This is the Shifting Principle.

TASK 4: Simultaneous Focus

SDM / SHI and CO: Variation tasks are used to bring about awareness of a connection between critical relationships observed and possible mathematical discourses (causal condition, formal/informal conjecture, concept, pattern, mathematical proof, etc.)

4.1 When A is being dragged to vary, vertices B, C and D either vary or not vary as consequence. Observe the behavior of B, C and D while A is varying to maintain ABCD looks like a rectangle.

4.2 Find a possible condition to relate the maintained-path and the varying configuration of B, C and D.

4.3 Use the condition found in 4.2 to construct the maintained path

4.1 and 4.2 are continuation of 3.3, the Shifting Principle continues with added attention to the consequential movements of the vertices A, B, C and D, thus the Co-variation Principle become in effect. In the process, the learner develops a DGE discourse for geometrical reasoning and construction. 4.3 is a consummation of the exploration in the form of a DGE soft construction (cf. Healy, 2000). The maintained-path takes the form of a circle centred at the midpoint of segment BC. The construction of this circle ensured D lies on the circle and when A is being dragged along this circle, ABCD becomes a rectangle (Figure 4).

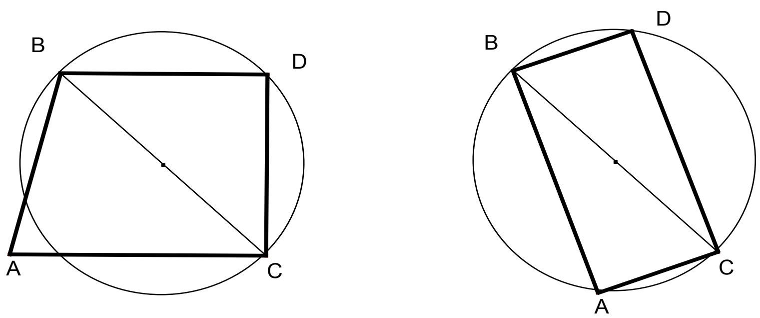

Figure 4

TASK 5: Conjecture and Proof (Development of Theoretical Reasoning)

SDM / CO: Development of DGE discourse to connect experimental reasoning and theoretical reasoning

5.1 Write a conjecture on what you have discovered in the form

GIVEN A DGE construction

IF (certain condition being maintained during dragging)

THEN (certain configuration appears to be maintained during dragging)

5.2 Drag A along the constructed maintained-path. Observe how different aspects of the figure vary together. Explain what you observe and formulate a logical argument to explain/prove your conjecture

4.3 (Figure 4) is a DGE representation of a conjecture, 5.1 asks the learner to write this in the form of a DGE-situated conditional statement, for example,

GIVEN Quadrilateral ABCD as constructed in TASK 1

IF A is being dragged along the circle centred at the midpoint of segment BC

THEN ABCD is always a rectangle

5.2 challenges the learner to formulate an explanation (or even a proof) for the conjecture just formed. I leave the readers to explore this discourse and to see how the Principles of Acquiring Invariant can be embedded in the reasoning process.

REMARKS

In the above I meshed together two epistemic frameworks, i.e. Principles of Acquiring Invariant and Task Design Epistemic Modes, to explore the mathematical concept formation process from experimental observation to discernment of abstraction using DGE as a context. A first remark is that these principles and epistemic modes form a nested network rather than follow a linear hierarchy. At any one instance during an exploration, any one of the principles and one of the modes can take dominance. These cognitive activities are pretty much learner driven but when designing a mathematical task, the designer can guide (as the five Tasks above) a learner to pay more attention to particular principle and mode while other principles and mode can be put in the cognitive background. This shifting between foreground and background is in fact a basic idea in the Theory of Variation. A second remark is that the task design model discussed in this paper is an attempt to crystalize a possible process bridging the experimental-theoretical gap in the DGE context. Specifically, the upshot of using variation and invariant is to drive an epistemic sequence that may look like:

> Constraint → Pattern Observation → Predictability → Emergence of Causal Relationship → Concept Formation → Explanation/Proof

This paper is an attempt to enrich the current research literature on the use of variation in mathematics education and to propose a perspective focusing on invariant that is pertinent to mathematics knowledge acquisition.

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HOW THE KNOWLEDGE OF ALGEBRAIC OPERATION RELATES TO PROSPECTIVE TEACHERS' TEACHING COMPETENCY: AN EXAMPLE OF TEACHING THE TOPIC OF SQUARE ROOT

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This study is the part of a larger study on investigating Hong Kong (HK) prospective teachers' (PTs) Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). In this paper, five HK PTs' SMK and PCK on teaching one topic regarding square root were investigated. The results suggest that insufficient understanding on the concept of algebraic operation is the major obstacle to limit those student teachers from teaching students with mathematical understanding. The results further echo with the viewpoint that SMK and PCK are two interrelated constructs and rich SMK leads to high quality of PCK.

BACKGROUND INFORMATION

Results from international comparative studies such as TIMSS and PISA indicate that students from East Asian regions (including mainland China, Hong Kong, Taiwan, and Singapore) outperform their Western counterparts (e.g., Mullis, et al, 2008; OCED, 2013). Educational professionals believe that the "curriculum gap" is not the sole explanation for the performance discrepancies between West and East, and that the "preparation gap" of teachers, as confirmed by the results of IEA-study Teacher Education and Development Study in Mathematics (TEDS-M) (Tatto et al., 2012), is a fundamental concern. The results from TEDS-M study showed that potential mathematics teachers from two participating East Asian regions – Taiwan and Singapore – ranked the top in their achievement in both CK and PCK assessments among other participating countries. It is intuitively believed that the good performance in such international assessment exercises would be the consequence of well-equipped and competent teachers in the two regions. However, less is known about the reasons behind this relationship, and more explorations on other East Asian regions might help. In this study, we aim to contribute to the current knowledge by studying a group of HK PTs' teaching knowledge. HK had undergone substantial educational reform at the turn of the millennium, which requires a paradigm shift of teachers' teaching from teachers-centered to students-centered; therefore the extent to which HK PTs are ready to deliver such kind of effective mathematics teaching becomes a crucial issue.

In his most cited article, Shulman (1986) set out the multi-dimensional nature of teachers' professional knowledge. He identified, among other dimensions, three

2014. In Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 4, pp. 97-104. Vancouver, Canada: PME.

aspects of professional knowledge: pedagogical knowledge, subject knowledge, and PCK. In the subject of mathematics, this knowledge is further conceptualized by Ball and her colleagues and categorized into two domains: mathematical subject matter knowledge (SMK) and PCK. In their mathematics knowledge for teaching (MKT) model (Hill, et. al., 2008, p.377), PCK and SMK are treated as two separate components. PCK includes knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of curriculum, yet all three constructs under PCK connect with content knowledge in various ways. Indeed, the relationship between PCK and SMK is very vague. Despite that some studies separated the constructs of PCK and SMK empirically, a deep connection between the two constructs was found (e.g., Krauss, Baumert, & Blum, 2008). The impacts of SMK on PCK have been explored by scholars, in particular, which types of SMK can equip mathematics teachers for effective teaching are attractive. For example, Even (1993) studied the SMK of pre-service secondary mathematics teachers from the U.S. and its interrelations with PCK in the context of teaching the concept of functions. The study showed that insufficient SMK might lead pre-service teachers to adopt teaching strategies that emphasize procedural mastery rather than conceptual understanding. By comparing the US and Chinese mathematics teachers' teaching competency, Ma's (1999) found that Chinese teachers possessed profound understanding of fundamental mathematics (PUFM), that their American counterparts lack, facilitates them to conduct more effective teaching, Ball and her associates also included both common content knowledge (CCK) and specialized content knowledge (SCK) into their construct of SMK. In particular, they defined SCK, different from CCK, "that allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide explanations of common rules and procedures, and examine and understand unusual solution methods to problem" (Ball, et al., 2005).

In a recent paper, Buchholtz et al. (2013) reemphasize and highlight Felix Klein's ideas "*elementary mathematics from an advanced standpoint*" (EMFAS) as another important category of teachers' professional knowledge. The results gained from their international comparative study indicate that the future mathematics teachers from top mathematics performing countries including HK still have the problems in linking school mathematics and university knowledge systematically. The construct of EMFAS looks different from SCK by definition. The former is stated as more mathematical, and the latter one is mathematics knowledge applied for teaching. However, we make the hypothesis that EMFAS and SCK should share some similarities in content, both of them lead to conceptual-understanding oriented mathematics teaching.

In this paper, we investigated HK PTs' PCK and SMK on one topic in lower secondary school algebra namely square root as one example. The fundamental to the learning of mathematics is the process of learning the abstraction (Mitchelmore and White, 2000). Therefore, when concerning the richness of their SMK and PCK in teaching this topic, we focus on if their SMK and PCK can facilitate them to teach students with algebraic abstraction. Specifically, the research questions we aim to answer in current paper are:

- What are the SMK and PCK that the HK PTs have for teaching the topic of square root?
- How PTs' algebraic thinking relates to the quality of PCK in teaching this topic?

METHODS

What is presented here is a portion of a larger project in which two groups of future secondary mathematics teachers in HK participated. They are either in the third or fourth year of their study towards a bachelor of Education (BEd) majoring in mathematics, or, during full or part time study in the program of postgraduate diploma in education (PGDE) in mathematics. The whole project comprises both quantitative and qualitative data collection, the former being a questionnaire tapping PTs' beliefs of the nature of mathematics and mathematics teaching and mathematics knowledge. Base on the results of this phase, five participants were selected to take part in the second phase which constitutes an interview aiming at capturing the PTs' PCK and SMK in teaching three topics. They were given the pseudonyms of Jack, Fanny, Mandy, Gary and Charles. In the second phase, video-based interview was employed. It was taken a TIMSS 1999 Hong Kong video which constitutes a 40 minutes lesson of Grade 8 class. During the interview, both researcher and interviewee sat next to each other. The researcher controlled the play of video and asked questions where appropriate. The interviewee watched video and sometimes wrote their responses on the whiteboard. The whole process was video-taped. The interview questions basing on Ball et al. (2008) MKT framework (with incorporation of EMFAS), are depicted in Table 1.

Figure 1: The context of video-based interview

		Interview questions	The context of video
PCK	KCT	What are your comments on this teacher's approach on how to introduce the topic of square root? If you were the teacher, how would you do?	The teacher in the TIMSS video told the students to find a number that, after multiplication of two identical numbers to give the resulting number. In general, the process of getting a square root as introduced by this teacher as a simple multiplication procedure.
	KCT	How to teach your students to find out the square roots of 9, i.e., $(a)^2$=9	The teacher in the TIMSS video put the focus on emphasizing the square roots of 9 could be either positive or negative, as for the value of square root is negative or positive, this can be judged by the sign in front of the

			surd.
	KCS KCT	What's the thinking behind the student when student treat "a" as a positive number in writing the expression: $(-a)^2$=9 for figuring out a negative value for the square root of a? (KCS); If you were the teacher, how would you respond to this student? (KCT)	The teacher in the TIMSS video wanted to enlighten student to find out negative square root of 9, so she wrote down a question: what is the negative square root of 9? One male student was invited to solve this problem on the blackboard. He immediately wrote down the expression: $(-a)^2$=9, thought for a while but could not find the answer. The teacher suggested him wipe out the negative sign in front of a.
SMK	CCK SCK (EMFAS)	What are your comments on this student's solution: $\sqrt{(-4)^2}$ = $(-4)^{2\times\frac{1}{2}}$= $(-4)^1$=-4? Is it correct or not? Please provide reasons to support your answere from the mathematical point of view?	In the textbook utilized in this video lesson, one exercise was to ask students think about if it is true that $\sqrt{(-4)^2} = -4$. To investigate the depth of student teacher's SMK, one hypothetical scenario was posted: One student demonstrated $\sqrt{(-4)^2}$= -4 is true, because $\sqrt{(-4)^2}$ = $(-4)^{2\times\frac{1}{2}}$= $(-4)^1$=-4

Table 1: Interview questions and how they correspond to PCK categories

FINDINGS

Some preliminary findings and analysis based on the HK PTs' responses to part of PCK and SMK illustrated in Table 1 will be presented.

KCT – Introduction the topic of square root

All informants tended to introduce the topic of square root by making a connection with the topic of square. There are two approaches of making this connection. The first approach is to start with introducing square numbers such as 4, 9, 16 and 25, for example, Mandy suggested to ask students,

> What is the square of 3? What is the square of 4? Thinking of 1^2, 2^2, 3^2… (Mandy)

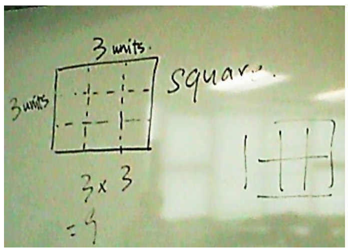

Figure 2: Fanny's picture to explain why 3^2 is not equal to 3×3

The second approach is to introduce the relationship between the area and the side of a square. Some student teachers tended to emphasize the notation of square and square root, and illustrated the concept of notations by a square image either explicitly or implicitly. For example, the picture presented below by Fanny demonstrates that 3 is the length of a side of the square with the area of 9. At the meanwhile, Fanny tries to help students to distinguish the concept of 3^2

from 3×2 (see Fig.2).

Not only Fanny, but other two student teachers Mandy and Jack mentioned explicitly that students might be confused the concept of the square of 3 with that 3 multiplies by 2. For example, Mandy justified the reason why she adopted the approach of introducing square numbers to students is to strengthen students' impression on the meaning of square, that refers to multiply by itself but not multiplying by 2.

KCT – Introduction of the notion "a^2= 9"

As for how to explain the equation: $a^2 = 9$, which is related to teach how to introduce the students about positive and negative square roots. The major approach that student teachers introduce this idea emphasizes the procedures. They highlighted the term "self -multiplication", and suggested to introduce students the concept that positive, positive turns out to be positive, and negative, negative turns out to be negative. Charles' approach is typical among other student teachers, since he wrote down 3×3=9 and (−3) × (−3) =9. However, not any visual representations were used by those student teachers to explain how to solve problems. Even through Gary was able to draw a picture to illustrate that "a" refers to the length of the side of a square whose area is 9, but he failed to use this similar image to explain why -3 is the square root of 9.

> Er… I might think of drawing a square. Three…three [is nine]…but I don't have ideas on how to draw the square with [the side] as negative 3? (Gary)

KCS and KCT – Reaction to a misconception

The student wrote down the negative sign in front of "a" in expression $(-a)^2$ =9 when he tried to solve a problem – what the negative value for square root of 9 is, what is the thinking behind him? The informants came up with two types of interpretations. One interpretation endorsed by three PTs – Charles, Fanny and Gary – is that the student is misled by the information "negative value". For instance,

> Em…he [the student] might not think that the value of this unknown number could be either positive or negative. He probably thought, taken it for granted, that a must be a positive number… because it is an unknown number, so the unknown number could be positive or negative? … but he did not think of the possibility that this number could be negative number. (Gary)

The second interpretation is endorsed by two PTs- Jack and Mandy and they attributed it as a piece of student's incorrectness.

> Well, in fact what he was thinking at that moment was what he had thought was totally wrong. I think he was empty in his mind. (Jack)

To respond this piece of student's thinking, the majority of informants just criticized this teacher's suggestion – ask the student to wipe off the negative sign. For example,

> The teacher should let him (the student) continue. In fact he (this student) was able to write this, why don't we let him finish it? That is … I think the teacher just wanted the student to write the equation…but I think that student have the whole plan in his mind,..., so we can

> talk about what was in his mind and helped him to clarify the misconceptions in terms of format. (Jack)

Other methods were regarding how to stimulate this student to think of "a" could be either a positive or a negative number. For example, Mandy tried to provide the student some hints,

How about the number a is -3? How about the square of (-3)? (Mandy)

CK – Mathematical explanations for why $\sqrt{(-4)^2} = -4$ is not true

All PTs can make a correct judgement that $\sqrt{(-4)^2} = -4$ is not true. The knowledge they apply is CCK, i.e., the radicand is non-positive so $\sqrt{(-4)^2}$ is not equal to -4.They also commented that there must be wrong in some steps in the expression: $\sqrt{(-4)^2} = (-4)^{2\times\frac{1}{2}} = (-4)^1 = -4$, however, no PTs was able to point out the mathematical reasons for why this method did not work. Some thought that it should solve $(-4)^2$ first, and then deal with $16^{\frac{1}{2}}$because the order of calculating matters.

DISCUSSION

The analysis of the five HK PTs' PCK in teaching this current topic shows that they adopted a procedural and a purely calculating approach to teach students square root. As evidenced in their approaches of explaining to students that $a^2 = 9$, the most of them try to explain in the way that 3 times 3 equals 9, and negative 3 times negative 3 equals 9, however, this approach cannot help students with algebraic thinking, since teaching them to substitute numbers 3 and -3 is kind of trial and error, yet nothing related to the generalization of patterns. Similarly, in responding students' question- adding one negative sign in front of a, the only approach that those student teachers employed was to provide the hints that, "-3 is the square root of 9" in order to emphasize that "a" could be either positive or negative. Their response to the KCS question demonstrated that those student teachers tended to interpret students' confusions from literal understanding; for some student teachers, it is even worse, they attributed it as students' incorrectness or lack of mind. Some evidences show that those PTs embrace students' previous knowledge in learning the topic square, that is, how to interpret the operational meaning of superscript 2 in 3^2, yet this is nothing to do with the content of square root.

The results from an analysis of those PTs' SMK show that they could have sufficient CCK in making judgment, yet the failure to answer student's enquiry why $\sqrt{(-4)^2} \neq -4$ reflects the weakness in their SCK in teaching this topic. It relates to their insufficient understanding of $\sqrt{\ }$, weak knowledge in how to apply index law and composite function. In the case: $\sqrt{(-4)^2} = (-4)^{2\times\frac{1}{2}} = (-4)^1 = -4$, those PTs seemed to overlook the fact that the index law cannot apply in the case $\sqrt{(\text{negative number})^2} = \left(\sqrt{\text{negative number}}\right)^2$, because of the properties of composite functions.

Here, $f: x \mapsto x^2$, and $f^{-1}: x \mapsto \sqrt{x}$, ideally by the use the concept of composite function, we have $f \circ f^{-1}(x) = f[f^{-1}(x)] = x$, but making x, $f \circ f^{-1}(x)$ and $f[f^{-1}(x)]$ equal only if $f^{-1}(x)$ is well defined. However, in this case $f^{-1}(x) = \sqrt{x}$ is undefined when $x < 0$.

The lack of adequate SMK especially SCK could be the major reason to explain why their PCK in teaching this algebraic topic is procedural. Algebraic thinking involves the understanding of roles and properties of variables, and relevant operations among those variables. Learning algebra we often go from a less abstract state to a more abstract state. Mitchelmore and White (2000) identified the learning stages in term of the intensity of abstraction, namely *familiarization, similarity recognition, rectification and application*. In this current case, knowing computationally that the square of the number 3 or -3 is 9 learners only reach the *familiarization* and *similarity recognition* levels. While, the rectification level is only reached when learners identify that we can only take the positive sign when taking square root of a number. It is because we treat *squaring-taking square root* as a pair of function and its inverse. Knowing what constraint is in there when writing $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$ will be in the level of *application* because the domain of the inverse function can only be applied to positive real numbers. However, when PTs' levels of abstraction cannot reach in rectification and application, how could they possess high quality of PCK that facilitates students to develop algebraic thinking?

CONCLUSION

In spite of the limitations, this study highlights the role of SMK especially SCK or EMFAS plays a significant role in those HK PTs' PCK in teaching the topic of square and square root. Consistent with the study conducted by Buchholtz et al. (2013), the results gained from current study show that those HK PTs could not connect relevant university mathematics with current topic. Lack of adequate knowledge of algebraic operation and functions leads those student teachers teach this algebraic topic in a procedural way. In addition, this study provided another perspective to evaluate the quality of PCK from the perspective of SCK and EMFAS. We hence rethink of the construct of PCK, which cannot be apart from CK especially SCK or EMFAS, which is more important than CCK in facilitating mathematics teachers to teach students with more conceptual understanding.

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EXPLORING THE RELATIONSHIP BETWEEN EXPLANATIONS AND EXAMPLES: PARITY AND EQUIVALENT FRACTIONS

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Examples and explanations are inherent elements of mathematics learning and teaching. This study explores the relationship between examples and explanations given for the same concept. Results indicated that for the concept of parity, fifth grade students offer different explanations for different examples. However, for the concept of equivalent fractions, algorithmic explanations were most preferred.

INTRODUCTION

Mathematical concepts are complex and multi-faceted. That is, there may be different equivalent ways of defining a concept, but they all necessarily lead to one set of critical attributes. While mathematically, all of the critical attributes of a specific concept are equally essential and should be equally attributed to that concept, psychologically, they may not be the same. On the one hand, students often associate non-critical attributes with some concept; on the other hand, students may associate a concept with a shorter list of critical attributes than it truly has (Hershkowitz, 1990). How can we help students attain a fuller range and more encompassing recognition of all critical attributes for a mathematical concept?

Several studies have focused on the roles of examples in concept formation and expanding a student's accessible example space (e.g., Watson & Mason, 2005). Yet, recognizing a wide set of examples as being instances of some concept, does not necessarily mean that the student will also recognize a wide set of critical attributes as belonging to that concept. Another seemingly separate line of research related to learning mathematical concepts, is research concerning explanations. Previous studies have focused on the types of explanations used by teachers and students (e. g., Bowers & Doerr, 2001) as well as the sociomathematical norms related to giving and evaluating mathematical explanations (Levenson, Tirosh, & Tsamir, 2009). While those studies are perhaps implicitly related to concept development, they do not focus specifically on how explanations may be related to examples and how this relationship may inform us of students' conceptualizations.

According to Watson and Chick (2011), in order for students to learn from examples and see what an example could be an "example of", a process must take place which includes seeking relations between elements of an example. This study explores the possibility that explanations can play a role in this process, that examples may help students "focus mindfully" on the examples (p. 285). It seeks to combine research related to examples with research related to explanations, and to explore the relationship between examples and explanations given for the same concept. Using two mathematical contexts, parity and equivalent fractions, the following questions are investigated: Do different examples give rise to different explanations, or are some explanations more prevalent than others, regardless of the different examples? Do students consistently give the same explanation for a concept despite being shown different examples of that concept or do they give different explanations for different examples?

EXAMPLES AND EXPLANATIONS

One key to using examples in concept formation may be variation. Zodik and Zaslavsky (2008) refer to Watson and Mason's (2006) discussion of variation in structuring tasks, and suggest a similar way of structuring examples. Just as some features of a task may vary while others are kept constant, so too with examples. The presentation of examples should be structured in such a way as to highlight relevant features while downplaying non-critical attributes. Rowland (2008) also claims that we learn from discerning variation and that "the provision of examples must therefore take into account the dimensions of variation inherent in the objects of attention" (p. 153). Learners need to be aware of which attributes of a concept can be varied and which cannot. At different times in the learning process, students may be aware of different dimensions which may be varied, sometimes claiming an unnecessarily restricted sense of possible variations (Goldeberg & Mason, 2008). In his study of examples in the teaching of elementary mathematics, Rowland (2008) found that some examples may obscure the role of variables, making it more difficult for students to learn from those examples. Recognizing the significance of teachers' choice of examples, Zodik and Zaslavsky (2008) investigated teachers' considerations in choosing examples. Their study pointed to several considerations, including choosing examples that will draw students' attention to relevant features.

Like examples, explanations are also used every day in the mathematics classroom and may have several functions. Explanations may answer a "how" question, and describe the procedure used to solve a problem, or they may answer a "why" question where the underlying assumption is that the explanation should rely on mathematical properties and generalizations (Levenson, Barkai, & Larson, 2013). Similarly, calculational explanations describe a process, procedure, or the steps taken to solve a problem. Conceptual explanations describe the reasons for the steps, which link procedures to the conceptual knowledge of the student (Bowers & Doerr, 2001). Hemmi, Lepik, and Viholainen (2013) inferred that explanations are an important part of problem solving and reasoning processes and may be used to make mathematical connections clear, even among young students. Explanations may also be given to rationalize actions, both for the giver of the explanation as well as for the receiver (Krummheuer, 2000). Nunokawa (2010) claimed that explanations not only communicate student's existing thoughts but may also generate new objects of thought by directing new explorations which may then deepen the student's understanding of the problem at hand. Thus, an underlying function of explanations is to expand students' mathematics learning.

METHODOLOGY

Tools and procedure

Two questionnaires, the parity questionnaire and the equivalent-fractions questionnaire, were handed out to 71 fifth-grade students. Both concepts had been introduced to students previously and it was expected that students would be familiar with both concepts at the time of the study. The parity questionnaire was handed out in the beginning of the year. On the parity questionnaire, students were asked to consider whether the integers 14, 9, 286, and 0 were even or odd and to explain their reasoning. The first two integers, 14 and 9, were chosen because they are relatively small natural numbers that elementary students can easily relate to, envision, and manipulate. By choosing one even and one odd number we could discern if the types of explanations given were related to the parity of the number. Zero was chosen because of the much researched difficulties, among students and teachers alike, conceptualizing and operating with this number (e.g., Anthony & Walshaw, 2004). The fourth number, 286, was chosen because it is not a number that is usually encountered by children in a day-to-day context and in order to see if there would be a difference between students' explanations for a two-digit number and a three-digit number. The fractions questionnaire was handed out approximately three months after the parity questionnaire. On the fractions questionnaire, students were asked to assess the equivalence of three pairs of fractions, 2/4 and 6/12, 5/15 and 10/30, 0/4 and 0/12, and to explain their reasoning. The first pair was chosen because it was thought that both fractions would be familiar to children, and like the numbers 14 and 9, they would be easy to relate to, envision, and manipulate. The second pair was chosen because they both reduce to one-third but students can also easily expand 5/15 to 10/30. The last pair was chosen because of its involvement with zero, as described above.

Analyzing the data

Students' assessments of the parity of the given numbers and of the equivalence of the given pairs of fractions were coded for correctness. Only explanations associated with correct responses were analysed further. On the parity questionnaire, four categories of explanations emerged from the data (See Table 1). Three explanations were mathematically-based (i.e., explanations based on mathematical definitions or previously learned mathematical properties, often using mathematical reasoning) and one was practically-based (PB). Some explanations were unequivocally wrong. For example, one student wrote that 9 is an odd number because "it is a prime number." Some explanations could not be categorized, such as "When you divide 14 by 7, nothing will stand by itself." These types of explanations were categorized as "other". On the fraction explanation, five categories emerged (see Table 2). As with the parity questionnaire some explanations (e.g., "2/4 is equivalent to 6/12 because all the numbers are even") were either invalid or incomprehensible and could not be categorized. Two experts in the field of mathematics education validated the categorization of explanations for each concept.

Theoretically, each type of explanation could be used for each example. For example, the parity of each number could be explained by claiming that it was or was not divisible by two. In order to assess students' tendencies to be consistent when explaining a concept, the explanations each student gave for the different examples were compared. For example, one student claimed that "14 is even because it's divisible by 2", "9 is not even because it is not divisible by 2", "0 is even because when you stand on 2 and jump backwards 2 steps you land on 0", and "286 is even because 6 is even." That student was consistent regarding the explanations given for parity of 14 and 9, but not consistent with regard to 14 and 0, and with regard to 14 and 286. If we consider all three even numbers together, we would say that the student was not consistent regarding the explanations given for even numbers. A similar comparison was carried out for the explanations given on the fractions questionnaire.

Categories	Students wrote…
Divisible by 2: An even number is divisible by 2, is a multiple of 2, or can be expressed as the sum of 2 equal whole numbers.	"14 can be written as 7 + 7", "14 is an even number because it's divisible by two without a remainder."
Number line: Even numbers are alternating whole numbers on the number line when you start with 0.	"When you start from 0 on the number line, jumping by twos, we end up standing on the number 14."
Last digit rule: If the ones digit of a number is even, then the whole number is even.	"14 is even because 4 is even."
Practically-based: Explanations that use daily contexts, drawing, and/or manipulatives.	"14 is even because if I want to give out 14 pencils to two children, each one would get the same amount of pencils."

Table 1: Categories of parity explanations

Categories	Students wrote…
Equal to the same number: Equivalent fractions are two fractions that represent the same number.	"5/15 and 10/30 are both equal to 1/3"
Algorithmic: Expansion or reduction of one or both fractions.	"If you multiply (both the numerator and denominator in 2/4) by 3, you get 6/12."
Numerator/denominator (N/D) relationship: The denominator is a multiple of the numerator.	"4 is twice as much as 2 and 12 is twice as much as 6."
Zero is nothing: relates zero to nothing.	"Because in both of them there is nothing."
Practically-based (PB): Explanations that use daily contexts, drawing, and/or manipulatives.	"Six and two are equal only with smaller pieces." (This explanation refers to the "pie" diagram where each "piece" is an equal fractional part of the whole pie.)

Table 2: Categories of equivalent fractions explanations

RESULTS

Parity questionnaire

As shown in Table 3, nearly all, of the students knew the parity of 14, 9, and 286. However, as expected, zero was a cause for confusion. Because one of the aims of this study was to investigate the variability or, conversely, the consistency of explanations for different examples, only the explanations of students, who evaluated correctly the parity of all four integers, were examined. This led to a final sample of 56 students.

	14	9	0	286
Correct evaluations	71 (100)	70 (99)	57 (80)	68 (96)

Table 3: Frequencies (in %) of correct evaluations per integer (N=71)

Table 4 summarizes the results of how children explained the parity of each integer. Taking into consideration a total of 188 valid explanations, 35% of those explanations were based on divisibility by two, 34% were based on the number line, 29% were based on the last-digit rule, and 2% were practically-based. In other words, none of the categories stood out as being truly dominant over the others. On the other hand, for each integer, there seemed to be one type of explanation which was used more frequently than the others. For example, most students explained the parity of 14 by writing that it was divisible by two, while most students explained the parity of zero by its placement on the number line.

	Divisible by 2	Number line	Last-digit rule	PB	Other
14	23 (41)	11 (20)	16 (29)	-	6 (11)
9	22 (39)	17 (30)	2 (4)	2 (4)	13 (23)
0	8 (14)	31 (55)	2 (4)	1 (2)	14 (25)
286	13 (23)	5 (9)	35 (63)	-	3 (5)

Table 4: Frequencies (in %) of types of explanations per integer (N=56)

With regard to consistency, only valid comprehensible explanations were considered, leading to a sample of 42 students. Explanations for the two, relatively small and familiar integers, 14 and 9, were compared first. Following that comparison, explanations for each two even numbers were compared, and finally the explanations for all three even numbers were compared. The highest consistency of explanations occurred when explaining the parities of 14 and 9 and 14 and 286. While over a third of the students offered the same type of explanation for 14 and 0, only 19% of the students offered the same type of explanation for 0 and 286, leading to a relatively low consistency rate for all three even numbers.

Examples	14, 9	14, 0	14, 286	0, 286	14, 0, 286
Consistent explanations	27(64)	18(43)	25(60)	8(19)	7(17)

Table 5: Frequencies (in %) of consistent explanations for groups of integers (N=42)

Fractions questionnaire

Out of the 71 students who filled in the parity questionnaire, 66 students also filled in the fraction questionnaire. Results of students' assessments are summarized in Table 6. Once again, introducing zero into an example seemed to cause difficulties.

	2/4=6/12	5/15=10/30	0/4=0/12
Correct evaluations	65 (98)	63 (95)	48 (74)

Table 6: Frequencies (in %) of correct evaluations of equivalent fractions (N=66)

There were 48 students that knew that all three pairs of fractions were equivalent. Their explanations are summarized in Table 7. Of the 120 valid explanations, 48% were algorithmic, 31% related to the fractions being equal to the same number, 10% related to the relationship between the numerator and denominator (N/D), 7% were PB, and 4% considered the "zero is nothing" analogy. In other words, there seemed to be a clear preference for algorithmic explanations. When looking at the explanations given for the different examples, algorithmic explanations were most prevalent for the example 5/10=10/30. However, no clear preference of one type of explanation was found for the other examples. Interestingly, more PB explanations were used to explain why 0/4=0/12 than for any other example, perhaps indicating the need for students to relate zero to something concrete in order to comprehend this equivalence.

	Equal to the same number	Algorithmic	N/D	Zero is nothing	PB	Other
2/4=6/12	17 (35)	19 (40)	4 (8)	-	1 (2)	7 (15)
5/10=10/30	8 (16)	25 (52)	8 (16)	-	1 (2)	6 (13)
0/4=0/12	12 (24)	14 (29)	-	5 (10)	6 (13)	11 (23)

Table 7: Frequencies (in %) of types of explanations per pair of fractions (N=48)

With regard to consistency, 32 valid explanations were considered (see Table 8). Surprisingly, few students explained why 2/4=6/12 in the same way as they explained why 5/15=10/30. Also note, the consistency rate for all three examples of equivalent functions (19%) was similar to the consistency rate for all three examples of even numbers (17%).

Examples	2/4=6/12, 5/15=10/30	2/4=6/12, 0/4=0/12	5/15=10/30, 0/4=0/12	2/4=6/12, 5/15=10/30, 0/4=0/12
Consistent explanations	8 (25)	15 (43)	17 (50)	6 (19)

Table 8: Frequencies (in %) of consistent explanations for equivalent fractions (N=32)

Comparing the contexts

Finally, when considering both mathematical contexts, 32 students correctly evaluated all of the tasks and offered valid explanations. Out of those students, only four (12%) were consistent in the types of explanations they gave for both contexts, giving the same type of explanation to explain the parity of the three even numbers and the same type of explanation when explaining why the three pairs of fractions were equivalent.

SUMMARY AND DISCUSSION

Do different examples give rise to different explanations, or are some explanations more prevalent than others, regardless of the different examples? The answer to this question may be dependent on the context. With regard to parity, different examples elicited different explanations. Within the context of equivalent fractions, nearly half of all the fractions explanations were algorithmic. While algorithmic knowledge is an essential element of mathematics knowledge (Fischbein, 1993), this type of explanation sheds little light on students' conceptualization of fractions and, specifically, equivalent fractions. When explaining why 2/4=6/12, over a third of the students pointed out that both of these fractions were equal to 1/2 and therefore equal to each other. Perhaps because this example is intuitive and close to the student's world, it was helpful in bringing out this explanation. This explanation focuses on the conceptualization of equivalent fractions as being different representation for the same quantity. On the other hand, the N/D explanation focuses on the conceptualization of a fraction as a ratio. Interestingly, this explanation hardly came up. Perhaps a different example, such as 3/9=5/15 would have elicited this explanation. Perhaps, in line with Watson and Mason's (2006) theory of task variation, different tasks might expose different relationships between examples and explanations.

Do students consistently give the same explanation for a concept despite being shown different examples of that concept or do they give different explanations for different examples? For both contexts, the general answer to this question was no, most students do not give the same explanation for different examples. However, upon a closer look, on the parity questionnaire, most students were consistent, not only when explaining the parity of 14 and 9, two relatively small numbers, but also when explaining the parity of 14 and 286. It was the introduction of zero, which caused students to think of other ways of explaining the parity of this number. On the fractions questionnaire, the opposite occurred. More research is needed in order to discern which examples will lead students to seek different explanations and which examples will lead students to use which explanations. Such research could be helpful to teachers planning lessons, as well as for researchers planning studies which involve the use of examples.

Mathematically, there is no reason to use different explanations when explaining why various instances of some concept are all examples of that concept. In fact, consistently using the same explanation may be seen as a sign of mathematical maturity. This study showed, however, that most students, at least during their younger years, do not consistently use the same explanation for each example. Zodik and Zaslavsky's (2008) study noted that teachers consider several issues when choosing examples to present in class. However, the types of explanations that different examples may elicit from students, was not considered. Students may need assistance in recognizing the generality of instances and in finding relationships and connections between examples. Recall that explanations may be used to help make mathematical connections clear (Hemmi, Lepik, & Viholainen, 2013) and deepen students' understanding (Nunokawa, 2010). Considering that different explanations may be based on different ways of conceptualizing a concept and may emphasize different attributes of a concept, teachers, as well as researchers, may consider the combination and integration of examples and explanations, when choosing which examples, and which explanations, to present to students in class.

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HOW A PROFESSOR USES DIAGRAMS IN A MATHEMATICS LECTURE AND HOW STUDENTS UNDERSTAND THEM

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Mathematics education literature suggests that diagrams should be included in mathematics lectures, however few studies have empirically studied the use of diagrams in the undergraduate classroom. We present a case study investigating the use of diagrams in a university lecture and how students in the class understood them. Three archetypes of student understanding of diagrams are described and illustrated.

INTRODUCTION

Although one of the main objectives of advanced undergraduate mathematics courses is to help students learn to construct and understand proofs, mathematics majors have difficulty constructing proofs (e.g., Weber, 2001) and determining if a proof is correct (e.g., Selden & Selden, 2003). One possible way of investigating the sources of these difficulties is to consider how students are taught proof in these courses. In particular, given discussions on the importance of diagrams and informal arguments in the learning of mathematics and the construction of proof (e.g. Alcock, 2010; Thurston, 1994), some have called for the use of diagrams in lectures for undergraduate students (e.g. Zimmerman & Cunningham, 1991; Alcock, 2010).

In their review of the literature, Speer, Smith, and Horvath (2010) highlighted the dearth of research on college-level classroom teaching practices in mathematics. While some studies on undergraduate mathematics classrooms exist (Weber, 2004; Mills, 2012; Fukawa-Connelly & Newton; in press), there is a lack of research on how mathematics professors use diagrams in their lectures and the extent to which diagrams enhance students' understanding. The present study addresses these issues.

Theoretical Perspective

The literature outlines various theoretical benefits of using diagrams when presenting both definitions and proofs in the classroom. Using diagrams in the presentation of new definitions may enable students to develop an intuitive understanding of the definition (Vinner, 1991), perceive the connections between the formal symbolism of a definition and conceptual understanding of the definition (Zimmerman & Cunningham 1991), develop intuition of whether or not related conjectures are true (Vinner, 1991), and prove related conjectures (Vinner, 1991). Using diagrams in the presentation of proofs may enable students to gain an intuitive sense of why a statement is true (Barwise & Etchemendy, 1991), understand steps within the proof (Barwise & Etchemendy, 1991), and prove similar theorems using similar diagrams (Tall, 1991). While we make no claims that this list is exhaustive, we used these potential benefits to frame our investigation into how students understood diagrams.

Research Questions

We consider the diagrams used in a lecture introducing the Riemann integral in an undergraduate real analysis course with the following research questions: 1) How did the professor use diagrams in this lecture and for what purpose? 2) What did the professor intend to convey by presenting these diagrams? 3) How did students interpret the diagrams and pictures that were presented in this lecture?

DR. A

The context for this case study is a real analysis course at a large public research university in the U.S. The course was taught by Dr. A (a pseudonym), a professor of mathematics with over three decades of teaching experience at the university level and a history of receiving high student evaluations. Dr. A had a reputation within the department of being a thoughtful and careful lecturer who frequently used diagrams in his lectures.

We videotaped a lecture in which Dr. A presented six diagrams. In this paper, we focus on the two diagrams presented in Table 1 (the diagram used when presenting the definition of upper and lower sums given a partition and the diagram used when presenting a proof of the claim that $\int_0^1 x\,dx = \frac{1}{2}$).

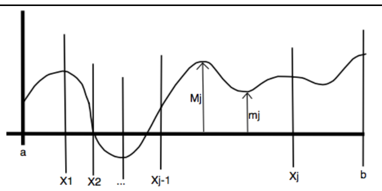
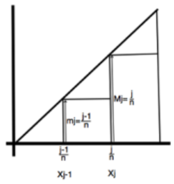
Diagram presented with the definition of upper and lower sums	Diagram presented with the proof of the claim that $\int_0^1 x\,dx = \frac{1}{2}$
	

Table 1: Diagrams presented by Dr. A

Dr. A was interviewed on his use of the diagrams in Table 1 and on his opinion on the use of the diagrams in mathematics in general. Dr. A was first asked why he chose to include the definition/proof and its associated diagram, and what he hoped to convey through their use. Dr. A was then was asked if he had hoped to convey each of the benefits discussed in our theoretical perspective, both through his general use of diagrams and, in particular, through his use of each of the two diagrams in Table 1.

Dr. A's Interview

In his interview, Dr. A reported having used the diagram illustrating the concept of upper and lower sums in order to help his students "associate concepts' symbols with geometrical pictures." He noted:

> The upper sum is approximation of the area by rectangles, which are larger than the area under the graph and the approximate by lower sums, again an approximation by rectangles, which have less area than the region under the graph of the function.

His goal of presenting this diagram was to convey the fact that upper and lower sums are approximations of area, since this concept will be essential when defining the

Riemann integral. When probed about the potential benefits of using diagrams, Dr. A agreed that he hoped the diagram would help his students develop a sense of intuition of the definition and prove related conjectures.

Next, Dr. A explained that he presented the proof of the proposition $\int_0^1 x\,dx = \frac{1}{2}$ as an example of using the approximation procedure that he outlined in the lecture. Dr. A reported that his goal in presenting this proof was to provide:

> A function where the areas a pretty clear, in the approximating rectangles can be easily seen to give the inequalities. … To give [the students] a concrete function to look at.

When probed about the benefits of using diagrams with proofs (listed in our Theoretical Perspective), Dr. A agreed that he hoped to convey each of these to his students through his use of diagrams including helping students write proofs about this concept. He suggested asking students to prove $\int_0^1 x^2\,dx = \frac{1}{3}$ would be an appropriate task to test students' understanding of his lecture.

STUDENT PARTICIPANTS

Five student participants for this study were recruited from Dr. A's class. Each of the students was pursuing either a major or minor in mathematics—the ages of the students varied from first to fourth years at the university. The goal of these interviews was to see how students understood the diagrams presented in the class and if the diagrams conveyed the mathematical insight that Dr. A intended. Each student was interviewed individually.

In the first task, we wanted to see how the participants understood the definition diagram use in lecture and whether this diagram conferred the benefits described in our theoretical perspective. Participants were first given a prompt with Figure 1 and were asked to draw the upper and lower sums on the partition, provide the definitions of upper and lower sums, and explain how the diagram was related to the definitions. Finally, to determine if the participants could use the diagram to infer properties about these concepts, each participant was asked what would happen to the sums if more points were added to the partition. If participants struggled with the first task, they were given the option to watch the video of Dr. A's presentation of the definition of upper and lower sums.

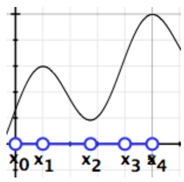


Figure 1: Diagram for the upper and lower sum task

In the second set of questions, we investigated how well students understood the proof that $\int_0^1 x\,dx = \frac{1}{2}$, particularly in relation to the diagram that Dr. A introduced in his lecture. After watching a video of Dr. A's proof presentation, participants discussed what they thought the professor was trying to convey in his presentation of the proof, what they remembered about the diagram, how the diagram affected their understanding of the proof, and what they thought the professor was trying to convey with the diagram. Each participant also was asked to explain Dr. A's diagram, including which parts of the diagram connected to which parts of the proof. Finally, each student was given the task of proving $\int_0^1 x^2\,dx = \frac{1}{3}$ which Dr. A thought students should be able to do if they understood the lecture.

Three Ways that Students May Understand Diagrams

We analyzed the student data to investigate how the students understood the diagrams from the lectures and to see the extent to which the students gained the insights the professor wished to convey. For the analysis of these data we followed the quasi-judicial procedure developed by Bromley (1986) for case study research, focusing on common patterns of student behavior to ultimately categorize them as cases of a certain type. The findings suggest three archetypes of student understanding of the diagrams presented in their course lectures: incoherent understanding, instrumental understanding, and integrated understanding.

Incoherent understanding

A student with an incoherent *understanding* of a diagram does not have a coherent understanding of how the components of the diagram relate to the formal mathematical theory. As a result, the student's responses to questions are geared toward imitating the behavior of the professor that he or she had previously witnessed. Three students evinced this type of understanding, which we illustrate with D3.

During the first task, D3 was able to correctly draw the upper and lower sums. However, when asked what information the graph provided, D3 responded:

> Well if I have both [the upper and lower sums], I could see that it will trace the function because if you put them on top of each other… It's basically this [upper sum] area minus this [lower sum] area and I feel like, I think it would give you this line [the function].

When asked to relate the graphs and definitions, D3 attempted to recall reasoning previously seen, "well, all I remember—all I keep thinking about is the function they give you, which is the upper sum minus the lower sum." Comments such as these reveal D3's belief that the difference of the upper and lower sums yields the function itself, illustrating D3's inability to connect the diagram to formal theory. Clearly D3 did not view the areas as approximations of the integral, as Dr. A intended.

Later, the student explained why the task was so difficult: "because I mean, during class we've never done any exercises like this. So I was really intimidated by like, I don't know, am I doing it correctly or not?" Feeling unfamiliar with the task, D3 had difficulty deciding how to respond to the task. D3's attempt to recall the reasoning presented in lecture and her inability to judge whether her responses made sense suggest D3 was relying on imitative reasoning.

Instrumental understanding

A student with an *instrumental understanding* of a diagram views the diagram as a tool to accomplish specific types of tasks, but does not understand the justification for why using this diagram yields the desired solution. Thus, following Skemp (1978), we say this student has an instrumental understanding of the diagram—the student knows what to do with the diagram to complete some tasks, but does not know why the solution is correct. In this archetype, the student does not have a strong understanding of how the diagram relates to the deductive mathematical theory. Hence, although the student may be able to flexibly use the diagram to accomplish some tasks, the student would not be able to draw novel inferences from the diagram, use the diagram to decide whether a statement is true or false, or connect the diagram to the logic of a proof that he or she observed. We illustrate this archetype with D2.

When asked how the upper and lower sums would be affected by a refinement of the partition, D2 reported that the upper sum would increase and the lower sum would decrease. When asked why this would occur, D2 explained, "as we increase… these areas [indicating areas between the lower sum and the curve of the function] will also increase, and also the denominator will also increase". This clearly illustrates a misunderstanding of how a refinement adjusts the upper and lower sums.

Next, when asked what the professor was trying to convey with the presentation of the proof that $\int_0^1 x\,dx = \frac{1}{2}$, the student responded "I think he's trying to show us how to prove that the… difference of the lower integral and the upper integral can be made small enough to show the area." When probed further:

Interviewer: Okay. Umm, is there anything else, or is that it?

D2: So that, that's it. Just the technique of how to show it.

We see that D2 believes the sole purpose of proof presentation is for the professor to communicate particular proving techniques to students.

Despite having a flawed understanding of how a refinement affects the upper and lower sums, D2 correctly produced a proof showing that $\int_0^1 x^2\,dx = \frac{1}{3}$. D2's description of how the diagram helped the proof construction highlights both the student's ability to relate the diagram to the proof and imitate reasoning:

> So for this I was just concentrating on the, how the curve would look like and what would be the relation of the upper and the lower, of the maximum and the min compared to the normal function, say like *x*. So like, since we could compare this function to *x*, I just had that in mind so we could use that partition.

The student further clarified that the professor's example had been in mind during D2's proof construction. D2 compared his diagram to Dr. A's proof diagram and appropriately adjusted the argument to construct a complete proof. D2 was successfully able to relate the diagram to the high-level ideas of the proof, utilizing Dr. A's reasoning to construct a similar proof. We note while D3 and D2 both illustrate imitative reasoning, they do so in different manners. D2 used Dr. A's reasoning and diagrams and adjusted the arguments to fit the new task, constructing a complete proof. This differs from D3's actions, which relied on mimicking exact actions and reasoning observed, regardless of the logical consequences.

Integrated understanding

A student with an *integrated understanding* of the diagram can use the diagram both to instantiate mathematical objects and mathematical logic; this student can form strong links between inferences drawn from the diagrams and deductive inferences drawn from the formal theory. One would expect that a student with an integrated understanding would be able to specify the components of a diagram, make inferences connecting the diagram and formal mathematical theory, and instantiate and apply the reasoning to proofs they observed and they wrote. So, not only is the student able to describe the mathematical objects being discussed at a basic level, but he or she is also able to build on the concepts. We illustrate this archetype with D1.

When asked how the student's diagram of the upper sum would be affected by a refinement, D1 was able to both relate the objects of the diagram to the formal theory and make inferences from the diagram. D1's responses throughout the first task demonstrated a clear understanding of upper and lower sums. However, despite D1's integrated understanding of the diagrams, D1 was unable to construct a complete proof of the claim that $\int_0^1 x^2\,dx = \frac{1}{3}$. D1's proof attempt began with choosing the partition of n^2 sub-intervals of length $\frac{1}{n^2}$. While correctly splitting the interval from 0 to 1 into sub-intervals with equal widths, this caused confusion when D1 did not correctly incorporate this when plugging in the maximums and minimums in the equations of the upper and lower sums, preventing D1 from constructing the proof. Nevertheless, when the interviewer asked what D1 was thinking while attempting to construct the proof, D1 explained why the integral should exist:

> Since your function is monotone increasing, every time you define a partition… [each rectangle is] going to the be upper for the one previous to it and the lower for the one after it… So as long as the partition is equidistantly spaced … You only have the last upper partition to consider. [Which] is just going to be the function value at that point, which is 1 times this infinitely thin slice, which is going to be $\frac{1}{n}$ as *n* goes to infinity so it should be nothing… it's monotone increasing so I'm always going to have this property.

D1 explained the monotonicity of the function leads the difference between the upper and lower sums to telescope to $\frac{1}{n}$, which goes to zero as n goes to infinity. Not only did this explanation demonstrate D1's ability to infer from the context of the diagram to the formal theory and to describe the relationship between them, but also justified the student's unusual choice of partition. Moreover, this monotonicity argument was not presented in lecture, illustrating D1's ability to make inferences and build further on concepts presented by the professor.

In Table 2, we present behaviors one may expect a student to exhibit as a result of their understanding archetype.

	Articulating the diagram's components	Inferring from the diagram	Linking the diagram to reasoning/proofs
Incoherent Understanding	One has an unstable or inconsistent interpretation of the diagram depending on the task evoked.	One may draw incorrect inferences, due to inconsistent understanding of diagram's components.	One cannot link the diagram to proofs in meaningful ways, since one may view proof as tightly tied to context.
Instrumental Understanding	One could specify the components of a diagram that relate to the mathematical objects discussed.	One may draw incorrect inferences, since understanding may not be integrated to the mathematical theory.	One may be able to relate the diagram to the high level ideas of the proof, but not specific logic of the proof.
Integrated Understanding	One could specify the components of a diagram that relate to the mathematical objects discussed.	One can draw inferences from the diagram, that are consistent with the formal theory.	One can use the diagram to instantiate reasoning and as a tool to construct proofs.

Table 2: Expected outcomes from the understanding archetypes

DISCUSSION

In this report, we presented a case study in which we studied how and why diagrams were used in a real analysis lecture by a highly regarded instructor, as well as how students understood the diagrams presented. In particular, we outlined three archetypes of how students may understand diagrams. There are two important observations that we have made. First, three of the five participants evinced an incoherent understanding of the associated diagrams. In particular, they were unable to see the partition diagram as representing an approximation of the area under the curve. Recall that Dr. A had a reputation as an excellent lecturer who valued diagrams. That three of five students had such a flawed understanding of the diagrams Dr. A used in his lecture illustrates the difficulties of incorporating diagrams into lectures and suggests that for many students, the presence of diagrams in lectures might not improve comprehension (cf., Alcock, 2010).

Second, we note that students might be able to use a diagram instrumentally to accomplish proving tasks without any deep understanding. We described D2, who flexibly used his diagram to prove a statement that he had not seen before. As proof is the primary means to assess performance in advanced mathematics, we imagine a professor would take D2's proof as evidence a deep understanding of the material. However, as we observed, he thought a refinement would increase the gap between upper and lower sums, implying that he could not possibly see how his proof established the existence of a Riemann integral. We contrast this with D1, who could not construct a proof despite seeming to have an integrated understanding of the diagrams. This reminds us that proof writing requires technical and algebraic expertise to complement the conceptual insights one might gain from a diagram.

Due to the small scale of the study, we make no claims of the exhaustive nature of the list of archetypes. We believe further research is necessary to investigate other archetypes for understanding diagrams and the proportion of students who fit each archetype. Such research would inform our understanding of the extent that diagrams can be used to improve understanding in lecture and how lectures might be improved.

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THE INCIDENCE OF DISAFFECTION WITH SCHOOL MATHEMATICS

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This paper reports the results of a study of affect and school mathematics, conducted with a whole cohort of pupils in year 9 (pupils aged 15 to 16 years) in a typical UK secondary school. This preliminary study provided contextual datafor a larger mixed methods investigation into disaffection. The study offers interesting insights into the incidence of affection/disaffection within this group. Further, since the school grouped pupils by ability, the study offered the opportunity to look at the distribution of aspects of affect and disaffection across the ability range. Results not only provide contextual data on the incidence of disaffection with mathematics amongst school pupils of this age, it also suggests some interesting tentative conclusions.

INTRODUCTION

Concern about disaffection with school mathematics is not new. There has been a widespread appreciation that it presents a problem for individuals and for society. The State of the Nation Report into Science and Mathematics Education by the Royal Society notes the widespread nature of current concern: "no decade since the 1970's…has seen so much being written about the disaffection young people appear to have for science and mathematics." (The Royal Society, 2008 p.171)

Many of these concerns relate poor attitude or disposition to mathematics to poor outcomes and achievement. In this way, the study of affect in mathematics education becomes important. In her own report, Vorderman (2011) talks about the corrosive effect of frequent failure, and the damage that this causes. The report espouses an approach to mathematics education that goes wider than the purely utilitarian. It talks about 'entitlement', and mentions not just achieving success, but also of 'satisfaction' and of 'increased confidence and motivation' (p.22). This wider rationale for studying mathematics in schools is endorsed by The Royal Society who suggest one of the purposes of learning mathematics is:

> To enable as many students as possible to participate in the scientific and mathematical elements of the conversation of humankind, in as many settings as possible. (The Royal Society, 2008, p. 21)

Considering the importance of the issue of disaffection as outlined here, there is not the volume of research that would seem appropriate to the social and individual impact that has been reported. The Royal Society State of the Nation report (2008) into mathematics points out that there has not been enough quality of research into this area, and cites only three studies (Brown, Brown, & Bibby, 2007; Mathews & Pepper, 2005), (Nardi & Steward, 2003) in relation to mathematics. Much of this evidence is

2014. In Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 4, pp. 121-128. Vancouver, Canada: PME.

concerned with progression, and thus with the incidence of disaffection, and is framed as the quantitative study of attitude or related constructs. The study by Nardi and Steward is the exception in that it goes further than other studies in addressing disaffection directly as an issue of significance, and in trying to characterise the construct in research terms.

In international terms, one of the starting points of the study by Zan and Di Martino (2007) is what they called an ‘alarming phenomenon’: the perceived negative attitude of students of mathematics to the subject. Three core themes emerged from the study, and these related to emotion (‘I like/don’t like maths), competence or efficacy (‘I can/can’t do maths’) and belief (‘Mathematics is…..’). Strong associations were found between liking and being able to do mathematics.

Other trends have also emerged more recently from the quantitative study of attitude. For instance, Noyes (2012) has remarked on the significant inter-group as well as inter-school differences found in data on affective variables studied in UK schools, suggesting that the teacher is a key influence in pupil’s experience of school mathematics. Further evidence of the important influence of individual teachers was also evident in a study I conducted with a colleague (Lewis & Forsythe, 2012). Other researchers, such as Boaler (2000) have pointed out that it is not necessarily the case that just low attaining pupils have negative attitudes to mathematics, since she also observed this with pupils in higher sets.

METHODS

My doctoral study was an investigation into the nature of disaffection with school mathematics. It was primarily a qualitative study into the subjective experience of students and pupils who report disaffection. It was conducted within an interpretivist and constructivist frame, and was focussed on issues of motivation and emotion as being central to young peoples’s experience of school mathematics. Preliminary results have been reported elsewhere (Lewis, 2013).

However, I had the opportunity to conduct a brief preliminary and quantitative study of a whole school cohort, in order to provide contextual data on the incidence and nature of aspects of affect. Since it was necessary for the study to be simple and bounded, I devised a simple instrument based on the core themes identified in the Zan and Di Martino study (2007), as described above.

The school is a comprehensive foundation 11-19 school in the UK with approximately 1300 pupils. The proportion of pupils who take free school meals is described as ‘average’ in the 2011 Ofsted report, and only 6% of pupils are from ethnic minorities. The school was rated as ‘good and rapidly improving’. In just two visits to the school I was able to survey the whole of the year 9 population of this school (n = 208).

Students were asked to rate the degree to which they agreed with these three statements, on a 4-point Likert scale (the points representing 1-‘not at all’; 2 - ‘a bit’; 3 - ‘sometimes’; 4 - ‘a lot’):

I like mathematics; I can do mathematics; I am satisfied that I get what I want from mathematics

RESULTS

Table 1 shows the number (and proportion) of students in each response category, for each question.

	Not at all (%)	A bit (%)	Sometimes (%)	A lot (%)	Mean
Like	32 (15)	60 (29)	95 (46)	21 (10)	2.5
Can do	7 (3)	31 (15)	116 (56)	53 (26)	3.0
Satisfied	9 (4)	56 (27)	90 (43)	52 (25)	2.9

Table 1: survey responses for whole cohort

How these figures are evaluated depend to some degree on the perspective. Only 10% of these pupils like maths a lot, but even that might be more than expected. 44% (29% + 15%) hardly seem to like it at all, with an additional 46% only liking it sometimes. More students appear to feel they can do mathematics than like it, with 26% reporting that they can do it ‘a lot’. But that still leaves 74% who can do mathematics at best only sometimes. The 25% of pupils who are satisfied ‘a lot’ is encouraging, but this also leaves 75% of pupils with at least a degree of dissatisfaction.

Since the school sets groups by ability in mathematics within each half year (labelled ‘K’ and ‘S’), we can address the question of whether, or to what degree, pupils in lower sets did (or did not) experience negative affect more than those in higher groups. This is an interesting question since it is sometimes assumed that lower attainment will lead to more disaffection, even though it is known that students in higher-attaining groups can also be disaffected with mathematics. A comparison can be made between the data from the groups in each half year. The scores below represent the percentage of pupils who reported ‘1’ (not at all), or ‘2’ (a bit) to the three items. This can be viewed as a blunt measure of negative affect.

	S1	K1	S2	K2	S3	K3	S4	K4	S5	K5
Don’t like	64	24	67	29	27	60	50	43	36	50
Can’t do	14	0	33	7	18	15	25	29	18	33
Not satisfied	59	7	33	4	18	40	38	48	45	58

Table 2: Percentage of pupils in each group reporting negative affect

K1 and S1 are groups at the same level, but they have very different scores. A higher proportion of students in S1 and S2 don’t like mathematics than in any of the other groups. Apart from groups K1 and K2, lack of efficacy (‘can’t do) appears to be evenly spread across the ability range. Also, whilst only 14% of S1 pupils report low efficacy, 59% report dissatisfaction suggesting a non-simple relation between these two variables.

It can be seen that the most negative affect in terms of attitude ('like') are in groups S1 and S2, with groups K1, S3 and S5 having the least. In terms of efficacy ('can do'), S2 and K5 score the highest (from the perspective of negative affect). For the satisfaction scale (where low scoring suggests dissatisfaction) pupils in K1 and K2 seem to be much more satisfied than other groups. S3 and S5 seem to have less negative affect than one might expect, but S1 and S2 seem to have significantly more than one would expect.

We can conclude that all three measures appear not to decline according to level of attainment. But since parallel groups at the same attainment level can have very different scores, this suggests that it is the class or group itself that is the major determinant of pupils affective experience of mathematics. The scores seem to relate to teacher/group more than level.

Qualitative data

Whilst administering the questionnaires I had the opportunity to ask the students to write briefly their answers to two questions:

The most frequent or strongest emotion that you feel in mathematics classes

One sentence that sums up your feelings about mathematics

The questions were not 'leading', since the pupils were only told that I was interested in their opinions about school mathematics. Since the number of students was 208, and all students in the year responded, the results can be said to be representative of mathematics students of this age. This data is useful in gaining an understanding of how prevalent aspects of disaffection with mathematics are within the population of that age.

For the single emotion-word response data, care had to be taken in organising and analysing the data. For instance, board, bord, bored, boredom and boring were all taken to refer to the single emotion of boredom. Multiple variations on other terms were also similarly consolidated. The words were then classified in a simple 'positive', 'neutral' or 'negative' manner. Although this is a fairly simplistic way to organise the data, it does have meaning within the context of this study. The results are shown below:

positive	neutral	negative
37	29	135

Table 3: Emotion word responses

This is a dispiriting result, and even more so since the cohort includes the full range of ability. It suggests, at the very least, that mathematics is not a pleasant experience for many students, for much of the time. Nonetheless it is also important to point out that it is not necessarily the case that pupils who report boredom are disaffected. To be strongly disaffected a pupil would have to report experiencing a whole range of negative or adverse affective responses.

Individual results include: Anger 11, Boredom 68, Confusion 10, Stressful 8, Depressing 5

On the other hand, 'Happy' was chosen 21 times, but 'Fun' only once.

The 'boredom' score here is consistent with boredom being the highest scoring negative emotion on another instrument used in the wider study, although the population there is very different. Such results confirm data presented in the literature on the incidence of negative affect in the school population as a whole.

In the two top groups (labelled K1 and S1) 30 pupils (17 +13 respectively) out of 51, which is well over half of pupils, reported negative emotions, of which 19 were 'bored', whilst 20 pupils (5 +15) reported positive or neutral emotions. It is worth noting the very different numbers of pupils in the two classes reporting positive emotions, suggesting that classroom climate is an important factor influencing students' affective experience of mathematics.

In the two bottom classes (labelled S5 and K5) 13 pupils (2 + 11) out of 23 reported negative emotions (about half) of which only 7 were 'bored', whilst 9 pupils (8 + 1) reported positive or neutral emotions. Note again the very different proportions of pupils choosing positive and negative emotions in the two classes.

Some caution needs to be applied in generalising from this data, however, due to the simplicity of the data, and the small numbers in each group. On the one hand, the cohort represents the full range of ability. On the other hand since only 208 pupils were surveyed, no attempt is made to underwrite the statistical significance of the results. In addition, a single one-word response does not represent a full examination of these pupils affect in relation to school mathematics.

The descriptive passages were also analysed by group, using the same categories as the quantitative data. Groups differ in the relative proportions who appear to 'like' and 'don't like' mathematics (and in terms of which one predominates in that group). Like the quantitative data, the evidence doesn't support the assumption that higher or lower groups like or don't like mathematics more than the other. Positive or negative affect (liking, not liking) and competence (can or cannot do) do not appear to be related to the level at which one is achieving. Put another way, students in higher groups appear to be as likely to not like, or feel they cannot do mathematics as students in lower groups.

Although the primary focus in this study is disaffection, it is worth examining evidence of genuine affection. In class K1 (a top group) there are only 4 responses that can be interpreted as indicating a condition of such affection for mathematics:

> It can be quite exciting in some lessons
>
> Maths is good for making you think (reported emotion – happy)
>
> Maths can be exciting and I learn a lot from it

> Maths is a tool we can use to solve problems (reported emotion – happy, relaxed)

Of course, it could be argued that this evidence is thin, and may not represent strong enough evidence to fully support a generalised claim. Nonetheless, what evidence there is in this study suggests that affection for school mathematics is very much the exception. Overwhelmingly, the comments of these pupils are negative, except in terms of utility. In the data, 6 statements use the word 'help' (as in the sense 'will help me'). A further 5 statements use utilitarian words like 'useful', 'essential', or 'important'. In class S1 this recourse to utility is mainly absent, and the picture that remains there is broadly negative. In the absence of utility, duty and coercion are mentioned:

> I have to do it

> I try to do the best I can to impress my parents

If this is the picture in the two top sets, it hardly gets any better in the lower groups. It can only be concluded that genuine affection for mathematics is a rarity.

The nature of mathematics

In most cases it was quite easy to identify those descriptive statements that related to the nature of mathematics rather than to affect or competence. These statements were split evenly between positive and negative. In terms of positive statements, the most common were about the general utility or value of the competence:

> It helps in life in some situations

> I think maths is life changing and it can help you in the future

A subset of these related directly to the exchange value of a good qualification in mathematics:

> It's an important subject and you need a good grade to succeed in further education

There were also some comments about mathematics being of value in its own right:

> Maths is good for making you think

> Maths is a tool we can use to solve problems

> Maths isn't very useful later in life but it challenges me which is a good thing

The negative statements include those that reflect the nature of mathematics as experienced by them. These include descriptions like 'hard', 'complicated', 'confusing', 'lists of tedious questions'.

Other negative statements related to the perceived lack of importance or utility:

> 80% of the time completely useless for my future (presumably said without irony!)

It isn't the primary purpose of this study to investigate in depth the epistemological beliefs about mathematics held by pupils, and no claim is made that this data represents a comprehensive examination in that way. However, pupils' views also influence their affective landscape, and it is interesting to have some idea of this broader picture, as exists in year 9.

DISCUSSION

The evidence from school K offers some insights into the incidence of negative affect in a whole-year population. This shows just how widespread the experience of negative emotions is. Boredom was identified as the single most common negative emotion. The data on disposition ('I like/dislike mathematics') and efficacy ('I can/cannot do mathematics') is broadly consistent with data from other studies (e.g. Zan & di Martino, 2007).

The evidence suggests that the experience of aspects of negative affect does not appear to relate in a simple way to ability grouping or attainment. Pupils in 'top' sets for mathematics exhibit dissatisfaction and aspects of disaffection with school mathematics as much as pupils in lower sets. The data here suggests that a key determinant of pupil's affective experience of mathematics is the classroom climate (or 'microculture' to used Hannula's (2012) term). There is very little data that relates such grouping to affect, but the data here is consistent with the finding of Noyes (2012). This is an important finding, and one that suggests that further research needs to be done to understand this phenomenon better.

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MATHEMATICIANS' EXAMPLE-RELATED ACTIVITY IN FORMULATING CONJECTURES

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This paper explores the role examples play in mathematicians' conjecturing activity. While previous research has examined example-related activity during the act of proving, little is known about how examples arise during the formulation of conjectures. Thirteen mathematicians were interviewed as they explored tasks that required the development of conjectures. During the interviews, mathematicians productively used examples as they formulated conjectures, particularly by creating systematic lists of examples that they examined for patterns. The results suggest pedagogical implications for explicitly targeting examples in conjecturing, and the study contributes to a body of literature that points to the benefits of exploring, identifying, and leveraging examples in proof-related activity.

INTRODUCTION AND MOTIVATION

Proof is a crucial aspect of mathematical practice, and researchers have emphasized its importance in the mathematics education of students across grade levels (e.g., Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002; Knuth, 2002; Sowder & Harel, 1998). However, there is much evidence that students at all levels struggle with learning to prove (e.g., Healy & Hoyles, 2000; Kloosterman & Lester, 2004; Knuth, Choppin, & Bieda, 2009; Porteous, 1990). One way to gain insight into how better to help students is to study the work of mathematicians, who are themselves successful at proof. Indeed, there is a history of research that studies mathematicians' thinking and leveraging those findings for possible pedagogical implications for students (e.g., Carlson & Bloom, 2005; Weber, 2008). Thus, given the essential role examples play in mathematicians' proof-related activities (e.g., Epstein & Levy, 1995), we examine mathematicians' work on conjectures and draw potential pedagogical insights. In this paper, we continue our previous work with mathematicians (Lockwood, Ellis, Dogan, Williams & Knuth, 2012; Lockwood, Ellis, & Knuth, 2013) by studying mathematicians' example-related activity as they engage in formulating conjectures. Our examination details the ways in which mathematicians systematically generated and used examples in developing conjectures and discusses implications for the teaching and learning of proof.

RELEVANT LITERATURE AND THEORETICAL PERSPECTIVE

In this paper, we follow Bills and Watson's (2008) lead by defining an example as "any mathematics object from which it is expected to generalize" (p. 78). In defining *proof*, we draw on Harel and Sowder's (1998) definition, which is "the process employed by an individual to remove or create doubts about the truth of an observation" (p. 241).

Harel and Sowder further distinguish between two kinds of activity associated with proving – *ascertaining* (removing one's own doubts) and *persuading* (removing others' doubts) (p. 241).

While much of the literature emphasizes limitations of example-based reasoning (particularly as a means of justification), a number of researchers have suggested the potential value examples may play in proof-related activity. As Epstein and Levy (1995) note, "Most mathematicians spend a lot of time thinking about and analyzing particular examples….It is probably the case that most significant advances in mathematics have arisen from experimentation with examples" (p. 6). Likewise, Harel (2008) notes that, "Examples and non-examples can help to generate ideas or give insight [about the development of proofs]" (p. 7). Other researchers have similarly reported that students and mathematicians display strategic uses of examples that benefit their proof-related activities (e.g., Ellis, et al., 2012; Garuti, Boero & Lemut, 1998; Pedemonte, 2007; Sandefur, et al., 2013; Weber, 2008). Our work builds upon such studies by seeking to identify potentially fruitful aspects of example-related activity in the development and proving of conjectures.

The study presented in this paper is situated within a framework developed by Lockwood, et al. (2012) and refined in Lockwood, et al. (2013) that categorizes *example types, example uses*, and *example strategies*. While the framework is not presented here due to space, it served as a broader context that guided data analysis.

METHODS

We conducted hour-long interviews with mathematicians in which they were presented with one or two mathematics tasks. A member of the research team (an advanced mathematics PhD student) conducted the interviews and participated in the analysis. During the interviews, the mathematicians were given time to work on the tasks on their own and were asked to think aloud; generally, the interviewer did not interrupt except to ask clarifying questions or to answer questions from the mathematicians. The mathematicians used Livescribe pens during the interviews, pens that both audio-record and keep live records of the mathematicians' written work. This technology allows for efficient data collection and facilitates rich analysis by providing both audio and written work of the interviews that can be re-played in real time, with the audio synced with the written work.

Participants

The participants were thirteen mathematicians from a large Midwestern university. The participants included seven professors, three postdocs, and three lecturers, with eight males and five females. Twelve participants hold a Ph.D. in mathematics, and one participant holds a Ph.D. in computer science. There were a variety of mathematical areas represented, including topology, number theory, and analysis.

Tasks

All thirteen mathematicians worked on the *Interesting Numbers* task while seven also tried an additional task that is not reported here. The *Interesting Numbers* task states, "Most positive integers can be expressed with the sum of two or more consecutive integers. For example, 24 = 7 + 8 + 9, and 51 = 25 + 26. A positive integer that cannot be expressed as a sum of two or more consecutive positive integers is therefore interesting. What are all the interesting numbers?" One approach to solving this task is as follows: It can be shown that the sum of any two or more consecutive positive integers has an odd factor greater than 1. Conversely, if a positive integer N has an odd factor $k > 1$, it can be shown that N can be written as the sum of either k or $2N/k$ consecutive positive integers, whichever is smaller. The interesting numbers are thus exactly those positive integers that have no odd factors greater than 1. In other words, the interesting numbers are the powers of 2.

Both tasks were chosen because: a) they were accessible (i.e., did not require specialized content knowledge and were easy to explore) but were not trivial (i.e., a solution was not immediately available), b) they were accessible to the interviewer, allowing her to ask relevant questions and engage with the mathematicians, and c) they involved open-ended questions that would facilitate conjecturing. These were not "prove or disprove" statements that already stated a conjecture, but rather these tasks required that certain numbers and sets be characterized. Through such activity, the mathematicians developed conjectures that they could then attempt to prove.

Analysis

As mentioned, the Livescribe pen yields both an audio record of the interview and a pdf document of the interviewee's written work (synced with the audio). In this pdf, the audio and the written work can be played back, so the researcher can see and hear what was written and said in real time. The interviews were also transcribed. To analyse these interviews, two members of the research team independently coded and then discussed four interviews using Lockwood, et al.'s (2012, 2013) framework for example types, uses, and strategies. In coding the interviews, the researchers also noted codes that emerged from their analysis and that were not captured by the previously developed framework. After the four interviews were initially coded, compared, and discussed, the remaining nine interviews were split up and coded. The two researchers came together regularly to discuss any issues or questions that arose in analysing these remaining interviews. After completing the coding of all the interviews, the researchers met to discuss phenomena and themes that pertained especially to conjecturing and revisited relevant episodes in the transcripts.

RESULTS

While we had previously (Lockwood, et al., 2013) reported on how mathematicians generated and used examples as they *proved*, here we report on their work with examples as they *conjectured*. In this section we elaborate a key phenomenon that we observed as mathematicians used examples while formulating conjectures. We call this phenomenon "Data Collection," in which the mathematicians systematically and, in some sense exhaustively, went through every example in a finite sequence in order to gather information. The mathematicians generated examples based on sequentially exhausting a small list of examples, and they then subsequently reflected back on these organized example lists in order to formulate a conjecture. This activity was productive for some mathematicians, as we explore below, suggesting that there is potential value in the methodical generation of examples in formulating conjectures.

To illustrate this phenomenon, we present Mathematician 1's (M1 – a professor) work on the *Interesting Numbers* task. M1 began by computing a sequence of small sums: 1+2=3, 2+3=5, 3+4=7, and 4+5=9. From these examples, he recognized that odd numbers greater than 1 could not be interesting. He proved this fact algebraically by showing that any odd number $2n+1$ is the sum of n and $n+1$. Continuing with algebra, he then looked at general sums of 3, 4, and 5 consecutive numbers beginning with n. Each case gave him an algebraic expression ($3n+3$, $4n+6$, $5n+10$) representing numbers that were not interesting, from which he tried to generalize.

After some time, M1 recognized that his algebraic manipulation had not illuminated a conjecture, and he said, "Okay. So at this point, I would start over and try and do something a little more visual." He then drew a number line and began to write out the numbers. Because M1 already knew that the odd numbers were not interesting, he crossed those out as he wrote. He then proceeded to go through the even numbers and cross out those of the form $3n+3$, $4n+6$, and $5n+10$ for some n (Figure 1). After working through the numbers 1 through 21, he concluded, "well, the answer does kind of pop out that it's the powers of 2, doesn't it?" By actually writing out the examples and then crossing out non-interesting numbers, the pattern of numbers not crossed out – 1, 2, 4, 8, and 16 – stood out in his figure. His construction of the complete table, and his subsequent reflection on it, suggest the "Data Collection" phenomenon – he systematically gathered a complete sequence of examples and deduced patterns from them.

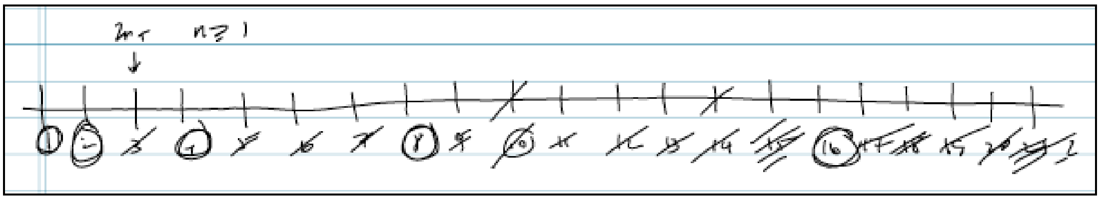

Figure 1: The "visual" list from which the powers of 2 conjecture emerges

We perceive that M1's prior knowledge and experience made him attuned to this sequence of numbers as powers of 2. M1 continued to pursue the powers of 2, saying, "Okay, so, um, so at this point I would maybe try the next one, 32," and he proceeded to write a conjecture that interesting numbers are powers of 2. To us, M1's careful construction of examples allowed for a common, familiar pattern to emerge visually on the page. M1's work suggests that the methodical generation of examples (what we call Data Collection) facilitated the efficient formulation of the conjecture.

As another, perhaps more extreme, example, we see in Figure 2 a table that M10 (a professor) created. This displays a great deal of care in detailing out a large number of cases. He also demonstrated Data Collection and formulated the correct conjecture by making note of the numbers that were not in the table.

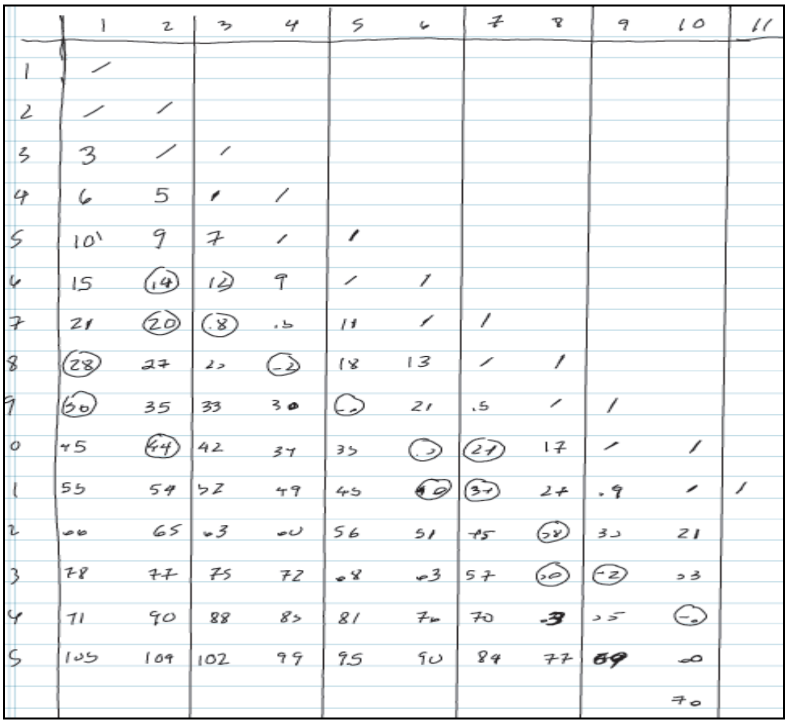

Figure 2: M10’s table

To see why the phenomenon of data collection was especially useful, we note that not all mathematicians engaged with examples in this way. In contrast to M1’s work, another mathematician, M6, did not generate data and detect a pattern. Instead, M6 developed an algebraic expression for a general non-interesting number, written as the sum of n consecutive integers starting with k. Starting with an arbitrary number (represented by $2^p \cdot q$ with q odd), M6 tried to find k, n (in terms of p, q) to make $2^p \cdot q$ non-interesting. Using only algebraic manipulation, M6 eventually found that this could be done if and only if $q > 1$. This result yielded the correct conjecture, but it took him more than twice as long (38 minutes) to find than the average time among the mathematicians that generated data (16 minutes). While the algebraic exploration was not an incorrect approach, we suspect that for conjecturing purposes, it did not so clearly illuminate potential patterns as the actual generation of concrete examples did. Indeed, unlike M1’s work, in which the powers of 2 conjecture fell out almost immediately upon exploring examples, the pattern of the interesting numbers was obfuscated for M6 by the algebraic manipulation.

In addition to helping mathematicians formulate a correct conjecture, we present two ways in which Data Collection was efficacious in supporting mathematicians’ conjecturing: Lemma Development, and Preliminary Conjecture Breaking. First, observations from generated data lead to lemmas, which in turn informed the development of conjectures. For example, M3 first looked at the numbers 1 to 14 and tried to write each one as a sum of consecutive numbers. He noticed that odd numbers were sums of 2 consecutive numbers, multiples of 6 were sums of 3 consecutive numbers, and numbers congruent to 2 mod 4 other than 2 were sums of 4 consecutive numbers. From these observations, M3 proved lemmas stating that these types of numbers were non-interesting. These lemmas allowed M3 to restrict his attention to multiples of 4, which led to the development of the full conjecture.

Second, the data collection also allowed the mathematicians to find examples that broke preliminary conjectures, which in turn led to the articulation of more accurate conjectures. This is seen in M4, who initially conjectured that the interesting numbers were the non-primes after looking at the numbers 1 to 6 (and incorrectly deciding that 6 was interesting). He continued on to look at the numbers 7 to 10 before he realized his mistake, saying about 6, "Oh, 1, 2... 1 plus 2 plus 3. Right. Revise conjecture. So far, so, the interesting numbers so far are 4, 8, [...] It looks like it's the [multiples] of 4." M4 revised his conjecture once more (to a correct conjecture) when he looked at 11, 12 and 13 and discovered that 12 was also not interesting.

DISCUSSION AND CONCLUSIONS

The results highlight ways in which specific example-related activity like Data Collection may play a valuable role in the development of conjectures. In this section, we discuss three aspects of the results and suggest potential implications for students. First, some mathematicians (as seen in M1 and M10) took the time painstakingly to catalogue a number of examples. The generation of sequences of examples and subsequent reflection on them enabled the mathematicians to formulate conjectures effectively and efficiently. Students may thus benefit from generating comprehensive sets of data that they can survey in search of patterns, which in turn could illuminate conjectures. It is important to emphasize for students that such work may take patience and care. Second, also notable is the fact that these mathematicians engaged in deliberate and strategic example generation, which stands in contrast to less systematic behaviour often found in students' work with examples. For students, then, there might be value in helping them learn to be more strategic and methodical in their use of examples, going beyond finding a few confirming examples that simply come to mind. Third, in some of the mathematicians (such as M6) we saw an immediate application of algebraic techniques that were less efficacious for conjecturing than the Data Collection was. We suspect that some students may put a premium on algebraic techniques and may assume that algebraic activity is more sophisticated than generating examples. Our findings suggest that students should be encouraged to engage with and see the value in finding concrete examples when conjecturing and not simply to apply algebraic formulas and techniques. As a final point of discussion, we note that the tasks in our study were well suited to facilitate Data Collection. Other tasks might be more or less effective in fostering conjecturing. Instructors should be aware of what kinds of activity and thinking certain tasks elicit and should expose students to tasks that might encourage Data Collection activities.

In this paper, we have reported on a beneficial phenomenon that emerged when mathematicians used examples during the activity of mathematical conjecturing. In this Data Collection phenomenon, mathematicians generated sequential lists of examples and used these lists in order to find patterns that might lead to conjectures. We saw this activity help mathematicians formulate correct conjectures, but it also helped with developing useful lemmas and also breaking initial conjectures to arrive at more accurate ones. These findings contribute to work that has been done previously that highlights the role of examples in mathematicians' proving, and it adds to the overall narrative that examples play a vital role in mathematicians' proof-related activity. The productive ways in which mathematicians use examples in formulating conjectures provide interesting and much-needed insights into proving and conjecturing into K-16 mathematics education.

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INFLUENCE OF EARLY REPEATING PATTERNING ABILITY ON SCHOOL MATHEMATICS LEARNING

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Recent studies in early mathematics education highlight the importance of patterning abilities and their influence on mathematics learning and the development of mathematical reasoning in young children. This paper focuses on young children's repeating patterning abilities and reports results from an ongoing four-year longitudinal study that investigates the development of early numeracy understanding of 408 children from one year prior to school until the end of grade 2. The analyses in this paper reveal a significant influence of young children's repeating patterning abilities one year prior to school on their mathematical competencies at the end of grade one.

INTRODUCTION

Mathematics has often been defined as science of pattern (e.g., Davis, & Hersh, 1980). It is also widely acknowledged that a general awareness of mathematical pattern and structure is important for mathematics learning at all stages (e.g., Mason, Stephens & Watson, 2009; Mulligan, & Mitchelmore, 2009). In this paper we focus on patterning abilities in early mathematics and adopt a differentiated understanding of pattern. We particularly focus on the question what different types of patterns and what kind of patterning activities might influence the development of which key mathematical concepts and processes in early years mathematics learning and later on.

REPEATING PATTERNS AND THEIR IMPORTANCE ON EARLY MATHEMATICS LEARNING

With Mulligan and Mitchelmore (2009) we define a mathematical pattern as any predictable regularity. In the work with Kindergarteners and primary school children, where our research is based, we distinguish three main types of mathematical patterns: spatial structure patterns, repeating patterns, and growing patterns. Examples for spatial structure patterns are spatial dot patterns and grids like the twenty field, both used in the early years to visualize numbers. Repeating patterns consist of a sequence of elements (the unit of repeat) that is repeated indefinitely (e.g., ABCABC…). In growing patterns a sequence of elements changes systematically (e.g., 1, 3, 5, 7, …).

In this paper we draw the focus exclusively on repeating patterns and their importance for mathematics learning.

Patterning activities with repeating patterns are supposed to develop general mathematical concepts in children such as ordering, comparing, sequencing, classification, abstracting and generalizing rules and making predictions (see e.g., Threlfall, 1999). These concepts then lead to the development of mathematical reasoning in young children (English, 2004; Mulligan, & Mitchelmore, 2009, 2013). It is mostly in the area of algebra (or pre-algebra) that repeating pattern work is seen as a conceptual stepping stone (Threlfall, 1999). National curriculums often consider repeating patterns (together with growing patterns) as a precursor for functional thinking and algebra (NCTM, 2000; Queensland Studies Authority, 2008). Mulligan and Mitchelmore (2009, 2013) highlight repeating patterns as important for measurement (which involves the iteration of identical spatial units) and as critical to the development of counting and multiplicative thinking (which involves the iteration of identical numerical units). However, it is important to note that these assumptions have been mainly derived from either observation, a experience, or are theoretical considerations. From the empirical perspective, in the last decade there is a substantial body of research, mainly qualitative studies, focusing on patterning strategies and looking at the level of students' awareness of or attention to pattern and structure (see e.g., Mulligan, & Mitchelmore, 2009; Papic, Mulligan, & Mitchelmore, 2011; Radford, 2010; Rivera, 2013; Warren, & Cooper 2006, 2008). Few studies however have tried to quantitatively measure the significance of patterning abilities in the early years for later mathematics learning.

FINDINGS FROM RECENT QUANTITATIVE STUDIES ON YOUNG CHILDREN'S PATTERNING ABILITIES

Mulligan and Mitchelmore (2009) tested 103 Australian Grade 1 students (5.5 to 6.7 years) on 39 pattern and structure items. They found a nearly perfect correlation between young students' general mathematical understanding and their pattern and structure competencies. A German study (Lüken, 2012) with 74 school starters (5.8 to 7.2 years) showed a significant correlation on a medium level between patterning competencies and early mathematical competencies and a slightly lower correlation with the mathematical achievement at the end of grade 2. Van Nes (2009) interviewed 38 Dutch Kindergarteners (four- to six-year-olds) on tasks on counting, subitizing, repeating and spatial structure patterns. As she used a small sample van Nes only very carefully suggests a correlation between a child's pattern and structure competencies and its mathematical competencies. However, all three studies either lack the use of statistically reliable instruments, or base their conclusions on rather small samples. Above all, all three studies did not discriminate between the different types of patterns. Thus, it is yet to be specified, if each of the three types of patterns in early childhood separately correlates with mathematical competencies and which key concepts and processes they effect.

To underpin the importance of patterning abilities regarding *repeating* patterns this paper focuses on the question, whether a child's ability to reproduce, extend, and explain a repeating pattern has a statistical effect on its mathematical competencies in kindergarten and the transition from kindergarten to school. Hence, the paper addresses the following questions:

- Is there a significant effect of young children's repeating patterning abilities on their mathematical competencies?
- To what extent do their repeating patterning abilities influence young children's mathematical performance one year prior to school, immediately before school entry and at end of grade 1?

METHODOLOGY

Context of this paper is a longitudinal study investigating the development of number concept development of 408 children from one year prior to school entry (5-year olds) until the end of grade 2. The study seeks to identify children that struggle with respect to their mathematics learning after the first (and second) year of school and compare their performance with their number concept development one year prior to school as well as immediately before school entry (i.e., grade 1).

Hence, the data collection involves four measuring points (MP1 – MP4) i.e., one year prior to school, immediately before school entry, at the end of grade 1 and grade 2 (which will be conducted in June 2014). At each measuring point the children performed on both a standardised test on number concept development that is suitable for their respective age (OTZ, DEMAT 1+ / 2+) as well as on a task-based one-to-one interview (EMBI-KiGa, EMBI). Table 1 provides an overview of the study design.

Measuring points	Instruments	Participants
June 2011 MP 1	OTZ	children participating in the study (n = 538)
	EMBI-KiGa	children participating in the study (n = 538)
June 2012 MP 2	OTZ	children participating in the study (n = 495)
	EMBI-Kiga	children participating in the study (n = 495)
June 2013 MP 3	DEMAT 1+	all grade 1 classes with children participating in the study (n = 2250)
	EMBI	children participating in the study (n = 408)
June 2014 MP 4	DEMAT 2+	all grade 2 classes with children participating in the study
(to be conducted)	EMBI	children participating in the study

Table 1: Measuring points, instruments and number of participants in the study

At MP3 and MP4 the whole learning group of children in the study is tested in order to compare the children's performance to their peers' and to diminish intra- and inter-group effects. When available, the instruments chosen for the data collection had been developed and trialled in international settings.

The OTZ (*Osnabrücker Test zur Zahlbegriffsentwicklung*) is a German adaptation of the "Utrecht Numeracy Test" (van Luit, van de Rijt, & Pennings, 1994; van de Rijt, van Luit, & Pennings, 1999) – a standardized individual test aiming to measure children's number concept development that involves logical operations based tasks as well as counting related items (van Luit, van de Rijt, & Hasemann, 2001).

The EMBI (*Elementarmathematisches Basisinterview*) is the German version of the Australian “Early Years Numeracy Interview” (DEET, 2001) developed by Doug Clarke an his colleagues in Melbourne – a task- and material-based one-on-one interview assessing children’s developing mathematical understanding in the four areas counting, place value, addition/subtraction strategies, multiplication/ division in grade one and two (Peter-Koop, Wollring, Grüßing, & Spindeler, 2013).

The EMBI-Kiga (*Elementarmathematisches Interview Kiga*; Peter-Koop, & Grüßing, 2011) corresponds with the “Detour for children starting the first year of school” of the Early Years Numeracy Interview (ibid, 24–26), that is also recommended for children in grade 1 and 2 who demonstrated difficulty in counting a collection of 20 objects. For a detailed description of the items and their development see Clarke, Clarke, and Cheeseman (2006).

The DEMAT 1+ (*Deutscher Mathematiktest für 1. Klassen*; Krajewski, Küspert, & Schneider, 2002) and the DEMAT 2+ (Krajewski, Liehm, & Schneider, 2004) are German curriculum based standardized paper and pencil tests to be conducted at the end of the school year with the whole class.

One instrument only, the EMBI-KIGA, uses an item on repeating patterns. We used this item at MP1 as a measure for the children’s repeating patterning abilities one year prior to school. The repeating pattern in this item is an ABCC pattern. The children are asked to reproduce, to extend and to explain the pattern. Figure 1 shows the complete item. The material used is coloured plastic teddies (counters).

Now watch what I do with the teddies.

The interviewer makes a ABCC pattern with the teddies (green, yellow, blue, blue, green, ...).

a) I have made a pattern with the teddies. Please make the same pattern.

(If the child’s pattern is a correct copy, point to it.
If not, point to your pattern.)

b) Please make the pattern go on a bit more.

c) How did you decide what came next in the pattern each time?

Figure 1: Repeating pattern item from the EMBI-Kiga/Early Numeracy Interview (DEET, 2001, 24-25)

For the data-analyses first a comparison of means in form of a one-way analysis of variance (one-way ANOVA) was conducted, because the item on children’s patterning abilities, which serves as the independent factor variable, can take three values (0, 0.5, and 1) and thus defines three separate groups based on the children’s performance on the item. Based on these groups the mean scores in all mathematics tests at all measuring points have been searched for significant differences between the groups. With the results of the one-way ANOVA the partial eta^2-values have been calculated in order to approximate the amount of variance in the mathematics tests that can be explained by the children’s performance on the patterning item. As a last step the linear correlation (Pearson’s r) between the item-performance and the mathematics test performances have been calculated to illustrate the linear dependencies of the two variables.

RESULTS

The results of the one-way ANOVA reveal significant ($p < 0.001$) differences in the mean values of the test and interview performances between the groups. Furthermore, the item on children's repeating patterning abilities shows substantial influences on their performances in all mathematics tests at each measuring point (see Table 2), i.e. one year prior to school, immediately before school entry as well as at the end of grade 1.

							Repeating patterning item one year prior to school	
			df	Mean Square	F	Sig.	Partial Eta^2	Pearson Correlation
Mathematics tests one year prior to school	OTZ total	Between Groups	2	2207,314	53,638	,000	,209	,457**
		Within Groups	405	41,152				
	N = 407	Total	407					
	EMBI-KiGa total	Between Groups	2	361,898	85,767	,000	,301	,547**
		Within Groups	399	4,220				
	N = 401	Total	401					
Mathematics tests at school entry	OTZ total	Between Groups	2	1293,345	43,497	,000	,177	,420**
		Within Groups	405	29,734				
	N = 407	Total	407					
	EMBI-KiGa total	Between Groups	2	60,155	26,151	,000	,114	,335**
		Within Groups	405	2,300				
	N = 407	Total	407					
Mathematics tests at the end of grade 1	EMBI total	Between Groups	2	214,795	18,515	,000	,085	,289**
		Within Groups	400	11,601				
	N = 402	Total	402					
	Demat 1+ total	Between Groups	2	1502,274	26,534	,000	,116	,340**
		Within Groups	405	56,617				
	N = 407	Total	407					

Table 2: One-way ANOVA results, Partial Eta2 and Pearson correlation (**correlation is significant on the 0.01 level)

One year prior to school the children's performance on repeating patterning abilities explains about 21% of the variance on the overall mathematics test performance (OTZ) and shows a significant medium correlation with $r = 0.457$. This also holds true for the performance on the EMBI-Kiga, where the influence is slighty stronger (30.1% explained variance, Pearson's r = 0.547), which can be explained through the inclusion of the item in the interview.

At the second measuring point immediately before school entry the item shows a medium but significant correlation to the children's performances on the OTZ (Pearson's r = 0.42) and still explains 17.7% of the overall mathematics test-performance (OZT). For the EMBI-KiGa performance the item-performance demonstrates similar effects and explains 11.4% of the variance with a significant correlation of Pearson's r = 0.335.

At the end of grade 1 the item-performance still explains 11.6% of the variance of their performance on the standardised DEMAT 1+ and shows a low but significant correlation (Pearson' r = 0.34). For the EMBI interview the children's repeating patterning abilities explains 8.5% of variance of the overall interview performance and correlates with Pearson's r = 0.289.

DISCUSSION

With respect to the question if each of the three types of pattern (see above) in early childhood separately correlates with mathematical competencies, this effect could be shown for repeating pattern abilities. Those children, who manage to solve the EMBI-Kiga item on repeating patterns one year prior to school, i.e., they can reproduce, extend and explain a repeating pattern of the form ABCC, are the children who demonstrate elaborate number concept development in kindergarten and who achieve best in a standardised mathematics classroom test at the end of grade 1.

This relationship appears to be stable over a period of two years and can be shown with different measuring instruments, i.e. individual (OTZ) and group tests (DEMAT 1+) as well as one-on-one interviews with a focus on strategies (EMBI-Kiga/EMBI). The explanation of variance for mathematics test performance provided by the pattern item as expected decreases until the end of grade 1 (a period over 2 years), but remains at a substantial level.

Looking closer at the mathematical concepts and processes of the applied instruments (see Table 1), significant positive linear correlations are found between repeating patterning abilities and computation skills (DEMAT 1+ and EMBI), i.e. children who demonstrate elaborate repeating patterning ability prior to school also show elaborate computation skills with respect to addition and subtraction at the end of grade 1. In addition, the data reveal significant positive linear correlations between repeating patterning abilities prior to school and addition and subtraction strategies other than counting (i.e., counting all, counting on and counting back) at all measuring points (EMBI). Furthermore, we cannot draw any conclusions with respect to other mathematical abilities or mathematical content areas e.g., geometry.

However, the question whether repeating patterning ability is a predictor for the development in specific domains of early numeracy learning yet remains open also due to limitations of the item used to assess patterning abilities in the EMBI-Kiga. Additional items considering different levels of difficulty with respect to repeating patterns as well as the documentation and analysis of children's explanations of the pattern would be necessary to further investigate that impact.

In summary the study reported in this paper indicates that it is important to differentiate the rather broad concept of pattern with respect to early mathematic learning (Papic et al., 2011). A correlation could be shown for repeating patterning abilities, but still needs to be investigated for growing patterning and spatial patterning abilities. Hence, a further large scale longitudinal study that involves several items on each repeating, growing, and spatial patterning abilities in order to increase reliability is desirable in the future.

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THE EFFECT OF AN INTELLIGENT TUTOR ON MATH PROBLEM-SOLVING OF STUDENTS WITH LEARNING DISABILITIES

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Reform-based math instruction calls for students' construction of conceptual understanding, solving challenging problems and explanation of reasoning. However, existing literature shows that students with learning disabilities (LD) easily get lost in reform-based instruction. As an outcome of collaborative work between math education and special education in instructing students with LD, we've developed an intelligent tutor (PGBM-COMPS) to nurture multiplicative reasoning of students with LD. The intelligent tutor dynamically models individual student's evolving conceptions and recommends tasks to promote her/his advancement to a higher level in the learning trajectory and solve complex word problems using mathematical model equations. This study evaluated the effect of this intelligent tutor on improving multiplicative reasoning and problem solving of students with LD.

INTRODUCTION

In line with the reform in math education, the Common Core State Standards for Mathematics (CCSSM) emphasizes conceptual understanding in problem solving, mathematical modeling, higher order thinking and reasoning, and algebra readiness (NGA & CCSSO, 2012). It also promotes student-centered learning as well as the use of technology.

New standards also stress "opportunity to learn" (OTL) for students. OTL refers to all students, including those with special needs or learning difficulties, have equal opportunity to get access to learning resources and meet the same high standards. According to National Council of Teacher of Mathematics (NCTM) Standards (2000), students with LD and without LD should be given the equal opportunities to solve meaningful and complicated mathematics problems. However, students with Learning Disabilities (LD) lag behind their peers without LD at least two grades levels (Wagner, 1995). Even though students with LD showed various problems in mathematics learning, they share some common characteristics (Goldman, 1989; Rivera, 1997). Students with LD are likely cognitively disadvantaged, particularly in the area of working memory (Richard, 2012), which lead to poor performance in acquiring math facts and solving mathematics problems (Kroesbergen & Van Luit, 2003). Due to these problems, students with LD often show difficulties in connecting the knowledge they have learned with new knowledge and generating new knowledge (Kroesbergen & Van Luit, 2003). Moreover, students with LD tend to have attention problems, which are often regarded as short attention span (Stevens, 1996; National Association of Special Education Teachers, n.d.). Students with LD who have a short attention span will easily get distracted if they see something, hear something, smell something or feel something and cannot focus on a task for more than several seconds (Stevens; National Association of Special Education Teachers). Facing with these disadvantages, students with LD have difficulties to fully get involved in mathematics problem solving, particularly in reform-based instructional environment (Miller & Hudson, 2007). Besides this, existing literature shows that students with learning disabilities/difficulties easily get lost in reform-based instruction and "seemed to disappear during whole class discussions" (Baxter, Woodward, and Olson, 2001, p. 545).

Given the characteristics of students with LD, new standards students with LD need to meet and today's inclusive classrooms, it is needed to develop intervention program to provide every student with optimal opportunities to learn and therefore meet the new standards. Computer-assisted instruction (CAI) may help teachers in meeting individual student's needs in the inclusive classroom. In fact, according to the National Council of Teachers of Mathematics Standards (NCTM, 2000), the Mathematical Science Education Board (1991), as well as the Mathematical Association of America (1991), current mathematics reform encourages the use of computer technologies for both teachers and students in the classroom.

The purpose of this study was to explore the effect of an intelligent tutor (PGBM-COMPS) on nurturing multiplicative reasoning of elementary students with LD. The specific research questions were: (1) Was there a functional relationship between the intervention delivered by the PGBM-COMPS tutor and students' performance on a multiplicative reasoning and problem solving criterion test; (2) did students improve their performance on solving word problems in various contexts with large numbers? And (3) did the intervention influence students' transfer of knowledge to performance on a norm-referenced standardized achievement test?

METHODOLOGY

Participants and Setting

This study was conducted within the larger context of the NSF-funded, *Nurturing Multiplicative Reasoning in Students with Learning Disabilities/Difficulties* project[1] (Xin, Tzur, & Si, 2008). Participants were three 3rd graders with school-identified LD, who enrolled in an urban elementary school in the United States. All three students (two boys and one girl) were included in the general education classrooms for 80% of the school day and they were all receiving Tier II and Tier III Response to Intervention

[1] This research was supported by the National Science Foundation, under a NSF grant [Xin Y. P., Tzur, R., & Si, L. (2008-2013) Nurturing Multiplicative Reasoning in Students with Learning Disabilities in a Computerized Conceptual-Modeling Environment. National Science Foundation (NSF),]. The opinions expressed do not necessarily reflect the views of the Foundation.

(RtI) support. All the instruction and testing were conducted in the school's computer lab early in the morning Monday through Thursday.

Dependent Measures

The *criterion test* used in this study was a researcher-developed 10-item test that assesses multiplicative reasoning (MR-test). Other tests included in this study were: a 12-item word problem-solving test (COMPS-test, Xin et al., 2008) that contains a range of multiplication and division word problems involving large numbers; Stanford Achievement Test (SAT-10, Pearson Inc., 2004)a norm-referenced test involving a subtest on mathematics problem solving. SAT-10 was used as a far-transfer measure. In addition, during the tutoring instruction, probes were given to assess students' mastery of the skills pertinent to each of the four modules of the PGBM-COMPS program.

Procedure

An adapted multiple-probe-design (Horner & Baer, 1978) across participants was employed to evaluate potential functional relationship between the intervention and participants' word problem-solving performance.

All three participants completed one MR-test during the baseline condition. Then one student (Lily) took another two equivalent MR-tests. Following the baseline, the intervention on Module A was first introduced to Lily. Once the data for Lily showed an accelerating trend, the intervention on Module A was introduced to the second student David immediately after he took two additional baseline MR tests. The same sequence was followed until all three participants were introduced to Module A intervention. Following Module A instruction, a probe on the criterion MR-test was taken before Module B instruction took place. After Module B, another probe was taken before Module C & D was introduced. Posttests were given following all modules' instruction.

Participating students worked with the intelligent tutor one-on-one on a laptop computer four times a week, with each session lasting about 20-30 minutes. Sessions were supervised by trained Research Assistants (RAs). Their roles included administering pre-post assessment, fixing/recording computer/program's "bugs" and guiding students to appropriate part of the program after any unexpected "interrupt." Participants received about a total of 20-28 sessions during the spring semester.

Intervention Components

The PGBM-COMPS tutoring program is composed of four modules (A, B, C, & D). Module A focuses on multiplicative double counting (mDC). When working with mDC tasks, students learn the concept of *composite unit* (CU). For example, in the following question, PGBM 7 towers with 3 cubes in each, how many cubes in all?, students learn to consider 3 cubes as a CU and count 7 times of such unit for solution.

Module B involves tasks to develop skills in *unit differentiation and selection* (UDS) and multiplicative *mixed unit coordination* (MUC), which make sure students know on which unit they are operating, whether it is the # of cubes [the 1'] or # of towers [the CU]. Module C and Module D present quotative division and partitive division tasks respectively. In this part, students learn to solve the problems either through mDC or dividing cubes into equal-sized groups for solution.

Following PGBM components in Modules A, C, and D, the COMPS component engages students in representing word problems in mathematical model equations (e.g., unit rate x # of units = product, Xin, 2012) and then solve for the unknown (could be any of the two factors or the product) in the equation.

RESULTS AND ANALYSIS

The figure in the Appendix presents three students' performance on the MR and COMPS tests during baseline, intervention, and post-assessment. Each student's performance in the PGBM-COMPS tutoring program is described as follows:

Lily

In the baseline, she used addition and subtraction for solving all problems and got 0 points for all the tests. After the intervention was imposed, she demonstrated a steady increase on the MR- test (See Figure 1, the blue diamond data points and its data path across the phases) and learned to use multiplication and division but the increase was not significant. Her performance in the probes following each module was relatively low except for the last phase (module C and D instruction) where she had a steady increase in performance. Within all the modules, her performance in module B was poorest. However, she got great increase on the COMPS posttests-t. The transcript of her working video supports several explanations for her difficulties in learning of each module and poor performance in the MR- test and probes. First of all, she could not concentrate on the tasks very well. She kept clicking on everywhere of the screen, which often caused the computer frozen and the program restarted. This wasted a lot of learning time, which caused her not go through the whole study in each module because time allotted to each module was limited.

David

David demonstrated a steady increase on the MR criterion test throughout the program and also had great improvement in COMPs test. Within all the modules, module B was the most struggling part for David and he had difficulty in module B UDS part (See the transcription and Figure 1 below):

Module B UDS, David, 04/24/2013

The problem is "Tom has 4 towers of 5 cubes in each, John has 4 towers of 10 cubes in each".

Program(P): How are these collections similar?

David(D): Chose the choice of "They have the same number of towers"

P: That's correct. How are these two collections different?

D: Chose the choice of "They have different number of towers"

P: That's incorrect.

D: Chose the choice of "John has more cubes in each tower"

P: That's correct. How many more cubes does John have?

D: Input "5"

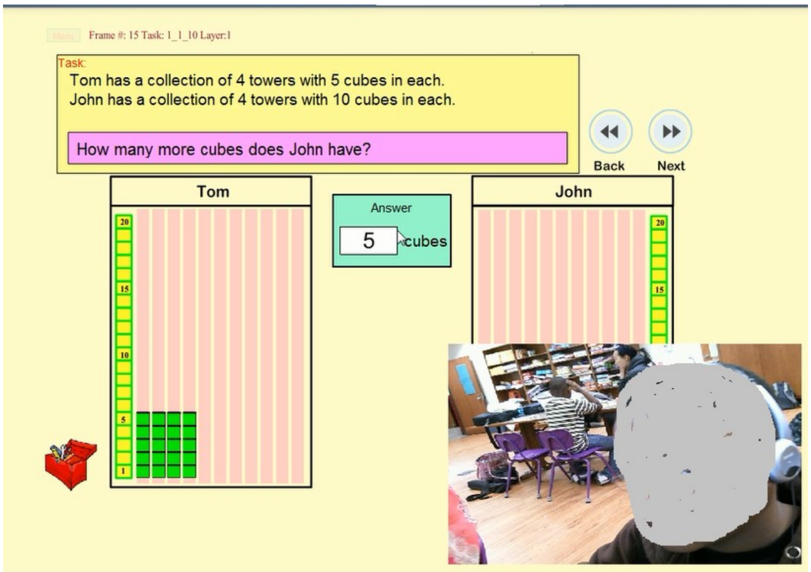


Figure 1

Neal

Similar to Lily and David, Neal also showed significant increase in COMPS-test, but he showed inconsistent and unsteady performance in MR-test. The probes scores indicated that module B was a struggling part for him. In addition, he had motivation problem. In the pretest and module A phase, his attitude was positive. However, from module B, where he faced strong struggle and had RA repeating prompting and instructing him, he became impatient and not concentrated on the tasks. In the post-test phase, he didn't what to take any test and often showed miserable look on his face and RAs had to provide cookies as a reinforcement to have him finish the tests.

DISCUSSION

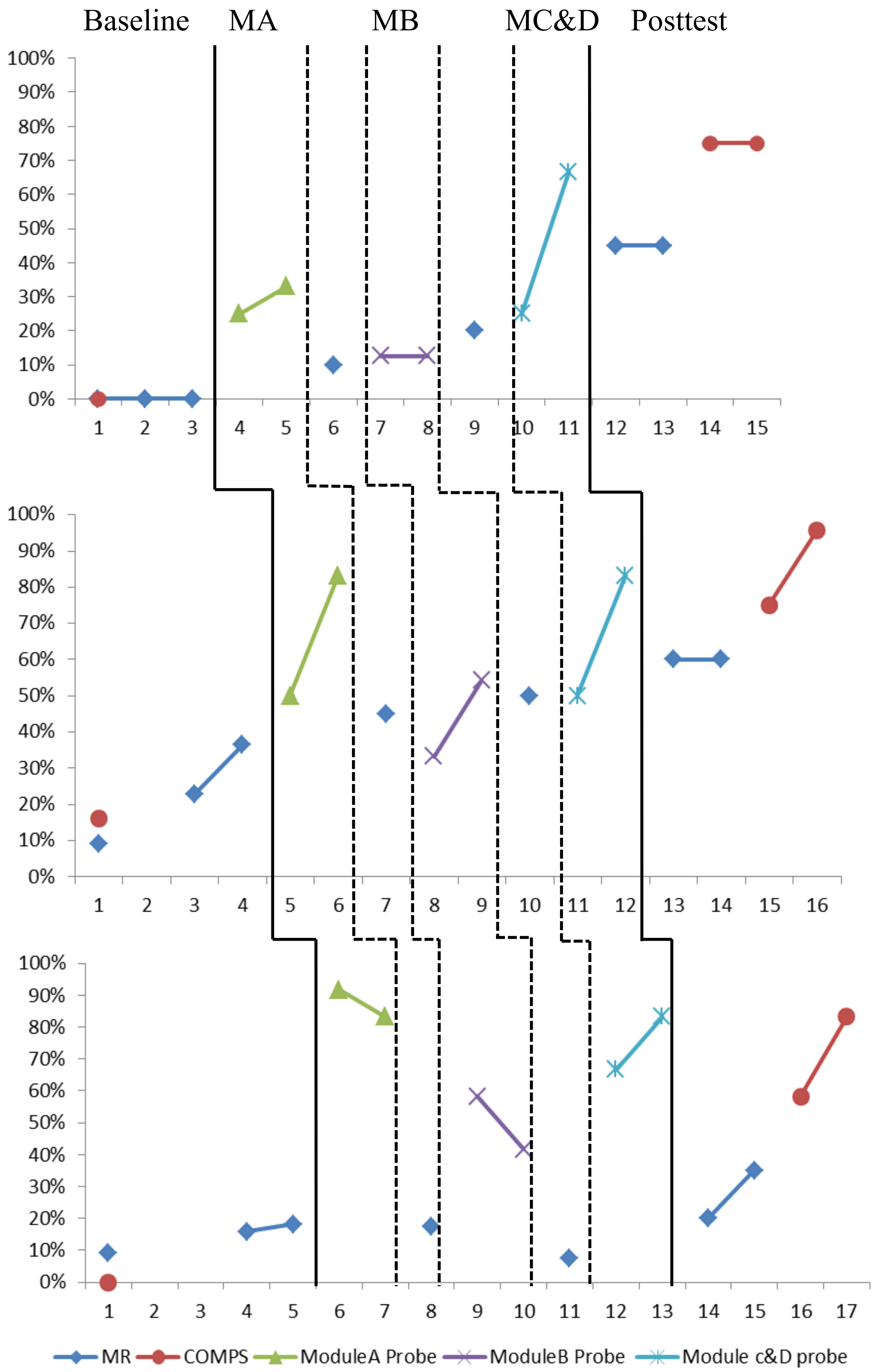
The PGBM-COMPS is probably the very first intelligent tutor that was created based on a research-based model of how students with LD develop multiplicative reasoning via reform-oriented pedagogy. Generally speaking, all three students' performance in module B was relatively poor than their performance in Module A, C & D. One explanation for this might be that the UDS part in Module B involved two-step problems, which posed a challenge for these participants. Since module B is the second part of the program, at this point, the students did not have enough ability to solve such challenging tasks. Also, to solve two-step problems, the participants needed to hold and simultaneously process much information in their mind. Since students with LD had poor working memory (Richard, 2012), it posed challenge on the participants to solve the two-steps problems. It is documented that students with LD performed at significantly lower levels than students without disabilities on multistep problem solving (Xin, 2005). So in the future research, it would be better to move UDS part to later phase.

In addition, two out of the three students (Lily and David) demonstrated a steady increase on the criterion test (i.e., the MR-test) administered throughout the intervention but the increase was not significant. The third participant, Neal, showed inconsistent and unsteady performance. On the other hand, it seems that all three students significantly improved their performance in solving contextualized word problems on COMPS-test. There are several reasons for this result. First, COMPS component was taught at the last stage of the program and when the COMPs tests were administered, the students may still have fresh memory of COMPS knowledge, which caused them perform well in COMPs test. Second, along the dimension of MR, the current support build in the system might not be sufficient to address the disadvantages of students with LD, which might have contributed to the relatively lower performance on the MR-test. By comparison, the COMPs component involved some elements that might have better addressed the disadvantages of students with LD. For example, in the COMPs part, the important words of problems were highlighted and three key elements (Unit Rate, # of Unit and Product) were mapped to mathematical model equation, which contributed to better catching students' attention to the crucial parts. Further, the model equation drove the development of the solution plan.

However, in the MR component, there were many items on the screen without the important parts stressed. Students were easily distracted and got lost in information "overflow." Session observation data indicated that two of the students (Lily and Neal) had trouble to concentrate on the tasks.

Thirdly, MR-test was designed to assess students' multiplicative reasoning ability. There are several types of problems in the MR-test. For most of the one-step problem, participating students had more success. For example, in the pretest, the participants mainly used addition and subtraction for solution. But after the intervention was imposed, they knew to use multiplication and division for solution. However, participating students had more difficulties in solving two-step problems in the MR-test. If provided with more supporting strategies in solving two step problems, these students might have more access to the mathematics problems involved and their multiplicative reasoning might be better assessed. In addition to providing more scaffolding to these students, the future research might consider modifying the instructional sequences of the modules in the intelligent tutor.

Appendix



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CHILDREN'S CONCEPTUAL KNOWLEDGE OF TRIANGLES MANIFESTED IN THEIR DRAWINGS

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When asked to draw different kinds of triangles, children reveal many creative ways to express variety. In this paper, the drawings of 81 children in the age between 4 and 6 will be examined and illustrated what kind of understanding of the concept "triangle" precedes the drawings. Therefore, different categories of the children's drawings were generated and also compared to their explanations of a triangle, which sometimes might not be in agreement with their drawings.

INTRODUCTION

"Shape is a fundamental construct in cognitive development in and beyond geometry" (Clements & Sarama, 2009, p. 199). According to Vollrath (1984), a comprehensive conception of geometric shapes is shown through different aspects like being able to name the shapes, give a definition of the shapes, show and illustrate further examples of this category and name all properties. Although this description was given for secondary school children and beyond, it is a good summary of what constitutes a comprehensive understanding of the concept of shape. In this paper it will be focused on the aspect of showing and illustrating many examples of geometrical shapes. This aspect is investigated through the drawings of the children.

THEORETICAL AND EMPRICAL BACKGROUND

It must be considered that in order to draw an object correctly, it demands the *knowledge* as well as the *ability* to put this knowledge down on paper, the so called drawing skills. If these are yet undeveloped, a child is not able to draw a geometric shape even though it might know how such a shape looks like. All developmental models concerning drawing skills (e.g. Piaget & Inhelder, 1967; Schuster, 2000) start with a so called "stage of scrawling", which becomes more realistic and more detailed on each stage. Piaget gives a very detailed description how the drawing skills of children develop: from the age of three the scrawls become more differentiated and shapes showing properties like "inside" or "outside" can be illustrated. Here, copies of a circle, square and triangle all look the same. From age four onwards, basic shapes such as square, rectangle, triangle, circle and ellipses can be drawn, but only from age 6 on, complex shapes can be drawn. If a child is not able to copy or draw a certain shape, Piaget interprets that it is due to a lack of knowledge, obviously not considering a lack of drawing skills, one of the reasons why his results are criticised (e.g. Freudenthal, 1983; Battista, 2007). In mathematics mainly the knowledge, which lies behind the drawing is important. Still, drawing a shape correctly demands knowledge and drawing skills and therefore, "wrong" drawings should not serve as indicator for lacking knowledge. Kläger (1990) highlights the importance to never regard drawings of children isolated but to always complement these through interviews, in order to gain insight in the perceptions of the children. For children cannot be "generalized", they draw what they see but also more or less than they see, they draw what they know but also more or less than they know (cf. Kläger, 1990, p. 15f.). The van Hieles (van Hiele & van Hiele, 1986), who also created a hierarchical developmental description, which other researchers prefer to interpret as levels (Battista, 2007), constitute that children realize shapes as whole entities from the age of four onwards and are not able to distinguish shapes by their properties before primary school and are consequently not able to draw specific properties before that.

There are several studies (e.g. Battista, 2007; Burger & Shaughnessy, 1986; Clements Swaminathan, Hannibal, & Sarama, 1999; Razel & Eylon, 1990) investigating young children's understanding of showing and illustrating further examples of geometric shapes not through drawings but by letting the children for example distinguish between examples and non-examples. These studies showed that children had more difficulties in recognizing triangles which were identified correctly in all of the different studies by approximately 60% of the children, compared to squares (80% - 90%) or to circles, which were identified correctly by nearly all of the children in these studies. There are no circles deviating from the prototype and square prototypes only occur concerning position, but there are several triangles deviating from the prototype and thus making it harder to be identified correctly in all variations. Therefore, it can be concluded that if it is harder to identify several types of triangles, it is also harder to draw several types of triangles – where knowledge is lacking it cannot be put into a representation. Some studies (e.g. Burger & Shaughnessy, 1986, Clements et al., 1999) indicate that children's prototype of a triangle seemed to be an isosceles triangle. They found that the majority of children did not identify a long and narrow, scalene triangle as a triangle, although they often admitted that it has three lines and three corners, something which might be seen in drawings as well.

The "drawing triangles task", which will be presented in the following, was already conducted by Burger and Shaughnessy (1986) with a smaller sample and a larger range between the ages (from preschool to college) than in the empirical study reported in this paper. They found that younger children often vary their drawings by ending up with "new inventions", as for example a triangle with "zic-zac-sides", older children vary their drawings more according to the nature of triangles (equilateral, isosceles, rectangular or general triangles). The study at hand complements these studies by examining whether children also prefer *drawing* prototypical triangles and what kind of triangles are drawn as variations. Additionally it will be examined whether these drawings are in line with their explanations of triangles. Furthermore, the competencies of the children are illustrated in the light of two different educational settings.

DESIGN

The study comprises 81 children, 34 from England and 47 from Germany in the age of four to six, who were interviewed at the beginning and at the end of one school year. The children in England were attending a primary school (for children from 4 to 11), where the children enter school in the year when they have their fifth birthday, but many children go to a reception class before that. The German children were attending a kindergarten where children from the age of three to six (up to primary school) can go to. In Germany (Baden-Württemberg), learning through play and an approach using "everyday mathematics" is at present the main concept for kindergarten education, whereas in England, elementary education is rather systematic and curriculum based and the expected competencies are described as "stepping stones".

The study was conducted in the form of qualitative interviews, taking about 30 minutes each. The order of the tasks – there were nine tasks altogether – as well as the material was predetermined but in accordance with the nature of qualitative interviews this order could be altered or complemented. There were two points of data collection, without a special intervention, one at the beginning of the school year 2008/ 2009 and one at the end of the school year. The English children, in contrast to the German children, were instructed in geometry during the year. Each child was interviewed individually, so copying the drawings from each other was not possible.

In this paper, the results of the "explaining triangles task" and of the "drawing triangles task" will be illustrated. First, the children were asked to "explain a triangle to someone who has never seen a triangle before". Later, after some other tasks, they were asked whether they could draw a triangle and afterwards a triangle that looks different than the first one and again a triangle that looks different than the first two, and so on. With this, the children's idea of triangles as well as their idea of diverseness was tested. Then they were asked to explain their drawings. Afterwards, it could be seen whether their first, general explanation of triangles were in line with their drawings and the explanations of their drawings.

In order to analyse the drawings and explanations of the children, different categories were generated and discussed. Besides the interpretation of the qualitative data as small case studies, also quantified details will be given to show tendencies and to suggest hypotheses because quantitative details can be one aspect of qualitative reality (Oswald, 2010, p. 186).

The underlying research questions are:

1. What kind of triangles do children draw when asked to draw a triangle?
2. In how far do they vary their triangles when asked to draw another one (and again another one and so on) that looks different than the first one (two,…)?
3. In how far do the explanations of the children match their drawings?
4. Can any differences be observed comparing the results of the two educational settings or the two points of data collection?

RESULTS

In the following, it first will be illustrated what kind of triangles the children drew as well as their way of varying different triangles, before it will be compared to the explanations of the children, all in the light of the two different educational settings. When examining the drawings of the children, it was first looked at what kinds of triangles the children did draw. It was distinguished between: (1) no triangle, (2) a "made-up" triangle (i.e. a non-triangle), (3) an equilateral triangle, (4) an isosceles triangle, (5) a rectangular triangle or (6) a general triangle (e.g. acute or obtuse angled). At both points of investigation, the majority of the children drew an isosceles triangle as their first triangle (38% of the English at the beginning and 47% at the end of the school year and 33% of the German at the beginning and 51% at the end of the school year). Only a few children drew an equilateral or a rectangular triangle as first triangle, but a general triangle (no specific one) was drawn by 24% of the English and 16% of the German children at the first investigation and by 38% of the English and 26% of the German children at the end of the school year. No child started with a "made-up" triangle, for example shapes with three corners but "zic-zac-sides", but often used such "inventions" in order to alter their triangle.

For the variations of the children, the following categories were generated.

Identity – Child draws the same or similar triangle again and again;

Area – Child draws triangles in different sizes;

Angular dimension – Child draws triangles with different angles;

Position – Child draws triangles in different positions and directions;

Combination – Child draws triangles that differ at least in two of the following attributes: area, angular dimension or position;

Objects from everyday life – Child draws objects from everyday life having geometric shapes (for example road signs);

Shape – Child draws different shapes (triangles and "own inventions");

Missing critical attributes – Child draws a shape that is missing some critical attributes of a triangle, as for example a third side.

The children's drawings were either grouped into one of the categories *identity*, *area*, *angular dimension*, *position*, or, if they drew at least two of these varieties, they were grouped into *combination*. Moreover, they could be additionally grouped into one of the other categories: *objects*, *shape* or *missing attributes*. Therefore, the overall percentages might be more than 100%.

It became obvious that most children connected "different" triangles with triangles that differ in their *area* dimension (see Table 1). Here, the triangles are all pointing upwards and are most of the time isosceles or equilateral.

	Identity		Area		Angular		Position		Comb.		Objects		Shape		Missing	
	E	G	E	G	E	G	E	G	E	G	E	G	E	G	E	G
2008	18	7	50	49	15	9	6	5	6	7	0	9	29	21	3	3
2009	12	14	71	44	24	28	9	12	6	12	0	9	9	21	3	3

Table 1: Triangle Drawings in Percentages

At the beginning of the school year, there were a few more English children than German children drawing triangles with different angles and positions. At the end of the school year, it was the other way round: now, slightly more German than English children diversified triangles according to angles and positions. Triangles as part of the geometric solids in everyday life (e.g. street signs or tents) were only drawn by the German children. At the beginning of the school year, there were more English children drawing triangles varying in their shapes, but later there were more German children drawing triangles varying in their shape. In both countries, only a few children left out critical attributes such as one side.

The children used different ways (like answering with gestures, through comparisons, and other informal or formal ways) to explain or define a triangle as reply to the question: “Could you explain a triangle to someone who has never seen a triangle before?”. It could also be that children used several ways for their explanations and so the overall percentage could again be more than 100%.

	No expl.		Informal expl.		Formal expl.		Other expl.	
	E	G	E	G	E	G	E	G
2008	15	30	9	30	62	16	12	30
2009	0	23	21	49	62	14	18	28

Table 2: Explanation of Triangles in Percentage

The results lead to the impression as if the concept knowledge of the English children is already more developed than that of the German children, because the English children explained the triangles more often, compared to 30% of the German children at the first and 23% at the second investigation, who did not explain the triangles at all. Moreover, the English children explained about four times more than the German children the triangles in a formal way, giving a definition as for example:”three straight sides and three corners”. Still, the drawings of the English children did often not fit the preceding explanations, as was the case for 24% at the first and for 12% at the second investigation. Here, only the informal and formal explanations were regarded (and no other explanations), because it is quite complicated to compare a gesture or a verbal comparison (e.g. “hat of a witch”), for example, with the properties of a drawing.

The “triangles” of Emma at the beginning of the school year for example (see Figure 1 below) do not all look like real triangles. Emma was explaining a triangle as having “three straight sides and three corners”, a correct definition not in line with two of her actual drawings:

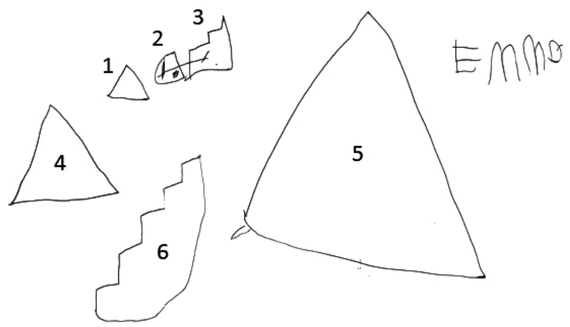

Figure 1: Emma (4,5 years), 2008, England

Interviewer:	Could you explain to me the difference between the triangles?
Emma:	This one is thinner (points to shape no. 6) and those two (points to no. 4 and no. 5) are fatter.
Interviewer:	Is there any other difference?
Emma:	No, they are all pointy. Some have two points and some have three points.

Emma might have seen the different spikes of the stairs as single triangles. So a triangle could also be part of any other object, her concept knowledge was still limited to that perception. However, at the end of the school year, Emma was drawing only triangles and no “made-up” shapes. Louis’ triangles also deviate into non-triangles in the end. He explained a triangle in an informal but correct way, saying that it has “three points and it comes straight up and it comes straight down”. He started with the triangle on the very left and ended with the shape on the right. When asked, whether all of these shapes are triangles, he answered:

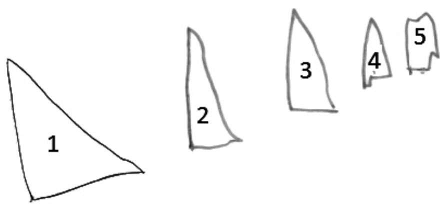

Figure 2: Louis (5,1 years), 2008, England

Louis:	Well, that one has got this bit (points to no. 4) and this one (no. 5) goes like that, it’s all a bit strange... not a proper triangle.
Interviewer:	How many corners does this shape have? (points to no. 5)
Louis:	(counts). 1,2,3,4,5!
Interviewer:	And how many corners does a proper triangle have?
Louis:	1,2,3 ...three!.
Interviewer:	But this is still a triangle? (points to shape no. 5)
Louis:	(nods) Yes!

It seems that Louis discerns between proper triangles who have three points and “other shapes” who have more points but can still named as triangles. Other children, who drew “made-up” triangles also diverted the sides of the triangles and drew “wavy” or “rocky sides”.

All the German children who were able to explain correctly (no matter whether this was informal or formal) what a triangle looks like, were also able to draw a triangle correctly. But there was no German child who knew a definition but was not able to connect it with a representation then. In contrast to this, to summarize the results above, the English children were often able to formulate a definition of a triangle but were not always able to connect this definition with a variety of representations.

DISCUSSION

The results revealed that the children in both educational settings mainly drew isosceles triangles, but it could not be detected whether their attempt was to draw an equilateral triangle and whether it just happened to limited drawing skills. It can be stated that prototype presentations were dominant not only for the first drawn triangle but also as varying triangles because most children varied their triangles through area size. It has to be discussed if the rare use of position as variation can be explained by the format of the paper and the horizontal orientation by drawing. Looking only on the drawings, no meaningful differences between the two educational settings can be asserted, except in drawing objects from everyday life or other shapes ("own inventions"). Although the English children were instructed in school and thus able to explain or define a triangle in most cases, their explanations did not always go in line with their drawings, presumably because they just knew the definition but could not connect it with a variety of representations.

CONCLUSION

Therefore, it can be concluded that instead of an isolated memorising of definitions and the limited use of only prototypical representations, which can rule children's thinking throughout their lives (Sarama & Clements, 2009, p. 216), already in preschool the focus should be more on the ability to connect a concept with many different representatives as examples. Teaching definitions should not be separated from showing different examples as well as from drawing shapes, otherwise it will be quite one-sided. Especially the drawing of triangles could be used not only as an assessment or research tool (as it was used here as well as in the studies before) but also as a teaching tool in everyday situations as well as in all kinds of teaching situations, because by drawing different triangles, different attributes can be easily demonstrated and consequently the drawing of triangles could help to build valid perceptions.

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CHARACTERISTICS OF UNIVERSITY MATHEMATICS TEACHING: USE OF GENERIC EXAMPLES IN TUTORING

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The aim of this paper is to report early findings from university mathematics teaching in the tutorial setting. The study addresses characteristics of the teaching of an experienced research mathematician and through interviews, her underlying considerations. An analysis of a teaching episode illuminates her use of generic examples to reveal aspects of a mathematical concept and links with the tutor's particular research practice, didactics and pedagogy emerge.

INTRODUCTION

Research on university teaching practice can inform mathematics education community's understanding of university mathematics teaching and produce resources that novice and experienced university teachers might access for professional development (Speer, Smith & Horvath, 2010). However, research regarding pedagogy and mathematics puts an emphasis on school mathematics teaching (Jaworski, 2003) and very little has been studied to date concerning the teaching practices and knowledge of university mathematics teachers (Speer & Wagner, 2009). Furthermore, the number of studies about teaching in the small group tutorial setting is still limited.

Petropoulou, Potari and Zachariades (2011, 2012) linked teaching practices in the format of lectures with research, teaching and studying experiences and argued that the process of thinking in mathematical research is used in university teaching (Petropoulou et al., 2011). The focus in this paper is on the characteristics emerging from one tutor's mathematics teaching and how they are linked with the sources of knowledge coming from her research practices and teaching (including epistemology, didactics and pedagogy). The aim of the wider study, on which this paper is based, is to produce understanding about mathematics teaching at university level; the setting of small group tutorials was selected since more opportunities of teacher-student interaction and dialogue emerge there.

THEORETICAL BACKGROUND

A systematic literature review from Speer et al. (2010) categorised published scholarship in university mathematics teaching and showed lack of research in actual university mathematics teaching practice. In particular, these authors report that most of the studies offer researchers' reflections on their own mathematics teaching and accounts of students' learning, and they insist that there is no systematic data collection and analysis focusing on teachers and teaching. They also make the distinction between teaching practice and instructional activities at university level, defining the

2014. In Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 4, pp. 161-168. Vancouver, Canada: PME.

former as teacher's judgements, thinking and decision making for the design, implementation and reflection on their teaching. Small group tutorials are seen as instructional activities along with lectures, whole class discussions, student's individual work on exercises and many other activity structures.

University mathematics education research is rapidly developing. A research focus has been the teaching of particular topics in undergraduate mathematics such as mathematical analysis (e.g. Petropoulou et al., 2011, 2012; Rowland, 2009) and linear algebra (e.g. Jaworski, Treffert-Thomas & Bartsch, 2009). The above studies are on the teaching of a large number of students in a lecture format; however, another focus of research is the teaching and learning of mathematics in alternative settings such as small group tutorials (e.g. Jaworski, 2003; Nardi, Jaworski & Hegedus, 2005; Nardi, 1996). In the context of small group tutorials, Jaworski (2003) investigated first year mathematics tutoring of six tutors. She distinguished tutors' exposition patterns as the main teaching aspect, with the most prevalent ones to accord with *tutor explanation*, *tutor as expert* and forms of *tutor questioning* and stressed that the teaching-learning interface is idiosyncratic to the tutor and to some degree to the particular students. Nardi et al. (2005) studied tutor's thinking processes collecting their interpretations of incidents from their teaching in small group tutorials concerning three strands: *tutor's conceptualizations of students' difficulties, tutor's descriptive accounts of pedagogical aims and practices with regard to these difficulties* and *tutor's self-reflective accounts with regard to these practices*. They produced a spectrum of tutor's pedagogical awareness with four dimensions namely *Naive and Dismissive, Intuitive and Questioning, Reflective and Analytic* and *Confident and Articulate* and indicated a development in tutor's readiness to respond in self-reflection questions over time. Nardi (1996) conducted her PhD research in undergraduate tutorials and explored the learning difficulties that first year students experienced in mathematics. Subsequently, Nardi (2008) investigated mathematicians' perceptions of their students' learning and reflections on their teaching practices based on data rooted in their small group tutorials and students' work. In the above studies, the tutor's teaching practice is examined in accordance with his or her students' learning outcomes and difficulties. In our study, we are interested in the tutor's teaching practice as well as the sources of knowledge that frame it. In this paper, we focus on the tutor's teaching on features of mathematical concepts by using generic examples.

Petropoulou et al. (2011) introduced the idea of an example used to illustrate critical characteristics of concepts as one of the lecturer's strategies to construct mathematical meaning in lectures. Our interpretation is that the use of an example to illustrate critical characteristics of concepts is what other researchers call a generic example. A generic example is an example that is presented so as to carry the genericity ("the carrier of the general") inherently (Mason & Pimm, 1984, p. 287). In other words, the general (argument) is embedded in the generic example "endeavoring to facilitate the identification and transfer of paradigm-yet-arbitrary values and structural invariants within it" (Rowland, 2002, p. 176). An example-of-a-generic-example that Rowland

(2002) routinely chooses for the introduction of the notion "generic example" is the calculation of the sum from 1 to 100 with Gauss' method. Gauss added 1 to 100, 2 to 99 and, so on, and computed fifty 101s. The genericity of his method is that it can be generalised to find the sum of the first $2k$ positive integers, which is $k(2k+1)$. The sum from 1 to 100 is a generic example of Gauss' method and as such it is "a characteristic representative of the class" (Balacheff, 1988, p.219, cited in Rowland 2002) of the sum of the first $2k$ positive integers. Nardi et al. (2005) reported that the use of generic examples was amongst the most discussed strategies that tutors used to enhance their students' concept image in tutorials.

METHODOLOGY

The context of the study

The study is being conducted in small group tutorials for first year mathematics students at an English University. Tutorials are 50 minute weekly sessions and a group include 5 to 8 students. Tutors are lecturers in modules offered by the mathematics department and conduct research in mathematics or mathematics education. The modules that are usually tutored are analysis and linear algebra, but tutors are sometimes flexible to provide assistance in other modules, as well. This study is part of a PhD project, which draws on data of tutorials of 26 tutors and data that systematically follow three out of the 26 tutors for more than one semester. Zenobia is one of the three tutors. She is an experienced lecturer, holds a doctorate in mathematics and does not prepare a design for her tutorial. Her tutees decide what questions or topics they all struggle with and usually select with her to deal with one or two relevant exercises from the problem sheets that follow each chapter in lectures. During the tutorial time and through the exercises, Zenobia put emphasis on concepts and mathematical thinking rather than computations.

Data collection and analysis

The first author observed, audio recorded and transcribed Zenobia's small group tutorials. Observation notes were also kept and a discussion with Zenobia about each tutorial was audio recorded and transcribed. The discussions concerned characteristics related to Zenobia's teaching practice and from her reflections we gained insight into her underlying considerations. The characteristics emerged through a grounded analytical approach of a small number of tutorials and were subsequently traced throughout the data. The following episode has been selected from a vast amount of data as a paradigmatic case that characterises the use of generic examples allowing us to reveal key issues in practice.

RESULTS

In Zenobia's tutorials, characteristics were identified through the process of coding and categorisation. These involved the use of examples to practice algorithms before tackling proofs in a more abstract setting; graphs to provide a visual intuition for formal representations; repertoires of strategies and techniques for the work on mathematics; minimal information for elegant proofs; questioning in students' valid or invalid definitions and claims for conceptual understanding; counterexamples to refute invalid arguments; and generic examples to reveal features of mathematical concepts.

In this paper, we analyse a teaching episode from a small group tutorial, which was about calculus revision for exam preparation purposes. One of the exam questions was to show that a function is bijective and for this exercise, they chose to work first of all on injectivity. This episode concerns the use of generic examples to reveal that "a strictly monotonic function (in other terminology monotonically increasing or monotonically decreasing function) on an interval of its domain is injective on this interval", a property that can be used as an alternative to the definition to prove injectivity on an interval. Before showing injectivity for the exam question, Zenobia used these examples and the following discussion occurred:

Zen: Are there any kinds of functions that you know are going to be injective, for instance? Is there anything about a function that you… Ok. So, let's draw some functions on the board, shall we? So, here's an example of a function. [The graph of $f(x)=x^2$]. And here's another example of a function. [The graph of $f(x)=sin(x)$.] And here's an example of a function [the graph of $f(x)=ln(x)$], and here's an example of a function [the graph of $f(x)=x$]. So, if you wanted to determine some domains on which all of these are injective, how would you do it? How would you do it for this one? [Zen. points to the graph of $f(x)=x^2$]. How would you find your domain of injectivity? Is it injective on anything?

S1: From 0 to ∞. [Zen. draws a red line from 0 to ∞ to show the domain on which $f(x)=x^2$ is injective.]

Zen: Right. This is definitely not injective on the whole thing, right? Because if I go off in opposite directions, I'm going to the same thing. Ok. But if I go from here on, that's injective, right? Ok. And what about down here? [Zen. shows the graph of $f(x)=sin(x)$.] Do you want to have a go at that? You're very close. I know you can do it. Just draw a little red line on the domain axis.

S2: I hope I'm right. I think. [S2 draws a red line from 0 to ∞.]

Zen: You think? Ok. So, what does "injective" mean? It means that there shouldn't be any two points that are at the same height. No, that's definitely not right.

S2: Can you have two parts to the domain? [S2 draws a red line from -π to π.]
Zen: I guess you could, sure. You just do it. I mean, it's conventional to choose a connected interval, but you don't have to.
S2: It must be from here to here. [S2 draws a red line from –π/2 to π/2.]
Zen: Excellent. Good, good. Right. So, what did you notice? You noticed that you can't have it go up and down, basically.
Zen: So, what can I say about… Ok, what about this function? [Zen. shows the graph of $f(x)=ln(x)$.] Is this injective? Is this an injective function?
S3: Yeah. It is injective.
Zen: It is injective. What about this one? [Zen. shows the graph of $f(x)=x$.]
S3: Yeah.
Zen: Ok. So, what can you say about this part of this function, this part of this function, this function and this function? [Zen. shows the previous functions restricted on the domain of injectivity.] What do they all have in common?
S4: They're monotonically increasing.
Zen: They're monotonically increasing, right. So, a function that's either monotonically increasing or… I could easily have chosen this, instead. [Zen. plots the graph of $f(x)=log_a(x)$ where $0<\alpha<1$.] I could have chosen this part instead. [Zen. shows the graph of $f(x)=sin(x)$ restricted on [π/2, 3π/2].] So, either monotonically increasing or monotonically decreasing is automatically going to be injective.

Using a range of examples, which included very simple and more complicated ones, Zenobia attempted to build up students' awareness of the feature of monotonicity on an interval: "a strictly monotonic function is injective". Through these examples, she introduced layers of generality of monotonicity on an interval so that her students could connect monotonicity on intervals with injectivity. In my discussion with her, she referred to the four functions as "standard" ones meaning that they "have many applications" and "are very special classes of functions; polynomial, trigonometric and logarithmic functions". On lines 4-6, we see the first three examples she devised (the graphs of $f(x)=x^2$, $f(x)=sin(x)$ and $f(x)=ln(x)$) each of which is a generic example of the feature of monotonicity on an interval. All four functions [lines 4-7] have the property that makes them monotonic on an interval; however, they should have a high level of generality about them in order to be generic examples. The linear nature of the graph of $f(x)=x$ indicates that it is not a generic example, since not all strictly monotonic functions are linear [line 7]. However, this function along with the parabola fit in the class of polynomial functions and also, the linear function is odd and the parabola is even. The logarithmic function carries more generality of monotonicity on the domain than the linear function, because it has not null curvature and its domain is not the whole ℝ. The periodic trigonometric function can be divided into intervals where it is either monotonically increasing or monotonically decreasing. As Zenobia argues:

> Everything you see in polynomials [regarding monotonicity on intervals] is already seen in these two functions so adding any additional polynomial you don't get anything new, whereas you never see periodicity or natural domain less than a whole axis in polynomials.

In the interview, she also informed us about her didactical and pedagogical intentions and links between the particular epistemology, didactics and pedagogy emerged:

The particular epistemology

In Zenobia's discussion with the first author, reflecting on her teaching approach with generic examples, she related it to the research mathematicians' practice of "decoding and encoding". She drew on her research practice experiences and explained:

> The first step [in doing research] is the decoding where you are given a problem and you have to understand what the problem is, what everything mean [e.g. by experimenting with images against definition], why it is a problem; the second step is with this picture that you have got from the decoding process, you get some intuition, you play around with things in your head a little bit and then you get this sort of 'aha I figured it out, I have got this idea now of why that works' and then you have got the encoding process [i.e. the third step] where you write it down [formally]. […] [In this teaching episode,] through examples I tried to extract from the complicated language that core intuition [of step 2]. I tried to teach them to decode the problem to something where they can sort of see 'oh of course that's how it works' and then figure out how to write it in their proof back into a formal language.

Zenobia also related her inductive thinking approach to a way of producing mathematical definitions in research informed by the history of mathematics.

> It's a situation where –from the set of examples that we have– we've come up with an ideal idea, and then we can actually rigorously then check that something is in that or not. […] We come up with new definitions any time we recognise that there are some sets of structures that have some relevance. But it really does emerge out of the examples. *And if you look at the history of mathematics, it's not that people have had the idea of a function. It's that they've had lots of examples of functions and they've tried to distil what the critical characteristics of a function are.* So, I think it's a very natural way to think about the relationship between examples and theories – it's that we don't define definitions just off the tops of our heads. We define them because they capture a behaviour we see in examples that have interesting kinds of properties.

The didactics

The teaching episode provides an example where through the use of the four functions Zenobia applies in her teaching the process of decoding of what it means to be strictly monotonic and what it means to be injective. The three out of four functions are carefully selected so as to be generic examples of monotonicity on intervals. Then, the observation of the commonality among the graphs of the functions leads to the "core intuition" (of step 2 in research) that 'a strictly monotonic function is injective'. The encoding process is the writing of the proof that 'a strictly monotonic function is injective'; however, this step is not included in this episode.

An interesting detail is that the last excerpt, concerning the history of mathematics, fits into Zenobia's discussion with her students in the tutorial studied, after the solution of the exam question. In the italic text of this extract, she explained to students the historical roots of the use of generic examples to reveal features of concepts. From a didactical point of view, through this discussion Zenobia enculturated students into this mathematical practice of the community of mathematicians and gave them historical evidence of its significance. Discussing with her, she reflects:

> I wanted to explain to them what it is to be a mathematician. […] It is important for students to learn this [encoding and decoding] process because that's a lot of the process of doing mathematics. And I think that's a lot of what mathematicians do on a daily basis.

The pedagogy

The teaching episode is also an example of her teaching in practice towards her aims (e.g. students' enculturation into being mathematical). The decoding process and the "core intuition" are collective among the students since different students showed the intervals of injectivity [lines 11, 26, 31, 33] and noticed the common property [line 38]. Zenobia focused all students' attention on the key features of monotonicity on intervals by explanations to each student concerning his or her claim [lines 13-15, 27-28, 39-44]; rhetorical questions [lines 20, 27] and the question about the commonality of functions restricted on intervals where they are monotonically increasing [lines 34-37]. Focusing on her explanations to students during the decoding process [lines 13-15, 27-28], we extract her use of everyday language. In the interview, she stresses:

> At this [intuitive view] point I am not trying to make them phrase things in a mathematical language. I do that quite a bit when I am trying to get into the intuition first and I really don't want to burden it with technical vocabulary. I bring up the vocabulary later and by the end I really make them put things in a very strict mathematical formulation.

CONCLUSIONS

In this paper, we reported characteristics of tutorial teaching, which occurred in one tutor's small group tutorials. The teaching episode discussed was "a particular moment in the zoom of a lens" (Lerman, 2001, cited in Jaworski 2003) and illuminated one of these characteristics. Zooming in we provided an example of the characteristic "generic examples to reveal features of mathematical concepts" and the suggestion of a path of informing: from the history of the development of mathematics to the research practices of the mathematician to his or her didactics to his or her pedagogy. This path contributes to mathematics education community's understanding of university mathematics teaching and offers a way to lecturers to consider their teaching, with regard to their epistemology, didactics and pedagogy, in order to enculturate students into the core of mathematics and being mathematical. An example of how a tutor implements her teaching towards this aim is revealed through the teaching episode. Zooming out and from analyses as a whole, we suggest that teaching practices are informed by research practices. This influence accords with findings in the format of lectures (Petropoulou et al., 2011, 2012) and has implications to the pedagogy of the tutor. In future studies, we will analyse data from the other tutors and search for common or different characteristics in teaching and underlying considerations.

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CARDINALITY AND CARDINAL NUMBER OF AN INFINITE SET: A NUANCED RELATIONSHIP

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This case study examines the salient features of two individuals' reasoning when confronted with a task concerning the cardinality and associated cardinal number of equinumerous infinite sets. The APOS Theory was used as a framework to interpret their efforts to resolve the "infinite balls paradox" and one of its variants. These cases shed new light on the nuances involved in encapsulating, and de-encapsulating, a set theoretic concept of infinity. Implications for further research are discussed.

This research explores the intricacies of reasoning about, and with, concepts of infinity as they appear in set theory – i.e., as infinite sets and their associated transfinite cardinal numbers. The APOS (Action, Process, Object, Schema) Theory (Dubinsky and McDonald, 2001) is used as a lens to interpret participants' responses to variations of a well-known paradox which invite a playful approach to two distinct ideas of infinity: *potential infinity* and *actual infinity*. According to Fischbein (2001), *potential infinity* can be thought of as a process which at every moment in time is finite, but which goes on forever. In contrast, *actual infinity* can be described as a completed entity that envelops what was previously potential. These two notions are identified with process and object conceptions of infinity, respectively (Dubinsky et al., 2005), with the latter emerging through the encapsulation of the former. Borrowing APOS language, this study explores the question of "*how to act*?" – a question which speaks to the mental course of action an individual might go through when reasoning with concepts of infinity, as well as to how an action in the APOS sense may be applied.

PARADOXES OF INFINITY

In this study, two versions (P1 and P2) of the infinite balls paradox are considered. P1:

> Imagine an infinite set of ping pong balls numbered 1, 2, 3, …, and a very large barrel; you will embark on an experiment that will last for exactly 60 seconds. In the first 30s, you will place balls 1 – 10 into the barrel and then remove ball 1. In half the remaining time, you place balls 11 – 20 into the barrel, and remove ball 2. Next, in half the remaining time (and working more quickly), you place balls 21 – 30 into the barrel, and remove ball 3. You continue this task ad infinitum. At the end of the 60s, how many balls remain in the barrel?

Briefly, the normative resolution to P1 compares three infinite sets: the in-going balls, the out-going balls, and the intervals of time. This resolution relies on two facts: (1) A set is infinite if and only if it can be put into a bijection (or one-to-one correspondence) with one of its proper subsets; and (2) Two infinite sets have the same cardinality (or 'size') if and only if there exists a bijection between them (Cantor, 1915). (The cardinalities of infinite sets are identified by transfinite cardinal numbers – a class of numbers that extends the set of natural numbers. While many of the properties of transfinite cardinal numbers are analogous to properties of natural numbers, there are some important exceptions, illustrated below.) Using facts (1) and (2), one may show that although there are more in-going balls than out-going balls at each time interval, at the end of the experiment the barrel will be empty – all of the sets are infinite, the cardinalities for all sets are the same, and since the balls were removed in order, there is a specific time interval during which each of the in-going balls was removed.

A variation to the paradox can easily be imagined. Consider the following, P2:

> Rather than removing the balls in order, at the first time interval remove ball 1; at the second time interval, remove ball 11; at the third time interval, remove ball 21; and so on… At the end of this experiment, how many balls remain in the barrel?

The difference between P1 and P2 is a subtle matter of which balls get removed – balls 1, 2, 3, … in P1, and balls 1, 11, 21, … in P2. The consequence is that although both experiments involve the same task (subtracting a transfinite number from itself), the results are quite different: P1 ends with an empty barrel; P2 ends with infinitely many balls in the barrel (balls 2-10, 12-20, etc.). Taken together, the two paradoxes illustrate an anomaly of transfinite arithmetic – the lack of well-defined differences.

BACKGROUND

Classic research into learners' understanding of infinity has centred predominantly on strategies of comparing infinite sets (e.g., Fischbein, et al., 1979; Tsamir & Tirosh, 1999; Tsamir, 2003). While a more recent trend has looked toward infinite iterative processes (e.g. Radu & Weber, 2011), power set equivalences (e.g. Brown, et al., 2010), and paradox resolution (e.g. Dubinsky, et al., 2008; Mamolo & Zazkis, 2008).

In the classic studies, participants were given pairs of sets and asked to compare their cardinalities. A common approach by participants was to reflect on knowledge of related finite concepts and extend these properties to the infinite case. For example, students were observed to compare sets in ways that are acceptable for finite sets, such as reasoning that a subset must be smaller than its containing set, but which result in contradictions in the infinite case (see fact (1) above). Furthermore, students were observed to rely on different and incompatible methods of comparison depending on the presentation of sets. If (e.g.) two sets were presented side-by-side, students were more likely to conclude the sets were of different cardinality than if the same sets were presented one above the other. Radu and Weber (2011) similarly found that students reasoned differently depending on the context of the problem – when infinite iterative processes were presented via geometric tasks, students reasoned about "reaching the limit", while an abstract vector task "evoked object-based reasoning" (p. 172).

In their work on power set equivalences, Brown and colleagues observed that while students "demonstrated knowledge of the definitions of the objects involved, all of the students tried to make sense of the infinite union by constructing one or more infinite processes" (McDonald & Brown, 2008, p. 61). These attempts were made despite the problem being stated in terms of static objects. Dubinsky et al. (2008) explored the process-object duality in a variant of P1, and observed a common strategy of "trying to apply conceptual metaphors" but noted that "the state at infinity of iterative processes may require more than metaphorical thinking" (2008, p. 119).

This study extends on prior research by using paradoxes to explore the nuances involved in reasoning with and about transfinite cardinal numbers. With APOS as a lens, this study offers a first look at participants' understanding of "acting" on transfinite cardinal numbers via arithmetic operations, focusing in particular on the challenges associated with the indeterminacy of transfinite subtraction.

THEORETICAL PERSPECTIVES

Due to space limitations, familiarity with the APOS Theory is taken for granted (see Dubinsky and McDonald, 2001 for details), and we focus on aspects which most closely relate to conceptualizing infinity. Dubinsky et al. (2005) suggest that "potential and actual [infinity] represent two different cognitive conceptions that are related by the mental mechanism of encapsulation" (2005, p. 346). Specifically, potential infinity corresponds to the imagined Process of performing an endless action, though without imagining every step. They associate potential infinity with the unattainable, and propose that "through encapsulation, the infinite becomes cognitively attainable" (ibid). That is, through encapsulation, infinity may be conceived of an Object – a completed totality which can be acted upon and which exists at a moment in time.

Brown et al. (2010) elaborate on what it means to have an encapsulated idea of infinity. Such an object is complete in the sense that the individual is able to imagine that all steps of the process have been carried out despite there not being any 'last step'. To resolve the issue of a complete yet endless process, Brown et al. introduce the idea of a transcendent object – one which is the result of encapsulation yet which is understood to be "outside of the process" (p. 136), that is, the object or "state at infinity is not directly produced by the process" (p. 137). Recalling P1, the empty barrel at the end of the experiment corresponds to what Brown et al. refer to as the state at infinity. As an object, it is transcendent since it is not produced by the steps of the process itself, but instead through encapsulation of the process. In accordance with APOS, Brown et al. consider encapsulation to be catalysed and characterized by an individual's attempt to apply actions to a completed entity. They offer an example to support Dubinsky et al.'s claim that while actual infinity results from encapsulation, the "underlying infinite process that led to the mental object is still available and many mathematical situations require one to de-encapsulate an object back to the process that led to it" (2005, p. 346). Brown et al. (2010) observed that the de-encapsulation of an infinite union back to a process was helpful in applying evaluative actions to an infinite union of power sets.

Weller, Arnon, and Dubinsky recently suggested a refinement to the APOS Theory, which includes a new stage they term *totality*. This refinement emerged from their analysis of students' understanding of 0.999… = 1. They observed differences among participants who "reached the Process stage but not the Object stage", and suggested an intermediate stage, wherein individuals would progress from process to totality and then to object. They noted that: “Because an infinite process has no final step, and hence no obvious indication of completion, the ability to think of an infinite process as mentally complete is a crucial step in moving beyond a purely potential view” (2009, p. 10). The authors suggested that while some of their participants could imagine 0.999… as a totality (e.g. with all of the 9’s existing at once), they were not able to see 0.999… as a number that could be acted upon. They suggest that the totality stage may be necessary for encapsulation of repeating decimals.

In this study, the paradoxes P1 and P2, when taken together, offer a situation similar to, but different in important ways from, previous lenses through which to interpret individuals’ understanding of infinity. As mentioned, prior research indicates that de-encapsulation has been connected to learners’ successes in applying actions to the object of infinity. The studies address different contexts of infinity, but share a common feature: they examine instances in which de-encapsulation makes use of properties of a process that extend naturally to the object. In contrast, P1 and P2 offer a way to explore transfinite subtraction, whose indeterminacy suggests a potential problem with de-encapsulation. The extent to which properties of the process may be relevant to properties of the object of infinity and the question of what other situations may or may not require de-encapsulation of an object in order to facilitate its manipulation are still open. This study takes an important step in that direction by exploring the following related questions: (1) How does one “act on infinity”? And (2) What can the “how” tell us about an individual’s understanding of infinity? As indicated above, the “how” refers to both the mental course of action an individual might go through when attempting to reason with actual infinity, as well as to how in the APOS sense an action (in this case transfinite subtraction) may be applied.

PARTICIPANTS AND DATA COLLECTION

For the purpose of this proposal, data from two participants with sophisticated mathematics backgrounds, Jan and Dion, will be considered. Jan was a high-attaining fourth year mathematics major who had formal instruction on comparing infinite sets via bijections. Dion was a university lecturer who taught prospective teachers in mathematics and didactics, the curriculum for which included comparing cardinalities of infinite sets. Neither participant had experience with transfinite subtraction.

Data was collected from one-on-one interviews, where participants were asked to respond to the paradox P1. Following their responses and justifications to P1, participants were asked to address the variant P2. A discussion of the normative resolution to P2 ensued, after which participants were encouraged to reflect on the two thought experiments and their outcomes. Jan and Dion were chosen for this study because they both resolved P1 correctly within the normative standards mentioned above, and because of their object-based reasoning which emerged in contrast to prior research (e.g. Mamolo & Zazkis, 2008; Dubinsky et al., 2008). As such, results and analyses will focus on their responses to P2 in comparison to their approaches to P1.

RESULTS

As mentioned, both Dion and Jan resolved P1 with appropriate bijections and language which referred to the sets as completed objects. When addressing the comparison between sets of balls and time intervals, both participants explained that the cardinalities were the same, and that "every ball that is put into the barrel is removed."

Jan's response to P2 was consistent with her approach to P1 – that is, she reasoned abstractly and deductively with the form of set elements, with sets as completed totalities, and with formal properties and definitions. She observed that "transfinite cardinal arithmetic doesn't work exactly like finite cardinal arithmetic" and connected her understanding of correspondences between infinite sets to explain the indeterminacy of transfinite subtraction. She elaborated:

> By assumption, only the balls 1, 11, 21, 31,.... are removed, (i.e. All balls of the form 10n+1 for n=0,1,2,3...). Now f(n) = 10n + 1 is not a bijection from the set of naturals to itself, since for example, there is no natural n such that f(n) = 2, so f(n) is not onto. So at first, one might guess that "the infinity of balls put in is somehow greater than the infinity of the balls removed". However! here we get into the indeterminacy of the "quantity" infinity minus infinity… The set of balls put into the barrel DOES have the same cardinality as the set of balls removed from the barrel, since there is a bijection between the set **N** of all naturals and the set [*writes*] S = {10n+1 | n is a natural number}, namely f(n) = 10n+1, which IS a bijection from **N** to S, but NOT from **N** to **N**. But even though there is a bijection… there are still an infinite number of balls left in the barrel after the minute is up! This is because **N** … is equinumerous with a proper subset of itself.

Thus, Jan realized that although the quantity of balls taken out of the barrel was the same as the quantity put in, this was not sufficient to conclude that *all* of the balls had been removed. She observed that remaining in the barrel was the set of balls numbered

> {10n+2 | n=0,1,2,...}. This set is clearly infinite, and represents a subset of the balls left... Since the set of balls left contains an infinite subset, it too must be infinite… we have changed the remaining balls from zero to infinity!

In contrast, Dion's response to P2 showed a shift in attention from describing cardinalities of sets to enumerating their elements. He used language consistent with a process conception of infinity, and overlooked the specific form of the set. While Dion commented on the similarities between P1 and P2 as well as the relevance of Cantor's work to his solutions, he reasoned with P2 informally, rather than deductively. When addressing P2, Dion noted that, as in P1, there existed bijections between pairs of sets of in-going and out-going balls and time intervals. He concluded that the variant and the "ordered case" should yield the same result: an empty barrel. When asked to elaborate, Dion argued for an empty barrel because "after you go [remove] 1, 11, 21, 31, …, 91, etc, you go back to 2" – language that describes a process of moving balls. During the interview, Dion struggled with the idea of a nonempty barrel. He stated:

> If ball number 2 is there, so is ball 2 to 10, etc… so, infinite balls there? I have trouble with that. (long pause) I have a strong leaning to Cantor's theorem (*sic*) and to use one-to-one… I want to subtract, but I can't.

Eventually, Dion conceded he was "convinced" of the normative solution to P2 since "you can't reason on infinity like you do on numbers", and he observed that while "on one hand infinite minus infinite equals zero, on the other it's infinite" – a property of transfinite arithmetic that was new to him.

DISCUSSION

This study delves into the conceptions of two individuals with sophisticated mathematics backgrounds, as elicited by variations of the infinite balls paradox, with the intent to shed new light on the intricacies of accommodating the idea of actual infinity. Dubinsky et al. (2005) proposed that the idea of actual infinity emerges from the encapsulation of potential infinity, and is recognised by an individual's ability to apply actions and processes to completed infinite sets. This study is a first look at individuals' understanding of 'action' given the nuanced relationship between an infinite set and its associated transfinite cardinal number. The issue of transfinite subtraction is explored and a first attempt is made to address the relationship amongst encapsulation of infinite sets and transfinite cardinal numbers, and the manner in which an individual applies actions to those entities.

How does a learner act on infinity?

In the context of set theory, actual infinity can be conceptualized in two ways – as the encapsulated object of a completed infinite set (to which bijections can be applied), and as the encapsulated object of a transfinite number representing the cardinality of an infinite set (to which arithmetic operations can be applied). Focusing on arithmetic operations, the data reveal two ways an individual may attempt to "act on infinity": (i) by deducing properties through coordinating sets with their cardinalities and element form, and through the existence of bijections between sets; and (ii) by de-encapsulating the object of an infinite set to extend properties of finite cardinals (elements of its conceptualization as a process) to the transfinite case. Exemplifying the former was Jan's reasoning with and resolution of the P2. Jan's ability to deduce consequences of a set being equinumerous with one of its proper subsets was showcased throughout her response. She consistently used language that referred to sets as completed totalities, reasoning with the form of elements (e.g. 10n+1) and bijections, rather than relying on enumerating elements (e.g. 1, 11, 21,…) to describe sets and relationships. Jan's response indicates that she consistently reasoned with the encapsulated object of an infinite set, using its properties to make sense of the paradoxes. Her approach allowed her to transition from acting on sets to acting on cardinals and contributed to her understanding of the indeterminacy of transfinite subtraction, allowing her to "act" – both by comparing sets and by applying arithmetic operations – in ways that are consistent with the normative standards of Cantorian set theory.

In contrast, Dion, who revealed a normative understanding of infinite set comparison in his resolution of P1, struggled during his engagement with P2. His attention to the process of removing balls ("go back to 2") suggests that Dion had de-encapsulated infinity (conceptualized as an infinite set) and tried to reason with properties of the process in order to make sense of applying the action of transfinite subtraction to the object of infinity (conceptualized as a transfinite cardinal number). This approach is consistent with other attempts to reason with infinity (e.g., Brown et al., 2010), however, in Dion's case, this lead to considerable frustration and self-described "trouble". Dion's struggle may be attributed to attempts to make use of properties of a process of *infinitely many finite entities* rather than make use of properties of an object of *one infinite entity*. In the case of subtraction, properties of the former are inconsistent with properties of the latter, whereas this is not necessarily the case with other actions. When Dion was faced with a non-routine problem regarding transfinite subtraction, he "acted" by de-encapsulating infinity, making use of the process and generalizing his intuition of subtracting finite cardinal numbers, and thus experienced difficulty with the indeterminacy of subtracting transfinite cardinals.

What can the "how" tell us about an individual's understanding of infinity?

Dion's difficulty and Jan's success with P2 suggest that:

- It is possible to conceptualize an infinite set as a completed object without conceiving of a transfinite cardinal number as one;
- De-encapsulation of an infinite set in order to help make sense of an encapsulated transfinite cardinal number is problematic; and
- In set theory, accommodating infinity involves more than being able to act on infinite sets, and includes knowledge of how to act on transfinite cardinals.

Further, Dion's difficulty highlights the importance of acknowledging the distinction between how actions or processes behave differently when applied to transfinite versus finite entities as an integral part of accommodating the idea of actual infinity. Through Dion's frustration that "I want to subtract, but I can't", and his insistence that "at some point we'll get back to 2" a conflict emerged that was difficult for him to resolve. Dion's realization that "you can't reason on infinity like you do on numbers", was important: it helped convince him of the normative resolution to P2.

Dion's struggle to re-encapsulate infinity in order to appropriately apply transfinite subtraction indicates that an understanding of how actions ought to be applied is relevant to the encapsulation of a cognitive object. Although Dion seemed able to consider the infinite sets of ping pong balls as completed entities which could be compared, his understanding of infinity nevertheless lacked one of the fundamental features that contributed to Jan's profound understanding: the knowledge of how to act on transfinite cardinal numbers. In Jan's words, "it is nearly impossible to talk about it [infinity] informally for too long without running into entirely too much weirdness".

An important contribution of this study distinguishes between the object of an infinite set and the object of a transfinite cardinal number, and identifies the significance of understanding properties of transfinite arithmetic in order to accommodate the idea of actual infinity. While there is substantial research focused on individuals' reasoning with cardinality comparisons, how individuals conceptualize transfinite subtraction has not previously been addressed. Jan and Dion illustrate two ways to try and make sense of transfinite subtraction: via deduction that coordinated completed sets and their cardinalities or via the use of properties of an infinite process through de-encapsulation. Taking into account the newly identified stage of *totality* in a genetic decomposition of infinity (e.g., Weller et al., 2009), questions also arise about the relationship and tensions between object, process, totality, and the de-encapsulation of an object to make use of properties of its conception as a process.

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MATHEMATICAL CERTAINTIES IN HISTORY AND DISTANCE EDUCATION

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An historical case is presented in which extra-mathematical certainties lead to invalid mathematics reasonings, and this is compared to a similar case that arose in the area of virtual education. A theoretical-methodological instrument is proposed for analysis of certainties. The article suggests the need for teachers to be aware that certainties of mathematics facts are not always based on mathematics understandings.

BACKGROUND AND OBJECTIVES OF THE PAPER

In Euclid's *Elements*, the author supported his theory of the parallels in the Fifth Postulate; there he established that two lines that are not equally inclined in relation to a third line will always have to intersect. Said proposal engages a behavior in the infinite (Kline & Helier, 2012), hence throughout history mathematicians resorted to different means to convince themselves of their truth—states Lovachevski (1974, pg. 2). Saccheri, for instance, decided to establish that truth by resorting to a double reduction to the absurd: denying the existence (no parallel to *l* crosses *P*), and denying unicity (more than one line crosses P). Denial of the existence produced a contradiction. From the second possibility, Saccheri deduced theorems that, albeit contradiction-free, seemed odd to him. This was sufficient for him to reject the second possibility, from which he derived the veracity of the Fifth Postulate as the sole possible option. According to Kline & Helier (2012, pg. 508), "when Saccheri concluded that the Fifth Postulate was the necessary consequence of the others, he was only able to show that when a person intends to establish something of which s/he is already convinced, s/he will be satisfied even if his/her demonstration has nothing to do with the facts."

Another attempt to demonstrate the parallels postulate arises in Legendre. In 1800, he published, according to descriptions by Lovachevski, that the sum of the angles of a triangle cannot be greater than 180°. He moreover argued that said sum could not be less than 180°. From his analysis, Lobachevski deducted that Legendre's reasons were incorrect and that "the biases in favor of the position accepted by all had probably induced him at each step to precipitate his conclusions or add what was still not legitimate to admit in the new hypothesis" (1974, p. 3).

In a critical reading of history, Lovachevski questioned the absence of logical rigor of the demonstrations of the Fifth Postulate; he objected to the ontology and idealistic epistemology that was the foundation of those attempts, by suggesting that "the concepts themselves did not encompass the truth that he wanted to demonstrate" (1974, pg.1) and by raising an empirical route as the alternative proof, by way of astronomical observations. With an open spirit, he built hyperbolic geometry, admitting with it "the existence of Geometry in a broader sense than what Euclid has presented" (1974, p. 1).

This passage through history illustrates how biases—taken on by Saccheri or Legendre- can disturb mathematics reasoning, and how certainty and convincement of mathematics facts can be strongly tied to extra-mathematical sources, such as ontological or epistemological commitments. The subsequent text contains arguments based on empirical evidence derived from a case study (Mariana), that that historical phenomenon associated with convincement and certainty also arises in mathematics instruction processes. The regularity of that phenomenon in such dissimilar arenas suggests, in one way or another, its generality, and raises the need for teachers to have knowledge of it and consider it in their didactic practices.

Research on certainty and convincement has been directed toward the professional arena of mathematics, such as that of the teaching of the subject. For the mathematician, convincement and certainty are drivers that boost its activity in the stages of heuristic development, and a guide for certifying its findings during proof processes (Tymoczko, 1986). The mathematics education community has carried out different studies that implicitly use the point of departure that, like what happens with mathematics, certainty is also important in building mathematics knowledge in the classroom. Some of those works have been recreated in extra-class environments and have focused either on the students (e.g., in Balacheff, 2000) or on the teachers (e.g., in Harel & Sowder, 2007); others, developed in classroom environments with intervention, have basically focused on the students (e.g., in Krummheuer, 1995). Unlike any of the foregoing, in this work the point of departure is an historical phenomenon associated with the building of certainties so as to take it as an epistemological laboratory that enables explaining the presence of the very phenomenon in current training environments using a virtual forum. This raises the challenge of having theoretical elements and analytical instruments that make it possible to distinguish states of certainty (with respect to the statements of mathematics contents that arise there) experienced by the students enrolled and that they express in writing. Below, the authors of this paper propose the instrument of analysis that has been developed for that purpose.

PROPOSAL OF AN INSTRUMENT TO DISTINGUISH EPISTEMIC STATES OF CERTAINTY AND OF PRESUMPTION OR DOUBT

This research deems that, associated with their assertions of mathematics content, subjects can experience internal states of certainty (when they associate the highest degree of probability to what they believe in) or of presumption (when they associate lower degrees of probability to what they believe in). Such states are known as "epistemic states" in Rigo (2013).

In the design of the theoretical-methodological instrument proposed below, there is a convergence of perspectives from different disciplines, namely: from philosophy

(Wittgenstein), psychology (Bloom, Hastings & Madaus) and sociology (Abelson). Of particular relevance to this study was the contribution of linguistics works, such as those of Hyland (1998), which made it possible to resort to analysis of the meta-discourse of the participants in the virtual forum so as to reveal the communicative intentions (many of which are unconscious) that they project through their writings.

The authors of this research consider that a person (who takes part in a virtual forum) experiences a degree of certainty, or of presumption or doubt, in a mathematics statement when one or more of the criteria that appear in Table 1 are met. Said criteria are sufficient, albeit not necessary.

Elements of speech	The person resorts to language emphasizers that can reveal a greater degree of commitment to the truth of what he is saying; for instance, when the person uses the indicative mode of verbs (e.g., I have).
Action	The subject carries out actions that are consistent with his discourse.
Familiarity	The person resorts to forms of sustentation based on familiarity (result of repetition, memorization and customs).
Cognitive formulation	The person resorts to forms of justification based on mathematics reasons.
Determination	The person spontaneously and determinedly expresses his adherence to the veracity of a mathematics statement, indicating some degree of determination. That degree may be higher when the subject maintains a belief, in spite of having the collective against him. He may even make efforts to convince others of the truth of his position.
Interest	The participations of a person who shows interest concerning a specific mathematics fact in a virtual forum are:
	-*Systematic*. That is to say, the subject answers all questions addressed to him in the most detailed manner possible.
	-*Informative*. His assertions, procedures and/or results are sufficiently informative.
	-*Clear and precise*.
Consistency	The person's varying interventions show consistency.

Table 1: Theoretical-methodological instrument for distinguishing states of certainty.

METHODOLOGICAL ASPECTS

The qualitative research reported here focuses on an interpretative-type case study (Denzin & Lincoln, 1994). The empirical study was carried out in the Diploma Program on Fundamental Themes of Algebra, the purpose of which was to strengthen the training of people who provide advice on algebra topics to adults in the process of obtaining their secondary school certificates. The teaching activities are carried out remotely by using the Moodle platform, through which the students receive support, are assessed and given feedback by a tutor. The episode analyzed here pertains to Module IV. It was selected due to the fact that the advisors tended to use sustentation in their responses. The episodes begin with the tutor asking the students to complete a task and they end with the agreement of the students on the solution to the task. For this report, the participations of three students were chosen given that those students appear to have experienced very different epistemic states when faced with the task proposed, despite the fact that none of the three answered the task correctly.

EPISODE: "THE MILLION DOLLAR PROBLEM"

The episode dealt with resolution of the following problem: You will get one million dollars if you can find a two digit number that simultaneously meets the following conditions: a) If you add to the first digit of the number we seek, a figure that is twice the second figure, the result is 5; b) If you add four times the second figure to double the first digit of the number we seek, the result is 7. The students were expected to conclude that no number could meet the problem's conditions, and that they would see that this was the case when they charted the equations in a graph that would produce two parallel lines.

1st Fragment. Mariana's first intervention. Presence of certainties

Mariana started with the following participation:

1.1	x+2y=5; 2x+4y=7
1.2-1.6	... Since the equations do not contain an equal unknown, the substitution method is applied … to eliminate one unknown, which leaves us with 2x+4y=10. After that step, you can do the operation.
1.7	2x+4y=10 <u>2x-4y=7</u> 4x+0=17
1.8-1.9.	We separate the terms and solve for "x". [So]... x=4.25
1.10-1.11	Obtaining the value, we substitute in one of the 2 equations: 2(4.25)+4y=7
1.12-1.13	We do the operation, separate the terms and solve for "y". 8.5+4y=7; 4y=7-8.5; 4y=1.5; y=0.375
1.14-1.15	We prove. First equation: 4.25+2(0.375)=5; 5=5. Second equation: 2x-4y=7; 2(4.25)-4(0.375)=7; 8.5-1.5=7; 7=7

During her resolution, Mariana used different equation systems. The first came from the translation from common to algebraic language (1.1); then she obtained an equivalent equation (at 1.5), and after that, at 1.7, she obtained a modified equation, by changing a sign (of the term 4y from the second equation). To obtain the value of x, she used the first equation at 1.7, and to obtain the value of y, she began with the second equation in 1.1 and ended with the second equation in 1.7. To prove the operation, she used the first equation from 1.1 and the second from 1.7.

In her resolution, Mariana liberally applied the rules of algebra, by capriciously changing the signs of the terms of the equations and by indistinctly using the equations that appear in those systems and combining them in an *ad hoc* manner, as they suited her purposes. It would seem that this responded to a specific objective, namely: to obtain values for literals *x* and *y*, an objective that may possibly have been derived from an interpretation of the literal only as an unknown, excluding the variable's other uses.

During this process, it would seem that Mariana experienced high degrees of presumption and even of certainty. Amongst other reasons, this is because of her *determination* to be the first to submit her answers and procedures to the judgment of the group; her use of *emphasizers*, specifically due to the indicative mode of the verbs (at 1.2 or 1.14); because her *actions* were the result of the procedures that she was announcing, for example when she announced that the substitution method was to be applied (1.2), all of her subsequent actions were aimed at trying to apply rules that she believed belonged to that method; because she sustained her assertions in *schemes* based on *familiarity* (such as the addition and subtraction method), at 1.7, or what she called the 'substitution method', at 1.2. She also demonstrated her certainty by showing *interest* in resolving the problem, by explaining her solution in a detailed manner, answering all of the questions in the problem, resolving the system raised without the tutor requesting it, and by presenting her resolution clearly.

2nd Fragment. José's questioning

José expressed the following to refute Mariana's answer:

2.1-2.2 Hello Mariana. You really surprised me. But you've [missed] a small detail.

2.9-2.11 2x-4y=7. In that step, you changed the sign (it should be ***+4***y or multiply by -1, but the entire equation), that's no longer the original equation. What do you think?

José realized that Mariana had not correctly applied the rules of algebra (changing the sign in the system at 1.7), and that that had consequences ("that's no longer the original equation", 2.9), and he informed her of it, waiting for her reaction.

3rd Fragment. Mariana's reply. Explicitation of reasons and ontological commitments, and strengthening certainty

Below is Mariana's reply

3.1-3.4 You are indeed completely right [José], the entire equation is affected. But the purpose of the system of equations is to arrive at the result by eliminating one of the unknowns. If I affect my entire equation, I would be left with 3 and I would not have an unknown to solve.

3.5-3.10 x+2y=5; 2x+4y=7. In this case, since the equations do not have one same unknown, the substitution method is applied where one of the two equations is multiplied by a number that serves to eliminate one unknown. 2(x+2y)=2(5), leaving us with 2x+4y=10. All is well so far.

3.11-3.12 After that step, you can do the operation
2x+4y=10
<u>-2x-4y=-7</u>
40+0=3

3.13-3.16 Once the value has been obtained, we substitute in one of the two equations: 2(3)+4y=7. We do the operation, and solve for "y"... y=0.25.

3.17-3.18 We prove it. First equation: x+2y=5; 3+2(0.25)=5 and I don't get 5.
Second equation: 2x-4y=7; 2(3)-4(0.25)=7; 6-1=5; nor do I get the 7.

3.19-3.20 So I only affect 4y, in order to not affect the whole equation, and much less my result. You may not see it as correct, but it is [correct] for me because the objective is to find the correct value.

3.22 Let's prove it. First equation: x+2y=5; 4.25+2(0.375)=5; 5=5.

At the beginning of the fragment (from 3.1 to 3.4), Mariana told José that he was right. Yet she subordinated those reasons to what she thought should be obtained from a system of equations: "to arrive at the result". This was probably because she believed that absurdities would be derived from her classmate's answer (such as "I would not have an unknown to solve" and "I would be left with 3", possibly referring to 3.12). At a second point (from 3.4 to 3.18), she followed José's suggestion, perhaps with the idea of 'mathematically showing him his error' by letting a contradiction arise from his suggestion: "I don't get 5" and "nor do I get the 7", without realizing that the mistake did not come from the resolution, but from the arbitrary nature of her manipulation of algebraic language (e.g., by assuming at 3.12-3.13 that x=3 or using the system of equations that best suited her ends). At the third point (3.19-3.20), she once again sustained the advisability of her method, once again subjecting it to the obtainment of her objectives: "to find the correct value" (3.20), and at the fourth (3.22) she proved its validity without realizing that she needed to substitute the values in the two equations at 1.1 and not just in the equation that best suited her interests.

In Mariana's second intervention, she very likely strengthened her epistemic states of certainty by being able to make her objectives and arguments explicit, and 'demonstrating' her classmate's error and the validity of her principles and her method, all of which she did with *determination* and with a *consistent attitude*. Her certainty can also be inferred from the use of emphasizers (not just due to the assertiveness of her language, but also due to the use of the indicating mode in "I [don't] get", at 3.18, "it is" at 3.2 or "much less" at 3.19). Her *interest* can moreover be seen in her reiteration of her resolution, clarification of her points of view, and public refute of her classmate despite her understanding that he was right, to a certain extent.

4th Fragment. Jeimy's participation. Doubt

4.2-4.4 I think I have a problem too. I'm trying to do the second exercise and cannot find the value for x or for y. My equations are: x+2y=5; 2x+4y=7

4.5 Then I change the sign in the first equation.

4.6-4.8 -x-2y=-5
<u>2x+4y=7</u>
x+2y=2; x=2-2y. Substituting in the first equation (2-2y)+2y=5 [so] 2=5

4.9 And I don't get any value ??????????????? What's going on? Help! ...

By rigorously applying the rules of algebra, Jeimy arrived at an absurdity that made her doubt her work. Without presupposing anything, she simply detected it and asked for help.

MAJOR FINDINGS

The case of Mariana is interesting. Although she shows her knowledge of some of the rules of algebra (see 3.5 to 3.12), her ontological commitments concerning the characteristics that must be possessed by mathematics tasks and, particularly by systems of equations –of having an unknown to solve and a precise and numerical solution that can be found- they appear to represent an obstacle that prevent her from fully applying those rules.

Mariana, like Saccheri or Legendre, was faithful to her ontological principles (or biases) and, just like them, those commitments and certainties lead her to "admit demonstrations that had nothing to do with the facts" and "they lead her to precipitate her conclusions or add things that were not legitimate" (see pg. 1 of this text).

Jeimy, like Mariana, faced a problem that jeopardized her beliefs (of the existence of a numerical and sole solution to all mathematical tasks) and her algebraic knowledge. But while Marianna obstinately held on, with not a trace of doubt, to an ideal of the mathematics object, subjecting the rules of algebra to those ontological commitments, Jeimy preferred to maintain her logical rigor—like Lovachevski did, taking due distance- by scrupulously following the rules of algebra. Unlike Mariana, Jeimy allowed herself to doubt the results obtained—like Lovachevski did—recognize her lack of knowledge and ask for help—a metacognitive openness that placed her in a position to learn.

An important didactic consideration stems from the analysis presented. And it has to do with the help that can be given to Mariana. José's participation reveals that it was not enough to demonstrate her algebraic errors because in one way or another she was already aware of them. What Mariana appears not to have realized, and perhaps she would need some help with this, is that her beliefs and ontological commitments (that she probably took as unquestionable and unmovable truths) lead her to lose logical rigor in application of algebraic rules and, in the final instance, represented an obstacle to moving forward in her learnings.

This texts show that certainty of mathematics facts can have deep roots in extra-mathematical considerations, such as ontological commitments, and that certainty is not always or necessarily tied to mathematical comprehension. Given the information presented here, it is important that teachers and their professors become aware of the phenomenon because it has significant consequences in the learnings of students.

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HIGH SCHOOL STUDENTS' EMOTIONAL EXPERIENCES IN MATHEMATICS CLASSES

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The aim of this qualitative research is to identify Mexican high school students' emotional experiences in mathematics classes. In order to obtain the data, focus group interviews were carried out with 22 students. The data analysis is based on the theory of the cognitive structure of emotions (Ortony, Clore & Collins, 1988) that specifies the eliciting conditions for each emotion and the variables that affect the intensity of each emotion. The participant students' emotional experiences are composed of 1) Satisfaction and disappointment while solving a problem, 2) Joy or distress emotions when submitting a test, 3) Fear and relief emotions in mathematics classes, 4) Pride and self-reproach emotions when grading a course and 5) Boredom and interest in mathematics classes.

INTRODUCTION

In the field of mathematics education, most of the research on students' emotions focuses on its role in mathematical problem solving (Corte, Op 't Eynde, & Verschaffel, 2011; McLeod & Adams, 1989; Schoenfeld, 1985; Madler, 1989; Op 'T Eynde, De Corte & Verschaffel, 2006; DeBellis & Goldin, 2006; Goldin, 2000; Goldin, Epstein, Schorr & Warner, 2011). Among other results, these studies have confirmed that people tend to experience similar emotions in the process of problem solving.

Research on emotions (Hannula, Pantziara, Wæge & Schlöglmann, 2010) has outlined the necessity to move beyond the simplistic view of distinguishing between positive and negative emotions. According to Lewis (2013) there are several reasons why this has not been done: 1) It seems more difficult to build a solid theoretical basis for emotions than for other affective constructs, 2) quantitative analysis, like survey methods, offer multiple possibilities to find cause and effect relationships between attitudes and beliefs. Furthermore, Hannula et al. (2010) and Hannula (2012) also noted the need for research focused on emotions during routine mathematical experiences because most of the research has focused on emotions and intense emotions in non-routine mathematical activities.

In this study we have attempted to analyse students emotions in routine activities and to go beyond a consideration of positive and negative emotions using the theory of the cognitive structure of emotions (Ortony, Clore & Collins, 1988). We are aware that the analysis of narratives of emotional experiences is quite different from the direct analysis of emotions, but as Ortony et al. (1988, p.8) we are willing, "(…) to treat people's reports of their emotions as valid, also because emotions are not themselves linguistic things, but the most readily available nonphenomenal access we have to them is through language". This is why we focused on the following research question: *What are high school students' verbal expressions of their emotional experiences in mathematics classes?*

THE THEORY OF THE COGNITIVE STRUCTURE OF EMOTIONS

We have chosen the *theory of the cognitive structure of emotions* (OCC theory from now on) to identify the students' emotional experiences. For Ortony et al. (1988) emotions arise as a result of interpretations of situations by those who experienced them: "[Emotions can be taken as] valenced reactions to events, agents or objects, with their particular nature being determined by the way in which the eliciting situations is construed" (Ortony et al., 1988, p. 13). Thus a particular emotion experienced by a person on a specific occasion is determined by his interpretation of the changes in the world:

> When one focuses on events one does so because one is interested in their consequences, when one focuses on agents, one does so because of their actions, and when one focuses on objects, one is interested in certain aspects or imputed properties of them *qua* objects. (p. 18)

Different types of situations that elicit emotions are labeled in classes according to a word or phrase corresponding to a relatively neutral example that fits the type of emotion (Ortony et al.,1988). For example, to refer to the emotion type "pleased about the confirmation of the prospect of a desirable event" they choose the emotion word *satisfaction* because it represents an emotion of relatively neutral valence among all those that express that you are happy about the confirmation of something expected.

The characterizations of emotions in the OCC theory are independent of the words that refer to emotions, as it is a theory about the things that concern denotative words of emotions and not a theory of the words themselves. From the distinction between reactions to events, agents, and objects, we have that there are three basic classes of emotions:

> Being *pleased* vs. *displeased* (reaction to events), *approving* vs. *disapproving* (reactions to agents) and *liking* vs. *disliking* (reactions to objects). (Ortony et al., 1988, p. 33)

Reactions to events breaks into three groups: one, the Fortunes-of-others group, focuses on the consequences for oneself of events that affect other people. The other two, the Prospect-based and Well-being groups, focus only on the consequences for oneself. Reactions to agents are differentiated into four emotions comprising the Attribution group. Reactions to objects lead to an undifferentiated group called the Attraction group. There is also a compound group of emotions, the Well-being/Attribution compounds, involving reactions to both the event and the agent simultaneously. It seems to be a general progression that operates the different groups of emotions in order: first reactions to events, then to agents, and finally to objects. From the previous considerations, the OCC theory specifies 3 classes, 5 groups and 22 emotion types. To illustrate in Table 1, we present the corresponding emotions to the Prospect-based group

Class	Group	Types (sample name)
	PROSPECT-BASED	Pleased about the prospect of a desirable event (*hope*)
		Pleased about the confirmation of the prospected of a desirable event (*satisfaction*)
		Pleased about the disconfirmation of the prospect of an undesirable event (*relief*)
		Displeased about the disconfirmation of the prospect of a desirable event (*disappointment*)
		Displeased about the prospect of an undesirable event (*fear*)
		Displeased about the confirmation of the prospect of an undesirable event (*fears-confirmed*)

Table 1: Emotion types according to the OCC theory (a extract)

To interpret emotional experiences in mathematics classes we have added two types of emotions in the Well-being group of emotions to the OCC theory. We call them *boredom* and *interest*. These emotional experiences are elicited by the appraisal that the students made of their own cognitive state: 1) states of *alertness* and *concentration* that produce understanding and learning in the case of *attention*, and 2) states of *distraction* and *deconcentration* that prevent understanding and learning in the case of *boredom*. Thus, we consider boredom emotions like “Displeased about an undesirable cognitive state of distraction” and *interest* like “Pleased about a desirable cognitive state of attention”.

METHODOLOGY

Context

The high school where the study was carried out lies to the west of Mexico City. Most of the students live in municipalities bordering the metropolitan area of Mexico City located in the State of Mexico, they come from low economic extraction and most of their parents did not attend college-level. Most students’ mothers are housewives.

Due to the inflexibility of the curriculum, all students have the same mathematics schooling path composed of six courses (one per semester) with five hours each class per week: 1) Algebra, 2) Geometry and Trigonometry, 3) Analytical Geometry, 4) Differential Calculus, 5) Integral Calculus and 6) Probability and Statistics. Generally, there is a traditional process of teaching and learning mathematics because mathematics classes focus primarily on the teacher's explanation and the subsequent resolution of exercises by the students.

Participants

We selected the 22 high school students (ages from 16 to 19 years old, 19 males and 3 females) who are the attending the Analytical Geometry course offered in a for students that have previously failed the course, and did not pass the "sufficiency test" for at least one time. The "ordinary tests" and the "extraordinary tests" are done during regular courses. If the student did not pass the "ordinary tests" has the right to take an "extraordinary test". If the student did not pass the course has the right to take a "sufficiency test" (done outside of regular courses) that is the mechanism by which students can accredit a course based on the demonstration of skills or knowledge through a unique test. The 22 students enrolled in this course agreed to participate in this research. As we had no gender distribution control, it was not taken into account in the data analysis.

Data gathering procedure

Methodologically, we decided to access to the students' emotions from their reports of experienced emotions because the focus of the research is on the students' subjective experiences of emotions. Thus, we carried out four focus group interviews of approximately one and a half hours during the mathematics classes in a regular classroom. We decided to use it because we observed during previous research at the same school that students feel confident and comfortable to express their thoughts, feelings and emotions about various topics in focus group interview.

The questions asked in the focus groups were: 1) Generally, how do you feel in mathematics classes? 2) How do you feel when solving a problem in a mathematics class? How do you feel when you cannot solve a problem in a mathematics class? 3) How do you feel when submitting a test? 4) How do you feel when you know that you failed a mathematics course? And 5) how do you feel when you pass a mathematics course? The role of the interviewers was to deepen on the use, meaning of words and phrases used by the students to answer the questions. Following the OCC theory, our questions intend to provoke students to talk about their emotional experiences in terms of the eliciting conditions.

Data analysis

The videotaped interviews were fully transcribed. In the transcript, students were identified as M*n*-G*k* or F*n*-G*k*. Where M and F indicate that the participant is male or female respectively, *n* (1 to 6) indicates the participant identification number and *k* (1 to 4) indicates the focus group number. Interviewers were identified as MI (male interviewer) and FI (female interviewer). We included explanations in square brackets in order to clarify some of the students' expressions. According to OCC theory to identify a type of emotion we consider three specifications:

1. ***Concise phrases*** that express all the eliciting conditions of the emotional experiences. We highlight with italic bold letters the concise phrases that shows the eliciting conditions of an emotion in the evidence.
2. *Emotion words* that express emotional experience. We highlight with italic letters the concise phrases that show the emotions in the evidence.
3. <u>Variables</u> that affect the intensity of emotions. We underlined phrases that express intensity of the variables in the evidence.

RESULTS

The participant students' emotional experiences are summarized in Table 2.

Eliciting conditions	Emotion types	Variables that affect intensity
Mathematics class	Fear/Relief	Effort Probability
Solve a problem / Not solve a problem	Satisfaction/Disappointment	Realization
Submit a test	Joy/Distress	Effort
Mathematics class	Boredom/Interest	Desirability Arousal
Grading a course / Not grading a course Solve a problem on the blackboard / Not Solve a problem on the blackboard	Pride/Self-reproach	Strength of cognitive unit Expectation-deviation

Table 2: The students' emotional experiences

Here as an example, the detailed evidence presented identifying satisfaction/disappointment emotions.

Satisfaction/disappointment

Students experience *satisfaction* emotions (pleased about the confirmation of the prospected of a desirable event) when they are able to solve specific problems. When this does not happen *disappointment* emotions appear (displeased about the disconfirmation of the prospect of a desirable event).

> **M2-G1**: *I feel good* ***when I understand***. I even want to go to the blackboard to answer the problem. But *I don't feel good* when I am trying to do something ***I don't even know***.
>
> **M1-G2**: *I feel happy* ***when I can solve the problem*** because I can do it. In fact, it is very difficult for me and *I feel good if* ***I can***.

M2-G2: *I am satisfied* if **I can solve a problem**. *I am more motivated with* ***the extra points***, I even want more problems.

Satisfaction emotions are affected by the "effort" variable (reflects the degree to which resources were expended in obtaining or avoiding an anticipated event). This occurs when the teacher gives favourable extra points in the assessment of the students that solve a problem, so the students are provoked to strive in order to obtain them. We noted this intensity when the students used the quantity adverb "more" to express a superlative degree of the experienced emotion.

M2-G2: *I am satisfied if* **I can solve a problem**. *I am more motivated with* ***the extra points***, I even want more problems.

The "likelihood" variable (the degree of belief that an anticipated event will occur) also appeared in the belief of a student that he will be able to solve a problem in the future because he has already solved similar problems.

M4-G2: *I feel really cool* because I have already learned how to solve the problem. <u>It will be easier to solve more problems like this</u>.

On the other hand, *disappointment* emotions are also affected by the "effort" variable, because it reflects the degree of sources employed by the students to solve a problem. There are two possible outcomes when students cannot solve a problem: look for help or quit the problem. In both cases, the experimented emotions are more intense. The emotion word associated for not solving a problem is *desperate* (we interpret this as a form of deep disappointment). The following dialogue shows this:

M1-G4: Sometimes I am desperate because **some of my classmates have finished the work and I don't even know what to do or how to begin**. <u>I ask the teacher for help</u> but it is useless because I don't know the previous subjects and the teacher says that I have to do the same. Then *I am desperate* ***when I see that everyone else has finished and I haven't***.

M2-G4: I am desperate if ***I cannot solve the problem and stop trying***. <u>I wait until the teacher explains it later</u>.

DISCUSSION, CONCLUSIONS AND LIMITATIONS

The results focused on the experienced emotions of students are an empirical contribution to mathematics education that helps to fill the gap in research about emotions in their daily lives with mathematics at school. By applying a complex theory of emotions, this research goes beyond a simplistic view that only considers positive and negative emotions.

Our analysis found eight (*fear*/*relief*, *satisfaction*/*disappointment*, *joy*/*distress*, *pride*/*self-reproach*) of the twenty-two types of emotions that the OCC theory considers and two additional ones (*boredom*/*interest*). Because eight of them (*fear*/*relief*, *satisfaction*/*disappointment*, *joy*/*distress*, *boredom*/*interest*) are reactions to events we can conclude that most of the students' emotional experience is related to achievement goals (learn in class, solve a problem, understand the teachers' explanations, interest to learn at class, pass a course, etc.). In contrast, as two types of emotion (*pride*/*self-reproach*) are reactions related to agents, a minority of experiences relate to "standards, principles, and values" (in the sense of the OCC theory). In addition, we did not find emotions arising from reactions to objects. This can be explained in several ways: First, it can be caused by a methodological limitation because we chose situations through which students had to recall triggering conditions. Well-Being/Attribution emotions may have been told if we had asked, for example: how do you feel when you collaborate with a classmate solving a problem? This methodological limitation is inevitable given the way the situation was presented to the participants. This leads us to consider a different implementation of the OCC theory in empirical research: ask participants for the situations where they experience a specific type of emotion. A possible question for this matter could be: What mathematical situations make you feel afraid/frighten? A similar question could be: In what situations have you been afraid of mathematics? Second, the emotional experiences that we found reflected the participants' circumstances: they are students focused on the goal of passing a course in which they are enrolled for the second time.

The proposed data analysis had the complexity to go beyond emotion words for students to focus on eliciting conditions. We consider that this is a methodological contribution, derived from the OCC theory, to analyze narratives of experienced emotions. Other studies could analyze students' and teachers' narratives as we proposed since it has been proved that *narrative inquiry* is relevant to an exploration of students' and teachers' affect (Di Martino & Zan, 2009, 2011).

A limitation of the OCC theory for an analysis of the emotions experienced by students is that it was originally formulated with no consideration of the specific settings where emotions are experienced. *Boredom* and *interest* are important parts of the emotional experiences in mathematics classes of the participants in this study, but they do not match with any of the 22 types of emotions established by the OCC theory. This shows the necessity to expand and adapt the OCC theory in order to consider specific emotional experiences in the mathematics classes. Future research could focus on other emotions that should be included to capture the complexity of emotions experienced in mathematics at school.

Acknowledgements

The research reported in this article was supported by several research grants from Research and Graduate Department of National Polytechnic Institute of Mexico (20131863, SIP20121519, SIP20111109) and the National Council of Science and Technology of Mexico (CONACYT, Basic Scientific Research 2012: 178564). We appreciate the help of the teacher Maria Patricia Colin Uribe for her help in conducting the field work and Marisa Miranda Tirado for her help in revising this manuscript.

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YOUNG LEARNERS' UNDERSTANDINGS ABOUT MASS MEASUREMENT: INSIGHTS FROM AN OPEN-ENDED TASK

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In response to an open-ended assessment task, 282 children of 6 to 8 years of age revealed their understandings of mass measurement. Each of the Year 1 and 2 children in 13 classes from 3 schools represented their knowledge of mass measurement in drawing and / or writing. Responses ranged from portrayals of activities they had undertaken or materials they had used in classes, to the more explicit articulation of key mathematical ideas. This paper presents samples of children's responses that illustrate a range of thinking and conceptual development about mass measurement revealed by the assessment tool.

INTRODUCTION

Although measurement is an important element of mathematics education, there is insufficient research in this area (Sarama, Clements, Barrett, Van Dine, & McDonel, 2011; Smith, van den Heuvel-Panhuizen, & Teppo, 2011). Smith et al. wrote of poor learning of measurement around the globe and called for the development of assessments that are more revealing of children's learning. In recent work, we have attempted to address some of the concerns related to both the teaching and learning of mass measurement (Cheeseman, McDonough, & Ferguson, in press). In a design experiment (Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003) we implemented rich learning experiences in mass measurement (McDonough, Cheeseman, & Ferguson, 2013) and evaluated children's understandings through use of a one-to-one interview (Cheeseman et al., in press). Our research has included assessment through the development and use of a pencil and paper test (Cheeseman & McDonough, 2013), and the administration of an open-ended task. Findings from administration of the latter are the subject of this paper. Our main purpose here is to present insights into the range and complexity of young children's reflections on their thinking about mass measurement.

In line with the philosophy of social constructivism, we hold "respect for each individual's … sense-making … [and children] … are seen as active and enquiring makers of meaning and knowledge" (Ernest, 1991, p. 198). Interpreting children's thinking from this perspective, we are interested not only in the common features in understandings as communicated by responses to the assessment task, but also in the differently constructed understandings that reflect the range and complexity of thinking exhibited by young learners when measuring mass.

We use the term mass rather than weight as it is the term used in the Australian Mathematics Curriculum (Australian Curriculum Assessment and Reporting Authority (ACARA), 2012). We recognize that among researchers and educators there are different interpretations of these terms and note that the English language adds complexity as we have no verb for the noun and we "weigh" objects to ascertain their mass. In our reading of research we found use of both terms mass and weight.

BACKGROUND

Young children are known to possess knowledge of mathematics, often informal knowledge, that is "surprisingly broad, complex, and sophisticated" (Clements & Sarama, 2007, p. 462) but research provides limited insights into young children's understandings of mass measurement prior to or during the early years of school. Children's expressions of their own perspectives on their knowledge of measurement can provide insights perhaps not otherwise available and can inform teacher interactions. The research reported here adds a layer to the education community's knowledge of young children's developing understandings of mass measurement.

OPEN-ENDED ASSESSMENT TASKS

Assessment is central to learning (Wiliam, 2010). In a review of research literature on formative assessment, Black and William (1998) discussed using student self-assessment as formative assessment and advocated greater use of formative assessment to improve student learning outcomes. They stated, "self-assessment by the student is not an interesting option or a luxury; it has to be seen as essential" (pp. 54-55). The student self-assessment protocol reported here is an open-ended task which offers insights into young children's thinking about the measurement of mass. Open-ended tasks provide opportunities for teachers to learn about individual student understanding (Sullivan & Lilburn, 2004).

MEASUREMENT UNDERSTANDINGS

In learning to measure, children develop skills such as how to use a balance scale and develop understandings of foundational ideas including awareness of the attribute, comparison, unit iteration, the need for identical units, precision, and number assignment (e.g., Lehrer, Jaslow, & Curtis, 2003; Wilson & Osborne, 1992).

Although research on the measurement of mass is limited, the literature does provide some insights into children's understandings at certain ages. Children play with ideas of mass from as young as 12 months (Lee, 2012), and there is evidence of children demonstrating awareness of the attribute from four to six years (MacDonald, 2012), identifying heavy and light objects prior to instruction at six to eight years (Cheeseman et al., in press), ordering three objects by weight at five years (Brainerd, 1974), quantifying with informal units in the second year of school (age six to seven years of age) and with formal units in the third year of school (Cheeseman, McDonough, & Clarke, 2011), and showing understanding of the relationship between the size of a unit and the number of units needed to measure the mass of an object at six and eight years (Spinillo & Batista, 2009).

However, with the exception of MacDonald (2012), we have been unable to locate literature on children's perceptions of their understandings of mass measurement that is informed by student self-assessment. The current study contributes to this field.

The research question addressed in this paper is: What understandings about measuring mass do young learners portray in response to the *Impress Me* open-ended assessment task?

METHODOLOGY

Research participants

Two hundred and eighty-two Year 1 and 2 students (6 to 8 years of age) and their teachers from three urban and rural schools in Victoria, Australia participated in the study. Each teacher taught a sequence of five lessons on mass measurement (provided by the researchers) to their class, following which they administered the *Impress Me* assessment task.

The assessment protocol

The teachers gave each child a blank piece of A3 paper then read the following prompt:

> We have been doing lots of weighing lately. I want you to show me on this piece of paper all you know about mass and weighing. You can write or draw or do both! Take your time and show your ideas and thinking as best you can.
>
> I want you to "impress me" with all you know about mass and weighing.

The researchers provided further information for the teachers:

> We expect no two responses to be the same and of course there is no one right answer! We want as much or as little as children are individually able to give. (If the issue arises, please note that we are happy to accept children's spelling.)

Children could choose to draw, write, or combine the two. For young children, drawing can potentially be a "powerful medium for discovering and expressing meaning [as it] brings ideas to the surface" (Woleck, 2001, p. 215).

Data collection and analysis

In analysing the children's representations, work samples were read and each element on the page was identified as a response. A grounded theory approach was taken to the data (Strauss & Corbin, 1990). Categories were derived by constantly comparing children's representations. Emerging patterns in the data were identified. In this paper, a selection of themes that reveal complexity in student thinking are discussed and illustrated by inclusion of sample responses. Any non-conventional spelling has been corrected to facilitate readability but sentence structure has not been altered.

FINDINGS

The *Impress Me* responses varied in and across classes and revealed various complexities in children's thinking about measuring mass. In this paper examples of responses are presented under three themes identified from the data: Equivalence, Measuring with precision, and Volume and mass.

Equivalence

Equivalence is a key understanding in mathematics (Charles, 2005). Equivalence of mass might be judged by hand (hefting), using balance or other scales, and using a range of objects and units. Responses dealing with ideas of equivalence are reported here from the simplest to the most complex levels of thinking.

- Awareness of equivalence with no explicit mention of mass

For example, seemingly referring to use of balance scales, a child wrote, "If you can't see if it's even or not you can look at the arrow. If it's in the middle it's even" and "If it is equal it stays in the same spot". The apparent reference to a balance scale, and to even and equal suggest attention to the attribute of mass but, without a conversation with the child, we cannot be certain.

- Emergent understanding of equivalence

Some children included more explicit mass terminology along with portrayal of balance scales. For example, one student wrote "Equal is things that are light and heavy" and drew a level balance scale labelling it "That's = the same".

- Equivalence with quantifiable materials

For example, a child drew six cubes in one bucket of a balance scale and four in the other, but showed herself adding two more cubes. She wrote: "I'm trying to make these buckets the same".

- Sophisticated understanding of equivalence with quantifiable materials

Children showed that two groups, each with a different number of objects, can be equivalent masses. For example, one student drew balance scales and wrote "10 tiny teddies and 3 Unifix blocks are the same weight".

- Equivalence using formal units

For example, (see Figure 1) a student wrote, "the playdough is 50 grams" (annotated by the teacher) and added, "they're equal" (transcribed by the teacher). The representation suggests also an understanding that two objects of different shapes can weigh the same amount, that is, an understanding of conservation of mass. Another child expressed this idea more explicitly: "Conservation means when you have the same amount but different shapes and make them into a different shape it will stay the same weight".

Figure 1: Student shows equality using grams

Measuring with precision

A further theme identified within the *Impress Me* data relates to children having concern for precision when undertaking mass measurement activities. Under this heading no hierarchy is implied.

- Demonstrated awareness of exactness and inexactness

For example, responses described children's fascination with the term, *approximately*. Explanations included: "close to your answer", "about", and "nearly the same".

Some children talked of the lack of precision of balance scales, in this example demonstrating keen observation of the scale and an awareness of possible limitations:

> Jack and I were [using] the scales. To make it even [we] did big ones and 4 tiny ones in one cup and in the other cup we put 1 pen and it was even!!!! But when we picked up the pen and put it in again and it did not equal so it depends what way you put it in.

- Evaluated the relative accuracy of different scales

For example one child wrote "The balance scale and the digital scale are maybe the best scales to use. Sometimes when you're hefting with the balance scale or the digital scale … the balance scale is wrong and the digital scale is right". Although the child used the term hefting incorrectly, and did not give an in-depth response, there appears to be attention to precision and some level of reflective thinking.

- Referred to the choice of unit and accuracy

For example, one child wrote "Mini teddies are more accurate because they're lighter and they're easier to stop the [balance] scale". As teddies are plastic and cannot be cut, some children combined larger and smaller teddies as informal units to get a more precise measure of the mass of an item. They reported the numbers and different teddies, thus giving a mathematically legitimate, non-conventional measure.

- Referred to weighing accurately in metric units

For example, "Kitchen scales tell you the exact weight something is" and "Digital scales are easy to know how much something weighs because on the bottom it tells the exact grams or kilograms".

- Demonstrated attention to precision

For example, one child wrote, “I learnt that some potatoes weigh approximately the same … some potatoes weigh 40 and another weighs 41 grams”.

Volume and mass

A further theme identified within the data was related to the identification that volume and mass are not related. Again, there were differences in responses.

The lack of relationship between volume and mass is a complex aspect of mass measurement that can present challenges for young learners, as expressed with clarity by one child: “There is something that is hard to understand, and that is, there are some things that are small that weigh more than a big thing and … big things are lighter than small things”. Some children seemed to be possibly developing an emerging understanding about mass and volume relationships or they were challenged in expressing their understandings, for example, “It doesn’t matter if it is small, it was the same”; “Some little things that are big same little thing weigh more grams”.

With possible consideration of volume, some children indicated the important understanding that mass cannot be judged by sight, for example, “I know you can’t tell something is heavy by looking”.

DISCUSSION AND CONCLUSION

The themes shared in this paper suggest complexity in young children’s mathematical knowledge; thus the study concurs with previous research (Clements & Sarama, 2007). However, the findings also extend that research by showing complexity of understandings specifically in relation to the measurement of mass.

It is apparent that, given suitable experiences, children of six to eight years of age can potentially engage with important mathematical ideas such as equivalence, precision, and the relationship between measurement attributes. Furthermore, the *Impress Me* assessment instrument provided the opportunity for the children to communicate their knowledge, record reasoning, and demonstrate reflective thinking.

Lehrer et al. (2003) wrote that “Developing an understanding of the mathematics of measure should originate in children’s curiosity and everyday experience … and children [should] develop a theory of measure rather than simply collecting measures” (p. 100), with the intention of developing generative and flexible learning. The examples in the paper show that measuring mass can require complex thinking, and that children can develop insights into big ideas of measurement that can potentially be transferred to other measurement attributes.

As illustrated in this paper, based on young children’s life experiences and limited formal study of five lessons on mass measurement in the year the study was conducted, there are many nuanced mathematical ideas that the children had come to understand, were developing, or potentially could develop. We recognise that there can be substantial differences in the meanings children construct from shared mathematical experiences and do not claim that all themes we have discussed apply to all children.

But we have shown that young children can potentially engage with sophisticated ideas of mass measurement. Like Stephan and Clements (2003), we question whether the complex mental accomplishments in measuring are always acknowledged in the teaching of mass. But to this end we also agree with the student who stated that "the more you do mass the better you get".

We propose that the *Impress Me* task can be a valuable self-assessment tool, and that its use can potentially benefit the children as well as researchers and teachers. While there may be limitations in its use, it can be one component of the formative assessment undertaken in mathematics classes.

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MAKING SENSE OF WORD PROBLEMS: THE EFFECT OF REWORDING AND DYADIC INTERACTION

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In this study we investigated the effect of the request to reword the text of problematic word problems on the occurrence of realistic answers. We proposed the activity of rewording word problems to fifth grade pupils either working individually or in dyads. We found that the rewording the problems while working individually had no effect, while rewording in dyads produced a strong increase of pupils' realistic answers. Moreover we analysed the pupils' reworded texts in order to characterize the kind of information added by pupils.

INTRODUCTION

The well known *l'age du capitaine* (Baruk, 1985) was one of the most popular studies that brought to the attention of international research the phenomenon known as 'suspension of sense-making' when solving word problems (Schoenfeld, 1991). Indeed many pupils' responses to this problem, as well as to other word problems of the same or a similar kind, have shown a tendency to unthinkingly apply arithmetic operations without critically considering the reality that the word problem is referring to. Explanations that have been raised for this phenomenon often refer to the stereotypical nature of the word problems typically used in school and to the implicit and explicit rules which govern educational practices surrounding word problems (the so-called didactical contract, see Brousseau, 1986) (Verschaffel et al., 2000).

In the study reported in this paper, we investigated the effect of inviting pupils to reword a given word problem – individually or in dyads – on the realistic nature of their answer to that word problem. Moreover, we analysed the information added in pupils' reworded problems to get a deeper understanding of their sense making process.

THEORETICAL FRAMEWORK

A word problem can be defined as:

> [...] a text (typically containing quantitative information) that describes a situation assumed as familiar to the reader and poses a quantitative question, an answer to which can be derived by mathematical operations performed on the data provided in the text. (Greer, Verschaffel, & De Corte, 2002, p. 271).

World problem solving still comprises an important aspect of mathematical school life. One of the main goals of word problems is to bring pieces of reality in the classroom in order to let pupils experience different aspects of mathematical modelling and problem solving processes, without the practical inconvenience to a direct contact with real world contexts (Verschaffel et al., 2000). In the 90's two pioneering studies (Greer, 1993; Verschaffel, De Corte, & Lasure, 1994) suggested that this goal of bringing mathematical modelling experiences into the classroom is very often not met. In these studies, it was found that upper elementary pupils only very rarely make realistic considerations when solving word problems. This was shown by a contrast between the very good performance on so-called standard word problems (S-items) – that can be solved correctly by straightforwardly applying operations with the numbers given in the word problem – and very low performance on so-called problematic word problems (P-items) – where peculiarities of the everyday life situation described in the word problem need to be taken into account. For example, the P-item *A man wants to have a rope long enough to stretch between two poles 12m apart, but he has only pieces of rope 1.5m long. How many of these pieces would he need to tie together to stretch between the poles?* was answered with "12 : 1,5 = 8 pieces" by virtually all fifth graders from Verschaffel et al.'s (1994) study.

Many authors have argued that this phenomenon can to a large extent be grasped by looking at the processes that occur at the beginning of pupils' word problem solution. Often, pupils seem to decide, based on a very quick and superficial reading of a word problem, which mathematical model leads to the solution. However, for the P-items as described above (and for word problems more generally), ideally there is an intermediate stage between the initial reading of the word problem text and the construction of a mathematical model. This intermediate stage, often described as the creation of a situation model (Kintsch & Greeno, 1985; Verschaffel et al., 2000), consists of representing the key elements and relations in the problem situation. In this stage, one's real-world knowledge about and personal experiences with the situation described in the P-item may help to construct a rich situation model.

The fact that pupils often do not succeed to create a(n extended) situation model of P-items and therefore fail to solve these problems realistically, may partly be explained by the scarcity of information available in the word problems themselves. As Zan (2011) suggested, word problems may show "narrative ruptures" when the question and the information needed for the solution are not consistent from the point of view of the narrated story. Voyer (2011) distinguished three kinds of information that may affect the extent to which this situation model is actually constructed: *Solving information* consists of the essential numerical data, the order of presentation of these data and the size of the numbers; *Situational information* is information which plays a role in the development of a context that anchors the mathematical question in a real life situation, like the initiating events, the setting details and temporal information; and *Explanation information* makes the relationships among the various pieces of information found in the text more explicit. Voyer (2011) posed different versions of the same frame problems to a sample of pupils to understand the relationship between the information presented in the problem text and the constructed situation model. He found that adding information that was non-essential for the mathematical solution of the problem, but still relevant to the problem context, had a positive influence on pupils' performance. Similarly, Palm (2008) observed that when information was added to word problems to make them more authentic (i.e., more closely simulating the real-life situation in which the problem occurred), a larger proportion of children makes proper use of their real-world knowledge in the problem-solving process.

RATIONALE AND RESEARCH QUESTIONS

So far, research has shown that reworded problems that provide more background information lead to better performance on word problems in general (Voyer, 2011), and to more realistic considerations vis-à-vis P-items specifically (Palm, 2008). In the current study, we did not make use of reworded problems in which we added information ourselves. Instead, we asked pupils to reword these problems by themselves and looked whether this would positively affect the realistic nature of their answers to these problems. Indeed, pupils may also be able to add to the word problems the various situational elements that were suggested by Voyer (2011). Using the argumentation by Zan (2011), the impact may even be stronger than when giving word problems with the information already added: One of the crucial differences between a real problem solving situation and a school word problem is that the latter is typically *hetero-posed*, i.e. the person posing the problem is someone different from who has to solve it (Zan, 2011). Therefore any word problem has to be expressed by a (generally written) text, to communicate to the solver *what* he/she has to solve, through an explicit request. Our proposal to ask pupils to reword the text of the problem may partially recover this loss of authenticity, and therefore positively affect the realistic considerations they make. In particular we wanted to see if this rewording helps pupils to consider aspects of reality in their situation model of the problem and consequently react more realistically to these items.

In order to strengthen the possible effect of asking pupils to reword given P-items, we asked some pupils to do this in dyads rather than working individually. Indeed we weren't sure that the rewording would be as effective if pupils work alone. Working in dyads creates a condition in which pupils are forced to make their proposals for rewordings (and possibly even the considerations that lead to them) explicit, and discuss to arrive to an agreement. Working in groups has been shown successful in eliciting realistic reactions (Verschaffel et al., 2000) and thus this kind of condition could make the rewording work more effective and the pupils react more realistically to the P-items.

So, in the current study, we investigated whether asking pupils to reword given P-items would lead them to solve these items more realistically afterwards, and if so, whether this would be more effective when pupils do this assignment in dyads rather than individually. Based on our theoretical framework, we expected a very low number of realistic responses in pupils who solve the task individually, a moderate increase in realistic responses when pupils were asked to reword the problems or could work in dyads, and the highest number of realistic responses in pupils who were asked to reword the problems while working in dyads.

METHOD

A total of 179 fifth graders (88 female and 91 male) were involved in this study. They came from three schools of Naples (South Italy, a region that is well-known for its weak results for mathematics both in national (INVALSI) and international (PISA) assessments). All pupils were randomly assigned to one of four conditions:

- IS condition (n = 19): Pupils individually (I) worked on solving (S) P-items.
- DS condition (n = 62): Pupils worked in dyads (D) on solving (S) P-items.
- IR condition (n = 38): Pupils individually (I) worked on solving P-items after being asked to reword (R) them.
- DR condition (n = 60): Pupils worked in dyads (D) on solving P-items after being asked to reword (R) them.

The DS-condition was added as an extra control condition in order to disentangle the effect of rewording the problems on the one hand and working in dyads on the other hand.

All pupils (or dyads) received a booklet with four word problems adapted from Verschaffel et al. (1994) and were asked to solve them. In the IR and DR conditions pupils were asked to first reword the problems, and they received detailed instructions to do so. More specifically, they were asked to rewrite each word problem by adding details that could help to figure out the underlying situation as in a story (who is involved, what is happening and why, where does the question come from?), while making sure that the operations and the answer to the problem would remain the same.

Pupils working in dyads (in the DS and DR conditions) received only one booklet per dyad and were instructed to work together on this assignment, to talk to each other and to negotiate about the rewriting of the word problem (in the DR condition) and to write down an answer only after agreement had been reached.

Due to space limitations, we will focus on one of the four P-items included in the study, namely the buses problem, adapted from Verschaffel et al. (1994): *450 soldiers must be bused. Each army bus can hold 36 soldiers. How many buses are needed?*

ANALYSIS

The key variable in this study is whether pupils' reactions to the P-problems were non-realistic (NR) or realistic (RR). For the buses problem above, answers were considered NR when the numerical answer was the mere reporting of the result of the operation 450 : 36 (i.e. "12.5 buses", or "12 buses remainder 18"), without any comment about the problematic nature of the problem and/or the given answer. Answers were considered RR when they showed some realistic consideration, either by rounding up the number of buses to the next number ("13 buses are needed"), or by adding any comment that indicated that realistic considerations were made (e.g., "12 buses are needed, and 18 soldiers are left", or "12 buses and perhaps a smaller one").

In addition to the nature of pupils' reactions to the word problems, we also did a deeper analysis of the reworded texts by the pupils in the IR and DR conditions. This analysis could reveal to what extent pupils had followed the instruction to reword the problem to a more meaningful one, and what kind of information pupils had added to do so. For this reason, we coded for every reworded problem in the IR and DR condition the number of information elements present in the reworded problem that were not yet in the original problem text, separated in descriptive (D) information (names, objects, places), intentional (I) information, action (A) information, temporal (T) information, and causal (C) information.

The latter categorization was based on Voyer's (2011) distinction among the different kinds of information that can be added in a word problem text, as well as on another study about the effect of different kinds of rewording of word problem texts on pupils' performance (Vicente, Orrantia, & Verschaffel, 2007). Here is an example of a pupil's reworded text together with our analysis:

> The head of the Italian army decided to make a war against the U.S. army. To do this, the Italian army has to train, so it must be transported in a proper military camp. The soldiers are 450 and must be transported by bus to 36 soldiers each. How many buses are needed?

D:	'The head of the Italian army', 'U.S. army', 'proper military camp';
I:	'he decided to make a war';
A:	'to make a war', 'to train';
T:	
C:	'they have to train'

RESULTS

As expected, there was a very low number of RRs to the word problem in the IS condition, where pupils worked individually and were not asked to reword the problem. Only 1 out of 19 pupils (5.3%) gave a RR. The same was true when pupils worked in dyads (DS condition): None of the 31 dyads gave an answer that could be considered realistic. Moreover, contrary to our expectations, we also did not find a positive effect of asking individual pupils to reword the problems: Only 2 out of 38 pupils in the IR condition (5.3%) gave a RR. However, asking dyads to reword the problems led to a spectacular result, as 22 of the 30 dyads in the DR condition (73.3%) gave a realistic answer. So, only the combination of rewording and working in dyads led to a dramatic increase in the number of RRs to the buses item.

As explained above, we also looked more carefully at the reworded texts that were produced by pupils in the IR and DR conditions, to get a better understanding of the effect of our experimental manipulations. Table 1 summarizes the mean number of elements added in the reworded texts for these two conditions.

	Total elements added	Descriptive	Intentional	Action	Temporal	Causal
IR	4.03	1.76	0.34	0.85	0.11	0.86
DR	6.32	2.38	0.16	2.55	0.19	1.03

Table 1: Mean number of elements added in the reworded texts in the IR and DR conditions

Some differences could be noted. Pupils working in dyads added on average 6.32 elements to the word problem, whereas pupils working individually only added 4.03 elements, which was a significant difference, $t(67) = 3.15$, $p = .001$. Additional tests showed that this was due to a difference in the number of added descriptive elements (1.76 vs. 2.38 on average, $t(67) = 1.79$, $p = .039$) and action verbs (0.85 vs. 2.55, $t(67) = 4.09$, $p < .001$).

We found many interesting and rich reworded texts built by pupils who answered in a realistic way. For example, the reworded text *450 soldiers must be transported in military bus where they will travel to go to a ceremony. The soldiers must be distributed by 36 in each bus. On the bus the soldiers eat tomato pizza and the generals talk. How many buses are needed?*, which was built by pupils who worked in dyads, shows the presence of some 'useless' details from the mathematical point of view, like "the soldiers eat tomato pizza", but that likely contributed to support pupils in imagining the situation, and afterwards giving the realistic answer "They used 13 buses, and in one of them there will be 18 soldiers". Other interesting considerations can be made regarding those pupils who decided to convert the story text in something closer to their own life. For example, *In the fifth grade classes of Madonna Assunta [pupils' school] there are 450 children who have to go to school camp in Puglia. The teachers have ordered the buses that can hold up to 36 children. Children pose themselves the problem of how many buses will be used to transport the 450 children*. The dyad who built this text answered "12 buses are needed and 18 children do not go to school camp". This kind of behaviour was consistent with the considerations developed in Davis-Dorsy et al.'s study (1991), where a problem personalization (i.e., personalizing the standard version of the problem with children's favourite food, and/or their friends' names) led to better results. In our study, these two pupils themselves proposed a personalization of the problem (they indeed just returned from a school camp), and the re-contextualization of the problem together with the personalized way of solving the realistic problem situation showed a flexible understanding of the arithmetical structure in the problem.

CONCLUSIONS AND DISCUSSION

Mathematical word problems are still an important aspect of mathematical school life (in terms of classroom activities, textbook exercises, evaluation tools, and so on). Undoubtedly, the stereotyped nature of word problems used in school leads pupils to routinely apply arithmetical operations based on superficial text processing, leaving in the shadow the mathematical modelling, in particular the building of a rich situational model. This has been evidenced by previous research about pupils' non-realistic answers to problematic word problems.

In this study, we attempted to deepen fifth graders' construction of a situational model of problematic word problems by asking to reword the word problem text. We also investigated the impact of working individually or in dyads. Combining these two manipulations resulted in four experimental conditions. We did not find any effect for the rewording activity when working individually (IR condition). However, asking dyads to reword the problems (DR condition) led to a spectacular result with almost three quarters of the dyads giving a realistic answer. Therefore, these results stress the importance to frame word problem solving processes as social activities. It would be interesting to investigate if this rewording experience in dyads can affect pupils' individual behaviour also afterwards.

Moreover we analysed the reworded texts produced by pupils in the IR and DR conditions. We found a major number of added descriptive elements and action verbs in the reworded texts of pupils who worked in dyads. Together with the previous results, this makes us to hypothesize that the descriptive elements and the action verbs are important elements in the building of situational models that carries pupils to develop mathematical correct and realistic considerations.

It has been argued that, when solving word problems, sometimes "*too much attention to the story will distract pupils from the translation task at hand, leading them to consider "extraneous" factors from the story rather than concentrating on extracting variables and operations from the more mathematically-salient components*" (Gerofsky, 1996, p. 37). While this may be true for standard word problems, the present study revealed that, as far as P-items are concerned, asking pupils to reword the problems and add various elements may lead to substantially more realistic reactions, at least if they are put in a meaningful communicative setting.

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STUDENTS' MANIPULATION OF ALGEBRAIC EXPRESSIONS AS 'RECOGNIZING BASIC STRUCTURES' AND 'GIVING RELEVANCE'

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Manipulating algebraic expressions is guided by students' structure sense and the individual process of relating subexpressions to each other. This paper presents a framework for identifying the underlying cognitive processes of manipulating algebraic expressions, based on 'basic structures' and 'giving relevance'. A design-research case study illustrates how different cognitive processes, related to each of these two constructs, lead to different activities of manipulating algebraic expressions.

MANIPULATION OF ALGEBRAIC EXPRESSIONS AS THE APPLICATION OF RULES

For several reasons, students need to learn to manipulate algebraic expressions in line with the appropriate rules. Not only does this help students to change the form of an expression in order to determine if two algebraic expressions are equal to one another, but it may also be a source for students' meaning making in algebra (Kieran, 2004). However, several studies show that it is hard for students to learn how to manipulate algebraic expressions, as it poses several, non-trivial challenges (e.g. Linchevski & Livneh, 1999).

In order for students to manipulate algebraic expressions in line with the appropriate rules, students must be able to see structures in an algebraic expression. Such a Structure Sense allows students to see, if and in what way a rule for algebraic manipulation can be applied to a given algebraic expression (Hoch & Dreyfus, 2005). However, there may be different ways in which an algebraic expression can be manipulated based on its structures. For example, $ab+ab$ can be transformed into $2ab$, but also into $a(b+c)$. Thus, not only is the manipulation of algebraic expressions guided by structure sense, but the rule-based manipulation of algebraic expression is also guided by cognitive processes, that might, more appropriately, be described as an amalgam of structure sense and of focusing on certain aspects of an expression.

This leads to the question, what characterizes such cognitive processes that lead students to focus on certain structures of an algebraic expression, while neglecting other structures? This paper attempts to characterize these cognitive processes.

MODEL FOR APPLYING RULES TO ALGEBRAIC EXPRESSIONS

Structure Sense and the manipulation of algebraic expressions

Studies about how students manipulate algebraic or even arithmetic expressions suggest that the rule-based transformation of such expressions is a complex interplay between the structure of an expression and the students' ability to see structures in the expression. Linchevski & Livneh (1999) introduced the concept of structure sense in order to grasp students' problems with the structure of algebraic expressions.

Hoch & Dreyfus refined the definition of structure sense and found that structure sense is a compound of students' abilities to

- "Deal with a compound literal term as a single entity. (SS1)
- Recognise equivalence to familiar structures. (SS2)
- Choose appropriate manipulations to make best use of the structure. (SS3)" (Hoch & Dreyfus, 2005, p. 146)

The starting point of Hoch and Dreyfus' model of structure sense is the structure of algebraic expressions, as the above list illustrates. It is the structure of a given algebraic expression that shapes the students activities to manipulate the expression (Fig. 1, right side).

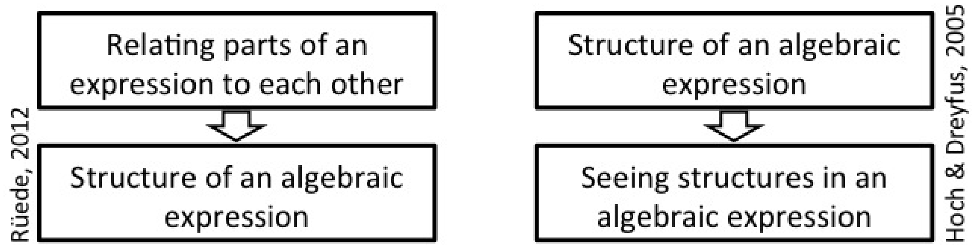

Figure 1: Comparison of Hoch & Dreyfus's and Rüede's models of structure sense

Rüede's (2012) notion of structure sense emphasizes that structure sense is the *individual* process of seeing structures and that the structure of an expression is constituted by the individual student. Rüede defines structure sense as the students' ability to see "different parts of the expression in relation to each other" (Rüede, 2012, p. 113). He empirically specifies four levels for allocating students' structure sense, with increasing degrees of elaboration for relating expressions to each other, e.g., on the first level, finding graphical similarities between subexpressions. Rüede concludes that "[choosing] the appropriate manipulation" for manipulating an algebraic expression is based on the students' ability

- to see subexpressions in an algebraic expression...
- …and to relate these subexpressions to each other and to the whole expression (Rüede, 2012) (see Figure 1, left side).

Synthesized model of cognitive activities of manipulating algebraic expressions

In this paper, it is assumed that the manipulation of algebraic expressions involves both processes of dealing with the structure of algebraic expressions, shown in Figure 1.

The first process is reconstructing the structure of an expression through relating its subexpression, the second process is identifying structures in an expression by "familiar structures". These are regarded as different processes, and are modelled with the notions of "giving relevance" and of "basic structures" respectively.

First, the role of a student's ability to relate the parts or subexpressions of an algebraic expression with each other (Figure 1, left side) is conceptualized as 'Giving Relevance'. Giving Relevance describes a student's individual ways of focusing on certain subexpressions or parts of an expression, while neglecting other parts. This definition follows Rüede's arguments of the importance of relating subexpressions; however, in contrast to him it is here proposed, that seeing a subexpression and relating it to others or to the whole expression is a matter of Giving Relevance to a subexpression. That is, for example, seeing the importance of a subexpression for applying an algebraic rule or for manipulating an expression with a certain aim.

Second, the role of familiar structures in an algebraic expression (Figure 1, right side) is conceptualized with the notion 'Basic Structures' and recognizing Basic Structures. Basic Structures are a student's individual knowledge of structures together with their symbolic manifestation. For example, $ab+ab$ may be a basic structure for a student, and may be associated with "the sum of two products with equal factors". Furthermore, a basic structure can, for a student, be associated with an algebraic rule, in the sense that it can establish the domain of applicability of a rule. For example, the application of the rule $ab+ac=a(b+c)$ might be guided by recognizing the basic structure $ab+ac$. Thus, there are basic structures that guide the application of already learned and conventionalized rules like $ac+ac=2ac$ or $a(b+c)=ab+ac$. However, there may also be basic structures that might lead to spontaneously invented and un-conventional transformations, as Demby (1997) suggests.

The manipulation of algebraic expressions is guided by both processes (also described below in Figure 2). For example, in an expression like $ab+ac+ab$ it might be that a student gives relevance to the two ab 's as a basic structure (sum of equal subexpressions), which might lead him to perceive the applicability of the rule $ab+ab=2ab$, and thus, might lead him to transform the expression into $2ab+ac$. On the other hand, if the student foregrounds the basic structure that underlies $ab+ac$, he might give relevance to all subexpressions (sum of subexpressions with one equal factor), which might then lead him to the transformation $a(b+c+b)$.

That is why in the theoretical approach of this article, both basic structures and giving relevance are conceptualized to moderate the students' activities of manipulating an algebraic expression. They allow framing the underlying cognitive processes of manipulating algebraic expressions (Figure 2). For that, the term cognitive activities is introduced. Cognitive activities describe the whole of cognitive processes underlying a manipulation and the activity of manipulating itself. Different cognitive activities can be distinguished by the way cognitive processes (upper line of Figure 2) and actual manipulations (bottom line of Figure 2) are related and interconnected. For example, there may be a cognitive activity "classifying". In such a cognitive activity, relevance is given to two or more subexpressions, according to some shared characteristic. A basic structure like $ab+ab$ might then lead to group equal subexpressions, in order to apply the underlying rule. It is the aim of this paper, to reconstruct and characterize the students' cognitive activities of manipulating algebraic expressions.

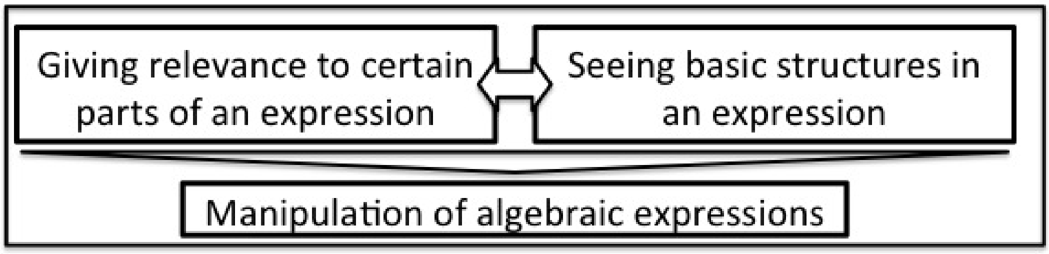

Figure 2: Synthesized model of cognitive activities of manipulating expressions

In the next chapter, the empirical part of a design-research study is presented, which is specifically designed to support such cognitive activities of manipulation.

METHODOLOGY

Design research as a methodology

The model of cognitive activities was used for a design research study on students' repertoire of basic structures and their abilities to give relevance. Design research intends to investigate learning processes of a given learning content by iteratively conducting design experiments. Each iteration of a design research experiment builds upon the empirical insights from the previous iteration. In order to do that, design research experiments start with a conjectured learning trajectory, which is based on findings about learning processes to the given content (Prediger & Zwetschler, 2013).

Design of the learning arrangement and the focus task

The learning arrangement in the case study presented here aims to enable students to see patterns in rule-based manipulations of algebraic expressions. To that end, it focuses on simple algebraic expressions with no more than four subexpressions and which can be manipulated by three previously given rules, namely the distributive law, the commutative law and the rule $ab+ab+ab=3ab$ (called "counting equal terms"). The case study presented here is part of the first design iteration.

This paper focuses on students' cognitive processes while working on one task of the learning arrangement, which was especially designed to support the cognitive processes in the model. In this focus task, students are asked to write down their manipulated expressions together with the original expression as an equation into the respective column. Each column stands for one of the above mentioned three algebraic rules (see Figure 3). The more algebraic expressions students manipulate, the more diverse become the equations in the columns, while, at the same time, these equations have the same underlying structure. This way, the task supports the cognitive processes and manipulation activities in the model of cognitive activities by:

- Helping students to gain awareness of rules by asking them to make rules explicit through writing the expression into the respective column.
- Supporting students to gain a more elaborate notion of basic structures underlying each rule. The more equations are written into a column, the more material students have to acquire more elaborate basic structures.
- Supporting students to give relevance, by enabling them to come back to previous manipulations in the column. This might help students to give relevance to those parts of an expression that constitute its underlying structure.

1. $\cancel{a \cdot b + a} \cdot c$	2. $c \cdot a + c \cdot a + c \cdot a$	3. $a \cdot (b + c)$
4. $c \cdot (a + a)$	5. $a \cdot c + a \cdot c$	6. $b \cdot a + c \cdot a$
7. $\cancel{b \cdot a + a} \cdot c$	8. $\cancel{(c + b)} \cdot a$	9. $a \cdot b + a \cdot c + a \cdot d$

Apply the above shown rules to these algebraic expression. Write the expression and its transformation into the table like in the example – in line with the rule you used for manipulating the expression.

Commutative law	Distributive law	Counting equal terms
$a \cdot b + a \cdot c = a \cdot c + a \cdot b$		

Figure 3: Task with table (with Bianca's notes in it), translation A.M.

Data gathering in design experiments and data analysis

The learning arrangement represents the first iteration of a design experiment, that is going to have three design circles. The learning arrangement in each iteration encompasses three school lessons (3 * 45min). Previous to each design experiment, the teacher uses specifically designed teaching materials in the classroom, which supports the description of geometric shapes with algebraic expressions.

The design experiment was conducted in a laboratory setting. Three 7th-grade students, Bianca, Daniela and Andrew worked – separated from the class – under the supervision of the researcher (author) on the tasks of the design research experiment. The students were chosen by the teacher according to their active participation in the mathematics classes. The sessions were videotaped. The resulting video was transcribed and the relevant sequences were translated to English by the author.

The data is based on the students work on the above described focus task (Figure 3). It stands exemplarily for the wider dataset of the design experiment. The method of analysis is a category-driven sequential discourse analysis. The three main categories of the analysis are based upon the two cognitive processes of the model of cognitive activities and the actual manipulation activities. The model is not an analytical tool, but was adapted in order to arrive at categories: The sequences, in which students manipulate algebraic expressions, are analysed for the nature of the underlying cognitive processes and of the actual manipulation, that is conducted or discussed. The aim of the analysis is to characterize the cognitive activities, which guide the students' manipulations of simple algebraic expressions.

RESULTS

At some point in the task, the students are confronted with the expression $c\cdot a+c\cdot a+c\cdot a$. They already have seen the rule $ab+ab+ab=3ab$ in the previous task. The students are now asked to manipulate this expression by applying one of the three given rules. Daniela, when confronted with the expression, immediately says:

319 D: There you could count equal terms, you could do 3 times c times a then.

It is apparent, that Daniela has no problems applying an already known rule to the slightly different expression. For her, the expression in itself seems to be a familiar structure, it is a basic structure in itself. Thus, the cognitive processes of giving relevance seems to be secondary, because relevance is given only in the sense that the subexpressions are recognized as being equal – and this is the precondition of the rule "counting equal terms". Thus, this cognitive activity, where an expression as a whole is associated with a known rule, is called "associating with a known rule".

At a later point, the students are confronted with the expression $ab+ac+ad$. The students already know the rule $ab+ac=a(b+c)$. The following exchange occurs:

334 B: Three times a times b times c…[…]

336 D: One could somehow everywhere, one could again exchange *[colloquiual for applying the commutative law, A.M.]*

337a A: One could count equal terms

337b B: One could exchange these *[said at the same time as 337a]*

338 D: Yes, but these are no equal terms.

When confronted with this expression, the students could not decide easily, which rule might be applicable. It seems that recognizing a basic structure in the expression is not guided by the expression itself, but by what most likely seem to be guessing processes: In turn 334, relevance seems to be given to the fact that there a three subexpressions and variables in alphabetical order, which might lead to the wrong rule $ab+ac+ad=3abc$. The next utterances (turn 337a and 337b) might hint at a process similar to the above example, where the expression at hand is wrongly associated with known rules.

In turn 338, Daniela counters Andrews proposal. She gives relevance to the different features of the subexpressions. This suggests that Daniela is aware of the declarative content of the rule "counting equal terms", which might suggest that her basic structure behind this rule is elaborated: she interprets this rule as a proposition about the relations of equal parts in an algebraic expression. Thus, the underlying structure of the rule "counting equal terms" in her basic structure includes the declarative content of the rule, which allows her to give relevance to the features of the subexpressions. The cognitive activity behind this manipulation is called "interpreting the declarative content of a rule".

The above described situation continues, and now Andrew takes the lead:

351	A:	A times left bracket b plus c plus d.
352	D:	What?
	A:	Bam!
353	B:	One could just b times a times…
354	D:	Yes, right, that is b plus c times d, that probably works.
355	A:	It is the same as a times left bracket b plus c, only with a number more, you know. This also works.

In turn 351 Andrew suggests a new transformation of the algebraic expression $ab+ac+ad$, namely $a\cdot(b+c+d)$. After a short interjection, Daniela approves this transformation (turn 354). Andrew also justifies his suggested expression in turn 355 using an analogy to a previously applied (and negotiated as a correct rule in various conversions of algebraic expressions) rule $a\cdot(b+c)$. He adds "with a number more, you know". In Andrew's view, the variables $b+c$seem to represent numbers – accordingly, he can see d as an additional number. Thus, he gives relevance to the individual variables in the subexpression $(b+c)$. At the same time, through the lens of this subexpression, he brings the variable d in relation to his basic structure underlying $ab+ac=a(b+c)$. This bridges the gap between the expressions $ab+ac$ and $ab+ac+ad$.

The cognitive process, which guides the manipulation of the expression, might be characterized with two features. Firstly, Andrew is giving relevance to one part of his basic structure of the rule $ab+ac=a(b+c)$, namely the expression $(b+c)$ and the variables ("numbers") in it. This allows Andrew, secondly, to build analogies between $ab+ac+ad$ and the rule $ab+ac=a(b+c)$, perhaps through the lens of the subexpression $(b+c)$ and relating d to this subexpression. This expands the basic structures that are available to him. Andrew's cognitive activity is thus called "building analogies by focusing on a subexpression".

CONCLUSION AND DISCUSSION

In this paper, a model for analysing the cognitive activities involved in the rule based manipulation of simple algebraic expressions is suggested and used to analyse a case study. Data from the first design iteration suggests different cognitive processes, which can be characterised by their underlying processes of recognising basic structures and giving relevance (Table 1).

The data from this first experiment does not cover, whether the found cognitive activities are generalizable; that is, if these cognitive activities are also employed in cases where students are confronted with more complex algebraic expressions. In further iterations of the here presented design experiments, the above shown model is going to be applied to design supportive means for manipulating more complex alge¬braic expressions. The here presented model of cognitive activities allows improving on future design experiments: the below illustrated cognitive activities (Table 1) can now be specifically initiated by supporting their underlying cognitive processes.

Basic structure	Giving relevance	Cognitive activity
Declarative content of rule is embedded	Giving relevance to subexpressions and their features (“not equal”)	Interpreting declarative content
a(b+c+d) reconstructed through *a(b+c)=ab+ac*	Giving relevance to one subexpression and its composition (“numbers”)	Building analogies by focusing on a subexpression

Table 1: Example of cognitive activities in the manipulation of algebraic expressions.

More generally, the here suggested framework has proven successful in gaining insight into the nature of students’ cognitive activities of manipulating algebraic expressions. In the here discussed focus task, the students employ three cognitive activities to manipulate algebraic expressions. In spite of the simple algebraic expression used in this study and their ‘simple’ structure, rather complex cognitive processes were identifiable. This might suggest, that existing models of structure sense might not have allowed grasping the students’ activities of manipulating expressions in such detail, because of their sole focus on the structure of expressions.

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AN INFERENTIAL VIEW ON CONCEPT FORMATION

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This paper focuses on an inferential view on introducing new concepts in mathematics classrooms. A theoretical framework is presented which helps to analyse and reflect on the processes of teaching and learning mathematical concepts. The framework is based on the philosophies by Ludwig Wittgenstein and Robert Brandom. Wittgenstein's language-game metaphor and especially its core, the primacy of the use of words, provide insight into the processes of giving meaning to words. Concerning the inferentialism by Brandom, the use of words in inferences can be regarded as an indicator of the understanding of a concept. The theoretical considerations are exemplified by the interpretation of a scene of real classroom communication.

INTRODUCTION

A lot of research on communication in the mathematics classroom has been done. Mathematical interactions have been analysed from many different perspectives (cf. Cazden, 1986). This paper focuses on the teaching and learning of mathematical concepts in classroom communication. By his theory of "language-games", Wittgenstein offers an alternative view on the introduction of concepts in mathematics classrooms. Elements of his perspective have often been used to discuss problems concerning communication in the mathematics classroom (e.g., Bauersfeld, 1995; Schmidt, 1998; Sfard, 2008). According to Wittgenstein, the expression of words does not constitute their meaning. Rather, it is the use of words, which constitutes the meaning, and therefore, the use of words constitutes the concept. On the basis of Wittgenstein's philosophy, the American philosopher Brandom worked out an inferential approach to the comprehension of the processes of concept formation.

On the basis of the theory of analysing arguments by Toulmin it will be described in this article in how far the processes of concept formation can be analysed in accordance with an understanding like this. In particular, the significance of judgments (the combination of subjects and predicates) and their connection among one another during their concept formation will be focused.

USE OF WORDS IN LANGUAGE-GAMES

Wittgenstein's concept of language-game is closely connected with the process of concept formation. It means that words do not have a meaning by themself. Therefore, a fixed, temporal lasting word's meaning does not exist:

"Naming is so far not a move in the language-game—any more than putting a piece in its place on the board is a move in chess. We may say: *nothing* has so far been done,

2014. In Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 4, pp. 217-224. Vancouver, Canada: PME.

when a thing has been named. It has not even got a name except in the language-game." (Wittgenstein, PI § 49)

Thus, Wittgenstein has a complex opinion on processes of concept formation. That means that the meaning of a word is solely put down to its use: "For a *large* class of cases—though not for all—in which we employ the word 'meaning' it can be defined thus: the meaning of a word is its use in the language" (PI § 43).

The meaning of a word shows and manifests itself in using the word in language. This might be a reason for the fact that Wittgenstein does not define what exactly he understands by speaking of "language-games". He uses the word "language-game" by describing the use of this word (e.g., by giving examples). That way, he gives meaning to this word. The theory of Wittgenstein of the attribution of meaning through the use of words is also closely connected with those of the language-game in another way: To this, let us have a look on the concept of numbers: When students understand numbers as a quantitative aspect of objects, then they can use this for calculating. But the handling of numerals is changing when numbers are regarded as ordinal numbers. Now, operations cannot be used in such an easy way anymore. The comprehension of the cardinal aspect of numbers is not sufficient either when negative numbers are introduced. Each of these changes entails an alteration of the language-game. In the changing language-games, the same numbers can be used in different ways. The way of use determines the current meaning. However, a well-developed concept of numbers needs different kinds of comprehensions – that is different ways of use – which are connected with *family resemblances*, to say it in Wittgenstein's words:

> And for instance the kinds of number form a family in the same way. Why do we call something a 'number'? Well, perhaps because it has a—direct—relationship with several things that have hitherto been called number; and this can be said to give it an indirect relationship to other things we call the same name. And we extend our concept of number as in spinning a thread we twist fibre on fibre. And the strength of the thread does not reside in the fact that some one fibre runs through its whole length, but in the overlapping of many fibres. (Wittgenstein, PI § 67)

The use of words in a language-game is by no means arbitrary. Rather, the use is determined by certain rules. These rules tell us how words can be applied:

> We can say that a language is a certain amount of activities (or habits) which are determined by certain rules, namely those rules that rule all the different ways of use of words in language. (Fann 1971, p. 74; my own translation)

Accordingly, observing the rules, that determine the use of words, is a considerable feature of our linguistic acting. A rule has the function of a "sign-post" (Wittgenstein, PI § 85) although each rule can be interpreted in a different way. Within the mathematics education research, a lot of rules, which determine the language-game "*mathematics education*", have already been reconstructed. The patterns of interaction and routines which were described by Voigt (1984) can also be counted as (combinations of) rules. For instance, *the pattern of staged-managed everyday* *occurrences* (in German: "*Muster der inszenierten Alltäglichkeit*") describes the *as if*-character of classroom situations in which the students' extracurricular experiences are taken up: if the students make too much use of these experiences, the teacher is going to disregard this use and highlights the mathematical contents. By using the word *demathematizing* (in German: "*Vermathematisierung*"), Neth and Voigt (1991) describe how teacher – while working on an situation which is open for different kinds of interpretations – makes a note on single words, formulas, signs, or the like of the students, in order to funnel the students' diversity of interpretation on mathematics as quick and purposeful as possible. Such rules make sure that the actions in class run smoothly by showing the agents, for instance, which actions they have to carry out, what they can achieve with them and where the limits of their actions are. Therefore, rules are constitutive for the classes, particularly as they determine the use of words or rather sentences on the one hand and support that the classes pass off smoothly on the other hand.

INFERENTIAL USE OF WORDS

Following Wittgenstein, a concept can be developed, if different ways of using the relevant word are well known. The definition of a word is just one possible way of using this word. Knowing different ways of using a word includes, among other things, knowing and using sentences that go with them:

> 'Owning' a mathematical term requires to know more relations and to know more about the handling with the term than it is expressed in its definition. […] Proofs help to explain the terms‘ inner structures as well as to link concept and with that to develop the purport of term. (Fischer & Malle 1985, S. 189f, my own translation)

We use words in situations of giving reasons for statements – also statements in which this word is used. For example, we can use "commutative law" to give reason for the similarity of 9+4 and 4+9. The aspect of reason of concept formation shows itself in the structure of the potential words' ways of use. Thus, every definition, for instance, has a conditional structure ("If…, then…"). Definitions are equivalence relations (or rather biconditional – "if and only if") which are also used in arguments. In short: The words' meanings are arranged in an inferential way. The American philosopher Brandom elaborated this inferential approach: "To talk about concepts is to talk about roles in reasoning." (ibid. 2000, p. 11). The understanding of a word is described by Brandom as follows:

> Grasping the concept that is applied in such a making explicit is mastering its *inferential* use: knowing (in the practical sense of being able to distinguish, a kind of knowing *how*) what else one would be committing oneself to by applying the concept, what would entitle one to do so, and what would preclude such entitlement. (ibid.)

The inferential use is carried out using reasoned arguments in situations of reason. To examine the students' corresponding arguments, the Toulmin-scheme – which has been already become established in mathematical education research – can be used. It also helps to reconstruct the implicit shares of arguments. In accordance with this, an argument consists of several functional elements. Undisputable statements function as *datum* (Toulmin 1996, p. 88). Coming from this, a *conclusion* (ibid.) can be inferred, which might have been a doubtful statement before. The *rule*[1] shows the connection between datum and conclusion. The rule legitimizes the conclusion. If the rule's validity is questioned, then the arguer could be forced to assure it. Within the reconstruction, such making safes are recorded as *backings* (ibid, 93ff) and can happen, for instance, in giving further details about the field where the rule comes from. As an example for the analysis by means of the argumentation-scheme by Toulmin, the following fictitious remark of a student is reconstructed, which functions at the same time as an example for the inferential use of the concept *bigger*: "As 3 apples are more than 2 apples, 3 is bigger than 2." According to this statement that - talking about numbers of apples – there is a smaller-bigger relation (datum), it can be concluded that there is a relation of size between the relevant numbers (conclusion). The conclusion is legitimized by a rule which is only implicit and which can be supported with the reference to the aspect of cardinal numbers (backing). Accordingly, the following Toulmin-scheme can be reconstructed:

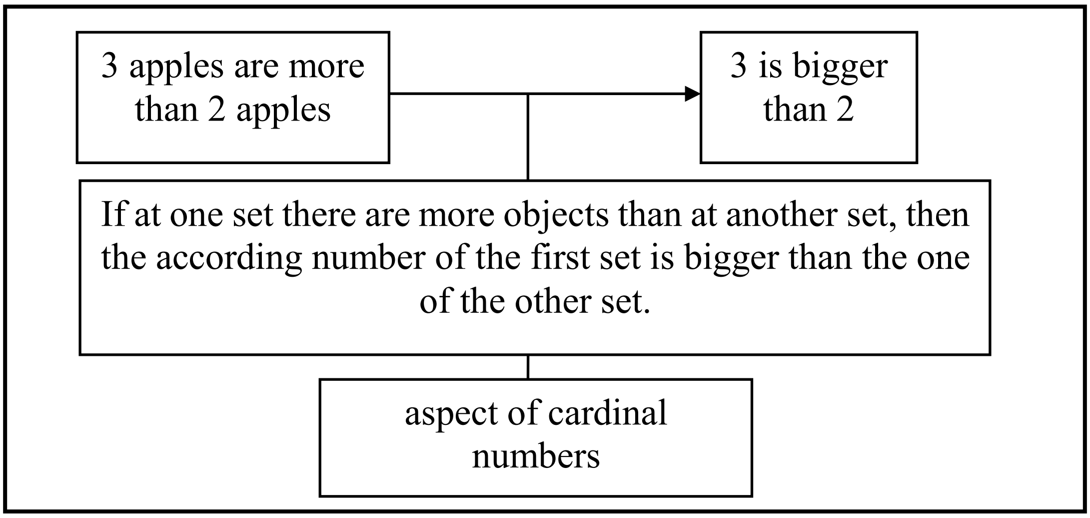

Figure 1: Application of the Toulmin-scheme

Following Wittgenstein, by means of such an argument a relation between two concrete numbers is expressed. In certain language-games, such an argument is surely regarded to be valid. But introducing negative numbers at school means that such kind of use of the word *bigger* is possibly no longer accepted. This change of the language-game causes a different use of numbers. Although 3 apples are more than 2 apples is true, it does not mean that -3 is bigger than -2. If the rule is applied on negative numbers in this way, it loses its' validity.

With regard to the theoretical consideration before, different important elements of the processes of concept formation can be recognized:

[1] The general connection which is described as warrant by Toulmin corresponds not entirely to the above rules by Wittgenstein (cf. the examples of (combinations of) rules given above).

- *datum* and *conclusion* consist of judgments, as a link between subjects and predicates,
- *rules* have a general character, in so far they connect general judgments in conditional or biconditional forms and
- the *backings* which are the basis for an argument.

Accordingly, the enormous significance of concrete and (combinations of) general judgments for concept formation is shown: Concrete judgments (datum and conclusion) are linked via more general connections (rules). The possibility of this connection is based on the knowledge of an area or rather of a context in which this connection is perceptible to the learners (backing).

METHODOLOGY

According to Wittgenstein we should not ask: What is the meaning of a word? Rather, we should analyse what kind of meaning a word gets (by its use) in the classroom. Therefore, we have to analyse social processes. Accordingly, Wittgenstein's approach enables a purely interactionist view on processes of concept formation which are a benefit for the interpretative researcher, particularly as they are not dependent on speculations concerning student's thoughts. If the use gives meaning to words (in the interaction), then the (linguistic) action is the sole criterion for the reconstruction. Thus, we have to follow the ethnomethodological premise: The explication of meaning is the constitution of meaning. By analysing the students' "languaging" (Sfard 2008) for mathematical concepts, the development and alteration of meaning by the use of the according words, we are able to reconstruct the social learning in the mathematics classroom. Therefore, the qualitative interpretation of the classroom communication is founded on an ethnomethodological and interactionist point of view (cf. Voigt 1984; Meyer 2007). Symbolic interactionism and ethnomethodology build the theoretical framework which will be combined with the concepts of "language-game" and "(inferential) use".

The main aim of the presented study is to get a deeper insight into the processes of giving meaning to words in the mathematics classroom. Therefore, alternative ways of introducing concepts are going to be considered. Comparing possible and real language-games can help to understand the special characteristics of the actual played language-game.

The empirical data are taken from several studies in which the *arithmetic mean* was introduced. The surveys were carried out in two classes (first class: fourth grade in primary school, age of students: 9 to 10 years; second class: fifth grade, secondary school age of students: 10 to 11 years) on the one hand and in interviews with two students of the third grade in primary school (age: 8 to 9 years, duration: 3 x 45 minutes) each on the other hand. It was the empirical studies' aim to get the students to collect different judgments on one concept and give reasons for their relationship (in this situation: their equivalence). This means that in relation to the arithmetic mean, there are two judgments: Firstly, the arithmetic mean is the quotient of the total and the number of the given values and secondly, the arithmetic mean is determined by the inversely arranging of the values to get an *adjusted value*. This can be expressed formally and briefly as follows ($\overline{x}$ is the arithmetic mean of $a_1,..., a_n$):

$$(a_1 + ... + a_n) = \underbrace{\overline{x} + ... + \overline{x}}_{n\ summands}$$

In every experimental setting, the series of tasks start with a reduction of the meaning of the arithmetic mean to the meaning of a "middle number" in a number series (the students were told that the second box contains the number of the first):

$$5 + 6 + 7 = \boxed{\quad} \qquad \boxed{\quad} : 3 =$$

By solving tasks like this the students should discover that the result of the division will be the "number in the middle", which could also be gained by modifying the summands to be equal to each other: 5+6+7 = (5+1)+6+(7-1) = 6+6+6. To get the general concept of the arithmetic mean, the summands, the amount of summands and the distance between the summands were varied gradually.

USE OF WORDS FOR CONCEPT FORMATION – EMPIRICAL EXCERPTS

By carrying out the empirical studies, learners were asked, among other things, to solve the following task – two different working-outs were to give: "Lisa weighs 14kg. Paul weighs 23 kg. Sarah weighs 25 kg. Marc weighs 26 kg. What is the middle weight?" Jule wrote about this:

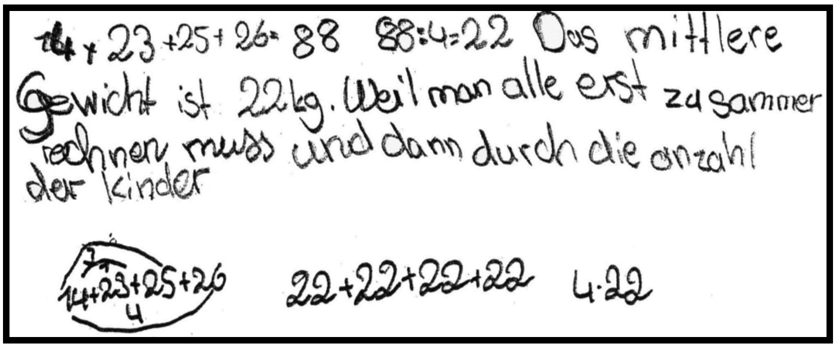

Figure 2: Jule (fourth grade) determines the arithmetic mean on two ways
(Translation: "The middle weight is 22kg. As you first have to calculate everything together and to divide it afterwards by the number of kids")

As suggested by the task, Jule speaks of "middle weight". Concerning this and other tasks, the students named the concept "number in the middle", "average", "balanced number", etc. Regardless of the name of the concept, the use of these words has been quite the same.

As an example for the different reasons for the equivalence of the two judgments of the arithmetic mean (first: quotient, second: inversely arranging), Malte's statements are given in the following. Malte explains why the inversely changing of 12+14+16 does not change the total (the transcript has been translated and linguistically smoothed).

Malte: Eh, with plus it is a team so to say. The result always stays the same – no matter what is changed. If one doesn't take away anything and doesn't add anything, either- but if one always swaps, swaps, swaps, the result will always stay the same. With minus, it is different.

Teacher: Could you explain to me the thing you said about the team – What do you mean? […]

Malte: This (*pointing left to right at the task 12+14+16*) is the team now. And if this one (*pointing at the summand 16*) is now so to say- or- these (*again pointing left to right at the task*) are <u>the students</u>. This one (*pointing at 12*) is missing two pens and then this one (*pointing at 16*) who has two pens too much gives- one pen to this one (*pointing at 12*) who is missing two pens.

Students: 2 pens (murmuring)

Malte's argument can be reconstructed as follows:

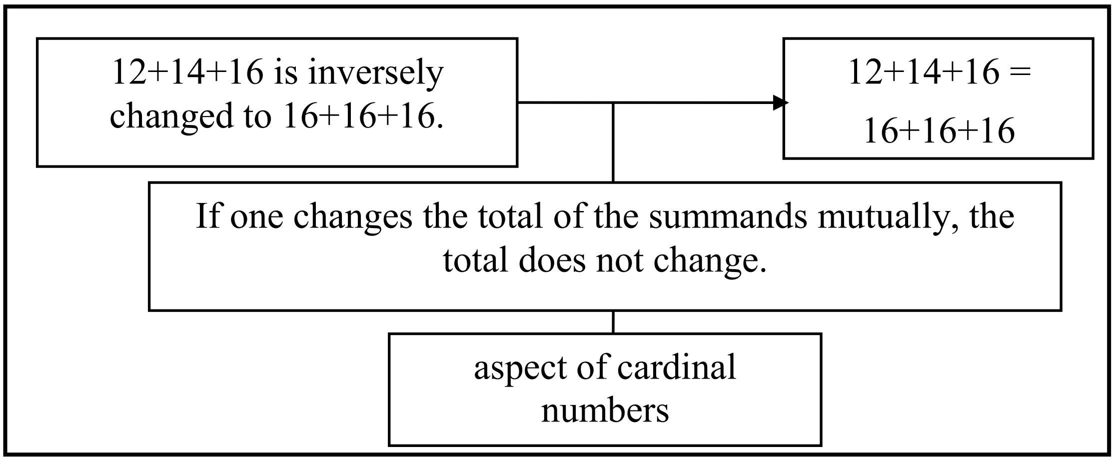

Figure 3: Reconstruction of Malte's argument

Malte uses words like *team* and *pen* to describe the remaining total of the inversely arranging. His given reasons can be put down to the aspect of the cardinal number in so far as he considers the change of the singular summands and not totals. In this way, he links both judgments of this pre-form of the *arithmetic mean*, which expands the concept's dimension. Later, Malte's argument (resp. its functional elements) is taken up by other students again and again (even there, where the students are able to distinguish between the arithmetic mean and the median) so that not only the connection, determined by Malte, is taken up constantly (cf. the above quotation by Brandom), but also the equivalence of both judgments is made clear by more elaborated arguments compared to those in Figure 3. Only the words *team* or *pens* as concrete objects of the change were not taken up. This can be interpreted as follows: The students refer to the way of the use and not only to the specific words which are used. In other words: The students seem to refer to Malte's implicit rule and his backing. Such moments are shown in the talk.

FINAL REMARKS

Wittgenstein's theory itself is not a theory of interpretation. Rather, he presents a theoretical framework which can be used on top of a theory of interpretation in order to understand processes of languaging for concept formation.

Corresponding to the given considerations based on the theories to Brandom, Toulmin and Wittgenstein concept formation can be understood as the (*inferential*) *use of judgments* and their (*general, regular*) *connections* (the *rules*). Throughout the arguments we commit ourselves to these judgments resp. to their connections which, if they are accepted, we can use continuously. This use can be, again, independent of the concrete words, but rather the more general way of using the words, the rules and backings, seem to be crucial for the (following) course of concept formation.

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FUNCTIONS OF OPEN FLOW-CHART PROVING IN INTRODUCTORY LESSONS OF FORMAL PROVING[1]

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Amongst important and under-researched questions are how introductory lessons can be designed for teaching initial proofs to junior high school students, and how such lessons enrich students' understanding of proofs. With a view to improving the learning situation in the classroom, in this paper we report on the various functions of introductory flow-chart proofs that use 'open problems' that have multiple possible solutions. Through an analysis of a teaching experiment in Grade 8, and by using a model of levels of understanding of proof structure, we identify the functions as enhancing the transition towards a relational understanding of the structure of formal proof, and encouraging forms of forward/backward thinking interactively that accompany such a relational understanding of the structure of proofs in mathematics.

INTRODUCTION

With proving and reasoning universally recognized as key competencies of mathematics education, it remains the case that students at the lower secondary school level can experience difficulties in understanding formal proofs (eg: Hanna & de Villiers, 2012; Mariotti, 2006). In order to enhance the capabilities of junior high school students with formal proving (from around the age of 14), it is important to have a clear framework to inform the design of introductory proof lessons. This is because such lessons aim to initiate inexperienced students into understanding the meaning of formal proofs fruitfully so that they can develop the competencies to construct proofs for themselves. We have previously reported that students who have experienced such introductory lessons can score around 10% better than expected on a question that involved choosing reasons to deduce a conclusion (see Miyazaki, Fujita and Jones, 2012). In this paper we report a further qualitative analysis that focuses on why the students did well in such mathematical proofs. Our research questions are as follows: how can introductory lessons for formal proofs be designed, and how do such lessons enrich students' understanding of proofs?

In order to enrich the introductory lessons of formal proving, our research study focuses on the students learning to use flow-chart proofs in 'open problem' situations where they can construct multiple solutions for congruent triangle tasks by deciding the assumptions and intermediate propositions necessary to deduce a given conclusion in a flow-chart format. Such proofs involve using the conditions for triangle

[1] This research is supported by the Grant-in-Aid for Scientific Research (No. 23330255, 24243077), Ministry of Education, Culture, Sports, Science, and Technology, Japan.

congruency as these are often used to introduce formal proofs in geometry in Japanese lower secondary schools (Jones & Fujita, 2013), and our discussions and analyses are related to this topic. The aim of this paper is to evaluate the introductory lessons designed on the basis of our theoretical framework by identifying their pedagogical functions and implications.

THEORETICAL FRAMEWORK: UNDERSTANDING PROOF STRUCTURE

We take as our starting point that a formal proof generally consists of deductive reasoning between assumptions and conclusions. Within this reasoning process at least two types of deductive reasoning are employed: universal instantiation (which deduces a singular proposition from a universal proposition) and hypothetical syllogism (where the conclusion necessarily results from the premises).

In order to understand the structure of proof, students need to pay attention to the elements of the proof and their inter-relationships. Research studies by Heinze and Reiss (2004) and by McCrone and Martin (2009) have identified that an appreciation of proof structure is an important component of learner competence with proof. In this paper we use the following levels of learner's understanding of proof structure initially elaborated by Miyazaki and Fujita (2010): Pre-, Partial- and Holistic structural levels. These levels are described in Table 1 and the overall framework illustrated in Figure 1.

Level	Description
Pre-structural	The basic status in terms of an understanding of proof structure where learners regard proof as a kind of 'cluster' of possibly symbolic objects.
Partial-structural	Once learners have begun paying attention to each element, then we consider they are at the *Partial-structural Elemental* sub-level. To reach the next level, learners need to recognize some relationships between these elements (such as universal instantiations and syllogism). If learners have started paying attention to each relationship, then we consider them to be at the *Partial-structural Relational* sub-level, with this sub-level being further sub-divided into a) universal instantiation and b) syllogism (see Figure 1).
Holistic-structural	At this level, learners understand the relationships between singular and universal propositions, and see a proof as 'whole' in which premises and conclusions are logically connected through universal instantiations and hypothetical syllogism.

Table 1: Levels of learner understanding of proof structure

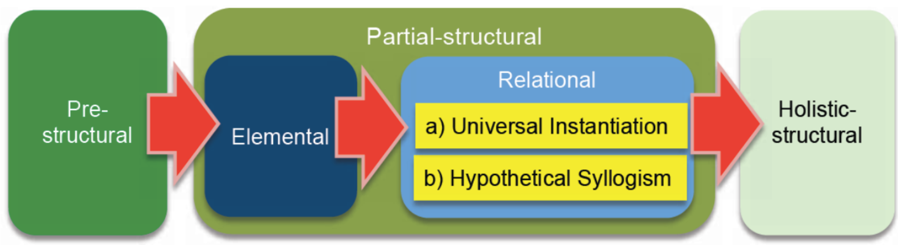

Figure 1: Framework of learner understanding of the structure of proof

To date we have utilized this framework to demonstrate students' explorative activity to overcome logical circularity in a proof problem (Fujita, Jones, & Miyazaki, 2011), and considered how a hypothetical learning trajectory for introductory lessons of formal proving could be designed so that students can be helped to develop their understanding of the structure of proof (Miyazaki, Fujita, & Jones, 2012). In this paper we focus on the design of introductory lessons of formal proofs.

INTRODUCTORY LESSONS USING OPEN FLOW-CHART PROVING

To design introductory proof lessons we used the following two pedagogical ideas: flow-chart proof format and 'open problem' tasks. A flow-chart proof shows a 'story line' of the proof. McMurray (1978) and others have provided accounts of the value of using flow-chart proofs prior to the use of formats such as the 'two column proof'.

Given the evidence that flow-chart proofs can help students to visualize the structure of proofs, in our research we are investigating how the power of flow-chart proofs might be enhanced at the introductory stage of proof learning by using 'open problem' situations where students can construct multiple solutions by deciding the assumptions and intermediate propositions necessary to deduce a given conclusion.

For example, the problem in Figure 2 is intentionally designed so that students can freely choose which assumptions they use to show the conclusion that ∠B=∠C. After drawing a line AO, for instance, students might decide ΔABO and ΔACO should be congruent to show ∠B=∠C by using the theorems "If two figures are congruent, then corresponding angles are equal." Based on AO=AO as a same line, ΔABO≡ΔACO can be shown by assuming AB=AC and ∠BAO=∠CAO using the SAS condition. However, other solutions are also possible. One approach might be to use the fact that ΔABO≡ΔACO can be shown by assuming AO=AO, AB=AC and BO=CO, using the SSS condition. As students can construct more than one suitable proof, we refer to this type of problem situation as 'open'.

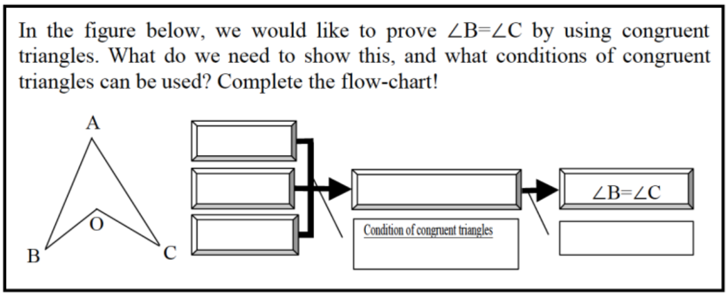

Figure 2: An example of flow-chart proving in an 'open-problem' situation

In accordance with our theoretical framework, in the introductory proof lessons it is particularly important to support transitions from the Partial-Structural to the Holistic-Structural level. The flow-chart format aims to help students to visualize that a formal proof consists of two kinds of propositional layers, one of which contains universal propositions (theorems) and the other contains the chain of singular propositions. Also, the flow-chart format can show clearly that a singular proposition is deduced by the universal instantiation of universal proposition, and that the chain of singular propositions between assumptions and conclusions would be established by hypothetical syllogism. Moreover, in order to show a given conclusion in the 'open problem' situation, students would be encouraged to seek out the necessary assumptions and intermediate propositions diversely. Then, they have a chance to originate alternative proofs by replacing the used theorems into others, and so on.

METHODOLOGY

To investigate the functions of open flow-chart proving in the introductory lessons of formal proving in Grade 8 (aged 14), we developed nine lessons based on the learning progression with three phases as follows (Miyazaki, Fujita, & Jones, 2012).

- Constructing flow-chart proofs in an 'open problem' situation (four lessons)
- Constructing a formal proof by reference to a flow-chart proof in a 'closed problem' situation (two lessons)
- Refining formal proofs by placing them into flow-chart proof format in a 'closed problem' situation (three lessons).

During the first phase of lessons, students constructed flow-chart proofs in 'open problem' situations. Through these tasks, the students were expected to learn how to think forward/backward between assumptions/conclusions and how to organize their thinking in order to connect assumptions and conclusions. Thus this phase aimed at supporting them to understand how to 'assemble' a proof as a structural entity. Note that they study proof in 'closed-problem' situations after the first phase.

Our main data are taken from one of our lesson implementations in which a teacher with 18 years of teaching experience conducted the set of the nine Grade 8 lessons in a junior high school in Japan during October 2013. The lessons were video-recorded and then transcribed. In the next section we report selected scenes from the fourth lesson in which students undertook the problem in Figure 2. By this data analysis, we identify the functions of open flow-chart proving during the introductory lessons designed using our theoretical framework of the understanding of structure of proof.

DATA ANALYSIS AND DISCUSSION

In reporting our findings from the fourth lesson, first we show the students' levels of thinking at this stage; in particular their incomplete understanding of universal instantiations. Then, we show how learning with 'open problem' proof tasks helped them to start to see proofs from a more structural point of view.

Enhancing the structural understanding of formal proof: universal instantiations

While prior to the lesson the students had used a one-step flow-chart proof to prove that two given triangles are congruent, during this lesson they tackled the problem in Figure 2. This has two steps; first deducing the congruence of triangles, and second, concluding the equivalence of angles. As one purpose of the lesson was to make students aware of the importance of universal instantiation (which deduces a singular proposition from a universal proposition), the teacher oriented the students to confirm the necessity of supplementary line AC to deduce $\angle B=\angle C$ by using the congruency of ΔABO and ΔACO, and wrote "$\Delta ABO\equiv\Delta ACO$" into the flow-chart on the board.

Thereafter, the students started to complete the flow-chart proof by themselves. After a suitable time the teacher asked student SA to say what he would put in the flow-chart box for the properties of congruent figures. SA answered "because of $\Delta ABO \equiv \Delta ACO$" (see line 6 SA in the transcript below); the teacher wrote this answer on the blackboard. Next, the teacher directed two other students to show their answer. One of them said, "Due to congruent triangles, angles are congruent", and another said, "In congruent triangles the corresponding angles are equivalent." The teacher also wrote these answers on the blackboard. At this time the teacher compared these three answers, and asked SA to explain more; their dialogue is shown as follows.

1	T:	SA, can you tell us why you wrote this?
2	SA:	Umm, I considered why the angles are equal; then I found an arrow is drawn.
3	T:	OK, because the arrow can be drawn (pointing the corresponding part of flow-chart on the blackboard).
4	SA:	I put 'it'.
5	T:	What is 'it'?
6	SA:	ΔABO and ΔACO are congruent.
7	T:	OK, if we can say these two are congruent, then we can use the arrow. So, SA, if two triangles are congruent, what can we show?
8	SA:	Angles are also equal.
9	T:	Good, angles are also equal? Anything else?
10	SA:	Sides are equal, too.
11	T:	Yes, sides are equal too. So, umm, in this case our conclusion is to say the angles are equal, so it is OK. But in general if two triangles are congruent, it can be angles but also sides as well, so we should add information generally about angles such as 'because angles are congruent or equal'.

Given that prior to this lesson the students could find the appropriate conditions of triangle congruency, and write them into the theorem box (universal proposition) given in the one-step flow-chart proof. It was expected that they would reach the *partial-structural elemental* sub-level (by paying attention to elements of proofs) during this lesson. Beyond this, some students might start reaching the *relational* sub-level (by understanding both universal instantiation and hypothetical syllogism) through examining the properties of congruent figures.

Nevertheless, during the early parts of this lesson it was evident that only a small proportion of the students could reach the *relational* sub-level. In fact, about half the students could not correctly write two boxes of flow-chart, each of which requested the condition of congruent triangles and the properties of congruent figures. Others just wrote a singular proposition "because of $\Delta ABO \equiv \Delta ACO$" into the theorem box (like student SA said). This singular proposition is not precise enough from a universal instantiation point of view. It is clear that such students remained at the *elemental* sub-level, and could not reach the *relational* one. In particular, the students who wrote the singular proposition could not understand that a singular proposition should be deduced by the universal instantiation of a universal proposition.

In order to resolve the student' lack of understanding, the teacher compared SA's answer with others answer in which universal propositions were correctly used (the relational sub-level), and pointed out that it was necessary to express the property of congruent figures generally because it was being used to deduce the equivalence of angles in this case (although it could be used to deduce the equivalence of both angles and sides). This resolution managed by the teacher might have supported the students to enhance their understanding of the universal instantiation that deduces a singular proposition with a universal proposition. This, in turn, could promote the transition from the elemental sub-level to the relational one.

From the above we can identify as the functions of 'open problem' flow-chart proving that it can enhance the transition towards a relational understanding of the structure of formal proof by helping student to visualize the connection of singular proposition to hypothetical syllogism and the connection with universal instantiation between a singular proposition and the necessary universal proposition. This 'open problem' flow-chart format can help visualize not only the connection of singular propositions by hypothetical syllogism but also the connections of a singular proposition with a universal one by universal instantiation. With this visualized format, students could be supported effectively to focus on the characteristics of the two kinds of deductive reasoning, by checking the expression of theorems and confirming their meaning and/or roles.

Encouraging thinking forward/backward interactively by using open proof situations

After most of the students made their own flow-chart proofs, the teacher picked up three answers, each of which used different conditions of congruent triangles (this was possible because of the 'open problem' situation). The teacher checked with the class if three pairs of angle/sides were necessary to deduce $\Delta ABO \equiv \Delta ACO$ with each congruent condition, and then also checked the reason why they chose these pairs on the basis of the words written in the box below each of the three pairs.

For example, student KA used the ASA condition and the teacher asked him why he chose the followings; '$AO=AO$', '$\angle BAO=\angle CAO$', '$\angle AOB=\angle AOC$'.

The student's explanation was as follows:

1	KA:	Because we can see AO=AO from the given figure.
2	T:	Can see it from the given figure?
3	KA:	And it is an assumption. I assumed by myself $\angle BAO = \angle CAO$, and also $\angle AOB = \angle AOC$ as well. And then we can show $\Delta AOB \equiv \Delta AOC$, and the condition is 'Two pairs of corresponding angles are equal and the included sides equal'. Due to congruent triangles, corresponding angles are equal and therefore $\angle B = \angle C$.

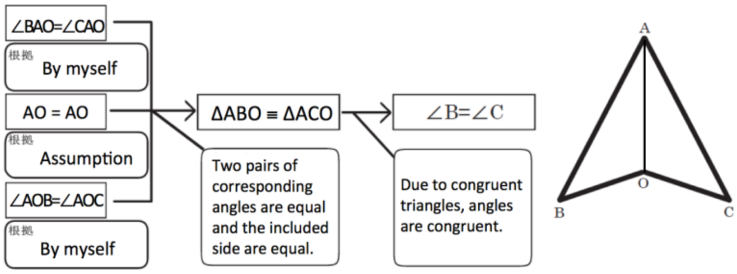

Figure 3: One of the flow-chart proofs by KA on the blackboard

As can be seen from the dialogue and the flow-chart proofs by KA shown in Figure 3, for the reason why “AO=AO”, KA wrote “Assumption” in the box and explained that this equivalence was apparent by means of the given figure (see line 3 KA). In contrast, for the reasons why “∠BAO=∠CAO” and “∠AOB=∠AOC” KA wrote “By myself” and explained that they were decided by himself (see line 3 KA). In this thinking process, there were the two ways of approach. One way is thinking forward, i.e. in order to find the conditions for ΔABO≡ΔACO, KA focused on the corresponding angles/sides of these triangles and judged that “AO=AO” could be one of the conditions. A second way is thinking backward, i.e. KA chose ASA as a condition and then looked for the other conditions (in this case “∠BAO=∠CAO” and “∠AOB=∠AOC”) which were necessary to satisfy this condition. It is the ‘open problem’ situation that made it possible for KA to use these two ways of thinking interactively. Furthermore, KA actually wrote in his worksheet two types of flow-chart proof. Each of these used different conditions: SSS and SAS. To complete these proofs he similarly determined the assumptions that were necessary to deduce the congruent triangles. Likewise, most other students in the class constructed three different proofs using similar thinking processes.

From the above we can identify as the functions of ‘open problem’ flow-chart proving that it can encourage thinking forward/backward interactively, accompanied by relational understanding of the structure of proof. The amplification of thinking backward, in particular, can be triggered by the ‘open problem’ situation. Moreover, the flow-chart proof format can support students to associate two modes of forward/backward thinking visually. This systematic learning with thinking forward/backward interactively is useful for the planning of formal proof that usually precedes its construction (Tsujiyama, 2012). Thus the learning of ‘open problem’ flow-chart proving in the first phase of introductory lessons of formal proving can be preparatory to the planning of formal proof in a ‘closed problem’ situation.

CONCLUSIONS

Within our focus on students understanding of the structure of proof, we can identify two functions of ‘open problem’ flow-chart proving. One is that it can enhance the transition towards the relational understanding of the structure of formal proof by visualizing both the connection of singular proposition by hypothetical syllogism and the connection with universal instantiation between a singular proposition and the necessary universal proposition. A second function is that 'open problem' flow-chart proving can encourage thinking forward/backward interactively, accompanied by relational understanding of the structure of proof. In particular, this study illustrates that 'open problem' flow-chart proving can give students a chance to find necessary conditions and combine them in order to connect assumptions with conclusions. This systematic learning with thinking forward/backward interactively is required to make the planning of formal proofs. We suggest that it is these functions that contribute to developing students' understanding of proofs, and that is why the students who experienced our introductory lessons scored 10% better than the national average of proof problems in general (Miyazaki, Fujita & Jones, 2012).

Due to page limitation we cannot show that some students, after finishing solving the assigned task, attempted to 'expand' and/or 'break' the given flow-chart proof format so that they could show their own way of proving. This further illustrates that the innovative use of 'open problem' flow-chart proving, as in our project, can cultivate students' productive thinking about formal proofs even in introductory proof lessons.

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TEACHERS KNOWLEDGE OF INFINITY, AND ITS ROLE IN CLASSROOM PRACTICE

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This paper considers infinity as an element of professional knowledge. We assume that teachers need a wider knowledge of the topic than that they possess as mathematics students. Using three models of professional knowledge, Mathematics Teacher's Specialised Knowledge (MTSK), Mathematical Knowledge for Teaching (MKT), and Knowledge Quartet (KQ), we discuss how the notions being proposed in these models, and on which they are constructed, might contribute to studying teachers' knowledge of infinity.

INTRODUCTION

Infinity as a learning item has been widely studied, from the seminal work of Fischbein, Tirosh and Hess (1979), to more recent contributions (e.g. Zoitsakos, Zachariades and Sakonidis, 2013; Dubinsky, Arnon, Weller, 2013). Worthy of note in this respect is the bibliographic review by Belmonte (2009), which considers over 300 published papers on the topic. The focus of these publications concerns mainly two aspects: the process of developing the cognition of infinity (e.g Lakoff & Nuñez, 2001), and differing conceptions of infinity (e.g. Belmonte 2009). Recently, the research community has begun to show interest in the understanding that prospective teachers have of infinity (e.g. Manfreda Kolar & Hodnik Cadez 2012, Dubinsky et al. 2013). Most of the research on these two aspects focuses on students or prospective teachers, leaving aside practising teachers and their knowledge, as well as their role in and for practice. Although we concur with the philosophy underlying previous research, and recognize the value of understanding the degree of teachers' cognitive development with respect to infinity, we feel an approach to teachers' knowledge of infinity from the perspective of professional knowledge should not be limited to an "*on its head*" knowledge (Thames and VanZoest, 2013, p. 592), but should rather consider in what way the teacher understands infinity in the teaching and learning context, and how he or she uses (or can use) this knowledge in their professional practice. This leads to the question of whether a teacher should understand infinity differently to the pupil (and if so, how), or should simply understand it to a more advanced degree. This paper draws together perspectives on this question, from both the literature on learning about infinity and the field of professional knowledge, with the aim of providing insights for future research into teachers' knowledge of the topic.

2014. In Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 4, pp. 233-240. Vancouver, Canada: PME.

INFINITY IN THE CLASSROOM

Infinity as a mathematical item is not typically explicit on primary and secondary syllabuses (e.g. NCTM, 2000), although it is to be found as a backdrop to certain mathematical notions, such as in the concept of limit or the measurement of area using integrals, and in others underlying basic processes, such as counting processes and number systems generation (Gardiner, 1985), both of which do have a place on the syllabus in various countries. A broad overview of curricular content leads us to wonder whether teacher training should contemplate the inclusion of mathematical aspects (the epistemological and phenomenological, amongst others) as well as didactic considerations, so that teachers know and understand infinity when it comes up in the curriculum, and, more significantly, are able to recognise it as a latent presence underlying a number of mathematical topics.

There are many approaches to infinity which take the pupils' point of view into account and tackle the topic intuitively. Many centre on the different types of reasoning called upon to understand iterative processes, or on pupils' own 'naturally' expressed definitions when dealing with concepts such as limit, density, and the periodicity of the decimal part of a number.

Recently, reviewing the notions put forward by different authors, Belmonte (2009) detected six different intuitive patterns underlying secondary pupils' understanding of infinity, for which he aimed to group the different classification systems deployed in previous research within a single system, including several novel notions about finding the sum of a series. Such studies are of even greater interest when considered alongside research into how topics such as limit are explored in class (e.g. Sierpinska, 1987), as direct classroom applications become apparent, with examples from real lessons involving discussions with pupils in which they reflect and articulate their own understanding of infinity.

From the point of view of teacher knowledge and training, it is not unreasonable to think that teachers should have a good working knowledge of the stages pupils need to pass through to achieve an understanding of infinity, as this will enable them to respond appropriately to their pupils, and to better select, organise and sequence classroom tasks. However, we would argue that in addition to understanding these developmental aspects of infinity; teachers should also know how to introduce the concept to their class in such a way as not to limit their pupils' development. Likewise, they should be aware of how certain conceptualisations of infinity limit the mathematical constructs that can be built, as shall be seen below.

AN EXAMPLE

The extracts presented below are taken from a discussion between the first author (R) and a secondary teacher (A) about an example given in Belmonte (2009). The three excerpts appear in chronological order and correspond to three different points in a continuous discussion:

> Example under discussion:
>
> *Everyone imagine a number. Halve it. Halve the results, and so on successively. What is the final result?*
>
> *A pupil answers:*
>
> *We don't know, because we don't know when to stop.*
>
> (Belmonte, 2009)

Extract 1

A: This is like the example of the jumping frog, which jumps towards the edge of the reservoir. I used to use it, but not any more.

R: Why not?

A: Because there were arguments. […] One person wouldn't accept it, while another would, and in the end they'd get angry and would say, "Well, I don't" and the other would say, "Well, I do," and they just wouldn't agree.

R: And why did one accept it and the other didn't?

A: Because of the physical aspect. You explain the sequences to them, how it works – half the length of the previous jump, then half again, and half again . . . and a lot of them say that the frog makes it. Others say when it gets close, it takes a bigger jump and gets there.

Extract 2

A: Sometimes they're given an example like the other day, the frog that jumps halfway. Does it reach, or not?

R: OK, and does it reach, or not?

A: No, no, it doesn't.

R: OK.

A: It does reach the limit, but it doesn't. In physical terms, it shouldn't reach it.

Extract 3

A: I'll give you the definition I give to my pupils. Infinity is something invented to explain the inexplicable. [. . .] Unknown, untouchable. Not invented, but it's there to explain something which doesn't have an explanation really.

We will use the teacher's statements to analyse aspects of the conceptualisation of infinity using notions drawn from various models of professional knowledge, specifically Knowledge Quartet (Rowland, Turner, Thwaites & Huckstep, 2009), Mathematical Knowledge for Teaching (Ball, Thames & Phelps 2008) and Mathematics Teacher Specialized Knowledge (Carrillo, Climent, Contreras & Muñoz-Catalán, 2013). We will organise the analysis in terms of the domains of

Mathematical Knowledge and Pedagogical Content Knowledge (PCK), derived from Shulman (1986), for their compatibility with the above models.

ANALYSIS

Mathematical knowledge

The teacher displays understanding of certain phenomenological aspects of infinity, such as the concept of limit, in that he expounds upon an example demonstrating an underlying notion of infinity as a gradual approach to a limit (from a clearly potential perspective), based on one of Zeno's paradoxes. Additionally, not only is he capable of establishing the connection between the limit and the example, but also, from his way of conceptualising infinity, he is able to discuss the example. This "*dealing with infinity*" is one of the Big Ideas (Kuntze, Lerman, Murphy, Kurz-Milcke, Siller & Windbourne, 2011) in relation to mathematical content. Considered in terms of KQ, it can be seen as pertaining to *Foundations* (Rowland et al., 2009), as it constitutes the theoretical background to various ideas, while at the same time forming part of *Connections* (ibid.), in that it puts the teacher's mathematical connections into action. Seen through the lens of MTSK, the Big Idea comes within the scope of *Knowledge of the Structure of Mathematics* (Carrillo et al., 2013), as it cuts across mathematical categories, and could be regarded as a foundation stone of school mathematics, lending theoretical support to a multitude of concepts. Regarding MKT, it is possible to argue for its inclusion in different subdomains. It seems clear that infinity cannot be regarded as pertaining to *Common Content Knowledge*, as it is beyond what might reasonably be expected of someone with mathematical schooling (given that, as mentioned above, there is not usually any specific focus on it), and as such it seems more appropriate, as an item exclusive to teaching, to assign it to the domain of *Specialised Content Knowledge*. In like fashion, it can be argued that an understanding of infinity, and the way in which this organises other mathematical concepts, fulfil the criteria for what Jakobsen, Thames and Ribeiro (2013) denominate "Familiarity with the discipline", as a characteristic of *Horizon Content Knowledge*.

In the case of our teacher, his conceptualisation of infinity as something unknown and artificial leads him to affirm that, although the concept of limit exists, the idea of its 'reachability' would not make sense in real life, illustrating a certain confusion between context and problem. As a result of his interpretation of infinity, the teacher fails to abstract the situation to a mathematical context. This process of modelling, which requires the teacher to be aware of the need to do so (for example, in terms of his objectives in employing a particular example), leads us to another component of mathematical knowledge, Knowledge of the Practice of Mathematics, (Carrillo et al, 2013), within the perspective of MTSK, or the Horizon Content Knowledge associated with the practice of mathematics (Ball and Bass, 2009). In this respect, given that the understanding involved is close to syntactic, we can understand the use of the notion of Foundations (Rowland et al. 2009).

Pedagogical content knowledge

Seen through the lens of PCK, defined in Shulman's (1986) seminal work as the knowledge which includes "ways of representing and formulating the subject that make it comprehensible to others" (ibid. p. 9), we note that the teacher chose a specific example to tackle a particular content associated with infinity, in this case the limit of a sequence. This choice, and the knowledge of the example itself as a means of representing the content, leads us to the need to consider PCK as applicable to infinity. In this case, the use of the three models above allows us to observe the teacher's knowledge from a very similar standpoint. Consistent with the observations made above, *Transformation* is present, this element of Knowledge Quartet being very close to Shulman's original definition of PCK. In the cases of MKT and MTSK, both models accommodate subdomains encompassing the choice of powerful examples for a particular content, *Knowledge of Content and Teaching*, in the case of MKT, and *Knowledge of Mathematics Teaching* in that of MTSK.

In MKT and MTSK, PCK is explicitly divided into three different subdomains, one for teaching mentioned above, another for the curriculum (in MKT, identical to *Curricular Knowledge* proposed by Shulman, 1986) or learning standards (representing an amplification in MTSK of the earlier work), and a final subdomain focusing on the students (MKT), or the characteristics of learning related to mathematics (MTSK). This kind of knowledge is visible in the case of the example above, in that the teacher is able to predict a typical answer, "it [the frog] makes a bigger jump and gets there," thus illustrating his understanding that some pupils are prevented from conceptualising the infinite reiteration of the process by the barrier which the context represents for them.

FINAL REFLECTIONS

Infinity is an item intrinsic to school mathematics, frequently non-explicit, requiring an approach beyond consideration of the process by which it is learnt, as has largely been the case to date. This paper represents a call to tackle the concept as an item of professional knowledge, applicable to the day-to-day work of teaching, while taking into account the cognitive aspects affecting the teacher's understanding of infinity (as a learner). The different frameworks that have been applied support this notion. Each, incorporating its own theoretical constructs, helps us to better understand the conceptualisation of infinity brought into play by mathematics teachers tackling the various topics which constitute the phenomenology of the concept. The notion of structural concept in mathematics, which derives from the MTSK model is of special interest for us, along with the consideration that knowledge of infinity is exclusive to teachers, and pertains to the specialised content knowledge subdomain of MKT. With respect to KQ, *Transformation* represents a powerful means by which to consider the pedagogical implications of infinity.

We recognise that the field under consideration, teachers' knowledge of infinity, is a recent innovation. We hope that these considerations are followed by others which enable them to be amplified. In the long term, we regard the inclusion of aspects of infinity in teacher training programmes as one of the challenges facing this field.

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PROBABILITY, UNCERTAINTY AND THE TONGAN WAY

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Problems teaching probability in Tonga (in the South Pacific) led to the question how language and culture affect the understanding of probability and uncertainty. The research uses a discursive approach to find the endorsed narratives which underlie Tongans' reasoning in situations of uncertainty. I aim to justify the claim that the Tongan Language and the Tongan way of life interact to make the concept of uncertainty unimportant and the concept of probability almost redundant in day to day discourse.

INTRODUCTION – A BAYESIAN ARRIVES IN POLYNESIA

As a teacher of statistics at 'Atenisi University in the Kingdom of Tonga (a group of Islands in the South Pacific) in 1994 and again in 2010, I experienced great difficulty explaining the concept of probability to students, who were otherwise proficient at learning mathematics. The students did not appear to relate to examples of uncertainty in the way in which I had expected.

Preliminary observations indicated that the Tongan language does not provide Tongan students with the tools and the intuitive ideas which are so important in developing the ideas of uncertainty and of probability. Although secondary and tertiary education is supposed to be in English, my students regularly switched to Tongan when discussing what I was teaching. Tongan is the language that mediates and organises these students' lives and activities. It is an integral part of their culture. These observations motivate the research question: *How do the Tongan language and culture shape discourses on probability and measuring uncertainty*? More generally, the hypothesis to be tested is that the linguistic tools provided by the Tongan language differ significantly from European languages and as a result the western concepts of probability and uncertainty do not exist in the community of native Tongan discourse (this said, as the result of the arrival of English language and Western forms of life, both the discourse on uncertainty and activities that require probabilistic thinking are in the process of developing in Tonga).

I should state from the start that my approach to statistics is that of a somewhat dogmatic Bayesian. Bayesian statistics is an axiomatic approach, which defines a rational way of making decisions in situations of uncertainty. My aim had been to try to teach this approach to my Tongan students, an attempt which ended in abject failure. The aim of this research is to explain, understand and learn from this failure.

My research (for which Anna Sfard serves as an advisor) has led me to the conclusion that discourses on probability are closely related to discourses on fractions but in this paper I will concentrate on the topic of probability and uncertainty and only mention

2014. In Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 4, pp. 241-248. Vancouver, Canada: PME.

findings about fractions in passing. In this paper I make use of a few samples from my field work in an attempt to explain the discourse on probability and uncertainty which I observed in Tonga.

THEORETICAL BACKGROUND

Language, culture, and mathematical thinking

The idea that language shapes people's view of the world dates back to the mid twentieth century and is known as the Sapir-Whorf hypothesis (Whorf, 1959). Initially there was great interest in the hypothesis and then for many years it did not receive much attention. Recently there has been a renaissance of interest (Cole, 1996, Deutscher, 2010, and many others). It is suggested that part of the reason that Whorf's ideas did not receive more support is that he over stated his case. He claimed that our mother tongue restricts how we think and prevents us from being able to think certain thoughts. The dominant approach today is "that when we learn our mother tongue, we acquire certain habits of thought that shape our experience in significant and often surprising ways" (Deutscher, 2010).

In his research, concerning the Oksampin communities in Papua New Guinea, Saxe develops the idea that not only language but culture and history are related to how mathematical ideas are understood. He proposes a methodological approach "rooted in the idea that both culture and cognition should be understood as processes that are reciprocally related, each participating in the constitution of the other" (Saxe, 2012, p. 16). My research in Tonga originated with the idea that language affects understanding but it quickly became clear that the cultural and historical background of the community had to be taken into account.

The development of probabilistic thinking – cross cultural studies

Little has been written concerning cross cultural studies of probabilistic thinking. In his survey of the literature Jones raises concern about "the lack of probability research outside western countries" (Jones, 2007, p. 944). In an earlier survey Shaugnessy stresses the need for “cross cultural comparison studies using in-depth interviews on decision making and probability estimation tasks” (Shaughnessy, 1992. p. 489).

I have only located two articles that deal with cross cultural studies on understanding probability. In the first (Amir and Williams, 1999), the authors compare two cultural groups within the same school in England. Language, beliefs, and experience were shown to influence the 11-12 year old children's “informal knowledge” of probability, which was defined as “the intuitive knowledge they bring to school and use in thinking about probabilistic situations" (ibid, p. 85). In the second article (Chassapis & Chatzivasileiou, 2008), the authors compare conceptions of chance and probability held by children who live in Greece and in Jordan. They also compare a group of Palestinian children living in Greece with children from the local Greek Christian community. They conclude that more religious Muslims tend to attribute random events to God, while less religious Greek Christians tend to attribute random events to chance.

Ian Hacking (1975) provides an historical dimension to the present study. The author looks at the preconditions for the sudden emergence of probability, as we know it today, in mid seventeenth century Europe. These preconditions included:

- The development of a simple notation for fractions
- Developments in the insurance industry and the theory behind annuities
- Changing attitudes to religion, fatalism and causality.

There are striking similarities between the historical process described by Hacking and the emergence of probability, which I observed, in Tonga. Understanding how the idea of probability emerged in Europe helps to explain what is happening in Tonga today.

CONCEPTUAL FRAMEWORK – A DISCURSIVE APPROACH

In this study I adopt the discursive approach proposed by Anna Sfard (Sfard, 2008) in which mathematics is defined as a form of communication or discourse. People are members of various overlapping "communities of discourse". A community of discourse is defined as those individuals participating in any given discourse and by the endorsed narratives which they use. An individual can be a member of a number of overlapping communities of discourse. This approach supplies a powerful framework through which to understand and explain the observations which I made in Tonga.

Using this theoretical framework the aim of the research is to identify, analyse and contrast the various communities of discourse that exist in Tonga, how they overlap and how they compare with typical western communities of discourse. In this paper I aim to analyse the endorsed narratives about uncertainty and probability, which I observed in Tonga.

SOME EMPIRICAL FINDINGS ABOUT PROBABILISTIC THINKING IN TONGA

The seeds of this research were planted while I was trying to teach basic ideas of uncertainty to a university level class. I (the "lecturer") was attempting to teach how future events can be assigned probabilities and had the following conversation:

Lecturer: What is the probability that the sun will be shining at this time tomorrow? (*in my way of thinking there was a reasonable chance of cloud cover*).

Student: It will be sunny.

Lecturer: Are you certain that it will be sunny?

Student: Yes.

Lecturer: Why are you certain?

Student: Because the angels told me.

The reader can imagine how frustrating I found this conversation when trying to teach basic ideas of probability from a Bayesian perspective. The student was an able student and spoke good English. The remainder of this paper should be seen as an attempt to understand what way of thinking lay behind my student's response.

The research started with a collection of anecdotes and progressed to include the use of questionnaires, semi-structured interviews, classroom observations, audio and video recordings of conversations with children and adults, as well as interviews with various professionals.

Vocabulary for uncertainty

In English, as in many other languages, there is a large spectrum of words for different levels of uncertainty: *Almost certain, ninety nine percent certain, very likely, probable, possible, conceivable, rare, slim chance, almost impossible* and many more. In Tongan, the only word in common use, which is similar to the above list, is 'mahalo', which is best translated as "maybe" or "perhaps".

An example to show how the dearth of suitable vocabulary affects the teaching of probability can be found in the Tongan version of the school curriculum for primary schools (Mathematics for Life Syllabus, 2009). The syllabus is written entirely in Tongan except for the section on probability, which is written together with an English translation. When I asked why the English translation was included, I received the explanation that the Tongan speaking teachers would not understand the Tongan and would need to refer to the English version to understand what they were supposed to teach!

Probability is measured using fractions but there were no words for fractions in Tongan until the missionaries introduced a rather complex way of expressing them during the second half of the nineteenth century. I found strong evidence that fractions are not understood in the same way as they are in the West, for example a large majority of Tongans (including some maths teachers) did not know how to answer "What is a half of a half?" My research led me to the conclusion that the Tongan discourse does not relate to fractions as numbers between zero and one. This has a clear effect on how probability is understood.

Answers to questions about the likelihood of future events

I conducted sixty structured interviews with a cross section of the population. The interviews took place near the main market. Unless my respondents spoke good English the interviews were carried out in Tongan by my assistant. Some of the questions were about the likelihood of future events such as "What are the chances that it will rain tomorrow?", "What are the chances of getting a head when tossing a coin?" and "What are the chances of a first child being a boy?" In all cases less than twenty five percent of the answers were in terms of uncertainty. All the other answers were in terms which to a Western way of thinking may appear dogmatic:

The chances of rain:	"The sky is cloudy so it will rain"
	"It is not going to rain"
	"Check at the internet [if it is going to rain]"
The chances of a head:	"You will know when you turn it over"
	"Depends who tossed the coin"
	"Toss it four times to get a head".
The chances of a boy:	"I believe they will get what they wish for"
	"Depends on the scan"
	"Sleep on time, wake up on time [then it will be a boy]"

The place of uncertainty in the Tongan community of discourse

In an attempt to understand the place of uncertainty in Tonga I interviewed a large number of people including teachers, church ministers, bankers, bingo players, government ministers, micro finance managers and Tongan language experts. The picture which emerged is of a community which, historically, has had little need for uncertainty. Definite answers are valued above uncertain answers and this is combined with a fatalism about the future. I have categorised some of the responses in terms of Hacking's preconditions for the emergence of probability.

Hacking suggests that developments in the insurance industry were one of the preconditions for the emergence of uncertainty in Europe in the seventeenth century. Insurance companies exist in Tonga but most people only take out insurance policies when the bank demands this as a condition for a loan. A typical response was "I insured my house until we had repaid the bank loan and then I stopped paying the premium". This is despite the fact that there is a serious danger of burglary, fire, flooding and hurricanes. Life insurance is also seen as unnecessary – while interviewing the minister of education, she said "I think they [Tongans] are quite certain that the processes and the checks and balances we have in the society insure that the future will be taken care of. .... If you ask a Tongan to pay out for life insurance they will think it is a total waste of money – I agree."

Another precondition, suggested by Hacking, concerns attitudes to religion, fatalism and causality. For more than a hundred and fifty years the Tongan way of life and value system has been dominated by what is known as the "Tongan Way", which aims to combine traditional Tongan values of respect and obedience with a deeply held belief in Christianity. In my research I explored how this belief system affects attitudes to free will, predetermination and the inevitability of future events. Most the people whom I interviewed expressed a strong belief that future events depend on the will of God and that we do not have free will to control them. A church minister suggested that "The might of God reinforced by concepts of monarchy and of culture and of respect and of dominance and of control [...] come through at the every person level as a sort of fatalism." I found evidence that the importance of obedience to God and to those of higher status led people to regard expressions of uncertainty in a negative light. I asked one of my respondents why she had given a definite answer and had the following conversation:

Interviewer: Why do you say that you are certain that it will rain?

Respondent: A 'maybe' answer would be dishonest. A definite answer is more honest. Not good to have doubts when you answer question. If you have doubts that is bad. If you ask the child and he answers 'maybe' the child will get a slap (you are being cheeky).

Interviewer: What about when you get it wrong [when it does not rain]?

Respondent: I feel good that I am positive about something that I believe in I didn’t have doubts.

DISCUSSION

I suggest two interpretations of the above findings and in particular, two explanations for what my student meant when he said he was certain that the sun would be shining. The first looks at the lack of vocabulary for uncertainty and the limited use of fractions. The second is based on an analysis of fatalism and predetermination. Finally I suggest a synthesis of these two approaches by comparing the observations made in Tonga with the emergence of probability in seventeenth century Europe.

Language, vocabulary and fractions (Sapir-Whorf)

Whorf claimed that "We dissect nature along lines laid down by our native languages". My observations in Tonga support the Sapir-Whorf hypothesis that language, and in our case the dearth of vocabulary, limit how uncertainty can be expressed or understood. Since there is no discourse on probability in Tongan, my student, when asked, what is the probability that the sun will be shining tomorrow, did not have the tools to give an answer in terms of uncertainty. Instead he understood the question as a prediction about the future – will it be sunny tomorrow? The concept of giving a reply in terms of probability by using fractions or percentages was not part of his discourse.

Religion, Fatalism and Predetermination

Through interviews and discussions I found strong evidence that the religious belief, which dominates Tongan society, includes a kind of fatalism by which an all powerful God controls our lives. Future events have been predetermined and we only have to wait to observe that future. Thus future events are not uncertain but are waiting to be revealed.

How does all this help us to understand the Tongan discourse on uncertainty? I suggest the following endorsed narrative:

God is almighty (a translation of the much used Tongan phrase 'Otua Mafi Mafi).

The future is predetermined because God is almighty and controls our destiny.

> It follows that there is no such thing as uncertainty regarding whether it will be sunny at this time tomorrow. It is no more uncertain than whether it was sunny yesterday.
>
> Thus any discourse about future events is a discourse about definite events which are known to God.
>
> When asked whether it will be sunny tomorrow there is no need for vocabulary describing uncertainty. Any question about what will happen in the future is a question about what has been predetermined by God.
>
> Given all of the above, what better way to answer the question about what God has decided than to go through the intermediary of "the angels"?

Both the above explanations combine to explain the developing discourse on probability and uncertainty, which I observed in Tonga. There are striking similarities with the process in seventeenth century Europe, described by Hacking (1975). In both cases the emergence of probability depends on a combination of a number of different preconditions.

This leaves us with the question: how would my student have felt if the sun did not shine at the same time tomorrow? (I was never able to answer this specific question as the angels provided correct information and the sun was shining at the same time the next day!).

CONCLUSION – A MORE MODEST BAYESIAN LEAVES POLYNESIA

This research originated with some surprising observations which I made while teaching at 'Atenisi, an institute founded in 1975 and dedicated to the encouragement of critical thinking amongst the young people of Tonga. The immediate purpose of this study is to contribute to the aim of encouraging critical thinking, not only at 'Atenisi but throughout the Tongan education system. Its broader purpose is to deepen our understanding of factors that shape mathematical thinking. The data already collected, only a small fraction of which has been presented on these pages, brought ample evidence for the strong interdependence between cultural practices, discourse (thus language), and thinking.

As noted above, I am a convinced Bayesian and over the years have made great efforts to convince family, friends and colleagues to make rational decisions in situations of uncertainty (usually with frustratingly little success). This research has led me to a more modest expectation from the Bayesian program, while also providing a framework on which to build a wider program to enable Tongans and people elsewhere to understand how to make rational decisions when faced with uncertainty.

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FUNCTIONS OF EXPLANATIONS AND DIMENSIONS OF RATIONALITY: COMBINING FRAMEWORKS

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This paper presents first results from a research project aimed at combining two theoretical frameworks, one concerning explanations and one related to rationality. The two theoretical lenses are used to understand one episode from a teaching experiment carried out in grade seven, concerning the construction of rectangles with a given perimeter. The combination of frameworks allows a finer analysis of the teaching episode and extends the original frameworks.

INTRODUCTION

Various theoretical frameworks are available to mathematics education researchers interested in analysing complex mathematical activities such as conjecturing, proving, and modelling. Sometimes, the networking of different theories may improve the understanding of data (Prediger, Bikner-Ahsbahs, & Arzarello, 2008). This paper combines two frameworks, one related to functions of explanations (Levenson, Barkai, & Larson, 2013) and one related to rational behavior in conjecturing and proving (Boero & Morselli, 2009), in order to investigate classroom tasks and didactical sequences which promote conjecturing and proving among students.

We chose to combine the two frameworks for several reasons. First, the framework related to the functions of explanation was previously used to analyse tasks found in national guidelines and curricula. We were interested in investigating the use of this framework when analysing classroom tasks and didactical sequences. The model of rationality was initially developed for the analysis of students' processes when faced with conjecturing and proving tasks. Those tasks usually took the form of "What can you tell about…?". We were interested in investigating the use of this model when students are faced with other kind of tasks, such as inquiry-based tasks, or tasks which require them to explain procedures. Finally, we wished to examine the possible links between functions of explanations and dimensions of rationality.

THEORETICAL FRAMEWORK S

Functions of explanations

Explanations are used every day in the mathematics classroom and are an integral part of learning and teaching mathematics. However, many research studies use the term 'explanation' in different ways, alluding to different functions of explanations (e.g., Hemmi, Lepik, & Viholainen, 2013; Yackel, 2001). Levenson and Barkai (2013), and Levenson, Barkai, and Larson (2013) set out to systemize and classify the possible functions of explanations which may arise from solving mathematical tasks in the classroom. Analysing curriculum documents in Israel and in Sweden led to six possible functions:

Function 1: Explanation as a description of one's thinking process or way of solving a problem (i.e., How did you solve the problem? Explain.)

Function 2: Explanation as an answer to a "why" question where the underlying assumption is that the explanation should rely on mathematical properties and generalizations (i.e., Why is this statement true/false? Explain.)

Function 3: Explanations as interpretations (i.e., Explain what this mathematical statement means in an everyday context. Explain an everyday occurrence in a mathematical context.)

Function 4: Explanations as a step in directing new explorations leading to generalizations (i.e., Find all possible solutions and explain.)

Function 5: Explanation as justifying the reasonableness or plausibility of a strategy or solution (i.e., Why did I choose to solve the problem in this way?)

Function 6: Explanations as a means of communication. This function may be a more general function considering that explanations, whether written down or expressed orally, are meant to be communicated.

It should be noted that the function of an explanation may depend on the task given as well as the context in which an explanation is requested or given. In the Israeli curriculum, for example, it was found that an investigative task may call for a child to explain a solution with the possible aim that this explanation leads to further investigation. A different task may call for an explanation which merely describes how to solve the task. It might be that the same task, implemented in different ways by the teacher, could lead to different functions of explanations. In Sweden, the functions of explanations seem to be tied in with major aims for teaching mathematics in primary school. The current study extends the study by Levenson, Barkai, and Larson (2013) by attempting to use their classification of the functions of explanations when analysing a series of classroom tasks given in an Italian mathematics classroom and by combining it with the framework of rationalization set out by Boero & Morselli (2009).

Rationality

Boero & Morselli (2009) developed a theoretical model for proving as a rational behaviour, derived from the construct of rationality proposed by Habermas (2003). According to the model, the discursive practice of proving may be seen as made up of three interrelated components: an epistemic rationality (ER), related to the conscious validation of statements according to shared premises and legitimate ways of reasoning; a teleological rationality (TR), inherent in the conscious choice and use of tools and strategies to achieve the goal of the activity; and a communicative rationality (CR), inherent in the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning, and the conformity of the products (proofs) to standards in a given mathematical culture. The construct was further developed to analyse specific phases within the conjecturing process, for instance the use of algebraic language (Morselli & Boero, 2011).

The following episode, taken from a proving task (Morselli & Boero, 2011) illustrates the three components of rational behaviour, highlighting teleological rationality. Seventh grade students were given the following problem: "The teacher proposes a game: Choose a number, double it, add 5, take away the chosen number, add 8, take away 2, take away the chosen number. Without knowing the result of the game, is it possible for the teacher to guess the number that you initially chose? If yes, in what way?". Students played the game and gradually discovered that the teacher always guesses ten. When explaining why the result is always ten, two solutions emerged: the expression (N×2+5-N+8-2-N-1=10) and the sequence of calculations (N×2 = A; A+5= B; B-N=C; C+8=D; D-2=E; E-N=F; F-1=10). Comparing the two representations sheds light on the dimensions on rationality. The two representations are correct from a mathematical point of view, thus fulfill the requirement of epistemic rationality, and are clear from the communicative point of view. Both representations lead to the result of the game, but the first one is more efficient in terms of the goal of the activity (showing that the result is always 10), since it shows that N is at first doubled and then taken away twice, making no contribution to the final result. From a teleological point of view, the first representation is more appropriate. The teleological component of rationality refers to efficiency and usefulness in relation to the final goal one wishes to achieve (here, showing that N does not affect the final result). We refer to teleological rationality for all those strategic choices that are linked to the final aim of the activity.

Combining the frameworks

As previously mentioned, the construct of rationality was initially adapted from a general description referring to any discursive practice (not only within mathematics), to the process of proving. The process of proving relates to only one specific function of explanation – explaining why a statement holds true. Yet explanations may have several functions. The aim of this study is to explore the possibility of extending the scope of the rationality construct and describe the rationality at issue when explaining. Can we see an expression of the three dimensions of rationality in explanation processes? If so, how can each dimension be described in relation to each function of explanations?

METHOD

The first step in combining two frameworks is to understand each one separately. Mutual understanding was achieved by reading previous research reports and by a first cycle of data analysis. Each author analysed the data by means of the two theoretical lenses. The two analyses were compared and divergent interpretations were questioned, so as to promote mutual understanding (of the frameworks) and a more complete interpretation of the teaching episode. Finally, we worked together in developing a combined description of rationality in explanation processes. In the following sections we describe some background of the project where this study was set and analyse some written productions in terms of the functions of explanations which arose and the dimensions of rationality observed.

TASK SEQUENCE: ISOPERIMETRIC RECTANGLES

The episode we refer to comes from a teaching experiment carried out in 2012 in a lower secondary school, within the project "Language and Argumentation", aimed at designing and experimenting task sequences with a special focus on argumentation and proof (Morselli, 2013). Core tasks are usually proposed as open-ended questions (*What can you tell about…?*) where, according to the socio-mathematical norms of the class, each answer must be justified.

The data below was collected from the beginning parts of the task sequence "*Isoperimetric rectangles*", implemented in grade 7 (age of the students: 13-14). At the core of this activity is the conjecture and explanation of the fact that, among all the rectangles with fixed perimeter, the square has the maximum area. The task sequence began with an explorative task in paper and pencil: "*Draw four rectangles with a perimeter of 20 cm*". After drawing the rectangles, students were asked to reflect on the construction of their rectangles. The second task, to be worked on in groups, was: *Compare the methods you used to draw the rectangles and synthesize*. Here we present our analysis of the group work.

FINDINGS

Students' written collective responses were collected by the researcher. Below, we analyse the results of four groups, using the dual lens of functions of explanations and rationality. For clarity of analysis, we break up the students' writing into segments (Seg a, Seg b).

Group One

Two students working together wrote the following:

Seg a: In order to make rectangles with a perimeter of 20 cm one must make 10 cm and then multiply by 2.

Seg b: With this method one can make 9 rectangles: 6+4, 7+3, 8+2, 9+1, 4+6, 3+7, 2+8 and 1+9, but the first, second, third and fourth one are equal to the last four. 5+5 cannot be done because a square is made.

Segment a: Regarding functions of explanations, the students describe their method. This is Function 1. It might be said that the first segment, in which the students explained their method, led them to explore several options, basically covering all the whole number options (Seg b). Evidence of this may be seen in what the students wrote in Seg b, "With this method one can make…" In addition, the students set out what is, in their opinion, all the possible rectangles taking into consideration the constrictions. In this sense, we claim that Seg a may be related to Function 4 in that it led to additional exploration. In addition, the students justify their strategy. They explain that they are looking for numbers which sum to 10 cm in order to find a perimeter of 20 cm. Therefore, Function 5 is present. Regarding modes of rationality, the students began their explanation by writing "In order to….". This clear indication of working towards a goal is evidence of Teleological Rationality (TR). The explanation is correct

(Epistemic Rationality – ER) and communicated in a comprehensible although not complete manner (Communicative Rationality – CR). For example, although the students noted that “one must make 10 cm”, this statement is rather general; it was not explicitly stated that it is necessary to take exactly two numbers whose sum is ten.

Segment b: Regarding functions of explanations, there might be evidence of Function 2 in that the students explain why they do not include 5+5. It is likely that students compare the properties of squares and rectangles and incorrectly conclude that a rectangle must have unequal sides. This incorrect property is used for justifying that the solution 5+5 is not acceptable. Regarding rationality, the students only list rectangles with whole number lengths and they do not include the square. They explicitly state that there are nine rectangles which fit the requirements of the problem, implying that these are all the possible rectangles. Because the square is not included, there is a lack of ER. Furthermore, their explanation consists mostly of examples without further elaboration (CR). However, there is a clear goal to list all possible rectangles and the students organize their discourse accordingly (TR).

Group Two

Two students working together wrote the following:

Seg a: We looked for a number that gave 10 and then we added it, for example 8+2. Afterwards we added the same number 8+8 is the length and 2+2 is the side thus giving a rectangle.

Seg b: Ex 9+9 and 1+1 = 20 and 3+3 [under the numbers it is written “sides”] and 7+7 [under the numbers it is written “bases”] =20 ex3 6+6 [under the numbers: “bases”] and 4+4 [under the numbers: “sides”] = 20.

Segment *a*: Regarding functions of explanations, the students describe what they are doing (Function 1). Unlike with Group one, it does not seem that the students expand their exploration beyond giving a few more examples (Seg *b*). Thus, there is no evidence of Function 4. Nor do they justify their strategy of looking for numbers that add to 10 cm. Therefore, Function 5 is not present. As in Group one, the students have a clear goal and they state so explicitly, “We looked for…” (TR). Their procedure is correct (ER), however, like Group one, their communication lacks the necessary details (CR). Instead of writing that they looked for two numbers which add to 10, they wrote, “We looked for a number that gave 10.” In addition, the word “gave” is not mathematical and does not convey that the students are looking for numbers which “add” to ten.

Segment *b*: Regarding functions of explanations, it seems that only Function 1 was present. Regarding rationality, the examples are correct and because the students do not claim that they have found all possible rectangles with perimeter 20 cm, we cannot comment on and evaluate the lack of additional examples (ER). On the one hand, the students attempt to communicate their ideas in a clear manner by designating “sides” and “bases” (CR) although perhaps, mathematically, it would be more precise to distinguish between “heights” and “bases” (ER). On the other hand, the statement “9+9 and 1+1=20" does not conform to mathematical standards of communication (e.g., using "and" instead of "+") (CR).

Group Three

Three students worked together and wrote the following:

> Seg a: We added two different sides, whose sum was 10 which multiplied by two the result was 20, that is to say the perimeter of the rectangle.
>
> Seg b: Other rectangles with perimeter of 20 cm, are not possible, unless with these sides 6cm+4cm= x2, 8+2cm= x2, 9+1cm= x2, 7+3cm= x2.
>
> 5+5+5+5cm=20 cm, it is not a rectangle, but a square. 10+10 cm= 20 cm but it is not a rectangle.

Segment *a*: Regarding functions of explanation, in addition to describing what they did (Function 1), the students justified why their strategy is valid – because it leads to a rectangle perimeter of twenty. Thus, there is some evidence of Function 5. As with Group One, if we look ahead to Segment b, we may say that the explanation in the first part led the students to explore what might be all possible solutions. Thus, Function 4 is present as well. Regarding rationality, once again we see students working towards a goal of finding two numbers whose sum is ten (TR). The procedure is correct (ER) and, as opposed to the first two groups, it is communicated with necessary details, such as stating that when the sum of the two sides is multiplied by 2, the result is 20 (CR). In addition, the communication employs relevant mathematical language using terms such as "sum" and "perimeter".

Segment *b*: As with the first group, both Function 1 and Function 2 of explanations are present. They describe what they did but they also explain why they do not include the square in their results, writing, "it is not a rectangle, but a square." From an ER point of view, the students' claim is incorrect. From a TR point of view, the students are working towards a goal, that is, they wish to show why only some examples are possible. They work towards this goal, ultimately reaching the example of the square, which seems to be, for them, the limit of the possibilities. Their written mathematical expressions, such as "6cm+4cm=x2" do not conform to acceptable mathematical convention (CR).

Group Four

This group of three students wrote the following:

> We found different ways of [drawing] rectangles. For instance 9cm1cm x2 times so we got 20 cm. Other examples are:
>
> Bartek: 2 cm, 8 cm; 6 cm, 4 cm; 3 cm, 7 cm
>
> Angelo: 6cm, 3.5 cm; 8 cm, 2 cm; 10.5 cm, 1.2 cm; 7 cm, 3 cm
>
> Manuel: 4 cm, 6 cm; 7 cm, 3 cm; 9 cm, 1 cm; 8 cm, 2 cm

As opposed to the other groups presented above, these students merely stated that they found different rectangles, some of which were incorrect, without describing the procedure or explaining the method that they used to find them. Thus, Function 1 is not present and ER is lacking. Their means of communication is deficient "9cm1cm x2 times" (CR). Finally, their list of examples seems to lack direction and purpose. Thus TR is missing as well. In comparison to the other groups, Group Four's results are lacking both in terms of the functions of explanations and rationality. Perhaps the lack of functionality of their explanation is related to their lack of rationality.

DISCUSSION

This paper explored the possibility of extending the framework of rationality to the process of explaining. We found that the three components of rationality may be found in students' explanations and that the three components may be expressed or described differently for explanations with different functions. We propose a first refinement of the framework of rationality, with reference to some functions of explanation.

Function 1: *When explaining one's procedure, ER relates to the correctness of the procedure (from a mathematical point of view), CR to the communication of the method (all passages must be communicated), TR to mentioning the final goal of the procedure and possibly to the links between aims and actions*.

Function 2: *When explaining why a statement holds true, ER refers to the mentioning of correct mathematical properties, TR to the choice of suitable properties according to the proving aim, and CR to the organization of an intelligible explanation*.

Function 4: *When looking for all the possible solutions, and explaining why they are finite/infinite, ER relates to referencing mathematical properties, TR to mentioning the final goal and to the organization of the exploration and explanation accordingly (for instance, showing all the possible sums in a regular order). CR refers both to the communication of the exploration and of the final explanation*.

Function 5: *When justifying the plausibility of a method/procedure, ER refers to referencing correct mathematical properties and CR to the communication of the explanation in an intelligible form. At this point, perhaps due to the nature of the task, we have not been able to describe TR as it relates to Function 5. Additional research is needed regarding this aspect of Function 5*.

What emerges from the above analysis is that the three dimensions of rationality are always present, and that each function of explanation requires rationality in action. Furthermore, we found that a single task may elicit explanations with different functions and that the shift from one explanation to another may be "natural", occurring even when explicit requirements to give an explanation are missing. As was shown, some explanations (Function 1) may pave the way to further explanations (Function 2, 4 and 5). These findings suggest that explanations with different functions may help promote students' rationality.

The interest in combining frameworks is twofold: reaching a better understanding of the teaching episode at issue and improving theoretical frameworks (Prediger et al., 2008). Combining the frameworks, allowed us to fine tune our analysis in terms of understanding how students' explanations may be intertwined with rationality. For example, as was shown in Groups 1 and 3, Function 5 of explanation seems to occur when there is a good level of TR evident in Function 1 of explanation. For the moment, we confined our analysis to the written group productions. The next step will be to use the refined and combined framework to analyse other parts of the teaching experiment, such as classroom discussions. From the point of view of functions of explanation, we expect that some functions of explanation, such as Function 1, 5 and 6 will emerge strongly in classroom discussions, while from the point of view of rationality, CR should have a major role, since communication to others is essential in discussions. An additional development concerns the planning and implementation of new tasks explicitly aimed at promoting different functions of explanation and associated rationality.

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TEACHERS' ABILITY TO EXPLAIN STUDENT REASONING IN PATTERN GENERALIZATION TASKS

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The purpose of this paper is to explore teachers' ability to explain student reasoning in linear and non-linear patterns and in different types of generalization tasks. A questionnaire consisting of student responses to different types of generalization tasks was developed and then given to a sample of 91 in-service mathematics teachers from 20 schools in Lebanon. Analysis of data shows that teachers' explanations exhibited variations in the extent to which they identified the elements and relationships found in students' responses. The results showed that teachers' ability to explain students' reasoning of linear tasks seemed to be higher than that of non-linear patterns. The findings also showed that teachers' explanations of students' reasoning for far generalization tasks exhibited a larger amount of data (elements and relationships) compared to the explanations for near generalization tasks.

BACKGROUND

Findings from previous studies reported that students' reasoning and strategies in pattern generalization is influenced by different factors. Of these factors is the function type of the pattern (linear and non-linear patterns). Krebs (2005) found out that while students are able to generalize linear patterns (constant difference between consecutive terms), they have difficulty in generalizing non-linear patterns (varying difference between consecutive terms). Another factor that has an impact on students' reasoning in patterns is the generalization type (near and far generalization tasks). Amit and Neria (2008) reported that while near generalization tasks (questions which can be solved by step-by-step drawing or counting) were accessible to the majority of students, those students faced difficulties in establishing and justifying a rule for the far generalizations (questions which are difficult to be solved by step-by-step drawing or counting).

On the other hand, few research studies aimed to explore teachers' ability to explain students' reasoning in pattern generalization. For example, El Mouhayar and Jurdak (2012) reported that teachers' ability to explain students' reasoning in far generalization tasks depend on their ability to explain students' reasoning in near generalization tasks. Other studies that focused on teachers' knowledge of pattern generalization used samples of prospective teachers. These studies indicated that prospective teachers recognize patterns in different ways. For example, some prospective teachers formulated rules from the sequence of numbers that are listed in a pattern, whereas others used relationships and cues that are established from the figural structure of a pattern (Chua & Hoyles, 2009; Rivera & Becker, 2007).

The present study extends the previous research on teachers' knowledge of pattern generalization and it attempts to understand teachers' ability to explain students' reasoning in pattern generalization.

Teacher knowledge of linear and non-linear patterns

Findings in the literature report that the difference between teachers' identification of students' rules in linear and non-linear is not significant. For example, El Mouhayar and Jurdak (2012) focused on the ability of inservice mathematics teachers from grades 7-9 to explain students' reasoning in linear and non-linear patterns tasks. The findings showed that there was no significant difference in teachers' ability to identify symbolic rules that best corresponded to students' pattern generalization processing.

Other findings from previous research showed that teachers are capable of using a variety of strategies to generalize linear and non-linear patterns in different ways. Rivera and Becker (2007) found that teachers generalized linear patterns using numerical and figural strategies. Similarly, Chua and Hoyles (2009) reported that teachers were capable of using a variety of strategies to generalize non-linear patterns in different ways resulting in a range of equivalent rules.

Teacher knowledge of near and far generalization tasks

Previous literature indicates that teachers are able to successfully generalize patterns using different strategies (Chua & Hoyles, 2009; Rivera & Becker, 2007). However, findings of previous studies reveal that teachers' explanations of students' reasoning in near and far generalization tasks seem to be lacking in terms of the elements which constitute a complete explanation. El Mouhayar and Jurdak (2012) showed that more than half of the in-service school teachers (grades 7-9) who participated in the study were unable to provide complete explanations for students' reasoning in near and far generalization tasks. More specifically, while teachers' explanations focused on constant-related counting elements that are not dependent on the step number of the pattern, the teachers' explanations did not include the variable-related counting elements that are dependent on the step number by relating the growing parts of the pattern to the step number.

RATIONALE OF THE STUDY

This study extends previous research on teachers' knowledge of student reasoning in pattern generalization in four directions. First, the present study aims at exploring teachers' ability to explain students' reasoning of linear and non-linear patterns and of near and far generalization tasks whereas previous studies dealing with teachers' knowledge in pattern generalization did not address function type (linear and non-linear) and pattern generalization type (near and far) simultaneously. Second, this study attempts to confirm and extend the results of few previous studies that investigated teacher knowledge of students' reasoning in pattern generalization across a larger range of grade levels than previous studies. Third, the present study aims to explore in-service teachers' ability to explain students' reasoning using authentic student work whereas previous studies have used contrived illustrative models of students' reasoning taken from the literature. Fourth, this study uses the "SOLO model" as a theoretical construct to explore the extent to which teachers use elements of student response to generalize patterns. Although previous studies have used SOLO in the context of teachers' knowledge, these studies did not explore teachers' knowledge of students' reasoning in pattern generalization in particular.

RESEARCH QUESTIONS

The present study aims at exploring teachers' ability to explain student reasoning in linear and non-linear patterns and in immediate, near and far generalization tasks. In this paper we address the following research questions:

- How well are in-service teachers able to explain students' reasoning of linear and non-linear patterns?
- How well are in-service teachers able to explain students' reasoning of near and far generalization tasks?

THE SOLO TAXONONMY

The Structure of the Learned Outcomes (SOLO) taxonomy was developed by Biggs and Collis (1982) to describe a hierarchy of different levels of knowledge ranging from lack of ability to proficiency. The lowest level is called prestructural and it represents the use of no relevant aspect of knowledge in a task. Responses at this level show little understanding of the task. The second level, unistructural, represents the use of only one relevant aspect of the task and therefore indicates some understanding of the task. The third level is the multistructural level whereby responses contain more than one aspect of relevance to the given task; however, those aspects remain separated without being unified or integrated into a coherent structure. The fourth level of the SOLO taxonomy is relational. Relational responses include all the characteristics of the multistructural level in addition to the use of aspects that are related and integrated into a coherent structure. The fifth level and highest level of SOLO taxonomy is called extended-abstract which represents knowledge that goes beyond the task requirements and generalizes its structure. Several studies (e.g. Groth & Bergner, 2006) applied SOLO taxonomy to describe the knowledge of teachers in different contexts.

METHOD

Participants

Ninety one in-service school teachers from different grade levels were selected from 20 schools in Lebanon, particularly Beirut and its suburbs. The majority of the participants (75.8%) had five or more years of experience in teaching mathematics. Of the 91 participants, 79.8% were females and 20.2% were males. All of the participants had either a university teaching diploma or a BA/BS with 52.4% of them having a degree in math education and 29.8% having a degree in pure mathematics.

Instrument

The instrument consisted of two parts. The first part collected information about teachers' background including teaching experience, teacher's major and teaching certificate. The second part consisted of 10 items designed by the researchers to examine teachers' ability to explain student reasoning in pattern generalization. A sample of students' responses were taken from a survey used in a previous study (Jurdak & El Mouhayar, 2013) involving 1232 Lebanese students from grades 4 to 11. The survey included four tasks (two linear and two non-linear). Each of the items displayed the problem (a linear or non-linear task showing the first four figural steps) and students' responses to the (1) near generalization (predicting steps 5 and 9) and (2) far generalization (predicting step 100 or step n). Teachers were asked to analyze one of the two generalization types (near or far) for each of the ten items. For example, participants were asked to analyze student reasoning for step n for item 8 from the questionnaire (Figure 1). For each item, participants were asked "How did the student think to get the number of squares?

The internal reliability of the questionnaire was calculated and Cronbach's alpha was found to be 0.788. The questionnaire was piloted with in-service mathematics teachers from different grade levels to make sure that all the items were understood.

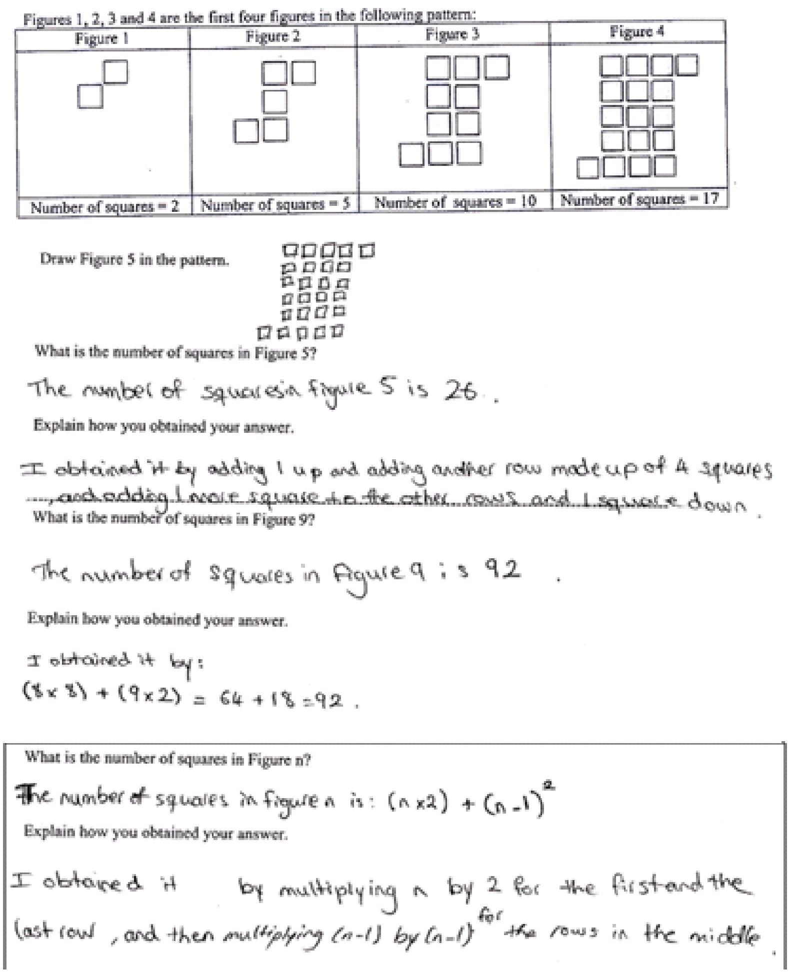

Figure 1: Item 8 from the questionnaire in a non-linear pattern

Data collection and analysis

In each of the 20 schools, participants filled out the questionnaire individually in the presence of the investigator. Filling out the questionnaire took around 90 minutes.

The data obtained were subjected to a series of analyses. First, for each item a rubric based on the SOLO taxonomy was constructed for the purpose of evaluating the teachers' responses to the questionnaire items. In particular, for each of the 10 items in the questionnaire, the investigator identified the elements and relationships that constituted a complete and coherent explanation of the students' responses. The scale points of a rubric were as follows: A teacher's explanation was given a score of 3 "relational" if the teacher identified all the elements of student's reasoning and connected them together; 2 "multistructural" if the teacher identified more than one element but did not address the relationships among these elements or did not address all the elements of student's reasoning; 1 "unistructural" if the teacher identified only one element and 0 "prestructural" if the teacher's explanation indicated a refusal or inability to become engaged in the problem. Two researchers coded the data independently and the discrepancies in coding were negotiated until consensus was reached. Second, cross-tabulations of teachers' ability to explain students' reasoning by (1) function type (linear and non-linear) and (2) pattern generalization type (near and far) were done to explore the possibility of significant differences. For this purpose, significant Chi-squared values and the adjusted residual values were examined. Third, a qualitative analysis was done on teachers' explanations in order to understand the differences in their ability to explain student reasoning by function type and pattern generalization type. Specifically, the qualitative analysis focused on the extent to which the elements and relationships were missing in teachers' explanation. For each item, the elements and relationships in each teacher explanation were identified based on the corresponding rubric. Consequently, the elements and relationships that were missing in teachers' explanations were identified from the rubric and the percentages for the missing elements and relationships were determined.

RESULTS

Teachers' ability to explain students' reasoning by function type

The cross tabulation of teachers' ability to explain students' reasoning by function type is shown in Table 1. Chi-squared was significant (χ^2 (3) = 22.424, p = 0.00) indicating significant differences in teachers' explanations.

For each of the linear and non-linear tasks, teachers showed different abilities to explain students' reasoning: pre-structural, uni-structural, multi-structural and relational with some variation in the relative frequencies across those levels (Table 1). Table 1 shows that teachers' explanations for linear tasks were mainly classified (based on the mode) as relational (29.7%) whereas teachers' explanations for non-linear patterns were mainly classified as either pre-structural (27.9%) or multi-structural (27.9%).

Function type	Prestructural	Unistructural	Multistructural	Relational	Total
Linear (%)	17.1*[1]	28.6	24.6	29.7*	100.0
Non-Linear (%)	27.9*	23.7	27.9	20.4*	100.0
Total (%)	22.5	26.2	26.3	25.1	100.0

Table 1: Cross tabulation of teachers' explanations by function type

A paired-samples t test was conducted to compare the mean of teachers' explanations of students' reasoning for the linear tasks (M=1.67, SD=1.077) with that of non-linear tasks (M=1.41, SD=1.101) and the difference was significant ($p<0.05$), which indicates that teachers' explanations of students' reasoning for linear tasks exhibited a larger amount of data (elements and relationships) than those for the non-linear tasks.

Teachers' ability to explain student reasoning by pattern type

The cross tabulation of teachers' ability to explain students' reasoning by pattern generalization type is shown in Table 2. For the four types of tasks, Chi-squared was significant (χ^2 (3) = 26.597, p = 0.00) indicating significant differences in teachers' ability to explain students' reasoning.

Table 2 shows that teachers' explanations for near tasks were mainly classified (based on the mode) at the multistructural level (30.1%) whereas teachers' explanations for far tasks were mainly classified at the relational level (32.3%).

Pattern type	Prestructural	Unistructural	Multistructural	Relational	Total
Near (%)	24.6	27.5	30.1*	17.8*	100.0
Far (%)	20.4	24.8	22.4*	32.3*	100.0
Total (%)	22.5	26.2	26.3	25.1	100.0

Table 2: Cross tabulation of teachers' explanations by pattern generalization type

A paired-samples t test was conducted to compare the mean of teachers' explanations of students' reasoning for near tasks (M= 1.41, SD= 1.045) with that for far tasks (M= 1.67, SD=1.131) and the difference was significant ($p<0.05$), which indicates that teachers' explanations of students' reasoning of far tasks exhibited a larger amount of data (elements and relationships) than those for near tasks.

A qualitative analysis of teachers' explanations focused on the amount of data (elements and relationships) that are used in explaining students' responses. The qualitative analysis suggests that teachers' explanations coded at the relational level used several elements of the students' responses and related them together. 50.2% (120 out of 239) of the teachers' explanations that were coded multistructural missed elements that were strategy specific whereas 49.8% (119 out of 239) of the teachers'

[1] * indicates that the adjusted residual was greater than |2|

explanations focused on several elements of student reasoning without relating them. Explanations that were coded unistructural missed at least two elements that were strategy specific. Exemplars were produced to illustrate different levels of teachers' explanations and to clarify discrepancies in explaining student reasoning in pattern generalization tasks.

The following excerpts are examples of teachers' explanations of a student reasoning in a far generalization task (step n) of a non-linear pattern (Figure 1). The pattern was perceived by the student as a large square in the middle with dimensions (n-1) by (n-1) and two additional rows at the top and bottom each of size equal to n.

Explanation at the relational level:

> The student separated the middle part of the figure from the upper and lower rows and noticed that the middle part is a square of dimensions equal to figure number minus 1. The area of the inside part is (n-1)2 such that n is the figure number. The number of squares in each of the upper and lower rows is equal to the figure number so it would be 2n for the two rows. For example, for figure 4 it would be 3×3 for the middle part and 2×4in the upper and lower rows.

Explanation at the multistructural level:

> The student related the figure number to the number of squares in the first and last row and to the number of rows in the middle.

The teacher's explanation explicitly referred to the different parts of the pattern and that there is a relationship with the figural step number, but did not explicitly point out the relationship of each part with the step number.

Explanation at the unistructural level:

> The student found a relationship between the figure number and the number of squares in each of the first four given examples. He/she applied the formula and found the number of squares in step 5. The same formula was applied in step n.

The teacher's explanation pointed out that there is a relationship between the number of squares forming the pattern and the step number; however, the teacher did not explicitly identify the different parts of the pattern and did not relate each part of the pattern with the step number.

DISCUSSION

One major finding in this study is that there were variations in teachers' ability to explain student reasoning in pattern generalization. Teachers' explanations fell into four levels: prestructural, unistructural, multistructural and relational. From a SOLO perspective, this finding indicates that teachers' explanations exhibited variation in the extent to which they used the elements and relationships found in student responses. This is supported by findings from other research studies that reported that teachers showed different abilities in analyzing students' reasoning (El Mouhayar & Jurdak, 2012) or in analyzing mathematical concepts and procedures (Groth & Bergner, 2006).

The findings in the present study indicate that teachers' ability to explain students' reasoning in linear patterns seems to be significantly higher than that of non-linear patterns. This finding does not parallel findings from previous research which indicate that differences between teachers' abilities to identify student rules for linear and non-linear patterns were not significant (EL Mouhayar & Jurdak, 2012). One plausible explanation for this finding is that linear patterns are less complex than non-linear patterns since the growth between the consecutive figural steps is constant whereas the growth in the latter varies.

The findings in the present study showed that teachers' ability to explain students' reasoning of far generalization tasks exhibited a larger amount of data (elements and relationships) compared to the explanations of near generalization tasks. This result does not parallel findings from other studies that indicate that while near generalization tasks were accessible to preservice teachers, they have difficulties in establishing and justifying a rule for the far generalization tasks (Rivera & Becker, 2007).

In conclusion, the findings in this study suggest that a move to help teachers in developing their abilities to analyze students' reasoning in pattern generalization is needed to ensure that teachers will have the ability to involve their students in pattern-based instruction as an approach for developing algebraic reasoning.

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USING MODELING-BASED LEARNING AS A FACILITATOR OF PARENTEL ENGAGEMENT IN MATHEMATICS: THE ROLE OF PARENTS' BELIEFS

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Being part of a larger research project aimed at connecting mathematics and science to the world of work by promoting mathematical modeling as an inquiry based approach, the present study aimed to: (a) describe parents' beliefs about inquiry-based mathematical modeling and parental engagement, and (b) explore the impact of a modeling-based learning environment on enhancing parental engagement. Results from semi-structured interviews with 19 parents from one elementary school classroom revealed strong positive beliefs on their engagement in their children learning, an appreciation of the modeling approach for bridging school mathematics and home, and their willingness to collaborate with teachers. Implications for parental engagement in mathematics learning are discussed.

INTRODUCTION

This study argues for an inquiry-based approach (IBL) in the teaching and learning of mathematics, one that is based on a models and modeling perspective (Lesh & Doerr, 2003). A modeling based IBL approach can serve as an appropriate means for bridging complex real world problem solving with schools mathematics (English & Mousoulides, 2011). This connection is necessary, as complexity gradually appears in all forms of the society and the education, and new forms of mathematical thinking are needed. Further, a modeling based IBL approach could contribute in enhancing students' abilities in designing experiments, manipulating variables, working in teams, and communicating their solutions with others (Mousoulides, 2013).

Integrating such an innovative approach in mathematics is not an easy process. It conflicts with various factors, including national curriculum requirements, teachers' beliefs and practices, and parents' beliefs and attitudes towards such innovations. The significance of parents' role has been documented in a number of studies (see Epstein et al., 2009), and parental engagement has been documented as a positive influence on children's achievement, attitudes, and behaviour. However, achieving appropriate parental engagement is a difficult and long-term process, and teachers should collaboratively work with parents to find the best appropriate methods. The present study targets the identified lack of studies, and examines parents' beliefs on their engagement in their children learning in mathematics, and on communication with the classroom teacher and students, by focusing on a teaching experiment on mathematical modeling.

THEORETICAL FRAMEWORK

The theoretical framework focuses on two strands: (a) instructional interventions to promote mathematical inquiry through a modeling perspective, and (b) parental engagement in the mathematics classrooms with an emphasis on parents' beliefs.

A Modeling Perspective in Inquiry Based Learning in Mathematics

In successfully working with complex systems in elementary school, students need to develop new abilities for conceptualization, collaboration, and communication. In achieving these abilities, a number of researchers propose the use of an inquiry-based approach in the teaching of mathematics, one that builds on interdisciplinary problem-solving experiences that mirror the modeling principles. In this study we adopt the use of *Engineering Model-Eliciting* Activities (EngMEAs); realistic, client-driven problems based on the theoretical framework of models and modeling (English & Mousoulides, 2011).

EngMEAs have been in the focus of our work for the last few years (see Mousoulides, Sriraman, & Lesh, 2008; Mousoulides, 2013). EngMEAs provides an enriched modeling approach by offering students opportunities to repeatedly express, test, and refine their current ways of thinking as they endeavour to create a structurally significant product for solving a complex problem. The development of the models necessary to solve the EngMEAs has been described by Lesh and Zawojewski (2007) in terms of four key, iterative activities: (a) Understanding the context of the problem / system to be modelled, (b) expressing / testing / revising a working model, (c) evaluating the model under conditions of its intended application, and (d) documenting the model throughout the development process. The cyclic process is repeated until the model meets the constraints specified by the problem.

Parental Engagement

Parental engagement has been documented as a positive influence on children's achievement in mathematics, regardless of cultural background, ethnicity, and socioeconomic status (Epstein et al., 2009; Ginsburg-Block, Manz, & McWayne, 2010). Active parental engagement, however, is quite difficult to be maintained. Therefore, programs of parental engagement should be carefully designed and implemented, taking into account all related variables and barriers (Vukovic, Roberts, & Wright, 2013). Musti-Rao and Cartledge (2004) suggest inviting parents' experiences in into discussion, and including parental engagement strategies in teacher professional development courses. They also propose a number of strategies for engaging parents, such as mathematics and science fairs, community involvement utilizing engineering experts, and the establishment of a clear communication between teachers and parents, in an attempt to bridge teachers' and parents' beliefs and expectations (Musti-Rao & Cartledge, 2004; Vukovic et al., 2013).

Epstein and Van Voorhis (2001) identify teacher and parents beliefs as an important barrier to creating effective relationships between home and school. Parents' beliefs on mathematics teaching and learning and the significance of their engagement might also impact parental engagement. Often, the beliefs and expectations between families and educators are not shared collectively, and in many cases parents might have negative beliefs that can lead to stereotypes regarding the relationship between them and teachers. In order for parents' beliefs to change into positive ones, parents should be open to invitations to be engaged in school mathematics, while more parental engagement training on how to work with parents and communities is needed for teachers (Epstein, et al., 2009; Ginsburg-Block, et al., 2010).

THE PRESENT STUDY

The Purpose of the Study

This study investigated parents' beliefs on their engagement in their children learning in mathematics, during the implementation of two complex modeling activities, in an elementary school classroom. Specifically, the study focused on parents' beliefs on the learning environment that was generated, parents' beliefs on their engagement, and their experiences with regard to collaboration and communication.

Participants and Procedures

The research presented in this study was part of MASCIL, a larger research design that includes: (a) inquiry-based mathematics and science instruction, (b) integration of engineering model-eliciting activities as a means to connect school mathematics to the world of work, and (c) examination of the appropriateness of various forms of parental engagement, including workshop participation, participation in classroom activities, and communication with teachers. During her participation in MASCIL, a longitudinal four-year project on Inquiry and Modeling Based Learning in Mathematics and Science, Nefeli (pseudonym) an elementary school teacher in a public K-6 elementary school in Cyprus, participated in a five-day professional development course on inquiry- and modeling-based teaching and learning in mathematics. Following her participation in the training, Nefeli organized the implementation of two modeling activities in her 6th grade (12 year olds) classroom.

Prior to the implementation of the modeling activities, the parents of all students in Nefeli's classroom (36 people) were invited to attend a presentation on the role of IBL and modeling in the learning of mathematics. Twenty-seven parents attended the presentation. Based on the feedback received by the participants, two three-hour workshops for parents were designed and delivered, prior to the implementation of the modeling activities. Nineteen parents and the classroom teacher participated in both workshops. During the workshops parents had the opportunity to work in groups in solving a modeling problem, and to discuss with the researchers and the teacher on how parental engagement could facilitate students' learning in mathematics. During the second workshop parents had the opportunity to familiarize themselves with the two modeling activities that were to be implemented in the classroom. Parents were also introduced to Twitter® and on the possibilities it could provide for the mathematics classroom, as an online technological tool which can break down the rigid classroom schedule barriers and allow teachers, students, and parents to collaborate. During the implementation of the modeling activities parents were encouraged to reflect on and comment on their children developments in the classroom, using Twitter®.

The Implementation of the Model Eliciting Activities

The two modeling activities (Water Shortage and Bridge Design) followed the design principles of the model eliciting activities, as these are described by Lesh and colleagues (Lesh & Doerr, 2003). Activities are not presented here due to space constraints; however a detailed presentation of the activities can be found elsewhere (see Mousoulides, 2013; English & Mousoulides, 2011). Each model eliciting activity entailed: (a) a warm-up task comprising a mathematically rich newspaper article, designed to familiarize the students with the context of the modeling activity, (b) "readiness" questions to be answered about the article, and (c) the problem to be solved, including complex tables of data. The *Water Shortage* activity asked students to assist the local authorities in finding the best possible country that could supply Cyprus with water. The *Bridge Design* activity required students to develop a model for calculating the cost for various bridge types. Both activities required students to develop their models for solving the problems by integrating both quantitative and qualitative factors.

The activities were implemented by the classroom teacher and the author. Working in groups of three, the students spent five 40-minute sessions on each activity. During the first two sessions the students worked on the newspaper articles and the readiness questions and familiarized themselves with software that was used for solving the problems (Google Earth & Spreadsheets) and for communicating their results (Twitter® & Wikis). In the next two sessions students worked on solving the problems. They developed a number of appropriate models for solving the problem, and shared these models with their teacher and parents. During model development students were prompted by teachers to share their ideas with their parents. To facilitate model sharing, a public Wiki was created, in which students could easily upload their files. Student then shared the links to their models with their parents, using appropriate tweets. The great majority of parents participated in the implementation of the activities, by following student groups' tweets and provided feedback and suggestions to students' models using Twitter® and the Wiki. All communication was held on an entirely anonymous basis, as to avoid only interactions between parents and their child; student groups were assigned random names (e.g., Aristotle, Plato etc.), and parents were also assigned names like parent 1, parent 2 etc. During the last session students wrote letters to local authorities (as required by the activities), explaining and documenting their models/solutions. Finally, a class discussion focused on the key mathematical ideas and relationships that students had generated took place.

Interviews

All nineteen parents were invited to participate in individual interviews. Seventeen parents (representing thirteen families) accepted the invitation and participated in individual semi-structured interviews. Three areas of interest were investigated: (a) parents' beliefs on the environment generated, which was based on mathematical modeling, and the implementation of the EngMEA, (b) participant's beliefs on parental engagement, and (c) her/his experiences during the EngMEA implementation with regard to collaboration and communication with the teacher and the students. The interviews were conducted right after school or in the early evening. Each interview lasted between 30 to 45 minutes and all interviews were audio recorded and later transcribed. Data were summarized through sequential analysis, and a grounded theory approach was adopted. Themes were identified and clustered through axial coding, which was conducted in AtlasTI software.

RESULTS

Results are based on the qualitative analysis of the interviews. The results are presented in terms of the themes that arose from the sequential analysis of parents' beliefs, with regard to the role of the mathematical modeling environment that was generated, and with regard to the collaboration and communication with the teacher and their children.

Parents' Beliefs on the Role of the Modeling Environment

Parents reported very positive beliefs with regard to the modeling activities, and the learning environment that was generated. They commented that the activities were interesting and challenging. They were also very emphatic on how positively their children worked on the activities. One parent mentioned: "It is not very often that we discuss at home in such an explicit and detailed way her (his daughter) work in mathematics […] she liked the bridge problem so much […] it (the activity) was very challenging also for me, and we spent like at home to explore various things on bridges." Another parent said: "Such activities could help our children to develop important skills, needed beyond school […] I am very pleased that Nefeli is using such innovative approaches."

The vast majority of parents mentioned that such activities were challenging, not only for their children, but also for them. One parent who was actively involved in the activity commented: "It was challenging to see interesting problems with no clear answers. I even discussed the activity with my husband a few times, and we both enjoyed the discussions with Andreas (their son)." She continued by clarifying: "I frequently visited the Wiki and commented on students' tweets. It was great! And my son also liked it very much. Believe it or not, he even discussed the activity with his cousins." Another parent expressed: "Students had a challenging opportunity to ask like professionals […] take into account various constraints, working with complex data, drawing assumptions, and looking for more data on the Internet […] such skills are so powerful and important."

Less positively, some parents mentioned that the activities were interesting, but rather difficult, especially the *Bridge Design*. Three parents mentioned that the activity was quite complicated, even for them. They expressed that their children experienced various difficulties in working with the problem, and that they would prefer their children to work with similar activities (model eliciting) but rather easier ones.

Parents' Beliefs on Communication and Collaboration

To improve their engagement, parents seemed to unanimously agree that good communication and active engagement was key. A parent noted: "I enjoyed the two workshops very much, although it was easy to participate […] workshops helped me a lot in understanding the concepts that were taught in the activities and in handling the Wiki." Another parent added: "Working with our children in this project is very promising […] we like it, and we also see that our engagement is appreciated by our teacher."

Although quite satisfied with the situation, parents explicitly mentioned that they expected from school and teachers to do more, in order to enhance their (parental) engagement. It was revealed that school's climate had a significant impact on the overall effectiveness of parental engagement. From parents' responses a number of factors were uncovered, showing what schools should do in order to encourage and enhance parental engagement. A parent mentioned that schools should promote parental engagement using various methods, and not only by expecting from parents to be engaged. She said: "Schools and teachers must actively seek and promote the parental engagement. Not all parents are engaged by default". The importance to implement initiatives that engage students was also mentioned by two parents. One of them mentioned: "Such activities are one of the best ways to engage parents, because their children are also much engaged. When children are excited and discuss their mathematics work at home, parents are more inclined to be engaged in mathematics.

All parents underlined the necessity for open communication in order to improve parental engagement. One parent noted that open communication was the key to accessibility. He commented that: "Parents should feel comfortable enough with the teachers to ask content-related questions, and even spend time on working on activities, if we are expected to assist our children at home." Another parent highlighted the importance of constant communication. She explained: "Every parent wants to be involved in her learning […] this should be welcomed by teachers and the school Head, and we should be able to freely communicate with them. An appropriate atmosphere is needed for successful parental engagement."

Although quite satisfied with the situation, parents explicitly mentioned that it was expected for school and teachers to do more in order to enhance parental engagement. It was revealed that a school's climate and culture impacted the overall effectiveness of parental engagement efforts in a significant way. From parents' responses a number of suggestions emerged for what schools should do in order to better encourage and enhance parental engagement. Parents explicitly highlighted how the activities assisted in building a partnership climate between parents and teachers. The activities opened a whole new space for fruitful collaboration and created better communication channels among parents, teachers, and children. "I had the feeling that we (parents and teachers) were equal partners," one parent commented. She continued: "It was far better than sitting at the back in the classroom and watching a lesson. We were actively involved and we had constant communication with our children and the teacher. It was really good." Another parent added: "I found those messages [tweets] a much more appropriate method of communication than signing tests […] I knew exactly what my child did in the activity, and even better I could now observe the process, not only the results."

DISCUSSION

The purpose of this study was to examine parents' beliefs on parental engagement in mathematics teaching and learning, with a focus on modeling as a problem based approach. The results supported the expectation that such an approach was likely to positively affect teachers-parents' partnership and possibly student outcomes (Epstein et al., 2009). The environment generated, provided opportunities for parents and teachers to establish appropriate communication and collaboration venues, which resulted in improved students' models (English & Mousoulides, 2011). The modeling activity implementation as a means to engage parents in school mathematics could be considered successful, while parents responded positively to their new roles as engaging partners in their children learning. During interviews, parents revealed positive beliefs towards innovations like a models and modeling perspective. Parents also reported significant positive beliefs towards their engagement in schools, indicated at the same time the necessity for the school and teachers to take actions. Parents identified that a clear and constant bidirectional communication venue was urgently needed and they stressed that the modeling environment could be a successful method to achieve this goal.

The findings from this study suggest a need for researchers to expand their definitions of parental engagement, beyond traditional ideas of school and classroom norms, to include a dimension related to active parental engagement and technology rich modeling environments. Despite its limitations, this study provides new insights into the importance of modeling related parental engagement practices in mathematics teaching and learning. It suggests that teachers and schools that have positive beliefs towards parental engagement and facilitate the use of inquiry- modeling-based approaches are more likely to have positive active parental engagement and probably better students' learning results. Unquestionably, students need high-quality instruction to improve mathematics learning. However, if schools, teachers, and parents work together in creating appropriate, collaborative environments, they are more likely to see higher students' learning outcomes.

Acknowledgement

This paper is based on the work within the project MASCIL – Mathematics and Science for Life (www.mascil-project.eu). The research leading to these results/MASCIL has received funding from the European Union Seventh Framework Programme (FP7/2007-2013) under grant agreement n° 320693. This paper reflects only the author's views and the European Union is not liable for any use that may be made of the information contained herein.

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DEVELOPING CONCEPTUAL UNDERSTANDING OF PLACE VALUE: ONE PRESERVICE TEACHER'S JOURNEY

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This paper reports on a portion of a research study that examined the development of 43 preservice elementary school teachers' conceptual understanding of place value, and highlights the experiences of one middle-performing preservice teacher. After participating in a research-based constructivist unit of instruction in place value, the findings showed that the preservice teachers demonstrated a statistically significant change in place value understanding. Common emergent mathematical qualities and qualities of disposition were identified in the qualitative analyses. These data provided insight into this preservice teacher's thinking strategies.

INTRODUCTION

It is widely documented in the research literature that many elementary teachers lack sufficient depth of understanding of the mathematics they are expected to teach (Ball, 1990; NRC, 2001). Oftentimes, elementary teachers can reproduce mathematical procedures, but they do not understand why the procedures make sense conceptually (Ma, 2010). Thus strengthening the mathematical content knowledge for teaching and improving constructivist-based pedagogical practices in teacher education programs should be explored (e.g., Hill, Blunk, Charalambous, Lewis, Phelps, Sleep, & Ball, 2008). Because place value is the foundation of number sense and the prerequisite to multidigit operational fluency (AMS, 2001; NCTM, 2000), it is an important topic for elementary teachers. Therefore the purpose of this research study was to examine the development of preservice elementary teachers' conceptual understanding of place value within a constructivist framework.

THEORETICAL FRAMEWORK

The learning theory of constructivism provided the framework and the lens through which the research was conducted, informed by the works of many researchers (e.g., Cobb, 1996; Noddings, 1990). The core components of constructivism in the mathematics classroom were explicit in the study's instructional sequence as follows: (a) role of student as active learner and as the authority on mathematical justification, (b) role of teacher as facilitator of learning and expert in questioning techniques, and (c) role of the classroom environment with a focus on discussion and problem solving.

The intervention for the research study was a constructivist instructional sequence designed by the researcher to develop conceptual understanding of place value. The place value instructional content was a blend of the works of Fosnot and Dolk (2005), McClain (2003), Safi (2009), and Yackel and Bowers (1997), in which place value was described in terms of three interrelated observable subconstructs: (a) quantification in the base ten numeration system, (b) invariance of number when composing and decomposing, and (c) the meaning of regrouping in multidigit addition and subtraction. The place value instruction built upon the theoretical conceptions of number (Fuson, 1990; Kamii, 1986), taking into consideration the complexity of place value, including key ideas such as the position of a digit, grouping, trading, and unitization. The researcher-developed assessment instruments as well as the interview protocols were anchored in this research literature on place value.

METHODOLOGY

Because the majority of current empirical research on preservice teachers' place value understanding has been purely qualitative, a mixed methods approach was used to collect data from 43 preservice elementary school teachers enrolled in the mathematics methods course. Quantitative place value data were collected from all 43 participants through administration of one pretest and two posttests. Data were analyzed using a repeated-measures analysis of variance (ANOVA) for correlated samples.

In the larger study, six participants were chosen to be interviewees based on their scores on the place value pretest—two low-performing, two middle-performing, and two high-performing. Qualitative data for these six participants, collected through two sets of interviews and document reviews in the form of homework and journal entries, were analyzed through a process of coding. The first cycle used provisional coding, using codes identified *a priori* adapted from Cobb and Wheatley's (1988) concepts of ten. Also included in the first cycle was initial coding, in which open-ended notes were made to characterize the preservice teachers' thinking strategies and record any salient affective observations. The second cycle used focused coding and inductive analysis to identify themes in the data (Patton, 2002), and the third cycle of coding added a layer of analysis to align with current national initiatives (e.g., Common Core Standards for Mathematical Practice, CCSS, 2011). As a result of this qualitative data analysis, six common emergent mathematical qualities and three common emergent qualities of disposition were identified, as shown in Figure 1.

Developing Quality
Mathematical Qualities Flexibility, reversibility of composition and decomposition
Connections made between mathematics topics
Efficiency
Development of self-created notation
Improved mental mathematics proficiency
Precise vocabulary, e.g., groups, unitization
Qualities of Disposition Comfort, trust, confidence in doing mathematics
Self-reflection, metacognition aided own understanding
Awareness of need for both procedural and conceptual knowledge

Figure 1: Common Developing Qualities of Six Interviewees

RESULTS AND DISCUSSION

Examining first the larger context of the study, prior to participating in the instructional sequence on place value, the 43 preservice teachers enrolled in the mathematics methods courses demonstrated *developing* levels of overall place value understanding but *limited* levels of base ten understanding. After participating in the place value instructional unit, the repeated-measures analysis of variance (ANOVA) of the pre- and posttest data showed that the preservice teachers' level of place value understanding had changed significantly, $F(2, 41) = 100.68, p < .001$, partial eta$^2 = .71$, placing the preservice teachers' level of place value knowledge between the *developing* and *full* levels of understanding. These ANOVA results suggest that the intervention of the constructivist instructional sequence was effective since the scores increased over time.

Though immersed in a larger study, this paper will focus on the journey of one preservice teacher, Liz, to illustrate a few of the emergent qualities identified in the qualitative analysis. In the original sampling, Liz was chosen as a middle-performing participant because her pretest score represented the median as compared to her classmates in her elementary mathematics methods course. In the present paper, Liz's journey is highlighted because of her ability to be self-reflective of her own mathematical learning.

Liz: Prior to Implementation of Instruction

Prior to Liz's participation in the instructional sequence, an initial interview was conducted to gain insight into the participants' thinking strategies on the pretest. This interview started with a focus on base ten items, beginning with pretest item 5, Figure 2. Liz's first few statements in this interview were characterized by her reliance on procedural thinking, as illustrated in the following excerpt.

5. Please consider the regrouped ones in the problem below:

$$\begin{array}{r} {\scriptstyle 1\,1} \\ 389 \\ \underline{+\,475} \\ 864 \end{array}$$

   a. What does the 1 above the 8 represent?
   b. What does the 1 above the 3 represent?

Figure 2: Item 5 on pretest (Thanheiser, 2010).

Interviewer: My first question is about number 5: Please consider the regrouped ones in the problem below. I'm hoping that I've asked you questions you've never thought about before, like about these [regrouped] ones.

Liz: Some of these tripped me up because no one has ever made me clarify them before. I just know that's how it is?

Interviewer: That's right. And you know how to get the answer.

Liz: Right, exactly.

Interviewer: And you never really thought about it deeply. And then as a teacher of mathematics, this is something we should be thinking about. So, what does the 1 above the 8 represent? What are your thoughts?

Liz: I think I put down that it represented 1. I guess I was thinking that because I know that it's 8 + 7 + 1, which comes to 16, so I just assumed that it represents 1. Now I'm analyzing everything…I guess it could represent 10 because you could be adding it to the—no, that wouldn't make sense. Never mind. Because it would be 10, then it would be 18 + 7 and that wouldn't work. So I still don't know!

Interviewer: Okay. What about the other 1 [over the 3]? Do you think it means the same thing?

Liz: Yeah, as of right now, yeah, because—it's 4 + 3 + 1.

Liz seemed to be following the verbal representation of the addition algorithm to describe her thinking for item 5. Even though Liz used metacognition strategies to rethink her answer, talking aloud reconfirmed her misconception of the values of the regrouped ones. In a subsequent discussion of a base ten subtraction problem during this interview, her response reflected a lack of understanding of the underlying base ten base ten concepts, as she was unable to see the unitization of a regrouped 1 simultaneously having a value of 100 and 10 groups of ten.

The last portion of Liz's first interview was focused on base eight addition and subtraction problems set in the "Candy Factory" context, adapted from Bowers, Cobb, and McClain, 1999. On her pretest, Liz had solved these by converting into individual candy pieces, calculating in base ten, then repacking the candy back into base eight. With very little guidance from the interviewer during this first interview, Liz was able to begin using a more symbolic, efficient method of recording her trades in base eight for one subtraction problem. After obtaining her answer, she exclaimed, "Oh my gosh! That's crazy! I would never have thought of it that way, though." It was at this point in the interview that she articulated commonalities between the written algorithms across different place values with different bases. Thus, even before formal classroom instruction, Liz was beginning to exhibit some of the identified emergent qualities: *connections made between mathematics topics; efficiency* (in base eight computation); and *self-reflection, metacognition aided own understanding*.

Liz: During Implementation of Instruction

Liz's journal responses to daily journal prompts provided rich descriptions of turning points in her understanding. On Liz's third journal prompt, she was asked to describe one thing about place value that she didn't know before the unit. She had written: "That sometimes in subtracting, borrowing from another number can represent a couple things. When we take from the hundreds column, it is actually a group of 100, but we treat it like a ten." Here, Liz alluded to unitization between the tens and hundreds columns, and how these concepts provide meaning to the standard written algorithm for subtraction.

Liz: Following Implementation of Instruction

Liz's second interview was conducted after the administration of the first posttest, which showed an improvement of her overall place value understanding, especially in base ten concepts. When asked during this second interview to explain her thinking on posttest 1 item 6b, Figure 3, Liz's journey of understanding took an interesting turn. Following is the dialogue that took place regarding item 6b.

6. Please read over Ryan's work then answer the question which follows.

Below is the work of Ryan, a second grader, who solved this addition problem and this subtraction problem in May.

Problem A Problem B

$$\begin{array}{r} {\scriptstyle 1} \\ 438 \\ +\ \ 47 \\ \hline 485 \end{array} \qquad \begin{array}{r} {\scriptstyle 2\,1} \\ \cancel{3}45 \\ -\ \ 52 \\ \hline 293 \end{array}$$

a. Does the 1 in each of these problems represent the same amount? Please explain your answer.

b. Explain why in addition (as in Problem A) the 1 is added to the 5, but in subtraction (as in Problem B) 10 is added to the 2.

Figure 3: Item 6 on posttest 1, adapted from Thanheiser (2010).

Interviewer: Now tell me about the 1 in subtraction and how it might be different [from the 1 in the addition problem].

Liz: For this one, it's 345 and we can do 5 − 2, but we can't do 4 − 5. So you borrow from the hundreds column. You're borrowing 100 and making this 200. And you're moving it over to this column, so what it really is, it's still 345 because it's 200 + 145. But when you move it over to this column, you treat it as a ten, but it's 10 groups of ten.

Interviewer: Aha, I think you've answered my next question. This 1 you said came from here so it means 100 this time, not 10. But even though it means 100, you said it means 10 groups of ten. My next question was why don't we go 1 + 4, but here you go 10 + 4?

Liz: Because it's ten groups of ten, not one group of ten.

Interviewer: Yes! Ten groups of ten plus four groups of ten that's 14 − 5 and you end up with 9. But you're still in the tens column, so it's nine groups of ten. You got it.

Liz: I think I finally got it!

Interviewer: I don't think you said it right though on the second time around [posttest 1]. I think the second time around, you speak to the procedure, how it makes it easier, carrying, you're over the limit so you've got to go the next one.

Liz: Did I not talk about all those groups?

Interviewer: But you didn't tell me what you just told me here….

Liz: I think it was because I think it's taken until right now.

Liz's self-reflection during the second interview led her towards a more conceptual understanding of place value. She now spoke enthusiastically in terms of like groups of numbers, and therefore was unitizing, even though she still used the traditional "borrowing" language. Hence, Liz's comments provided evidence of three developing qualities: *precise vocabulary, e.g., groups, unitization; flexibility, reversibility of composition and decomposition* (of base ten numbers); and *comfort, trust, confidence in doing mathematics.*

Also during the second interview, Liz was asked to identify which classroom activities were most helpful in guiding her towards deeper understanding of base ten concepts. She cited that counting in base eight and finding base eight sums mentally were also helpful classroom activities:

Liz:	I thought the counting itself takes a while to get used to just because you don't go to 10, you go to 8 and it starts over.…And I guess adding could be—I felt like I was doing grouping more in my head almost. When you had to add numbers, because if it was over—
Interviewer:	If it was over a rod?
Liz:	I was comfortable between 1 and 7, but once it went over 8, it was like wait, what does that represent now?
Interviewer:	Right, like if it was 7 + 2?
Liz:	Yeah, and then didn't we say it was like *one-ee-one*?
Interviewer:	One rod and an extra one: one-ee-one.
Liz:	It was weird because one-ee-one in my mind is eleven, but it wasn't eleven.
Interviewer:	Because it's one group of eight and an extra. That's good. So it made you think?
Liz:	Yeah, definitely made me think.

When Liz stated that she was beginning to group in her head, this was evidence that she was developing the quality *improved mental mathematics proficiency* (in base eight).

Near the end of the second interview, Liz reflected on her experiences thus far. The following excerpt illustrates the quality *awareness of need for both procedural and conceptual knowledge*. A distinction is made here that this does not refer to the preservice teachers' *acquisition* of both procedural and conceptual knowledge, which is indeed important (Hiebert & Lefevre, 1996). Instead, this quality was designated if the participant expressed an *awareness* (newly discovered for most) of the need for conceptual knowledge underlying the procedures with which they were proficient. In this excerpt, Liz's statements provide rich insight into her perception of mathematics.

> It's just funny how much I *didn't* know about those values [the regrouped ones]. Apparently, I didn't know anything [laughs] because I think I just did it! We used to play with those cubes, and I did that right I think. But I guess I never knew how it translated to the algorithm.

This was a very powerful statement for two reasons. First, Liz had never realized, prior to this experience, how physical actions with manipulatives are directly connected to the written algorithms that they represent. But even more importantly, Liz's statement about being able to correctly perform a written algorithm implied that she had previously thought that knowing how to do a procedure meant understanding the underlying mathematical concepts, which is not necessarily true.

Liz's Place Value Understanding: A Summary

Prior to the instructional sequence, Liz exhibited a procedural knowledge of place value operations that lacked a conceptual foundation. As Liz participated in the constructivist instructional sequence, her place value understanding shifted from procedural to conceptual, exhibiting improved place value conceptual understanding in all three subconstructs and in the unifying themes of place value: unitization, grouping and trading rules, and the position of the digit determines its value. By the end of the study, Liz's posttest scores placed her near *full* understanding of place value concepts.

CONCLUSION AND IMPLICATIONS

Preservice elementary teachers need rich mathematical experiences in their methods courses that provide them opportunities to discuss, invent, conjecture, and problem solve to increase their own conceptual understanding of place value. This place value understanding consequently provides a structure for the concepts underlying the written algorithms for whole number addition and subtraction. The participants' thinking strategies articulated in these qualitative analyses not only provide insight into the quantitative data, but these strategies can also help mathematics teacher educators anticipate their preservice teachers' place value misconceptions.

The results of this study have the following implications for possible future research: a longitudinal study could be designed to explore connections between preservice teachers' experiences and their students' achievement in place value, an instructional model for constructivism could be developed to allow mathematics educators to readily implement constructivist strategies, or the common emergent mathematical qualities could be further explored to develop more robust descriptions in the context of cultivating mathematical habits of mind in preservice teachers.

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FUNCTION NOTATION AS AN IDIOM[1]

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Functions play a large role in mathematics education beginning in middle school. The aim of this paper is to investigate the meaning teachers hold for function notation; namely, we suggest that many teachers view function notation as a four-character idiom consisting of function name, parenthesis, variable and parenthesis. Many of the teachers who engaged in tasks aimed at exploring teachers' meanings for function notation responded in a manner suggestive of viewing function notation idiomatically.

INTRODUCTION

The function concept permeates mathematics education beginning in middle school. Studies show that learning the function concept is challenging, even for high performing undergraduates (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Carlson, 1998). While research continues to extend our understanding of function, there is still a need for exploring *meanings* associated with function. Wilson (1994) draws attention to the understanding of function for one particular pre-service teacher; this case study of the pre-service teacher's thinking about functions as "computational activities" sets the tone for the goal in this paper of delving into the meanings teachers hold. Our focus shifts from the function concept in general, however, to function notation in particular.

THEORETICAL FRAMEWORK

The use of function notation is ubiquitous in mathematics beyond middle school. It is also commonly a teacher's experience that, at some moment during its introduction, some student will ask, "Why use $f(x)$ when all we really mean is y?" (Thompson, 2013b). Teachers' abilities to answer this question will be based in their meanings for function notation, its conventions, and its uses. As such, our goal in this paper is to explore teachers' meanings for function notation.

We consider meanings to be constructed by an individual to organize his or her experiences. Creating meaning entails constructing a scheme through repeated reasoning and reconstruction to organize experiences in a way that is internally consistent (Piaget & Garcia, 1991; Thompson, 2013a; Thompson, Carlson, Byerley, & Hatfield, in press).

[1] Research reported in this article was supported by NSF Grant No. MSP-1050595. Any recommendations or conclusions stated here are the author's (or authors') and do not necessarily reflect official positions of the NSF.

The function concept is a significant and complex element of mathematical fluency that permeates school mathematics beginning in grade 8. We choose to focus on the notational aspect of functions for the sake of brevity. In particular, we recall that a function's definition consists of the components shown in Figure 1.

$$\overbrace{\overbrace{\overbrace{f}^{\text{Name}}\;(\overbrace{x}^{\text{Input}})}^{\text{Output}} = \overbrace{x(3x+1)(x^2-14x)}^{\text{Rule for how to produce output}}}^{\text{Function definition}}$$

Figure 1: A scheme for defining a function using function notation (from Thompson, 2013b)

For the working mathematician, the notation $f(x)$ itself represents the value of the function f when given a value of x, with or without the associated defining rule. One may refer to the function f, specify the input variable x which is to be used to describe the rule, or call the entire $f(x)$ to stand for the output of the function. The latter notation may be used to introduce a rule or to hold the place of an unknown or complex rule in the definition of another function. We suspect that many teachers do not have this meaning for function notation, and instead employ the four-character idiom—function name, parenthesis, variable, parenthesis (e.g. $f(x)$), in its entirety, as the name of a function.

METHODOLOGY

Thompson (2013a) argued the need for understanding teachers' mathematical meanings because those meanings are passed on to the students. Investigating teachers' in-the-moment meanings serves as a starting point for professional development to help teachers develop meanings that are more productive for teaching for coherence. In light of this, we designed tasks to explore teachers' meanings for function, specifically focusing on whether function notation holds the same meaning for teachers as what we as researchers think of when using function notation. The tasks were administered to 100 high school teachers as part of a larger assessment. A team developed scoring rubrics to characterize meanings revealed in teacher responses (Thompson & Draney, under review).

Tasks

The first task we will discuss consists of two parts. Part A asks the teacher to complete a function definition by filling in blanks and Part B presents sample student work for the teacher to explore (Figure 2).

Part A addresses function notation as an idiom directly. We suspect that teachers who view function notation as a four-character idiom will read "c of v" as the name of the new function, and fill in the blanks with r's and u's because the functions referenced are read "w of u" and "q of r". It is unlikely for teachers who see function notation idiomatically to tease out the function name "*w*" from the notation $w(u)$ because they view the "*u*" as part of the name.

Here are two function definitions.

$$w(u) = \sin(u-1) \text{ if } u \geq 1$$

$$q(r) = \sqrt{r^2 - r^3} \text{ if } 0 \leq r < 1$$

Part A. Here is a third function *c*, defined in two parts, whose definition refers to *w* and *q*. Place the correct letter in each blank so that the function *c* is properly defined.

$$c(v) = \begin{cases} q(\_\_) \text{ if } 0 \leq \_\_\_ < 1 \\ w(\_\_) \text{ if } \_\_\_ \geq 1 \end{cases}$$

(on next page)

Part B

James, a student in an Algebra 2 class, defined a function *f* to model a situation involving the number of possible unique handshakes in a group of *n* people. He defined *f* as:

$$f(x) = \frac{n(n+1)}{2}$$

According to James' definition, what is $f(9)$?

Figure 2: Item 1 Variable mismatch

In the case where a teacher fills in the blanks of Part A with "*v*", we included Part B to explore how the teacher addresses variable mismatch in student work. The sample work provided gives an ill-defined function $f(x) = \frac{n(n+1)}{2}$ and requests the value of $f(9)$. In particular, Part B reveals the degree to which teachers' take notice of and can explain the problem of variable mismatch in a the definition of a function. A teacher who reads function notation idiomatically will be unbothered by the mismatched variables *x* and *n* in the function definition, and will compute $f(9)=45$. Teachers who do not read function notation idiomatically will, at least, observe that James' function always produces the output $\frac{n(n+1)}{2}$ regardless of the input *x*. Ideally, a teacher would be able to further explain to James that his function is ill-defined because the variable *x* has not been defined.

The second task we include in this paper was designed to evaluate teachers' tendency to use function notation in the rule of another function's definition. In particular, we wanted to see if function notation served the purpose of representing a varying quantity for the teachers. So we designed a situation that necessitated the use of function notation to model a quantitative situation that was familiar to teachers (Figure 3).

Hari dropped a rock into a pond creating a circular ripple that spread outward. The ripple's radius increases at a non-constant speed with the number of seconds since Hari dropped the rock. Use function notation to express the area inside the ripple as a function of elapsed time.

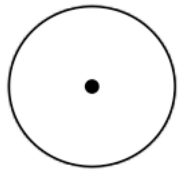

Figure 3: Task 2 Rock into pond

We anticipated that teachers might use function notation on the left-hand-side of the function's definition but not use function notation to represent the circle's radius as a function of time. We included the phrase "at a non-constant rate" to describe the growth of the circle's radius so that teachers would not assume unthinkingly that the radius increases at a constant rate and hence model the scenario with $r = kt$.

RESULTS

Ninety-seven of the 100 teachers who were given Task 1 responded. Table 1 shows the distribution of responses. The table has two main points of interest. First, 48 of 97 teachers filled in the blanks of Part A with *u*'s and *r*'s or otherwise did not use the variable *v* (e.g. some teachers wrote *q* or *w* in the blanks). This supports our suspicion that many teachers read function notation idiomatically, as explained in the task design. They saw what was written to the left of the equal sign as the name of what was written to the right of the equal sign. They did not parse the definition according to the scheme in Figure 1. For those teachers who filled in the blanks with something other than *u, r* or *v*, we suspect that the "*c* of *v*" on the left hand side of the function definition is read idiomatically by the teachers. These teachers are unlikely to identify "*c*" as the function name and "*v*" as the input variable, instead reading "*c* of *v*" as the entire function name.

Fill in Blanks \ Variable Mismatch	$f(9)=45$	f is constant but $f(9)=45$	$\frac{n(n+1)}{2}$	f is ill-defined	***Totals***
Other	12	0	2	1	**15**
u's and *r*'s	26	3	2	2	**33**
v in 1-3 blanks	2	1	1	3	**7**
v in all blanks	17	1	12	12	**42**
Totals	**57**	**5**	**17**	**18**	**97**

Table 1: Teacher responses to Task 1

The next number that stands out in Table 1 is that 57 of 97 teachers responded to Part B by substituting 9 for *n* in James' definition to obtain a value of 45 handshakes. This number shows that most teachers were untroubled by the variable mismatch in James' definition and used the function definition as if it were written $f(n) = \frac{n(n+1)}{2}$. Of the 33 teachers whose Part A response suggested idiomatic thinking of function notation, only 7 wrote responses that suggested an awareness of something awry with James' definition. Only 2 of those 33 were explicit about the variable mismatch being problematic to the function definition.

Moreover, almost half of the teachers who filled in the blanks of Part A with *v*'s substituted 9 in the right side of James' definition in Part B. We suspect that these teachers are aware of the practice of using a variable consistently in function definitions, but this practice did not keep them from overlooking the variable mismatch in James' definition.

While the goal of Task 2 was to reveal teachers' usage of function notation in modelling scenarios, it became evident in scoring that this item could be used to gain insight into whether teachers view function notation idiomatically. In Table 2, we compare teacher responses in Task 1 and Task 2. For Task 1, we categorized the pairing of responses in Part A and Part B as Low, Medium or High based on the degree to which responses revealed a tendency to use variables consistently and identified the problematic nature of James' function definition. Likewise, teacher responses to Task 2 are ranked as Low, Medium or High based on the use of function notation within the model described by the teacher.

Fifty of 87 teachers gave Low responses, meaning that (1) they did not use function notation or (2) they used function notation only on the left-hand-side of their model and used variables inconsistently (see Figure 4). Another 16 of 87 gave Medium responses, which include responses that used function notation on both sides of the model but used variables inconsistently.

Task 1 \ Task 2	Low	Medium	High	***Totals***
Low	39	9	9	**57**
Medium	1	2	2	**5**
High	10	5	10	**25**
Totals	**50**	**16**	**21**	**87**

Table 2: Teacher responses to Task 1 and Task 2

We suspect that teachers who gave solutions like that in Figure 4 view function notation idiomatically. In particular, this teacher might read "*f* of *x*" as the entire name of the function describing the area of the circle, making the variable mismatch on the right-hand-side of the definition a non-issue for this teacher. In fact, if we look to the same teacher's response on Task 1, he filled in the blanks with *r* and *u* and computed *f*(*9*)=45 as if the variable discrepancy was not present. In this manner, this teacher seems to consistently read function notation idiomatically.

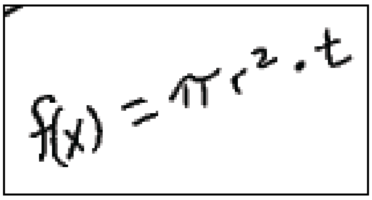

Figure 4: Sample teacher response to Task 2

We look at another teacher's responses to both items to try and describe a viable model for his thinking with regard to function notation (Figure 5). Notice this teacher gave the highest-level response to Task 1 Part A, filling in all the blanks with *v*. However, the teacher computed the value of *f*(9) in Task 1 Part B by substituting 9 into the right hand side of the James' function definition. It is possible that this teacher is loosely aware of the need for consistency in variable usage in defining a function, but this consistency is not required to evaluate a function value. We suspect that this teacher sees the left hand side of a function definition as a label for the function name and the right hand side of a function definition as "where the math happens". Looking on to his response to Task 2, we see the teacher uses "A_{circle}" to introduce his model for area. This unconventional notation to reference the circle's area reinforces the idea that what appears on the left side of the equal sign is a label.

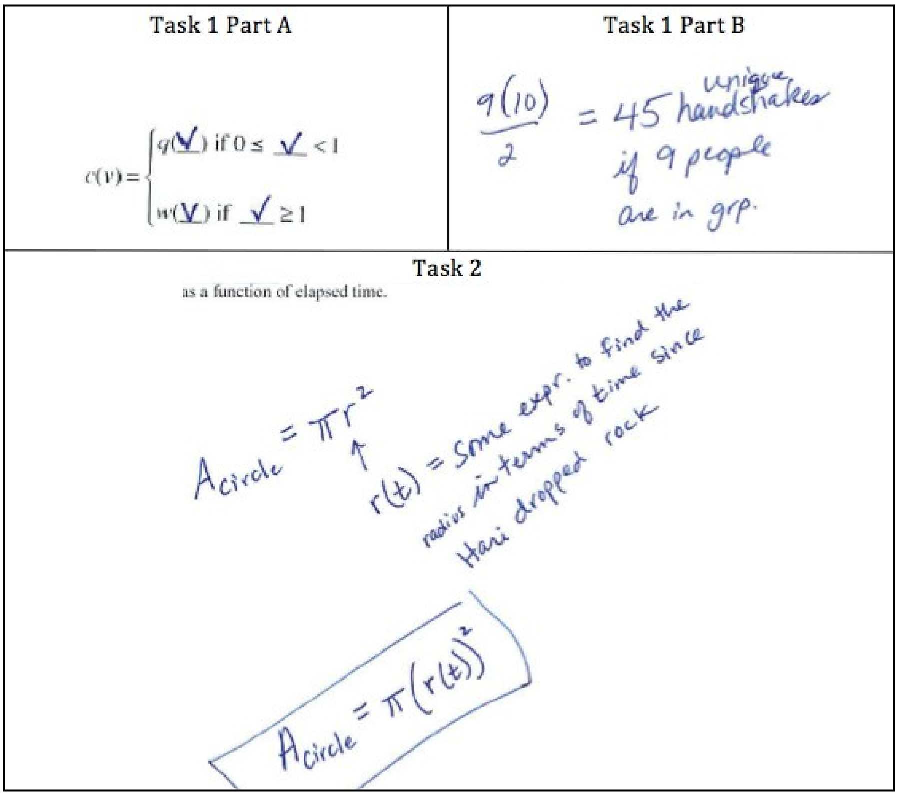

Figure 5: Sample teacher responses to Tasks 1 and 2

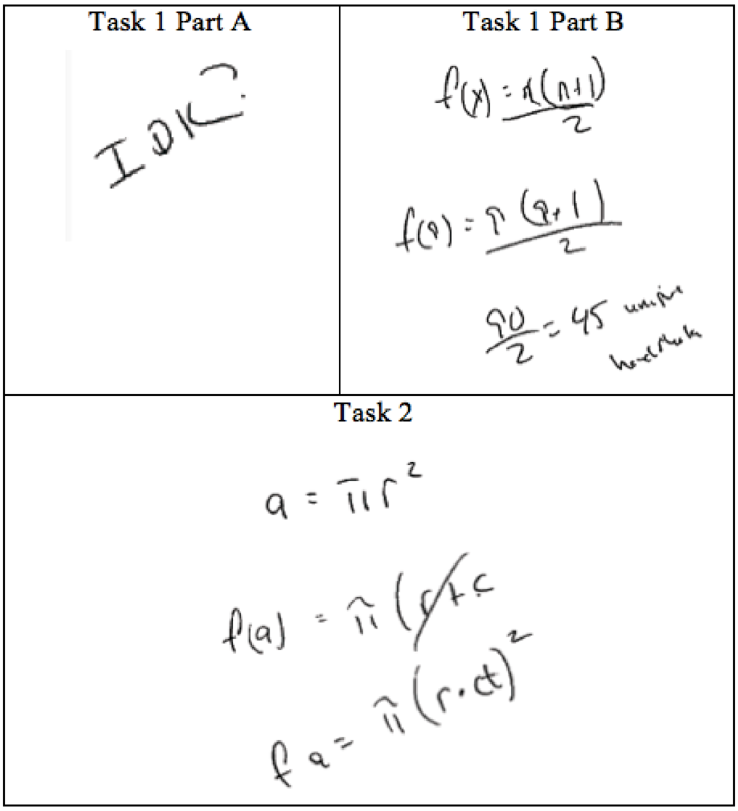

Figure 6: Sample responses from teacher on both tasks

Figure 6 gives one final example of one teacher's responses to both tasks. This teacher did not give a response to Task 1 Part A, used James' definition by evaluating the right hand side by substituting n=9 in Task 1 Part B and struggled to introduce function notation in his response on Task 2. In fact, the teacher appears to have attempted to convert the area formula for a circle into a model using function notation by introducing the phrasing "*f*-parenthesis-variable-parenthesis" on the left hand side. Collectively, this teacher's responses suggest a lack of importance, from the teacher's perspective, to consistent usage of variables (as seen in Task 1 Part B and Task 2—in which the variables *a, r* and *t* are all utilized in the model) and the possibility of holding a meaning for function notation as nothing more than a conventional label of the left side of an equal sign.

DISCUSSION

Our goal for this research was to explore teachers' mathematical meanings about function notation. We suggest that our results, though specific to meanings for function notation, support Thompson's claim that attending to meanings must be central to our work as mathematics educators (Thompson, 2013a). Responses to our tasks reveal that many teachers read function notation idiomatically. Consequently, we suspect these teachers view only the content to the right of the equal sign as the mathematically relevant portion of a function definition as described in Figure 1. This type of reasoning leads to a need for describing a rule to model scenarios rather than employing function notation to represent a varying quantity. We suggest further research be conducted to explore the extent to which teachers who view function notation idiomatically use function notation to represent varying quantities as opposed to developing rules to model scenarios.

Another area of interest is to look at what meanings teachers hold for the notational devices used for operations on functions. For instance, what meaning do the following equations have for a teacher who views function notation idiomatically?

$$(f \pm g)(x) = f(x) \pm g(x)$$
$$(f \circ g)(x) = f(g(x))$$

Since the content on the left hand side of the equal sign is no longer of the simple form—letter, parenthesis, variable, parenthesis—does the left hand side still serve as a label? Does it add confusion, is it ignored, does the teacher simply focus on the right hand side? Under these conditions, what meanings do these teachers convey in their classrooms while teaching function notation and operations on functions? Further research ought to investigate such questions, as classroom discussions regarding function and operations on functions are likely impoverished when the teachers leading such discussion hold idiomatic meanings for function notation.

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THE INTERPLAY BETWEEN LANGUAGE, GESTURES, DRAGGING AND DIAGRAMS IN BILINGUAL LEARNERS' MATHEMATICAL COMMUNCIATIONS

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This paper provides a detailed analysis of the mathematical communication involving a pair of high school calculus students who are English language learners. The paper focuses on the word-use, gestures and dragging actions in the student-pair communication about calculus concepts when paper-based static and then touchscreen dynamic diagrams. Findings suggest that the students relied on gestures and dragging as multimodal resources to communicate about dynamic aspects of calculus. Moreover, examining the interplay between language, gestures, dragging and diagrams made it possible to uncover English language learners' competencies in mathematical communications. This paper points to an expanded view of bilingual learners' communication that includes gestures, dragging and diagrams.

INTRODUCTION

The goal of my research has been to extend Moschkovich's (2007) sociocultural view of bilingual learners, to "uncover" bilingual learners' mathematical competencies when they communicate about significant calculus concepts. Although some research has shed light on bilingual learners' non-linguistic forms of communication such as gestures and diagrams (Gutierrez, Sengupta-Irving, & Dieckmann, 2007; Moschkovich, 2007, 2009), this work has not addressed the use of digital technologies, and dynamic geometry enviornments (DGEs) in particular—which have been shown to facilitate student communication by providing visual and dynamic modes of interaction (Ferrara, Pratt, & Robutti, 2006; Falcade, Laborde and Mariotti, 2007)—and the interplay between these multimodal resources for analysing bilingual learners' mathematical communications.

The current research questions concern the kinds of multimodal resources that bilingual learners use to communicate about certain calculus concepts using a touchscreen-based DGE. In particular, I investigate:

1. What characteristics of communications, and what kinds of *mathematical discourse practices* (Moschkovich, 2007) do bilingual learners engage in, when working with touchscreen-based DGE?
2. How may this analysis uncover bilingual learners' competencies and resources in mathematical communications?

2014. In Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 4, pp. 289-296. Vancouver, Canada: PME.

THEORETICAL FRAMEWORK

Moschkovich's (2007) sociocultural view of bilingual learners questions the efficacy of the *vocabulary* and *multiple meaning* perspectives for understanding bilingual mathematics learners because such perspectives focus on what learners don't know or can't do. The vocabulary perspective views the acquisition of vocabulary as a central component of learning mathematics for bilingual learners. The multiple meaning perspective focusses on learning to use different meanings appropriately in different situations. In contrast to these deficit perspectives, the *sociocultura*l view focuses on describing the resources that bilingual learners use to communicate mathematically.

The sociocultural view draws on a situated perspective of learning mathematics. From this perspective, learning mathematics is a discursive activity "that involves participating in a community of practice, developing classroom socio-mathematical norms, and using multiple material, linguistic, and social resources" (p. 25). In the sociocultural lens, bilingual learners are seen as participating in *mathematical discourse practices*—practices that are shared by members who belong in the mathematics or classroom community. In general, "abstracting, generalising, searching for certainty, and being precise, explicit, brief, and logical are highly valued activities across different mathematical communities" (p. 10). Moschkovich argues that analysing the extent and type of mathematical discourse practices can highlight the competencies of bilingual learners: "even a student who is missing vocabulary may be proficient in describing patterns, using mathematical constructions, or presenting mathematically sound arguments" (p. 20).

Complementary to Moschkovich's sociocultural view of bilingual leaners, I adopt Sfard's (2008) communicational theory, which conceptualises learning as a change in one's mathematical discourse. Sfard's approach highlights the way in which thinking and communicating (for Sfard, this includes talking and gesturing) stop being but 'expressions' of thinking and become the process of thinking in itself. In terms of bilingual learners' use of multiple resources in their mathematical discourse, Sfard (2009) suggests that utterances and gestures are two modalities that serve different functions in the thinking-communicating process. Namely, gestural communications ensure all interlocutors "speak about the same mathematical object" (p. 197). Moreover, gestures and diagrams are forms of visual mediators that learners may utilise as resources in mathematical discourse. Although Sfard has not adequately addressed the distinction between dynamic and static gestures, diagrams and visual mediators in general, the distinction is important for this paper because of the potential for the dynamic visual mediators in DGEs to evoke temporal and mathematical relations in calculus concepts. In addition, bilingual learners who are still grasping the English language may draw on dynamic visual mediators such as gestures and DGEs as multimodal resources to communicate.

In summary, I use Sfard's communicational theory to analyse bilingual learners' thinking as they communicate about calculus concepts given two types of visual mediators, static and dynamic. I focus on their word use and gestures as features of their mathematical discourse and their mathematical discourse practices within the activities. This enables me to analyse the interplay of resources situated in their use of static and dynamic diagrams and to uncover their competencies in mathematical communications.

METHODOLOGY OF RESEARCH

The participants of the study were three pairs of 12th grade students (aged 17 to 18) enrolled in two sections of the AP Calculus class in a culturally diverse high school in Western Canada. The participants were selected for their relatively low English ability–all of them have only been studying in Canada in an English-speaking schooling environment for two to three years. The detailed data analysis that follows focusses on one pair of bilingual learners, Ana and Tammy, whose native language is Mandarin and who had the lowest English language ability amongst the three pairs.

The study took place at the end of the school year in the participants' regular calculus classroom, outside of school hours. At the time, the participants had just finished enrolling in a year-long AP Calculus course where key concepts in calculus were taught using an iPad-based DGE called *Sketchpad Explorer* (Jackiw, 2011). Therefore, the students have experienced with exploring and discussing, in pairs, concepts such as the definition of a derivative, derivative functions, related rates, and the Fundamental Theorem of Calculus through geometrical, dynamic sketches.

Each pair of participants was asked to discuss ten different diagrams—five static diagrams shown in PDF form and then five dynamic diagrams presented in *Sketchpad Explorer*. The five static diagrams (see Figure 1a) were taken from students' regular calculus textbook (Stewart, 2008), and the five dynamic sketches (see Figure 1b) were minimally adapted from the ones that the students had used in class during the school year. For the purpose of comparing patterns of communications, each of the five static diagrams had a corresponding dynamic sketch that involved the same target concept. After giving the instructions, the researcher turned on the camera located in front of and facing the student-pairs, and then left the room, until the students finished talking about all the diagrams. Each student-pair took around 25 minutes to complete the task.

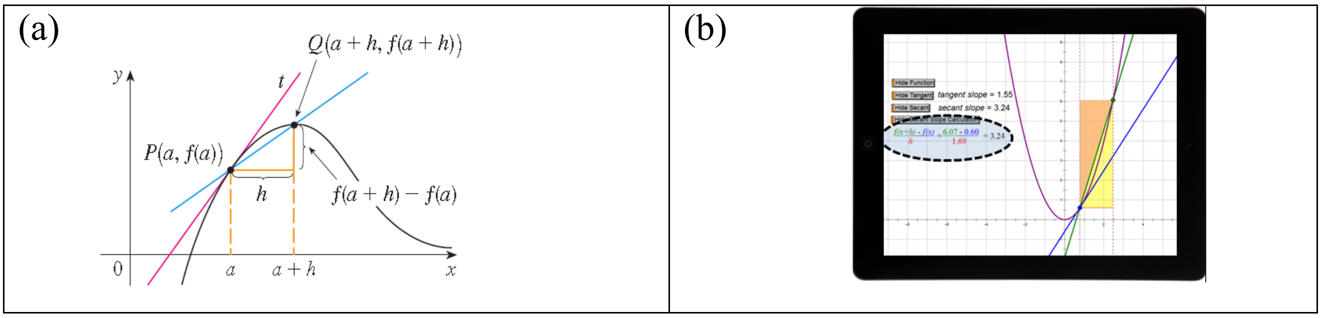


Figure 1(a): A static and (b): dynamic diagram conveying the definition of a derivative.

Figure 1b shows the screenshot of the dynamic sketch related to the definition of a derivative (with *Hide/Show* buttons "show function", "show tangent", "show secant" and "show secant calculation" all activated). As either point on the secant line is dragged, the corresponding numerical values of the tangent slope, secant slope, and secant slope calculation, *[f(x+h)–f(x)]*/*h*, are displayed with each value colour-coded.

ANALYSIS OF DATA

Below, I provide a detailed analysis of Ana and Tammy's discussion around the dynamic sketch described above, relating to the definition of a derivative. Prior to the episode, the students have already talked about the corresponding static diagram; some key analysis of that episode is discussed alongside. I divided the episode into two parts, each beginning with a transcript, for the purpose of identifying themes in each part.

Episode Part 1: Interplay between language, gestures, dragging and diagram

1 T: From zero to positive <*Dragging/gesture 1s start* (Figure 2a)>, the slope is…

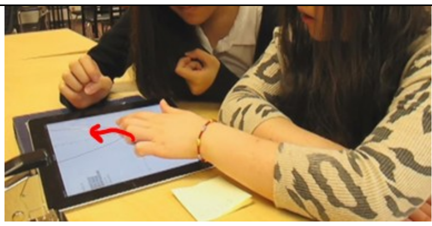
2 A: The tangent line is increasing.

3 T: Tangent line is increasing <*Dragging/gesture 1s start end*>. And from here to zero, it's decreasing.

4 A: <*Dragging/gesture 2s start (Figure 2b)*> And at zero, the tangent line is zero. <*Dragging/gesture 2 end*>.

(a)

Dragging/ gesture 1s:



(b)

Dragging/ gesture 2s:

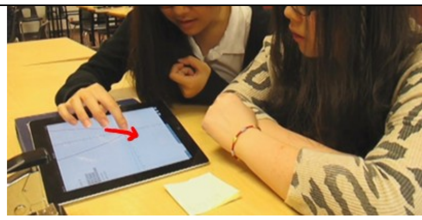


Figure 2 (a) and (b): T and A's dragging and gesturing actions in Episode Part 1.

When the students opened the sketch, two buttons were already in the "show" position; therefore, the graph of a parabola, $y = x^2$ and its tangent line at a given point appeared on the sketch. Ana and Tammy explored the dynamic sketch using the dragging modality. In the first exchange, Tammy's utterances, "tangent line is increasing" (line 1) was accompanied by dragging the point of tangency from left to right (dragging 1s), although technically it was the tangent *slope* that was increasing and not the tangent line. Following that, Ana seemed to mimic Tammy's utterance/dragging combination with "the tangent [slope of the] line is zero" (line 4) while she performed a similar dragging action to move the point of tangency towards the vertex (dragging 2s). These are two of the five series of dragging actions spanning between 2 to 5 seconds observed in the episode.

A further analysis suggests that these dragging actions were not merely dragging but also gestural communications—to communicate the dynamic features and properties in the sketch as obtained by dragging. To illustrate why the dragging actions are also considered gestures, it would be possible to imagine a static environment where the dragging modality is not available. If a speaker moves his/her finger along a graph while referring to the tangent slope as "increasing" or "decreasing", this action can be considered a kind of dynamic gesture for communicating the idea, "*as x varies along this graph*". In the current episode, the dynamic environment allows the dragging with one finger on the touchscreen and the gesturing with the index finger to blend together as one action. Hence, I refer to this action as *dragsturing*. The importance here is that *dragsturing* is one action subsuming both dragging and gesturing characteristics, in that it allowed the point to be moved on the screen (dragging), and it fulfilled a communicational function (Sfard's definition of gesturing). My purpose here is not to objectify an action but to present the dual functions of dragging and gesturing in the *dragsturing* action for analysing the students' thinking-communicating process. Prior to this, Ana and Tammy had used deictic gestures, complemented by words like "this" and "here", for naming various mathematical objects when discussing a static diagram.

Furthermore, during the first exchange, Tammy used phrases "is increasing" and "is decreasing" to describe the tangent slope. Her utterances were accompanied by her *dragsturing* (dragging/gesture 1s) which was immediately mimicked by Ana (dragging/gesture 2s). The use of the present continuous tense "is [verb]–ing" was a change from their previous discussion over a static diagram, where the girls used the verb form "is [noun]" four times when discussing the same topic. The word use "is increasing" and "is decreasing" were accompanied by dynamic *dragsturing* to *communicate* the change of tangent slope as the point was being dragged. This shows the interplay between *dragsturing*, language and diagrams in the two students' discourse. Thus, in the present episode, dragging and gesturing transformed the way Ana and Tammy communicated about the tangent slope. The verb forms suggest that "something is happening" at the very moment. This analysis is made possible by studying the interplay between *dragsturing*, language, and diagrams in the students' mathematical communication.

Episode Part 2: Engagement in valued mathematical discourse practices

A: *<T presses "show secant" button>* Secant. *<A presses "show secant calculation" button.>*

T: For… *<A starts performing dragging/gesture 3s (Figure 3a)>* if you want to get the secant line… you have to find two points to, to, *<A's dragging/gesture 3s ends and immediately starts performing dragging/gestures 4s (Figure 3b)>* calculate change of y and change of x.

A: I think, when the two points get closer *<dragging/gesture 4s end>*, the tangent line is… there is less different between the tangent line and secant line. *<T starts performing dragging/gestures 5s (Figure 3c)>*

T: And *<dragging/gestures 5s ends>* they will be together.

A: And if there are the same point, they will be the same, the two lines.

(a) Dragging/gesture 3s: (b) Dragging/gesture 4s: (c) Dragging/gesture 5s:

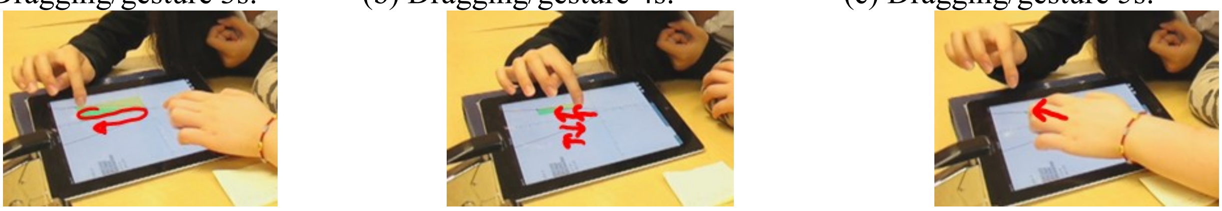


Figure 3(a),(b), and (c): T and A's dragging and gesturing actions in Episode Part 2.

As the episode unfolds, Ana and Tammy began to explore the other two *Hide/Show* buttons, and continued to drag the points. They moved from discussing procedures to talking conceptually about the definition of a derivative. This can be observed through the evolution of different mathematical discourse practices they engaged in. Upon *exploring* the change of tangent slope in the early part of the episode, Tammy suggested that "if you want to get the secant line… you have to find two points to, to calculate the change of *y* and change of *x*" (lines 9 to 11). At this point, Tammy's mathematical discourse practice focused on *calculating*.

However, the students' talk did not end with a formula as observed in the static environment; Tammy's *calculating* was followed by Ana's *comparing*, evident in her word use "closer" (line 12) and "less different" (line 13) to describe the state of the two lines when the tangent approaches the secant. Her comparing led to *predicting* and *generalising* about the tangent line in Tammy's "the two points *will be* together" (line 15) and Ana's "they *will be* the same, the two lines" (line 16). The use of the future tense in "will be" in both statements indicates that both students had moved from a procedural and algebraic way of thinking about derivative to a conceptual and geometric one. Tammy's *dragsturing* (dragging/gesture 5s) at the end to bring the secant line towards the tangent line can be taken as confirming her *generalization* that the two slopes will eventually be the same.

Out of the sociocultural view, the vocabulary perspective would criticise Tammy for incorrectly stating that "*tangent line* is increasing... and from here to zero, it's decreasing," (line 2) in the earlier part of the episode when it is really the *tangent slope* that is changing. Likewise, the multiple meaning perspective would point to Ana's inability to grasp the meaning of "function" later in the episode. Hence, neither perspective would view Ana and Tammy as engaging in valued mathematical discourse practices like *comparing*, *predicting* and *generalising*.

Since gestures are taken as communicational acts in Sfard's term, it was interesting to observe that the girls incorporated gestures in responding to each other. For example, while Tammy talked about the two points on the secant line, Ana was *dragsturing* the points on the secant line around, which seemed to be responding to Tammy's utterance. Then, the two exchanged roles when Ana suggested that the secant line will get "closer" to the tangent line. Tammy seemed to have responded by her *dragsturing* to bring the lines "together". These gesture-utterance correspondences were noted in the analysis of other pairs of bilingual learners' conversational pattern involving dynamic sketches as well.

DISCUSSION

The detailed analysis provides strong evidence that bilingual learners utilised a variety of resources, including language, gestures and visual mediators in their mathematical communication—with gestures taking on a prevalent role. These included deictic gestures accompanying static visual mediators as well as dynamic gestures for communicating temporal relationships such as the "change of *x*". Moreover, a new form of gesture emerged in the touchscreen dragging action with the dynamic diagrams. These *dragsturings* fulfil the dual function of dragging and gesturing.

The presence of dragging and gestures transformed word use. As illustrated in the episode, Ana and Tammy resorted to verb forms that imply motion while they used dragging to change the tangent slope. This was a change of verb-form from their earlier discussions around the static diagrams, where the students used the "is [noun]" form to communicate a static sense of calculus ideas. In a sociocultural view, the bilingual learners engaged in significant mathematical ideas on both static and dynamic environments, but they participated in *different* mathematical discourse practices. With the static diagrams, the students communicated about calculus procedurally by defining mathematical objects and developing a formula for tangent slope. With dynamic diagrams, their communication was characterised by comparing, predicting and generalising practices, as shown in the episode. The analysis is made possible by studying the interplay between word use, dragging, gestures and diagrams. I argue that these elements must be accounted for in the full set of resources that bilingual learners utilise in mathematical communication. As Sfard (2009) explains, utterance and gestures take on different roles in mathematical communications. I would go further in suggesting that language, gestures, and diagrams serve *complementary* functions in mathematical communications.

New conversational patterns were introduced by the students in the current episode. With a static visual mediator, the students mainly communicated with utterances accompanied by deictic gestures. This conversational pattern evolved in the presence of *dragsturing* over a dynamic visual mediator, where gestures-gestures and gestures-utterances sequences were observed in the conversation. This observation supports that bilingual learners make use of gestures as important forms of communication, and in this case, to respond to each other in mathematical communications. Also in the study, I observed one person *dragsturing* simultaneously as the other spoke; this allowed the two students to communicate simultaneously without interfering with each other. Using Sfard's communicational framework–which defines gestures as communicational acts–is especially useful for understanding the mutual *communications* involved in these new kinds of conversational patterns.

It could be said that the design of the dynamic sketches has a significant role in facilitating students' mathematical communications. The *Hide/Show* buttons allowed the students to talk about their ideas gradually one button at a time, while the *dragging* affordance enabled them to attend to dynamic relationships and connect algebraic with geometric representations of calculus. In tune with previous studies on DGEs-mediated student thinking (Falcade, Laborde and Mariotti, 2007), the students may have communicated about derivatives geometrically and conceptually as they exploited the functionalities offered in the sketch. As Chen and Herbst (2012) contend, "the constraints of diagrams may enable students to use particular gestures and verbal expressions that, rather than using known facts, permit students to make hypothetical claims about diagrams" (p.304).

CONCLUSION

In this paper, I showed that bilingual learners utilise language, gestures, dragging and diagrams as a full set of resources to communicate mathematically. I also addressed the interplay between these resources for uncovering bilingual learners' competencies engaging in significant calculus ideas. In my analysis, *dragsturing* emerged as a new, significant form of communication which gave rise to new conversational patterns. This study points to an expanded view of bilingual learners' communication that includes gestures, dragging and diagrams. In particular, future research should consider examining the kinds of gestures and the interplay of resources, which are situated in the mathematical activities, in order to identify mathematical competencies for bilingual learners.

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INVESTIGATING STUDENT PARTICIPATION TRAJECTORIES IN A MATHEMATICAL DISCOURSE COMMUNITY

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This paper details the analysis of the participation of individual students in a teaching experiment in which the researcher aimed to facilitate a mathematical discourse community. This involved positioning students as mathematical authorities capable of generating and evaluating mathematical thinking. The extent to which students acted as mathematical authorities was investigated by tracking their participation across a number of lessons. Students' use of discourse community practices such as explaining and justifying thinking, evaluating the thinking of others and asking questions was documented and Wenger's (1998) trajectories of identity were used to describe their participation. The profiles of four students of different achievement levels with contrasting participation practices will be presented and discussed.

INTRODUCTION

In recent years, theoretical and empirical research has amassed which demonstrates the benefits of participation in classroom mathematical discussions (Walshaw & Anthony, 2008). Learning mathematics can be conceived of as becoming a participant in progressive discourse. This follows Sfard's (2001) conception of learning mathematics as developing a discourse and Bereiter's (1994) arguments for science as progressive discourse. Bereiter lists the 'moral commitments' that facilitate progressive discourse. These involve a willingness to work toward common understanding, a willingness to pose questions and propositions so that they can be tested by others, a willingness to expand the set of collectively accepted propositions, and a willingness to subject any belief to criticism in order to advance the discourse. Bereiter also argues that classroom discussions can and should have these characteristics.

One example of his ideas in the context of the mathematics classroom is the Math Talk Learning Community (MTLC) framework (Hufferd-Ackles, Fuson & Sherin, 2004). This was developed as part of a year-long study in an elementary class where the focus teacher successfully implemented reform-orientated, discussion-intensive teaching practice. The framework charts the progress of the class and describes developmental trajectories in the areas of questioning, explaining mathematical thinking, source of mathematical ideas, and responsibility for learning. These trajectories detail changes in teacher and student actions as the class began operating as a discourse community and generally involve devolution of mathematical authority from teacher to students in each of the areas listed above. This results in lessons consisting of community negotiation of mathematical meaning where students' "math sense becomes the criterion for evaluation" of mathematical ideas (Hufferd-Ackles et al., 2004, p. 88).

2014. In Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 4, pp. 297-304. Vancouver, Canada: PME.

The Irish primary mathematics curriculum emphasises mathematical discussion as a key part of a child-centred, constructivist approach (Government of Ireland, 1999). In the Irish context, Dooley (2011) has explored the potential of the discursive approach for harnessing learner agency and devolving mathematical power to students. The aim of my research was to facilitate a discourse community as described by the MTLC framework and to explore the nature of student learning over time in such a community. As such, I was following Mercer's (2008) calls for a renewed focus on the temporal aspects of the teaching and learning process. Analysis of student participation in whole class discussion at group level was conducted using the MTLC framework (Hufferd-Ackles et al., 2004) and has been discussed elsewhere (NicMhuirí, 2012; 2013). This analysis of the teaching experiment showed that I was successful to some extent in devolving mathematical power to students by positioning them as mathematical authorities capable of generating and evaluating mathematical ideas. However, it also became apparent that the nature of student participation varied from pupil to pupil. For this reason, and to better understand how the discourse community worked in practice, analysis was carried out on the participation of individual students. This analysis will be detailed and the trajectories of four students of different achievement levels and participation styles will be presented.

THEORETICAL FRAMEWORK

The research was conducted from a sociocultural perspective and my focus was on what students might learn from the discourse community approach in terms of transformation of participation (Rogoff, 1994). Dreier's (1999) notion of a trajectory of participation in social practice through both time and space was used as a means of a conceiving of individual students' participation in the discourse community over time. The concept of a community of practice (COP) (Lave & Wenger, 1991; Wenger, 1998) was also used. Engagement in a joint enterprise within a COP requires negotiation and "creates among participants relations of mutual accountability that become an integral part of the practice" (Wenger, 1998, p. 78). The concept of identity is central to theories of participation in a COP and Lave suggests that becoming 'knowledgably skilful' (1993, p. 65) and developing an identity as a community member are part of the same process. Because identity is constantly renegotiated in practice, identities form trajectories within and across communities (Wenger, 1998). Identity is developed in participation with others so the teacher and the classroom community are key influences for students (Grootenboer & Zevenbergen, 2008). It can be argued that within a discourse community, there is a different understanding of what it means to be 'knowledgeably skilful' (Lave, 1993) than in traditional mathematics classrooms. In traditional classes, students are often positioned as 'received knowers' who reproduce teachers' methods (Boaler, 2003) but in a discourse community, students are positioned as mathematical authorities capable of generating and evaluating mathematical ideas. For this reason, it was envisaged that participation in a mathematical discourse community might influence students' mathematical identities.

METHODOLOGY

Design research has been applied to the classroom in the form of the 'classroom design experiment' (Cobb, Gresalfi & Hodge, 2009) which has grown from the teaching experiment research approach. The aim of this classroom design experiment was both the facilitation of a discourse community and the study of this instructional design. Fractions, decimals and percentages were chosen as focus areas as these have been identified as problematic in Irish primary classes (Eivers et al., 2010). The experiment was carried out at fifth class level (10 – 11 years old) with 18 students in a designated disadvantaged boys' school in which the researcher taught fulltime. Schools are designated as disadvantaged by the Department of Education based on indicators of socio-economic status in the population of parents. Lessons were recorded using a digital voice recorder based on their perceived potential for interesting classroom discourse. 'Interesting' should be understood to mean relevant to the research because of predicted participation patterns of students in whole-class discussion. Digital records of board work from the interactive board were also saved. In all, 31 recordings were collected over the course of a school year. Thirteen recordings were transcribed so as to be representative across mathematical topics and over time. The ethical issues of conducting teacher-research in ones' own classroom are complex (NicMhuirí, 2012) but the university mandated guidelines were followed at all times.

There are methodological issues relevant to examining the student experience through time and suitable data interrogation techniques were not easily identified. Mercer asserts that "the same act repeated cannot be assumed to be 'the same' act in repetition, because it builds historically on the earlier event" (2008, p. 36). This creates problems about using coding schemes that do not acknowledge the temporal nature of discourse. This is problematic when considering one student's participation over time but particularly problematic when considering the influence of the community on the individual. The contribution of any individual student may build on the historical contribution of a different student. For this reason, it proved unmanageable to devise a systematic coding scheme that circumvented the embedded nature of student contributions in specific times and contexts. Instead, the transcripts were interrogated to discover to what extent students engaged in practices of the discourse community.

Boaler defines classroom practices as "the recurrent activities and norms that develop in classrooms over time" (2003, p. 3). The key student practices in a discourse community were extrapolated from the MTLC framework (Huferd-Ackles et al., 2004), rather than the empirical study because the community of the classroom design experiment might not exhibit all the practices that were envisaged in the design. The key practice that was envisaged for students was to act as mathematical authorities by engaging in generating and evaluating mathematical ideas as discussed earlier. When investigating the participation of individual students in lesson transcripts, attention was paid to how the student came to speak i.e. invited or unprompted. Similarly the nature of their contributions were studied to determine whether the contribution was mathematically correct or incorrect; the degree to which it was confidently and coherently stated; whether it built on the solutions of others or came to be built on by others; whether it contained a question for me or for another student; and whether it gave any indication of ability or emotion. In this way, a description of the nature of student participation over time was created.

The resulting participation profile was then examined with reference to Wenger's (1998) trajectories of identity within a COP. These trajectories reflect the positioning of the person within the COP. Wenger presents five types of trajectories: peripheral, inbound, insider, boundary and outbound trajectories. A peripheral trajectory suggests less than full participation in community practices. An inbound trajectory may indicate current peripheral participation but a commitment to future full participation. An insider trajectory indicates full participation in community practices. A boundary trajectory indicates that identity is located in the nexus of communities of practice and an outbound trajectory indicates outward movement from one community to another. The participation of 10 out of the 18 students was investigated. These students represented different achievement levels according to standardised test results.

STUDENT PROFILES

The participation profiles of four students of different achievement levels and participation styles will be presented here. Pseudonyms have been used.

Darragh

While all higher achievers were active contributors to class discussion, none contributed to quite the same extent as Darragh. He consistently, and from the very beginning of the teaching experiment, contributed significant mathematical ideas and vocabulary to class discussions that other students later used. For example, he used the terms 'simplify' and 'equivalent' before I did and explained them to his peers when questioned. His many contributions were confidently and coherently stated and he was mathematically correct more often than he was incorrect. He regularly commented unprompted on the solution efforts of his peers, sometimes building on their suggestions (5 transcripts). In fact, he sometimes interrupted me or other students to share his thinking. He questioned students about their strategies (2 transcripts) and also directly questioned me in two lessons. Questioning of any kind was not a common student practice. It appeared that Darragh had an awareness of his own role and ability and once questioned his peers on whether they understood a mathematical explanation he had offered. On another occasion, he referred to not wanting to "confuse people." He appeared to be alert to the nature of our mathematical activity, commenting when we had discussed at length whether 1/25 is equivalent to 25%, that it had been "a big discussion for a little question." On his own initiative, he once described mathematical links between the activities in different lessons. On another occasion, again unprompted, he followed up on a question I had posed to another student. Darragh appeared to act as a mathematical authority, regularly contributing ideas and engaging in determining what was mathematically correct. In Wenger's (1998) terms, the nature of Darragh's participation could be described as an *insider trajectory*.

Jake

Jake's score on the standardised test placed him in the middle of the range of class achievement. He generally contributed to discussions on my invitation. His contributions, though often mathematically correct, were at times faltering and hard to follow. This is particularly true of his contributions to lessons early in the school year. His apparent increased ability to articulate his thinking after this may be due to the experience gained in teaching experiment lessons. It may equally reflect a greater competence with the later lesson topics. Jake often referenced other students' work sometimes to agree with it or to suggest a new approach when a peer made an error (3 transcripts). On other occasions, he shared significant ideas that went against contributions previously made by his peers (2 transcripts). Some of Jake's contributions were significant both in relation to their mathematical content and their role in shaping the classroom discussion (2 transcripts). He also appeared to be willing to make an attempt at solution and share his ideas when faced with challenging problems. For example, on one occasion the class were attempting to convert 23/25 to a percentage and Jake suggested unprompted that it might be 22 10/10 %. While this contribution shows a gap in his knowledge of percentages, it also shows that he was willing to take risks and attempt to apply some of his previous knowledge about fractions to the new situation. This pattern of participation suggests a genuine effort to act as a mathematical authority. It is tempting to describe the nature of Jake's participation as indicative of an inbound trajectory because of his growing confidence observable across the course of the year. However, he engaged in the practices of the discourse community from the first lesson of the teaching experiment. In this lesson, he displayed high levels of responsibility for learning when he disagreed with previous contributors to present his own understanding of the problem situation. This suggests that a description of *insider trajectory* is more suitable.

Kevin

Kevin was a lower achieving student who regularly displayed a willingness to contribute when invited to do so. His contributions were generally comprehensible but were not always mathematically correct. On a number of occasions, he used language which lacked mathematical precision. For example, he once suggested an alternative solution to sharing pizzas by saying, "You can put one slice in a half to get the same way… but not like one big half." In this case it seemed that he was referring to cutting a half in half but lacked either the mathematical knowledge or language to identify what the result would be. He sometimes commented on the ideas of others, generally to agree with them rather than disagree (3 transcripts). On two occasions his contributions appeared to influence the ideas of others. On the first occasion, he described a pattern he had noticed in a group of equivalent fractions which another student built on in later contributions. On the second occasion, he correctly identified the largest of a group of decimal numbers and when another student suggested that it may be a different number, Kevin successfully explained his reasoning to him. Though Kevin's ideas sometimes lacked the mathematical complexity and precision of some of his peers, his pattern of participation indicates a genuine engagement in the community. However, the extent to which Kevin acted as a mathematical authority is debateable, particularly in regard to his evaluations of the thinking of others. Although he sometimes was observed to agree with the ideas of others, he was not observed to disagree with his peers or ask questions of their methods or my explanations in whole class discussion at any time. Such actions could be useful learning practices for all children and valuable next steps to a fuller participation in the discourse community for Kevin. I would argue that there is not enough evidence to describe Kevin's participation as an insider trajectory. Instead I would suggest that he may have been on an *inbound trajectory*.

Steven

Steven was a lower achieving student who contributed regularly to whole class discussion both unprompted and by invitation. His contributions were coherently stated but were often not correct and on a number of occasions he struggled with mathematical language (4 transcripts). He frequently admitted to not understanding explanations of mine or of his peers (5 transcripts). In fact, he asked questions in the whole-class setting in 8 out of the 13 transcribed lessons. In one lesson, Steven asked 7 out of the 14 recorded student questions. These questions tended to consist of requests for restatements of explanations rather than questions probing the mathematical content. As mentioned above, this mode of participation is different from other students, the majority of whom posed few if any questions. It appeared that Steven did not attach meaning to the commonly perceived social risk of asking questions or making mistakes. On one occasion, toward the end of the year, despite questions and hints from his peers, he persisted for a number of minutes in attempting an incorrect fraction-subtraction method on the whiteboard. He appeared to appreciate the attention of the class but did not engage with their comments about the mathematics involved in his method. In this sense, Steven's contributions to discussions often appeared to be made with the aim of gaining the attention of the class, rather than with the aim of developing mathematical understanding.

Despite his regular contributions, the nature of Steven's participation appeared to be limited. Like Kevin, the extent to which he acted as a mathematical authority is debateable. While the admissions of incomprehension suggest that he was following the mathematical discussion and self-monitoring for understanding, they also suggest that he may have been struggling with mathematics at the class level and may indicate that he was appealing for more explicit direction in an effort to lessen the cognitive load. His actions suggest that he did not view participation in whole class discourse as a community effort to negotiate mathematical meaning. In fact, there is little evidence to suggest that his identity is invested in future full participation, a necessary condition of an inbound trajectory (Wenger, 1998). For this reason, a description of his participation as a *peripheral trajectory* may be best.

DISCUSSION

The analysis of participation trajectories shows variation in the extent to which students engaged in discourse community practices. Those on insider trajectories, like Darragh and Jake, may have demonstrated positive practices to their peers as their participation styles consisted of many of the desirable student actions for a discourse community. Such students might be considered as 'old timers' (Lave & Wenger, 1991) from whom other students may have learned ways of acting in the community. The role of the teacher cannot be described in the same way as teacher actions in the discourse community are not necessarily suitable models for student actions. For example, the teacher will often refrain from evaluating mathematical contributions. Instead students are expected to take on this role. Boaler (2006) discusses how some effective teachers explicitly draw attention to and promote valuable learning practices in whole class discussion. It is likely that the incorporation of this teaching practice within my own approach would have improved students' use of discourse community practices, particularly students like Kevin and Steven. It is possible that such a teaching practice would result in more dynamic trajectories of identity.

The analysis of individual lessons (NicMhuirí, 2013) showed students appeared to be positioned as 'active knowers' (Boaler, 2003) and used their own agency to generate mathematical ideas. The analysis presented here attempts to go beyond a simple snapshot of a student's actions in any one lesson, and is focused on how the relationship between the student and the discipline is developing over time irrespective of the particulars of mathematical topic or content. Although the teacher and the classroom community are key influences, the involvement of the teacher is central only for a limited time in a student's life. It is a student's identity and relationship with the discipline of mathematics that will remain an influence on the student's learning over time (Grootenboer & Zevenbergen, 2008). The participation trajectory analysis gives some insight into that relationship and also gives an illustration of mathematics learning as participation in progressive discourse (Bereiter, 1994).

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THE INFLUENCE OF SYMMETRIC OBJECTS ON SPATIAL PERSPECTIVE-TAKING – AN INTERVIEW-STUDY WITH YOUNG ELEMENTARY SCHOOL CHILDREN

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Symmetric objects are known to be perceived easier than asymmetric objects, because less information has to be processed. Therefore, symmetric objects are often used for spatial tasks. However, in perspective-taking the use of symmetric objects can also cause difficulties, as two side-views of these objects are mirror-images of each other. To examine this influence, 95 children at the beginning of first grade were asked to solve a systematically varied set of tasks in interview sessions. It was assumed that they have more difficulties to solve the tasks with symmetric objects than with asymmetric ones. Against expectation, this effect could not be confirmed based on the number of correct answers. However, the types of errors and the children's explanations show the difficulties of perspective tasks with symmetric objects.

INTRODUCTION

Spatial perspective-taking is an essential component of spatial ability. Children's spatial perspective-taking was initially studied in the famous "three-mountains-task" of Piaget and Inhelder (1999; French first edition in 1948). Subsequent studies varied many different task characteristics; the effects of these variations on the ability to coordinate perspectives are well known (for an overview see Fehr 1978 and Newcombe 1989). However, the effects of symmetry, which are suggested by some results of Lüthje (2010) in a spatial perspective task with preschool-children, have not been studied yet – although in research as well as in school symmetric objects like animals or vehicles are used often.

This study examines, with a systematically varied set of tasks, if the use of symmetric objects influences spatial perspective-taking and under which circumstances this can be observed. Since symmetric objects have two side views that are mirror-images of each other and differ only in their left-right-orientation, we suppose that spatial perspective tasks with symmetric objects are solved less often or less well than tasks with asymmetric objects. In symmetric tasks, we also assume that the two side views, which are symmetric to each other, are more often confused with each other.

THEORETICAL FRAMEWORK

Spatial perspective-taking is defined as the ability to imagine how objects appear from another point of view than one's own (see Cox 1977). There are two essential components for successfully solving spatial perspective tasks. First, one needs to know that one particular view of an object corresponds with one particular position. Therefore, two persons in the same position perceive the object in the same way. If they differ in their positions, their views also differ from one another. Second, spatial perspective-taking also requires the ability to figure out mentally how exactly the other view looks like. In other words, one must be able to imagine *what* can be seen from the other position and especially *how* and *where*, in relation to other objects, this particular object is seen (see Coie et al. 1973; Fishbein et al. 1972; Salatas & Flavell 1976).

To succeed in perspective tasks with symmetric objects, the subject must distinguish between the two side views that are symmetric to each other with respect to a vertical axis. These side views differ only in their left-right-orientation. However, the discrimination between left and right is difficult even for adults (see e.g. Ofte & Hugdahl 2002; Storfer 1995) and develops later than the discrimination in the two other dimensions, front-back and top-bottom (see Shepard & Hurwitz 1984).

Studies about the perception of orientation observed that subjects often confuse an oriented object with its mirror-image, for example the letters “b” and “d” or “p” and “q”, as well as pictures of common objects (see Davidson 1935; Gregory et al. 2011; Gregory & McCloskey 2010). Interestingly, the confusion of mirror-images appears especially with respect to a vertical axis (but see for a different interpretation of research results Gregory & McCloskey 2010) whereas the perception or construction of symmetry is seen as particularly easy if the axis is vertical (see Grenier 1985). In reproduction tasks with dot pattern, symmetry (especially with respect to a vertical axis) even seems to be a facilitative factor (see Bartmann 1993; Bornstein & Stiles-Davis 1984; Liu & Uttal 1999).

DATA AND METHOD

Subjects and context

95 first-graders (average age: 6 years 8 months) in Germany participated in this study. During individual interviews, every child was asked to solve 32 perspective tasks and to explain its solution after each task. The interviews took place in a separate room during lessons, lasted about 15 to 25 minutes, and were videotaped.

The tasks

For all tasks, a square base (40cm×40cm) with four differently coloured toy figurines, placed in the center of each side, was used. In the middle of this plate, the interviewer placed 16 different objects one after another. Four photographs of the object (depicting the object’s four different sides) were positioned between the child and the plate. For every task, the child was asked which of the pictures would correspond to the view of one of the toy figurines, saying: “Which of the photographs did the green man take?” When the children gave their answers, they were invited to explain their decisions, before being asked about another toy figurine’s view. In every situation, only two views were tested to limit subsequent mistakes that are rooted in a previous mistake.

Two different types of objects were used for the study: toy animals that are well-known to children and have clearly determined sides (front, back, left side, right side) and arrangements of two differently colored cuboids of the same size as abstract objects without distinguishable front/back and sides. For each object type, eight symmetric and eight asymmetric objects were used. The natural symmetry of the animals was abolished by lifting one leg and adding an item from a circus-context. The symmetric animals were also extended with items from the circus-context to minimize the differences between symmetric and asymmetric animals (see figure 1). For cuboid tasks, first a symmetric arrangement was built; then an asymmetric arrangement was created by sliding one cuboid orthogonal to the mirror plane (see figure 2).

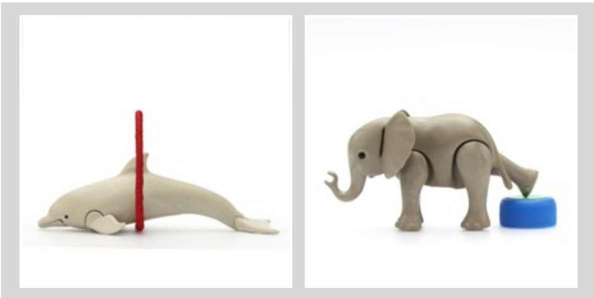

Figure 1: Examples for symmetric and asymmetric animals

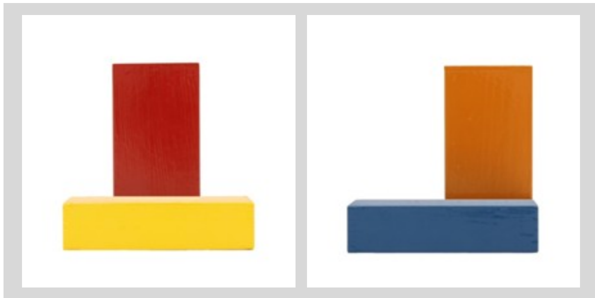

Figure 2: Examples for symmetric and asymmetric cuboid arrangements

Besides symmetry and type of object, two other factors were varied: the object's orientation (parallel or orthogonal to the child's line of sight) and the type of view (side view or front/back view). To ensure comparability, every symmetric object was paired with an asymmetric object of the same object type and within such a pair of tasks all other variables were kept constant: the orientation, the arrangement of the pictures, and the two toy figurines, whose views should be figured out.

Procedure of analysis

The interview data was analysed in two ways: the children's decisions were classified with respect to the type of mistake made, and the explanations were transcribed and categorized by qualitative content analysis.

The answers of the children were classified as follows:

- Correct answer: the child chose the picture that shows the toy figurine's view.
- Egocentric mistake: the child chose the picture that shows its own view in-stead of that of the toy figurine.
- Inversion mistake: being asked about a side view the child chose the wrong one.
- Ambiguous mistake: if the child was asked about the view opposite to its own of an orthogonal aligned object (so it was a question about a side-view) and it chose the picture that shows its own view, this mistake could either be classified as an egocentric mistake or as inverting the side views. Therefore, this mistake was named "ambiguous mistake".
- Other: all other mistakes.

The children's explanations were transcribed, including their gestures. In a second step, these explanations were sorted by likeness. The analysis of difficulties and similarities lead to a category system that is explicated in the following section.

RESULTS

Solution rates and error rates

On average (∅), children solved 70% of all items correctly (this corresponds to 22 out of 32 items) with a minimum of 37% and a maximum of 100%. The solution rates of the different items are similar: on average, the items were solved correctly by 70% of all children (see table 1 for further information).

animals	symmetric	asymmetric	sum
front/back views	92.3%	93.8%	93.1%
side views	65.8%	62.8%	64.3%
sum	79.3%	78.3%	78.8%
cuboids	**symmetric**	**asymmetric**	**sum**
front/back views	82.5%	70.5%	76.5%
side views	45.8%	45.8%	45.8%
sum	64.2%	58.2%	61.2%
animals and cuboids combined	**symmetric**	**asymmetric**	**sum**
front/back views	87.0%	82.1%	84.6%
side views	56.1%	54.3%	55.2%
sum	71.6%	68.2%	69.9%

Table 1: Average solution rates

As known from other perspective-taking studies, differences between the two objects types were significant ($p<.001$): children solved the tasks with animals (∅78.8%) more often than the tasks with cuboids (∅61.2%). It was also hypothesized that the side views (∅55.2%) would be more difficult than the front and back views (∅84.6%) ($p<.001$). However, a significant difference in the solution rates could be observed between symmetric and asymmetric objects in the other direction than expected ($p<.005$): the symmetric tasks (∅71.6%) were solved slightly more successfully than the asymmetric ones (∅68.2%). Further analyses showed that this difference is grounded in the tasks with front and back views of cuboids. The symmetry of the views seemed to help children to concentrate on relevant details of the views and to distinguish them from the asymmetric side views. In other groups of tasks (side views of cuboids, side views and front/back views of animals) no significant differences between tasks with symmetric and tasks with asymmetric objects were found.

The assumption that children would solve spatial perspective tasks less often with symmetric objects than with asymmetric objects cannot be confirmed. However, the comparison of the error rates shows an interesting difference: the side views of parallel-aligned symmetric objects were inverted more often than those of asymmetric objects, with animals as well as with cuboids (see table 2).

animals	symmetric	asymmetric
egocentric	36.1%	57.1%
inverted side views	56.9%	36.4%

cuboids	symmetric	asymmetric
egocentric	45.0%	71.0%
inverted side views	35.8%	19.4%

Table 2: Rates of mistake-types in relation to all mistakes at side views with parallel-aligned objects[1]

Thus, the difficulty that symmetric objects pose can be seen in the inversion of the side views. A more thorough analysis of the children's explanations will give hints for better understanding these difficulties and differences.

The children's explanations

In analysing the transcripts of the children's explanations, three main categories could be extracted for both conditions. Since the children's explanations differed according to the objects types, the categories are presented separately for animal and cuboid tasks before a comparison as well as frequencies are given.

Categories for the animal tasks

Category 1: Reference to details In statements subsumed under this category, the children referred to details of the animal and said what can be seen from a special view: "He sees the face."; "She takes a picture of the tail." Usually they name that part of the animal that is nearest to the toy figurine and therefore appears in the front.

Category 2: Reference to intrinsic alignment of the animal This category contains statements that do not refer to special details, but to particular sides of the animal: "You see it only from behind."; "Because he is standing in front of it."; "He takes a photo of the side." In this category, the words "behind", "in front" and "side" could also refer to other frames of reference than the intrinsic alignment of the animal: to the child's view or to the toy figurine's view. However, in most cases the children referred to the intrinsic alignment of the animal: if they said "He stands behind the animal," they did not mean the toy figurine opposite to themselves, but the toy figurine that is standing next to the animal's tail.

Category 3: Reference to extrinsic alignment of the animal This category includes all explanations about how the animal was oriented without referring to details or the intrinsic alignment of the animal: "The elephant looks to the man."; "It walks there (showing the direction of the animal)."; "Because it stands that way (making a motion from the back to the front of the animal)."

Categories for the cuboid tasks

[1] The side views of the orthogonal aligned objects were excluded because in this condition the mistakes could not be classified as egocentric or inverting of side views (see above "ambiguous mistake").

Category 1: Reference to the front-back-relation The explanations in this category showed awareness of what can be seen in the front from the position of the toy figurine or if a part of one cuboid is hidden by the other cuboid: “The red one is in front of the yellow.”; “He couldn’t see the blue block well.”; “She can see the black one better than the green one.”

Category 2: Reference to different sides of the cuboids Although cuboids have no intrinsic alignment, some of the children’s statements are similar to statements of category 2 of the animals in that way that they refer to different sides of the cuboids: “Because there the block is narrow.”; “The block appears wide.”

Category 3: Reference to the extrinsic alignment of the cuboid building This category includes all explanations by which children described how the cuboids were positioned: “There is the black one, and there is the green one (showing both cuboids either at the arrangement or at the photographs).”; “Because this is here (pointing to one of the cuboids) and the other one is beside it.”; “The blue block is on THAT side (pointing to the blue block) and not there (pointing to the other side of the second block).”

Apart from some categories that occurred very rarely and are therefore not presented, there is one frequently observed category in both object-groups, which may be called “Just because!” Statements subsumed under this category are not arguments but rather claims: “Because it is the same.”; “I know it.”; “They fit together.”

<u>Frequencies of categories</u>

The analyses of frequency distributions showed that all described categories are re-presented in both object-type conditions (animals and cuboids) and also in symmetric as well as in asymmetric tasks. The following descriptions focus on the comparison between symmetric and asymmetric items.

Front/back views of animals: There was no difference between symmetric and asymmetric tasks. Both were predominantly justified with statements of category 1 (reference to details): ∅51.5% [2] (symmetric) and ∅52.9% (asymmetric) of all explanations could be assigned to this category. The second most common category was category 2 (reference to intrinsic alignment) with a frequency of ∅34.9% (symmetric) / ∅31.6% (asymmetric). Every other category accounted for less than 8%. The dominance of the first two categories is comprehensible because of the intrinsic alignment of the animals, which does not differ between symmetric and asymmetric animals.

Side views of animals: For these tasks, differences in reasoning could be observed. The tasks with asymmetric animals were mostly justified with category 1 (reference to details) (∅41.7%), whereas with symmetric animals, category 1 accounted for only ∅7.9%. With symmetric animals category 3 (reference to extrinsic alignment of the

[2] The following percentages refer to all explanations that could be assigned to one of the categories. Situations, in which the child gave no explanation or the explanation could not be allocated, are not included in the frequencies.

animal) was the most frequent one (∅30.8%) and statements like "Just because!" were also very frequent (∅25.8%). This result reflects the problems that children often have in distinguishing the side views of symmetrical animals.

Side views of cuboids: In this condition, a difference between symmetric and asymmetric tasks appeared as well. Category 1 (reference to details) occurred more often with asymmetric arrangements, in which one cuboid appears at the front or back of the other one (∅45.7% vs. ∅15.6%). For symmetric arrangements, category 3 was the most frequent one (∅40.0%). Interestingly the category "Just because!" was not as frequent as in the side-views-condition of animals (∅15.0%).

Front/back views in cuboid tasks: Children justified their decisions in these tasks predominantly by category 1 (reference to the front-back-relation), but more often in symmetric (∅66.1%) than in asymmetric arrangements (∅52.9%). Asymmetric arrangements lead more often (∅14.4%) to category 3 (reference to extrinsic alignment) than symmetric arrangements (∅6.1%). These results confirm the interpretation of the solution rates, that the symmetry of the views helped the children to concentrate on relevant aspects (here: the front-back-relation).

CONCLUSION

Two different directions of the influence of symmetric objects on spatial perspective-taking could be observed in this study: On the one hand symmetry has a simplifying effect, probably because it helps perceiving relevant aspects of the task. This could be seen in the condition of front and back views of cuboids, in which tasks with symmetric objects were solved more often than tasks with asymmetric objects. On the other hand symmetric objects complicate the solution of the tasks, if the two side views that are symmetric to each other have to be distinguished. This effect was not reflected in the solution rates, but could be observed in the distributions of types of mistakes and of the children's explanations: In symmetric conditions, the children more often interchanged the side-views and had more difficulties to justify their decisions than in the asymmetric ones. The statements for the asymmetric side views showed that the differences between the two side views, which lay in the front-back-dimension, helped the children to distinguish between them.

For further research of perspective-taking the influence of symmetric objects should be considered carefully, especially if asymmetric and symmetric objects are used in the same study. For working with perspective-taking tasks in school, teachers should be sensitive to the difficulties of side views of symmetric objects. It could be useful to start perspective-taking with young pupils with asymmetric objects. Teachers should also stress the importance of considering the orientation of the objects, especially the left-right relation, because the strategy of referring to what can be seen is not always successful.

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SURVIVAL OF THE FIT: A BOURDIEUIAN AND GRAPH THEORY NETWORK ANALOGY FOR MATHEMATICS TEACHER EDUCATION

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Supervision of student teachers in their field experience is one of the practices that characterizes the work of many teacher educators. This paper takes up the issue of teacher education field experience and associated faculty supervision, drawing on the conceptual tools of Bourdieu's social field theory and a graph theory network analogy to interpret data from a self-study research project. In this brief paper, one data storyline is presented to convey narratives of a teacher educator's efforts to disrupt and reconceptualize the network of relations in teacher education field experience.

PURPOSE OF RESEARCH STUDY

Field experience supervision constitutes a key aspect of the work of many teacher educators. The specific nature of this work varies significantly across teacher education program contexts, with varying efforts to enact supervision in ways that reflect the complexity of teaching and learning to teach. However constructed and conceptualized, supervision of student teachers in their field experience (also called practicum or internship) is one of the practices that characterizes my work as a teacher educator and faculty advisor. In addition to constituting one of my realities as a teacher educator, I also see it as an opportunity for studying my own learning about what shapes my identity as teacher educator, faculty advisor, and researcher.

This paper takes up the issue of teacher education field experience, with a particular focus on the role of teacher educator as faculty supervisor 'in the field'. Having felt less than satisfied over the years with my role as a faculty advisor, I have been drawn to experiment with various models and visions for enacting my role differently (Nolan, 2011). Without describing these models in detail, this paper focuses on the tensions and disruptions erupting as I endeavored to move my role as a faculty advisor beyond tokenism in the field (Nolan, *under review*).

The paper draws on the theoretical framework of Bourdieu's social field theory—and his 'thinking tools' of habitus, field, capital and doxa—put forth as a way of visualizing the networks of social relations in the field of field experience. Also in this paper, I draw on the ideas and language of mathematics graph (network) theory (Clark & Holton, 1991) as a way to draw analogies between the two theoretical constructs. Both theories, when interlinked in this unique and playful manner, lend themselves to a way of conceptualizing how networks of relations feature prominently in (re)constructing the field of teacher education, and token faculty advisors within.

RELATED LITERATURE

The field of teacher education is being researched extensively from diverse perspectives. The study of theory-practice transitions from university courses to school practicum has been a prominent one, including those interested in making the transition a smoother one (Jaworski & Gellert, 2003) as well as those resisting the existence (or at least the language) of a theory-practice dichotomy (Zeichner, 2010). In addition, there are numerous and theoretically diversified studies on becoming a teacher, from those with a poststructural focus on identity constructions (Brown & McNamara; 2011; Nolan & Walshaw, 2012; Williams, 2011) to those with the more technical concern of understanding the skills and content knowledge required by teachers (Ball, Thames & Phelps, 2008; Chapman, 2013). More recently, the field of teacher education research has been paying much closer attention to the structures and roles of that specific component of teacher education programs referred to as the school practicum or field experiences (Cuenca, 2012; Falkenberg & Smits, 2010).

CONTEXT AND METHODOLOGY

In my university's four-year undergraduate teacher education program, the culminating field experience is a four-month internship (practicum, field) experience in schools. Each prospective teacher (intern) is paired with a cooperating (mentor) teacher in the school and assigned a university supervisor (faculty advisor). Each faulty advisor works with approximately four interns over the internship semester, and are expected to visit, observe and conference with each intern 3-5 times during this four-month internship. From my perspective, the model is problematic and 'deficient' in a number of ways, not the least of which is that a mentorship relationship between faculty advisor and intern based on only 3-5 visits over four months is not adequate to disrupt and challenge the view that teacher education programs merely train and prepare prospective teachers for the *real* experience of school classrooms. As a faculty advisor, my role in this internship model has felt superfluous, even token over the years. Thus, I was drawn to design and implement new ways of being a faculty advisor and doing internship supervision.

This paper is based on a self-study of my practice as a faculty advisor, working with interns during their internship conducted each year over a period of approximately six years (2007-2012). As a methodology, self-study can be defined as the intentional and systematic inquiry into one’s own practice (Loughran, 2007). In teacher education, self-study is powerful because of the potential to influence prospective teachers, as well as impact one’s own learning and practice as a teacher educator. Drawing on self study approaches in my research highlight my conviction that the boundaries between research, teaching, and learning are blurred (Nolan, 2014). In fact, self study embeds the learning acts of teacher educator as both researcher and learner. By studying my own professional practice, I am in a better position to reflect on the relationships between research, teaching, and learning and to interrogate the discourses shaping my roles and practices as a teacher educator. I accept that a key “aim of self-study research is to provoke, challenge, and illustrate rather than confirm and settle" (Bullough & Pinnegar, 2001, p. 20).

THEORETICAL FRAMEWORK

The research study informing this paper challenges and disrupts traditional discourses of teacher education programs and associated field experience, tracing the intersections of identity, agency and reflexivity in mathematics teacher education using Bourdieu's sociological theory (Bourdieu, 1977, 1990; Bourdieu & Passeron, 1977). The key concepts of Bourdieu's social field theory confirm the complexities of becoming a teacher by focusing on the dynamic relationships between structure and agency within a social practice. Such an approach highlights the network of relations and discursive practices that support (and (re)produce) traditional practices in field experience models, acknowledging the normalized practices and dispositions of schooling as strong forces in shaping teacher educator (faculty advisor) identity and agency (Nolan, 2012). In this research, I draw on Bourdieu's social field theory (specifically, the concepts of *habitus*, *field*, *capital* and *doxa*) to expose the discursive productions of the network of relations constituting field experience.

Bourdieu (1990) claims that a person's habitus, or set of dispositions, in a social practice field (that is, a socially instituted and structured domain or space) are tightly bound up in and by the network of practices and discourses (relations) within that field. Field and habitus are central to understanding this social network of relations since the two concepts are produced and reproduced in a dialectical relation to each other through social practice. Grenfell (1996) clarifies these relations by offering the following:

> Individuals are embedded, located in time and space, which sets up relations. These relations are not simply self-motivated and arising from individual choices but immanent in the site locations in which they find themselves. Such relations are differential and objectively identifiable. They are structured structures, but, equally, structuring structures in a generative sense. (p. 290)

In this brief paper, it is not possible to provide a comprehensive overview of Bourdieu's key concepts or thinking tools. The larger research study draws more extensively on these conceptual tools of Bourdieu's sociological theory to understand social relations in networks of practices, specifically those relations produced through teacher education field experience and supervision models.

METHODS AND DATA SOURCES

The study has taken on various characteristics as it has evolved over the years, and as I have adapted my internship 'supervision' approaches in response to research data. During each year of this self-study, a Professional Learning Community (PLC) was sustained 'virtually' through the use of desktop video conferencing and through 'real' face-to-face professional development sessions with interns and their cooperating teachers. The professional development aspect of the project focused on lesson study approaches that incorporated the recording and analysis of classroom teaching videos. By creating a multi-dimensional model for internship, my aim was to construct an expanded faculty advisor role that would enhance opportunities for sustaining a mentorship relationship between myself and my interns. Data collection for this self-study included interviews and focus groups with interns during six internship semesters (2007-2012). The interviews and focus groups were conducted in person and through video conferences. Also, as researcher, I kept a self-study journal to better understand and reflect on my role as a faculty advisor.

While the key aim of my evolving model for internship supervision focused on strategies for expanding my role as faculty advisor, that aim merely serves as the subtext for what I attend to in this paper. As alluded to earlier, the intent of this paper is not to present, analyze, and discuss large amounts of the research data per say, but more to reflect on the self-study data in the context of illuminating (and interrogating) the network of field experience relations within which my own identity and learning as a teacher educator and faculty advisor is being (re)produced. This paper draws on data from that larger research study, along with Bourdieu's social field theory, to conceptualize the network of relations that are shaping me as a faculty advisor in reconceptualizing secondary mathematics teacher education field experiences.

PRESENTATION AND DISCUSSION OF DATA

Elsewhere (Nolan, *under review*), I present and analyze five (5) data storylines that convey narratives of my efforts to disrupt and reconceptualize the network of relations in teacher education field experience, with the ultimate goal of understanding how (or, if) my professional practice might shape and influence a more dynamic view of these networks. I use the language of nodes and links to playfully highlight the metaphorical connections between Bourdieu's concept of social networks in a field and the mathematics field of these storylines in detail. Then, I briefly refer to the other storylines and present a network diagram to visualize the relations through one possible configuration of a directed graph, or “digraph” (Clark & Holton, 1991, p. 230).

For the purposes of this paper, I refer to the term *nodes* to stand for the sources, actors or agents in the network (of which there are 5, plus myself as faculty advisor (FA)) and *links* to reflect the pathways or relations connecting the various network nodes (represented by a directed graph with single or double arrowheads). The data storyline is presented as constituting a node and connecting pathway of the network. Playfully linking this research analysis to graph theory draws attention to how a mathematical structure such as a graph can be used to model key coupling relations between objects/agents, providing a way to imagine the interactions and links between the structuring structures in Bourdieu's social networks.

Data Storyline: Metaphorically Speaking

This storyline conveys my efforts to understand my interns' perceptions of my role as a faculty advisor in their professional development as interns and becoming teachers. My self-study initiative set out to expand my role as a faculty advisor—that is to move beyond tokenism (Nolan, *under review*). My model for enacting my role as a supervisor included many more contact hours than what is typical. During a focus group session with a group of three interns one semester (2010), I questioned them on my role as faculty advisor and its overall value to them in terms of their professional development during the internship semester. The following quote from one intern speaks to an illustrative response to this question:

> If our coop is doing their job right they really *should* be doing that professional development process with us, so having you there is just kind of extra, I guess. I don't know if it's completely necessary. But if you *were* to do it, I would probably still prefer that you come out and see me... like, if I had had problems with [the coop] then I would want you there, I would *need* someone else, but since we got along then the roles kind of seem the same to me. (Intern, Dec 2010)

I pursued this line of questioning a bit deeper in the focus group, but the underlying message of their responses remained: I was "just kind of extra." Later, I reflected in my self-study journal how I was taken aback by their comments:

> *Wow. That's harsh. My efforts to disrupt the token and remote role of the faculty advisor have been constituted by the interns as 'extra' and much the same as the cooperating teacher has to offer. In their eyes, I've not expanded and redefined my role in the manner I set out to. Instead, the interns have constructed an identity for me as liaison, mediator, umpire, even peacemaker. So, as long as there are no "problems" with the cooperating teacher, I am not needed. Hmmm*. [Researcher journal entry]

In another year of the study (2011), an intern suggested that my role was like that of "a fine tooth comb":

> I think it's good that you're distinct from the cooperating teacher. I feel like with my cooperating teacher, I come to class the morning of, we do a quick little preconference, I teach, and then we post conference. Whereas with you, I feel like it's very specific, focused on one specific lesson and looking for perfection almost. So I think you're more the fine tooth comb of the operation, and [my coop] is more of the overseeing almighty part of the operation, if you know what I mean? [Intern, Dec 2011]

This storyline of 'metaphorically speaking' confirms that interns value their cooperating teacher's experience and perspective first and foremost, and that the role of the university supervisor takes on a distant second, or even unnecessary 'extra'. In a review of research on the ways in which cooperating teachers participate in teacher education, Clarke, Triggs, and Nielsen (2013) also found that the roles of cooperating teachers and university supervisors are valued quite differently. They echo other research in confirming how "the role of the cooperating teacher has always been regarded as important within teacher education" (p. 4), whereas perceptions on the role of the university supervisor is less uniform and agreed upon in the literature. It is interesting to note that Clarke, et al. (2013) also report that "cooperating teacher feedback remains largely fixed on the technical aspects of teaching" and tends to be "more confirmatory (positive) than investigative (reflective) in nature" (p. 13), which leads me to propose a 'survival of the fit' mindset. That is, I propose that a positive, confirmatory approach to interacting with the interns in their process of becoming (a teacher) is a much better fit with their own habitus (set of dispositions) than one which challenges them to engage in deep and substantive reflections which may actually challenge their habitus-field fit. In other words, cooperating teachers provide interns with feedback in the form of practical tips and techniques, whereas I am asking interns to spend time in what Grenfell (2006) refers to as a *nowhere* space, that is, "areas in which they could engage with the contradictory elements of teaching and respond in line with their own developing pedagogic habitus" (p. 301).

Once in the schools for their field experience, prospective teachers are “confronted with the task of learning the discursive codes of practice” (Walshaw, 2007, p. 124) in the secondary mathematics classroom, and no longer in my own university classroom. Interns identify their cooperating teachers as being much better positioned to initiate them into these practices, and hence the practices themselves often remain unquestioned and misrecognized. These discursive codes of classroom practices, in part, constitute the network of relations that Bourdieu puts forward. The pathways of already well-established classroom practices represent cultural capital that holds considerable value in the field, and thus preservation and normalization of these well-established practices are important in becoming a teacher. In the language of network theory, it is easy and convenient to follow the shortest path or the path of least resistance when it comes to participating in one's field experience.

Bourdieu and Networks: The Work of Interpretation

Since it is only possible to present and discuss one storyline, it is worth at least naming each of the other storylines and constructing a visual network to convey one possible configuration of pathways and nodes (Figure 1). The five storylines (nodes) are: (1) not sitting in the back of the classroom (interns), (2) metaphorically speaking (interns), (3) "I appreciate the opportunity but..." (interns), (4) intern placement protocols (program structure), and (5) "If the process becomes disruptive to students or the intern's growth..." (cooperating teacher). Each of these storylines and the directed links (edges) are further elaborated on in the full paper and presentation.

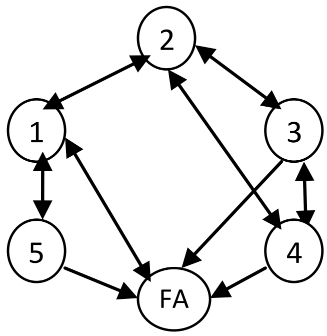

Figure 1

The five storylines constituting a network of relations (consisting of nodes and pathways) all relate to the social practice of teacher education field experience and supervision—drawing attention to how tightly woven together the network of relations within a field are. They reaffirm the sources/nodes and links/pathways that form the core of established and taken-for-granted social practices of teacher education and field supervision—what could be referred to as teacher education and supervision doxa. In many ways, the storylines also highlight my failed attempts to bring about significant disruptions to the traditional model of supervision, including (as conveyed in storyline #2) cooperating teachers' and interns' constructions of the university supervisor as 'other' or 'extra'.

CONCLUDING THOUGHTS

Reflections on the research presented in this chapter means disrupting the storylines and pathways sustaining the current networks of relations, working to reveal their arbitrary and contingent nature. In connection to my own professional learning, I am coming to terms with the challenges facing me as I attempt to trouble the discursive network of relations in field experience. At times I am drawn toward abandoning my research efforts aimed at reconceptualizing secondary teacher education through an alternative field experience (internship) model. It is hard for me to believe that different and multi-directional pathways can be successfully introduced to trouble the current network.

Adopting a reflexive stance in teacher education would aim to expose the socially conditioned and subconscious structures that underlay the reproductive nature of the network of relations (examining the interactions between and among nodes). What is unique about the approach I take up in this research study is how I acknowledge my own complicity in (re)producing the network of relations in the field experience and for supervision. While I seek to disrupt and reconstruct the network, it is evident that I also comply with its structures and relations. It could be said that I have learned how to be strategic—I am deliberate in striving not to disrupt the game of supervision so much so that no one will want to play with me anymore. In other words, my own 'survival of the fit' as a faculty advisor comes into play in this network analysis.

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AN INFERENTIALIST ALTERNATIVE TO CONSTRUCTIVISM IN MATHEMATICS EDUCATION

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The purpose of this paper is to draw attention to a relatively new semantic theory called inferentialism as developed by the philosopher Robert Brandom. We argue that it offers a better alternative to the still present representational view of mind than does (socio)constructivism. After a discussion of the shortcomings of (socio)constructivism, we summarize the key features of inferentialism that make it worth thinking through more carefully in mathematics education research.

REPRESENTATIONALISM AND THE MYTH OF THE GIVEN

This paper invites mathematics educators to study a semantic theory that Brandom (1994, 2000) has elaborated in recent years. He critiques a representational view of mind that has long been criticized in several disciplines but is still common among cognitive scientists and, in our experience, in how mathematics educators and teachers talk about teaching and curriculum (Bakker & Derry, 2011). Rather than seeing representation to be the basis for reasoning, Brandom explains the meaning of representations through their origin in reasoning practices.

Representationalist theories describe the activity of learning as that of the modification or construction of internal representations. The student is assumed to possess a pre-existing internal faculty of representing external phenomena which the teacher is supposed to expand in a given, pre-determined direction. The goal of teaching is accordingly characterized as bringing about the correct representations in the student's mind, while its success is determined by the measure of correspondence which exists between the internal representations and external reality. In recent decades, representationalism has come under attack. Cobb, Yackel, and Wood (1992) argue that the central problem of representationalist theories is their implicit appeal to a *dualism* between individual representation and external reality. They write:

> At the outset, mathematics in students' heads (internal representations) is separated from mathematics in their environment (external representations that are transparent for the expert). The basic problem is then to find ways of bringing the two back into contact.(p.14)

This dualism is endemic to representationalism: Once representations are cut off from the world and the practices that contribute to their constitution, they cannot be brought together again except by the assumption that mind and world do stand in some sort of primitive relationship, for example through a fundamental form of immediate sense experience. Sellars (1956/1997), one of Brandom's inspirations, attacked this assumption of immediate sense experience, which he called the "Myth of the Given." He argued that the normativity inherent in conceptual content – the ways in which

2014. In Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 4, pp. 321-328. Vancouver, Canada: PME.

concepts and speech, in contrast to objects and causes, can be correct and incorrect – cannot in any way be derived from a nonconceptual, nonnormative reality which is simply “given” to the mind. His argument was amplified by Brandom’s colleague McDowell (1994/1996), who pointed to a complementary and, he claimed, equally pernicious assumption. Once the failure of appeals to the given is recognized, it is natural to reject any form of foundationalism. But this response would go too far and may reject the need for any relation between mind and world at all, because it is thought that a form of foundationalism is the only way to make internal representations square up with the world. In McDowell’s term, one becomes a *coherentist*, rejecting any form of constraint on one’s thinking imposed by external reality and preferring instead pragmatic, deflationary notions such as the internal consistency of one’s ideas.Such an approach risks being like a “frictionless spinning in a void” (McDowell, 1996, p. 11).

One way to escape from the “oscillation” (p. 17) in the history philosophy between the given and coherentism is to accept Sellars’ view that, though humans have access to external objects, these objects or representations of them can only play a normative role (be used in assertions which can be correct or incorrect) by being taken up into the language games of people: They must become subject to norms which were already in place in human practices. In his influential phrase, they must be placed in the “space of reasons” (Sellars, 1997, §36). Brandom systematically elaborates this idea (Bransen, 2002). He pictures humans’ interactions as engaging in a the essentially social “game of giving and asking for reasons”; objectivity and representation are not basic but must be mediated through this game. This also necessarily involves an emphasis on the *holistic* and *social* nature of concepts.

Cobb et al. (1992) presented constructivism as an alternative to representationalism. We summarise some of the problems faced by constructivism and suggest inferentialism as an alternative that is more convincingly rooted in philosophical traditions. In brief, inferentialism has been argued to be compatible with Vygotsky’s ideas (Derry, 2013). It also connects to a “domesticated” Hegelianism (e.g., the aforementioned notions of mediation, holism and social nature of concepts) which maintains links with analytic philosophy (Bernstein, 2002). Inferentialism is more explicitly concerned with concept use and reasoning practices than much published work arising from sociocultural and activity-theoretical perspectives, which makes it especially interesting to mathematics education research. However, due to space limitations we concentrate on why we think it is a better alternative to variants of constructivism.

CONSTRUCTIVISM

A basic constructivist presupposition is that learners “create their own understandings” through the process by which they are immersed in meaning making (Rogers, 2011, p. 178). In this way, the constructivist approach to knowledge overcomes the problematic representational tie between mind and world by downplaying the world element. As such, the constructivist hopes to reject the correspondence relation that sits at the heart of representationalist theories by undermining the implicit *dualism* of the representational doctrine. Consequently, constructivism is able to make room for two desiderata which representationalism was shown to lack: It can offer accounts of individual learning and the social embedding of learning processes.

The central problem the constructivist alternative faces, however, is how to come to terms with accommodating these central desiderata without postulating that the learner is in direct contact with the world. This section aims to show that constructivism has been unable to offer a satisfactory account of the learning process because it adheres to a thinly veiled neo-Kantian pre-supposition, which dictates that on some level mind and world must be kept apart if we are to avoid the problematic representationalist idea that mind and world stand in some sort of primitive relationship to one another. We hold that this neo-Kantian dogma undermines the constructivist's position because, in trying to avoid falling into the Myth of the Given, it embraces a dualism of its own.

There are many variants of constructivism, but for our argument it is sufficient to recognise the general distinction between the cognitive constructivist and socioconstructivist approaches, because most individual manifestations of constructivist theory can broadly be categorized under one of these constructivist positions (Mason, 2007). Cobb and Bowers (1999) delineate this general distinction by considering two metaphors:

> In the case of the cognitive perspective, a central organising metaphor is that of knowledge as an entity that is acquired in one task and conveyed to other task settings. In contrast, a primary metaphor of situated learning perspective is that of knowledge as an activity that is situated in regard to an individual's position in a world of social affairs. (p. 2)

This conceptual difference affects the methodology of both approaches, because, "If from the cognitive point of view, knowing means possessing, from the sociocultural perspective it means belonging, participating, and communicating" (Mason, 2007, p. 2). In this sense the focus is *either* on the "construction of internal knowledge or meaning" *or* the "construction – if it can even be called that – of new communities of discourse and social practice" (Kaartinen & Kumpulainen, 2002).

Thus there is a tension between cognitive and socioconstructivism resulting from their contrasting attempts to satisfy the desiderata lacking in representationalist theory, whilst simultaneously adhering to implicit and problematic neo-Kantian presuppositions, which dictate that on some level mind and world are disconnected. Consider cognitive constructivism (e.g., Von Glasersfeld, 1980). By focusing upon the *individual's* internal cognitive mechanisms it explains the process of learning in terms of the individual's construction of internal knowledge or meaning. The problem is that in attempting to circumvent the issue of a nonconceptual reality which the mind is primitively able to represent, cognitive constructivist's take a coherentist approach, which tries to conceptualise knowledge claims and articulate standards of objectivity on the basis of the internal coherence of beliefs. In this way, the individual's loses touch with the world. So, even if cognitive constructivism satisfies one desideratum – that of individual learning – it can only do this by neglecting objective external constraint and falling prey to relativism. Learners, therefore, are left spinning in McDowell's frictionless void.

Socioconstructivism, on the other hand, appreciates that learning cannot be segregated from social practices and must take place within certain communities of discourse, but typically offers no account of how individual's go about constructing internal knowledge claims. If we cast learning as that mental phenomena that take place only inside of the learner's head, then we ignore the now well established idea that a theory of learning should include an account of the reflexive relation between an individual student's reasoning and the evolution of the classroom practices that constitute the immediate social situation of their mathematical development (e.g., Bowers, 1996). On the socioconstructivist view, however, whilst we sidestep the coherentist pitfall, we have no account of how differing contexts and discourse practices provide the sufficient basis for *individual* understanding. We have no explanation of how the same knowledge – of simple mathematical truths, for example – can be internalised by differing individuals with often markedly different learning processes. Subsequently, if learning concerns only the construction of social practice, then we lack a convincing account of how individuals come to know anything. How, for example, can I know that my knowledge of the mathematical truth, $1 + 1 = 2$, squares with yours? In Cobb et al.'s words, students "have no way of knowing whether their individual interpretations of a situation actually correspond to those of others" (1992, p. 17-18). For the socioconstructivist, therefore, learning refers solely to an individual's integration into a particular environment to such an extent that the role of the mind – in forming and modifying knowledge claims and meaning designation – plays almost no part.

To attend to this problem and so illuminate why there exists a tension between cognitive and socioconstructivism, we must look to constructivism's neo-Kantian presuppositions about the mind-world relation. How can individual learning be any more than the coherence of my own beliefs and so satisfy the notion of objectivity? And how can the constructivist make an appeal to sociocultural practices whilst still accommodating individual learning? We claim that these two desiderata cannot be *jointly* satisfied within a rigidly neo-Kantian constructivist theoretical framework, because the constructivist maintains that the best way to deal with representationalism's Myth of the Given inspired deficiencies is to adopt the neo-Kantian doctrine that says, on some level, mind and world must be kept apart. But this will not do, because it functions as an incomplete renunciation of the dualism at the heart of representationalism and causes problems for constructivism's own account of learning processes. The constructivist wants to renounce the representationalist assumption that the world is simply given to us, but to do so he endorses the neo-Kantian notion that the world is not entirely open to the mind. But then, instead of *overcoming* dualism, the constructivist merely has a choice between deciding in favour of the mind and discounting the world (cognitive), or in favour of the environment (socioconstructivist) and losing his way back to the mind. Just as in Kant's transcendental story, the constructivist has – perhaps implicitly – decided that there exists a gap between mind and world, so that we can have a theory of learning centered on mind *or* environment, but not both.

It is clear from the preceding discussion that if the constructivist is to offer a theory of learning that conceives of the learning process as one in which individual learners have "taken-as-shared mathematical interpretations, meanings, and practices institutionalized by wider society" (Cobb et al., 1992, p. 16), he needs additional theoretical tools to overcome the neo-Kantian gap described above. He requires an *evolution* of constructivism's theoretical commitments.

In our view, one interesting evolution of constructivism, which can be found in the ideas of Cobb et al., is the introduction of normativity. For Cobb et al., mathematical learning is multidimensional; it contains individual and sociocultural elements, because individual learners are subject to intersubjective sociomathematical norms (Cobb & Yackel, 1996). These norms, as instantiated via social practice, bind individual learning to the collective in a way inconceivable from within the traditional constructivist picture by regulating "what counts as an acceptable mathematical explanation and justification" (p. 461). Sociomathematical norms thus provide the basis by which constructivism can answer to both desiderata lacking in representationalist theory without succumbing to the Myth of the Given or coherentist tendencies. But Cobb's account has not provided an explication of the functionality of the normative process that underpins the learning process; the *how* and *why* concerning normativity. As such, Cobb's evolution only goes so far. Moreover, we wonder whether a perspicuous and systematic description of this process could be given by making further use of the metaphor of construction. As an alternative or supplement to this evolution in Cobb's work, we argue that Brandom's inferentialism offers an approach that permits of a clearer articulation and expansion of what Cobb is trying to achieve.

INFERENTIALISM

Inferentialism differs from representationalism and constructivism in privileging the metaphor of inference over those of representation and construction. Following Sellars, Brandom espouses an inferentialist semantics, which sees the meaning of a word as determined by the inferences in which the word plays a role – a Hegelian idea also found in Vygotsky (see Bakker & Derry, 2011). This distinguishes him from typically representationalist theories, which understand meaning in terms of reference to objects. For example, the meaning of "red," for Brandom, is defined by that one can, *inter alia*, derive "p is colored" or "p is not blue" from "p is red." From this it follows that inferentialism must be *holistic* in nature: To understand one concept, one must understand many. So, to

> grasp or understand […] a concept is to have practical mastery over the inferences it is involved in – to know, in the practical sense of being able to distinguish (a kind of

> know-*how*), what follows from the applicability of a concept, and what it follows from. (Brandom, 2000, p. 48, his italics; cf. Sellars, 1997, §36)

This is a *practical* mastery because the assessment of one's concept use is not up to oneself (or to the way one's concepts mirror reality). Rather, it is up to the people one engages with in the aforementioned social game of giving and asking for reasons. Brandom (1994, Part I) develops a social scorekeeping account of language, in which one keeps track of the claims one takes oneself or other persons to be committed to on the basis of their utterances, their actions, or what one takes to follow from those utterances or actions. These practices are constrained by norms, as in Cobb; but we wish to emphasize that, contrary to Cobb, Brandom gives a perspicuous account of their provenance from within scorekeeping practice. Brandom's account is essentially socially perspectival (1994, Section 8.6). It is primarily important what one *takes* people to be committed to; only later does the question arise whether they are or should *really* be committed to those inferences. One's commitments are, however, genuinely subject to external constraint, in that they may be challenged or endorsed on the basis of facts about reality; Brandom allows for noninferential access to the external world through perception. This access is not immediate, however, because it is filtered through the game of giving and asking for reasons.

In sum, Brandom understands concepts in terms of the set of inferences in which they play a role. Concepts are not "in the head," nor are they completely "out there" – rather, they reside in the game of giving and asking for reasons. Though concepts do refer to the external world and may be said to be "constructed," in a loose sense, by the discovery and articulation of new inferences which involve them – a process which may occur both on an individual and a supra-individual level – both representation and construction can be explained in terms of inferences. The individual develops new concepts by becoming aware of new possible inferences. It is the task of the teacher to support and guide this process, while being aware of the individual learning which manifests itself in the situative teacher-student webs of reasons, and without being able to retreat to an external vantage point outside of the social game.

The upshot is that Brandom gives a *dynamic*, *holistic*, *nondualistic*, and *social* picture of human rationality. Reasoning is not primarily an internal phenomenon which depends on essentially private cognitive structures, but takes place in the social game of giving and asking for reasons. This game is dynamic in that the inferences a given word is engaged are not fixed, and in that Brandom sees the status of utterances as depending on how they are recognized (in the Hegelian sense of *anerkennen*) by one's conversational partners. The way they understand your utterance informs the content it is eventually seen to possess. This is one of the ways in which Brandom's account of linguistic activity is *relational*. Moreover, it is inherently practical, stressing the concrete contexts in which learning can only take place, and allowing us to see, for example, how the teaching of concrete, meaningful examples may aid teaching more than the abstract transfer of structural knowledge that does not help students to make inferences. Finally, it is well suited to being applied at different levels of grain. By introducing the notion of a web of reasons, which is the manifestation of the game of giving and asking for reasons in a particular situation (Brandom, 1994, p. 5), it is possible to accommodate both individual learning as well as more general features of classroom learning.

IMPLICATIONS AND CONCLUSION

It is worth emphasizing recommendations that inferentialism offers mathematics educators (Bakker & Derry, 2011). Firstly, it asks them to understand concepts primarily in *inferential* rather than representational terms, as the set of moves available in the social game of giving and asking for reasons. Secondly, it privileges holism over atomism, emphasizing the interrelations between concepts. This has implications for curriculum and teaching, not only in statistics education but also in, for example, vocational education, Bakker and Akkerman (in press) suggest that inferentialism has something important to offer when we intend to overcome common dichotomies such as between school-based and work-based knowledge, and between mathematical and contextual knowledge. In particular the concept of webs of reasons can help to do more fine-grained analyses of the many types of reasons in making claims or decisions. We assume that more areas within mathematics education can benefit from an inferentialist perspective, for example research into reasoning, proof and argumentation, but also research into the role of signs and representations. One key area for further research may be to study the relation between reasoning and representing. Brandom's reversal of philosophical methodology of explaining the meaning of representations in terms of inference may be too extreme for explaining learning. It may turn out that a co-evolvement of reasoning and representing, as in Peirce's theory of diagrammatic reasoning (Bakker, 2007), is more convincing.

Inferentialism has been applied in educational settings and found useful, also for the analysis of mathematical learning (e.g., Huβmann & Schacht, 2009). This paper has sought to urge its theoretical virtues. We do not claim that it surpasses all its competitors, but we have given some reasons for studying if the metaphor of inference is preferable to those of construction or representation – both of which can be explained in terms of it. We also think that an inferentialist theory of education provides the systematic resources needed to solve some of the problems which have plagued educational researchers. Because of its systematicity and roots in both analytic and continental philosophy, it may prove a more perspicuous alternative to many proposals currently on the market. Inferentialism escapes the oscillation between the given and coherentism we discussed and avoids the dichotomies between, for example, mind and world, and the individual and the social. It promises a systematic and coherent way of dealing with these dichotomies without reifying them as is often done. Though promising *evolutionary* developments of the constructivist paradigm are found in Cobb's work, we propose that a *revolutionary* account – a replacement by an inferentialist framework – may ultimately be more useful. We hope that mathematics educators feel invited to study inferentialism and related philosophical perspectives.

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TECHNOLOGY-ACTIVE STUDENT ENGAGEMENT IN AN UNDERGRADUATE MATHEMATICS COURSE

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In this paper we describe the design and implementation of a technology-active introductory first year university mathematics course. The design principles underpinning the course are presented. The results of the implementation show some areas where the technology-active approach has proven of value, as well as improvements that can be made for the next cycle. Some implications for the integration of technology in large lecture undergraduate teaching are presented.

BACKGROUND

This paper describes the implementation of a study comprising one of three components of a wider research project, led by a research team at the University of Auckland, entitled *Capturing Learning in Undergraduate Mathematics*. The *Intensive Technology Innovation* reported here investigates a digital technology initiative in an entry-level mathematics course where technology is employed in four major ways, as described in the course design principles below.

The use of technology in undergraduate mathematics is well-established with respect to lecturer use of mathematical environments (Thomas & Holton, 2003; Drijvers, 2012), and students who have used technology within components of particular courses, for example as laboratory assignments (Oates, 2011). However, it is less common for students to use technology as intensively as in this study, especially in the sense of having integrated, unrestricted use of mathematical environments and websites (such as *Matlab* & *Wolfram Alpha*) in lectures, tutorials and assessments, and being able to access these ubiquitously (in all coursework except the final exam) through smart-phones or portable computers (Hoyles & Lagrange, 2010; Oates, 2011). Such technology use in tertiary education is strongly indicated (Hoyles & Lagrange, 2010; Stewart, Thomas, & Hannah, 2005). Potential benefits include increased student engagement in mathematical activities and discourse (Scucuglia, 2006); improved inter-representational versatility (Thomas & Holton, 2003); and improved understanding in particular content areas (Thomas & Holton, 2003, Oates 2011). There are also studies that describe potential difficulties with the use of technology (e.g. see Drijvers, 2012), and Oates (2011) identifies a complex range of factors that should be considered to achieve an effective and integrated technology-active learning environment. For example, Gyöngyösi, Solovej and Winslow (2011) found evidence that weaker students commit more errors when technology is present, while Stewart, Thomas and Hannah, (2005) note that students need time for *instrumental genesis* (Artigue, 2002). Particular factors considered in the design of this study include teacher privileging (Kendal & Stacey, 2001); access to technology and congruence between learning and assessment (Oates, 2011); student instrumentation (Stewart, Thomas, & Hannah, 2005); technology-active, -neutral and –trivial assessment (Oates, 2011); students' use of lecture recordings (Yoon, Oates & Sneddon, 2013); and the *pragmatic, pedagogic* and *epistemic* value of technology for particular topics in the curriculum (Artigue, 2002; Stacey, 2003).

METHODOLOGY

This research follows a design experiment methodology where "a primary goal for a design experiment is to improve the initial design by testing and revising conjectures as informed by ongoing analysis of both the students' reasoning and the learning environment". (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003, p. 11). The focus of this study is a technology intervention introduced to a university mathematics course during the second semester of 2013. The research questions ask if the course design was effective in engaging students in learning mathematics, and if so how, and what improvements might be made for future cycles. What we present here are some results from the initial cycle in this continuing experimental process.

Course design principles

There are at least four guiding principles employed in the initial cycle of course design and construction. First, technology should be integral to the assessment process. Hence, each student was required to register and enrol into *MathXL* – a web-based homework, tutorial and assessment system, which was used for five skills quizzes (1% each) and the mid semester test (10%). The *MathXL* program provides instant feedback by marking student answers, identifies topics where the student needs to focus their attention and directs them to sections in an online textbook as well as creating a personalised Study Plan. The quiz and test questions were largely free-response, exercising the *MathXL* facility for numerical, algebraic and graphical input of solutions, in contrast to static multiple-choice style questions. The quizzes are a time-limited, non-supervised assessment where students have three attempts and their best score is recorded. The mid-semester test is also time-limited but held in a supervised computer lab with one attempt per question. Students were allowed access to CAS-calculators if they had them as well as online resources, although time factors would have made this impractical for most. There is still some debate about whether the test should be technology-free (skills-based) or technology-active.

The second principle was that the lecturers should model a range of appropriate technology including: a web-based graphing calculator; *YouTube* clips; applets to demonstrate critical features of mathematics; and mathematical websites. In addition to the importance of teacher-privileging (Kendall & Stacey, 2001), another reason for this was to minimise any disadvantage to students who did not have access to specific technologies. At the end of each lecture students were directed to webpages that illustrated the concepts at the heart of each topic, and a video-recording of each lecture was available to students within 24 hours. Third, students were encouraged to use any technology platform they had access to, including all calculators, mobile phones, computers, tablets, etc. and any e-resources they could access with these. The final design principle was that technology should be actively used in the one-hour weekly tutorials that all students were expected to attend.

The course consists of 36 one-hour lectures and 10 one-hour tutorials, which are worth 8% of the final grade. At the end of the course students were asked to complete three questionnaires: a technology questionnaire; an attitude survey; and the standard university student course evaluation. Three volunteers also worked on technology-active group tutorial tasks held in a computer lab. As they worked on these tasks field notes were made on an observation schedule. Each of these students had their own CAS-calculators and could use any technologies they thought appropriate. Figure 1 shows examples of the questions used in the online technology questionnaire, which contained a mix of 19 open and closed questions, and investigated student use of technology in general; mathematics-focused technology use; and the student pattern of technology use during the course. The open questions had unlimited response space.

1. Which mathematical-learning technologies did you observe the lecturers or tutors using and modeling in their teaching of MATHS 102? Please select all that are appropriate.

   MathXL ❑ Graphics or CAS calculators ❑ *Autograph* ❑ Wolfram Alpha ❑

   GeoGebra ❑ *Khan Academy* ❑ Smartphone or Tablet App ❑

   Other Internet Use (specify) ❑ ___________ Other Technology (specify) ❑ ___________

2. Which mathematics learning technologies did you personally use in the course? Please indicate your frequency of use, and whether this was the first time you had used them.

   Often = almost daily; Sometimes = 1 or 2 times per week; Seldom = a few times in the semester

   MathXL Often ❍ Sometimes ❍ Seldom ❍ Never ❍

7. What activities did you use technology for? Please specify which technologies you used for each of the following activities: [Lectures, assignments, tutorials, quizzes]

11. Describe the kind of activities you used technology for when working on mathematics problems in the course.

Figure 1: Examples of the open and closed questions from the questionnaire.

Attitude to learning mathematics with technology	Suggested goals
I like using technology to learn maths	My primary intention in using technology in maths is to check my work
Using technology in maths is worth the extra effort	My main purpose in using technology is to get the answer to the problem I'm working on
Maths is more interesting when using technology	When I use technology I aim to finish as soon as possible
Using technology hinders my ability to understand maths	My main goal in using the technology is to get a better grade in the course
I prefer working out maths by hand rather than using technology	I use the technology to find more than just the basic answer to the question

Table 1: Examples of the scale items

For the attitude survey, a Likert scale was constructed with five subscales in 29 randomised items and a range of five possible responses (strongly agree, agree, neutral, disagree, and strongly disagree). The subscales measured: attitude to maths ability; confidence with technology; attitude to instrumental genesis of technology (learning how to use it); attitude to learning mathematics with technology; and attitude to versatile use of technology. The versatility subscale had four questions and the others five. In addition, there were five questions covering possible goals in technology use, which was not a subscale. Table 1 gives examples of some items.

RESULTS

22 students (out of 131 in the course who sat the final exam) participated in the study; thirteen of these completed the questionnaire and nine the attitude survey. Responses were anonymous so it was not possible to tell how many were in the intersection of the two groups. Although this is a relatively small number of responses, we still believe it gives a reasonable indication of the student reaction to the course. In addition, 50 students completed the online course evaluation. In the questionnaire, ten of the 13 students (76.9%) agreed that the lecturers had made sufficient use of the technologies in the lectures, and recognised the use of a range of platforms. They agreed there was a wide use of technology during the course. All used *MathXL*, seven almost daily and six once or twice a week; 11 used *Desmos*, six of them daily, two once or twice a week; and six used Wolfram Alpha, five of them daily. *Khan Academy* was used daily by five students, *Autograph* by two and *GeoGebra* by one. In addition ten students made daily use of a graphic or CAS calculator.

All the students used *MathXL* for the assessment quizzes, at least once or twice a week, with a mean of 4.72 out of five quizzes. Similarly, all used it for homework, ten at least once or twice a week and twelve for revision, ten at least once or twice a week. Furthermore, nine used it in their study plan and ten for help with solving problems, mostly at least once or twice a week. For the assignments, along with various internet sites, six students mentioned using calculators, five *Desmos*, five *Khan Academy* and two each *Autograph* and *MathXL*. Nearly all the students owned a laptop (12) and a smartphone (11), with ten also having a home computer and four a tablet. Nine (69.2%) had external access to *Desmos*, five (41.7%) to *Autograph* and two (16.7%) to *GeoGebra*. On average they found *Desmos* useful (3.9 out of 5), *Autograph* slightly useful (3.33) and *GeoGebra* not useful (2.2). In response to the summary questions, twelve (92.3%) said that they thought the technology use had helped their learning of mathematics, eleven (84.6%) liked the extensive use of the technology and twelve (92.3%) wanted the technology to be available in future courses. Some comments they made included:

> I learnt a lot from this course through the many technologies made available to me. I spent several hours each week practicing using various websites, apps and online tutorials, as well as recorded lectures. Highly recommended.

> *MathXL* helped me to focus on areas of maths I needed help with.

There was a broad use of mathematical technology throughout this course, enabling students to feel supported in the learning process…technology (for) visual learners like myself (makes) maths seems less daunting.

Particularly in year one mathematics, the use of technology has helped me gain a quicker and deeper understanding as to how various equations behave and being able to quickly look up a mathematics problem on the internet also assisted greatly.

[It should be used in future] Because it is really useful for understanding concepts, for practising them and learning them.

25 of the 42 open responses to the course evaluation item "What *was most helpful for your learning?*" specifically cited technology, with positive references to *MathXL* and the quizzes (17), recorded lectures (7) and access to the web (8), for example *Desmos*, *Khan Academy* and *WolframAlpha*. Comments included:

Mainly the recorded lectures – I've found them very useful for going over when I haven't understood something or forgotten something.

MathXL was extremely helpful for my learning. Being able to check my answers instantly was a great encouragement and stimulant.

MathXL: The website was amazing – the instant feedback on answers and also the facilities to learn what I did wrong, as well as how to do it correctly were fantastic.

Being prompted during lectures of other sources of information available such as *Desmos* and *Khan Academy*, to be able to be used concurrently with *MathXL*'s resources.

I think the quizzes online are the best method for cementing your knowledge of the math.

Data from the *MathXL* website and the lecture recording access also support a high level of student engagement with the technology. While we would expect a high proportion of students to access the quizzes and the test because they are assessed (average of 95 across the 5 quizzes; 124 for the test out of 130 students), a significant number of students still engaged with revision exercises and individual study plans (e.g. 93 & 84 respectively for the test revision and 41 for an exercise on differentiation). Similarly, the lecture recordings were well used, with an average of more than 100 student-accesses to each individual lecture, and a peak of more than 200 for two lectures, one on logs and exponential functions and one on trigonometry. However, not all comments and experiences were positive. Two students in the questionnaire presented forceful reasons for a negative perspective:

MathXL was a disastrously unfair method of assessment as it was difficult to formulate your thoughts when a test is in such a different format to what you have always done. I have personally always been rather good at maths but I have done very poorly in this course as I have struggled with everything being computer/technology based.

...too reliant on technology without understanding the core foundations of mathematics. It is like designing a bridge without first knowing fundamental engineering principles.

These sentiments were echoed in a few responses to the course evaluation item "*What improvements would you like to see?*", where comments were mostly about syntax or the use of *MathXL* in assessment, for example: "I didn't like using *Math XL* for the mid-term test." and "I also strongly disliked *MathXL*, for multiple reasons, and I lost marks on a few questions in quizzes for incorrectly entering something rather than for getting the wrong answer." The means of the attitude survey subscales, shown in Table 2, indicate student agreement that they have a positive attitude to their maths ability, to learning maths with technology and to versatile use of technology. The level of agreement rises in terms of their confidence in technology use and their attitude to learning how to use the technology (instrumental genesis).

Subscale	**Mean* (Low-High)**	**Cronbach Alpha**
Attitude to maths ability	3.89 (3.33-4.56)	0.695
Confidence with technology	4.42 (4.33-4.44)	0.910
Attitude to instrumental genesis	4.40 (4.11-4.56)	0.820
Attitude to learning mathematics with technology	3.93 (3.11-4.22)	0.838
Attitude to versatile use of technology	4.11 (3.67-4.44)	0.872

*Scores on negative items were reversed. 5 represents Strongly Agree.

Table 2: The means of the subscale responses and the reliability measures.

To gauge the internal consistency of the subscales the Cronbach Alpha (CA) measure of reliability was calculated. Four of the subscales show good or excellent reliability. The consistency of the subscale Attitude to maths ability is marginal (a CA of 0.7 is considered acceptable) but the CA would rise to 0.801 if the item 'I can get good results in maths' (mean = 3.33) were excluded. This may indicate that even those who see themselves as good at maths may be less confident of getting good results. The levels of agreement with the suggested goals for technology use were: My primary intention in using technology in maths is to check my work (4.00); My main purpose in using technology is to get the answer to the problem I'm working on (3.11); When I use technology I aim to finish as soon as possible (2.78); My main goal in using the technology is to get a better grade in the course (3.56); and I use the technology to find more than just the basic answer to the question (4.11). So while students are using technology to check their by-hand work it is often not just a basic answer. They are mostly neutral on whether they only use the technology for their current task or whether they try to finish as soon as they can, and probably do want to use the technology to get a better course grade. In Q16 of the questionnaire the students were asked 'Describe what you see as your main goals in using technology in the course', with space for up to three goals. Without any suggestions to lead them, nine students contributed 20 goals, most commonly: to improve learning and understanding of mathematics (6); to apply the mathematics, especially in the real world (3); and to practise mathematics (2).

Data from the group of three students working on the specially designed technology-active tasks has yet to be fully analysed. While space prohibits reproducing the tasks here, the tasks were designed with two main purposes and goals in mind:

firstly, they were non-directed problems to be worked on as a group; and secondly to facilitate and encourage active use of technology. All three volunteers were clearly enthusiastic about the use of technology; all had their own CAS-calculators, and made frequent use of the computer while working on the tasks. They were all enthused about the online graphing package *Desmos* especially its free availability and ease of use. An interesting observation came in the second tutorial, where one of the students, after they had effectively answered the set questions, used the internet to explore the nature of their findings (fitting a polynomial curve through a number of points). One negative observation was that on several occasions one or more of the students became disengaged from the group to work individually on their calculators, although the computer acted more as a focal point for the group.

The examination results at the end of the course showed a pass rate of 74.6% with 23.1% A grades, which compared well with previous corresponding semesters when technology was not integrated, such as 2012 (76.6%, 21.0%) and 2011 (77.4%, 26.4%). Thus the students were not disadvantaged by the course changes in terms of results. The student course evaluation, completed by 50 students, confirmed satisfaction with the course, with 77.1% satisfied overall with the course quality.

DISCUSSION

The evidence from this first implementation of the technology-active undergraduate mathematics course supports the value of this kind of intervention. In particular, most of the students enjoyed the experience, especially the use of *MathXL* for revision and quizzes, and were highly engaged with the mathematics through the technology. Their confidence in using the technology and attitudes to technology use of all kinds, and, importantly, to learning mathematics through the technology, were all very positive. The examination results confirm that the effect on assessed learning was at worst neutral, with clear indications that the technology had both pragmatic and epistemic value (Artigue, 2002) in facilitating understanding. There were two factors that appear to have significantly enhanced student engagement, as suggested by Scucuglia (2006). One was the relative ease of instrumental genesis of some of the technology, especially the *Desmos* program. The second was the crucial role of lecturer example, privileging the use of the technology in learning (Kendal & Stacey, 2001). This was not only noted and commented on by students but seems to have led to a wider and increased level of participation in technology use. We have learned that the attitude scale used is robust and reliable, with a minor adjustment needed to one subscale. Other lessons include the need to increase student participation (especially in the surveys and collaborative tutorials), providing information and requesting volunteers early in the course, and scheduling interviews earlier too. The positive outcomes described here, and the lessons learned from this implementation, will be taken forward into a second cycle of the course in semester one 2014. Integration of an intensive, technology-active intervention in a large undergraduate mathematics class is relatively rare. This research has demonstrated that implementing such a programme is not only feasible and can be done smoothly, free of problems, but also that it has considerable potential benefits.

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EXPERT MATHEMATICIANS' STRATEGIES FOR COMPARING THE NUMERICAL VALUES OF FRACTIONS – EVIDENCE FROM EYE MOVEMENTS

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There has been a controversial debate if individuals solve fraction comparison tasks componentially by comparing the numerators and denominators, or holistically by considering the numerical magnitudes of both fractions. Recent research suggested that expert mathematicians predominantly use componential strategies for fraction pairs with common components and holistic strategies for pairs without common components. This study for the first time used eye movements to test if this method allows distinguishing strategy use on specific problem types in expert mathematicians. We found the expected fixation differences between numerators and denominators in problems with common components but not in problems without common components.

THEORETICAL FRAMEWORK

Numerous studies have shown that students at all age levels experience large difficulties with learning of and dealing with fractions (e.g., Vamvakoussi & Vosniadou, 2004). A typical mistake is to consider a fraction as two separate natural numbers (the numerator and the denominator) rather than as one rational number. Accordingly, students have been found to compare two fractions by comparing their components separately rather than by comparing the holistic fraction values. As a consequence, many students make typical mistakes when componential comparison is not in line with holistic comparison (Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013). For example, they believe that 1/4 is larger than 1/3 because 4 is larger than 3.

Further studies have suggested that adults also tend to base their comparison of fractions on the fractions' natural number components ("natural number bias"; Vamvakoussi, Van Dooren, & Verschaffel, 2012), and that even expert mathematicians do so in special cases of comparison problems, namely when the two fractions have the same numerator or the same denominator (Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013). In fact, in such cases, it can be an effective strategy to compare only the non-equal components of the fractions, rather than taking into account the fraction magnitudes.

These experiments with fraction comparison problems also contributed to the debate on whether individuals generally process fractions componentially by focusing on their components, or holistically by taking into account the fraction values. While Bonato, Fabbri, Umiltà, and Zorzi, (2007) suggested componential processing, Schneider and Siegler (2010) showed that holistic processing is also possible, and Meert, Grégoire,

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and Noël (2010) proposed a hybrid model of fraction processing including componential and holistic characteristics. As suggested by Obersteiner et al. (2013), the question whether a person applies a componential or holistic strategy might crucially depend on individual factors and specific task characteristics. In a computerized experiment involving expert mathematicians, these authors could show that response times on fraction comparison items depended on the numerical distance between the numerators when the denominators were equal (e.g., 16/21 vs. 20/21), and on the numerical distance between the denominators when the numerators were equal (e.g., 4/17 vs. 4/39). When the two fractions had no common components (e.g., 11/23 vs. 19/31), response times depended on the difference between the fraction values rather than on the differences between the components. These results led to the conclusion that expert mathematicians take into account the fraction values only when no easier strategy (comparing the components) is applicable.

The conclusions above were based on response time data that were recorded in a computerized experiment and averaged across participants. Although such a method has certain advantages, it is only an indirect measure of individual strategies. The reason is that it is not possible to control for all factors that might have influenced response times in addition to the numerical distances between fractions or fraction components, so that alternative explanations (e.g., specific task features) for the observed response time patterns cannot be completely ruled out. Also, response time data on the group level do not take into account that strategy use might vary largely between individuals. In fact, there is a large number of strategies that can be applied to fraction comparison problems, and there is empirical evidence that individuals indeed make use of a wide range of strategies (Clarke & Roche, 2009).

Eye Movements as a Method for Assessing Individual Strategies

Assessing individual strategies on cognitive problems is a methodological challenge. Individual reports have been used in previous research, but the reliability of this method can be questioned, in particular in younger participants (Robinson, 2001). Recently, recording eye movements has become more and more attractive to researchers to assess individual strategies on mathematical tasks. For example, Green, Lemaire, and Dufau (2007) could show that eye movements were a reliable measure of individual strategies in multi-digit addition problems; Sullivan, Juhasz, Slattery, and Barth (2011) successfully used eye movements to assess adults' strategies on positioning numbers on a number line; and Dewolf, Van Dooren, Hermens, and Verschaffel (2013) used eye movements to validate students' strategies on mathematical word problems. In these and many other studies, recording eye movement has been considered a promising tool for investigating individual strategies, because eye fixations and eye movements are assumed to correspond to mental operations (Grant & Spivey, 2003).

THE PRESENT STUDY – QUESTIONS AND HYPOTHESES

The aim of the present study was to test if recording eye movements could be a suitable method for assessing individual strategies on fraction comparison problems. Following the results of the above-mentioned study by Obersteiner et al. (2013), we addressed the questions if expert mathematicians indeed solve fraction comparison problems by comparing the numerators when the denominators are equal; by comparing the denominators when the numerators are equal; and by comparing the fraction magnitudes when the fractions do not have common components; and if these strategies could be measured through eye movements. We involved expert mathematicians in this study, because for these people it was possible to establish clear hypothesis concerning the strategies they would use for comparing fractions, based on previous studies. This would not have been the case for students who have been found to apply a variety of strategies, many of which are actually invalid (Clarke & Roche, 2009). As this was – to the best of our knowledge – the first time eye movement was used for assessing strategies in fraction comparison, the aim of our study was to show that this method was in principle suitable for this purpose.

We recorded individual fixation times of both eyes and hypothesized that fixation times would be longer for numerators than for denominators when the fractions have common denominators (Hypothesis 1), because participants would have to spend more time on comparing the numerical values of the (non-equal) numerators than to verify that the denominators are equal. For the analogue reason, we hypothesized that fixation times would be longer for denominators than for numerators when the fractions have common numerators (Hypothesis 2). Finally, we expected that fixation times would be equally long for denominators and numerators when the fractions do not have common components (Hypothesis 3), because the participants would need to take into account the numerical values of all numbers involved to determine the fraction magnitudes, and it is not sufficient to compare the components separately.

METHOD

Participants

There were eight participants in this study with high expertise in mathematics. Six of them were staff members of a German university who had an academic degree in mathematics, and two were students majoring in mathematics. The mean age of these eight participants (five female) was 26 years (*SD* = 3.9).

Design and Procedure

The participants sat in front of a computer screen, which was connected to a binocular remote contact free eye tracking device (SensoMotoric Instruments) with a sampling rate of 500 Hz. The eye tracking device was placed underneath the screen. The participants were asked to avoid head and body movements as far as possible. First, calibration was performed through fixations of nine small dots on the screen. After that, two practice trials were presented to make the participants familiar with the procedure. Then the experiment started, and two fractions at a time appeared next to each other. Participants were instructed to choose the larger fraction as fast and accurately as possible by saying aloud the word “left” or “right”. Their answers were noted down by a researcher who supervised the experiment. After each trial, a fixation cross appeared in the middle of the screen for two seconds.

All in all, there were 32 fraction comparison items, half of which had common components (eight pairs with common numerators, eight pairs with common denominators). To be consistent with the experiment conducted by Obersteiner et al. (2013), we presented the items with common components and the items without common components in two separate blocks. Within each block, items appeared in pseudo-randomized order.

RESULTS

Data from one participant had to be excluded from the analysis due to low calibration quality. To analyse fixation times on numerators and denominators, we defined rectangular-shaped same-sized areas of interest (AOI) that surrounded the numerators (AOI “Num”) or the denominators (AOI “Denom”) of both fractions (Figure 1). We then compared fixation times between these AOIs for each fraction type.

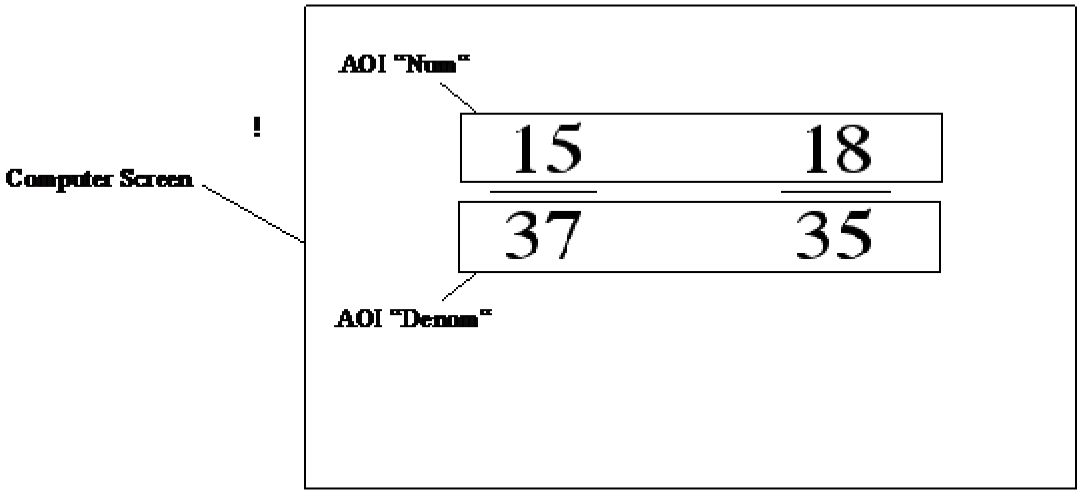

Figure 1: Sample item and areas of interest (AOI).

Table 1 displays the mean fixation times in ms for numerators and denominators for each fraction type. For the statistical comparison of fixation times between AOIs, we used a generalized estimating equation model that takes into account repeated measures within subjects.

	Numerators		Denominators	
Type	*M*	*SD*	*M*	*SD*
Common Denominators	359	264	185	189
Common Numerators	391	459	561	504
Without Common Components	1053	836	820	586

Table 1: Mean fixation times (in ms) for numerators and denominators, depending on comparison type. Note: *M* = Mean, *SD* = Standard deviation.

As expected, for items with common denominators, fixation times were significantly higher for numerators than denominators, Wald $\chi^2(1, N = 7) = 21.47$, $p < .001$, suggesting that participants paid more attention to the (unequal) numerators than to the (equal) denominators, which supports Hypothesis 1. For items with common numerators, fixation times were significantly higher for denominators than for numerators, Wald $\chi^2(1, N = 7) = 5.76$, $p = .016$, suggesting that participants focused more on the (unequal) denominators than on the (equal) numerators. This is in line with Hypothesis 2. For the items without common components, the difference between fixation times for numerators and denominators was not significant, Wald $\chi^2(1, N = 7) = 2.28$, $p = .131$, supporting Hypothesis 3.

These results are in line with our expectation that the participants in our study would apply componential comparison strategies to fraction comparison problems with common components, and holistic comparison strategies to comparison problems without common components. Indeed, fixation patterns as illustrated by heat maps (Figure 2) lend further support to this assumption. Figure 2 displays heat maps for three selected items. In heat maps, reddish colours indicate longer fixation times. For the items with common denominators (2a) or common numerators (2b), fixations were predominantly placed on the non-equal parts of the fractions. For the item without common components, (2c), the heat maps indicate that fixations were more equally distributed among the fractions' components, and they suggest that participants spent more time on comparing each fraction's numerator and denominator, hinting to a holistic approach, in which the numerical value of each fraction is determined through the numerical relation between numerator and denominator.

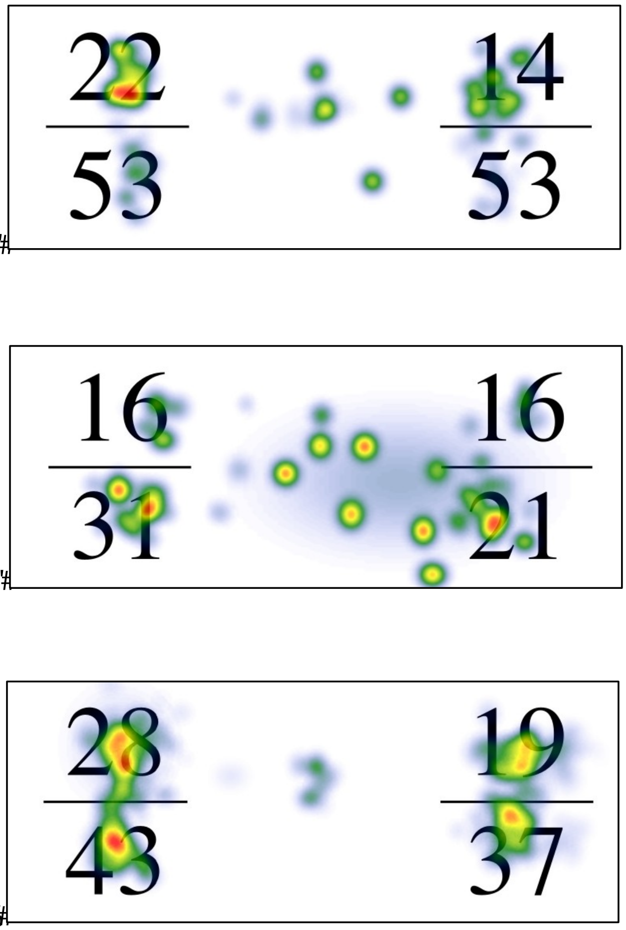

Figure 2: Heat maps for sample items with common denominators (a.), common numerators (b.) and without common components (c.) *Note*: Reddish colours indicate longer fixation times.

DISCUSSION

This study was the first to report eye movement data during fraction comparison. We involved adults with high expertise in mathematics so that we could establish clear hypotheses concerning their strategies on specific types of fraction comparison problems, as reported in the literature. We distinguished comparison problems with fraction pairs that have common numerators, common denominators, or no common components. The purpose of this study was to investigate whether the expected differences in strategy use between problem types could be assessed through eye movements.

In line with the results of the computerized experiment conducted by Obersteiner et al. (2013), the data suggest that the participants in our study focused on the non-equal components of the fractions when the two fractions had common components, but that they used a holistic approach when the two fractions did not have common components. This result helps understanding the controversial conclusions that have been drawn from studies about individuals' strategies in fraction comparison (e.g., Bonato et al., 2007; Meert et al., 2010; Obersteiner et al., 2013; Schneider & Siegler, 2010; Vamvakoussi et al., 2012; Van Hoof, Lijnen, et al., 2013; Van Hoof, Vandewalle, & Van Dooren, 2013). It supports the assumption that adults with high expertise in mathematics use different strategies for comparing the numerical values of two fractions, and that these strategies depend on the specific type of fraction comparison task at hand. When the fractions have common components, they prefer componential strategies; when the fractions do not have common components, they prefer holistic strategies. As the participants in our study were expert mathematicians, this conclusion might not generalize to other individuals such as primary and lower secondary school students. However, when studying performance on fraction comparison problems, one should always be aware that individuals could apply different strategies depending on the type of item.

Concerning the method of our study, we can conclude that recording eye movements is a promising tool to assess individual strategies. It might be used especially fruitfully with participants and on tasks for which self-reports are less reliable. More specifically, recording eye movements on fraction comparison could allow assessing the large variety of strategies that students have been reported to use on such problems (Clarke & Roche, 2009). The present study can be seen as a first step towards further investigations of eye movements on fraction problems in adults without mathematical expertise and – more importantly – in school students. Assessing these strategies can also be useful for identifying typical misconceptions about fractions that students might have. This could eventually lead to teaching approaches that are tailored to the individual needs of students.

A limitation of the present study is certainly the low sample size, which limits the generalizability of our findings. We are currently conducting a follow-up study with very similar items in a larger sample of students of mathematics to replicate the results presented here. Moreover, further analyses on the individual level could allow deeper insight into individuals’ fraction comparison strategies.

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THE DEVELOPMENT OF SOCIOPOLITICAL CONSCIOUSNESS BY MATHEMATICS: A CASE STUDY ON CRITICAL MATHEMATICS EDUCATION IN SOUTH KOREA

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This is a case study on critical mathematics education lessons in South Korea. This study explores the development of the social consciousness of students via critical mathematics lessons in which students use mathematics as a tool to analyse social issues and to justify their claims about social issues. However, student's rarely demonstrated much development of their agency for social change after taking part in such lessons. This phenomenon is interpreted in the light of the students' Korean sociocultural background and is explained as 'reserved agency.'

INTRODUCTION AND THEORETICAL BACKGROUND

Critical mathematics education (CME) is a compelling field of mathematics education research (Stinson & Wager, 2012). However, CME has primarily been studied among students of a Western sociocultural background; CME research conducted in countries with an Eastern cultural background (including South Korea) is difficult to find. The South Korean educational environment emphasizes social mobility via educational achievement, a phenomenon which is sometimes called 'education fever,' and constitutes the distinguishing feature of education in South Korea. Thus we expected to find some interesting outcomes from CME research conducted in a South Korean sociocultural context. Based on this assumption, I conducted a case study of CME lessons given in a Korean context. Frankenstein (1983), Gutstein (2003, 2006), and Turner (2003) have all conducted research on CME lessons. Their studies about CME are based on Freire's educational theory (1972), particularly with regard to his notions of conscientization, a problem-posing pedagogy, and generative themes. The lessons in this case study are also designed based upon Freire's theories (1972). The focus of this study is the development of the sociopolitical consciousness of students through CME lessons. The development of the sociopolitical consciousness of students is explained as a developing sense of agency (Gutstein, 2003). Gutstein (2006) separates the concept of a student's agency into 'using mathematics' and 'going beyond mathematics'. 'Using mathematics' means that, through mathematical analysis, students develop an understanding of and critical mind toward social issues. 'Going beyond mathematics' means that students have a positive perception about their sociocultural background, and realize that social change can be made by collective action. This study explores the development of social consciousness in students via CME lessons through interpretation of the agency of the students.

RESEARCH METHOD

The CME lessons which serve as the basis of this study were conducted at a middle school in South Korea where the researcher worked for two years as a teacher. The lessons were conducted for five days during the students' summer vacation. The research participants consisted of students of various levels of academic ability and from a variety of socioeconomic backgrounds. The lessons were designed and conducted by the researcher. Each lessons contained a generative theme related to social issues in South Korea. The themes of the lessons were 'the gap between the rich and the poor', 'the minimum wage and the minimum cost of living', 'lookism', and 'school violence'. Each theme was chose in hopes of garnering student interest. Of the previously mentioned themes, two lessons ('The gap between rich and poor' and 'The minimum wage and the minimum cost of living' lessons) were analysed for the purposes of this study. The tasks to be solved in the lessons contained two social issue contexts and mathematical problem solving. The students participated in small group activities and whole group discussion repeatedly throughout the lessons.

Audio recording data for each group and video recording data for the whole classroom were collected. Each student's worksheets, survey and interview data were also collected as well. All data was analysed using ground theory methods. Triangulation to obtain validity of research was carried out using various sources of data and participant checking. An audit trail and constant comparison method were used while conducting data analysis.

RESEARCH RESULT AND ANALYSIS

Identifying social conflict situations through numerical analysis

The task that the students engaged in the classroom was to compare the wealth held by the six Americans, who share 59% of the world's wealth with that held by the twenty people worldwide who share 2% of the world's wealth. The students obtained detailed values related to social conflict while they solved the task.

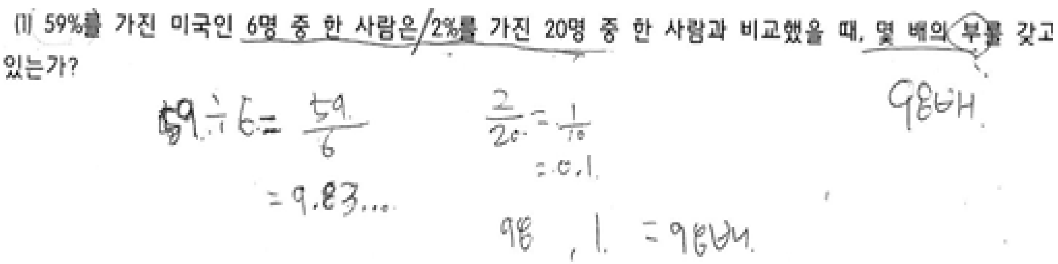

Figure 1: Student's written work

Students compared the value obtained by dividing 59% into 6, and the value obtained by dividing 2% into 20 (Figure 1). They noticed that the wealthy have 98 times more than the poor. After the specific value was obtained, the students talked in small groups, as transcribed in the conversation that follows (all names of this paper are pseudonym):

Su-bin: Oh, 90 times – that's crazy! They're earning 900,000 Won while I'm earning 10,000 Won. And they're earning 1,800,000 Won while I'm earning 20,000 Won.

Gun-yung: Wow. That's a really big difference.

Su-bin: Hey, this is about 100 times.

A-reum: Yeah, it's an amazing gap.

When students first undertook this task, they didn't show any emotional response to it. However, after they understood the social conflict situation through mathematical analysis, they spoke about their feelings. Students' comments like "90 times – that's crazy!", "Wow", and "It's an amazing gap" are evidence of how impressed they were by the information. Specific numbers like "about 100 times" help students to become aware of and understand more concretely the problem of the gap between the rich and the poor.

(3) 내가 속한 대륙의 사람들과 느낌을 공유해보자. 그리고 간단하게 기록해보자.

내가 속한 대륙 : 남미

사탕을 나눠받았을 때의 느낌 : 나쁘지 않았다. 그리고 아시아와 아프리카의 내전의 원인도 알겠다

Figure 2: Student's comments about intercontinental resource (wealth) distribution

Some students applied the understanding of social issues that they acquired through the CME lessons to a broader context. While students solved the task related to intercontinental resource (wealth) distribution, one student remarked, "It helped me to understand the causes of the civil wars in Asia and Africa" (Figure 2). This concrete awareness that arose from mathematical analysis helped this student expand his thinking about social issues.

Justifying assertions using mathematical investigation

Students tried to justify their assertions using mathematical investigation. One student analysed the annual data about the minimum wage and the minimum cost of living by proportional thinking (Figure 3).

연도 (1월기준)	2001	2003	2005	2007
최저임금	1,865원	2,275원	2,840원	3,480원
최저생계비	956,250원	1,019411원	1,136,332원	1,205,535원
	1 : 512	1 : 448.	1 : 400.	1 : 346.

마침내 수정이는 2015년도의 최저임금을 결정할 수 있었다. 수정 적고, 그 이유를 설명하시오.

Figure 3: Ye-jun's worksheet

This analysis was conducted to account for the rate of increase of the minimum wage by showing it as a ration of the minimum wage and the minimum cost of living. Because the ratio gradually reduced, (1:512, 1:448, 1:400 and 1:346), though, the student's analysis couldn't be used to account for the change. However, his use of data was meaningful because it emerged from his own thought. After a while, he changed the focus of analysis from the ratio of the minimum wage and the minimum cost of living to the rate of increase of the two components.

Ye-jun: Look! In 2000, the minimum cost of living was 928,398 Won, and now, it is 1,546,000 Won. The minimum cost of living has increased 1.7 times, but the minimum wage has only increased 1.08 times. So the minimum wage should be higher.

He said that the rate of increase of the minimum cost of living and the minimum wage are 1.7 and 1.08 each, so he claimed that the minimum wage should be increased in proportion to the minimum cost of living. This demonstrates how he used mathematics to justify his assertions with regard to a social conflict issue, i.e. his 'using mathematics. (Gutstein, 2006).

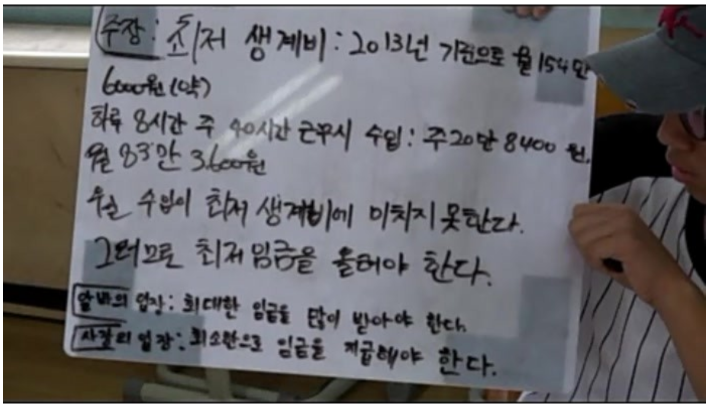

Figure 4: Student's assertion regarding minimum wage

Another student analysed one's total monthly income when one works for minimum wage (Figure 4). He found that the monthly income of a labourer who receives minimum wage is 833,600 Won (about 800 U. S. dollars) in 2013. He thus showed that this level of income was lower than the minimum cost of living and asserted that the minimum wage should be increased. It is another example of a student using mathematics to justify his own claim.

Sense of agency for social change

One's sense of agency is a person's reaction to a social conflict situation as a member of society. The agency for social change of students can be seen in their 'going beyond mathematics' (Gutstein, 2006). The previous examples show that mathematical analysis was actively occurring in the classroom. However, the students didn't take an active stance with regard to the individual's role in social change.

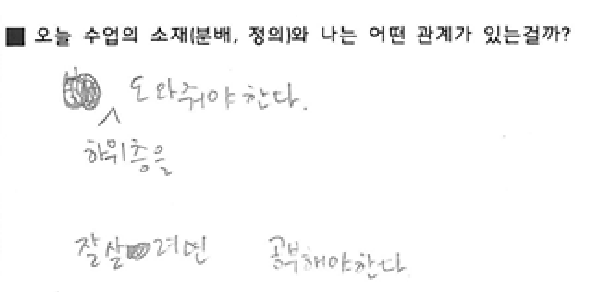

Figure 5: Student worksheet #1

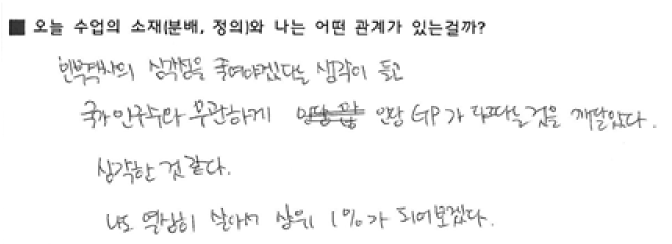

Figure 6: Students worksheet #2

In the first student response (Figure 5), the student asserts that, “We should help the lower class. In order to have a good life, I have to study.” These sentences present the student’s awareness of the gap between the rich and the poor, but it doesn’t mention any concrete actions one can take to reduce that gap. The only action the student mentions is that “I have to study” to achieve social success. The second student response (Figure 6) repeats a similar sentiment. Although the student mentioned “The severity of the gap between the rich and the poor,” the student also said, “I will live in the top 1%, so I have to work hard.” In short, the student’s perceptions about the social conflict situation and their vision for their role in bringing about social change are not consistent. This is quiet different from the results of previous studies by Gutstein(2003, 2006) and Turner(2003) which showed growth in the critical agency of the students involved. In the case at hand, the students didn’t mention any actions that they could take to bring about social change, but instead focused on their own personal success. However, interviews with students showed that students are not just selfish. But that they already understand the prerequisites for social change.

Teacher:	Why do you say that nothing is going to change?
Ye-won:	Umm... I have seen many social movements, but such movements failed to reach a critical mass and bring about real social change… Still, many people have no interest in social change, they just want to become upper class. Some people’s struggle for social change will fail. Now, only people with wealth and power can change our society. Otherwise, every endeavour for social change will fail.
Teacher:	So, you don’t think that we can solve such socioeconomic problems?
Ye-won:	I think it’s possible to make a movement of a small number of people. But a small number of people has no power.
Yoon-seo:	Our government doesn’t do anything for (poor) people.
Ye-won:	We have a government that is only interested in money…
Yoon-seo:	I think the gap between the rich and the poor is a serious problem in our society. Our government frequently talks about welfare policies and how to fix this problem, but they do nothing.

The students saw people’s participation in social movements and support of polices to help unprivileged groups as preconditions for social change. However, in their view, the Korean socio-political situation, so the students couldn’t see a way to bring about social change via collective social action. In response to this reality, the students chose a more individualistic way. They decided to "study hard." If we take a superficial view of this notion, such a choice seems far from affecting social change. This differs from previous studies where the participating students wanted to act to bring about social change, a sentiment which appears to be more proactive on the surfaces. The results of CME methods are closely connected to the social environment in which the research is conducted. Therefore, an attempt to interpret the students' ideas as to how to affect social change based upon their sociocultural background is reasonable and consistent with the purpose of this study. Thus the interpretation of the students' responses with regard to their Korean sociocultural background is as follows.

First, we should consider several features of the study's participants. The participating students chosen in Gutstein(2003, 2006) and Turner's (2003) studies were typically social minorities. Because the identity of each group was largely homogenous, the participants expressed a common stance toward social conflict situations. However, the participants in the present study were students of various socioeconomic level and members of the ethnic majority of their country. Accordingly, the participants did not have a common position on social problems, and their awareness of social issues varied by student. Therefore the students tended to choose the more individual way of "studying hard" rather than collective means of seeking social change.

Second, we should consider the sociocultural context in Korea. In Korea, educational achievement is emphasized as a means of social mobility (Lee, 2006). Students are pressured to to enter a prestigious university, get a well-paying, respectable job, and work for the advancement of their family. Therefore, students are encouraged to focus on trying to get higher grades rather than participate in social movements. Meanwhile, the participation of secondary school students in social movements is considered unnecessary, and most parents of such students would not approve of such pursuits. Based on this background, the students that participated in this study thought that they couldn't act for social change because they are students. Students regard taking action to affect social change as the exclusive property of adults. Consequently, the range of their imagination as to their role in affecting social change was limited to simply "studying hard".

As previously noted, students understand the prerequisite conditions for social change and believe in the need for social change. However, the possible avenues for social change in which they can participate is restricted by their sociocultural background. To their minds, the students' agency can be manifested after they acquire educational achievements and social mobility. Although their sense of agency is already developed, its manifestation is delayed. I have termed this kind of later-appearing agency 'reserved agency.' Figure 7 below provides an explanation of reserved agency.

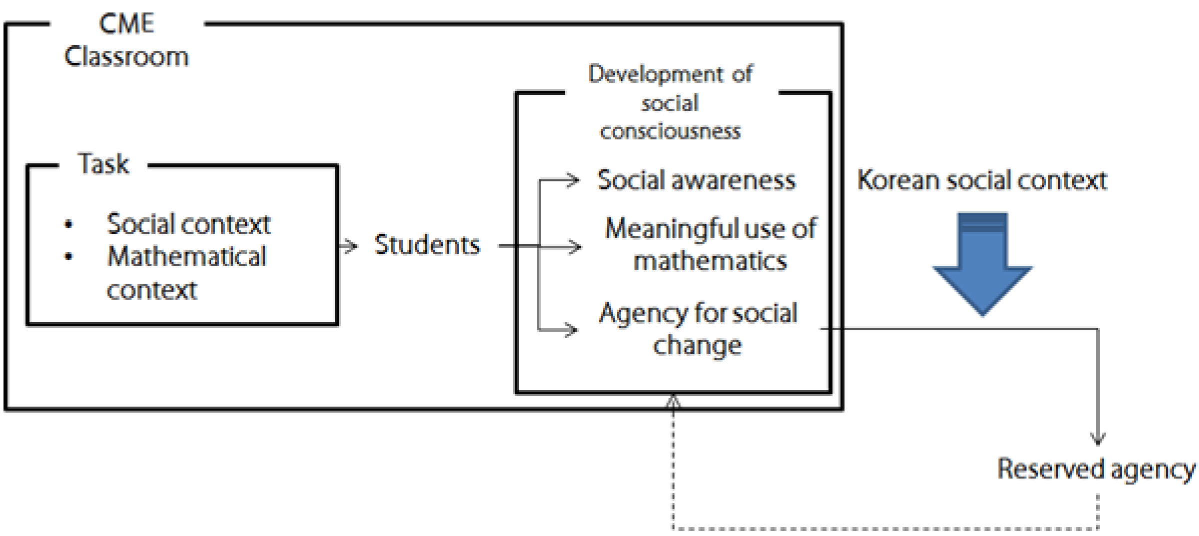

Figure 7: Emergence of reserved agency

We can conceive of reserved agency as a result of the conflict between the student's agency for social change that emerged from CME lessons and sociocultural restrictions. However, reserved agency partially affects the student's agency for social change at the present point because reserved agency gradually induces changes in the student's perspective on social issues. One student stated how her own view on labour strikes changed. The following interview was conducted 4 months after the CME lessons.

Teacher:	Have you experienced any change in your attitude towards labourers after the CME lessons? For example, with regard to labour strikes?
A_reum:	I think I feel bad for them now.
Teacher:	How did you used to feel?
A_reum:	I never gave them much thought before. After [the CME] lessons… I understand the social situation and I know why they go on strike, so I understand their position. I listen carefully to what they say.

The dotted line from 'reserved agency' to 'agency for social change' in Figure 7 means the affection of reserved agency. Thus, we can regard a change of view as an evidence of the existence of reserved agency.

CONCLUSION

This study sought to examine the effect of CME lessons in Korea. In particular, this study focused on the development of social consciousness among the study's middle school student participants. This study sought to explain why the student's sense of agency for social change is limited and reserved sociocultural restrictions. This phenomenon, termed 'reserved agency,' is a consequence of the conflict between the students' development of agency and the restrictions of their sociocultural background. However, reserved agency subtly induces the development of the students' sense of agency at the present point as well. This study show not only how CME can induce the development of social consciousness of students in the Korean sociocultural context, but also how the Korean sociocultural background of such students influences their experience of CME lessons. This implies that practice and interpretation based on social context is important for the implementation of CME. Considering the methodological limitations of this study, the conclusions of this study may not be easily generalized to other contexts. However, this study can contribute positively to a greater understanding of how CME might be used in Korea by suggesting the phenomenon of reserved agency.

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EXAMINING THE COHERENCE OF MATHEMATICS LESSONS FROM A NARRATIVE PLOT PERSPECTIVE

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This paper aims to clarify how coherence of 'structured problem solving' mathematics lessons can be produced by comparing the lessons of three teachers from a narrative perspective. Results of our analysis showed three main coherence characteristics of lesson teaching sequences: (1) sequence scenes are recursively developed based on the previous scenes, (2) there is a scene of setting a learning goal in terms of the conflict between what students know and what they do not know, and (3) coherent plots are grounded in certain mathematical content knowledge. We conclude by introducing a metaphor of living theatre to better understand the coherence of lesson structure.

INTRODUCTION

This paper aims to clarify the coherent qualities of mathematics lessons commonly referred to as "structured problem solving" (Stigler and Hiebert, 1999) as conducted by effective teachers. Since the TIMSS video study, Lesson Study has drawn global attention as a means for improving the quality of mathematics lessons and teachers' knowledge for teaching. Stigler et al. (1999) identified a pattern, or script, in effectively taught mathematics lessons in Japan: reviewing the previous lesson, presenting the problem for the day, students working individually or in groups, discussing solution methods, and highlighting and summarizing the main point. This script has been historically developed by Japanese teachers for cultivating students' mathematical thinking abilities and attitudes as well as their knowledge and skills.

However, we should not directly equate the above teaching pattern with an effective mathematics lesson, because there is a range of teacher efficacy from effective to ineffective, and a range of lesson success from successful to unsuccessful, even if the pattern is indeed adopted by most of the primary school teachers in Japan. Namely, for teacher development it is not effective to simply use this pattern. Rather, it is important to know how lesson coherence can be produced. We believe that there is a substantial difference in lesson quality depending on whether a lesson is developed like a narrative or in isolated steps. It has been reported that Japanese mathematics lessons can be characterized as coherent accounts of a sequence of events and activities that comprise the classes, as if they were a story or drama (Stigler and Perry, 1988; Shimizu, 2009).

We believe it is essential to explore how such coherent accounts are created for studies of teachers' knowledge. Ball et al. (2008) provide a framework for mathematical knowledge for teaching (MKT) that elaborates on subject matter knowledge and pedagogical content knowledge. They mention several research tasks in situating such knowledge in the context of its use, such as how different categories of knowledge come into play over the course of teaching. On this point, Silverman and Thompson's (2008) MKT framework based on 'key developmental understanding (KDU)' as "a conceptual advance that is important to the development of a concept" (Simon, 2006, p. 363) seems useful for planning lessons. However, it remains unclear what processes of a lesson a teacher can practically realize using such knowledge, particularly to produce coherence in teaching.

THEORETICAL BACKGROUND

Several researchers have noted that children's learning is narrative in nature. Dewey (1915, p. 141), for instance, stated, "(Children's) interest is of a personal rather than of an objective or intellectual sort. Its intellectual counterpart is the story-form…Their minds seek wholes, varied through episode, enlivened with action and defined in salient features—there must be go, movement, the sense of use and operation—inspection of things separated from the idea by which they are carried. Analysis of isolated detail of form and structure neither appeals nor satisfies." This suggests that even if we collect all of the parts that constitute a lesson structure, it will not attract the attention of children unless it is in a story-form. Mathematics education studies have also seen effective lessons as being in story-form. Krummheuer (2000) understood classroom situations as "*processes of interaction*: students and teachers contribute to according to their sense and purpose of these events" (p. 22) in terms of classroom culture; this view was influenced by Bruner's (1990) view of narrative as having the following characteristics: sequentiality, a factual indifference between the real and the imaginary, a unique way of managing departures from the canonical, and a dramatic quality. Zazkis and Lilijedahl (2009) tried to shape mathematics learning as storytelling to enhance students' interest in, and to engage them with, mathematical activities. They listed the following general elements of good stories: plot, beginning, conflict and resolution, imaginary elements, human meaning, wonder, and humour.

We consider the concept of plot as being crucial to analyzing a quality lesson. Krummheuer stated that "a plot characterizes the sequence of action in its totality: it describes something that is already fixed… But an unfolding plot connotes something fragile, not yet entirely executed, still changeable. Both aspects are essential and the tension between these two dimensions of this concept is crucial for its adaptation for classroom interaction and its function for learning" (p.25). It seems that there are two of these aspects that correspond to the planning and the practicing of a lesson, respectively. We consider the following script, identified as a Japanese lesson structure (Stigler and Hiebert, 1999), as a way of adding the role of plot to a narrative structure.

- Reviewing the previous lesson;
- presenting the problem for the day;
- students working individually or in groups;
- discussing solution methods; and
- highlighting and summarizing the main point.

However, as we stated above, even if teachers use this pattern in their teaching, there is a range of possible lesson evaluations from very good to very poor. Shimizu (2009) suggested that lessons conducted by effective teachers can be compared to stories or dramas. A coherent account of a lesson can be explained as a well-formed story which "consists of a protagonist, a set of goals, and a sequence of events that are causally related to each other and to the eventual realization of the protagonist's goals. An ill-formed story, by contrast, consists of a simple list of events strung together by phrases such as "and then…", but with no explicit reference to the relations among events" (Stigler and Perry, 1988, p. 215; cf. Shimizu, 2009). Thus, it is important to examine how a coherent plot in teaching can be produced during a lesson. In addition, it is important to remember that the protagonists are the students, and that thus their ideas and feelings are central components of the story, and that a teacher may assume that as many students as possible will play active roles. On the contrary, the lesson may not be effective if the only active persons are a teacher and just a few capable students.

METHODOLOGY

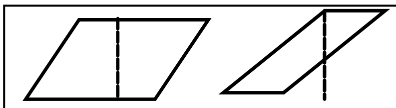


Figure 1

We asked six teachers to conduct lessons: A) 2 experienced teachers who specialize in mathematics teaching, B) 2 experienced teachers who do not specialize in mathematics teaching, and C) 2 teachers who have a few years' experience. We selected the content 'area of a parallelogram for which the height cannot be known from a straight line on its inside' from a fifth grade mathematics textbook (Fig.1, right). We assumed that children would have difficulty in the height, and that differences in teaching among the teachers would appear when dealing with this difficulty.

During a preliminary meeting with each teacher, we introduced multiple methods for finding the area of the parallelogram. Then, we asked him/her to conduct their lesson to help students find multiple solutions to the problem and to understand the concept of area beyond simply understanding how to solve the problem. We also interviewed each teacher to better understand what he/she valued most in designing and practicing his/her daily lessons.

The lessons were recorded with video cameras and field notes. We made transcripts of the video data. In our data analysis, we first extracted all meaningful interactions to examine whether a teacher's questioning or instruction evoked student responses, and how he/she subsequently responded to the students. Next, we conceptualized each interaction unit in terms of the teacher's intention and the interaction's practical effects, before trying to reconstruct the entire picture of the lesson structure, that is, the 'plot', by examining how the interactions were connected to each other. Finally, we compared the reconstructed lesson structures of the six teachers' lessons and tried to clarify the characteristics which comprised the creation of a coherent plot.

Below, we present the results of our analysis of the lessons conducted by three of the teachers: Mr F (the above type A, 35 years of experience), Ms Y (type B, 18 years of experience), and Mr S (type C, 3 years of experience).

RESULT 1: THE CASE OF MR F'S LESSON

First scene: Reviewing the formula for the area of a parallelogram

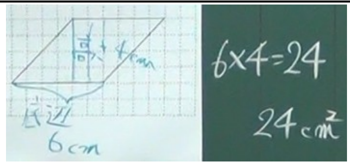


Figure 2

Mr F began by reviewing the formula for finding the area of two parallelograms (base 6, height 4; base 3, height 1) (Fig. 2). Here, the interactions between Mr F and the students showed a pattern. First, Mr F asked the value of the area and the students answered 24 by counting the unit squares or by using the area formula. Next, Mr F asked what formula they used, and they answered 6×4. Moreover, Mr F asked what 6 and 4 referred to in the figure, and one student indicated the base and height locations by tracing along the figure with her finger on the blackboard. In particular, Mr F made her check the vertical relationship between the base and height and trace the height of the shape in several places. The pattern of interaction here was: answer formula meaning of the values in the formula arbitrary places of height of the shape. We found that this pattern of interaction was also used in the case of the 3×1 formula.

Second scene: Setting a problem through an experience of the conflict

Mr F presented a problem as follows.

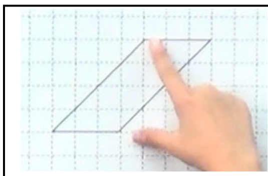


Figure 3

Mr F: I have one issue with this. I am bothered by this parallelogram. Do you understand my trouble?

Student 1: The previous parallelograms had this line. This time, we can't draw this (line) (Fig. 3).

Mr F: I tried to find the height, but there's nothing there! Oh, there's no height!

Students: But, but… (Several students raised their hands to respond.)

Mr F: But, does the parallelogram have an area?

Students: Yes, it has an area.

Mr F: Yes, it does. This is a parallelogram. But we can't use the area formula because we don't know the height. Don't you feel like crying?

The problem setting was like the beginning of a narrative in which the students were involved in an issue troubling Mr F, where the two circumstances ('there is no height' and 'the area formula can't be used')were given as the problematic aspects of the issue. We note that the problem was set based on the preparation conducted in the first scene.

Third scene: Setting a goal by comparing between the known and the unknown

Mr F next proposed setting a learning goal for the students. One student said, "Let's find the height", but the task at hand was not simply to find the height of the shape. Mr F then tried to direct the students' interest to figuring out a formula to find the area of a parallelogram of unknown height based on the student's statement.

Mr F: Oh, yes. The height of this is dubious. If you know the height, then…

Students: We can know the formula!

Moreover, he clarified the task by aligning three parallelograms and confirming that the formula could now be used only for 6×4 and 3×1 parallelograms (Fig. 4). Then, the students were able to set a goal: to find the area of a parallelogram of unknown height using a formula. We note that this aligning of the three parallelograms implicitly prepared the students with insight for two ideas to solve the problem by seeing it as half of a 6×4 parallelogram and as four 3×1 ones.

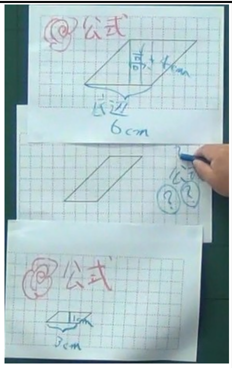


Figure 4

Fourth scene: Individual activities and redefining the goal

The students individually tried to solve the problem. However, Mr F found that some students just wrote the formula 3×4=12 procedurally (Fig. 5), which was different from the set goal of understanding the situation based on the known parallelograms using the area formula. Mr F then stopped these students and restated the task for all the students again.

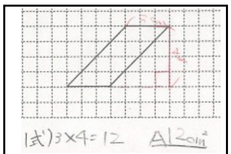


Figure 5

Mr F: Some of you may be thinking of this as the height. As it is now, we don't know whether this is the height or not, because it doesn't meet the base. So, you can't set this as the height (Fig. 6). Consider using the formulas you already know.

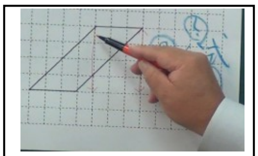


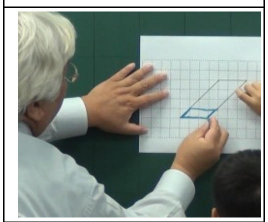
Figure 6

Mr F's redefining of the task in this way seemed to work successfully because all the students then started considering the problem using the known parallelograms. We found a total of 13 distinguishable solutions in the students' notebooks.

Fifth scene: Class discussion (1): Sharing the fundamental idea

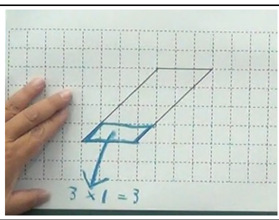
We found that Mr F employed one particular type of interaction in which he tried to deepen one basic idea by using plural voices during the class discussion. First, Mr F invited the students who had come up with the idea of using four 3×1 parallelograms to present their idea to the class. Mr F's writing on the blackboard gradually became more detailed as he interacted with the different students. We characterize this series of interactions as multi-layered.

S2: Here is 1, 2, 3 and 4.
Mr F: What is here? (He circled the bottom one.)



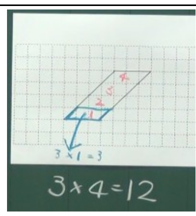
⇒

S3: The small parallelogram is 3×1=3. As there are 4, the answer is 12.
Mr F: Can anybody else explain in the same way? (He wrote the formula.)



⇒

S4: The bottom one is 3. There are 4 parallelograms, 3 times 4 is 12. (He wrote the numbers.)



⇒

Mr F: What is the case of one step?
Ss: Three.
Mr F: What about for two steps?
Ss: Six. …

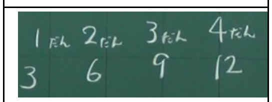
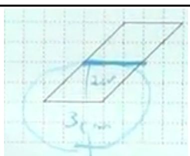


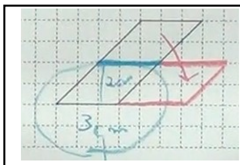
Figure 7

Sixth scene: Class discussion (2): Sharing various ideas

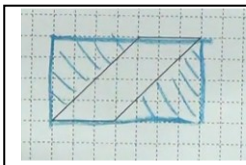
Mr F then invited the students to share their other ideas, and the following four ideas (Fig. 8.1-8.4) were presented.



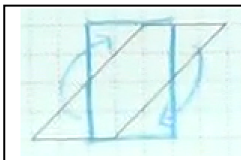
3×2×2
Figure 8.1



6×2
Figure 8.2



4×7-(4×4÷2×2)
Figure 8.3



4×3
Figure 8.4

Here, how to transform the parallelogram into known figures and the relevant formulas were confirmed. Then, Mr F classified these ideas into two categories, ‘parallelogram based’ (Figures 8.1, 8.2) and ‘rectangle based’ (Figures 8.3, 8.4).

Seventh scene: Class discussion (3): Rethinking the goal

Mr F then proposed a rethinking of the main goal, and asked the students again what the height of the shape was. The students answered that it was 4, but they were not confident about their answer. Here, Mr F told them to reflect on the idea shown in Figure 7, saying together with the students, “The height of the smallest one is 1 cm, the height of the parallelogram one step higher is 2 cm…” while circling each parallelogram as they spoke (Fig. 9). Moreover, he modified the table by changing the word ‘step’ to ‘cm’ and newly adding cm2, indicating the area of each smaller shape (Fig. 10). As a consequence, the students could reinterpret one ‘step’ as 1 cm of height and then understand that the area formula that they already knew was actually applicable to all parallelograms.

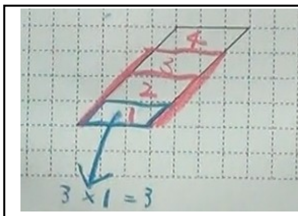


Figure 9

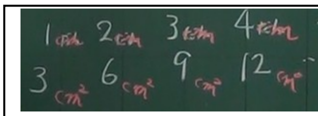


Figure 10

The lesson ended by applying the formula to other figures and summarizing the main learning points of the lesson. This was the final, eighth scene of the lesson.

RESULT 2: THE CASE OF MS Y’S LESSON

Ms Y’s lesson followed the 5 steps of the typical Japanese teaching ‘pattern’ as identified by Stigler et al. (1999). However, a crucial difference from the class conducted by Mr F was that the students did not experience any conflict and did not share a common, explicitly stated learning goal. Ms Y simply presented the problem of ‘finding the area of a parallelogram with base BC’ (Fig. 11). She tried to prevent the students from considering side CD as the base, but this resulted in the following interactions.

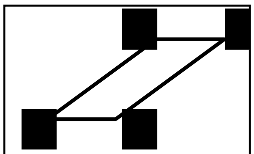


Figure 11

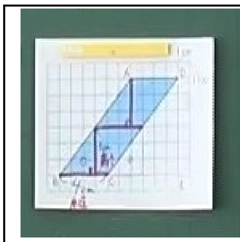


Figure 12

Student 5: I cut it horizontally and made it into two parallelograms. The formula is thus 2 of 4×3. It is 12×2, so the answer is 24 (Fig. 12).

Student 6: But, in the upper parallelogram the middle line changes to become the base.

Student 7: Yes, student 5 is wrong, because we must set BC as the base.

In fact, a similar series of interactions occurred twice. Ms Y did not try to redefine the goal, as was seen in Mr F's lesson. The lesson thus progressed in a disconnected way with respect to Ms Y's original intention and the students' actual thinking processes.

RESULT 3: THE CASE OF MR S'S LESSON

Mr S's lesson also followed the previously mentioned five teaching steps. However, we found two main differences in comparison with Mr F's lesson. First, the units of interaction often never exceeded one return consisting of the teacher's questioning, a student's response, and the teacher's approval. Additionally, one interaction unit was often not connected meaningfully with another. Indeed, Mr S often used the expression "and then" when shifting between scenes.

The second difference was related to subject matter knowledge regarding height. In Mr F's case, the height was reconstructed by reflecting on how many parallelograms of 1 cm height were stacked up together. On the other hand, in Mr S's lesson, the height was summarized as the length of the segment which lies at a right angle to the base, similar to the length of the pillar of a house. We believe that these differences had substantial effects on the students' ability to understand the height of the parallelogram; indeed, some of the students in Mr S's class asked him "So, in the end, what is the height in this case?" at the last scene of the lesson when the main points were summarized.

DISCUSSION

To discuss how lesson coherence can be produced, here we take Mr F's lesson as an exemplary case and compare it with those of Ms Y and Mr S. While all three lessons went through the five steps identified previously as the Japanese pattern, we observed that the eight scenes comprising Mr F's lesson formed a coherent plot: 1) Reviewing the formula for the area of a parallelogram; 2) Setting a problem through an experience of the conflict; 3) Setting a goal by comparing what is known and what is unknown; 4) individual activities and redefining the goal; 5) Sharing the fundamental idea (Class discussion); 6) Sharing various ideas (Class discussion); 7) Rethinking a solution for the goal (Class discussion); and 8) Applying the formula to other problems and summarizing the main point(s).

One characteristic of Mr F's class was that one scene was recursively developed based on the previous scenes. For example, setting up a problem in the second scene was based on the preparations performed in the first scene; similarly, setting a goal in the third scene was conducted by comparing the problem (unknown) in the second scene with the known parallelograms and the formulas discussed in the first scene. Thus, we believe that this recursive characteristic represents a crucial aspect of coherence.

A second characteristic consisted of the students' experiences of the conflict between what was known and what was unknown, including the goal-setting activity for coping with the conflict and the final attainment of the goal. These all combined to make the lesson into a coherent story. Without such goal-setting, Mr Y's lesson would not have been a well-formed, coherent story, and as a result the students may have tried to refute the correct method of finding the area of the parallelogram.

A third characteristic is that MKT based on KDU (Silverman and Thompson, 2008) increased coherence because the lesson was developed around the idea of how many parallelograms of 1 cm in height would need to be stacked. It seems that Mr F had understood beforehand that the idea would help lead the students to understand the formula for finding the area of parallelograms. This contrasts with Mr S's teaching, in which the height was summarized as just the length of a segment in his class.

Lastly, to focus on developing a sense of lesson coherence, we propose the term "living theatre" as a more appropriate metaphor. From this perspective, we can interpret the actions of Mr F to get as many students involved in the lesson as possible as his way of constructing a living theatre, with the students as the main actors (or role-players) on the classroom 'stage'. In addition, the teacher is also one of the main characters in this theatre; in this case Mr F began the lesson with a story describing his problem in trying to find the height of the parallelogram. We believe that such a spirit is the very nature of successful efforts to construct coherence in mathematics lessons.

Acknowledgement: This work was supported by JSPS KAKENHI Grant Number 23501019.

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TEACHER METAPHORS – DIFFERENCES BETWEEN FINNISH IN-SERVICE AND PRE-SERVICE MATHEMATICS TEACHERS

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This article reports what kind of metaphors do Finnish pre-service (n=72) and in-service (n=65) mathematics teachers use for teacher's role, how do these metaphors differ and do in-service teachers metaphors differ due teaching experience. Data was gathered via questionnaires in years 2010-2013. Metaphors provide insights into beliefs that are not explicit or consciously held and show teacher's beliefs about themselves. Changing teachers' beliefs can help to change teachers' behaviours and in such way improve teaching and learning process. Metaphors were classified into five categories. The most common metaphor used by pre-service teachers was self-referential 46% (n=33). In-service teachers used by far didactical metaphors (51%, n=33) and only 15% (n=10) presented a self-referential metaphor.

THEORETICAL BACKGROUND

Metaphors are not just words or expressions. They enable people to understand one phenomenon by comparing it to something else. Metaphors are also a valid tool for gaining insights into teachers' thoughts and feelings regarding their teaching (Zhao, 2009). According to Kasten (1997) metaphors would seem to have an important place in the provision of explanation. Metaphors capture and model teachers' understanding of teaching and learning and provide insights into beliefs that are not explicit or consciously held (Beijaard, Verloop and Vermunt, 2000).

The potential power of metaphors as a "master switch" to change teachers' beliefs was realized in 1990, when Tobin investigated how the use of metaphors helped teachers to conceptualize teaching roles. He found the possibility that significant changes in classroom practice are possible if teachers are assisted to understand their teaching roles in terms of new metaphors. When the teacher's role changes also the metaphor describing it changes. Reflection is assumed to play a key role in change of practice. Many researchers see a cyclical relationship between changing beliefs and changing practices. It is therefore important to study how pre-service and in-service mathematics teachers describe their views of mathematics teacher's role with metaphors and do these metaphors differ. (Kagan, 1992; Lerman, 2002; Wilson & Cooney, 2002)

Mathematics teachers' beliefs

Teachers' beliefs about mathematics, its learning and teaching are reflected strongly in their practice. Beliefs affect on what gets taught in the mathematics classroom and how. Pehkonen and Törner (1998) summarized that an individual's mathematical beliefs are compound of his subjective, experience-based, implicit knowledge on

2014. In Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 4, pp. 361-368. Vancouver, Canada: PME.

mathematics and its teaching and learning. The spectrum of an individual's beliefs is very large, and its components influence each other.

We base our construction of beliefs and referring terminology on the article of Op't Eynde, De Corte, and Verschaffel (2002), who have strived for making a synthesis regarding previous belief researches. In the paper Op't Eynde and others (2002) define mathematical beliefs to be implicitly or explicitly held subjective conceptions people hold to be true, that influence their mathematical learning and problem solving.

Exploring mathematics teachers' beliefs with metaphors

The Beijaard, Verloop, and Vermunt's (2000) model of teacher identity identifies three distinct knowledge bases of teacher knowledge. Teachers' professional identity can be described in terms of *teacher as a subject matter expert, teacher as a pedagogical expert,* and *teacher as a didactical expert*.

Löfström, Anspal, Hannula and Poom-Valickis (2010) studied what metaphors first, third and fifth year university students' in Estonia used and how much agreement there was between metaphors and the scores on the teacher identity measure by Beijaard model. The results indicate that the model by Beijaard and colleagues can be applied as an analytical frame of reference when examining metaphors, but that it would be useful to develop and expand the model further to include metaphors categorized as *self-referential* and *contextual metaphors*.

Oksanen and Hannula (2012) used the new Löfström et al. (2010) model of teacher identity to classify Finnish 7-9 grade mathematics teachers' (n=70) metaphors about teacher. According to these results the new model makes the metaphor classification more clear. Only 2 metaphors (3%) were not classified into any category. The most common metaphor used by in-service teachers, was by far teacher as *didactics expert* (n=33, 51%).

Portaankorva-Koivisto (2012) studied prospective mathematics teachers' metaphors (n=16) for mathematics, teaching and the teachers' role. She found out, that 44% of the pre-service teachers used *self-referential metaphors*. This indicated that further study and comparison to in-service teachers' metaphors was needed.

Metaphor categories

In this study we use the Löfström et al. (2010) model to analyse teachers' metaphors for their profession:

Teacher as subject expert. Teacher has a profound knowledge base in his subject(s). Teaching is concerned with getting across information to the students. Typical metaphors in the subject expert category describe the teacher as a source of knowledge. For example: a book, a radio, a computer.

Teacher as didactics expert. Teachers need knowledge about how to teach specific subject-related content so that pupils can capitalize their learning. This kind of knowledge is referred as knowledge of didactics, and is integrated with an understanding of how learning experiences are facilitated in a particular subject. The teacher is described as a person who is responsible for designing her pupils learning process. For example: a coach, an engine, a lighthouse.

Teacher as pedagogical expert. The understanding of human thought, behavior, and communication are essential elements in the teacher's pedagogical knowledge base. Emphasis is on relationships, values, and the moral and emotional aspects of development. The teacher is seen as someone who supports the child's development as a human being. These metaphors stress teacher's role to raise or educate the child. For example: a mother, a second father, an older brother, a firm tree.

Self-referential metaphors. These metaphors describe features or characteristics of the teacher's personality, with reference to the teacher's characteristics (self-referential) without reference to the role or task of the teacher. One might say that the metaphors describe who the teacher is. For example: a machine, a candle.

Contextual metaphors. These metaphors describe features or characteristics of the teacher's work or work environment, or in other ways referred to characteristics of the environment (contextual). One might say that the metaphors described where (physically, socially and organizationally) or in what kind of setting or environment the teacher works. These metaphors mostly described teachers' work as too demanding, multifunctional, including too many responsibilities (pupils, parents, colleagues, heads and society). For example: a king or an actor.

Hybrids. These metaphors include elements of more than just one of the above categories.

Unidentified. Unidentified metaphors could not be categorized in any of the categories presented above.

RESEARCH QUESTIONS

- What kind of metaphors do in-service mathematics teachers and pre-service mathematics teachers use for teacher's role?
- How do pre-service teachers' metaphors differ from in-service teachers' metaphors?
- How do in-service teachers' metaphors vary across the length of their teaching experience?

METHODOLOGY

Instrument and procedure

Pre-service teachers. In Finland, secondary teacher education is a 5-year programme (3 BA and 2 MA, 300 ECTS). The students major in one school subject and minor in one or two others. Prospective mathematics teachers have pedagogical studies (60 ECTS) as their minor subject and these studies can be taken within one academic year.

Pedagogical studies combined to subject studies give qualifications to teach at the secondary level.

Data for this study was gathered from 81 mathematics teacher students in the University of Helsinki in two cohorts. The first cohort (n=38) had their pedagogical studies academic year 2011 – 2012 and the second cohort (n=43) a year later, 2012 – 2013. The data was collected after the spring semester at the end of teacher students' pedagogical studies.

The assignment was: the students were asked to write a metaphor and expand the statement "as a mathematics teacher I am ...", and to continue with explanation for their statement. Only the metaphors with students' permission to use as data were gathered for this study.

In-service teachers. A questionnaire with 77 statements was built in connection with an international NorBa study (Nordic-Baltic Comparative Research in Mathematics Education). The last part of the questionnaire is qualitative and includes one item: "Please think and write down a metaphor characterizing a teacher. Please explain your metaphor. Teacher is like … My brief explanation of the metaphor is as follows…"

The respondents were 94 Finnish mathematics teachers teaching grades 7-9 from different regions of Finland with different teaching experiences and ages. The average age of respondents was 41 ranging from 25 to 61 years of age. The average duration of teaching experience of the respondents was 14.5, ranging from 1 to 35 years (1-5 years teaching experience n = 23, 6-20 years teaching experience n = 19, over 21 years of teaching experience n = 26). Teachers filled in the survey and 70 of them presented also the metaphor.

Analyses

The metaphor categorization was judged on a case-to-case basis using two independent raters, whose coding was compared at the end. The three authors worked as two pairs, one pair coding the in-service teachers' metaphors and the other pair the pre-service teachers' metaphors. As the agreement rate was somewhat lower in the case of pre-service teachers' metaphors, the third author was invited to also code those metaphors where no consensus was found. The metaphors and their explanations were analyzed as a unit, as the metaphor itself may be used to express different meanings. The raters analyzed the metaphors "from pure towards complex".

83% (58/70) of the in-service teachers' metaphors were categorized completely identically. In case of 13% (9/70) the metaphors were coded partly identically. If the unit of analysis contained elements of two or more aspects, the one category used by both raters became the final category. It both raters used two or more same categories, were these metaphors classified as hybrids 9% (6/70). Only 4% (3/70) were coded differently and 2 metaphors (3%) could not be identified in any category. Those five metaphors were removed (finally n=65).

After categorizing the in-service teachers' metaphors, 77% (62/81) of the pre-service teachers' metaphors were categorized completely identically. In case of 5% (4/81) the metaphors were coded partly identically. In the case of no consensus 19% (15/81), the third rater was used. When at least two coders agreed on coding, their coding was recorder. At this stage four metaphors, where categorized as hybrids (5%, 4/81). Two metaphors (3%) were left unidentified in agreement and for seven metaphors 9% (7/81) no agreement was found. Those nine metaphors were removed (finally n=72).

RESULTS

When pre-service teachers were asked to describe themselves as mathematics teachers, the most common type of metaphor 46% (n=33) was *self-referential* (see Table 1). In comparison, only 15% (n=10) of in-service teachers presented a *self-referential* metaphor.

The most common metaphor used by in-service teachers was by far *didactics expert* 51% (n=33) and also 38% (n=27) of the pre-service teachers presented a metaphor in this category. In all three professional-age-groups (1-5 years of teaching n=20, 6-20 years of teaching n=17 and over 21 years of teaching n=28) teacher as *didactics expert* was the most commonly used category. After that *pedagogical expert* (n=9, 14%) and *self-referential* (n=10, 15%) metaphors were almost similarly used regardless of teaching experience.

Teachers	***n***	**Subject expert**	**Didactics expert**	**Pedagogical expert**	**Self-referential**	**Context-ual**	**Hybrids**
pre-service teachers	72	2/72 (3%)	27 (38%)	5 (7%)	33 (46%)	1 (1%)	4 (6%)
in-service teachers	65	4/65 (6%)	33 (51%)	9 (14%)	10 (15%)	3 (5%)	6 (9%)
1 – 5 years experience	20	0/20 (0%)	11 (55%)	3 (15%)	4 (20%)	0 (0%)	2 (10%)
6 – 20 years experience	17	2/17 (12%)	8 (47%)	2 (12%)	2 (12%)	2 (12%)	1 (6%)
21 years or more experience	28	2/28 (7%)	14 (50%)	4 (14%)	4 (14%)	1 (4%)	3 (11%)

Table 1: In-service and pre-service teachers metaphors categorized in 6 categories (no unidentified metaphors included)

A closer analysis of pre-service teachers' *self-referential* metaphors shows that these metaphors can be classified into four different categories. Metaphors describing *personality or characteristics* (n=8, 24%): "As a mathematics teacher I am a clock. Punctual." Metaphors describing *hesitation* (n=11, 33%): "As a mathematics teacher I am a ship in fog. Hopefully, I will find my way to harbor." Metaphors describing *a new* *beginning, a new era* (n=6, 18%): “As a mathematics teacher I am a young foal. Bouncing around everywhere.” Metaphors describing that *something “big” is waiting ahead* (n=8, 24%): “As a mathematics teacher I am a final leg runner, who has received the baton from my own teachers.”

In-service teachers’ *self-referential* metaphors differ a lot from pre-service teachers’ *self-referential* metaphors. Only one big category was found when analysing these metaphors: over half (n=6, 60%) of in-service teachers’ *self-referential* metaphors describe the *variability of mathematics teachers’ job*. These teachers who presented a metaphor in this sub-category have all more than 6 years experience. For example, teacher with 17 years experience: “Teacher is like an amoeba. Adjusts into every situation. You can never know how your day at work will be.” Teacher with 13 years experience: “Teacher is like a rollercoaster. He has good and bad lessons, success and failure even with parallel classes. Excitement is always present.” The rest of the teachers (n=4, 40%), who only have 1-5 years teaching experience, presented a *self-referential* metaphor, which did not describe the variability of the job but instead something else: persistence or suitability to the job. It can be seen, that when teachers gain more experience their *self-referential* metaphors start to describe the variability of the job.

Metaphors describing teacher as *didactics experts* can be classified into two categories: *active* and *passive*. Those teachers who presented *an active didactic metaphor* among *didactical metaphors* (pre-service teachers n=17, 63% and in-service teachers n=23, 70%) are genuinely present in the learning process and constantly strive for better results both in teaching and learning. A pre-service teacher: “As a mathematics teacher I am a shepherd. Guiding my flock through varying terrain even when it is difficult. I lead the way to the new green pasture with my whistle.” An in-service teacher, 26 years experience: “Teacher is like an actor, who changes the role when needed. There is not just one correct way to teach – it depends on the a) subject and theme b) students c) occasion and d) teachers’ persona.”

Teachers who presented a *passive didactic metaphor* among *didactical metaphors* (pre-service teachers n=10, 37% and in-service teachers n=10, 30%) see themselves mostly as someone who is there to support the students when needed. A pre-service teacher: “As a mathematics teacher I am a compass. Showing the way but can’t take anyone there.” An in-service teacher, 1 year experience: “Teacher is like a Guide. Gives information and helps to survive in problem situations, but the student and his parents decide where the student goes.” At this time, in-service teachers’ teaching experience did not have an influence whether an active or passive didactical metaphor was presented.

DISCUSSION

It is remarkable, that when in-service teachers gain more teaching experience, it does not change the metaphor describing mathematics teacher’s role. In all three in-service teachers’ professional-age-groups *teacher as didactics expert* was the most common used metaphor (1-5 years of teaching n=11, 55%, 6-20 years of teaching n=8, 47% and over 21 years of teaching n=14, 50%) but only 38% of the pre-service teachers presented a metaphor in this category. This refers that the biggest change in metaphors and in such way also in teachers' beliefs occurs during teacher studies. This is an important result and message to those who plan prospective teachers' studies at the university level.

Presence of hybrid metaphors could be explained by complexity of the teacher's job. Four (6%) pre-service and six (9%) in-service mathematics teachers provided hybrid metaphors. The variability of teachers' job was expressed also in other categories. The number of unidentified metaphors was low, only 4 metaphors from 151 (3%) were not categorized into any category of the model extended from Beijaard, Verloop, and Vermunt's (2000) framework (Löfströn et al., 2010).

Because the pre-service teachers were assigned to write a metaphor and expand the statement "as a mathematics teacher I am ..." it might have result into more *self-referential* metaphors. In-service teachers continued the following sentence "teacher is like …" which is not that subjective. Although a closer look into the *self-referential* metaphors reveals that pre-service teachers are more insecure and suspicious and maybe that's why they presented metaphors in this category. An experienced teacher focuses more on the didactical side of the job and concentrates on how to manage the varying situations every day.

Looking at all the respondents, teacher as "didactics expert" was the most common metaphor used (60/151, 40%). According to these teachers it is important to create learning environments that support the students learning process and to use different teaching and learning methods. Learning may occur when students are actively involved and critical thinking is pursued. According to this metaphor analysis Finnish mathematics teachers' beliefs seem to be constructivist. Also Wilson and Cooney (2002) pointed that students learn mathematics most effectively when they construct meanings for themselves, rather than simply being told. A constructivist approach to teaching helps students to create these meanings and to learn. However, the latest PISA (2013) and two recent national assessments show reduction in students' mathematical skills (Hirvonen, 2011 and Metsämuuronen, 2013). If teachers' beliefs are constructivist, what kind of teaching approach do they actually use and how are their classroom practices? As the NorBa-project continues, we will find answers to these questions. It would also be interesting to collect metaphors from applicants applying for mathematics teacher studies and to follow these students trough their studies and see how the metaphor describing mathematics teacher changes.

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PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' CONSTRUCTION OF BOX PLOTS AND DISTRIBUTIONAL REASONING WITH THREE CONSTRUCTION TOOLS

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In this study we examined two prospective secondary mathematics teachers' constructions of box plots and their understanding of the distribution they were representing. The participants constructed box plots with paper-and-pencil, graphic calculator and TinkerPlots during clinical interviews. The study indicated that prospective mathematics teachers recognized that using technology to construct box plots provided affordances compared to creating a box plot by hand.

INTRODUCTION

In statistics, data can be represented in many different ways such as graphs and tables that have the potential to provide new understandings of the characteristics of the data (Myatt, 2007). Bakker, Biehler, and Konold (2004) emphasize that box plots provide rich representations since they give information about both measures of center and spread of the data, and can facilitate making comparisons of distributions. Although box plots are viewed as effective representations, it has been documented that students struggle with understanding the data they convey. Box plots can be challenging for students to understand because data is presented as aggregate instead showing individual points and understanding the median and quartiles is not as intuitive as once suspected (Bakker, Biehler, & Konold, 2004). Additionally, delMas (2004) stresses that "understanding how the abstract representation of a "box" can stand for an abstract aspect of a data set (a specific, localized portion of its variability) is no small task" (p. 87).

These problems could be minimized with the availability of technology in statistics. Chance et al (2007) outline many effective uses of technology in the learning of statistics. Three of these categories are automation of calculations, emphasis on data exploration, and visualization of abstract concepts. Automation of calculations allows for timely calculations with high accuracy and emphasis on exploration suggests that many graphs can be produced quickly. Visualization of abstract concepts is the idea that technology helps students to "see" statistical concepts. These uses of technology can potentially help students with the challenges of box plots.

Although there are many statistical packages/technologies that can help students create box plots, two widely used options are *TinkerPlots* (*TP*) (Konold, & Miller, 2005) and graphing calculators (GCs). Burrill (1997) studied the roles and potential of using GCs and remarked that, using GCs, students could be able to see if a data set contained an outlier, which could allow them to exclude the outlier from the data set and reexamine

2014. In Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 4, pp. 369-376. Vancouver, Canada: PME.

the distribution. On the other hand, Garfield and Ben-Zvi (2008) found that *TP* allowed students to perceive individual data values of a box plot which facilitates students' understanding. These studies focused on how the technology helped students understand box plots, but it is also important to focus on if teacher notice and appreciate these allowances when working with these technologies.

In this study, we examined two prospective mathematics teachers' thinking about box plot constructions by paper-and-pencil, GC, and *TP*. Using the above-mentioned categories of effective uses of technology (Chance et al., 2007) as a framework, we examined how prospective mathematics teachers reasoned while representing a data set. Accordingly, we identified the challenges and understandings of each prospective mathematics teacher as well as highlighting the teacher's recognition of the affordances of each type of technology.

METHOD

Participants

The participants of this study consisted of two prospective secondary mathematics teachers (1 male, 1 female) who were enrolled in a course about teaching mathematics with technology. These participants were selected based on recommendations from the instructor and their availability to meet with the researchers. Pseudonyms (Amy and John) are assigned to the participants. Both participants were seniors and their ages were 21. Neither participant had experienced using *TP* before the interviews but both had used GC.

Task and Interviews

The task used was taken from the Number of Rope Jumps data (Lappan et. al 2003, p. 40), which describes the maximum number of rope jumps for each student of a 28-person class. The data had a large variation and contained an outlier. Semi-structured clinical interviews were conducted individually with the participants. A *TI-84 Plus Silver Edition* GC, a laptop with *TP* software, a ruler, and paper were provided for the interviews. The data set was already entered as lists in the GC and available as a set of data cards in *TP*.

The data for this paper comes from a larger interview about multiple data sets. Each interviewee was asked to construct a box plot using paper-and-pencil first, then a GC, and lastly using *TP*. In addition, the interviewees were asked to construct a box plot after the outlier (300) was excluded from the data set by hand. The interviews, which took about an hour and a half with each participant, were videotaped and voice recorded. The interviews were transcribed and the transcribed data was analyzed descriptively. We analyzed the data by three main categories; which were box plot constructions with paper-and-pencil, using the GC, and using *TP*. In each category instances of reasoning with box plots and the issues or affordances of the technology were identified. Data matrices were constructed (Benard & Ryan, 2010) for each category in order to compare and contrast the interviewee's responses.

RESULTS

Box Plot Constructions with Paper-and-Pencil

At the beginning of the interviews, both interviewees were given the data in a table, and asked what is needed to construct a box plot. Both interviewees mentioned the requirement of a five number summary, and each used 1-Variable stats from graphing calculator to find the five number summary then constructed the box plot by hand. Amy constructed a vertical box plot while John constructed a horizontal box plot (Figure 1 and Figure 2).

In both cases the interviewees provided a number line with a scale but acknowledged that their scales were only estimates and not exact. This is important to note because having imprecise representations makes reasoning about the data more difficult. In fact Amy was aware of her inaccurate scale by saying: “the scale is gonna be kinda off” while constructing her box plot.

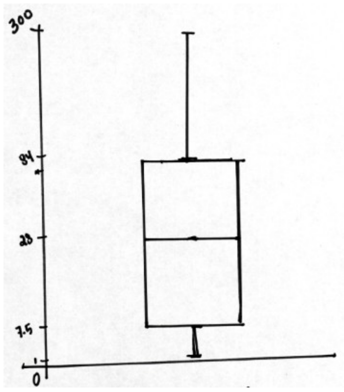

Figure 1: Amy’s box plot

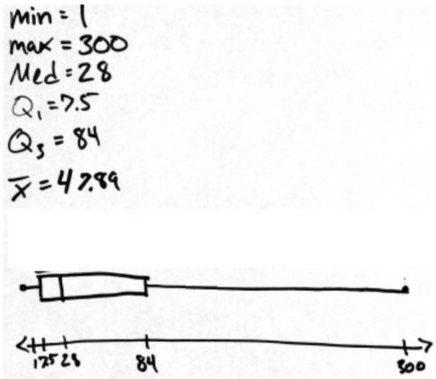

Figure 2: John’s box plot

After constructing the box plots, each interviewee was asked to reason about the distribution. Amy (A) said it, “looks like the data is probably skewed. I guess to the right”. Then she explained:

A: ... And if I look at this sheet [data table], I can kinda see. That most of it, 300, is kinda out by itself, but I have like a few 90’s 93, 96 that’s still pretty close to 84 [upper quartile] So, most of the data is right around the median besides this one 300 which is way out here [points to the upper whiskers] Oh, there is a 113 [in data table], that’s OK, that is kinda in there [shows a point on the upper whisker]. But I’d say it is pretty accurate [the box plot] based [on], like, the skewness of it, everything is pretty accurate, but kinda skewed.

Although Amy was able to determine the shape of her distribution, Amy referenced the individual cases from the table to decide on the shape of the distribution instead of using her box plot. This suggested that Amy did not understand how her box plot described the shape of the data or that she is was sure about the accuracy of her representation. When John was asked to reason about the distribution of the data with his box plot (Figure 2), he attended to an aspect of the variability by noting “It is definitely clustered before the median, before 28 because there is much smaller range between the minimum and the median in that case." Since John did not refer back to the data and used the aspects of a box plot to discuss the distribution, John demonstrated a better understanding of a box plot. However, unlike Amy, John did not address the shape of the distribution.

Next, both interviewees were asked to identify any outliers and create a box plot without the outlier (Figure 3 and Figure 4). Amy had difficulty identifying whether 300 was an outlier. She could not clearly express how she could identify outlier(s) in a data set, and she said that she forgot the formula for identifying an outlier. On the other hand, John did not have such difficulty. He applied the typical 1.5*IQR + Q3 formula of identifying outlier(s) in a data set.

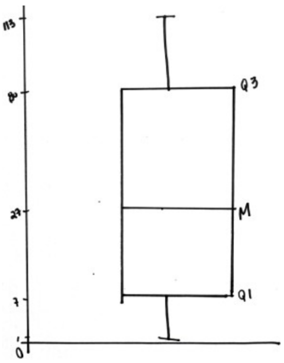

Figure 3: Amy's box plot after excluding the outlier

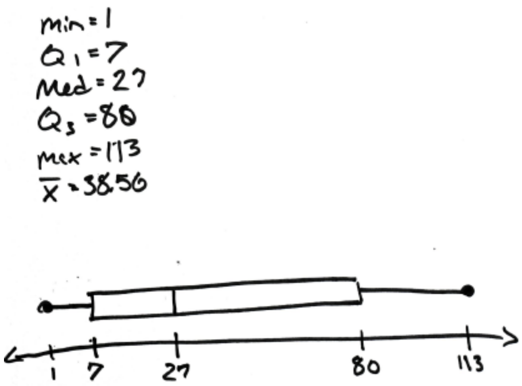

Figure 4: John's box plot after excluding the outlier

While both interviewees were capable of creating a box plot for the data with little trouble, each had their own challenges. First, the constructions with paper-and-pencil were not accurate since both interviewees constructed box plots with a poor scale. Amy demonstrated difficultly with the aggregate nature of the box plot and needed to refer to the individual cases to describe the shape of the distribution. Additionally, Amy had a problem identifying the outlier of the data set. On the other hand, John used the aspects of the box plot to reason about the variability demonstrating a better understanding of the representation.

Box Plot Constructions Using the GC

Next, interviewees were asked to construct box plots using the GC. Both interviewees chose to create a modified box plot with 300 denoted as the outlier. The researcher asked the interviewees to compare these to their previous box plots. Since Amy constructed all vertical box plots with the paper-and-pencil environment and the GC only constructed horizontal box plots, she rotated her paper to view her previous graphs horizontally while comparing them with GC's. When asked how the box plots were alike and different, she answered as follows:

A: Theirs [GC] is much more accurate scale-wise…You can see…how they have it set up scale-wise. You can barely see that little whisker but it is really close 1 and 7, which is easy to see. And then, you can see that—I mean mine is just more spread out. The scale is much better [GC].

On the other hand, John stressed the differences of his previous and current box plot constructions as follows:

J: It is different because it doesn't, mine doesn't show that there is, doesn't consider the outlier not a part of the actual box plot. So, if I wanted to remedy that, then I would have my maximum here but I would have a lone dot….

Also, the researcher asked the interviewees whether there was anything they wanted to change about their early understanding of the data after they constructed a box plot using the GC. John stated that "no my understanding stayed pretty, pretty [un]impacted. But it is nice to immediately be able to tell about outliers instead of having to calculate them for myself". On the other hand, Amy believed that her understanding changed a lot. She addressed the accuracy of construction of the GC saying "this'd [showing the GC] tell a lot more versus this [showing her box plot]. This is, just looking at it [hand drawn box plot] looks more deceiving whereas this one [GC] it's very accurate…"

In both cases, the calculator's ability to quickly and accurately create a box plot was considered helpful. For Amy, the accuracy of the scale helped her to better understand the distribution. In fact, she believed that her hand drawing was misleading. John appreciated the ability of the calculator to find and denote the outlier quickly.

Box Plot Constructions Using *TP*

For the final box plot, the interviewer constructed a box plot (Figure 5) within *TP* because interviewees were not familiar with the tool. Interviewees were asked to reason about the data and compare this graph with the previous box plots. The first impression of John about the representation was as follows:

J: This does not consider outliers although I would assume that we could make it consider outliers. This is really, really nice being able to show or because it shows where the data points lie. So you can clearly see that the data is clustered average of the left side (rope jumps) and as you and as you go to the right there is less and less data except, except for like around 90 apparently they get tired at 90. Yeah this is really useful.

Interestingly this is also the only instance in both interviews where the jump rope context was addressed.

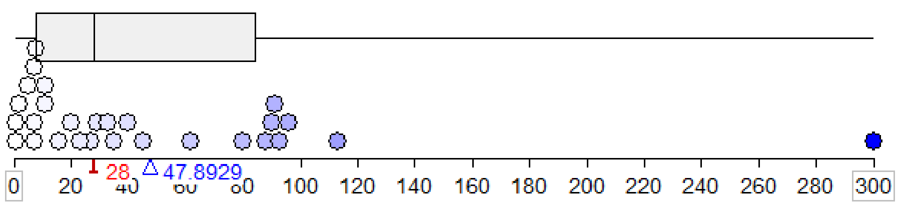

Figure 5: The box plot representation in TP

Compared to the GC representation of the box plot, John thought *TP* was superior since it could show the box plot as well as the individual data points. This gave him information to better understand his data and changed his understanding:

J: I think that this actually did change [his understanding] a little bit. We can see there is another cluster around 90, but with the other methods that I have been using there's no way to turn that just by looking at it, you have to actually examine the data itself.

In recognizing the there was another cluster around 90 that the box plot did not show, John was also gaining a better understanding of the pros and cons of the box plot representation. Amy provided similar expressions in terms of visualization of *TP* and being able to see individual data points, and addressed that the box plot representation in *TP* is better than the previous representations.

A: This, this is awesome [TP]…I know there was a lot of people that I was in school with that struggled and probably this kinda software--probably help them, a lot, with learning box plots and so, just visually you can see it. I mean I like this (showing the GC) and obviously, like, paper-and-pencil, probably, you can do it but it is not as effective. I mean like, like as I said earlier, like, mine were kinda skewed (refers her all previous box plots using paper-and-pencil) and you could see, when you saw it on the calculator and then, like, putting this information to this (showing *TP*) it's just even more--visual. I really liked it.

At the end of the interview, the researcher asked for the interviewee's last comments about the three different technology representations. John's thoughts summarize many of the affordances of each technology.

J: I'll start with the paper. The pros, you can very clearly tell whether or not a student understands what minimum, umm the quartiles, the median, and the max represent…The paper method does make it more difficult to recognize when there are outliers, however. Calculator, it is nice in most cases because you're able to use the trace to tell where each important area is--quartiles, and median, minimum, max. But, there is not really, there is not necessarily the understanding of what is going on behind the graph. You can't tell for sure whether the students know without talking to them directly and in a large class that (inaudible) confusing and hectic. As for *TP*, it's pretty much, I can't come up with any cons, but it's, it's really nice to be able to see the data points to see where they lined at any point in time, it gives a really good visual representation of what each of the four areas or four quadrants represent.

DISCUSSION AND IMPLICATIONS

This study indicated how prospective mathematics teachers reasoned about distributions with box plots while using different tools. Amy initially had trouble understanding the distribution of the data set without the use of the individual cases while John demonstrated a better understanding of the box plot representation. In both cases, the use of technology changed their understanding of the distribution. For Amy, technology provided an accurate representation because her scale was poor. Additionally, Amy and John found that having the individual cases in *TP* gave them new insights into their distribution. Finally, John explicitly expressed that having the outlier identified and marked on the box plot was helpful to him. Accordingly, we could conclude that these prospective teachers recognized the ability of technology to create box plots accurately and with additional information (denoted outliers or individual data points).

Interestingly, all three of the aforementioned Chance et al (2007) categories for effective uses of technology were observed by the prospective teachers. Both interviewees recognized the strengths of using technology for creating graphs accurately and quickly, automation of calculation and emphasis on data exploration were acknowledged by the prospective teachers. Finally, since box plots are an abstract representation (delMas, 2004) and the teachers expressed appreciation for technologies ability to help visualize the box plot, the prospective teachers are recognizing technologies ability to visualize abstract concepts.

Although more research needs to be conducted in this area because of the small sample size of this study, the study findings suggest that prospective teachers should have experience with different types of technology to produce box plots. This exposure may help to produce prospective teachers that both develop deeper understanding of box plots and that are more likely to use different type of technology in their future classrooms.

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RECONSTRUCTION OF ONE MATHEMATICAL INVENTION: FOCUS ON STRUCTURES OF ATTENTION

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The goal of the study was to reconstruct and dismantle a sequence of events that preceded an insight solution to a challenging problem by a ninth-grade student. A three-week long solution process was analysed by means of the theory of shifts of attention. We argue that concurrent focusing on what, how and why the student attends to when working on the problem can adequately explain his insight.

INTRODUCTION

The goal of the case study presented in this paper was to reconstruct a sequence of events that preceded an insight solution to a challenging problem by a 9th grade student, Ron, who worked on it with his classmate, Arik. Solving the problem required from the students to re-invent the Gauss' formula of the sum of the first *n* integers. The case of interest occurred in the framework of an on-going study that explores the affordances of a particular project-based learning instructional approach (Palatnik, in progress).

The study aims at contributing to research concerned with demystification of insight in mathematical problem solving. Cognitive psychologists frequently refer to an insight problem as one, which solution includes restructuring the initial representation of the problem followed by a sudden realization of the solution – so called *aha*-experience (e.g., Knoblich, Ohlsson & Raney, 2001). Cognitive mechanisms involved in restructuring the initial representation are still relatively uncertain (e.g., Cushen & Wiley, 2012). Furthermore, research on insight problem solving usually explores processes that last for minutes rather than weeks, as it happened in the case presented in this paper. In our study, the three-week-long solution process is analysed through the lenses provided by the Mason's (1989, 2008, 2010) theory of shifts of attention, which, as we argue below, can (partially) explain how the insight occurred.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

Mason (2010) defines learning as a transformation of attention that involves both "shifts in the form as well as in the focus of attention" (p. 24). To characterize attention, Mason considers not only *what* is attended to by an individual (i.e., what objects are in one's focus of attention), but also *how* the objects of attention are attended to. To address the *how*-question, Mason (2008) distinguishes five different structures of attention. Four of them have shown up in our data analysis.

According to Mason (2008), *discerning* details is a structure of attention, in which one's attention is caught by a particular detail that becomes distinguished from the rest of the elements of the attended object. Mason (2008) asserts that "discerning details is neither algorithmic nor logically sequential" (p. 37). *Recognizing relationships*

2014. In Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 4, pp. 377-384. Vancouver, Canada: PME.

between the discerned elements is a development from discerned details that often occurs automatically; it refers to specific connection between specific elements. For instance, when attending to the string of numbers 6, 2 and 3 one can effortlessly recognize that they are connected by the relationship $6 \div 2 = 3$. Recognizing the same relationship, however, is more effortful when one looks at the string of numbers 1, 2, 3, 4, 5 and 6. *Perceiving properties* structure of attention is different from *recognizing relationships* structure in a subtle, but essential way. In words of Mason (2008), "When you are aware of a possible relationship and you are looking for elements to fit it, you are perceiving a property" (p. 38). To stretch the above example, when one searches the string 1, 2, 3, 4, 5 and 6 for the numbers that can fit a division relationship, one can effortlessly discern the numbers 6, 3 and 2. Finally, *reasoning on the basis of perceived properties* is a structure of attention, in which selected properties are attended as the only basis for further reasoning.

Since our study concerns the phenomena of insight problem solving, we choose to consider not only *what* is attended and *how*, but also *why* the solver's attention shifts. We found it useful to address a *why*-question by identifying obstacles embedded for the solver in attending to a particular object and discerning the possible "gains and losses" of the shift to a subsequent object. Three research questions guided the study:

1. What were some of the objects of attention for the pair of middle-school students in due course of re-inventing the Gauss formula in the context of coping, for three week, with an insight problem related to numerical sequences?
2. For each identified object of attention, what was the structure of attention?
3. Why did the students move from one object of attention to another?

METHOD

Context

The case of interest occurred in the framework of a project "Open-ended mathematical problems", which is conducted by the authors of this paper in 9th grade classes of one of schools in Israel. At the beginning of a yearly cycle of the project, a class is exposed to a set of about 10 challenging problems. The students choose a problem to pursue and then work on it in teams of two or three. The students work on the problem practically daily at home and during their enrichment classes. Weekly 20-minute meetings of each team with the instructor (the first author) take place during the enrichment classes. At the end of the project, the teams present their work at the workshop.

One of the mathematical problems proposed to the students was Pizza Problem (Figure 1). It is a variation of a problem of partitioning the plane by *n* lines (e.g., Pólya, 1954; Wetzel, 1978). When introduced to the problem, the students are briefly explained mathematical notation as well as the meaning of terms "recursive formula" and "explicit formula". It is of note that 9th graders in Israel, as a rule, do not possess any systematic knowledge on sequences; this topic is taught in 10th grade.

Every straight cut divides pizza into two separate pieces. What is the largest number of pieces that can be obtained by n straight cuts? A. Solve for n = 1, 2, 3, 4, 5, 6. B. Find a recursive formula for the case of n. C. Find an explicit formula. D. Find and investigate other interesting sequences.

Figure 1: Pizza Problem

The choice of the case, data sources and analysis

During the three years of the project, five groups of students choose to work on Pizza problem. One group out of five did not produce any explicit formula. Four groups did so, and in three of them the students were able to explain us how. In this paper, we focus on the remaining group, the team of Ron and Arik. This is for two reasons. First, it is a particularly illustrative case of successful learning (cf. Simon et al., 2010, for the rationale of focusing on successful learning cases). Second, Ron and Arik could hardly explain us, at least not straightforwardly, how they invented the formula. Moreover, the process of invention looked serendipitous to us. Thus, we found particularly interesting and important to attempt to dismantle this serendipity.

The data included the audiotaped and transcribed protocols of the weekly meetings, intermediate written reports that the students prepared for and updated during the meetings, and authentic drafts produced between the meetings. These data were juxtaposed to initially reconstruct the whole story. Pencil marks on the students' drafts were particularly informative for making suggestions about the occurrences of the shifts of attention. The initial reconstruction was shown to Ron, who took the leading role in the project, during a follow-up interview. (The interview was conducted six mounts after the events described.) In the interview, Ron provided us with additional information that supported most of our interpretations and rejected some of them. This information helped us to refine the initial reconstruction.

RECONSTRUCTION

At the beginning, the students produced about 30 drawings of circles representing a pizza, which were cut by straight lines. They counted the number of pieces on the drawings and observed that the maximum number of pieces is obtained if exactly two lines intersect within the circle. The answers for 1, 2, 3, and 4 cuts were found: 2, 4, 7 and 11 pieces, respectively. It was difficult for the students to find a number of pieces for 5 cuts from the drawings as they became overcrowded.

To overcome this difficulty, Ron created a *GeoGebra* sketch and found that the maximum number of pieces for 5 cuts is 16. The students recorded their results as a horizontal string of numbers. They noticed that the differences between the subsequent numbers in the string form a sequence 2, 3, 4, 5 and used this observation to solve the problem for 6 cuts. The next goal for the students was to find a recursive formula. After several unsuccessful attempts to think of the strings of numbers, the students organized their findings vertically and eventually drew a table (see Figure 2).

2a 2b 2c 2d 2e

Figure 2: The drafts produced during the first week

From this point, the students shifted their attention to exploring the tables. The students' way for so doing can be described as looking for the arithmetic relationships between the numbers in the tables and marking them. One of the first relationships that they attended to was a *zigzag pattern* (see Figure 2d). At this stage they introduced the notation: P for the number of pieces, n for place of P in the table (only later they noticed that n represents also the number of cuts) and, eventually, P_n. A formula $P_n = P_{n-1} + n$ was written as a symbolic representation of *zigzag* pattern.

Then the students began looking for an explicit formula, which would enable them, in words of Arik, "to find P_{100} without finding P_{99}"[1]. The students tried to find it on the Internet and did not succeed. They also considered finding the explicit formula in Excel since "there are a lot of formulas in Excel." When this plan did not work, they asked the instructor for help. The instructor only helped the students to build a spreadsheet based on their recursive formula and encouraged them to keep looking.

In a week, the students brought to the meeting five tables with marked patterns: a *diagonal pattern* corresponding to the previously obtained formula $P_n = P_{n-1} + n$ (Figure 3b), a *horizontal pattern* summarized by the formula $P_n = (P_{n-1} + n - 1) + 1$ (Figure 3c), a *mixed pattern* accompanied by (incorrect) formula $P_n = n + P_{n-1} + n - 1$, and a *vertical pattern* corresponding to the formula $P = \sum n + 1$.

The instructor noted that the first three formulas were algebraically identical; the students had not noticed it and were surprised. Surprisingly to the instructor, the students presented a *vertical pattern* and formula $P = \sum n + 1$ just as one of their results, and not as a milestone on the way to the explicit formula. He said:

Instructor: [Let's] focus on this way [*vertical pattern*]…Tell me, how do I get, for example, 22?

Ron: Twenty two without 16? It goes ... I make one plus zero and one and two and three and four and five and six.

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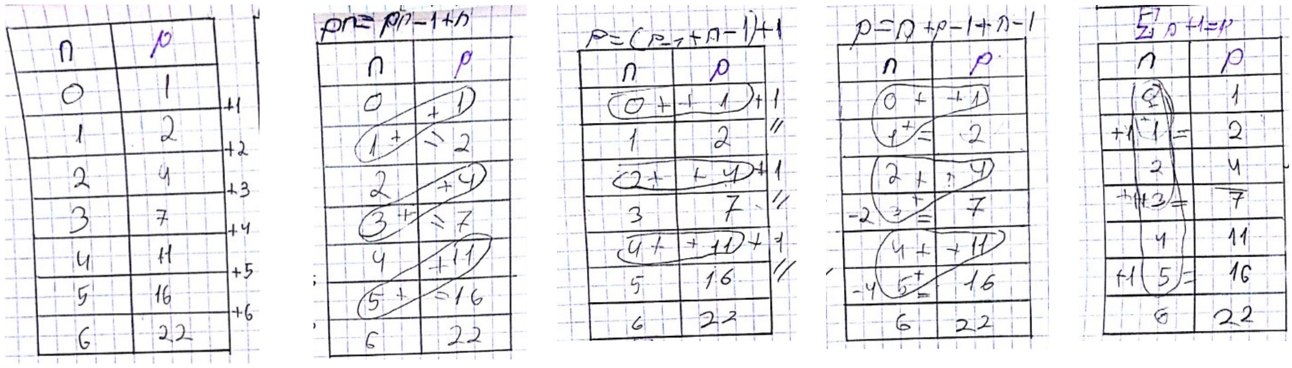
[1] All the excerpts are our translations from Hebrew.

Instructor:	One and two and three and four and five… There is some formula for calculating it.
	…
Arik:	So, [you ask] how to calculate it? Without summing the numbers?
Instructor:	Yes, without summing the numbers. You know, there is a formula that can give you an answer [instantly]. Do you understand why this is important?
Arik:	Because it takes time to calculate [by the formula $P=\sum n+1$].



3a 3b 3c 3d 3e

Figure 3: The drafts produced during the second week

At the next meeting, the students introduced the desired formula: $P_n=\frac{n}{2}(n+1)+1$. The instructor was astonished by the students' success and asked them to explain their invention in as much detail as possible. Ron took the lead. In his words: "I was stuck in one to six. And I just thought…six divided by two gives three. I just thought there's three here, but I could not find the exact connection [to 22]. I do not know why, but I multiplied it by seven, and voila – I got the result." This explanation along with the data from the follow-up interview enables us to offer, with some certainty, the following reconstruction of the events immediately preceding Ron's "voila".

Ron focused on the left column of a table similar to Table 3e. He experimented with the vertical string of numbers attempting to somehow, mostly by using the operations of addition and subtraction, create an arithmetic expression that would return a number from the right column. He asked his parents and the older sister for help; they tried and did not succeed. Then he came back to exploring the table, and this time he also tried to multiply and divide. One of these attempts began from computations $6\div 2=3$ and $3\times 7=21$. Ron realized that 7 in the second computation is not just a factor that turns 3 into 21, but also a number following 6 in the vertical pattern. He noticed (not exactly in these words) the following regularity: when a number from the left column is divided by 2 and the result of division is multiplied by the number following the initial number, the result differs from the number in the right column by one. He observed this regularity when trying to convert 6 into 22, and almost immediately saw that the procedure works also for converting 4 into 11 and 5 into 16. He observed that even when division by 2 returns a fractional result (5:2=2.5), the entire procedure still works. The *aha*-experience occurred at this moment. To verify the invention, he calculated P_{100} by the discerned procedure and compared the result with the corresponding number in his Excel spreadsheet. The last step was to convert the invented procedure into the formula. From the follow-up interview:

Instructor: How did you convert it [the observed regularity] into the formula?

Ron: It was a difficult part…I did it really in line with the arithmetic operations that I've used. I divided *n* by 2, and then I like multiplied by *n*+1, which is the next *n*, and then plus one.

SUMMARY OF FINDINGS

The answer the first research question straightforwardly steams from the above reconstruction. Namely, the students attended, among others[2], to the following objects: handmade sketches of a pizza, a GeoGebra sketch, strings of numbers, two-column tables, and a left column of a table similar to that in Figure 3e. For each of these objects, we now answer the second and third research questions. The answers for the first four objects are summarized in Table 1.

The last object of attention was identified as "The left column of the table similar to that in Figure 3e." The structures of attention for this object can be described as follows. Ron *discerned* sub-sets of the set of numbers 1, 2, 3, 4, 5 and 6, *recognized* various relationships in the sub-sets, *perceived* the division property and *discerned* a sub-set "2, 3, 6" that fits it. He *recognized* the relationship $6 \div 2 = 3$, *discerned* a subset "3, 22", *recognized* the relationship $3 \times 7 + 1 = 22$ and *perceived* numbers 6 and 7, which have been *discerned* in the above relationships, as numbers that belong to the vertical pattern. Ron then *perceived* the relationship "$3 \times 7 + 1 = 22$" for additional triples of numbers, namely, $(4 \div 2) \times 5 + 1 = 11$ and $(5 \div 2) \times 6 + 1 = 16$. (This was his *aha*-experience). Solution to the problem was concluded by means of *symbolic reasoning* with the perceived property, that is, converting "$3 \times 7 + 1 = 22$" into the formula $P_n = \frac{n}{2}(n+1) + 1$.

Objects of attention	Structures of attention: How is the object attended to?	Why did the students move to the next object?
Handmade sketches	*Discerning* the bounded areas in order to count the pieces. *Perceiving* that the maximum number of pieces is obtained if exactly two lines intersect within the circle.	When there are more than four cuts, some areas become small and it is difficult to count them.
A GeoGebra sketch	*Discerning* the areas bounded by the circle and five cuts in order to count the pieces. Counting is supported by the easiness of moving the cuts so that small areas can be enlarged.	The drawings, even dynamic, are not convenient for the larger numbers of cuts; results of counting are not ordered.
Strings of numbers	*Discerning* the neighboring numbers of the string and r*ecognizing* the relationships between them: the differences of the neighboring numbers form a sequence 1, 2, 3,	The number of pieces (P_n) is visible in the string, but the number of cuts (n) is not; realization that producing an

[2] Additional objects of attention include an Excel spreadsheet and more. These objects were attended to, but turned to be secondary rather than primary objects of attention in due course of solving the problem.

	4, 5 etc.	explicit formula requires both n and P_n to be visible.
Two-column tables	*Recognizing* various numerical relationships between the numbers (including diagonal, horizontal, mixed and vertical patterns). *Symbolic reasoning* on the basis of the perceived properties: $P_n = P_{n-1} + n$, $P_n = (P_{n-1} + n - 1) + 1$, $P = \sum n + 1$	Realization, partially based on the instructor's prompt, that an explicit formula can be produced by looking at the vertical pattern, which is visually situated in the left column of the table.

Table 1: Structures of attention for the first four objects

DISCUSSION

Pizza Problem appeared to be extremely difficult for Ron and Arik, and one can wonder: why so? The research literature on algebraic reasoning provides us with some initial answers. In line with Radford (2000), we observe that the problem was difficult because it required from the students to shift from pattern recognition to algebraic generalization. In terms of Duval (2006), the problem required from the students to shift the representational registers for many times. In line with Zazkis and Liljedahl (2002), we conclude that the problem was difficult because in the course of its solution the recursive approach was dominant, and this approach is known to prevent the students from seeing more general regularities. Furthermore, Ron's *aha*-moment could usefully be analysed in terms of the representation theory of insight (e.g. Knoblich, Ohlsson & Raney, 2001): the insight occurred when a particular representation was put forward among many other representations.

However, considering the problem's difficulty due to the students' under-developed algebraic reasoning and explaining the insight by identification of shifts in representations is compatible only with one venue of the presented analysis, the one concerned with Mason's *what*-question (i.e., *what* objects are in the focus of attention?) An added value of our analysis is in putting forward also a *how*-question – this is in line with the Mason's theory – and a *why*-question. We argue that concurrent focus on these three questions is pivotal for explaining the observed phenomena. Specifically, focusing on the *how*-question enabled us to better understand the interplay of the structures of attention that lead Ron to his main insight. Focusing on the *why*-question enabled us to identify a pivotal sub-sequence of shifts of attention in a (seemingly) serendipitous chain of attempts.

Our last point is about possible pedagogical implications of the presented case study. Liljedahl (2005) found that *aha*-experiences have positive impact on students' attitude towards mathematics. He then raised a question of how to organize learning environments, in which such experiences might occur. An instructional format outlined in this paper can serve as an example of such an environment[3]. Let us point out

[3] We claim so based not only on the case of Ron and Arik, but on the fact that four out of five teams, who worked on the same problem, also experienced aha-moment when inventing the explicit formula.

its central characteristic. On one hand, the students had enough room for autonomous learning. On the other hand, the chosen format included opportunities for the instructor to focus the students' attention on the most promising idea from the pool of their ideas.

Acknowledgements: This research is supported, in part, by the Ministry of Science and Technology, Israel by the Israeli Science Foundation (grant 1596/13) and the Technion Graduate School.

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REVISITING MATHEMATICAL ATTITUDES OF STUDENTS IN SECONDARY EDUCATION

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The aim of this paper is three-fold. First, we review and briefly synthesise the main points of the recent debate around the concept of 'attitudes to mathematics'. We then present the measurement methodology we employed to capture 'attitudes to mathematics' in the context of a large scale UK project with secondary school students, and how these results inform the theoretical debate. Finally, we report some substantive results about how the resulting attitudinal constructs, namely 'maths disposition' and 'maths identity' change during one academic year, and between various groups of interest (e.g. gender). We conclude with a brief discussion of methodological and educational implications.

INTRODUCTION

The importance of mathematics to students' access to Science, Technology, Engineering and Mathematics (STEM) subjects in Higher education, and hence to their educational and socioeconomic life opportunities, as well as the need to promote a mathematically engaged society is well documented in literature and recent policy documents (Ofsted, 2006; Roberts, 2002; Smith, 2004). In a recent report ACME (ACME, 2009) recognises this important issue and advocates 'tackling the perceptions of mathematics" as a particularly important issue in the current economic climate, placing emphasis on the importance of mathematics as a "powerful analytical tool", with inherent "pervasiveness" and a "key workforce skill".

The paper focuses on 'attitudes to mathematics' with three particular aims: (a) to review and briefly synthesise the main points of the recent debate on the issue of 'attitudes to mathematics', (b) to present the measurement methodology we employed to capture 'attitudes to mathematics' in the context of an on-going ESRC project with secondary school students and their teachers, and (c) to report some preliminary descriptive substantive results about how this attitudinal construct changes during one academic year.

THEORETICAL PERSPECTIVE

The study of students' attitudes towards mathematics has gained considerable interest over the past 40 years. A lot of instruments (e.g. Lim & Chapman, 2013) have been proposed and used since then with a key influence the widely used Fennema-Sherman scales (Fennema & Sherman, 1977). Each of those instruments attempted to capture one of the many 'dimensions' or constructs associated with 'attitudes towards mathematics': beliefs, values, identities, engagement, affect, emotions, motivation, confidence, self-efficacy, dispositions, are only a few on the list (Ruffell, Mason, & Allen, 1998). This complexity, as well as the lack of agreement on the definition of the construct has led researchers (e.g. Watson, 2011) to recently revisit the established instruments of the 1970s and 1980s looking for alternative universal definitions or more parsimonious instruments. A useful starting point to this conceptualisation is probably Ruffell et al.'s (1998) decomposition of attitudes into three sub-components, namely cognitive, affective and conative. Their reflective analysis, as well as others that followed did not manage to reach consensus on the topic.

Despite these controversies, the study of students' attitudes and/or dispositions is very important because this may reveal key influences on their choices and decision-making and hence future engagement with STEM (Archer, Halsall, Hollingworth, & Mendick, 2005) . Previous studies had also identified a plethora of socio-cultural factors which are significant in shaping students' dispositions and choice-making in education in general, and in STEM subjects and mathematics in particular: class, gender, nationality, ethnicity, parental and peer cultures are just the beginning of the list . In our earlier work with post-secondary students we had also contributed with instruments for measuring what we called dispositions and self-efficacy in mathematics (Pampaka, et al., 2011; Pampaka et al., 2013).

Our current work, reported here, also attempts to add to this debate by a new concise instrument of students' attitudes and dispositions towards mathematics. The overall aim of this study is to understand (i) how learners' dispositions to study mathematics develop through secondary school, (ii) how mathematics pedagogies vary across different situations and contexts and (iii) how these pedagogies influence learning outcomes (including attitudinal ones).

METHODOLOGY

Project Design: The Teleprism Study

The paper is empirically based on initial findings from an on-going ESRC funded study of teaching and learning secondary mathematics in UK (www.teleprism.com). The project is designed to capture the five years of students' progression in Secondary Education (Year 7 to 11, i.e. students aged 11 to 16) in about one year of data collection: From October 2011 to December 2012. This design poses a series of methodological challenges around the combination of longitudinal and cross-sectional analyses, which go beyond the scope of this paper. The research question we seek to answer in this paper regards measuring 'dispositions' and attitudes to mathematics.

Instrumentation and Sampling

The nature and design of the study (i.e. longitudinal at school level for selection purposes) make it necessary to employ a varied sampling frame to ensure maximum coverage of the schools of England. We invited schools, drawing on various sources (including national databases), with an initial requirement for them to take part with all their Year 7 to 11 mathematics teachers and classes and be willing to follow this up at two more data collection points (hereafter DPs). In total, we approached over 2200 schools and we were able to establish collaboration with 40 of them. We note here issues around self-selection bias in this type of studies, which limits the representativeness of the achieved sample.[1]

Data collection in these schools involved a student questionnaire (at all three data points, as shown in Table 1) about students' attitudes to mathematics, confidence at various mathematical topics, future aspirations, and their perceptions of the teaching they encounter. The latter was also captured through a teacher survey administered to their mathematics teachers (twice during the course of the first academic year of the study, 2011-2012, i.e. along students' DP1 and DP2). Student questionnaires are based on different versions of the same instrument to reflect the age and level of students (i.e. 5 different Year Groups, from hereafter Y7 to Y11). Background variables and measures of students' attainment are also being collected including gender, ethnicity, language of first choice, proxies of socioeconomic status, and earlier National Curriculum level records. The various sections of the questionnaire capturing teaching and learning perceptions have been constructed and expanded based on our previous TransMaths framework (www.transmaths.org) where we validated and used instruments for students aged 16 and older (Pampaka, Kleanthous, Hutcheson, & Wake, 2011; Pampaka et al., 2013; Pampaka et al., 2012). The achieved sample size at each data point, from the participating 40 schools is summarised in Table 1, with the different completion patterns. It should be noted that some schools dropped out during the study.

Sample Description N=student numbers	**DP1** Oct - Dec 2011	**DP2** June/July 2012	**DP3** Oct – Dec 2012
Matched at all DPs	3744	3744	3744
Completed only one DP	5358	1186	2127
Completed DP1 and DP2	3051	3051	-
Completed DP1 and DP3	1172	-	1172
Completed DP2 and DP3	-	771	771
Total cross sectional sample	**13325**	**8752**	**7814**

Table 1: Sample Description [based on preliminary matching, unique cases: 17448]

For this analysis we focus on the instrument developed to capture students' mathematical attitudes, with the items, and the response format, shown in Figure 1.

[1] However we have plans in place to investigate the comparability of our sample to the national one

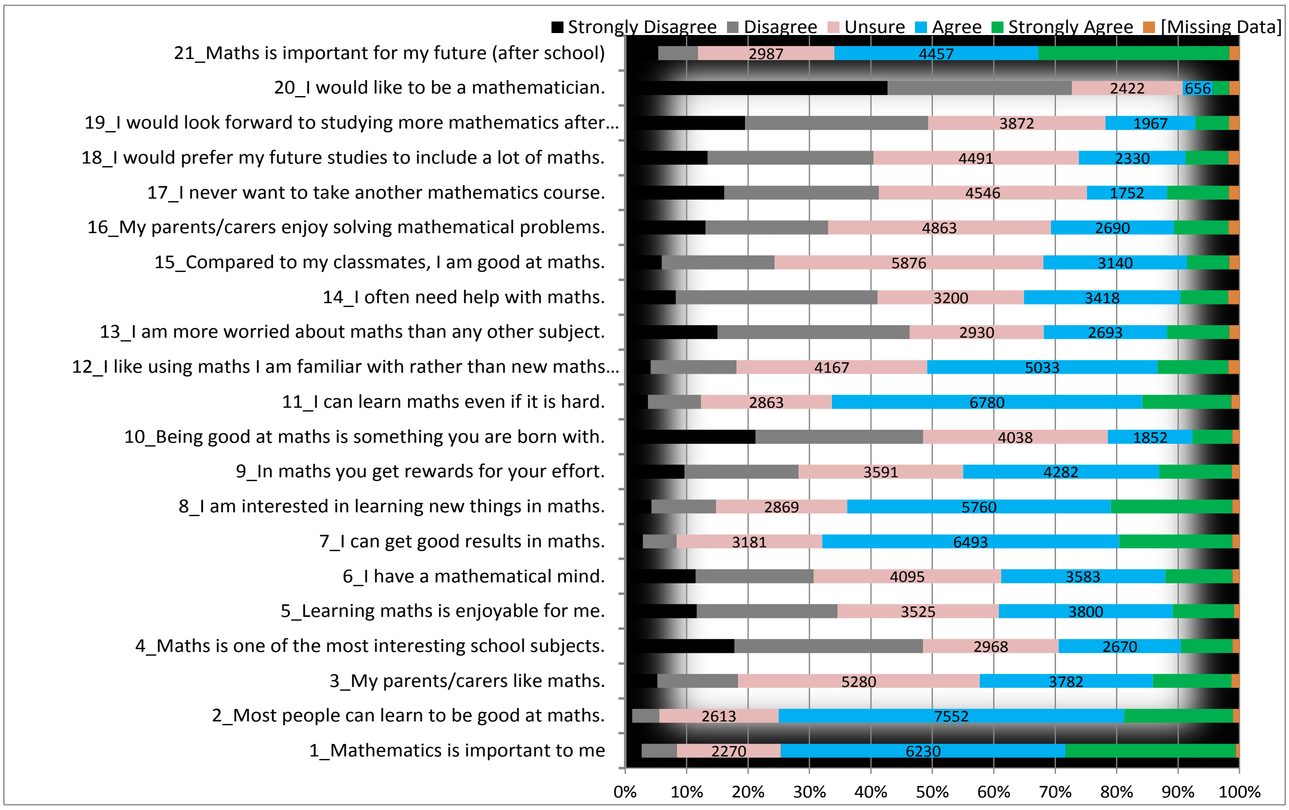

Figure 1: The items of the instrument for students' 'attitudes' towards mathematics, with the distribution of their responses at Data Point 1 (DP1, N=13325)

For the validation of the constructed measures (outlined in the next section) we draw on data from the cross sectional samples at each DP, whereas for some comparative, substantial results based on these measures, we limit analysis here to the 3744 matched cases who completed all DPs.

A measurement approach to construct validation

The validation process refers to the accumulation of evidence to support validity arguments regarding the students' disposition measures. Our psychometric analysis for this purpose is conducted within the Rasch measurement framework, following relevant proposed guidelines (Wolfe & Smith Jr., 2007) based on Messick's definitions of validity (Messick, 1989). The Rasch model is preferred because it provides the means for constructing interval measures from raw data. We have been extensively employing this approach for the validation of our constructed measures (Pampaka, et al., 2011; Pampaka, et al., 2013; Pampaka, et al., 2012). The Rasch rating scale model (using the Winsteps software) is considered the most appropriate for the scaling problems we have in this particular paper (i.e. a common Likert type scale). Our decisions about the validity of the measures are based on the following statistical indices (all these have been examined but cannot be all presented in this limited space):

- ***Item fit statistics*** to indicate how accurately the data fit the model, providing evidence in support (or not) of the unidimensionality assumption.

- ***Category Statistics*** are examined for the appropriateness of the Likert scale used and its interpretation by the respondents (i.e. communication validity).
- ***Person – item maps*** and the item difficulty hierarchy provide evidence for substantive, content and external validity.
- ***Differential Item Functioning*** (DIF) suggests potential group differentiation, which is important when an instrument is used with different groups or at different occasions (e.g. gender, year group and DP for time invariance).
- Qualitative data from interviews with students (in two case study schools) are used along the survey results, for validation, and deeper insight.

Further Statistical Modelling

Eventually, once the measures' validity is established we proceed with further statistical modelling to investigate and model change in attitudes and its association with other measures of pedagogy (Pampaka, et al., 2012) or attainment. We limit the presentation here to some descriptive results.

SELECTED RESULTS

Measuring 'attitudes' towards mathematics

As mentioned earlier, our instrument was intended to measure a general attitude in mathematics, as defined by the mixture of items. Following the measurement framework described above would provide us evidence of this hypothesis in regards to the unidimensionality of this construct. The evidence for this in the Rasch context is given by fit statistics which are local indicators of the degree to which the data is cooperating with the model's requirements. Inconsistent data (e.g. those departing from the ideal of 1) may become a source of further inquiry. For the purposes of this paper we take any number above 1.3 (of infit MnSq) as possible cause of concern, whereas infit values below 1 are considered as overfits and are not discussed. The results from our initial analysis with all the items to define a measure of 'mathematical attitudes' were not supportive for this hypothesis and operationalization: a few items were signified as misfitting (i.e. Items 10, 12, 13, 14 and 21). A unidimensionality test also suggested the existence of two dominant dimensions, with the following split of items which we explored further:

- Sub-dimension 1: Items 1, 4, 5, 8, 17, 18, 19, 20 and 21
- Sub-dimension 2: Items 2, 3, 6, 7, 11, 12, 14, 15, and 16

Separate Rasch rating scale models were performed on these two sub-dimensions, with all available data at each data point, combined together (resulting in a sample of 30000+) in order to check for DIF between DPs to ensure measure invariance over time. The fit statistics of these two measures are presented in Tables 2, for what we call mathematics disposition and Table 3, for mathematics 'identity'. For the former, two items are found to be misfitting (Item 17: I never want to take another mathematics course, and Item 21: Maths is important for my future). The coding of Item 17 was reversed for this analysis, and this might be causing its misfit. Item 21, seems to be one of the most difficult items of this measure (as indicated by its low measure value). Both are considered useful for this construct, so it was decided to keep them in the model. The psychometric properties of the second construct of 'identity' do not present any problems in regards to fit statistics.

ENTRY NUMBER	TOTAL SCORE	TOTAL COUNT	MEASURE	MODEL S.E.	INFIT MNSQ	INFIT ZSTD	OUTFIT MNSQ	OUTFIT ZSTD	PT-MEASURE CORR.	PT-MEASURE EXP.	EXACT MATCH OBS%	EXACT MATCH EXP%	ITEM
1	119826	30547	-1.57	.01	.93	-8.8	.91	-9.9	.69	.69	59.3	54.0	statement1
2	82107	30418	.59	.01	.85	-9.9	.85	-9.9	.78	.73	53.8	47.6	statement4
3	91439	30454	.09	.01	.80	-9.9	.80	-9.9	.79	.73	54.2	47.6	statement5
4	109298	30395	-.93	.01	.84	-9.9	.83	-9.9	.74	.71	56.6	50.4	statement8
5	97413	30170	-.29	.01	1.48	9.9	1.72	9.9	.62	.73	47.6	48.2	statement17
6	82637	30182	.53	.01	.68	-9.9	.70	-9.9	.80	.73	59.0	47.1	statement18
7	77153	30180	.83	.01	.77	-9.9	.76	-9.9	.78	.73	57.3	47.9	statement19
8	57508	30194	2.04	.01	1.18	9.9	1.13	9.9	.66	.70	56.3	56.0	statement20
9	114163	30198	-1.28	.01	1.53	9.9	1.51	9.9	.60	.70	43.6	52.5	statement21
MEAN	92393.8	30304	.00	.01	1.01	-3.2	1.02	-3.3			54.2	50.1	
S.D.	18822.5	139.2	1.08	.00	.30	9.3	.34	9.3			5.0	3.1	

Table 2: Item fit statistics for "Mathematics Disposition"

ENTRY NUMBER	TOTAL SCORE	TOTAL COUNT	MEASURE	MODEL S.E.	INFIT MNSQ	INFIT ZSTD	OUTFIT MNSQ	OUTFIT ZSTD	PT-MEASURE CORR.	PT-MEASURE EXP.	EXACT MATCH OBS%	EXACT MATCH EXP%	ITEM
1	117106	30440	-.83	.01	.97	-3.8	1.01	1.7	.48	.56	56.8	51.9	statement2
2	99364	30384	.11	.01	1.01	1.5	1.03	3.9	.61	.60	47.5	45.5	statement3
3	92493	30433	.44	.01	.85	-9.9	.86	-9.9	.75	.61	46.7	43.7	statement6
4	113759	30438	-.63	.01	.73	-9.9	.70	-9.9	.70	.57	59.8	50.1	statement7
5	110610	30295	-.49	.01	.79	-9.9	.76	-9.9	.67	.58	57.1	49.2	statement11
6	103150	30164	-.11	.01	1.30	9.9	1.40	9.9	.43	.60	45.0	46.8	statement12
7	92695	30153	.39	.01	1.26	9.9	1.43	9.9	.51	.61	40.1	43.6	statement14
8	92089	30181	.43	.01	.72	-9.9	.75	-9.9	.67	.61	53.3	43.5	statement15
9	86303	30165	.69	.01	1.22	9.9	1.26	9.9	.57	.62	39.7	42.7	statement16
MEAN	100841	30295	.00	.01	.98	-1.4	1.02	-.5			49.6	46.3	
S.D.	10319.2	122.8	.51	.00	.22	8.7	.26	8.8			7.0	3.2	

Table 3: Item fit statistics for Mathematics 'Identity'

Further investigations of DIF as well as category statistics are in support of healthy measures (these results will be provided for the interested reader at www.teleprism.com/PME2014 and will accompany the presentation).

Using the constructed measures in further analysis

The corresponding resulting scores (in logits) of the students in these measures were extracted and added in the datasets for further analysis: higher score indicate higher disposition and more mathematical 'identity'. Some descriptive results with these measures are shown next with the matched sample (N=3744), in relation to change over time, by year group and gender.

Figure 2 shows students dropping mathematical attitudes as well as some gender and year group differences. It should be noted that Year 11 was excluded from this analysis due to the limited matching sample (<100). The other sample sizes are as follows: Y7=1249, Y8=856, Y9=734 Y10=742.

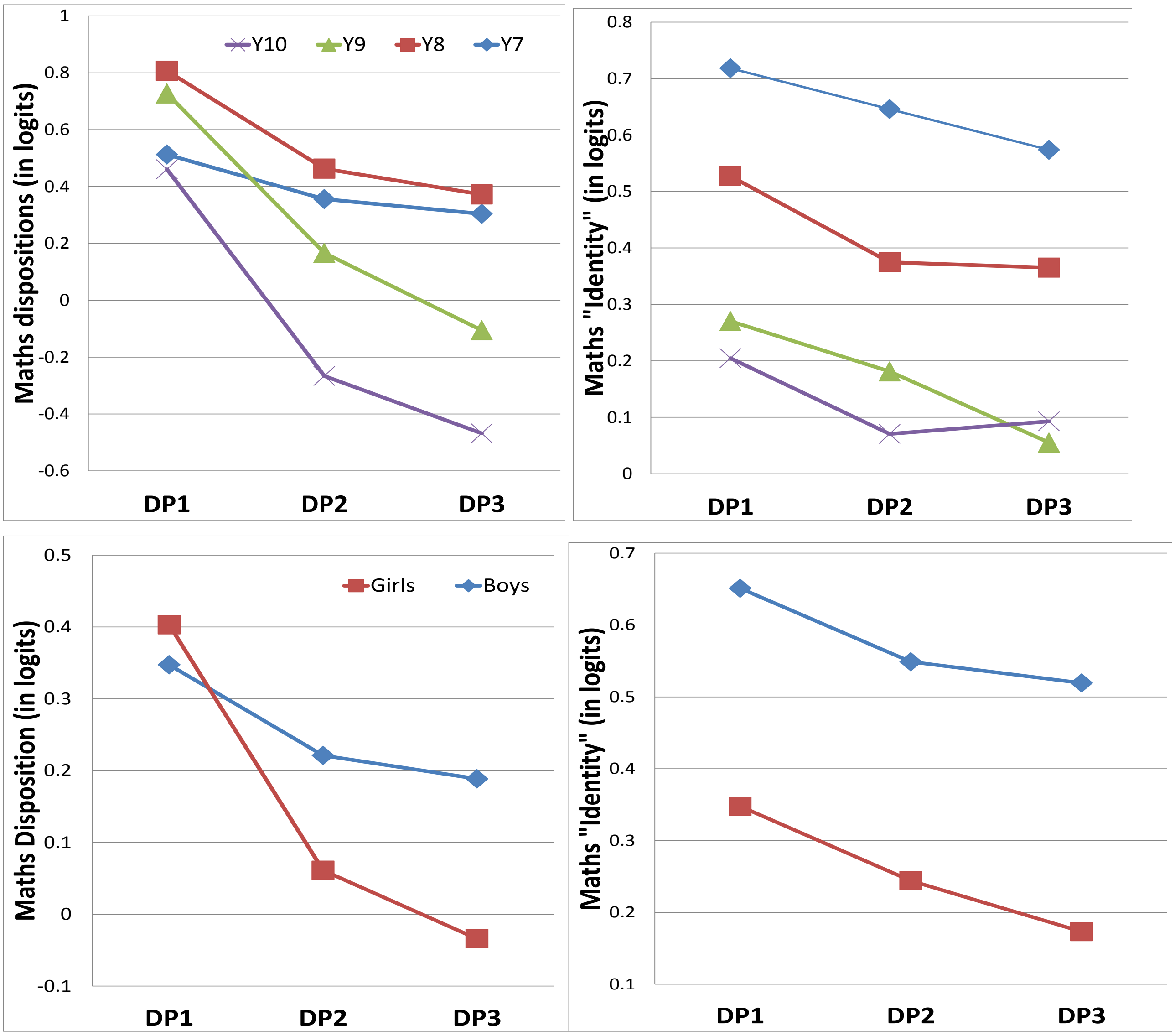

Figure 2: Changes in maths disposition (left plots) and ‘identity’ (right plots), by year group (top plots) and gender (bottom plots).

CONCLUDING REMARKS

Our results in regards to the dimensionality of mathematical attitudes are in agreement with earlier conceptualisations (Ruffell, et al., 1998) of attitudes as a multidimensional construct that could be decomposed into the affective, conative and cognitive components: Our ‘identity’ measure is constructed based on ‘expressions of feelings towards mathematics, thus is closely related to the affective component. Disposition is constructed based on expressions of behavioural intention, thus it corresponds to the conative component. To this we should add that our instruments include a contextualised self-efficacy instrument, which we believe is linked to the cognitive aspect, and we intend to test in the near future.

Results with these measures (Figure 2) are in support of previous findings in regards to students’ dropping dispositions and engagement with mathematics (e.g. Pampaka, et al., 2012). However, further analysis needs to be performed to account for school effects (multilevel modelling) and associate with other measures of interests such as pedagogical practices in mathematics.

Acknowledgement: The authors would like to acknowledge the support of the ESRC grant with reference RES-061-25-0538.

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‘VALUE CREATION’ THROUGH MATHEMATICAL MODELING: STUDENTS’ DISPOSITION AND IDENTITY DEVELOPED IN A LEARNING COMMUNITY

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This study examined how mathematical modeling activities within a collaborative group impact on students' perceived 'value' of mathematics. With a unified framework of Makiguchi's theory of 'value', mathematical disposition, and identity, the study identified the elements of the value-beauty, gains, and social good-with the observable evidences of mathematical disposition and identity. A total of 60 college students participated in 'Lifestyle' mathematical modeling project. Both qualitative and quantitative methods were used for data collection and analysis. The result from a paired-samples t-test showed the significant changes in students' mathematical disposition. The results from the analysis of students' written responses and interview data described how the context of the modeling tasks and the collaborative group interplayed with students' perceived value.

INTRODUCTION

Studies reported that when students see themselves as capable of doing well in mathematics, they tend to value mathematics more than students who do not see themselves as capable of doing well (Eccles, Wigfield, & Reuman, 1987; Midgley, Feldlaufer, & Eccles, 1989). To see the value in mathematics, it is essential for students to believe that mathematics is understandable, not arbitrary; that, with diligent effort, it can be learned and used; and they are capable of figuring out mathematical problems based on their experiences. Kilpatrick and his colleagues (2001) introduced "productive disposition" as one of key components of mathematical proficiency and defined as the "habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy"(NRC, 2001, p. 131). "Mathematics disposition" (NCTM, 1989, p233) was also included in the National Council of Teachers of Mathematics Evaluation Standards. Developing such a disposition toward mathematics requires frequent opportunities to recognize the benefits of perseverance, and to experience the rewards of sense making in mathematics. It becomes a question of what learning environment supports students to engage in meaningful learning of mathematics and to develop positive disposition as well as self-concept. A number of studies demonstrated that mathematical modeling, which plays a prominent role in the new Common Core State Standards for Mathematics (CCSSM), promotes socially situated learning environments with group collaboration, classroom discussion, initiative, and creativity and it has the potential to develops positive disposition toward mathematics and strengthen their mathematical identity (Ernest, 2002; Lesh & Doerr, 2003). The studies highlight that learning mathematics extends beyond individuals' learning concepts, procedures, and learners learn to be a part of a community of practice and become participants in the mathematics being practiced (Boaler, 2002). How a student learns mathematics involves the development of the student's identity as being a part of a mathematics classroom community (Anderson, 2007). A mathematical identity consists of a participative "mode of belonging" related to one's participation in a mathematical community of practice typically, the mathematics classroom (Wenger, 1998).

THEORETICAL FRAMEWORK

As a unified framework of Makiguchi's theory of value creation (1930), mathematical dispositions outlined by NCTM Evaluation Standards 10, and identity (see Table 1), this study identified the elements of the value with the observable evidences of mathematical disposition and identity.

Mathematical Disposition, Identity, Sense of belonging	Makiguchi's Elements of Value
• Interest, curiosity, and inventiveness in doing math	Beauty
• Confidence in using math to solve problems and communicate ideas • Willingness to persevere and become persistent in math tasks • Flexibility in exploring math ideas and trying alternative methods in solving problems • Appreciation of the role of mathematics in our culture and its value as a tool and as a language • Inclination to monitor and reflect on their own thinking and performance • Valuing of the application of mathematics to situations arising in other disciplines and everyday experiences	Gains
• See oneself as a learner, and doer of mathematics • Sense of belonging in a learning community, global citizenship	Identity Social value (Social Good)

Table 1: Theoretical framework (Makiguchi's theory of value, disposition, and identity)

The concept of value in the notion of Makiguchi (1930; Bethel, 1989) takes into account the subject and object relationship (students' relationship with mathematics in this study), which reflects human creativity. In the notion of Makiguchi (1930; Bethel, 1989)'s value creation, it is critical that students feel happiness, enjoyment, and pleasure in their own processes of investigating and understanding mathematics, as a result, students construct meaning, and value is created. In Makiguchi's concept of value, the three elements of the value are the following:

Beauty is perceived to be an emotional and temporary value. The value of *Gain* is an individual value and self-development, and beneficial aspect that is related to the whole of man's life. *Social good*, however, is a social value and is related to the life of the group. The value of good is the expression given to the evaluation of each individual's voluntary action, which contributes to the growth of a unified community composed of the individuals (Makiguchi, 1930; Bethel, 1989).

RESEARCH QUESTIONS

The purpose of this study is to develop and evaluate a model for students to *create value* in learning mathematics. With the unified framework of Makiguchi's theory of value, mathematics disposition, and identity, this study examines how 'socially-situated' mathematical modeling activity within a collaborative learning community can contribute to students' development of their mathematical disposition, identity, and sense of community as well as students' creating mathematical meaning. The guiding questions for this study are as follows:

1. What changes (if any) are observed in students' mathematical disposition that results from learning mathematics through mathematical modeling within a learning community?

   Specifically, How do students perceive value of beauty and gains, in Makiguchi's notion, of learning mathematics before and after experiencing mathematical modeling activities within a collaborative group?

2. How are students' mathematical identities transformed from their involvement in mathematical modeling activities within a collaborative group?

3. How are students' perceived social values, in Makiguchi's notion, of learning mathematics observed during mathematical modeling activities within a collaborative group?

   1) How does the collaborative group create a sense of belonging to the group that can be realized through engaging in mathematical modeling activities with group members?

   2) How do students interpret mathematical results within the socially situated context of modeling activities?

METHODOLOGY

Both quantitative and qualitative methodologies were used in data collection and analysis, investigation, and interpretation. Multiple data sources including surveys, interview data, students' written tasks and journals were collected (see Figure 1).

These data sources provided participants with multiple opportunities of their reflecting and sharing thoughts about how these experiences impacted their disposition and identity. The participants were a total 60 students who enrolled in college algebra courses taught by the researcher. The curricular task for the study is a modified version of the mathematical modeling project developed by the Center for Discrete Mathematics and Theoretical Computer Science (DIMACS) at Rutgers University. The project introduces the ecology of humans as a topic, and ecological foot printing is developed as a tool for assessing human impact and as a decision-making tool. These topics are relevant to social and environmental issues in which students engage in everyday lives. The investigator attempted to provide students with the tasks that require everyday knowledge, critical thinking, and a collaborative work. The mathematical modeling project was conducted within groups of four or five for four consecutive weeks. After completing the first week of conducting the project, students were asked to collect their own data. The Mathematical Disposition Survey (MDS) was conducted at the beginning of the study and the end of the study, and the results were compared. The mathematical disposition survey instrument is a modification of the one developed by Kisunzu (2008). Students' written tasks and journals were collected after each class. A total of eighteen focal students were selected for interview based the results from the analysis of Mathematical Disposition Survey and students' journals. Semi-structured interview offered students the opportunity of giving detailed statements on their written tasks, questionnaires, and journals. The researcher took field notes and audio-taped all the activities in classroom and interviews.

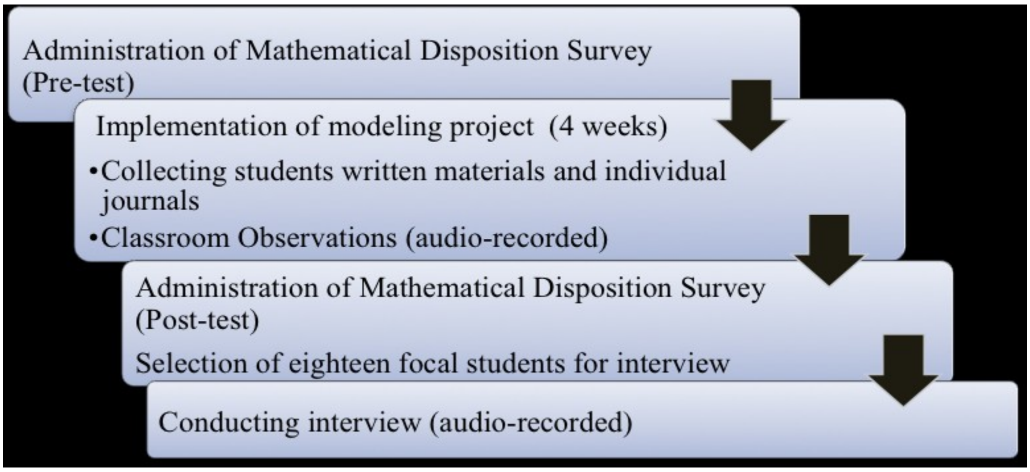

Figure 1: Procedures (Multiple methods)

RESULTS

The result from a paired samples t-test showed the significant changes in students' mathematical disposition between pre and post survey. There was significant difference in the mean scores for Mathematics Disposition Pre-test (Mean =132.57, SD= 23.65) and for Post-test (Mean=138.97 and SD= 24.52 with condition of t=-3.25 with $p < 0.01$) (See Table 2.)

	Pre-Test	Post-Test
Mean	132.57	138.97
Variance	567.12	607.65
Observations (N)	47	47
Pearson Correlation	0.85	
df	46	
t Stat	-3.248	
P(T<=t) two-tail	0.002	
t Critical two-tail	2.013	

Table 2: Descriptive Statistics and Paired Samples T-test (Mathematics Disposition)

For further investigation, the pre and post survey mean scores in each aspect of mathematics disposition (confidence, flexibility, perseverance, interest and inventiveness, meta-cognition, usefulness and appreciation) were analyzed by a paired samples t-test. Students' gain score from pre to post test was statistically significant for the aspect of flexibility (Pre: Mean=10.53, SD=3.69; post: Mean=12.09, SD=2.51 with $t=-3.28$, $p<0.01$), for the aspect of appreciation (Pre: Mean=16.49, SD=5.34; Post: Mean=18.06, SD= 4.61 with $t=-2.62$, $p<0.01$). Interview data were analyzed searching for evidence of 'changes' in disposition, identity, and students' perceived value resulting from engaging in modeling activities in a collaborative group.

Students' changes in disposition and identity

As the main parts of the modeling project involves collecting their own data, mathematizing the data, and interpreting the data, the aspect of modeling with students' own data promoted autonomy among students and alternative ways of solving problems (the aspect of flexibility) by reflecting their own thinking through collaborative group work:

Excerpt 1. Change in the aspect of flexibility and meta-cognition (Value of gains)

Chloe: Group members shared different ways of doing it, when they found different solutions from others, they thought about it and came back to talk about it. If you work with three other people, you can get different perspectives. For instance, there was a case that we had same answers but we found everyone solved them in different ways. Something like that, it was cool. Also no one depends on anyone since I myself had to live my days and collect my own data.

The aspect of modeling that was relevant to students' everyday lives seemed to have contributed to their development of positive disposition and personal identity as doers of mathematics:

Excerpt 2. Change in the aspect of interest and confidence, and identity transformation (Value of beauty, gain, and identity)

Ella: I had difficult in doing math entire my life until to this day. I feel like this project would be beneficial for the students like me. I think it is important to think analytically and think outside of box through this kind of project. I changed my view of math in the sense that it became enjoyable since this project gave me some excitement. The project used math but it was interesting.

Sense of belonging and social value: what it means to understand mathematics

There was the evidences indicating socialization in a group through emotional connections by asking for help and sharing stories of events with particular topics:

Lisa: We talked about our data, also our personal lives, why we had these numbers, what electric devices have used. I had a big number and she had a smaller number than mine. Then we talked about why and talked about the details in our personal lives. Especially with Alexis, cause we both live with family and others lived on campus so our numbers were pretty close but others' were very different from ours. We talked about it at the personal level. Generally, as for a group work, some people do not do their parts but in our case, everyone contributed their parts.

Interpreting mathematical results and social value: what counts as mathematical argumentations in modeling tasks

'What constitutes mathematics argument' was related to the affiliation with the group, and the real life context of modeling tasks helped them to establish socio-mathematical norms.

Interviewer: While working on this project in a group, how did you guys decide the solution is correct?

Deana: we took a look at them to see if they make sense, like realistic number not too high or too low, one girl's number was so low everybody else was high, so we told her "you did something wrong". Then we found that she forgot to add something.

With regards to the norms of what counted as an acceptable or valid mathematical explanation, students justified it based on the real life contexts by comparing data and examining the process of measuring footprints with group members.

CONCLUSION

In Makiguchi's theory of value, benefit or gain as advancement of the life of the individual in a holistic manner and that is beneficial aspect of the interactions with an object (mathematics in this study), for example, one did develop the level of confidence in doing mathematics and was able to express his or her idea within a group or developed one's willingness to navigate alternative ways of solving problems and monitor own thinking. The individual creates value through contributing to the well-being of the larger human community and society (Ikeda, 2001). Social value was created through students' interactions with the external context to mathematics and also with other members while working in a group. A mathematical identity and norms were related to students' sense of belonging related to students' participation in the mathematical community of practice (Wenger, 1998). Students deeply engaged with mathematics through modeling activities by sharing mistakes, listening to and offering suggestions about other's work, and thinking about rationales behind why particular decisions were meaningful. The development of dispositions can be understood as being shaped by the interrelation between the context of mathematics tasks and interactions with others. For further study, By examining students' modeling activities and interactions with peers in the classroom, one can understand better, how these elements interplay with students' construction of disposition and identity.

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IMPROVING PROBLEM POSING CAPACITIES THROUGH INSERVICE TEACHER TRAINING PROGRAMS: CHALLENGES AND LIMITS

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The paper presents the results of a study based on a training program for in-service mathematics teachers, targeting to improve their skills of problem posing and qualitative appreciation of problems. During this training program, we found an improvement in participating teachers' availability to discuss and analyse math problems, but also resistance to adapt posed tasks to the students' thinking.

INTRODUCTION

The knowledge unique for an effective teaching of mathematics has been a research focus in mathematics education ever since Shulman (1986) introduced the concept of pedagogical content knowledge. Ball, Thames and Phelps (2008) further refined Shulman's initial framework and proposed a structuration of mathematical knowledge for teaching into components as common content knowledge (CCK), specialized content knowledge (SCK), knowledge of content and students (KCS), and knowledge of content and teaching (KCT). From these, the SCK represents the mathematical knowledge needed for teaching and it is needed in tasks as "modifying tasks to be either easier or harder, finding an example to make a specific mathematical point" (Ball et al., 2008, p. 400).

Ball et al. were primarily interested in operationalizing the acquisition of SCK in pre-service teacher training. However, research shows that task adaptation - as a form of problem posing – is a challenging activity for in-service teachers, independently of their teaching experience (Silver, Mamona-Downs, Leung, & Kenney, 1996). A second factor adding on the complexity of the in-service teachers' case is the fact that many of the teachers obtained their initial training under a different educational paradigm. A third element worth mentioning is the teachers' in-depth experience with curriculum materials, textbooks and evaluation systems. This experience can lead to situations where teachers assign differentiated importance to elements of the content to be taught, whether this is expressed in problem types, concepts or strategies for problem solving and thus limit their adaptations. Consequently, professional development programs for in-service teachers aiming to create conditions for acquiring or refining their SCK might require a completely different approach from the one employed in the case for pre-service teachers.

In this paper, we look at an implementation of a teacher-training program and analyse the impact (benefits and limitations) of its design on in-service teachers' ability to pose multiple-choice problems relevant for students' learning with understanding.

In particular, the questions we ask in this paper are: Which elements of an in-service teacher training program might prove to be useful in helping teachers improve their problem posing skills? Which aspects of problem posing (PP) might prove to be effective to achieve this goal? More specifically, our hypothesis was the following: if, based on structured strategies, we systematically expose teachers to PP contexts, then their willingness to discuss and analyse problems will increase, and their ability to pose and solve problems focused on students' understandings will improve.

BACKGROUND

In-service teacher training programs (ITTP), or professional development programs, are seen as part of the teachers' learning process throughout their career (Broad & Evans, 2006). Their purpose is to provide, between others, the context for acquiring deep and broad content knowledge and knowledge about teaching and learning. Traditional ITTP were organized as "formal, highly structured activities outside the context of teachers' actual work" (Schlager, Fusco, & Schank, 1998, p. 2). However, Broad and Evans, synthesize some characteristics of effective ITTP that contrast traditional ITTPs. Based on their literature review, the authors enumerate the features of ITTP that makes them useful: they link teacher and student learning; they must be personalized, and, the key element for its success is collaboration, shared inquiry and learning from and with peers. Under these conditions it is more likely that teachers will adopt the newly learned approaches in their classroom teaching.

A second aspect relevant to our paper is problem posing (PP), in particular posing multiple choice questions. Literature on PP in mathematics education has increased significantly in the last two decades. Here we adhere to the definition given by Silver et al. (1996) according to which PP is defined as the creation of new problems or modification of an existing one. As far multiple choice problems are concerned, the literature consists mainly of tips, hints and recommendations on how to build distracters with no indications for a more systematic approach. The task of creating multiple-choice questions is challenging since the distracters need to consist of answers to which the teacher can (clearly) associate an interpretation in terms of the student's knowledge and understanding. However, hint and feedback formulation is a challenging task for teachers (Singer & Voica, 2013).

METHODOLOGY

Participants

The sample used in this experiment consisted of 51 in-service mathematics teachers at junior and high level, participants in a training program. The ITTP was organized by the authors of this paper and consisted of 5 days training within a summer institute. The purpose of the training was to enhance teachers' assessment related competences. More specifically, the training targeted at improving teacher's competency to pose multiple-choice type problems and analyse them based on a set of criteria.

Tasks and design of the study

This study is based on detailed observation of the participants' behaviour during discussions and their written work submitted during the training period. During the program, overall organized as a sequence of interactive workshops, participants had to solve a series of tasks such as: Discussion and comparison of problems' formulation from the point of view of mathematical coherence and consistency, degree of difficulty and their usefulness in teaching particular concepts; Formulating distracters for a given problem; Modifying problem elements such as data, the problem question or the distracters; Formulating hints and feedback for choosing certain distracter. By the end, participants were required to pose a multiple-choice problem along with: feedback to students on each distracter, a hint for solving the problem and two modifications of the initial problem (one easier and one more difficult). The problems were presented and their qualitative aspects discussed in-group, during a final session.

For each task performed during the training program, participants completed work sheets where they noted the solution of the tasks, further proposals originating from their colleagues and comments/observations raised during the group discussion. All these files were scanned and posted on an e-learning platform associated to the training program. The final session, during which each participant presented his/her problem in front of the group, was audio and videotaped.

We analysed, holistically, the problems generated during this workshop and the whole group discussions related to them. In this analysis, we looked at various aspects, as: the didactic potential of the proposed problems – comparisons with easier and harder version included; the quality of the feedback given on distracters; quality of hints for solving the problem; the focus of the collectively conducted analysis. Here we present a qualitative study of those data.

RESULTS AND DISCUSSIONS

The ITTP presented few challenges. We see these challenges originating from two sources. On one hand, in Romania there are no specialized programs on PP and, consequently, this approach is a novel offer in ITTPs. On the other hand, many of the participants in our study were supervisors involved in the organization of mathematics Olympiads. The specific goal of these contests is to select highly trained students by facing them with difficult/advanced problems. In contrast, a teacher is more interested in advancing their students' understanding through the problems they offer them. From this point of view, Olympiad problems are more performance oriented, while the problems that teachers need in their classroom teaching are learning oriented. The need to change participants' vision about the finality of problems proved to be a challenging task for the organizers at the beginning of ITTP.

The training program: challenges and failures

We started the program with the optimistic expectation that teachers' competency in posing and solving problems (by having as reference point students' understanding) will improve. However, the conclusions must be nuanced.

The ITTP aimed, among other goals, to develop a self-reflexive attitude of the participants in what concerns the quality of the proposed/chosen problems for class work. In order to create the circumstances for the development of a reflexive attitude, the workshops were designed to be interactive, with systematic feedback from peers and instructors. Our impression, as instructors, from the workshops was one of real progress in participants' ability to pose and analyse problems from the point of view of its affordances to promote student learning with understanding. However, when the participants performed their final products, in an individual manner, their old conceptions proved stronger in influencing the PP process than group discussions.

In the following, we present the analysis of some components of trainees' posed problems; the selection highlights key-points of recurrent situations and each example is representative for the general case.

Example 1. The following problem was proposed by M:

Consider the following sequence of numbers: $a_1 = \frac{7}{3}$, $a_2 = \frac{11+2\sqrt{7}}{\sqrt{7}+2}$, $a_3 = \frac{17+\sqrt{70}}{\sqrt{10}+\sqrt{7}}$. *Then*, $a_4 = \ldots$

In solving the posed problem, M starts from the following sequence of equalities: $a_2 = \frac{11+2\sqrt{7}}{\sqrt{7}+2} = \frac{7+4+\sqrt{7}\cdot\sqrt{4}}{\sqrt{7}+\sqrt{4}} = \frac{\sqrt{7}^3-\sqrt{4}^3}{7-4}$. M considers as natural the idea that, after processing as above the term a_2 and observing the other two given terms, the solver could "guess" the following rule: $a_n = \frac{\sqrt{3n+1}^3-\sqrt{3n-2}^3}{3}$. Therefore, he can identify the "correct" answer ($a_4 = \frac{13\sqrt{13}-10\sqrt{10}}{3}$).

During the analysis of the problem's elements (formulation and distractors; feedback to solver; hint for solving), we have not witnessed even a minimal concern for the solver. In designing the problem, its author has not thought of how the problem might be understood or seen by others. It seems that the poser started from a pre-existing idea he wanted to put forward, and the concern for the solver was hindered by an attitude such as: "If I thought about it, others will also do."

Such behavior was not a singular one; several participants displayed a proudness stemming from the fact that they proposed competition problems which could be solved by very few students. In quite a few problems proposed at the end of the ITTP we could identify the same attitudinal pattern.

Yet, teachers' attitude has to be considered in perspective: in a culture dominated by school competitions aiming at strict mathematical performance, problem solving often is reduced to a formal game with mathematical concepts.

Example 2. The following problem was proposed by D:

Mircea and Cristi have together 28 years. Victor and Mircea have together 26 years. Andrew is half of Cristi's age. Victor, Mircea and Cristi were together 42 years. Which of the four boys is the second, if they are considered in increasing order of the ages? a) Mircea b) Victor c) Cristi d) Andrei e) It is impossible to know.

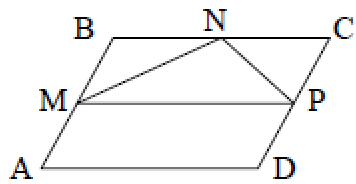
The background topic of this problem is an artificial one, being only used as a kind of "cover" for a system of linear equations. However, as such, it is just a reflection of the kinds of problems often encountered in the textbooks.

We focus another aspect now. The feedback to student on answer e) was formulated, as "The answer is absurd." If we evaluate the feedback from the point of view of the usefulness for the student, we have to conclude that this is not informative at all. The feedback doesn't help students to realize what was wrong in their solving. For example, if the student fails to translate the problem into a system of equations and solve the system, then for that student it is really impossible to know the answer! A better choice for this distracter might be: "all the four persons have the same age": a minimal understanding of the meaning of the given data could lead immediately a solver to decide that this answer is (really) absurd.

In fact, the feedback should be informative to the student; and teachers' activities should start from imagining difficulties students might encounter in contrast with mistakes they could commit. In this respect, comments like "your solution is not right" or "you are wrong" (seen relatively often in proposals of the participants) are useless for the solver and might have a negative impact on their confidence. Comments as such made by D (and others in our sample) might originate from a traditional view of teaching: where the teacher "delivers" methods and content and, thus, the sole responsibility for failure is on the student who "didn't try enough".

Example 3. V proposed the following problem:

The parallelogram from the next represents a garden, whose area is 24 dam^2. M, N, P are the mid-points of the sides AB, BC and CD. The area of MNP (cultivated with flowers) is equal to:



a) 6 b)3 c)9 d) 12 e)18.

V's intention was to propose a "realistic" problem. In his conception, the problem should get this characteristic just from using words that are from everyday use. However, the problem is purely mathematical: he even uses a geometric drawing and geometrical terms (midpoint).

From a mathematical point of view, the problem is correctly formulated and might be a useful problem for students studying area in context of special quadrilaterals. However, the quality of problem formulation, due to the mixture of language, is poor. What might explain the participants' inability to "see" this aspect of the problem? We hypothesize, again, that their experience as teachers is marked by strong focus on content (and the reasons for this are so varied that we shall not dwell on them at this point). Therefore, they perceive the problem already beyond its formulation: as to what it is once in pure mathematical terms. Our hypothesis is sustained by the fact that, once the problem is subject of discussion, they realize the shortcomings – however, when they pose the problem they seem completely "absorbed" by the mathematical aim of it.

The above examples reveal some important shortcomings in teachers' PP, such as: certain "blindness"/"short-sightedness" about problem formulation; un-informative feedbacks; artificial problem contexts and failed attempts to connect to everyday situations. These aspects were recurrent in most of the participants' problems.

In comparison with a relative isolation at the beginning, progressively during the workshops, participants expressed their willingness to engage in discussion of the problems proposed by peers and by themselves.

Their availability for involvement in the problem analysis process could be assessed through the dynamics of interventions during the workshops, but also through records from the final assessment, where the participants pointed out the usefulness and relevance of ITTP for the work of the teacher in a class.

We insert below some comments of participants from this last category.

"Even after 30 years of teaching, I learned a lot."

"Most interesting parts were the discussions about changing distracters."

"The part of maximal interest was about distracters and feedback to student."

If at the beginning, most teachers did not want to expose their products to group discussion (mainly of fear of value judgments), towards the end we witnessed participants' openness in this regard and even a desire to get feedback on personal creations. Furthermore, we found that by the end of the seminar, the quality of problem analysis has improved. If at first discussions often slipped to collateral subjects, towards the end the themes addressed were converging towards key aspects of PP. Moreover, even the supervisors - initially reluctant to the idea of a training program with focus on PP - became actively involved in the process; not as much eager to share their own experience, but to learn more about the proposed methodology. We selected one more point that further illustrate the nature of discussions generated during the analysis of the posed problems.

A participant, M, suggested the problem below, which generated a discussion about the possibility of applying them to different classes:

Find out how many numbers in the sequence: 1, 4, 7, 10, ..., 301, are divisible by 5.

In presenting his solution, M used an algebraic method, which involves writing the numbers from the sequence as $3k + 1$ and identifying the numbers of the given form that are multiples of 5. Then, M indicated a second method of solving that goes more towards exploration: writing more terms of the sequence, one can observe that in each group of five consecutive terms, exactly one is divisible by 5.

Based on these solutions, participants discussed the possibility of proposing this problem for different classes, how distracters can be adjusted, or how the problem's difficulty level can be controlled. For example, the discussion revealed that a more difficult problem might be: "*Find out how many numbers in the given sequence are divisible by 5 or by 2.*" On this proposal, discussions continued on the possibility to adapt the initial solutions to the new problem. More specifically, it was concluded that it can be solved by the principle of inclusion and exclusion, or by an exploratory approach (in any sequence of 10 consecutive terms of the series, there is the same number of terms divisible by 2 or 5). Therefore, in the context of the ITTP sessions, even a "classical" problem led to relevant discussions on its exploitation in different class contexts.

Other presentations have also generated extensive discussions, touching aspects of the nature of the context of a problem, its veracity, correctness in a strict mathematical sense, and the possibility to correlate mathematical correctness and the need to create attractive contextual problems.

CONCLUSION

In this paper, we presented a study on the benefits and limitations of an ITTP as teachers' PP skills of multiple-choice problems are concerned. Two major aspects were identified. On the one hand, we observed a certain resistance from the behalf of teachers to shift in their problem posing process towards interpretations of students' thinking. Next, we synthesized the different manifestations of this resistance. During problem posing, teachers gave a superficial attention in formulating the problem: often, the problem formulation is elliptic or full of ambiguity, while background topic is irrelevant for students' motivation. Some formulations reflect a cognitive behaviour of the type: "if I had thought of this, surely the student will have the same idea". We interpreted such case as one of "blindness/short-sightedness" of the problem poser since it prevents him/her from seeing the problem objectively, from the readers' point of view. This attitude caused strong resistance to any suggestion of a need for change.

On the other hand, we document a change of participants' behaviour that happened on two dimensions, both observable in group interactions: an openness to discuss and analyse the quality of own posed problems, as well as a capacity to focus on some key-aspects of PP. The training program, thus, contributed to the development of a reflexive attitude as far as problem appreciation is concerned. These results illustrate that our starting hypothesis was only partially confirmed and that it needs further refinement.

We found a positive impact on group interaction, while individual products still exhibited thinking patterns linked to traditional views of teaching and learning. We interpret this situation as indication for changes to be brought to the training program. It seems that a follow up that systematically combines group interactions with individual tasks exploited during further interactions could significantly influence a PP approach focused on students learning with understanding. An ITTP built on such design principle might be a topic for future research.

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TEACHER SEMIOTIC MEDIATION AND STUDENT MEANING-MAKING: A PEIRCEAN PERSPECTIVE

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To interpret in detail the meaning-making in the classroom and the corresponding teacher semiotic mediation, we have resorted to Peirce's triadic Sign theory, interpreted by Sáenz-Ludlow and Zellweger. We present an example of the use of a few elements of that theory in the analysis of a classroom episode in which meaning is constructed with the teacher's semiotic mediation.

Meaning-making in the classroom (e. g., Antonini & Maracci, 2013) and the corresponding teacher semiotic mediation (e. g., Mariotti, 2012; Samper, Camargo, Molina & Perry, 2013) have gained importance as theoretic constructs for describing and explaining mathematics teaching and learning. In our case, they are fundamental due to the teaching approach (see details in Perry, Samper, Camargo & Molina, 2013) with which the pre-service mathematics teacher plane geometry course is developed in the Universidad Pedagógica Nacional (Colombia). This course has been the setting of our research, concerning teaching and learning proof, since 2004 (e. g., Camargo, Samper, Perry, Molina & Echeverry, 2009; Molina, Samper, Perry & Camargo, 2011). With the intention of interpreting in detail both phenomena, we have started to recur to elements of Peirce's triadic Sign theory based on Sáenz-Ludlow and Zellweger's (2012) elaborations. In this paper we analyze, in the light of such theoretical elaborations, a classroom episode in which geometric meaning is constructed with the teacher's semiotic mediation. We want to contribute to the determination of what it means to adopt a semiotic perspective of teaching and learning, inspired on Peirce's triadic sign idea. The analysis presented is part of an ongoing research on the use of conjectures as class content organizers, a project which is financed by the Colombian national science foundation, Colciencias.

SPECIFYING THEORETIC ELEMENTS

Peirce's distinctive contribution is to conceive SIGN activity (semiosis) as one in which three components are related: sign-object (*so*) that which is alluded to in a communication or thought, sign-vehicle (*sv*) the representation with which the object is alluded to (e. g. a word, gesture, graph), and sign-interpretant (*si*) that which is produced by the sign-vehicle in the mind of whoever perceives and interprets it.

Succinctly, we describe the semiosis that takes place in a verbal exchange constituted by two turns: in an *intra-interpretation* act (self-self), a person Y selects a particular aspect of a *sign-object* that is part of his *sign-interpretant*, encodes it and expresses it in

2014. In Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 4, pp. 409-416. Vancouver, Canada: PME.

a *sign-vehicle* addressed to a person X; in an *inter-interpretation* act (self-others) that takes place within his knowledge and experience, X decodes the *sign-vehicle* emitted by Y and constructs a sign-*interpretant* which determines a sign-object.

Delving deeper into the sign-object, there are three subcategories: Mathematical Real Object (*MRO*), immediate object (*io*), and dynamic object (*do*). The Mathematical Real Object is a historic-cultural object constructed by the community of mathematicians, which serves as reference for the community of mathematical discourse. The *sender's immediate object* is constituted by the specific aspect of the Mathematical Real Object that he wants to represent with a sign-vehicle. The *receiver's dynamic object* is constituted by the aspect interpreted from the sender's sign-vehicle. The immediate object is expressed in the sign-vehicle that carries it while the dynamic object is generated in the receiver's sign-interpretant. For this reason, for the analysis, it must be inferred from one or more sign-vehicles. This makes clearly distinguishing the dynamic object from the immediate object harder when the person changes his role from receiver to sender and there has not been a substantial change in the Real Object's aspect that the person is referring to.

In a dialogic interaction (a collective semiosis) in the classroom, which purpose is to make sense of an *MRO*, a sequence of SIGNS from different semiotic systems is naturally used. The *do*'s that emerge in the students' minds from the interpretation of these SIGNS will be, in a lesser or greater degree, in *accordance* with the teacher's intended *io*. The teacher's intentional *semiotic mediation* is constituted by all his deliberate actions that facilitate and guide the convergence of the students' evolving *do*'s to the intended *io* of the SIGNS. For this to happen, the teacher infers, interprets, and integrates, into one dynamic object, the most significant aspects of the *do*'s articulated by the students' that, in one way or another, he deems necessary in the evolution of their *do*'s as they try to make sense of the intended *MRO*. We call *didactic dynamic object* (*ddo*) this emerging and evolving dynamic object that is inferred and constructed by the teacher as a result of an intentional classroom interaction. The teacher uses his constructed *ddo*'s to make those didactical decisions necessary to facilitate the evolution of students' *do*'s so that they will approximate the intended *io*.

EPISODE CONTEXT

The episode took place in the plane geometry course developed during the second semester of 2013. The course is a second semester course of the pre-service teacher program and one of its intentions is that the students learn to prove and widen their view of proof. The teacher, coauthor of this paper, has ample experience in the respective curricular development.

The students, in groups of three, after solving the problem, "Given three non-collinear points A, B and C, does there exist a point D such that $\overline{AB}$ and $\overline{CD}$ bisect each other?", using Cabri, formulated a conjecture related to the construction carried out to solve the problem. One of the groups constructed the three non-collinear points A, B

and *C*, $\overline{AB}$, the midpoint *M* of $\overline{AB}$, line *CM*, the circle with center *M* and radius *CM*, and determined *D* as the intersection of the circle and line *CM*.

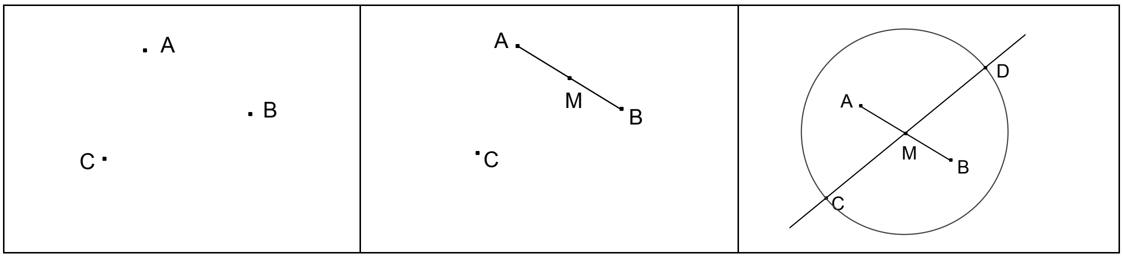

Figure 1

The conjecture the group presented, taking into account the teacher's request to specify in the conditional statement's antecedent the conditions for *D*, was: "Given three non-collinear points *A*, *B* and *C*, if *D* belongs to line *CM*, such that $\{D\} \neq \{C\}$, *M* midpoint of $\overline{AB}$ and $MD = MC$, then $\overline{AB}$ and $\overline{CD}$ bisect each other". Interacting in an instructional conversation (Perry, Samper, Camargo & Molina, 2013), teacher and students proved the conjecture. They then analyzed how the proof would change if the condition *D* belongs to line *CM* is substituted for *D* belongs to ray *CM*, concluding that different warrants would be involved which lead to different possible betweeness relations of points *D*, *C* and *M*. Immediately, the teacher questioned the existence of a point *D* with all the conditions imposed in the antecedent of the conjecture; due to this, they began the task of specifying what in the theory they then had to permitted them to construct each geometric object involved (i. e., validate the construction). Specifically, they could guarantee the construction of the segment, its midpoint and the ray or line, but not the construction of the circle used to determine *D* because the available theoretic system did not include geometric facts about circles.

The scene of the episode that we analyze in this paper is solving this difficulty, motivated from the theory. It starts with the teacher asking the class how to substitute the construction of the circle, that is, how to obtain point *D* with the required conditions without using the Cabri option "circle". The following are possible appropriate answers[1]: (i) Having points C and M, construct line *CM* to assign to them coordinates *y* and *x*, respectively, with $y > x$, use the defined metric to find the distance from *C* to *M* ($y - x$) conviniently construct the number *z* ($z = x + (2y - 2x) = 2y - x$); assign to *z* a point which turns out to be *D*. (ii) The procedure is the same as the previous one except that zero is assigned as *C*'s coordinate, and therefore, the real number conveniently constructed is 2*y*. (iii) Having points *M* and *C*, determine (without using coordinates) the distance (*MC*) between them; construct the ray opposite to ray *MC*; use the latter ray and the distance determined to locate point *D* on that ray.

[1] They are based on the Real-Number-Line Postulate which establishes that: (i) to each point of the line there corresponds a unique real number and (ii) to each real number there corresponds exactly one point of the line.

EPISODE ANALYSIS

With his *sv*, "How can we make point *D* appear without using the circle?", the teacher sets a task with the purpose of helping the students begin to make meaning of the Point-localization Theorem: "Given ray *CT* and a positive real number *z*, then there exists only one point *X* such that *X* belongs to ray *CT* and $CX = z$, reason why we consider this theorem and its proof, which is not part of the available theoretic system, as the *MRO* of the semiotic activity when the task is carried out. Now, the teacher's *MRO* is the process described in (iii). Specifically, the teacher wants the students to experience constructing the conditions that permit localizing a point *X*, since in each future situation in which the theorem is used they will have to begin by establishing the real number *z* and the ray on which the point will be located. The teacher's *sv* carries as *io* the possibility of defining a procedure to find *D* with a certain betweeness and at a certain distance from a specific point, without using circle. Taking into account the question's context, the *io* can be specified as: the possibility of defining a procedure to find *D* such that *M* is between *C* and *D*, and M is equidistant from *C* and *D*, without using a circle.

María[2] answers: "There we already **have a ray** [*CM*, (or a ray on line *CM*)], we could take measurements, first take a measurement from… *C* to *M* and then we take **the measurement from M to…** that is, take another measurement and **up to where that measurement gives us** (with the fingers of the extended hands, and these in perpendicular planes, she lets the right hand fall over the left one twice, **gesturing cutting**), there put point *D*". María's *sv* includes enunciation and gestures with her hands as she verbalizes. It carries as *io* a procedure which consists of measuring the distance between *M* and *C* and copying it starting at *M* on ray *CM* (which corresponds to transferring the measurement on the opposite ray of ray *MC*) to determine *D* such that *M* is between *C* and *D*. Above we have used boldface letters for the specific signs which back María's *io* description. It seems María's *si* includes an image of a physical compass capturing a distance and transporting it, and her *do* is consistent with the teacher's *io* in so far the student defines a procedure with the required conditions.

The teacher interprets María's answer as appropriate to continue constructing the details of the *io* he expects the students to approach; so his *ddo* includes the coordinates as resource to obtain the distance between two given points. The teachers' *sv*, "If we take measurements, what do we automatically need?" carries as *io* the same aspect of the procedure –find measurement– which María made reference to, but it also indicates that she must specify which geometric object she will use to obtain distance, request that although it does not carry, for any listener, explicitly an aspect of the procedure they are talking about, it can be understood by whoever knows that the use of coordinates together with a metric are required to find the distance between two points or the use of the Two points-number Postulate that declares the existence of the distance between any pair of different points. Ángela interpreted the first option. Her

[2] This and the other student names are pseudonyms.

sv, "Coordinates", carries as *io* the use of the points' coordinates to determine distance; in this moment of the conversation, her *do* seems to coincide with the teacher's intended *io*, who accepts her idea to use the coordinates to find the wanted distance. From the expression "take measurements" with which the teacher echoes María's proposal, we see that he accepts the path she suggests, using coordinates, although using the afore mentioned postulate is a more direct way to introduce the Point-localization Theorem. This is a teachers' didactic decision.

The next intervention is Dina's with the *sv*: "we can say […] that twice the distance from A to M is equal to the distance from A to B, right? Because it is a midpoint. So, using Ángela's idea, with coordinates, we can say that twice CM is equal to the distance from C to a D that I am going to place somewhere. Then, already there we are placing…". This *sv* carries two immediate objects: first, a relation between distance measurements implied by the midpoint of any segment ($2AM = AB$); second, the possibility of applying the relation mentioned to C and M, having obtained the required distances using the points' coordinates. We see that Dina's *si* includes a different condition to the one used so far ($CM = MD$), to characterize the midpoint of a segment ($2CM = CD$), condition she wants to use with coordinates to find D. Her *do* is a procedure to determine D as the endpoint of a segment with C as the other endpoint and M as midpoint, using the relation established by the Midpoint Theorem (i. e., If M is the midpoint of $\overline{AB}$ then $2AM = AB$ in terms of coordinates. Dina's *do* is relatively close to the teacher's intended *io* because it satisfies the condition of not using a circle to locate D, and also, when proposing the use of the afore mentioned theorem she is constructing the z mentioned in the Point-localization Theorem. That is, she substitutes the use of the circle with the use of the midpoint of $\overline{CD}$ not determined yet (which leads to having M between C and D) and using C as the initial point from which the constructed distance is put. As will be seen, this *do* is lacking proximity to the teacher's *io* in what concerns how to use the coordinates and for what.

The teacher points out that Dina and Ángela's ideas are pertinent. Besides, he clarifies that: "Then, we use coordinates to guarantee the distance that I want it to be", *sv* that carries as the teacher's *io* the role the coordinates will have in the procedure for finding D without using circles. The teacher's *si* includes the idea that the procedure that he aspires to establish in the class is developing adequately, and with his *ddo* he emphasizes that the procedure without circles requires coordinates not only for obtaining a distance but also for using it in determining the coordinate that will turn out to be that of the point that they want to locate.

With his next *sv*, "[…] what you (Dina) want is to use concrete numbers as coordinates. What concrete numbers do you want?", the teacher initiates the construction of an example of the obtainment of the coordinate that they want to determine using the coordinates of C and M. Here we do not analyze the interaction through which the example was constructed because its content is mainly of an arithmetic nature. It is enough to know that the coordinates of C and M were 2 and 4, respectively, and that they concluded, not without some difficulty for some students, that the coordinate of point D had to be 6. Molly explained the reason for this result with the following *sv*: "Because the distance from C to M is… two… units. That from M to D must also be two units, therefore D's coordinate must be six". Molly's *do* seems to be a procedure consistent with: calculate CM and take into account the equidistance condition ($CM = MD$) to obtain D's coordinate by adding CM to M's coordinate.

Having finished developing the example, on the blackboard, written by the teacher, can be read: $c(C) = 2$, $c(M) = 4$, $c(D) = 6$ (i. e., the coordinate of C is two, etc.). With regard to this, the teacher emphasizes the difference between what is represented with the first two notations (having points C and M, to each a coordinate is assigned) and what is expressed in the third (having coordinate 6, to it a point is assigned which is precisely D). This description corresponds to his next *sv*: "These points (signaling the points in the notation $c(C) = 2$, $c(M) = 4$) already exist; we can give them those coordinates, right? After that we would have to say, this number exists, six, (points at the notation $c(D) = 6$) and to this number six, what do we do to it?" Various students respond the question correctly as they say: to six we have to "associate a point, D". From the teacher's intervention, we infer that his *si* includes an image of the difficulty students have to distinguish the conditions under which each of the items of the Real-Number-Line Postulate can be applied and the corresponding effect. His *ddo* emphasizes the distinction of each of the item's application; specifically he highlights that to determine point D it is necessary to first give the real number that will be his coordinate. The *do* of each of the various students who completed the teacher's comment is relatively consistent with the teacher's intended *io*. Later, already having assigned D to the coordinate 6, teacher and students agree that the equidistance condition alluded to by María at the beginning of this episode is satisfied.

Next, the teacher indicates that the procedure carried out in the example must be generalized. Various students propose designing as x and y the respective coordinates of points C and M. Antonio wants to designate D's coordinate but the teacher changes the course of the conversation towards the number z, "First the number… Which one shall it be? A number will appear… then there exists the number z, and what condition should that z have?". The teacher's interaction with various students trying to refine an answer can be summarized as follows: z is a positive number because it is an absolute value, $|x-y|$. When the teacher asks whether what they have said about z is sufficient, Molly responds: "No, we must say that that number belongs to [is associated to] point D". With respect to such an answer, the teacher asks if they agree with that statement and although various students disagree, the comments they make indicate that they find it difficult obtain a general expression for z. From this we see that the teacher's *io* is closely related to, on the one hand, Molly's explanation on why, in the example, D's coordinate should be 6, and on the other hand, the comment he made to emphasize that they have points C and M and coordinates are assigned to them, but with respect to the point they are searching for, first a positive number must be determined and then, the point assigned to it is precisely the one searched. The *do* that appears as a collective construction by the students that participate in the verbal exchange seems to be far from the teacher's intended *io* in three issues: (i) designating *D*'s coordinate without taking into account that they are looking for *D*; (ii) believing that the number z represents the distance between *M* and the point searched for; (iii) believing that the distance between *M* and the point searched for coincides with the coordinate that this point must have.

Seeing the difficulty the students have to obtain a general expression for the number z, makes the teacher simplify the situation by proposing that *C*'s coordinate be zero and *M*'s coordinate y, with $y > 0$. We also gives the value of z ($z = 2y$) and shows that effectively the distance from *C* to *M* is the same as that from *M* to the point to which z should be assigned to.

With the former explanation, they are ready to continue validating the construction, specifically the existence of *D*. In this process, orchestrated by the teacher, students are given the opportunity to respond correctly very punctual questions related with the procedure to determine *D*. These correct answers permit us to see signs of the beginning of a convergence of the *do*, constructed communally by those that participated in the exchange, towards the teacher's intended *io*.

234	Teacher:	[…] So, we can say the coordinate of point *C* is going to be equal to whom?
235	Juan and others:	To zero.
242	Teacher:	[…] Ready, coordinates for points *C* and *M* appeared. What do we have to do afterwards?
243	Jack:	Create the number *z*.
244	Teacher:	[…] What would we do with that number *z* afterwards?
245	Student:	Assign a point to it.
250	Teacher:	A unique point. What will it be called?
251	Various:	*D*
252	Teacher:	*D*, okay, such *D*…
253	Ángela	belongs to the line
254	Student:	*CM*
255	Teacher:	belongs to the line *CM*, okay, and…
256	Juan:	*D*'s coordinate is twice y.
257	Student:	*z*
258	Teacher:	*D*'s coordinate is equal to z. That is the correct way to write it. […]

FINAL REMARKS

The analysis presented is illustrative of the teacher's semiotic mediation characterization, using the elaboration that Saénz-Ludlow and Zellweger have done of Peirce's triadic Sign theory. In this paper we present an extension of that elaboration to include, as a central aspect of the mediation, the *didactic dynamic objects* (*ddo*'s) that emerge and evolve in the course of semiotic mediation, and that seek the convergence of the student's dynamic objects to the teacher's intended immediate object. Conceiving the teacher's semiotic mediation this way permits us to identify in greater detail the teacher's role in students' meaning-making.

It is important to mention that in the analysis carried out here we focused on the discursive student sign-vehicles but not on the gestures with which they accompany their interventions. This is due to the fact that the teacher's *ddo*'s seem to emerge principally from what the students say and not from what they do. Particularly, in María's first intervention with which the communicative exchange begins (when she proposes "take measurements") directs the semiosis through a path that permits advancing in meaning-making of the Point Localization Theorem, relating it to the Real-Number-Line Postulate. However, the gesture with which María accompanies her proposal seems to be a parody (acted out) of the Point-Localization Theorem, which if it had been discussed in class, would have led to a different semiosis and probably to meaning-making of the theorem relating it to the Two Points-Number Postulate as well as to the Real-Number-Line Postulate. This observation leads us to point out the importance of the teacher's didactic decisions, in the course of *semiotic mediation*—the semiosis that takes place in the classroom.

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ANALYZING STUDENTS' EMOTIONAL STATES DURING PROBLEM SOLVING USING AUTOMATIC EMOTION RECOGNITION SOFTWARE AND SCREEN RECORDINGS

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Emotions play important part in mathematical problem solving, yet the theories of their role are still at their preliminary stages. In our study, we introduce a method, where screen recordings and automatic emotion recognition software are used to study the emotional states of five upper secondary school students during a solitary GeoGebra problem solving session. Common emotional states during problem solving were neutral (40 % of time), sad (34 % of time), happy (15 % of time) and angry (8 % of time). Different phases of problem solving were emotionally different, non-neutral emotional states being most prevalent during decision points such as using the undo button. The method used opens possibilities for new kinds of research designs for studying the role of emotions in problem solving.

INTRODUCTION

Affective elements have received much attention in the literature of mathematics education in general and problem solving in particular (e.g. McLeod & Adams 1989, Schoenfeld 1985, Leder, Pehkonen & Törner 2002, DeBellis & Goldin 2006). However, the majority of studies have focused on relatively stable traits in the affective domain, such as attitudes, beliefs and values. Considerably less attention has been given to emotions, defined as "rapidly changing states of feeling experienced consciously or occurring preconsciously or unconsciously during mathematical (or other) activity" (DeBellis & Goldin 2006, p. 135).

Emotions are influential in the key moments that determine the success of solving a problem. Goldin (2000) uses the concept of *affective pathways* to describe how the typical patterns of emotional states lead to successful or unsuccessful problem solving behavior, and in the long run partly shape one's attitudes, beliefs and values concerning mathematics. However, proper understanding about the role of emotional states requires further research.

Studying momentary emotional states is typically work-intensive and therefore the number of subjects in such studies is often small. In this report, we introduce a method that automatizes part of the work and thus opens possibilities for new kinds of research designs: we use screen recording technology to capture student's computer-aided problem solving process and automatic emotion recognition software to analyze student's emotions during the process.

2014. In Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 4, pp. 417-424. Vancouver, Canada: PME.

Phases of problem solving

For research purposes it is often useful to identify different phases of the problem solving process. Several models of phases or stages of problem solving have been introduced by different authors (e.g. Polyá 1945; Mason, Burton & Stacey 1982; Schoenfeld 1985; Hähkiöniemi, Leppäaho & Francisco 2013). Polyá's (1945) model with four phases – *understanding the problem, devising a plan, carrying out the plan and looking back* – is most widely recognized, but it is intended to be rather a guide to a problem solver than a research tool. Although the same applies to Mason's et al. (1982) model, they importantly point out that problem solving process doesn't necessarily proceed linearly along the phases but the solver might for example return to make sense of the problem after some new information has occurred to him or her.

In his studies, Schoenfeld (1985) used transcriptions of students' discussions during problem solving to identify the phases of problem solving. He distinguishes six phases in problem solving: *reading, analyzing, exploring, planning, implementing*, and *verifying*. Essential to his model are also decision points – moments in the problem solving process when a student should use metacognitive skills to decide on further actions. For example, when new information concerning the problem occurs to the student, he or she must consider whether the current attempt to solve the problem is still valid or should a new strategy be used instead.

The two main approaches to collect data on student thinking during the process of problem solving have been *think-aloud* and *stimulated recall* methods. However, as Hähkiöniemi, Leppäaho and Francisco (2013) indicate, it is possible to study problem solving with computer software using only screen recordings as the base for analysis. Hähkiöniemi et al. (2013) investigated problem solving with dynamic geometry software GeoGebra. They found none of the previous models of problem solving directly applicable, and thus developed a new classification, consisting of five phases: *framing the problem, exploring the solution, conjecturing, investigating the conjecture*, and *justifying*. There was great variation between students on how they moved through these phases: some students proceeded linearly from phase to phase, but most of them skipped phases and/or returned to previous phases during the process.

In present study, an adaptation of Schoenfeld's (1985) model is used. Instead of discussion or think-aloud transcriptions, screen recordings are used to conduct the analysis, similarly to Hähkiöniemi's et al. (2013) study.

Emotions and problem solving

There is a general agreement that emotions have an important role in human learning and mathematical problem solving specifically (Hannula 2012). However, theories about the role of emotions in the process of problem solving are still at their preliminary stages (Lehman et. al. 2008, Goldin, Epstein, Schorr & Warner 2011).

It is well established that emotions direct attention and bias cognitive processing. For example, fear (anxiety) directs attention towards threatening information and sadness (depression) biases memory towards a less optimistic view of the past (Power and Dalgleish 1997). There is also indication that positive emotions promote the creative aspects of problem solving, while negative emotions facilitate the reliable memory retrieval and performance of routines (Pekrun & Stephens, 2010). In mathematical problem solving, curiosity, puzzlement, bewilderment, frustration, pleasure, elation, satisfaction, anxiety, and despair have been observed to have important self-regulative functions (DeBellis & Goldin 2006).

There are several theories concerning emotion, emerging from different research traditions. In this paper, we follow the Darwinian tradition, where emotions are seen as products of evolution, they can be categorized to a small number of universal basic emotions (anger, disgust, fear, happiness, sadness and surprise) and these emotions can be identified from facial expressions (Ekman & Friesen 1971; Ekman 1992). This approach allows us to use Facial Action Coding System (FACS; Ekman & Friesen 1978) to identify emotions from students' facial expressions. The process of identifying emotions has traditionally required trained human coders, but during last few decades, automated computer methods have been developed (Bettadapura 2012).

There is some evidence that basic emotions would be rare in learning context. (Craig, D'Mello, Witherspoon & Graesser 2008; Lehman, D'Mello & Person 2008). Therefore, alternative emotion classifications that would suit better the learning setting have been developed. Both Craig et al. (2008) and Lehman et al. (2008) used a learning-centered emotion classification consisting of anxiety, boredom, confusion, contempt, curiosity, eureka and frustration. Pekrun and Stephens (2010) describe a model for emotion in achievement setting, which includes enjoyment, relaxation, anger, frustration, boredom, hope, joy, relief, anxiety, hopelessness, pride, gratitude, contentment, shame, sadness and disappointment.

Hannula (2012) suggests that achievement emotions can also be looked through six basic emotions. Enjoyment, hope, joy, pride, and gratitude are different variants of happiness, whereas boredom, hopelessness, sadness, and disappointment are variants of sadness. Frustration is a variant of anger and anxiety a variant of fear. Relaxation, relief and contentment do not present any basic emotion, but they can be seen as removal of a negative emotion. (Hannula 2012).

In this study, our aim is to find out, how different basic emotions occur in different phases of problem solving. Another important aim is to investigate, how our research methodology, which combines screen recording and automatic emotion recognition, works in the context of mathematical problem solving.

METHODS

Data collection

The data for this study was collected in a Finnish upper secondary school in Helsinki. Participants were five students (two girls, three boys) participating in the advanced mathematics syllabus. Each student separately participated in a session lasting about one hour, of which 20 minutes was devoted to problem solving. The session consisted of getting acquainted with the dynamic geometry software GeoGebra (Hohenwarter, 2002) using a practice applet, working with a two-part mathematical problem using two GeoGebra applets, and discussing the process with the researcher.

The mathematical background of the problem was the circumscribed circle of a triangle. First part of the problem was to find out, whether a circle could be drawn so that it goes through three points, in three given situations. The following instructions were given:

> Let A=(-2,0), B=(0,2), C=(2,0), D=(0,0) and E=(3,-2). Is it possible to draw a circle so that the circle goes through a) A, B and C, b) B, C or D, c) C, D and E?

The GeoGebra applet provided with the problem included the points mentioned in the instructions, a coordinate system and a customized toolbar with following tools: *Move, Delete, Point, Perpendicular Line, Perpendicular Bisector, Angle Bisector, Polygon* and *Circle with Center through Point*.

In the second part of the problem, the GeoGebra applet contained the same toolbar but no coordinate system. Students were given a statement "It is always possible to draw a circle through three given points" and asked to either show that it is true or show that it is false.

Each student's screen was recorded during the use of GeoGebra applets. Integrated webcam of the laptop computer running GeoGebra was used to record a video of student's face. After the problem solving session, a video-based stimulated recall interview (e.g. DeBellis & Goldin 2006), was conducted and audio-recorded, but that data is not analyzed in this report.

Analyzing the screen recordings

We analyzed only the screen recordings of the two problems. Events on the screen were transcribed to a table with time codes. Transcriptions were read multiple times and similar events were grouped together. We ended up with the following event classes: *changes to next or previous applet, chooses a new tool, clicks the undo button, deletes an object, draws an object, explores the menu, holds the pointer still, moves a point and moves the cursor*.

To identify the phases of problem solving, the screen recordings were watched again, looking for phases described in the literature. By making interpretations of students' actions (Table 1), events representing Schoenfeld's (1985) characterizations of reading, analyzing, exploring, implementing and verifying could be found in the data. Moments when a student used GeoGebra's delete tool or the undo button, or changed to the next or the previous applet, were considered decision points. Each event in the transcription was encoded to belong to one of the phases or to be a decision point.

Student action	Phase
Student holds the pointer still above instructions	Read
Student holds the pointer still or moves it around without concentrating on any particular object	Analyze
Student draws objects, moves points or objects or explores the menu	Explore
Student performs a series of actions	Implement
After accomplishing a goal, student holds the pointer still or redoes the actions that lead to success	Verify
Student uses the delete tool, the undo button, or changes to next or previous applet	Decision point

Table 1: Interpretation rules for student actions.

Automatic emotion recognition

Noldus FaceReader 5 was used to recognize emotions from student's face videos. The software is based on the theory of basic emotions and it has been used e.g. in studies of usability (Goldberg 2012), intelligent tutoring systems (Harley, Bouchet & Azevedo 2012) and consumer behaviour (He, Boesveldt, Graaf, & Wijk 2012). FaceReader identifies key points in subject's face and classifies the emotions using an artificial neural network trained with manually notated images (Loijens & Krips 2014). In addition to six basic emotions, the classification includes an emotionally neutral state. As an output, FaceReader produces the intensity of each emotional state 10-30 times per second and the dominating emotional state at each time.

In our analysis, for each face video we first used FaceReader's automatic calibration and then ran the analysis. FaceReader state logs were used to add emotional data to the event transcription. Whenever multiple emotional states occurred during a single event, all the states were recorded (Table 2).

Time code	Event	Emotional state
15:34	Student draws an angle bisector at the intersect of the perpendicular bisectors of AD and BD	Sad
15:38	Student clicks the undo button	Sad/Angry
15:45	Student explores the menu	Angry/Neutral

Table 2: GeoGebra event transcription with emotional states.

All event-emotion combinations were extracted from the transcriptions and separately cross tabulated with the classifications of the event classes and phases of problem solving. A chi-squared test was calculated for both contingency tables.

RESULTS

Most common emotional states during problem solving were neutral (40 % of time) and sad (34 % of time). Happy (15 % of time) and angry (8 % of time) were also common, whereas surprised (3 % of time), disgusted (1 % of time) and scared (0% of time) were rare.

Cross tabulation of event classes and emotional states is presented in Table 3. Surprised, disgusted and scared with 12, 7 and 0 event-emotion combinations respectively were so rare that they were excluded from further analysis. Small expected frequency prevented using a chi-squared test.

Event-emotion combinations	Neutral	Happy	Sad	Angry	Total
Changes to next or previous applet	3	1	5	1	10
Chooses a new tool	25	16	30	7	78
Clicks the undo button	0	5	10	10	25
Deletes an object	3	2	3	3	11
Draws an object	24	17	29	7	77
Holds the pointer still	27	11	19	3	60
Moves a point	19	11	17	1	48
Moves the cursor	21	9	19	4	53
Total	126	73	139	41	379

Table 3: Contingency table of the event classes and the emotional states

The cross tabulation of the phases of problem solving and the emotional states is presented in Table 4. Different phases were emotionally different beyond coincidence (chi-squared test p=0.004).

Event-emotion combinations	Neutral	Happy	Sad	Angry	Total
Read	11	7	12	2	32
Analyze	21	3	10	3	37
Explore	58	42	70	20	190
Implement	19	1	20	4	44
Verify	10	6	11	2	29
Decision points	7	14	16	10	47
Total	126	73	139	41	379

Table 4: Contingency table of the phases of problem solving and the emotional states

DISCUSSION

Our results suggest, that although neutral was the most common emotional state in this study, also some of the basic emotions (happiness, sadness and anger) are common in problem solving. If achievement emotions, described by Pekrun and Stephens (2010), are looked through basic emotions as Hannula (2012) suggests, most of the achievement emotions are covered by these three emotional states.

Results also indicate that different events and the phases of problem solving are emotionally different. However, more research is needed to investigate these differences in detail. The method for classifying phases of problem solving used in this study, although being straightforward, includes multiple presumptions about student thinking.

Goldin (2000) suggests that emotions are in an important role during the key moments of problem solving process. This is in line with our findings about the emotional states associated with decision points (Schoenfeld 1985): the proportion of non-neutral emotional states (happiness, sadness and anger) was greatest during the decision points (Table 4).

CONCLUSIONS

The aim of this study was to use a combination of screen recording analysis and automatic emotion recognition to analyze students' emotions in different phases of problem solving. Our results, indicating that different phases are emotionally different, are encouraging and suggest that this kind of methodology can be used to study mathematical problem solving. More studies with a larger number of subjects are needed to further investigate the potential of this research design.

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DIAGNOSTIC COMPETENCES OF MATHEMATICS TEACHERS – PROCESSES AND RESOURCES

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Reviewing the research on teachers' diagnostic competences shows that most findings focus on the correspondence between teachers' diagnostic judgments and students' actual achievement, while cognitive processes and cognitive resources of teachers in diagnostic situations have been examined much less. We intend to extend this state of research from a domain-specific point of view by empirically identifying and theoretically describing processes and resources of mathematics teachers while judging tasks and students' solutions. In an interview study with expert teachers and mathematics educators (n=6) it was possible to deduce typical steps in a diagnostic process to identify resources (i.e. aspects of teacher knowledge) they relied on.

INTRODUCTION

In mathematics teaching we find many different diagnostic situations, which can be characterized according to their position in the learning process and their respective objectives (e.g. Ingenkamp & Lissmann 2008; Wiliam 2007):

- *Initial assessment* aims at gaining information about the students' conditions for future learning (e.g. previous knowledge of students).
- *Formative assessment* is needed for supporting individuals or for adapting instructional choices during the learning process.
- *Summative assessment* is needed for assessing learning results and can be used for certification or placement of students.

(Depending on the authors the terms 'assessment' and 'diagnosis' are considered either synonymous or contrasting in certain aspects. In this paper we assume no difference). Diagnostic situations can also be differentiated by the level of formality: In addition to formal diagnostic tests, there are also informal diagnostic situations which influence instruction. In mathematics teaching such diagnostic situations are often linked to the activity of working with tasks, e.g. (i) Teachers analyse and select tasks with respect to their potential diagnostic value and (ii) teachers evaluate students' solutions to a task.

Current and recent research focuses on the precision of teachers' diagnostic judgements (dubbed the 'veridicality-paradigm') (cf. Hoge & Colardaci 1989, Südkamp, Kaiser & Möller 2012), while many questions regarding the cognitive processes of teachers during the assessment process and the domain specificity of diagnostic competence remain unsettled (Schrader, 2011). In a similar way that Ball, Thames & Phelps (2008) investigated mathematical knowledge for teaching by analysing teachers activities, we intend to create some insights into teachers'

2014. In Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 4, pp. 425-432. Vancouver, Canada: PME.

diagnostic competencies by analysing their cognitive processes and their use of resources during assessment.

THEORETICAL FRAMEWORK

Concepts of diagnostic competence

Diagnostic competence most often is defined as the ability of a person to judge people appropriately (Schrader, 2011) and measured by numerical indicators for the precision of such diagnostic judgements. Three aspects of precision are frequently studied, each of them being related to specific diagnostic activities and situations (Spinath 2005, Lorenz & Artelt 2009, Schrader & Helmke 1987): (1) The judgement of a *level* of an attribute of a student or a task relates to the situation of selecting tasks with an appropriate content or level of difficulty. One can ask if teachers underrate or overrate such attributes. (2) Judging the *variance* of some attributes within a group of students is necessary for deciding about individualisation strategies. Finally (3) correctly estimating the *rank* (a) of the difficulty of tasks or (b) the abilities of students can tell something on the use of content knowledge for selecting tasks or the knowledge on the relative strengths and weaknesses of the class. It seems obvious that the numerical precision of such judgments can only be regarded as an indicator for diagnostic competence at work. Within this approach knowledge about the structure and the influencing factors of diagnostic competence (pertaining to the task, the student, the context or the teacher) is based on studying the reasons for judgment biases (Südkamp, Kaiser & Möller 2012).

Still there are many open questions left, such as in which way teachers generate diagnostic judgements in the pedagogical context. There is a lack of understanding of cognitive processes of teachers guiding their judgement. Also the domain-specificity or even topic-specificity of diagnostic competence and how diagnostic competence is composed would be of interest. By correlating the above-mentioned indicators Spinath (2005) showed that diagnostic competence should not be considered as general ability but rather as construct that consists of several sub-competences. Still we do not possess any fairly coherent theoretical model of diagnostic competence and empirical evidence for it (Schrader 2011; Anders et al. 2010).

For mathematics education it is a fruitful task to contribute to a better understanding of the processes and the knowledge connected to diagnostic situations with respect to the domain of mathematics. This can be seen as embedded within the broader challenge of constructing a theory of teacher knowledge in mathematics. For example, within the framework of Ball et al. (2008) competences needed for diagnostic activities can be located in several areas: *Common content knowledge* (CCK) is needed to evaluate the correctness of a student's solution for instance, *specialized content knowledge* (SCK) is used for example to vary the degree of difficulty of tasks and *knowledge of content and students* (KCS) helps to understand students (mis-)conceptions and approaches.

Approaches for modelling diagnostic processes

Understanding cognitive processes in diagnostic situations can also be seen as a question within the field of research on expertise. Here one can find several models for cognitive processes in diagnostic situations which can be used as a framework for further research: Croskerry (2009) proposes a model for diagnostic reasoning in the medical context by integrating previous efforts of promoting diagnostic competence of physicians: (1) the intuitive approach leaning on experience and gestalt effects and (2) the analytic approach using knowledge and systematic information gathering. To describe diagnostic judgements she proposes a dual process model (in the sense of Kahneman, 2003) where patterns are processed by an unconscious system and by rational processes of a conscious system which interact in specific ways (practice, override and calibration processes) to reach a diagnostic judgement. The fact that Croskerry (2009) calls this a "universal" model already indicates that this can be considered a broad framework which leaves many space for specification (such as by modelling the conscious system by critical thinking, training, logical competence etc.). Nickerson (1999) proposes a model to describe the process of rating other people's knowledge. First a model of own knowledge is used as an anchor to describe the knowledge of others (default model). In several steps this model is refined by including information on the particularity of one's own position, on the random other and on more and more information on specific others. This way the process of gaining insight in other people's knowledge can be seen as an alternation of anchoring and adjustment (Tversky & Kahnemann 1974). In this model Nickerson can explain frequent tendencies of overestimating knowledge of others. Nickersons model appears to be very general and especially refers to factual knowledge. It should be transferred into pedagogical context with caution. Morris et al. (2009) on the other hand construct a model very specific to a diagnostic situation in mathematics teaching. They show that "unpacking" the sub-goals of a task can be considered an important facet of diagnostic competence with regard to the planning and evaluation of learning processes. The ability to decompose mathematical content within a task can be useful in diagnostic situations to locate students' mistakes. However is doubtful if the "unpacking competence" is enough to master diagnostic situations which require identifying deficient conceptions of students, since misconceptions (such as the "division makes smaller" error in calculating with fractions) cannot be deduced by analysing correct solution processes.

These examples of very different scope show, that there are indeed different frameworks available for modelling cognitive processes and knowledge resources of teachers during diagnostic activities. To substantiate these models it seems desirable to have a concrete picture of cognitive processes of mathematics teachers. It is our goal not to test these general models but to create knowledge on processes in the concrete domain of mathematics teaching that can connect to the more general models and inspire further research in this area.

RESEARCH QUESTIONS

In our study we focus on informal diagnostic situations and teachers working with mathematical tasks in a diagnostic way, such as when judging tasks or evaluating students' solutions. These diagnostic situations often occur when tasks have to be selected and embedded in existing material or when the teacher has to react towards students mistakes spontaneously. For an in-depth investigation of teachers' diagnostic competence in these situations we assume a double focus on processes and on resources during the formation of diagnostic judgments and pursue the following research questions: (1) What kind of *processes* can be identified in teachers´ diagnostic judgements? (2) What kind of *knowledge* do teachers rely on during these processes? By these questions we intend to create a deeper understanding of diagnostic processes but also to further clarify possible components of diagnostic competence of mathematics educators. A long-term objective connected with our research is to derive consequences for teacher education and professional development.

DESIGN OF THE STUDY

As a method to gain information on cognitive processes and knowledge of teachers we decided to capture their reasoning by means of two phased think-aloud interviews (Ericsson & Simon 1993). In the first phase we initiated diagnostic processes by first presenting two tasks and afterwards three students' solutions to each task and asking the participants to evaluate each of them. In the second phase teachers had to reflect their own diagnostic process by describing the process and additionally by giving reasons for their judgement. By this combination of parallel and retrospective think we expected to capture a large part of the relevant processes.

As participants we chose three experienced mathematics teachers and three scholars in mathematics education. The latter had experiences as mathematics teachers *and* as teacher educators (for at least three years in each of their professional phases) and so we could draw on practical experience and reflected theoretical knowledge likewise. The aim of selecting this sample was to find a maximum variety of different processes. Think-aloud-protocols of the diagnostic processes and the reflections of their own processes supplied the data for the analysis in the present study which amounted to 12 evaluations of tasks and 36 evaluations of students' solutions. For the interviews we chose the tasks from the topic "fractions", because of the broad systematic knowledge about students' conceptions, errors and misconceptions in this field. The tasks and the interview guideline were developed and optimized in a pilot study. The students´ solutions were selected so that they represented typical mistakes and frequent misconceptions. Figure 1 shows the tasks and solutions we used.

In the first phase the participants had to analyse tasks. They were asked: "Which challenges do you see? Which difficulties do you expect?" Then the participants were given the three students' solutions to each task and had to evaluate them by answering the question: "Which conclusions do you draw?"

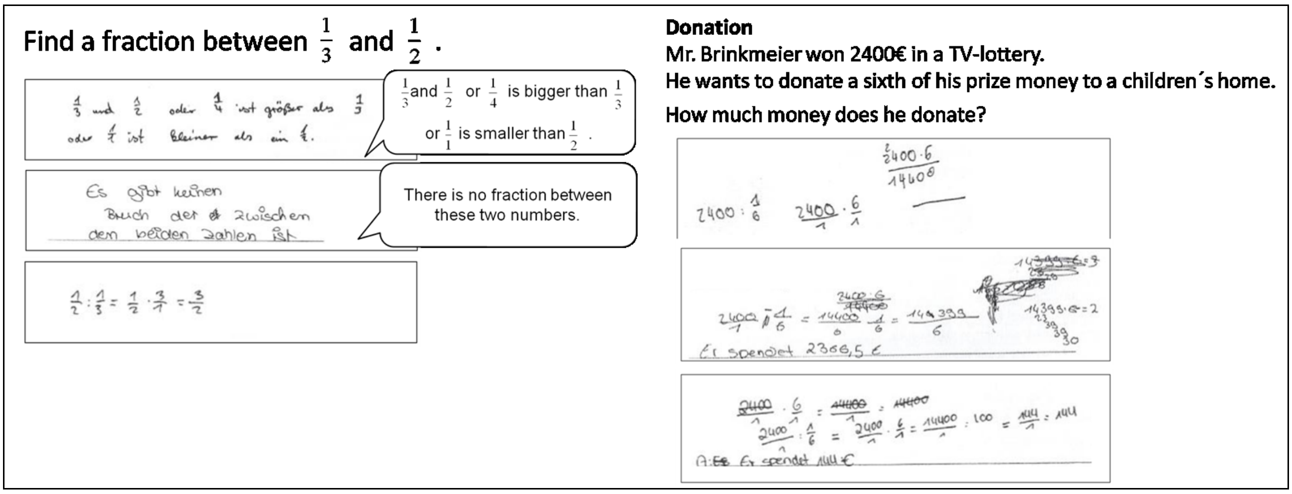

Figure 1: Tasks and students´ solutions (from: Wartha, 2007).

Interpretative content analysis techniques were used to analyse the data (Ericsson & Simon 1993, Mayring 1983) with the objectives were to reconstruct types of diagnostic processes and to generate a theoretical overarching structure. In the analysis we first focused on assessment processes (see research question 1). In the next step we analysed the same data with a focus on the kinds of knowledge underlying these processes (see research question 2).

RESULTS

We present some exemplary results of the two interpretative cycles described above. Focussing on diagnostic processes (see research question (1)) resulted in more than 15 Processes, of which we present three important ones. Table 1 shows the name of the process (code), a description of the code and excerpt of an interview to illustrate the category.

Code	Description	Representative teacher statement
Standard solution	Design a solution for a given task.	"[...] you can solve it by division."
Identify deficits	Discover and name an incorrect approach.	"1/4 is bigger than 1/3. This is typical. When the numbers are in the denominator. With bigger and smaller."
Identify strengths	Discover and name skills.	"[...] this is great. He writes down the number 2400 as fraction."

Table 1: Excerpt of identified assessment processes.

The category "standard solution" refers to the process of designing a solution on your own or mentioning a common solution approach by its name. "Identify deficits" refers to recognizing an incorrect approach in a student solution. Finally to "identify strengths" means to see students´ competences in their solutions. When analysing the same data with a focus on the kinds of knowledge underlying these processes (see research question (2)) we found among others the following knowledge categories: In the first example in Table 1 (see above) the interviewee refers to the mathematical correctness and therefore draws on his mathematical knowledge. In the second example the interviewee explicitly refers to a typical mistake and therefore uses a component of pedagogical content knowledge that refers to systematic knowledge (gained in educational research). In the third example one can see a reference to a concept of a fraction as a rational number – although it remains unclear whether this should be assigned to explicit knowledge on students' development of number concepts or merely to the recognition of a mathematical fact.

When analysing the results of the coding process (which could only be indicated by few examples here) it is possible to develop a "bigger picture": Some of the processes are essential; they show up frequently and can be interpreted as steps in an assessment process. In every step different qualities of individual processes were observable.

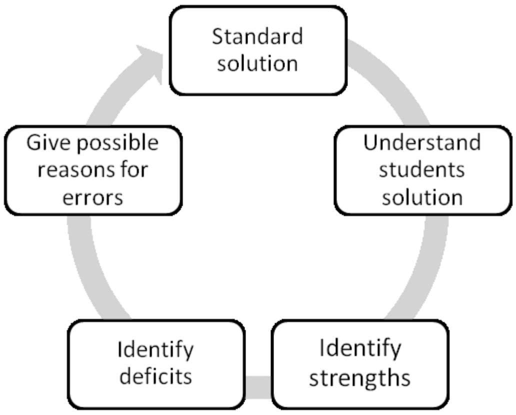

Figure 2: Idealized five-step model for the diagnostic process.

Figure 2 shows an idealized model of steps during informal assessment: The initial point often is a standard solution or an approach. Then own solutions are compared with students' solutions. Thereby strengths and deficits of the solution can be identified. The last step is to find a (hypothetical) reason (or several reasons) for errors – if they occur. As a very common strategy across all steps we observed that interviewees spontaneously decompose tasks or solutions and to analyse them step by step – just as Morris et al (2009) advise the participants in their study.

The cognitive resources the interviewees rely on when moving through the diagnostic process as described above can be characterized as different types of knowledge: We could identify content knowledge (CK) and pedagogical content knowledge (PCK). For example, the participants of our study used knowledge on *mental models* of mathematical concepts, on *typical errors* and on typical *misconceptions*, Furthermore *mathematical correctness* was evaluated and *student strategies* were identified.

CONCLUSIONS AND DISCUSSION

The main objective of our study was a deeper understanding of diagnostic processes. Although such processes followed quite individual patterns, they could be categorized as different types of "steps in the diagnostic process". Furthermore it became evident that the participants showed different degrees of flexibility, for example in the number of possible approaches to solve a task they mentioned (often combined with more than one representation). In future analyses this flexibility may serve as an indicator for the quality of the diagnostic process and/or the diagnostic competence of the teacher. To clarify this connection remains an open question for further study.

Another objective was to identify types of knowledge, which teachers use while forming their diagnostic judgement – this amounts to delineate different components of diagnostic competence. A provisional interpretation of our results with regard to aspects of diagnostic competence is that three different aspects can be identified: (1) *Knowledge*: the use of content knowledge (CK) as well as pedagogical content knowledge (PCK) was observable. (2) *Abilities*: we observed the ability to decompose mathematical tasks but also to analyse tasks and solutions step by step and the ability to take the students' perspective. (3) *Attitudes*: we state that a kind of readiness for assessment is necessary e.g. for taking students perspective – although this aspect did not emerge directly by our systematic analysis but is inferred rather generally from our experience during the interviews.

We regard our results as modest extensions to theoretical frameworks which only partially focus on diagnostic competence, First our results regarding aspects of knowledge can be integrated into the theoretical framework of Ball et al. (2008) but still need further foundation, e.g. by efforts to quantitatively measure the aspects described here. Second, the decomposing of tasks and students solutions found within our study can be considered close to the research by Morris et al. (2009). However, while Morris et al. refer only to teachers "unpacking" mathematical concepts one should also consider the process of teachers identifying misconceptions that cannot be deduced by starting from correct mathematical concepts.

Finally our research also uncovered certain differences in diagnostic processes of teachers with different levels of experience. For example, in our analyses some striking differences showed up, which also should be investigated further: Experts (with a scholarly background) seem to use a variety of different approaches to analyse a task. They appear to be more focused on strengths in their assessment than teachers. Experts draw more explicitly on subject-based knowledge (PCK). Because of the sample size of our study these differences can be seen as tendencies only. They should be regarded as hypotheses which need a more rigorous treatment and may be tested in a different design.

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THE PREDICTIVE NATURE OF ALGEBRAIC ARITHMETIC FOR YOUNG LEARNERS

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The present study revalidated a measurement model describing the nature of early number sense. Number sense was shown to be composed of elementary number sense, conventional arithmetic and algebraic arithmetic. Algebraic arithmetic was conceptualized as synthesis of number patterns, restrictions and functions. Two hundred and four 1st grade students were individually tested on four different occasions. Data analysis suggested that elementary number sense follows a logarithmic growth, while conventional arithmetic and algebraic arithmetic adopt a linear growth rate until the third measurement and then they accelerate. Analysis showed that the growth of algebraic arithmetic directly predicts students' mathematics achievement in second grade and the growth of conventional arithmetic and indirectly the growth of elementary number sense.

INTRODUCTION

Researchers and organizations have documented the importance of enhancing students' early number sense (National Council of Teachers of Mathematics, 2000; National Mathematics Advisory Panel, 2008; Pittalis, Pitta-Pantazi & Christou, 2013). The development of students' number sense is considered as an important outcome and key ingredient of school curricula and a foundation for developing formal mathematical concepts and skills in elementary school (Yang, Li & Lin, 2007). Research findings support that number sense is a powerful predictor of mathematics outcomes and a vital prerequisite to success in mathematics (Malofeeva, Day, Saco, Young, & Ciancio, 2004).

The present study builds on previous studies asserting that early number sense consists of three distinct, but interrelated components (Pittalis, Pitta-Pantazi, & Christou, 2013). In particular, it was theoretically established and empirically validated that early number sense is a general theoretical construct that consists of three components (a) elementary number sense, (b) conventional arithmetic and (c) algebraic arithmetic. It was suggested that elementary number sense is comprised of key elements of numbers sense (see Jordan, et al., 2006), such as counting and number knowledge. Conventional arithmetic refers to story problems and number combinations that encompass number transformation situations. Finally, the proposed new component, algebraic arithmetic extends the two-dimensional model proposed by Jordan and her colleagues (2006) and incorporates number patterns and number equations.

In this study, we revalidated the structure of early number sense by encapsulating algebraic arithmetic component in a more comprehensive and systematic way, traced

2014. In Liljedahl, P., Oesterle, S., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 4, pp. 433-440. Vancouver, Canada: PME.

the development of number sense components and examined the way in which number sense growth factors relate to mathematics achievement. In particular the aims of the study were to: (a) validate the nature of early number sense components, (b) propose a growth model describing the development of number sense and (c) examine the relation between number sense growth factors and mathematics achievement.

THEORETICAL BACKGROUND

Research indicated that number sense is one of the most important concepts to be developed in early mathematics (Baroody, Eiland & Thompson, 2009). The quality of young children's number sense is a key predictor of later mathematical success; both in short and long term (Aunio & Niemivirta, 2010). For instance, research findings suggest that early number sense development contributes in learning more complex mathematics concepts; it promotes numerical fluency and is foundational to all aspects of early mathematical skills (Baroody, et al, 2009; Jordan et al., 2010). A number of research studies showed that inadequate development of number sense in early grades may be related to mathematical learning difficulties (Jordan et al, 2007). Moreover, Jordan and her colleagues (2010) showed that number sense is a powerful predictor of mathematics outcomes at the end of first grade and at the end of third grade, while number sense at the beginning and at the end of kindergarten was highly correlated with first grade mathematical achievement (Jordan et al., 2007). Locuniak and Jordan (2008) showed that kindergarten number sense was a strong predictor of calculation fluency in second grade, while Yang and her colleagues (2007) showed that the mathematics achievement of students in 5th grade was correlated with number sense performance. In addition, it is supported that students who enter school with strong number sense are more likely to benefit from teaching in the elementary grades and that the effect of weak number sense may be cumulative (Jordan et al., 2010).

A well-accepted and broad definition of number sense refers to a coherent understanding of what numbers mean, numerical relationships and the ability to handle daily life situations which involve numbers (Yang, 2005). Pittalis, Pitta-Pantazi, and Christou (2013), based on a synthesis of the literature, empirically validated a measurement model hypothesizing that number sense is a general second-order theoretical construct comprised of three first-order latent factors, namely (a) elementary number sense, (b) conventional arithmetic and (c) algebraic arithmetic. The proposed nature of number sense defines a more dynamic and flexible construct that may facilitate students' advancements and transition to a more abstract and relational system of thinking.

The foundation of algebraic arithmetic component of number sense lies on the research findings suggesting the introduction of students to algebraic reasoning at a much earlier age (Lins & Kaput, 2004). It should be noted that early algebraic reasoning conceptualizes algebra as a specific type of activity that builds on bridging arithmetic and algebra by promoting (a) understanding of the function of operations, (b) generalization and justification, (c) extension of the number system and (d) notation with meaning. These kinds of activities may contribute to the (a) transition of students from arithmetic towards algebra and (b) in the empowerment of arithmetic operations and computational fluency (Russell, Schifter, & Bastable, 2011).

The conceptualization of algebraic arithmetic component of number sense builds on Drijvers and his colleagues (2011) description of algebra as an amalgamation of (a) patterns and formulas, (b) restrictions and (c) functions. Examining the relations of these three stands with number sense, it can be conjectured that patterns, restrictions and functions may contribute in sustaining and further enhancing two major dimensions of number sense, namely, the relations among numbers and the conceptualization of the effect of operations of numbers. Moreover, this kind of activities may activate self-awareness mechanisms regarding the relations among numbers and promote self-reflection about the function and the properties of operations. Thus, in an attempt to provide a comprehensive and functional description of early number sense, we could suggest that “algebraic arithmetic” encompasses the development of a more sustainable and abstract understanding of the relations among numbers and of the effect of operations on numbers.

THE PRESENT STUDY

The purpose of the present study was to describe the nature of early number sense and explore the predictive validity of number sense growth factors on mathematical achievement. Specifically, the aims of the study were to (a) to revalidate the model proposed by the authors in PME36 suggesting that algebraic arithmetic is a component of early number sense, (b) to trace the development of six year old students’ early number sense and (c) to investigate the relations among number sense latent growth factors and mathematics achievement.

In the present study, algebraic arithmetic component captures number patterns, functions and restrictions (equations and balance scale restrictions), as proposed by Drijvers and his colleagues (2011). The parameter of number patterns involves researching for regularity and patterns to recognize a common algebraic structure. The dimension of restrictions describes students’ ability to find which value(s) of the unknown satisfies the required conditions in various situations; balance scale tasks or in more formal setting, such as equations. Finally, the function component involves students’ ability to investigate arithmetic relations between quantities/variables.

Measures

The majority of the test items were adopted from the Curriculum Based Measurement (Fernstrom & Powell, 2007) and the rest ones were developed based on the theoretical considerations of the study. Six types of tasks were used to measure elementary number sense: (a) counting tasks, (b) number recognition, (c) quantity discrimination, (d) number knowledge, (e) enumeration, and (f) non-verbal calculation. In the counting tasks, students were asked to enumerate objects, in the number recognition tasks, students had to read numbers, in the quantity discrimination tasks students were asked to decide which was the largest number, in the number knowledge tasks, students were asked to find smaller and bigger numbers of a given one and in the non-verbal calculation tasks students had to add or delete objects in a given set so the number of objects corresponds to a given number. Conventional arithmetic factor was measured by three types of tasks: (a) story problems, (b) understanding of operations and (b) number combinations. In story problems tasks, students had to select the appropriate number sentence for a list of story problems. A set of new items were developed for understanding of operations in which students were presented with simple addition, subtraction and multiplication word problems and were asked to select out of 4 mathematical sentences, the one that fitted the problem. In number combinations tasks students had to find mentally the result of addition, subtraction, multiplication and division combinations. Finally, the proposed component algebraic arithmetic number sense was measured with four types of tasks: (a) number patterns, (b) restrictions-equations, (c) restrictions-balance scale and (d) functions. In number patterns tasks, students had to extend or complete number patterns, such as 5, 8, 11, … The ability to solve number patterns implies that a student can conceptualize the relations among numbers to fill in or extend a number pattern. For the assessment of students' abilities in number restrictions two types of tasks were used, number equations and balance scales. In the number equations, students were asked to complete the missing terms of equations, such as $3+5=4+\square$. The other restriction task appeared in the form of a balance scale. Students had to identify the value of two or three shapes which balanced in a balance scale with a given number. Finally, regarding student's abilities with functions, students were presented with function machines and a table which showed the input and output values. Students were requested to provide the input or output numbers which were missing. The task was an adaptation of a task presented by Drijvers and his colleagues (2011).

Mathematics achievement was measured with the Screening Assessment for Gifted Elementary and Middle School Students (SAGES-2). The SAGES-2 assesses aptitude and achievement in order to identify gifted students. We used the Mathematics subtest measures (K-3) which required not only recall but also understanding and application of mathematical ideas and concepts. In the present study, we used the mathematics score of SAGES-2 to measure students' mathematics achievement.

Participants, Procedure and Data Analysis

Two hundred and four first grade students were the subjects of the study. Students were assessed on the number sense measures four times during the period October to June (approximately one administration per two months) in the school year 2012-2013. During each measurement students were interviewed in two sessions of approximately 30 minutes each. Students had a time restriction for each type of task (one minute for the majority of tasks). Students were individually tested in all four occasions. The order of the parts was rotated in the four time series. The SAGES-2 test was administered a year later, in December 2013, when the subjects of the study were in the second grade.

The use of confirmatory factor analysis made sense because we wanted to examine the validity of an a priori model, based on past evidence and theory. CFA was conducted by using MPLUS, which is widely popular for its robust parameters (Muthén & Muthén, 2007). To trace the development of number sense components we used growth models. Growth models examine the development of individuals on one or more outcome variables over time. In order to evaluate model fit, three widely accepted fit indices were computed: The chi-square to its degree of freedom ratio (x2/df should be <2); the comparative fit index (CFI should be >.9); and the root mean-square error of approximation (RMSEA should be close to or lower than .08).

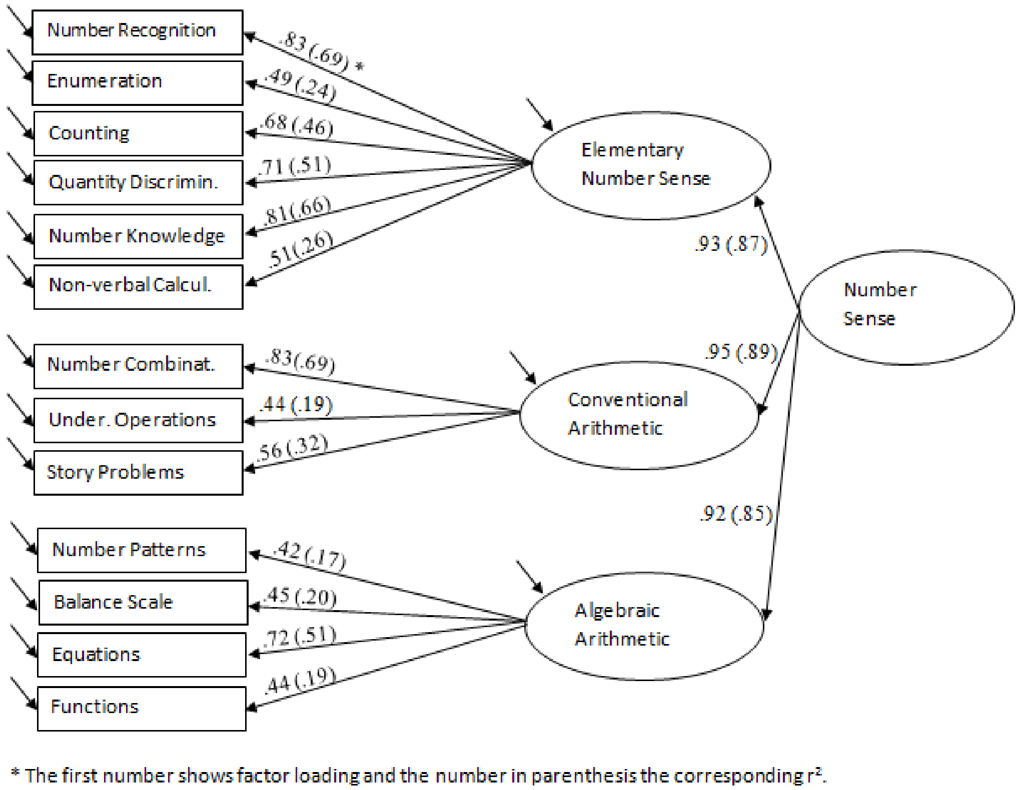

Figure 1: The nature of early number sense components.

RESULTS

The results of the analysis gave strong evidence to revalidate the construct validity of the hypothesized model describing the nature of early number sense (χ^2/df=1.60, CFI=.96, and RMSEA=.05). The results of the study showed that early number sense is a general, higher-order latent construct and might be described as a synthesis of three dimensions, namely, elementary number sense, conventional arithmetic and algebraic arithmetic. Figure 1 presents the standardized solution of the analysis and indicates that all factor loadings were statistically significant and most of them were rather large, ranging from .42 to .83 (see Figure 1). In addition, the factor loadings of the three first-order factors, elementary number sense, conventional arithmetic and algebraic arithmetic to the second-order factor, early number sense, were extremely high (.93, .95 and .92 respectively). Moreover, the predictive validity of the three components of number sense on the higher order early number sense factor was almost identical ($\underline{r}^2_{element}$. =.87, $\underline{r}^2_{algebraic}$ =.89, and $\underline{r}^2_{conventional}$=.85).

To examine the relation of number sense growth with mathematics achievement, first we examined the validity of alternative growth models. Longitudinal data in four time waves were used. The best fitting model (with the smallest AIC and BIC, χ^2/df=2.54, CFI=.98, and RMSEA=.08) was the one hypothesizing that elementary number sense follows a logarithmic growth rate while conventional arithmetic and algebraic arithmetic follow a linear growth rate until the third measurement and then they accelerate (see Figure 2). Thus, the results of the study showed that the growth of elementary number sense progressively is reduced, compared to the growth rate of the other components that increase. Elementary number sense had the largest mean latent slope (3.55), while conventional arithmetic mean latent slope was the second largest (3.28) and algebraic arithmetic mean latent slope was the smallest one (3.23).

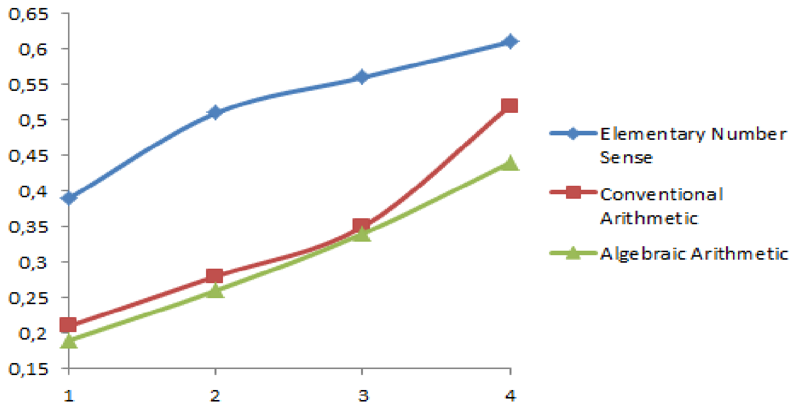

Figure 2: The development of number sense components.

Then, to examine the relation among number sense growth factors and mathematics achievement we tested alternative structural models. The adopted model (χ^2/df=2.18, CFI=.97, and RMSEA=.07) showed that the latent slope factor of the algebraic arithmetic component is a strong predictor of mathematics achievement ($\underline{r}$=.64) and of the latent slope factor of conventional arithmetic ($\underline{r}$=.70). Figure 3 shows that the slope factor of algebraic arithmetic had also an indirect effect on the latent slope factor of elementary arithmetic ($\underline{r}$=.32), through the latent slope of conventional arithmetic. In addition, the latent slope factor of conventional arithmetic had a direct effect on the latent slope of elementary arithmetic ($\underline{r}$=.46). The analysis showed that the intercept and the slope factors of algebraic arithmetic had a very high positive correlation, indicating that a student with a higher initial value in algebraic arithmetic would have a higher growth. On the other hand, the intercept and the slope factors of elementary arithmetic had a moderate negative correlation, indicating possibly a ceiling effect.

DISCUSSION

The results of the study reaffirmed the model describing the nature of students' early number sense, suggesting that early number sense consists of elementary, conventional and algebraic arithmetic components. Algebraic arithmetic includes number patterns, restrictions and functions. The proposed component of algebraic arithmetic adopts Drijvers and his colleagues (2011) definition of algebra. Elementary number sense develops with a logarithmic rate, while conventional arithmetic and algebraic arithmetic develop with a constant rate and accelerate progressively.

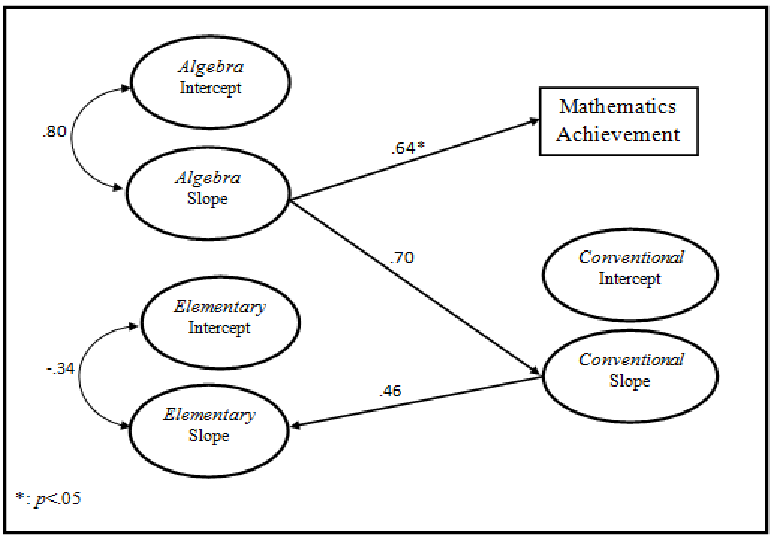

Figure 3: The relations among number sense and mathematics achievement.

The innovative aspect of the study lies on the exploration of the relation among the growth factors of number sense and mathematical achievement. The results of the study showed that students' algebraic arithmetic growth rate in the first grade predicts their mathematical achievement in the second grade. What is noteworthy is that neither the conventional arithmetic nor the elementary arithmetic could predict the second year's mathematical achievement. On the contrary, the growth rate of algebraic arithmetic proved to have a direct effect on the growth of conventional arithmetic and an indirect effect on the growth of elementary number sense through the growth of conventional arithmetic. This means, that algebraic arithmetic predicts conventional arithmetic and elementary arithmetic growth rates. Moreover, the intercept of algebraic arithmetic relates positively with its slope, suggesting that a student entering first grade with a good understanding of algebraic arithmetic will result in a significant growth rate of algebraic arithmetic. Thus, students entering primary school with a high value in algebraic arithmetic might exhibit high growth rate in algebraic arithmetic and consequently high growth rate in the two other components of number sense and high mathematics achievement in general. These findings highlight the dynamic nature of algebraic arithmetic and underlie the potential of integrating algebraic arithmetic situations in kindergarten. This algebraic arithmetic may be of the form of simple number patterns, restrictions and function activities appropriately developed for kindergarten children that enhance their understanding of the relations among numbers and flexibility with numbers.

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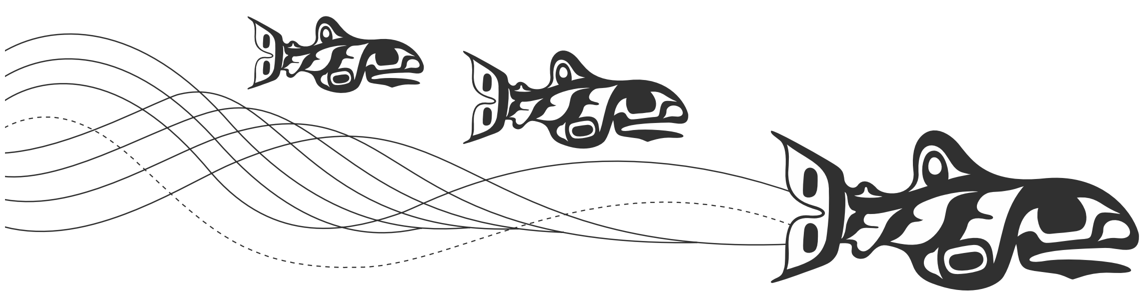
INDEX OF AUTHORS AND COAUTHORS
Volume 4



A

Adams, Anne 1

B

Bakker, Arthur 321
Beitlich, Jana T. 337
Benz, Christiane 153
Biza, Irene 161

C

Camargo, Leonor 409
Carrillo, José 233
Cheeseman, Jill 193
Christou, Constantinos 433
Clay Olson, Jo 1
Cui, Chen 337

D

Derry, Jan 321
Ding, Lin 97
Ding, Rui 145

E

El Mouhayar, Rabih 257
Ellis, Amy B. 129
Ely, Rob 1

F

Fujita, Taro 25, 225
Fukawa-Connelly, Tim 113

G

Gao, Yang 41
García González, María del Socorro 185

H

Hannula, Markku S. 361, 417

J

Jaworski, Barbara 161
Jones, Keith 25, 225

K

Khmelivska, Tetiana 337
Kimura, Keiko 353
Knott, Libby 1
Knuth, Eric 129
Koichu, Boris 377
Kollhoff, Sebastian 9, 137
Komatsu, Kotaro 17
Kondo, Yutaka 25
Kortenkamp, Ulrich 33
Kosko, Karl W. 41
Kumakura, Hiroyuki 25
Kunimune, Susumu 25
Kwon, Oh Nam 345

L

Ladel, Silke 33
Lake, Elizabeth 49
Lange, Diemut 57
le Roux, Kate 65
Leatham, Keith R. 73
Leikin, Mark 81
Leikin, Roza 81
Leuders, Timo 425
Leung, Allen 89
Leung, Allen Yuk Lun 97
Leung, Issic Kui Chiu 97
Levenson, Esther 249
Levenson, Esther S. 105
Lew, Kristen 113
Lewis, Gareth 121
Liu, Jia 145
Lockwood, Elise 129
Lüken, Miriam M. 137
Lynch, Alison G. 129

M

Ma, Xiaojun....................................145
Maier, Andrea Simone 153
Mali, Angeliki.................................161
Mamolo, Ami 169
Martínez Navarro, Benjamin...........177
Martínez-Sierra, Gustavo 185
McDonough, Andrea 193
Mejia-Ramos, Pablo 113
Mellone, Maria 201
Meyer, Alexander............................209
Meyer, Michael................................217
Miyazaki, Mikio 225
Molina, Óscar 409
Moll, Gabriele..................................337
Montes, Miguel................................233
Morris, Noah....................................241
Morselli, Francesca..........................249
Mousoulides, Nicholas G. 265
Murawska, Jaclyn M. 273
Musgrave, Stacy 281

N

Nardi, Elena.......................................49
Ng, Oi-Lam.....................................289
NicMhuirí, Siún...............................297
Niedermeyer, Inga 305
Nolan, Kathleen...............................313
Noorloos, Ruben..............................321

O

Oates, Greg 329
Obersteiner, Andreas 337
Oh, Kukhwan...................................345
Okazaki, Masakazu..........................353
Oksanen, Susanna............................361
Okumuş, Samet................................369

P

Palatnik, Alik..................................377
Pampaka, Maria...............................385
Park, Joo Y. 145
Park, Joo young...............................393
Pelczer, Ildikó 401
Perry, Patricia.................................409
Pesonen, Joonas A...........................417
Peter-Koop, Andrea 9, 137
Peterson, Blake E. 73
Philipp, Kathleen.............................425
Pittalis, Marios 433
Pitta-Pantazi, Demetra 433
Portaankorva-Koivisto, Päivi..........361

R

Rapone, Ben 1
Reiss, Kristina 337
Ribeiro, Miguel 233
Rigo Lemini, Mirela........................177
Ruwisch, Silke 305

S

Sáenz-Ludlow, Adalira 409
Sakamaki, Aruta................................17
Samper, Carmen.............................409
Schmidt, Maria...............................337
Shaul, Shelley...................................81
Sheryn, Louise................................329
Si, Luo 145
Singer, Florence Mihaela 401
Stockero, Shari L...............................73

T

Taylor, Sam 321
Thomas, Mike 329
Thompson, Patrick W. 281
Thrasher, Emily..............................369
Tzur, Ron.......................................145

V

Van Dooren, Wim 201
Van Zoest, Laura R. 73
Verschaffel, Lieven.........................201
Voica, Cristian................................401

W

Waisman, Ilana 81
Watanabe, Keiko 353
Weber, Keith...................................113
Wo, Lawrence..................................385
Wong, Ngai Ying..............................97

X

Xin, Yan Ping................................. 145

Y

Yang, Xuan 145