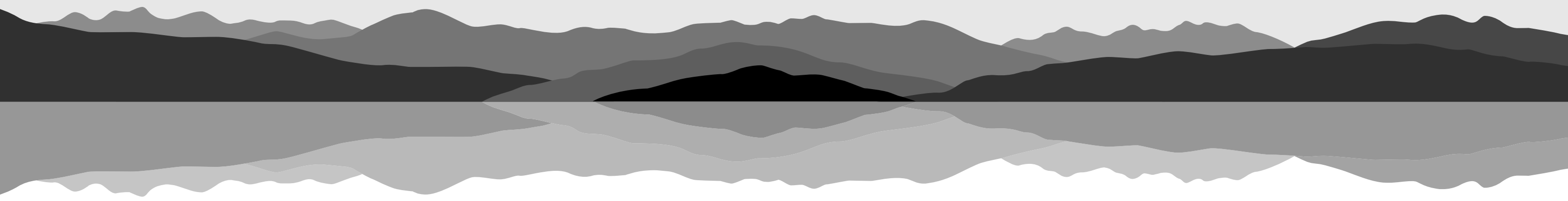

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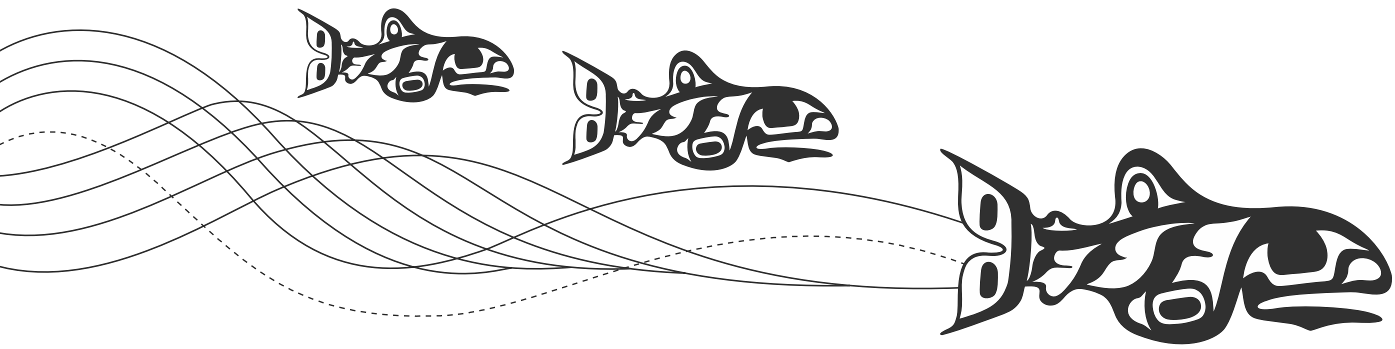

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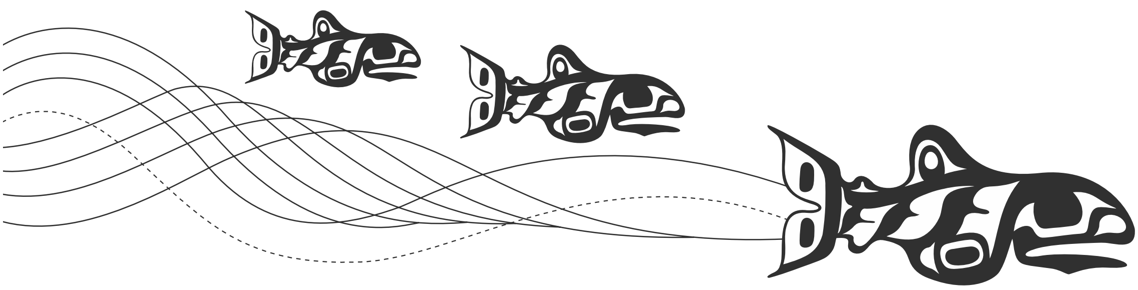

EFFECTS OF PROMPTING STUDENTS TO USE MULTIPLE SOLUTION METHODS WHILE SOLVING REAL-WORLD PROBLEMS ON STUDENTS' SELF-REGULATION

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In the project MultiMa (Multiple solutions for mathematics teaching oriented toward students' self-regulation), we investigated the effects of prompting students to use multiple solution methods while solving real-world problems on their learning. In this quasi-experimental study, we compared three treatment conditions. In one condition, students solved real-world problems by using multiple solution methods. These solution methods consisted of a solution using a table and a solution using differences. In the other two conditions, the same real-world problems were solved using only one of the methods. About 307 ninth graders from twelve middle track classes took part in this study during four lessons. Before and after a teaching unit, students' self-regulation was tested.

INTRODUCTION

The development of multiple solutions is an important component of school curricula in different countries. Encouraging students to use multiple solution methods improves students' mathematical knowledge. However, we do not know much about the influence of the use of multiple solution methods on students' self-regulation, which is crucial for lifelong learning. As solving real-world problems is an important part of mathematics education, we chose this type of task to investigate the effects of prompting students to use multiple solution methods on students' self-regulation while solving real-world problems.

THEORETICAL BACKGROUND

Self-regulation

Boekaerts (2002) defines self-regulation as "students' attempts to attain personal goals by systematically generating thoughts, actions, and feelings at the point of use, taking account of the local conditions." Thus, self-regulation is divided into three main parts: (1) students' orientation toward the attainment of their own goals, (2) the thoughts, feelings, and actions that can help them to attain these goals, and (3) working toward the attainment of their goals. It is further set within the framework of local conditions.

Self-regulatory processes can be acquired from and are sustained by social as well as self-sources of influence. Zimmerman (2000) describes four developmental levels of self-regulatory skills. The development of self-regulation begins on the first level, which is called an observational one. On this level, learners vicariously observe and imitate skills from a proficient model. On the level of emulation, learners imitate these

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skills with social assistance before they can work independently under structured conditions on the next level (the level of *self-control*). A *self-regulated* level is achieved when learners can flexibly and systematically adapt their performance to changing conditions.

Multiple solutions and self-regulation

Heinze, Star, and Verschaffel (2009) claim that the ability to use multiple representations (or multiple solution methods) and to flexibly switch between a range of representations is a critical component of the skills needed to solve mathematical problems. Recently, some experimental studies were carried out to identify the influence of prompting students to construct multiple solutions on students' learning in mathematics (Rittle-Johnson & Star, 2007). Students who developed two solution methods for the same task outperformed students who developed one solution at a time. Comparing two solution methods for the same problem or presenting two solution methods using different problems improved students' procedural flexibility. Students who developed two solution methods were more flexible in their choice of the appropriate solution method. In addition, Große and Renkl (2006) state that reflecting on various solution methods helps learners to apply methods more flexibly and effectively. Furthermore, Tabachneck, Koedinger, and Nathan (1994) found that it was more effective to employ a combination of strategies than to rely on a single strategy for solving algebra problems. Flexibility and adaptivity are important parts of self-regulatory skills. Prompting students to construct multiple solutions can improve their flexibility and adaptivity and thereby also improve their self-regulation.

The influence of prompting students to construct multiple solutions while solving real-world problems with missing information on students' self-regulation was investigated in the study by Schukajlow and Krug (2012). The results showed that, while controlling for self-regulation on a pre-test, students in the condition in which multiple solutions were prompted reported significantly higher self-regulation on the post-test than students in the condition in which they were instructed to develop one solution only.

Multiple solutions, modelling, and self-regulation

We distinguish between three types of multiple solutions that can be constructed in solving real-world problems (cf. a similar approach by Tsamir, Tirosh, Tabach, & Levenson, 2010). First, multiple solutions may result from variability in mathematical solution methods. The second type of multiple solutions can be developed if students have to make assumptions about missing data and thus arrive at different outcomes/results. The third one includes variability in mathematical solution methods as well as in different outcomes/results. The effects of prompting the second kind of multiple solutions on students' self-regulation were examined by Schukajlow and Krug (2012). In the current paper, we explored the effects of prompting the first type of multiple solutions on students' self-regulation.

The important activities that need to be implemented while modelling consist of simplifying a complex situation that is presented in the task, mathematizing, and working mathematically to reach a mathematical result. While solving a real-world problem, there are several ways in which the learner can simplify the problem, mathematize, or work mathematically. Solution methods can be pre-formal or formal ones while the outcome/results stay the same. Whereas formal solution methods are the final stage in a genetic development, pre-formal solution methods refer to a certain basis of formal argumentation, but are codified in a non-formal way (Blum, 1998).

To illustrate a solution process and to exemplify two pre-formal solution methods, we will analyze the solution of the task "BahnCard," which was developed in the framework of the project MultiMa. First, the problem solver has to understand the problem "BahnCard" and construct a model of the situation. Then the model of the situation needs to be simplified and structured and the important values need to be identified. These values are the costs per year for each card and the amounts for the outward and return journeys that would be paid using each card.

BahnCard

Mr. Besser lives in Hamburg. His parents live in Bremen. For the outward and return journeys with the "Deutsche Bahn" (German Rail), Mr. Besser has to pay 100 €. There are two special offers, the so-called "BahnCard 25" and the "BahnCard 50." The prices for each year and the prices for the outward and return journeys from Hamburg to Bremen for owners of the "BahnCards" are listed below.

BahnCard 25	BahnCard 50
Price per year: 59 € Price for the outward and return journeys: 75 € Number of customers: 3.1 million	Price per year: 240 € Price for the outward and return journeys: 50 € Number of customers: 1.6 million

Mr. Besser is going to buy a "BahnCard." When is it worth buying the "BahnCard 25" and when the "BahnCard 50"? Write down your solution.

Figure 1: Modelling task "BahnCard"

Next, the simplified situation needs to be mathematized, and different mathematical solution methods can be applied to solve the problem. One solution method that can be applied is a "pre-formal solution method using differences." In order to solve a real-world problem using differences, one has to understand the meaning of the important values and to transfer information between reality and mathematics several times. Whereas the "BahnCard 50" is 181 € (= 240 € - 59 €) more expensive than the "BahnCard 25," each outward and return journey with the "BahnCard 25" is 25 € (= 50 € - 25 €) more expensive than with the "BahnCard 50." Obviously, one has to calculate a difference for the costs per year and a difference for the cost per journey as well as to interpret the mathematical results. The question is how often one has to take a trip with the more expensive "BahnCard 50" until the cheaper prices for the journeys pay off. This is exactly after 7.24 (= 181 € ÷ 25 €) journeys per year. This result has to be interpreted—for example, "For up to 7 journeys per year, the "BahnCard 25" is cheaper."—and validated. Another way to solve this problem is to apply a "pre-formal solution method using a table." Students can make assumptions about a possible number of journeys per year (e.g. 1, 3, 6…), calculate the total cost for owners of the "BahnCard 25" and the "BahnCard 50," compare the costs, identify up to what number of journeys owners should take the "BahnCard 25", and write a recommendation about which offer is preferable for a certain number of journeys.

This analysis of solving the task "BahnCard" shows two ways to solve a real-world problem. Specifically, using different solution methods leads to the same result.

Being able to choose between different solution methods grants problem solvers the ability to solve tasks more flexibly and monitor their own solution process. Therefore, we assumed that similar effects as by Schukajlow and Krug (2012) could be found in our present study in which we prompted another way to provide multiple solutions to real-world problems: multiple solution methods (MSM). In addition, we assumed that the effects on self-regulation would not differ between our one solution method (OSM) conditions.

RESEARCH QUESTIONS

1. How many solutions will students develop in the MSM-condition and will there be differences in the number of solutions developed between the MSM-condition and the OSM-conditions?
2. Will students' self-regulation differ according to the opportunity to develop multiple solution methods? In particular, will students in the MSM-condition report more self-regulation than students in the OSM-conditions?
3. Will students' self-regulation differ between the different types (i.e., table vs. differences) of prompted solution methods? More precisely, will there be differences in the reported self-perceptions of students in the OSM-conditions?

METHOD

Design and sample

307 German ninth graders (48.26% female; mean age=14.6 years) were asked about their self-regulation before and after a teaching unit (see Figure 2). The teaching unit consisted of two sessions: the first and second lessons as well as the third and fourth lessons. Four schools with three middle track classes each took part in this study. Each of the twelve classes was divided into two parts with the same number of students in each part. The average achievements in the two parts did not differ, and there was approximately the same ratio of males and females in each part. There were three different treatment conditions "multiple solution methods" (MSM), "one solution method (table)" (OSM1), and "one solution method (differences)" (OSM2). At each school, there were six different groups, which were evenly assigned to the three treatment conditions. Furthermore, each part of a class was assigned to a different treatment condition. In total, there were 24 groups: eight groups in the MSM-condition, eight in the OSM1-condition, and eight in OSM2-condition. The students in MSM, OSM1, and OSM2 were taught in different classrooms.

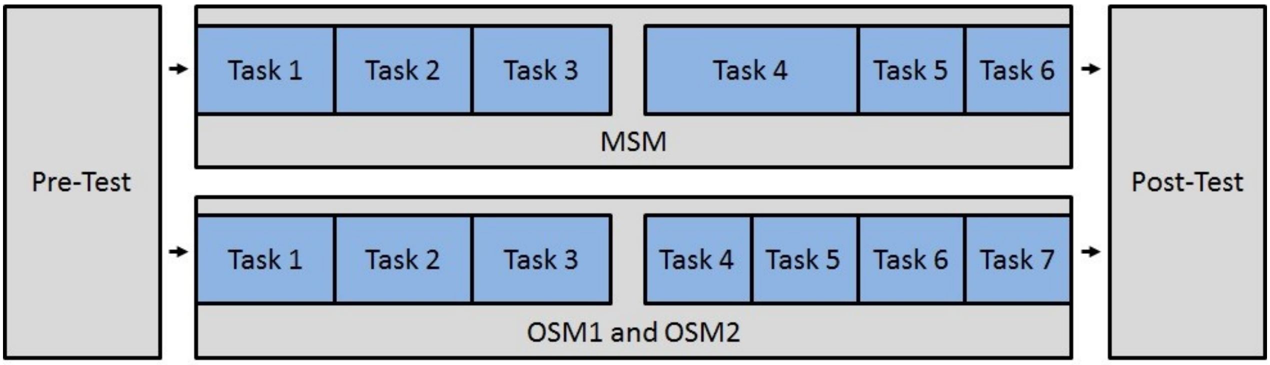

Figure 2: Overview of the study design

To implement the treatment, which consisted of solving real world problems using different solution methods, three teaching scripts were developed. Six teachers who participated in this study received these scripts with all tasks to be administered and a detailed plan for each teaching unit. Each teacher taught the same number of student groups in each treatment condition, so the influence of a teacher on students' learning did not differ between conditions. In each lesson, at least one member of the research group was present to videotape and to observe the implementation of the treatment.

Treatment

In the recent study, we used the student-centered learning environment from the DISUM project, which was complemented by direct instructions for the teaching unit. In all treatment conditions, the same methodological order was implemented for the first session. In the first session, a teacher first demonstrated how real-world problems can be solved using one specific method (in the OSM-conditions) or using multiple solution methods (in the MSM-condition). Then students solved tasks using the demonstrated solution methods according to a special kind of group work (alone, together, and alone) and discussed their solutions with the whole group in the classroom. The teacher summarized the key points of each treatment condition. Furthermore, in the MSM-condition, the teacher emphasized the development of two different methods.

In the second session, four problems were solved in the OSM-conditions and three tasks were addressed in the MSM-condition by applying the same kind of group work. After the fourth task in the MSM-condition, the teacher highlighted and summarized the link between the two methods and fostered discussions about students' preferences for one or the other solution method, whereas in the OSM-conditions, an additional task was given. Finally, in the MSM-condition, students had the opportunity to choose their preferred solution method to solve the last two tasks and discussed their choice in the classroom.

Four out of six tasks given in the MSM-condition required the development of the two solution methods: "Use two different solution methods to solve this problem. Write down both solutions." In the OSM-conditions, students solved a standard version of this task (see e.g. Fig. 1) using the demonstrated solution method.

Measures

After the second and third lessons, students were asked to report their self-regulation using a 5-point Likert scale (1=not at all true, 5=completely true) before and after a teaching unit (see Figure 2). The sample item is "While learning mathematics, I set my own goals that I would like to achieve." This scale consists of 6 items and was adapted from the longitudinal PALMA study (Pekrun et al., 2007). Reliability values (Cronbach's Alpha) for self-regulation were .66 and .75 on the pre-test and post-test respectively. The number of solutions developed (0=no solution; 1=one solution; 2=two solutions; 3=more than two solutions) was estimated by two independent raters for 20% of the tasks. The values for inter-rater agreement (Cohen's Kappa) were between .89 and .94.

RESULTS

For statistical analysis, we used t-tests, and examined that our data met the statistical assumptions for applying these tests. Levene's tests showed that there was heterogeneity of variance for some measures. For these tests, we used the adjusted values for degrees of freedom and t-values.

Number of solutions developed

First, we investigated how many solutions were developed across all problems in the MSM-condition. The analysis of students' answers showed that 1% of the students did not solve any of the posed problems, 5% of the students used one solution method, and 94% used two or even more than two solution methods. Thus, nearly all of the students in the MSM-condition used two or more solution methods (mean=1.92, standard deviation SD=0.25) as intended in our study. In the OSM-conditions, students did not or rarely used two or more solution methods (mean=1.01, SD=0.08 and mean=1.04, SD=0.24). The t-tests (MSM-OSM1: $t(116)=34.0$; $p<0.001$; effect size Cohen's $d=4.97$ and MSM-OSM2: $t(194)=25.2$; $p<0.001$; $d=3.61$) indicated that there were highly significant differences between the numbers of solution methods that were used in the respective conditions. These results revealed that nearly all students will use multiple solution methods while solving real-world problems if one prompts them to do so.

Multiple solutions and self-regulation

To examine the influence of prompting students to use multiple solution methods while solving real-world problems on students' self-regulation, we compared self-regulation on post-tests while taking into account the respective pre-test measures. The t-tests indicated that there were no significant differences between the MSM-condition and the OSM-conditions (MSM-OSM1: $t(185)=0.33$; $p=0.78$ and MSM-OSM2: $t(169)=0.36$; $p=0.72$). Thus, students in the MSM-condition did not report more self-regulation on the post-test than students in the OSM-conditions when controlling for self-regulation on the pre-test.

Self-regulation	Pre mean (SD)	Post mean (SD)	Adjusted post[a] mean (SD)
MSM	3.64 (.59)	3.47 (.63)	3.47 (.57)
OSM1	3.60 (.67)	3.48 (.73)	3.50 (.58)
OSM2	3.62 (.57)	3.54 (.70)	3.50 (.56)

a Adjusted by the pretest.

Table 1: Students' self-regulation on the pre-test, post-test, and adjusted post-test.

Different solution methods and self-regulation

To investigate the potential impact of prompting students to use different types of solution methods (i.e., table vs. differences) on students' self-regulation, we compared self-regulation in the one-solution conditions on the post-tests, taking into account the pre-test measures. The adjusted post-test means for self-regulation in the two OSM-conditions were identical with just a minor difference in the SD. A t-test showed that there were no significant differences between self-regulation in the OSM-conditions ($t(170)=0.36$; $p=0.97$). Thus, in the present study, we were able to confirm our assumption that students' self-regulation does not differ according to the type of solution method applied.

DISCUSSION

The results indicated that there were significant differences in the number of solutions developed between the MSM-condition and the OSM-conditions, as intended in our recent study. Furthermore, there was no difference in the impact of prompting different solution methods on the self-perceptions of students' self-regulation. However, we did not find any effects of prompting students to use multiple solutions on students' self-regulation. Although prompting the use of multiple solutions has previously been shown to increase flexibility (Rittle-Johnson & Star, 2007) and also self-regulation (Schukajlow & Krug, 2012), we could not find any effects of prompting students to use multiple solution methods on self-regulation in the recent study. One explanation for this result may be that students in the MSM-condition were not instructed to use certain solution methods according to the specific task but were rather instructed to use their preferred method to solve all tasks of this type. The highest level of self-regulation in Zimmerman's hierarchal order – flexibly and systematically adapting one's performance to changing conditions – was not achieved in the MSM-condition. The ability to choose a solution method based on individual-, task-, and context-specific criteria is an important part of being flexible and adaptive (Heinze et al., 2009). These criteria should be taken into account in future studies.

Compared to the results by Schukajlow and Krug (2012), where significant differences in students' self-regulation were found, students did not have the opportunity to make assumptions about missing information and to apply their assumptions to the task. This lack of autonomy could be a reason for the failure to find an increase in students' self-regulation in the current study.

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AN ANALYTIC FRAMEWORK FOR DESCRIBING TEACHERS' MATHEMATICS DISCOURSE IN INSTRUCTION

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We illustrate an analytic framework for teachers' mathematics discourse in instruction (MDI). MDI is built on three interacting components of a mathematics lesson: a sequence of examples and related tasks; accompanying talk; patterns of interaction. Together these illuminate what is made available to learn. MDI is grounded empirically in mathematics teaching practices in South Africa, and theoretically in socio-cultural theoretical resources. The framework is responsive to the goals of a particular research and professional development project with potential for wider use.

INTRODUCTION

Recent reviews of research on mathematics teachers, teaching and teacher education evidence the growth of this work (e.g. Sullivan, 2008). In their review of such research in thirty years of PME, Ponte & Chapman (2006) conclude with a call for future research that attends to "…innovative research designs to deal with the complex relationships among various variables, situations and circumstances that define teachers' activities" (p. 488). The framework offered in this paper responds to this call. Our central concern is a framework that illuminates the complexity of teaching mathematics in ways that are productive in professional development research *and* practice; a framework that characterise teaching per se, across classroom contexts and practices, *and* captures shifts in practice.

The framework we present developed within the Wits Maths Connect Secondary Project (WMCS), a five-year research and professional development project aimed at improving the teaching and learning of mathematics in ten relatively disadvantaged secondary schools in one education district in South Africa, through ongoing engagement with what we have come to describe as teachers' MDI. MDI characterises the teaching of a mathematics lesson as a *sequence of examples/tasks* (which we distinguish below), and the *accompanying explanatory talk* - two commonplaces of mathematics teaching that occur within *particular patterns of interaction* in the classroom. In previous work in WMCS and a similar project in primary schools, we conceptualised MDI to examine coherence within a task, and so between the stated problem or task, its exemplification or representation, and the accompanying explanations; and more recently to examine coherence across a sequence of tasks/examples and accompanying explanatory discourse within a lesson, and in relation to the intended object of learning (e.g. Adler & Venkat, forthcoming). It was our empirical data that emphasized the need for coherence, and teaching that mediates towards mathematics viewed as a network of scientific concepts (Vygotsky, 1978), and so towards generality (Watson & Mason, 2006), and objectification (Sfard, 2008).

There are clear commonalities with other frameworks, particularly aspects of the Mathematical Quality of Instruction (MQI) framework (Hill, 2010) and that of Borko et al (2005), both of which include attention to language/discourse (depending on their orientations to language), and to justification and/or explanation. In particular we share the concern of MQI to foreground the importance of generality in mathematics, and so what mathematically is made available to learn. Neither pay attention to examples, and so the specificity of example/task selection. This is a key element of the MDI framework, and we hope the elaboration that follows below illustrates its salience.

SOCIAL CONTEXT

It is common cause in South Africa today to hear that school mathematics is "in crisis". Learner performance in local, national and international comparative mathematics assessments are poor across levels, and while explanations increasingly acknowledge system wide failure, considerable 'blame' is placed on the knowledge of practice of mathematics teachers (Taylor, Van der Berg, & Mabogoane, 2013)

Of course, Teachers' MDI is only a part of a set of practices and conditions through which performance is produced, not least of which is social class and related material and symbolic resources in the school. That said, our concern from both a research and professional development perspective is to understand how teachers' MDI is implicated in what is made available to learn. In the majority of schools in South Africa (as is the case in schools serving disadvantaged learner populations in many parts of the world), schools provide the sole sites of access to formal learning. Within this, learners' access to mathematical learning resources is through the teacher's discourse. Understanding how teachers' MDI supports mathematical learning matters deeply. We want to be able to describe whether and how teachers' MDI shifts over time, in what ways, and how MDI is related to what is made available to learn in school.

SOME THEORETICAL ROOTS AND RESOURCES

MDI has its roots in analytic tools developed for describing the constitution of mathematics in mathematics teacher education practice (e.g. Adler & Davis, 2006). Based on Bernstein's insight that evaluation is "key to pedagogic practice" (2000 p.36), and following Davis' elaboration of this through the notion of evaluative judgment (Davis, 2005), we described three key features of mathematics pedagogy (school or teacher education). First, for something to be learned/taught, it has to be presented in some form. In mathematics, this is always a representation rather than the thing itself, one that as yet has to be invested with particular mathematical meanings. What then follows is reflection on this 'object' – semiotic mediation – so as to fill out its meaning. At some point reflection will need to end, and meaning fixed as to what can/does count as legitimate with respect to the 'object'. Description, while important, is not sufficient for linked research and development. In the first year of WMCS

(2010), we observed that teachers typically selected, sequenced and explained some examples for the announced focus of a session, often with poor levels of coherence between the example and its elaboration, and/or across a sequence of examples. Many lessons began and ended with teacher-directed whole class interaction. In some lessons there was opportunity for independent learner work on set problems. Across classroom contexts, opportunity for learner ideas to enter the discourse varied from none to substantive, with the former dominant.

The detail of our responsive professional development practice is not the focus here. Our position was that we needed to start where we all were – the teachers themselves, and their well-oiled practices; and the project team, with its goal of enhancing opportunities to learn mathematics. We constructed a simple framework foregrounding the intended object of learning: improved coherence, in our view, rested firstly on appreciation of that which was to be learned. We found further resonance with the work on examples (e.g. Watson & Mason, 2006) and variation theory (e.g. Runesson, 2006) as resources for bringing the object of learning into focus. This broad framing is operationalised into an analytic framework for describing teachers' lessons over time.

AN ANALYTIC FRAMEWORK FOR MDI

Table 1 below presents the framework. It is not possible here to elaborate it in full, nor illustrate it in detail. We briefly discuss each of the analytic resources, and how we have assigned levels in the example and explanation spaces constructed – increasing generality in examples; increasing complexity in tasks; towards objectified talk in naming; and towards generality and use of mathematics in legitimating/substantiating – and with respect to participation, towards increasing opportunity for learners talk mathematically, and teachers' increasing use of learners' ideas. We illustrate our use of this framework through a WMCS Grade 10 Algebra lesson.

Our unit of study is a lesson, and units of analysis within this, an event. The first analytic task is to divide a lesson into events, distinguished by a shift in content focus, and within an event then to record the sequences of examples presented. Each new example becomes a sub-event, as illustrated in Table 1 below. Our interest here is whether and how this presentation of examples within and across events brings the object of learning into focus, and for this we recruit constructs from variation theory (Marton & Pang, 2006). The underlying phenomenology here is that we can discern a feature of an object if it varies while other features are kept invariant, or vice versa, and different forms of variation visibilise the object in different ways. Variation through *separation* is when a feature to be discerned is varied (or kept invariant), while others are kept invariant (or made to vary); *contrast* is when there is opportunity to see what is not the object, e.g. when an example is contrasted with a non-example; *fusion* is experienced when there is simultaneous discernment of aspects of the object is possible; and generalisation is possible when there are a range of examples in different contexts so that learners can discern the invariants – an expanded form of separation. These four forms of variation can operate separately or together, with consequences for what is possible to discern – and so, in more general terms, what is made available to learn. In WMCS we are interested in analysing the teacher's selection and sequencing of examples within an event and then across events in a lesson, and then whether and how, over time, teachers expand the example space constructed in a lesson.

Object of learning – mediation towards scientific concepts				
Exemplification		**Explanation**		**Learner Participation**
Examples	**Tasks**	**Talk/Naming**	**Legitimating criteria**	
Examples provide opportunities within lesson for learners to experience **Level 1-** separation or contrast **Level 2-** any two of separation, contrast, and fusion **Level 3-** fusion and generalization	**Level 1** – *Carry out known operations and procedures* e.g. multiply, factorise, solve **Level 2** – Apply level 1 skills;& learners have to decide on (explain choice of) operation and /or procedure to use e.g. Compare/ classify/match representations; **Level 3** – *Multiple concepts and connections.* e.g. Solve problems in different ways; use multiple representations; pose problems; prove; reason.etc	**Level 1** – *Colloquial language i*ncluding ambiguous referents such as this, that, thing, to refer to objects **Level 2** – *Some math language* to name object, component or simply read string of symbols when explaining **Level 3-** *Uses appropriate names* of math objects and procedures	**Level 1**NM (Non- Math) *Visual:* Visual cues or mnemonics *Metaphor:* Relates to features or characteristics of real objects **Level 2M** (Math) (Local) Specific /single case (real-life application or purely mathematical) Established shortcuts; conventions **Level 3M** (General, partial) equivalent representations, definitions, previously established generalization but explanation unclear or incomplete, principles, structures, properties but unclear/partial **Level 4M** (General full)	**Level 1** –Learners answer *yes/no questions or offer single words* to teachers unfinished sentence **Level 2** –Learners answer (what/ how) questions in phrases/ sentences **Level 3-** Learners answer why questions; present ideas in discussion; teacher revoices / confirms/ asks questions

Table 1: Analytic framework for mathematical discourse in instruction.

Of course, examples do not speak for themselves. There is always a task associated with an example, and accompanying talk. With respect to tasks, we are interested in its cognitive demand in terms of the extent of connections between and among concepts and procedures. Hence, in column two we examine whether tasks within and across events require learners to carry out a known operation or procedure, and/or whether they are required to decide on steps to carry out, and/or whether the demand is for multiple connections and problem solving. These three levels bear some resemblance to Stein et al's (2000) distinctions between lower and higher demand tasks.

With respect to how explanation unfolds through talk, and again the levels and distinctions have been empirically derived through examination of video data, we distinguish firstly between naming and legitimating, between how the teachers refer to mathematical objects and processes on the one hand, and how they legitimate what counts as mathematics on the other. For the latter, we have drawn from and built on the earlier research discussed above, together with aspects of Sfard's (2008) word use and endorsements as key elements of mathematical discourse. Specifically, we are interested in whether the criteria teachers transmit as explanation for what counts is or is not mathematical, is particular or localised, or more general, and then if the explanation is grounded in rules, conventions, procedures, definitions, theorems, and their level of generality. With regard to naming, we have paid attention to teacher's discourse shifts between colloquial and mathematical word use.

Finally, all of the above mediational means (examples, tasks, word use, legitimating criteria) occur in a context of interaction between the teacher and learners, with learning a function of their participation. Thus, in addition to task demand, we are concerned with what learners are invited to say i.e. whether and how learners have opportunity to use mathematical language, and engage in mathematical reasoning, and the teacher's engagement with learner productions.

A LESSON

The illustrative lesson, as stated by the teacher, is a Grade 10 revision lesson on algebraic fractions leading to a focus on the operation of division of algebraic fractions. The lesson consists of five events, with a new event marked by a new key concept in focus. The first event focused on a review of the meaning of a *term* in an algebraic expression. The teacher presented six examples of expressions (sub-events) in increasing complexity, with each next example of an expression produced by her performing an operation on the present expression. The task for learners was to agree to the number of terms in the new expression. The second event reviewed a common factor using just one example of a binomial expression. Event 3 signals new work. The teacher presented a sequence of four examples (sub-events) of algebraic fractions. The task was simplifying (through factorization) the expressions in each of the numerator and denominator to produce a single term. Complexity increased in terms of the type of factorisation required in successive examples. The task in events 4 and 5 was division of algebraic fractions. The examples in event 4 were of positive algebraic fractions only and event 5 included examples with negative algebraic fractions. We illustrate the use of our framework through detailed analysis of Event 4, particularly sub-event 4.3. in the box on the following page

Our analysis of Event 4 shows the Teacher operating at Level 3 with respect to examples, Level 1 with respect to tasks (which remain at the level of learners carrying out known procedures), and interaction (learners answers yes, no questions, and provide words/phrases in response to teachers questions on what to do). With respect to explanatory discourse, the teacher's words while frequently including ambiguous referents, move on to rephrase using mathematical language to name objects and processes, and thus at level 2; criteria shift between emphasis on visual features of expression, conventions, with some reference to structure and generality and so across levels 1 - 3.

Event 4: Sub-events 4.1 – 4.4 — Examples and tasks

T writes example 4.1 on the board, asks questions mainly requiring yes/no answers, completion of sentences by learners in unison, leading to the solution. Occasionally learners respond with a phrase or sentence to a *what* or *how* question. Any *why* question she answers herself. Examples 4.2 and 4.3 follow the same form. The transcript extract below details the talk leading to the solution for 4.3. Example 4.4 is then given for learners to do independently.

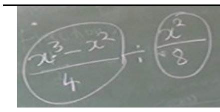
4.1 $\frac{2}{6} \div \frac{2}{3}$ 4.2 $\frac{2x}{6x} \div \frac{2x}{3x}$ 4.3 $\frac{x^3-x^2}{4} \div \frac{x^2}{8}$ 4.4 $\frac{x^2-x}{x^2+x-2} \div \frac{x^2+4x}{x^2-4} \times \frac{3x+12}{1}$

Examples: Level 3 - Variation is by separation, generalization and fusion. The structure of the division of one fraction by another is kept constant and terms varied (Separation). These range from simple to complex; from numerical to algebraic. Eg 4.4 extends to three fractions and a product (Generalization). Egs 4.3 and 4.4 require associating common factor with fraction division (Fusion).
Tasks: Level 1 - Perform the indicated operations to simplify expressions

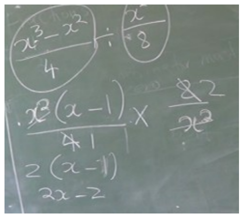
Sub-Event 4.3 — Talk and legitimating criteria

Analysis of explanatory talk highlighted as follows: *italics for colloquial* and underlining for formal language; and **bold type for criteria/legitimations;**

1.T: It's *one and the same thing.* They give you *something like this* (writes symbols on board),.... x cubed minus x squared *the whole thing over, over* four divided by x squared *over* eight...ok?
2. Ls: Yes
3. T: So it's, it's *one and the same* concept. *Over here* (points to number 4.1 ($\frac{2}{6} \div \frac{2}{3}$)) you just have two numbers, a fraction divided by a fraction, ok?
Ls: Yes



4. T: *Over here* (pointing back to 4.3) is *the same thing*. I've got, here's one fraction divided by one fraction (circles each fraction). So the examiner is just making your life difficult, ok?
5. T: So....what are we going to do *over here*? (points to first fraction)



6. Ls (some): we are going to divide
7. T: ...**remember the rule** that we learnt *over there*? (points to similar expression, Event 2,factors obtained to simplify fraction)
8. Ls: Yes.
9. T: For before we can go and divide, **what must I do**?
10.Ls: *Take out* the common factor.
11.T: *Take out* the common factor, ok?
12.Ls: Yes
13. T: So, the same thing applies here. It is everything that you, that you have learned, but they just put it into one thing to make it look a bit complicated. It's actually very simple...ok?
14. Ls: Yes
15. T: So*, over here* we need the common factor. Why? Because **we want to have one, one term *at the top* and one term *below***, ok?
16. Ls: Yes
17. T: So, what is common factor to the two terms?

[18-36] – not shown; includes reference to "**change the sign"** shift from division to multiplication

37. T: **So, you just apply the same principle, it's just that when it looks complicated just pause and say *what must I do* here?** Because I know terms *like this* (points to $\frac{x^3-x^2}{4}$), I cannot just...go and say *this* (pointing to $x^3 - x^2$) divided by *this* (points to 4) ...ok?

38. Ls: Yes 39. T: Make sure that you have got <u>one term</u> *at the top* and <u>one term</u> *below*. So from here I can, what must I do? … [T, together with Ls and with similar interactional pattern, produce the answer.]
Talk: Level 2 – Uses some math language (e.g. ln 3) to name individual components or simply read string of symbols when explaining **Legitimation: Level 1** Reference to visual features (e.g. ln 3, 4, 13) and **Level 2M (Local)** Established shortcuts; conventions (e.g. lns 7, 10, 11, 30) and **Level 3M (General)** Makes reference to structure/principle but not clear due to naming (e.g. ln 37)
Event 4: Interaction pattern
Interaction pattern: Dominantly Level 1 Ls answer yes/no questions or supply words to T's unfinished sentence; **Occasional Level 2** Ls answer what/how questions in phrases or sentences

DISCUSSION

Our MDI framework allows for an attenuated description of practice, prising apart parts of a lesson that in practice are inextricably interconnected, and how each of these contribute overall to what is made available to learn. It co-ordinates "various variables, situations and circumstances" of teacher activity (Ponte & Chapman, op cit) There is much room for the teacher to work on learner participation patterns, as well as task demand (and these are inevitably inter-related); at the same time her example space even in sub-event 4.3, evidences awareness of and skill in producing a sequence of examples that bring the operation of division with varying algebraic fractions into focus, hence the value of this specific aspect of MDI. While not within scope here, contrasting levels in earlier observation of this teacher indicates an expanded example space and more movement in her talk between colloquial and mathematical discourse. The MDI framework is thus helpful in directing work with the teacher (practice), and in illuminating take up of aspects of MDI within and across teachers (research); e.g. our analysis across teachers suggests that take-up with respect to developing generality of explanations is more difficult. We contend further that content illumination through examples is productive across pedagogies and so across varying contexts and practices.

The MDI framework provides for responsive and responsible description. It does not produce a description of the teacher uniformly as in deficit, as is the case in most literature that works with a reform ideology, so positioning the teacher in relation to researchers' desires (Ponte & Chapman, op cit). We have illustrated MDI on what many would refer to as a 'traditional' pedagogy. MDI works as well to describe lessons structured by more open tasks, indeed across ranging practices observed.

CONCLUSION

In this paper we have communicated the overall framework, and illustrated its potential through analysis of selected project data. What then of its wider potential? While we have suggested this in pointing to our use across a range of practice in our data, we recognize that MDI arises in a particular context. Its potential beyond the goals of the WMCS project needs to be argued. Analytic resources are necessarily selective, reflecting a privileged reading of mathematics pedagogy. We have made these visible and explicit, and hold that its generativity lies in their theoretical grounding.

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MATHEMATICAL COMPETENCIES IN A ROLE-PLAY ACTIVITY

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This paper concerns a competencies-based analysis of the outcomes of a role-play activity aimed to foster conceptual understanding of mathematics for first year engineering students. The teacher role has been considered in order to investigate the competencies addressed by the questions created by the students and their matching with the activity's educational goal. The analysis shows that the quality of the posed questions made by the students highlights the moving from the instrumental approach, the students are used to, towards a relational one.

INTRODUCTION

In this paper we focus on the analysis of the outcomes of a role-play activity aimed to foster *conceptual* understanding of mathematics for first year engineering students. The design of the activity was suggested from the fact that, during some interviews, some students ascribed their poor performance to *strange* and *unexpected* questions. This suggested the idea to support the students by on-line, time restricted activities based on role-play, which actively engage them and induce them to face learning topics in a more critical way. Students are expected to play the role of a teacher in order to force them *to ask questions*.

In the following we are going to investigate the outcomes produced by the students and to discuss the findings with respect the goal of the activity. We have used the Niss competencies framework (2003), also referred by the European Society for Engineering Education – SEFI (2011), to analyse the questions created by the students assuming the teacher role. Our research questions were:

a. what competencies are addressed by the questions posed by the students?
b. does the posed questions address relational knowledge/conceptual understanding rather than instrumental ones?

Finally, we try to draw some ideas for further work concerning the other roles played by the students.

THEORETICAL BACKGROUND

Competence in mathematics is something complex, hard to define which requires the students not only knowledge and skills, but at least some measurable abilities, which Niss names *competencies* (Niss, 2003). He has distinguished eight characteristic cognitive mathematical competencies. The following table lists them in two clusters:

The ability to ask and answer questions in and with mathematics	***The ability to deal with mathematical language and tools***
[1] Mathematical thinking competency	[5] Representation competency
[2] Problem handling competency	[6] Symbols and formalism competency
[3] Modelling competency	[7] Communication competency
[4] Reasoning competency	[8] Tools and aids competency

Table 1: Cluster related to cognitive mathematical competencies

Mathematical thinking competency includes understanding and handling of scope and limitations of a given concept; posing questions that are characteristic of mathematics and knowing the kinds of answers that mathematics may offer; extending the scope of a concept by abstracting and generalizing results; distinguishing between different kinds of mathematical statements (theorems, conjectures, definitions, conditional and quantified statements).

Problem handling competency includes possessing an ability to solve problems in different ways; delimitating, formulating and specifying mathematical problems.

Modelling competency includes analysing the foundations and properties of existing models, and assessing their range and validity; decoding existing models; performing active modelling in given contexts.

Reasoning competency includes understanding the logic of a proof or of a counter-example; uncovering the main ideas in a proof, following and assessing other's mathematical reasoning; devising and carrying out informal and formal arguments.

Representation competency includes understanding and utilising different kinds of representations of mathematical entities; understanding the relations between different representation of the same object; choosing, making use of and switching between different representations.

Symbols and formalism competency includes decoding symbolic and formal language; understanding the nature of forma mathematical systems; translating back and forth between symbolic language and verbal language; handling and manipulating statements and expressions containing symbols and formulas .

Communication competency includes understanding other's mathematical texts in different linguistic registers; expressing oneself at different levels of theoretical and technical precision.

Tools and aids competency includes knowing and reflectively using different tools and aids for mathematical activity.

EXPERIMENT SETTING AND METHODOLOGY

The setting is a University with a 3-year BSc degree in Electronic Engineering and first year students taking part in a two trimester intensive modules in mathematics. Our research focus on the second module, which concerns topics from linear algebra and calculus. The module has ten hours per week in face-to-face traditional lectures/exercises sessions, supported by an e-learning platform which provides the students with various learning resources and communication tools. The experiment has been performed with voluntary students, who were liked to be involved in a massive and more interactive use of the e-learning platform.

From the viewpoint of the theory of mathematics education, the online experimental activity, we are going to describe, can be framed within the so-called 'discoursive' approach (Kieran et al., 2001). The activity is based on role-play and has been organized as follows. The course contents have been split into different parts and each part into as many topics as the involved students. For each part a cycle of activities based on role-play has been created. Three topics have been assigned to each student, corresponding to three roles played by the student. Each cycle took nine days, three per role. For the *first topic*, the student acts as a teacher who wants to evaluate the topic's learning so he/she has to prepare some suitable questions – at least six questions. For the *second topic*, the student has to answer to the questions prepared by a colleague. Finally for the *third topic*, the students again acts as a teacher, checking the correctness of the work made by the previous two colleagues. At the end of each cycle, the files produced by the students were revised by the teacher-tutor of the course and the revised files were made available to the students. All the produced worksheets were stored in a shared area of the platform in order to be available to all the students.

A COMPETENCIES-BASED ANALYSIS

In the following we want to analyse students' work concerned the first role using the framework of the above Niss competencies. The methodology used for the analysis has been adapted from Jaworski (2012, 2013).

Let us see some examples (the number in the square brackets refers to the table 1).

In the first role we find questions asking for:

The definition of some concepts involved in the topic at stake:

Q1: "What is an Euclidian space?"

Q2: "Which means "*f* differentiable in *x*"?"

Q3: "Given the basis B = $\{u_1,\ldots,u_n\}$ of V, you can write v = $x_1u_1+\ldots+x_nu_n$ for suitable $(x_1,\ldots,x_n)$ ϵ to..?"

Q4: "Which relation does exist between vector space and Euclidean space?"

We note the different formulation of the first two questions, which refers to different expectations and then different competencies. While in all the cases the expected answers require the student to recognize and the scope and limitation of the mathematical concept [1], in the second case the ability to deal with mathematical language seems to be stressed [5, 6, 7]. In fact, questions such as “what is…?” let the students to answer using for instance only formal language, reproducing a definition in a textbook; questions such as “which means…?” require more deeply understanding which allow the student to use various mathematical representations, including verbal, to understand formal language and to translate it to verbal language and finally to express oneself mathematically in different ways. The third question refers to the span property of a basis. It requires the students to includes to handle symbolic expression [6], recognize the concept/property [1] and knowledge its scope and limitations [1]. The fourth question, instead, concerns the scope and the limitation of the two concepts at stake [1], but it also requires to make connections between them, recognizing for instance if and how one extends the properties of the other class of objects.

The understanding of the steps in a given proof:

Q5: “In which steps of the proof the linearity of the function is used?”

Q6: “Why the Lagrange theorem is applied in the interval $[x,x+h]$?”

Q7: “The equation $y'(x) = g(x)$ for which theorem has solution in $[a, b]$?”

The above questions refer to proof of theorems seen by students during lectures and available in their textbooks (Q5 – differential theorem, Q6 – dimension theorem, Q7 – Cauchy problem for differential equations). All of them require the students first of all the ability to understand already existing chain of logical arguments in order to prove a statement starting from fixed hypotheses [4]. Moreover, questions such as Q5 require the student to make his own chain of arguments in order to justify the application of a given theorem [4] and also to express himself mathematically [7], whilst questions such as Q7 require to make connections with previous knowledge to justify a statement in the proof. Finally, we note that, in order to answer the questions, students need to recognize some mathematical concepts (homomorphism in Q5) and to understand their scope and limitation (Lagrange theorem in Q6) [1].

The recognition of the main ideas in a proof:

Q8: “Which are the main steps in the proof of the differential theorem?”

Q9: “In the proof of the Steinitz lemma, which is the fundamental step allowing to prove the thesis?

Both questions refer to the ability of uncovering the central ideas in given proofs [4]. At the same time the answer requires the student from one hand the ability of express himself mathematically in different ways [7], also using verbal language, and thus it requires the capability to understand symbolic language in formal proof and translate in verbal language [6]. Moreover, the answer to Q9 requires the student to connect the existence of non-trivial solutions of a suitable homogeneous linear system to the existence of non-trivial solutions of the vector equation in the definition of linear dependence of vectors [1].

The construction of their own proofs:

Q10: "In the proof of the Steinitz lemma, why the rank of A is less or equal to *n*?"

Q11: "In the proof of the differential theorem, prove that all the hypotheses needed to apply the Lagrange theorem are verified"

Q12: "Is in R^n (n>1) differentiability equivalent to continuity?"

The above first two questions refer to the ability of constructing informal and formal own arguments in order to justify and make clear some steps in a given proof [4]. This require the capability of handling and manipulating symbolic statements and expressions and switch between them and verbal language [6] and the ability of express himself at a certain level of theoretical and technical precision [7]. The difference between the two questions seems to be a greater formality of Q11 with respect to Q10, made evident by the use of the word "prove", highlighting different weights of [6] and [7] for each of them. The last question requires the student to identify the scope of the equivalence between differentiability and continuity – just R [1] – and it is expected that the student is able to prove the true implication and to give a counter-example in the other case [4, 7].

The conversion among various semiotic representations:

Q13: "In the Cauchy's problem which means the expression $y'(x_0) = y_0$ graphically?"

Q14: "Explain in words the Cauchy problem"

Q15: "Write the Cauchy problem (in mathematical language)"

The above questions refer explicitly to the ability of using different kinds of semiotic representation systems of mathematical entities, including verbal language, and the capability of passing from one to another, which is the Duval conversion process (Duval, 2006). Even if we have already noted that such process is implicitly required in other questions, here it is the main focus and it seems us important since Duval states that such capability has to be trained and suggests to make such kind of explicit activities. The pre-requisite of such questions concerns symbol and formalism competency and the answer to Q13 and Q14 requires communication competency.

According to the methodology shown by the above examples, all the questions made by the students has been analysed and for each of them the addressed mathematical competencies have been individuated. The following table resumes the outcomes of this analysis – L* refers to linear algebra topics and C* to calculus ones.

The course setting does not make use of tools, so the related competency has not taken into consideration.

Looking at the outcomes, we can note that the quite predominant competencies addressed by the questions concern the ability to ask and answer questions in and with mathematics, in particular *thinking* and *reasoning mathematically*. Interviews have give evidence that it depends on the teacher role played by the students, which have emulated the way their teachers act with them during exam sessions.

Competency\ Topic	L1	L2	L3	L4	L5	L6	L7	L8	C1	C2	C3	C4	C5
Thinking math.	3	7	6	5	11	9	5	5	3	6	6	3	4
Problem solving	1	2	1	1	5	0	2	1	0	0	3	1	2
Modeling math.	0	1	0	0	0	0	0	0	0	0	0	0	0
Reasoning math.	3	1	4	3	0	5	2	3	2	8	4	1	0
Representation	4	1	0	0	0	5	0	0	2	0	1	4	5
Symbols	6	2	0	0	0	5	2	0	0	2	1	5	3
Communication	6	8	10	6	10	13	8	6	6	7	6	9	9
Aids and tools	-	-	-	-	-	-	-	-	-	-	-	-	-

Table 2: Analysis of mathematical competencies in posing questions.

Moreover, also the *communication* competency is strongly addressed, for the nature itself of the activity which requires the students to express mathematically each other.

Considering the above remarks, we can state that most of the questions address relational knowledge/conceptual understanding rather than instrumental ones, and thus the goal of the activity seems to be achieved from our point of view.

This conclusion has been also supported by:

- students' feedbacks, which reports their appreciation of the teacher role, because it has allowed them to be in the teacher's perspective, so getting able to understand the educational goals which are more conceptual than instrumental;
- students' marks at the next exams, which have obtained a better advancement, due to the fact that this kind of activity has given the students a sort of guidance for the organization of their study, providing time constrictions, topics to revise, indications of the relevant activities.

Moreover, the students report that to *ask questions* have helped them to study in a more critical and deeper way, with greater care, because it is not simple to pose a question due to the fact that there is no method to do that. At the same time, the request of a certain minimum number of questions on a topic requires to range over all the programme, not only concentrating on the specific and restricted topic but also paying attention to all the other linked topics. It is also interesting to note that some students has used this role to make critical points clear (posing as questions exactly their own doubts). Finally we noticed some non-cognitive aspects such as the trend to pose non trivial questions, also for pride reasons, and this has required the mastery of the topics.

CONCLUSIONS AND FUTURE WORK

In this paper we have began to analyze the outcomes of a role-play activity aimed to foster conceptual understanding of mathematics for first year engineering students. The analysis has been performed using the Niss competencies and SEFI framework and has concerned the work of the students in the teacher role.

We plan to continue the analysis of the second role, in particular we are interesting to see what competencies are addressed by answering to the posed questions and its matching with the expected competencies revealed by the questions.

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USING THE PATTERNS-OF-PARTICIPATION APPROACH TO UNDERSTAND HIGH SCHOOL MATHEMATICS TEACHERS' CLASSROOM PRACTICE IN SAUDI ARABIA

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In this paper, patterns-of-participation theory serves as a lens to interpret and understand Saudi high school mathematics teachers' practices during the current reform movement and the role the new textbooks play in influencing teachers practice. The data presented is about Haya and Nora, two experienced, high school mathematics teachers. Generally, Nora's and Haya's practice as high school mathematics teachers reveals patterns of tension and confusion with regards to understanding the current reform movement in Saudi Arabia.

INTRODUCTION

One of the main goals of most education reform initiatives has been to change teachers' classroom practices. The most recent reform curricula focuses on highlighting teacher practices that promote and evoke students' understanding of mathematics alongside the changes in content (Tirosh & Graeber, 2003). Changes to a teacher's role that are included in the education reform movement call for more research in order to understand and theorise about teachers' classroom practices. The Saudi Arabian education system has undergone major changes in the past decade. Government agencies involved in education have introduced new policies, standards, programs, and curricula with the expectation that teachers incorporate the changes seamlessly and without consideration of existing beliefs and practices. My main research goal is to gain a better understanding of how high school mathematics teachers in Saudi Arabia are coping with recent education reform including how their practices are changing in response to the changes that are happening in the education system in general, and specifically, to the introduction of the new mathematics textbooks. In this article, patterns-of-participation (PoP) (Skott, 2010, 2013) approach will serve as a lens to interpret and understand Saudi high school mathematics teachers' practices during the current reform movement and the role the new textbooks play in influencing teachers practice.

Textbooks in mathematics classroom:

For a long time, school mathematics has been associated with textbooks and curriculum material (Remillard, 2005). According to *Trends in International Mathematics and Science Study* (TIMSS), textbooks and documents such as exercise resources for use in classrooms as teaching aids, remain important elements in mathematics classrooms in many countries. Textbooks play an important role in shaping the curriculum experiences of mathematics (TIMSS 2011). This fact is especially true in Saudi Arabian high schools. Textbooks provide teachers with a basic outline for thinking about what mathematics should be taught, when, and how. In 2010, the Ministry of Education introduced new mathematics textbooks, the primary, and sometimes only, resource for teachers. The Ministry sees this initiative as a major step towards creating change in teaching practices.

In Saudi Arabia textbooks have official status clearly reflecting official curriculum. The new approved mathematics textbooks in Saudi Arabia are based on the curricula published by McGraw Hill Education learning company. According to the ministry of education in Saudi Arabia, the new mathematics curriculum aims to (a) help students to develop higher-order mathematics thinking skills, (b) develop ways of mastering these skills, (c) construct a strong conceptual foundation in mathematics that enable students to apply their knowledge, (d) make connections between related mathematical concepts and between mathematics and the real world, and (e) apply mathematics logically to solve problems from daily life (Ministry of Education of Saudi Arabia, 2013).

Traditionally, curriculum materials or textbooks have been a center agent of policies to regulate mathematics practice in ways that parallel instruction with the reform perspective (Remillard, 2005). Textbooks are often the main resource for students and teachers in the classroom, offering the everyday materials of lessons and guiding the activities teachers and students do. As a result, educational policy makers use textbooks as an essential means to decide what students learn (Battista & Clements, 2000).

Research on teachers' curriculum use focuses on understanding how teachers "interact with, draw on, refer to, and are influenced by" curricular materials when designing their lessons (Remillard, 2005, p.212). While effective student learning is one expected outcome of textbook use, the development of teachers' techniques and practice is an additional desired outcome. Researchers have only recently started to shed the light on the impact of curriculum materials on teachers and how teachers use them (Remillard, Herbel-Eisenmann, & Lloyd, 2009). The focus of how teachers interact with and use curriculum materials has not been always considered significant to studying curriculum. Historically, research about school curricula relied mainly on examining the textbooks to restructure the contents of classroom practice (Love & Pimm, 1996). Reform efforts in mathematics education are the product of curriculum development supported by standards adopted by the National Council of Mathematics Teachers (NCTM, 2000). Teachers face the demand of applying new curriculum materials, and adopt new conceptual and pedagogical approaches to teach new standards-based curriculum (Remillard, 2005).

THEORETICAL FRAMEWORK

Skott (2010, 2013) introduced PoP as a promising framework, which provides coherent and dynamic theoretical understandings of mathematics teacher practices. Skott's (2009, 2013) main motivation in developing this framework was to overcome the conceptual and methodological problems of belief research. Although some researchers such as McLeod and McLeod (2002) note that there has been significant advancement in the study of beliefs and affect in mathematics learning, the progress can be more noticed in relation to theoretical aspects, researchers still call for more extensive studies to assure that progress exists in the quality of instruction. However, Skott (2009, 2013) views the call for more work to do, after all what has been done, in the field for beliefs research as a negative sign. "To a large extent, then, belief research is still conceived of as a promising field of study. Phrased negatively, however, its still-promising character suggests that after 20 years of persistent effort, the field has still not lived up to the expectations of its founders" (Skott, 2009, p. 28).

The challenges and complexity associated with beliefs research has led some researchers, such as Skott (2009, 2013) and Gates (2006), to call for more social approaches to beliefs research. Gates (2006) indicates that there is a need to take a social approach when studying teacher belief systems because it will shift focus from cognitive constructs. A change toward sociological constructs will balance existing views about the nature and genesis of beliefs. Skott (2010) also supports this view indicating that taking a context – practice approach by adopting PoP as a framework provides more coherent and dynamic understandings of teaching practices. Furthermore, it will help in resolving some of the conceptual and methodological problems of a belief–practice approach while maintaining an interest in the meta-issues that constitute the field of beliefs.

The social approach of research in mathematics education has progressively promoted the notion that practice is not only a personal individual matter; it is in fact situated in the sociocultural context. Although the relationships between individual and social factors of human functioning have generated much debate in mathematics education, it is mainly in relation to student learning (Skott, 2013). Therefore, PoP is a theoretical framework that aims to understand the relationships between teachers' practice and social factors. To a considerable degree, PoP adopts participationism as a metaphor for human functioning more than mainstream belief research. Therefore, PoP draws on the work of participationism researchers, specifically Vygotsky, Lave and Wenger, and Sfard.

Skott (2010) initially developed the patterns-of-participation framework in relation to teachers' beliefs. However, in order to develop a more coherent approach to understand teachers' practices, Skott (2013) extended the framework to include knowledge and identity. Research on teachers has mainly focused on studying three relatively distinct domains: teachers' knowledge, beliefs, and identity. This leads to some incoherence that negatively influences the understanding of the teachers' role in classrooms. Skott presents PoP as a coherent, participatory framework that is capable of dealing with matters usually faced in the distinct fields of teachers' knowledge, beliefs, and identity.

METHODOLOGY

This paper is part of an ongoing study that intends to develop more coherent understandings of Saudi high school mathematics teachers' practices during the current reform movement. The data presented in this paper comes from two experienced high school mathematics teachers Nora and Haya. Nora has 13 years of experience teaching mathematics in public and private middle and high school in Saudi Arabia. She has a Bachelor degree of Science with a specialization in mathematics. Nora has never taken any education courses. Haya has 10 years of experience in public high school. She has a Bachelor degree of Education with a specialization in mathematics. The education courses Haya had in university focused on general issues related to teaching, such as lesson planning and classroom management.

I conducted a 60-minute, semi-structured interview with both participants. I invited them to reflect on their experiences with mathematics and its teaching and learning during their years of experience. During the interview, they expressed their views about the recent reform movement in Saudi Arabia. I also asked them to reflect on their experience teaching mathematics using the old and new textbooks. Interviews were audio recorded and transcribed. As used by Skott (2013), I used a qualitative analysis approach based on grounded theory method.

DISCUSSION

Being a teacher in an era of educational reform

Nora shows her deep personal commitment to current educational reform in Saudi Arabia. She believes that the pace of educational reform has been increasing at the global level and Saudi Arabia needs to join the global movement of education reform. She emphasizes the need to be reasonable and fair when we talk about recent reform efforts. She explains, "reform is one of the controversial topics among people who are interested in educational issues in Saudi Arabia...but we have to admit, changing is difficult and complicated". In the interview, Nora indicates that success of reform movements depends, at least in part, on the degree of match between teachers' perceptions of the teaching practice and their role as teachers, and the demands of the reform movement. She states that "creating a positive change starts with creating a motivated teacher".

Haya on other hand has more skeptical view abut recent reform movement in Saudi Arabia. She states, "I think reform ideas are something nice to read about in a book or something. These ideas usually are not applicable to a real world classroom". She argues that many teachers are confused when it comes to understanding the goals of the recent reform movement. She blames the Ministry of Education for this confusion. She explains that, on one hand, the Ministry introduces new mathematics curriculum which they claim will change the culture of mathematics learning in schools towards a focus on reasoning and problem solving, but on the other hand, the Ministry established new standardized tests for high school students which maintains a traditional teacher-centered and exam-based educational environment. She claims that the ultimate teaching objective “was and still is” to improve students’ exam marks and the recent reform movement failed to change this objective.

New mathematics textbooks impact

Nora expresses that before the introduction of the new textbooks, she was very enthusiastic. She believes that the new textbooks are generally better than the old textbooks. She also believes that the new textbook supports student learning and creates more positive and engaging environment in the classroom. However, Nora indicates that she feels isolated and unsupported in her use of the new curriculum materials. She states, “Very often I have questions about the textbook, but I don’t know where I can’t find answers”. She complains that the Ministry of Education did not take teachers’ preparation of the use of the new textbooks into account. She indicates that the only other resource she has besides the teacher's guide is her communication with other mathematics teachers in her school. Conversations Nora has with other teachers provide support and a rich resource for Nora’s practice. After the implementation of the new textbooks, Nora and her colleagues started talking more about teaching mathematics.

Nora comments about her teaching using the new textbooks; “although I feel that the new textbook could offer a better learning experience to the student.., I am not sure if I am using it effectively...I’ve been trying to change since we adopted the new textbooks, but sometimes I feel that changes are not obvious in my practice.” She indicates that the textbooks motivate her to reflect on her own teaching practice. She explains that teachers need to learn not only from textbook but also from their own teaching practice.

Nora argues that one of the most positive aspects about the new textbooks is that many of the activities ask students to explain and express their understanding. Nora says that her students find it difficult to put their understanding into words because they are simply not used to talking in the mathematics classroom as they do in other classes. However, Nora indicates that some of the activities presented in the textbook do not make sense. She explains, “I honestly don’t feel that I should let the textbook control what I do in the classroom all the time”. It seems to me that Nora struggles with eliminating the authority of the textbook on her practice.

Haya’s unsettling sense of confusion regarding curriculum change is noticeable in her remarks. While she indicated that the new textbooks are better than the old textbooks, she is sceptical about the impact these textbooks could have on teaching practice. Haya believes that most mathematics teachers have outdated perceptions of mathematics teaching and learning and merely changing the textbooks is not going to change teachers’ perceptions. She expresses her frustration about the big gap between society’s high expectations towards teachers and teachers’ real capability of meeting these expectations.

When Haya describes her teaching practice after she started using the new textbooks, she explicitly indicates that change is something she thinks about more than she actually applies in her classroom. She explains that sometimes her classroom seems more interactive and engaging, but what students actually learn is very limited. She also criticizes the new textbooks because they do not address students' different mathematical ability levels. She thinks the textbooks are designed for students with strong mathematical skills but students with low skills find many activities of the textbooks confusing.

What does it mean to do mathematics?

During the interview, Nora discusses the issues of classroom culture around what it means to "do mathematics". She believes that there is a common culture in school mathematics which views doing mathematics as sitting quietly at a desk, finishing a worksheet, using the textbook as a main resource and turning in the completed work prior to class ending. The new textbooks, in Nora's view, challenge this old lasting culture. Nora comments on the textbook's presentation of situational problems which are connected with real life situations. She believes that the textbook surely make some positive transformations compared to old textbooks, which simply delivered mathematical concepts in a very isolated manner. However, she also indicates that making the connection is not always easy.

Both teachers consider examinations as being powerful force in forming and directing how teachers and students do mathematics. Doing mathematics in Haya's classroom is extremely influenced by students' achievement and tests marks. She argues that high school students care most about doing well in school exams and standardized tests. She explains her job as a teacher is to help her students "know how to do mathematics". Haya also comments on some activities in the textbook which encourage students to explain their understanding and justify their solutions. She indicates that she tries as much as she can to include these activities in her classroom practice, but at the same time, she claims that high school is too late to start encouraging students to master these types of communication skills in the mathematics classroom.

RESULTS AND CONCLUSION

Both teachers show their commitment to the profession and express their concern for doing what is best for their students. They both indicate that the content and structure of the textbooks changed significantly from the former textbooks. Students' achievement and tests marks are significant to the classroom practice of both teachers. Also, Nora's and Haya's practice as high school mathematics teachers reveals patterns of tension and confusion with regards to understanding the current reform movement in Saudi Arabia. The lack of support and guidance both teachers received before and after the implantation of the new textbooks has a negative impact on their use of the textbooks in their classrooms. Both teachers developed a sense of isolation in the current reform movement. Also, part of the tension both teachers are experiencing comes from their struggle with eliminating the authority of the textbook in their practice.

Both teachers have developed a sense of obligation and stress to improve their teaching practice. Nora seems more motivated about improving her teaching practice; she uses the new textbook as a tool for self-directed professional development. The different perspective the new textbook offers about mathematics learning encourage her to reflect more on her teaching practice. Also, Nora's interaction with other teachers in her school is significant to her classroom practice.

Haya on the other hand appears less motivated about the making changes to her teaching practice. It seems that her perceptions on teaching and learning are being compelled to change in order to keep up with reform demands. She is uncertain about the meaning of the change and has some resistances to making changes in her practice. With her struggles to make some changes in her practice, her conceptions of teaching and learning mathematics seem to remain the same. Although she indicates she was very supportive about the implementation of the new textbooks, a sense of uncertainty about the value of the new textbooks started to emerge in her practices.

Understanding the patterns in the ways in which the two teachers participate in these practices and contribute to their constant reconstitution and renegotiation of their teaching is a complex task. Using the data I collected from the interviews, I was able to get a sense of some of the practices that are significant to the two teachers' classrooms participation. However, to develop a better understanding of both teachers' practice as mathematics teachers, more data is needed. The use of multiple open interviews in combination with observations of classroom and staff-room interactions may allow access to practices and figured worlds beyond the classroom (Skott, 2013). Skott (2013) also suggest that "teacher's narratives about her own schooling; about formal and informal collaborative activities with her colleagues; and about discursive manifestations such as the reform" provide deeper understanding to the meanings teachers bring to their classroom practice (p. 552).

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PARENT-CHILD MATHEMATICS: A STUDY OF MOTHERS' CHOICES

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Research on mathematics found in 'everyday' interactions (e.g., Walkerdine, 1988) often relies on analysis of parent-child talk during studies of social interactions and/or literacy events more generally. In contrast, from the outset of the current study, parents were aware that mathematics was the focus of study and that each of them would determine the activities to be video-taped in their home. In this paper, we report on the types of activities six middle class mothers perceived as opportunities to engage their preschool child with mathematics. Analysis also included the patterns found within and across families. Overall, the mothers documented play-based events, many of which were common across four or more homes and entailed 'less conventional' mathematics. Parental styles of mathematical engagement are discussed.

BACKGROUND

Gifford (2004) argued that the formal and informal pedagogy that supports children's development of early mathematical competence has not been well documented. While we know children enter school with considerable mathematical knowledge, we know much less about the ways in which parents and significant other family members support them in developing that knowledge prior to school. Much of the recent research, into young children's mathematics learning within the context of home and family relies on parent reports (e.g., Lefevre et. al. 2009) or observations of parent-child interactions during activities or tasks, using materials provided by the researchers (e.g., Vandermaas-Peeler, Nelson, Bumpass & Sassine, 2009; Anderson, Anderson & Shapiro, 2005: Anderson, 1997). This research on young children's engagement in mathematical activities at home demonstrates considerable diversity across families in terms of the frequency and types of math, although findings across studies indicate that families tend to emphasize counting and number concepts (e.g., Anderson, Anderson & Thauberger, 2008). Of particular relevance to the current study are the few studies (Walkerdine, 1988; Aubrey, Bottle & Godfrey, 2003; Trudge & Doucet, 2004), which investigated parent-child interactions during 'naturally occurring' events at home through audio or video recording or direct observations. In these studies, although researchers identify the mathematics evident in activities and events or in parent-child interactions, it is unclear whether the parents construed the activities as mathematical. In contrast, in the present study, we were interested in having parents identify the activities that they believed were examples of ways in which they engaged their young children in mathematics. Thus this study investigated the types of activities parents view as contexts for mathematics learning, and the ways in which the activities evolved when parents knew the focus of the study was on

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parent-child mathematical activities and events and the interactions that occurred therein. We documented parents' self-selected mathematical activities in the context of their home, providing insight into the nature of activities that mothers perceive as opportunities to engage their preschool child with mathematics. By asking participants to identify and document mathematical activities and events in the home, the current study augments previous studies, which rely on parent reports alone. Likewise, unlike studies where researchers provide the materials and tasks for the parents and children, the current study observed the ways in which the parent-child dyads constructed activities using resources found in their homes.

Our research is informed by socio-historical theory (e.g. Vygotsky, 1978; Wertsch, 1998) and the notion that learning is social, as well as individual. Children learn to use the "cultural tools" such as mathematics of their community and culture inter-psychologically as they are guided and supported by parents and significant other people. As they practice using these "tools" and support is gradually withdrawn, children learn to use them intra-psychologically or independently.

METHOD

Six families were recruited from an unaffiliated Child Study Centre located on a university campus. The children (5 girls, 1 boy) were two and a half years old at the beginning of the study. The parents were well-educated, middle to upper-middle class, and lived in relatively affluent neighbourhoods adjacent to the university campus. On mutually convenient occasions spread over two years, we videotaped parent-child dyads (4 mother-daughter, 1 father- daughter, 1 mother-son) participating in everyday 'at home' events of their choosing (e.g., baking cookies, viewing photos). As indicated earlier, we informed parents at the outset that the research focus was on children's early mathematics in the home. At the beginning of each home visit, the mother designated the shared activity that was to be videotaped. Four of the families were video taped by the same research assistant, who remained the field researcher for the duration of the study. Two of the mothers opted to carry out their own videotaping, an option made available to all parents. The number of video taped sessions varied across the families (i.e. 4-10), with all sessions lasting at least 15 minutes.

To analyse the data, we viewed the videotapes of each family three times, and wrote comprehensive notes during each viewing. We then transcribed each videotaped session for each family in its entirety. The first author read the transcripts three times, referring to the initial notes each time so as to provide thorough documentation and understanding of each episode. Each transcription was then analysed in terms of the types of activities and events in which each family engaged. Patterns across and within families and similarities and differences across activities and events were determined. Secondly, parent-child interactions were analysed according to the extent to which mathematics was explicitly present in the interactions as the activity evolved. For that analysis, we used five a five point scale, namely mathematics was deemed: (a) prevalent b) a major focus c) an equal focus with other aspects d) a minor focus or e) incidental throughout the activity, from an observers' perspective.

RESULTS

Because each mother individually identified the activities and events to videotape without consulting any other participants, diversity across families was anticipated. Overall, 44 specific activities (e.g., playing Snakes & Ladders; viewing Photos) were documented. When these were clustered according to general defining features (e.g., board games; family time), thirteen categories emerged (see Table 1). Eleven of the 13 categories contained activities chosen by at least half of the mothers. The most common categories were puzzles, pretend play, board games, story time, family time and playing with toys. Closer examination revealed that the activities chosen were mainly those we intuitively associate with children's play (i.e., using stickers to create pictures) and child's at-home participation in family routines (i.e., baking cookies) with minimal examples of school-like activities. Indeed, these mothers predominately chose to videotape adult-child play of one sort or another.

As might be expected from the design of the study, each videotaped activity within the categories was unique to the parent, child and materials involved. For instance, the 'number' puzzles that the Adam (pseudonyms are used throughout) family used incorporated puzzle pieces with a numeral fitting onto a background space showing the same number of objects. On the other hand, the jigsaw puzzles that the Pimm family used involved a picture broken into a number of irregular, interlocking pieces. Such contrasts led us to consider to what extent the mothers' choices were based on overt mathematical features of some materials marketed to homes. We labelled an activity 'conventional', when numerals, shapes and counting were key features of the material (e.g., BINGO) and 'less conventional' when mathematical elements or features were not deemed key to the typical use of the material (e.g., Hungry Hippos). Analysis revealed that about one-quarter of the activities (11), which the mothers chose to videotape, involved commercially produced mathematical materials, while over half of the activities (27), were characterized as less conventional, mathematically.

Finally, to describe the extent to which mathematics was explicit during the chosen activities, we used a 5-point scale described earlier (See Table 2). For example, as the Beet dyad played checkers, the mother often explained her moves and what might happen if the child moved one way or another. After a checkmate, the mother counted the checkers, and on one occasion the child made (and named) a square with four checkers. Here, counting and shape recognition were deemed explicit attempts to include mathematics, whereas the mother's explanations appeared to illustrate "how to" play the game. Thus we assigned "(d) Math occupies a minor portion of the activity but seems conscious" to best describe the minor role explicit mathematics seemed to play here. (See Table 2, Beet, board game.)

Category	Each family's activity					
	Adam	Star	Penn	Pimm	Liu	Beet
Puzzles	Number		Jigsaw(2)	Jigsaw	Number	Jigsaw
Play	Store	Stickers		Pegboard	Traintracks	Tea party
Board game	Snakes & Ladders	Hungry Hippos	BINGO			Checkers
Story time	Number & shapes	Felt story board		Sounds of World		Matching objects
Family time		Lunch		Photos	Videos	Baking
Toys		Traintracks	Cars	Food/dolls	Pop up	
Playdoh	Sharing pizza	Happy face				Making food
Physical games	Hopscotch			Follow the leader		Water sprinkler
Matching games	Cards:word numeral, dots			Cards: images	Rods: "Ten" family	
School like	Word problems			Yearbook entry	Compu-ter game	
Songs	1,2,buckle my shoe	Row, row your boat			ABCs	
Other games				Dreydel	Macaroni	
Miscellaneous				Penny tracing		

Table 1: Activities mothers chose to videotape

Activity	Adam	Liu	Penn	Star	Beet	Pimm
Puzzle	a	a	c		e	c
Play	a	b		a	d	d
Board game	a		a	b	d	
Story time	a			e	d	e
Familytime		b		d	a	e
Toys		b	b	e		e
Playdoh	a			d	d	
Physical	a				e	e
Matching	a	a				e
School	a	a				d
Songs	a	e		e		
Other games		a				e
Misc						e

a: Math is the core and goal of the activity.
b: Math occupies a major portion of the activity but was not the original goal
c: Math occupies an equal part of the event, other aims and content are achieved.
d: Math occupies a minor portion of the activity but seems conscious.
e: Math is incidental or subtle for the most part and may/may not be apparent

Table 2: Activities ranked on a continuum of mathematical involvement

Once each activity/family was coded, the analysis revealed that while one mother (Adam), chose the same type of activity each time, the other mothers chose activities, which varied somewhat according to the explicitness of the mathematics. In addition, we were surprised to see that the families' profiles appeared to fall along a similar continuum (See Table 2). That is, these mothers' choices suggested parental styles of engagement whereby for two families (Adam, Liu) the majority of the activities video taped were explicitly mathematical, for two other families (Pimm, Beet) the majority of activities videotaped were incidentally mathematical and for the remaining two families (Penn, Starr) an eclectic style of engagement was evident.

DISCUSSION

Due to the small size and the homogenous nature of the sample, some caution is called for interpreting the findings of this study. However, we believe the findings from the study are significant and contribute to the literature on young children's engagement in mathematics at home. In previous research using self-reports and surveys (e.g., Lefevre et. al.), families have reported similar activities as these mothers chose to videotape (e.g. baking cookies; playing board games). Thus the current study provides further evidence of the "myriad of ways in which everyday practices common to many home environments" (Benigno & Ellis, 2008, p. 298) may support children's mathematics development. However, although many of the families in the current study engaged in playing board games for example, analysis revealed that families played an array of board games with different affordances in terms of mathematical learning. Thus the present study provides a more nuanced insight and understanding of these 'taken-for-granted' activities (i.e. 'doing puzzles'), as a site for children's early mathematics learning.

That the mothers chose mainly play based activities appears to reaffirm that "[m]uch of young children's exposure to math does not occur during explicitly didactic interactions" (Benigno & Ellis, 2008, p. 294). Likewise, it seems the mothers in the current study concur with the majority (77%) of parents interviewed by Canon and Ginsburg (2008) who believed "children's mathematical learning should be incorporated into their daily living" (p. 250). However, further research with families from diverse backgrounds is needed to determine the extent to which this holds for those outside the mainly Euro-Canadian, middle class homes represented here, especially since some cultural groups favour a more didactic form of teaching and learning with an emphasis on rote memory.

Unlike the somewhat dichotomous "instrumental or pedagogical typifications" put forth by Walkerdine (1988) and Aubrey et al. (2003) for mother-child mathematical interactions, in the current study, at home mathematical activity appeared to fit along a continuum as identified by the 5-point scale. Thus in addition to those activities at the extremes where mathematics was core or mathematics was incidental, we documented activities where mathematics played a major or minor role as well as those where mathematics seemed equally important to non-mathematical goals or aspects of the activities. Looking across the six families, the results of this study suggest that in addition to mothers who take on a "pedagogic stance", and others who share more instrumental activity with their child, some mothers adopt a more eclectic style of mathematical engagement than a dichotomous view permits us to see. Of course, it remains to be seen if preferences seen in the video taped sessions reported here, will prevail in other data sources (parent interviews, diaries) yet to be analysed or if an eclecticism within all families might be revealed as more of the families' everyday practice is examined. While we concede that a continuum of parental practice complicates our search for a definitive explanation as to how and why children enter school with a range of mathematical knowledge, it likely better represents the complexity of the ways in which parents and children engage in mathematics at home. Further research, which accounts for the breadth and depth of mathematical experiences in families, is needed to better understand the nature of children's mathematics learning prior to school.

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METRIC EQUIVALENCE IN INTERNATIONAL SURVEYS: CULTURAL EDGES

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This is a critical methodological paper concerning the translation and cultural adaptation processes of an international mathematics education survey questionnaire. Metric equivalence concerns not only language, but also content and activities chosen as indicators in the survey. We here focus the challenges when making cultural, historical and societal considerations when adapting a survey to a new language and cultural context. We conclude that the recommended back translation is not enough to ensure metric equivalence when adapting surveys to a new country. Therefore, we suggest an elaborated method for cultural adaptation. Regarding our survey, this resulted in a survey translation that is better culturally adapted for respondents.

INTRODUCTION

The background for this paper is the now reflected and elaborated answers to an important question posed at the discussion after our presentation at PME37 (Andersson & Österling, 2013): "What were your considerations during the translation process?"

Cross-cultural surveys imply translations of questionnaires to new languages and cultural contexts. To be able to compare results across the borders, the translations need to obtain metric equivalence. The aim of this paper is to document and describe the methodology we developed for translating and adapting a questionnaire from an Australian-Asian context into Swedish language and school culture. We here account for our experiences and critical reflections after the translation and adaptation of the international survey questionnaire within The Third Wave Project, "What I Find Important" (WiFi) (Seah & Wong, 2012), a survey that across cultures investigates what students value as important when learning mathematics. This large-scale quantitative investigation consists of a web-based questionnaire with 89 questions to be distributed to 11 and 15-year old students in 19 different countries. Our task was to translate the questionnaire, into Swedish with possibilities to, first, research what Swedish student value and, second, to be able to make international comparisons.

In a quantitative study, a good measure of values is hard to obtain (see Andersson & Österling, 2013). The problems can be compared to the methodology of attitude surveys, where indicators of attitudes are used instead of posing direct questions (Sapsford, 2007). To obtain metric equivalence, it is crucial that an indicator indicates the same value after a translation. We aim in this study to keep the metric equivalence by conserving the intended meaning of each indicator after translation. Hence, we need to choose either culturally neutral indicators, if such exist, or we need to adapt indicators that conserve the intended meaning across cultural borders.

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The WiFi-study is based on value categories from different theoretical frameworks, mainly mathematical values (Bishop, 1988) and cultural values (Hofstede, Hofstede & Minkov, 2010). The value diversity meant a need to differentiate amongst the many dimensions and layers of values that are portrayed in the classroom. To give some examples; Seah & Wong (2012) take the stance that "values are regarded in [the Third Wave project] from a sociocultural perspective rather than as affective factors." This sociocultural perspective may imply that values can be found in relationships, languages and available discourses. Hofstede et al. (2010) instead define values as "the core of culture", explaining that culture reproduces itself and its values through cultural practices. Those practices can be what parents say and do when fostering their children, or what activities teachers choose to do in the classroom. How activities are values is decided by the members of the cultural group.

The questions in the WiFi-survey questionnaire consist mainly of activities from mathematics classrooms. Respondents are asked to answer how important each activity is when learning mathematics. The different activities were chosen as value indicators in the WiFi-questionnaire. Therefore, we need to address cultural practices in the mathematics classroom to validate that the intended meaning of our indicators was culturally stable. In this validation process, we used several methods: repeated pilot tests, interviews with targeted students and educational and historical research to understand the cultural background of Swedish mathematics education.

Historical, Societal and Cultural Background of Swedish Mathematics Education

From the results of WiFi-study we will learn more about what students express as important learning activities in mathematics. To obtain a cultural adaptation while maintaining metric equivalence during translation, we needed deeper knowledge about societal and historical facts that form mathematics educational practices. Otherwise, it is hard to determine what value a value indicator indicates. To give an example, Lundin's (2008) work shows that when Swedish schools became public and mandatory in 1842, teachers had to deal with a large number of children that were the first generation attending school. The first early schoolbooks had two aims; to support the learning of mathematics and support teachers to cope with disciplinary problems. "This need led to the promotion of schoolbooks filled with a large number of relatively simple mathematical problems, arranged in such a way that they (ideally) could keep any student, regardless of ability, busy – and thus quiet – for any time span necessary." (Lundin, 2008, p.376). Mathematics was used as a medium for fostering children. The School Inspectorate's research report (2009) concludes that teachers are still relying on textbooks when planning their teaching, hence trust the textbook to fulfil curriculum objectives. Lundin's (2008) explanation of the historical development might explain the School Inspectorate's (2009) results. This particular way of organising mathematics education is believed to support teachers in managing non-homogeneous student groups so that each student can work according to his/her previous learning and needs. It is likely that parents and students expect mathematics classes to be conducted this way. Hence, working quietly in the textbook has become part of the culture of Swedish mathematics classrooms.

METHODOLOGY

As commonly practiced, the WiFi-study Research Guidelines (not published) suggested translation and back translation as a way of obtaining metric equivalence. However, after having done successful back translations, we conducted a pilot test of the translated questionnaire with a sample of 11-year-old students. It turned out that there was several questions the targeted students did not understand. Therefore, we needed to consider how we best could adapt the questionnaire to a Swedish context, and how best choose contents and activities as indicators that Swedish students are familiar with. A back translation did not serve our purposes. We needed other methods for the cultural and linguistic adaptation.

Exploring methods of cultural adaptation

Translation and back translation can be conducted to investigate problems in the target text. However, this produces limited information of the quality of the target text – which also, as described, became our experience. Harkness, Villard & Edwards (2010) criticizes the use of back translation as a standard method, drawing on research that shows that appraisal of the target text directly is more efficient.

We explored, evaluated and adapted the guidelines for cross-cultural research, published by the Survey Research Center (2010). Harkness et al. (2010) suggests "The TRAPD Team Translation Model" as current best practice. The steps in this model are; **T**ranslation, where two translators make two independent translations; **R**eview, where the translations are compared and refined; **A**djudication, where the translation is separated from review with focus on, amongst other things, a cultural adaptation; **P**ilot test and finally **D**ocumentation of every step in this process. A team should include translator, reviewer and adjudicator. Adjudication is suggested to follow these steps; linguistic mistakes in the translation process, cultural adaptation problems, questions that do not work in the intended group and generic problems from the source version. Each survey is unique, and we adapted this model to suit the circumstances of our project. The frames of this project did not allow for hired professional translator or to organize extensive pilot tests. But we had a team, consisting of three mathematics teachers' educators and researchers. We used the different stages iteratively, and went back to new translations, reviews and adjudications. During this process, we added scoping interviews with students as well as knowledge from earlier educational research to improve the cultural adaptation. Below we describe how this adapted model was used to improve the quality of the translated questionnaire and to keep the metric equivalence.

RESULTS

Results from the adapted TRAPD-process

Scoping interviews: We needed to learn more about how the intended group of students themselves expressed their valuing and interpreted our questions. Semi-structured scoping interviews (Bryman, 2012) were hence conducted. In the translation process, this was intended to help us use students wording and examples in our translation and to facilitate the understanding of the questions.

1st translation: In this stage, the translators, three persons in our case, made a close translation of the WiFi-questionnaire from English to Swedish.

1st review: The translators compared and reviewed each other's translations in review meetings to decide on the best translation. We focused at this stage to keep the translation as close to the original version as possible for a successful back translation.

Back translation: Two persons, who had not previously seen the questionnaire, conducted the back translation from the Swedish translated questionnaire to English.

1st adjudication: In our project, also the adjudication was a team work. We compared the original and the back translated questionnaires and used colour codes to grade the similarities/differences between them. Since the 1st translation was close to the source questionnaire, the back translation was acceptably similar to the source questionnaire.

1st pilot test: In this pilot test, a group of 28 eleven-year-old students were asked to answer the questionnaire, and when doing so, indicating what questions they found difficult to understand or interpret.

2nd adjudication: When analyzing the pilot test, there were too many questions students found difficult to understand. We concluded that we needed to improve the cultural adaptation as well as the adaptation to the intended group. We looked up items in research texts and in the curriculum to check for meaningful and proposed activities in a Swedish context. An example can illustrate the process so far:

Example 1: Q9 focuses ”Mathematics debates”. In the 1st translation, this was easily translated to "*Debatter med matematik*", and the back translation was close enough, "debating maths". However, when trying out the questionnaire in the pilot test, eleven students out of 28 did not understand the question. And when discussing "Mathematics debates" in the 2nd adjudication, not even we as adjudicators were sure about how such a debate is enacted in the classroom. "Mathematics debates" are in the WiFi Research Guidelines (not published) classified as an indicator of valuing openness and exploration. Mathematics debates is not an activity that is common in Swedish classrooms, so out of what it is supposed to indicate, we tried to adapt the indicator, and describe an activity that children could recognize. In the 2nd translation, the question was formulated "*Debattera och ifrågasätta lösningar i matematik*" (Debate and question mathematical solutions), a cultural adaptation so respondents can visualize a situation while still relating to valuing openness and exploration.

Documentation was kept during the whole process of all the different versions of each question. It supported our evaluation of the improvement of quality.

This process made us realize that translation and back translation is not a good instrument to ensure metric equivalence when researching students valuing when learning mathematics. We need to use other methods and decided to take the adaptation one step further. Consequently we followed up the pilot test with interviews of participating students in order to better understand the intended meaning of their answers to some of the questions.

Understanding respondents' intended meaning

Respondents obviously need to understand survey questions. Therefore, we asked them how they interpreted the questions and what their intended meaning was when answering our questions.

Example 2: According to the pilot test a large proportion of students valued Q36 "Practicing with lots of questions" as important or absolutely important. However, Sara, 11, did not. We discuss this result in particular, since it aligns with research results, which show that this is an important trait of Swedish mathematics education.

This question was not hard to understand or to translate. Still, we got contradictory answers in the interviews. We wanted to find out what students valued when they responded that "practising a lot" (*öva genom att göra många uppgifter*) is important or not. Sara, 11, expressed:

Interviewer: - Do you think you need to practise a lot to learn mathematics?

Sara: - Well, if you are already good at it… no!

Her reasoning and intended meaning of this response was more elaborated and very different from what we predicted. She here stated that "good" students don't need to practice that much. However, later in the interview, she gives us examples of mathematical content one always needs to practise a lot, which is practicing the times-tables. She also recognises that there is a different learning process in learning times-tables from learning problem solving, but she cannot express what she finds important for learning problem solving. Her rating of "Practicing with lots of questions" was "neither important nor unimportant". Therefore, using "Practicing with lots of questions" as an indicator becomes hazardous, since respondents make connections and reflections we cannot predict. Interviews with students allowed us to discover some of those unpredicted responses, thus allowing us to problematize conclusions from the data.

3rd adjudication: We worked further on finding expressions and concepts from Swedish classroom contexts. We used previous educational and historical research, as well as our years of experiences as teachers and teacher educators to find the best expressions that could fit classroom cultures and the selected age group of the respondents. At this stage, the team used all information we had gathered to reconsider our translation and adaptation. We used results from the pilot test, from interviews, from a curriculum analysis and the back translation. This method allowed us to evaluate our translation from several perspectives.

2nd translation: We moved away from our initial intention of keeping the target questionnaire (the translated version) as a close translation to pass a back translation. Instead, we put a lot of effort in analyzing what activities that could be the best indicators of the requested value. The use of indicators in the WIFI-study has previously been discussed by Andersson & Österling (2013). We give an example to show how we worked through the whole process.

Example 2: Q11 focuses "Appreciating the beauty of maths" and Q60 "Mystery of maths" were not comprehensible for the Swedish students due to the pilot test. The version we tried out was a close translation. In the 2nd translation we chose to give examples to illustrate what "beauty and mystery of maths" can be. Q11: "*Uppleva att matematik kan vara vacker (som mönster i konst, arkitektur och natur*)" (Experience that mathematics can be beautiful (like patterns in art, architecture and nature) and Q60: " *Undersöka gåtfulla matematikexempel (till exempel kan du lätt mäta en tredjedel av 9 cm exakt med linjal, men en tredjedel av 10 cm går inte att mäta exakt*)", (Exploring enigmatic mathematical examples (e. g. you can measure a third of 9 cm exactly with your ruler, but you cannot measure a third of 10 cm exactly). If those questions were to be back translated, a comparison would say that they are quite different. But the intended meaning is easier for respondents to understand. Therefore, this way of adapting questions to what is familiar of respondents conserves the intended meaning, and thus improves the metric equivalence, since the new question works as an indicator of the values intended.

To sum up, there were a large proportion of questions where the mathematical content and/or the mathematical activities in classrooms were not familiar to Swedish eleven-year-old students. There were also questions that could be interpreted differently, due to cultural differences or due to individual experiences amongst respondents. Therefore, we made some clarifying examples, or even chose a different activity, to try to improve the metric equivalence and construct validity.

CONCLUDING DISCUSSION

Quantitative cross cultural surveys and assessments like TIMSS or PISA are increasingly important aspects of policy making decisions about mathematics education. Those investigations pose the same questions in all countries, since the aim is to compare knowledge between countries.

Recognizing that there are historical and cultural differences between participating countries make it problematic to compare the assessed knowledge, since it is based on the assumption that mathematical content is valued equally everywhere. The WiFi-study is different; it surveys what students find important and does not assess students' mathematical knowledge. But the survey still suffers from the same difficulties, that we are not sure if mathematics or mathematical activities are valued equally across the participating countries. Translating a questionnaire with questions about learning mathematics does not imply only linguistic aspects. The indicators need to be evaluated out of what mathematical content students recognize when being part of the subject of mathematics and what mathematical activities students from different countries or cultures are familiar with.

The WiFi Research Guidelines (not published) suggested translation and back translation. However we could conclude that a successful back translation is not enough to ensure metric equivalence. Having our minds set on how to translate questions so that they would suite the back translation resulted in a too close translation, and respondents in the pilot test did not understand all the questions. Therefore, a back translation did not help us neither with the meaningfulness of item content to each culture, or with the metric equivalence. Instead, an adapted TRAPD-model (Survey Research Centre, 2010) gave us useful tools to improve the cultural adaptation. However, a cultural adaptation cannot be drawn too far without affecting the instrument validity across languages. We had to pay careful attention to maintain the metric equivalence in order to have the possibility of making cross-cultural comparisons of students' values, as intended in the WiFi-project (Seah, 2013). From the results from a finished WiFi-study we can learn more about differences between cultures and values in mathematics learning. However, our dilemma is that at the same time, we depend on some of this knowledge when adapting a proper questionnaire.

Until our larger research study shows us where edges of cultural values can be found in mathematics education, we recommend the other seventeen teams within the WiFi-project, or similar cross cultural projects, to reflect on the translations and cultural adaptations and maybe adopt and adapt further the team translation process. Within the adjudication stages, there are rich opportunities to critically reflect on cultural adaptations through interviews, pilot tests and previous research to improve metric equivalence in cross-cultural research.

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‘I SENSE’ AND ‘I CAN’: FRAMING INTUITIONS IN SOCIAL INTERACTIONS

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In this article we examine intuitions as they emerge in groupwork activities. We provide a framework and a methodology to code various aspects of the activity, social and mathematical. Focusing mostly on students’ gazes, we explore how affective moves give rise to, and determine, students’ interactions and thoughts. We argue that intuition does not take place in the mind of the individual, it is not a matter of ‘I think’, but it arises from actions and reactions, in relationships with others and with artefacts. Data from a 50 minutes groupwork activity of four grade-9 students allows us to further discuss our framework.

INTRODUCTION AND BACKGROUND

Dewey (1938) states that intuitions and illuminations are not "part of the theories of logical forms" (p.103). Illumination is the phenomenon of sudden clarification arriving in a flash of insight and accompanied by feelings of certainty (see Liljedahl, 2012, and references therein). Intuition, as well, is a form of thinking that provides the learner with a sense of certainty (Fischbein, 1987): it is perceived as global (rather than analytical), coercive and self-evident. Sometimes intuitions from everyday experience contrast with mathematical knowledge and can impede learning: misconceptions are such kind of intuitions (Fischbein, 1987).

Andrà & Santi (2013) underline that intuitions are a way of establishing a relationship between the learning subject and the object of knowledge, they are a mode of existence of the consciousness which intertwines with perception, sensorimotor activity, emotions (which provide the learner with a sense of likelihood of success, see Roth & Radford, 2011), and mathematical generalization. They conclude that intuitions can start in a private, individual moment, but it is in the communitarian self (Radford, 2012) that they develop towards mathematical generalizations. If so, which is the relationship between the individual moment of illumination (see also Liljedahl, 2012) and the emerging of shared intuitions in the communitarian self, which can develop into mathematical deductive forms of thinking and proving? In order to answer to this question, we have developed a methodological framework (Liljedahl & Andrà, in press) that helps us capturing and decoding the turbulent undercurrents of groupwork mathematical activities. After briefly presenting the framework that informs our research, we apply it to the analysis of an episode in a grade-9 class working on basic concepts in probability. We will discuss illuminations that emerge and develop in the

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social interactions, as well as how they inter-relate with other modes of existence of the consciousness.

FRAMEWORK, THEORETICAL AND METHODOLOGICAL

Groupwork activities in the classroom have gained more and more attention in the last decades. In such activities, communication plays a primary role. Sfard (2001) points out that "communication may be defined as a person's attempt to make an interlocutor act, think or feel according to her intentions" (p.13). Following this view, thinking is thus subordinated to and informed by the demand of making communication effective. Within this domain (called interactionist or participationist) learning is seen as becoming participant in a mathematical activity. Activity is sensitive to context and allows the growth of mutual understanding and coordination between the individual and the rest of the community. Accordingly, each activity has its roots in our cultural heritage and can be shaped and re-shaped by the group of practitioners. It is within this framework that thinking is conceptualized as a case of communication, since interactionist research postulates the inherently social origin of all human activities (Sfard, 2001).

Sfard (2001) suggests that in learning processes, seen as initiations to become skillful participant in mathematical discourses, two key factors need to be considered: the tools that mediate the communication and the meta-discursive rules that regulate it. The focus of this paper is on the latter.

Meta-discursive rules have an implicit nature, they are tacit, and it is within the system of such rules that culturally-specific norms, values and beliefs are encoded (Sfard, 2001). According to Merlau-Ponty (2002), awareness is not a matter of 'I think that' but of 'I can': before the reflective, the positing thought, there is an act ('I can do this'). Specifically, since learning "occurs in and through relations with others in the pursuit of collectively motivated activity" (Roth & Radford, 2011), motivation is the orientation of the activity. Emotions express the student's current state with respect to the motive of the activity, they express her sense of likelihood of success in realizing such motive (Roth & Radford, 2011). Given the social environment in which the students act, interact and determine the moves of the activity on the ground of their emotions, we methodologically exploit the idea of interactive flowchart.

Interactive flowcharts were introduced by Sfard and Kieran (2001) as a way to capture "two types of speaker's meta-discursive intentions: the wish to react to a previous contribution of a partner or the wish to evoke a response in another interlocutor" (p.58). A conversation can be coded as being comprised of a series of invisible arrows aimed at specific people and/or specific utterances. The scheme follows two basic structures: (a) a vertically or diagonally upward arrow is called a *reactive arrow* and points towards a previous utterance; (b) a vertically or diagonally downward arrow is called a *proactive arrow* and it points towards the person from whom a reaction is expected. Add to this a distinction between arrows that are on task or mathematical in nature (solid) and off-task or non-mathematical in nature (dashed). Sfard and Kieran

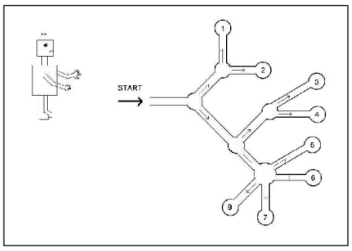
(2001) developed this scheme to code conversations between two people. Ryve (2006) extended this scheme to account for more than two people by assuming that a proactive utterance is meant to address each of the other participants. Table 1 in our example is read as follows: M in 1 makes a proactive statement to L and D, D reacts in 2, and so on. In our earlier research (Liljedahl & Andrà, in press), we found it was necessary to consider the flow of conversation, but also who the participants are looking at. As such, we introduce new set of arrows, meant to represent where someone is gazing during each utterance. We use red arrows to represent the speaker and blue arrows to represent non-speakers. In Table 1, for example, M looks at the paper in 1, D looks at the paper in 2.

METHODOLOGY

At the core of the research presented here is a 45 second video clip of a group of four students working on a mathematics problem.

The problem was inspired by the work of Iversen and Nilsson (2005), who used a similar task to see how students make sense of random phenomena. The problem is:

> A robot walks along a corridor, it turns right with probability 1/3 and it turns left with probability 2/3. The map shows the labyrinth where the robot has to move. Compute the probability for the robot to be in each of the rooms.



Iversen and Nilsson (2005) asked the students to say which is the room with the highest probability. Our problem was crafted so as to use the representation provided by the task in order to introduce the concepts and the algorithms related to the tree diagram: why should one multiply subsequent branches? Why and when should one add? The task was presented like a game, and the students seemed willing to work on it as such.

The task was used as part of a series of four lessons on probability in a grade-9 (14-15 year olds) class in Bologna, Italy. The task formed a significant portion of the second lesson. Four students, Luca (L), Fabio (F), Davide (D), and Marco (M) were selected to be videotaped while they worked on the task as a group. They worked on the task in a separate room and were filmed by a grade-12 student from the same school. The entire session lasted 50 minutes. The first 5 minutes of this video were transcribed. From this, the first 45 seconds were selected to constitute the data for the research being presented here. This subset of the data was selected because it exemplified some very interesting and turbulent undercurrents of group interactions. We also introduce a new interlocutor to the interaction – the paper (P) with the problem on it. This paper holds the gaze of the participants at different times of the conversation (we do not code blue arrows when the students are looking at P). Unlike the arrows representing utterances all of the gaze arrows are diagonally downward to represent the passage of time.

READING DATA

Table 1 presents the transcript and interactive flowchart with the blue-red gaze arrows. Figure 1 shows some snapshots from the video overlaid with some gaze arrows (for ease of reading, each student has assigned a color: yellow for L, blue for F, green for D and red for M); the arrows help the reader to focus on gazes and do not follow the blue-red coding used in Table 1. We first present the data codified according to our methodological framework, then we analyze the codified data.

			L	D	M	P
00:00	M:	To the left two thirds, to the right one third.	o	o	o	o
00:01	D:	Yes, I don't remember. (*speaks over M*)	o	o	o	o
00:03	M:	Then it goes two thirds, two thirds.	o	o	o	o
00:06	M	Can you give me a pen, please?	o	o	o	o
00:07	L:	No, let's do the first case, which is the one where it goes always …	o	o	o	o
00:10	M:	… left. You have two thirds here …	o	o	o	o
00:11	L:	That is the most probable one. (*speaks over M*)	o	o	o	o
00:13	M:	…and here is one third.	o	o	o	o
00:15	L:	Should you erase?	o	o	o	o
00:16	M:	Yes, bravo!	o	o	o	o
00:17	D:	I'm cute!	o	o	o	o
00:19	M:	Two thirds and here one third, hence these two thirds…	o	o	o	o
00:21	F:	... they g ... they go ….	o	o	o	o
00:22	M:	Two thirds of two thirds.	o	o	o	o
00:25	D:	But … but what are you saying? Then no …	o	o	o	o
00:27	M:	Of these two thirds you should do …	o	o	o	o
00:28	D:	We have … but what do we have to compute? (*speaks over M*)	o	o	o	o
00:30	L&M:	The probability that the robot will arrive in each one …	o	o	o	o
00:34	M:	of these rooms.	o	o	o	o
00:35	D:	In the meantime, let's see …	o	o	o	o
00:36	L:	Why don't we first compute how many probabilities there are in all?	o	o	o	o
00:37	M:	To me this is the room with the highest probability.	o	o	o	o
	D:	Why?	o	o	o	o
00:42	L:	There are 8 in all.	o	o	o	o
	M:	Because here there are the highest number of probabilities, and then …	o	o	o	o
00:45	D:	Of course	o	o	o	o
	M:	… the probability is higher.	o	o	o	o

Table 1: Interactive Flowchart with Gaze Arrows

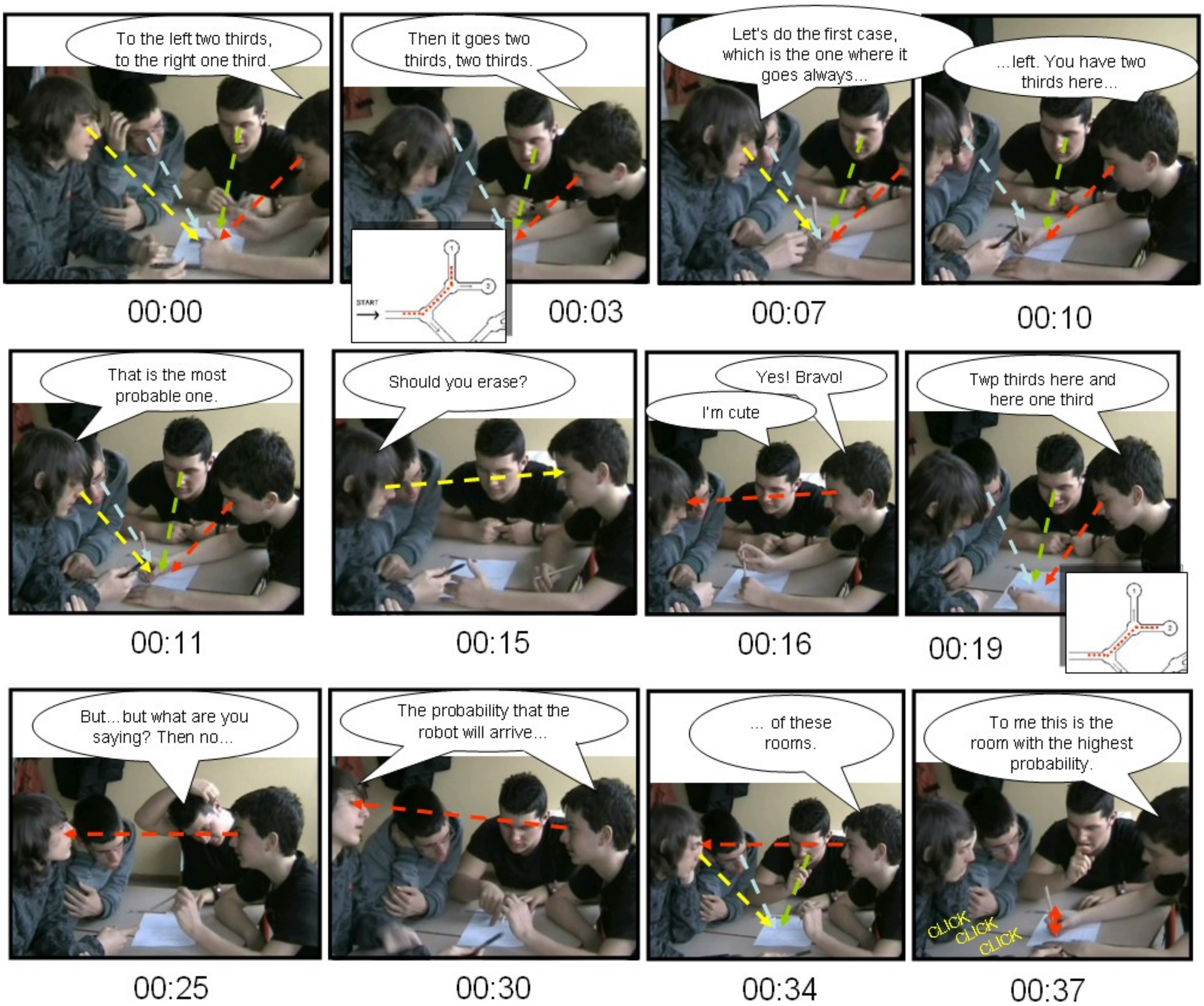


Figure 1: Some snapshots from the video overlaid with gaze arrows.

Data codified with our methodology

If we look at the verbal transcript, we see that the students are making sense of the task. Both L (00:11) and M (00:37) come to notice that the highest probability is related to the first room, an observation (coming from the students) which is in line with the original formulation of the task by Iversen and Nilsson (2005).

The interactive flowchart shows that M is contributing the most proactive statements (n=7) as opposed to L (n=3) or D (n=0). M and D responds to the most number of proactive statements (each n=5) as compared to L (n=1 not counting the self-talk as a reaction). Finally, there is a marked difference in the number of proactive statements that each person makes that are reacted to – M (n=6), D (n=3), and L (n=1, not counting the self-talk).

The gaze arrows show that D never looks at L. D doesn't look at anyone – he only looks at the paper when he is speaking. Figure 1 tells us that the students spend a lot of time looking at the paper, indeed. M, on the other hand, spends more time looking at L (n=6 in Table 1) than at the paper (n=5). At 00:25 D is asking a question while gazing at the paper. But M is not looking at D – he is looking at L. Then, while M responds to D's question at 00:27 he continues to look at L. This happens again at 00:34. At the same time L only looks at M three times. Once at 00:15, then again at 00:25 while D is asking a question, and finally 00:36 while M is looking at the paper.

ANALYSIS

Gaze arrows in Table 1 and Figure 1 tell us that, as much as M is attending to L, L is ignoring, maybe even avoiding, M. Why is M so intent on L and why is L ignoring M? We can see something interesting happening at 00:25. While D is asking the question, L and M are looking at each other. But these are not looks of equal intensity. In the video M is clearly more intense in his gaze upon L, who, after a while, glances away from M (see also Figure 1). From that moment on M continues to be very intensely focused on L. L seems to sense this and diverts his gaze from M, only looking back at him while M is looking at the paper (00:36). Clearly there is an affective aspect to the interaction between L and M. There are emotions, efficacy, will, and motivation in how L and M are interacting with each other.

True, all the students express their will to solve the task: D's questions aimed at letting him follow M's reasoning, his posture, his repeated and attentive gazes at the paper speak to D's will to be part, to contribute to the solution. On the side of both M and D there are many attempts to make their interlocutors act, think or feel (Sfard, 2001). M addresses mostly L, D prompts M. Power relationships are established: *power to do*. We see that an 'I can' and an 'I sense' intervene in this groupwork activity: M's and L's ones, respectively. M is working with fractions, he is interested in the procedure. We see that an 'I can' ('I can deal with this kind of computations', 'I can do this kind of math') emerge in his speech, in his interactions with his classmates. L, instead, seems more interested in understanding the overall sense of the activity ("Why don't we first compute how many probabilities are there in all?" 00.36). We rather see an 'I sense' in L's words. We have already commented that both L (00:11) and M (00:37) come to notice that the highest probability is related to the first room, but seemingly from different standpoints: L makes his conclusion based on the fact that room 1 is arrived at by always going left, which has a higher probability than right. We can say that L has an illumination, a rapid coming to mind of the features of the room with the highest probability, coming out of the blue, few seconds after the beginning of the activity. M, on the other hand, arrives at the same conclusion much later, by means of computations. Only after considering fractions can he say that room 1 has the highest probability.

There is a tension between L and M, between conceptual 'I sense' and operational 'I can'. Moreover, we see that each of these stances prevents each student from seeing the other's point of view. 'I can' might be inclusive: in our example, M is trying to pull L in. On the other hand illumination ('I sense') is rather individual and private, it does not need to pull others into it: after the moment of illumination, in fact, there is a distinct phase of validation—aimed at put such an 'I sense' into sharable, communicable, terms (see Liljedahl, 2012). Communication takes place in order to stimulate a reaction: L's illumination at 00:11, in fact, takes the form of a rather self-thought, and it is not reacted. L's illumination is as sudden as private.

M's intensive gazes on L speak to M's 'I can': he can go on with his reasoning if L is with him. L's avoiding, expressed by (absence of) gazes to M, tells us that L is avoiding this kind of 'I can': L 'cannot' use fractions, he prefers to reason at another level, more theoretical. In Figure 1, at 00:37, M taps with his pencil on the paper, pointing at room 1. M is sharing his 'I can', his claim about the room with the highest probability. L is reacting to M, neither verbally nor with gazes, but with his own pencil, opening and closing it repeatedly (CLICK CLICK CLICK in Figure 1). Interaction is taking place at another level: M is expressing his 'I can' while L is again expressing his avoidance of fractions, his 'I can't use fractions'. At the same time, we see the will to participate, to solve the task, expressed by all the students—in different manners.

DISCUSSION AND CONCLUSION

Stemming from findings in interactionist research (Sfard, 2001), we have explored how affective moves give rise to and determine groupwork activity. Affective moves are meant as meta-discursive rules that shape actions, motivation, and interactions of students, thus directing learning (see also Roth and Radford, 2011). Participation in a groupwork activity is social, but it is also mathematical: we can distinguish the social and the mathematical in our analysis, but we cannot separate them. Many moves of the activity we have analyzed are both social and mathematical in nature.

According to our framework, we can also say that even L's 'I sense' originates from an 'I can'. In other words, we can see that it is from L's 'I can see a structure' that the illumination about room 1 at 00:11 starts, and it is from M's 'I can use fractions', 'I am good with fractions', that all his proactive statements arise. L's 'I can', more conceptual in nature than M's one, is expressed by an 'I sense' at 00:11 ("That is the most probable one"). The initial 'I sense' at 00:11 mirrors another 'I can', more operational, at 00:36 ("why don't we count how many probabilities are there in all?"). M also expresses an 'I sense', which is rather procedural and it is linked to the fractions involved: M's 'I can' is thus operational. Following Merleau-Ponty (2002), we can say that intuition is first an 'I can', it is originated by will and power to do. Intuition is socially communicated, expressed, as an 'I sense'. In social interactions, sometimes there emerge mostly the 'I can' (which is also more involving, as we have argued), other times the 'I sense' is predominant.

'I can' is conveyed by gazes in our methodological framework: M, in fact, expressed his 'I can' by looking intensively to L, and L's avoidance of fractions is mirrored by his avoidance of glancing at M. Also D's absence of gazes to M and L speaks to a consonant absence of 'I can': D is not good in math, while M and L are (we know this from the teacher). Looking at the paper expresses D's need to adhere to the task. His prompts to M express his need to go slow.

The 'I can', might become shared with others when the nature of this 'I can' is involving. For example, when it entails actions (operations with fractions, in our example). Illuminations of different nature need a subsequent moment to become sharable. Our findings also confirm the unavoidably central role of emotion and motivation in learning processes—especially in interactionist researches.

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COLLECTIVE PROBLEM POSING AS AN EMERGENT PHENOMENON IN MIDDLE SCHOOL MATHEMATICS GROUP DISCOURSE

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This naturalistic case study investigates the problem posing patterns that emerge as four small groups of 12 year old students in Western Canada work collectively on a structured mathematics task. A method of data analysis is introduced that blurs the data to create transcript "tapestries" providing visual evidence of collective patterns of posed problems that emerge over time. Results in progress suggest that groups vary widely in terms of the problems posed, and in terms of the patterns in which these problems emerge in their discourse. The reposing of problems helps to structure each group's discussion, with the role that each problem plays in the conversation evolving as it reemerges in the discourse.

INTRODUCTION

Problem posing has been defined as "the creation of questions in a mathematical context and… the reformulation, for solution, of ill structured existing problems" (Pirie, 2002). Working from this definition, one might argue that there are two kinds of problem posing, depending on the purpose of the problem being posed (Silver, 1994), and where it occurs in relation to the problem solving process. In the first half of the definition, a new problem is generated from a situation, a problem, or an experience. In the second half of the definition – the "How can I (re)formulate this problem so that it can be solved?" type – a related problem is generated in response to the original problem, as a way of making that original problem more accessible. This study focuses on this second kind of problem posing, describing the behavior of small groups in a mathematics classroom who pose their own problems in the process of solving an assigned problem task. My research question is: What problem posing patterns emerge as small groups of students work collectively on a mathematics task?

THEORETICAL FRAMEWORK

The current National Council of Teachers of Mathematics' Standards document (2000) notes that problem posing is an important component of problem solving, recognizing it as an indication of a "mathematical disposition." Students can be supported as they move from a novice level to an expert level through various forms of instructor intervention ranging from introductory activities to specific problem posing strategies (Bonotto, 2013; Singer, 2009; Singer & Mascovici, 2008) to participating in problem posing (and solving) programs (Brown & Walter, 2005; Crespo, 2003; Crespo & Sinclair, 2008; English, 1997, 1998; Leung, 1993; Pirie, 2002).

Many studies of problem posing rely on their subjects' written work, a static product, as the focus of analysis. While this has the advantage of allowing researchers the ability to draw on a large pool of subjects, it also has the effect of (appropriately enough) triggering yet more questions about the research itself. In an excellent discussion of the results of one such study (Silver & Cai, 1996), the researchers wondered if middle school students only recorded problems they knew they could solve; perhaps they were able to generate more complex questions, but hesitated to write them down because they were not able to solve them. The researchers also questioned the trend of simpler questions being posed before the more complex ones were. Perhaps the subjects originally had the more complex question in mind first but decided to record the simpler questions at the beginning of their written responses. All of this points to problem posing being difficult, and perhaps simply inappropriate, to capture with an end product consisting of a written list of problems.

Some argue that group work has the potential to provide a safe structure for building problem posing competence (Kilpatrick, 1987; Silver & Marshall, 1989), and offers the opportunity for students to work together less competitively and more productively (Brown & Walter, 2005). Yet, despite these and similar recommendations (English, 1997; Lester, 1994; Silver, 1994; Silver, Mamona-Downs, Leung, & Kenney, 1996), there is little in the literature about how problem posing works on a collective level.

Little documentation exists about the group itself as a learner, how its understanding unfolds (Martin, Towers, & Pirie, 2006), and how it thinks. Although in casual conversation, a teacher might refer to what a certain group thinks or, for example, describe the personality of the class in period three (Bowers & Nickerson, 2001), it can be difficult for researchers to conceptualize the group as a unit of analysis, even a small group. Thus, studies of small groups have often tended to focus on how working within the group affects the learning of the individuals within the group rather than on the group itself (Stahl, 2006). The concept of group learning is "a difficult, counter-intuitive way of thinking for many people" (Stahl, 2006, p. 16) due to the strong association of cognition with an individual psychological process.

There is a benefit for the researcher who studies groups: the group's discourse may be considered to represent its thinking (Stahl, 2006). However, the discourse cannot "be analyzed by solely considering a sequence of statements that are made''(Yackel, 2002, p. 424). One might even argue that the individual pathways of growth of understanding within the collaboration do not exist at all (Martin et al., 2006). An utterance is linked to the past in that it is a response to another utterance, or utterances. An utterance is also a response to what has been, or what is currently, happening and the utterance is connected to the future, in that it is formed in anticipation of an impending utterance. The "conversation" of a group "is crisscrossed by other places and temporalities, by absent third parties, who may express their voice through the participants' discourse" (Grossen, 2009, p. 266) and also by the uptake and reuptake of individual threads of ideas. One might envision the utterance as a part of a tapestry that comes from the past and stretches into the future, an idea I will connect to in my methodology.

METHODOLOGY

The research took place at a middle school (ages 10 – 13 years) in a large suburban school district in British Columbia. Sixteen students from each of two classes of 30 12 year old students (i.e. just over half) participated in the study for a total of 32 students. The groups were composed of students who were all working at grade level but who had mixed levels of ability in mathematics. The study occurred in the spring of the school year, with session tapings taking place roughly every two weeks depending on the school schedule, for a total of five sessions for each class, with each session lasting approximately 40 minutes. As I was using a grounded theory approach (Glaser & Strauss, 1967), I selected groups "for their ability to contribute to the developing/emergent theory" (Miles & Huberman, 1994, p. p. 28) – namely those who were working collectively on the tasks. Participating groups were videotaped by stationary cameras and also audiotaped. I took field notes throughout the sessions from a location at the back of the classroom, and compared these notes to the video and audio recordings to clarify events captured in the tapings. Other data sources included the task sheets where group members recorded their work and solutions, and the class whiteboard where some groups chose to write their ideas while presenting their solutions to the rest of the class. I refer to the groups through the acronyms JJKK, REGL, NIJM and DATM.

The task that is the focus of this case study reads as follows:

> **The Bill Nye Fan Club Party**
>
> The Bill Nye Fan Club is having a year-end party, which features wearing lab coats and safety glasses, watching videos and singing loudly, and making things explode. As well, members of the club bring presents to give to the other members of the club. Every club member brings the same number of gifts to the party. If the presents are opened in 5 minute intervals, starting at 1:00 pm, the last gift will be opened starting at 5:35 pm. How many club members are there?

DATA ANALYSIS

As this study involves elaborating upon and building theory about problem posing as a process, I analyzed the data using a constant comparison method (Glaser & Strauss, 1967). The process of determining whether or not a group had posed a problem was necessarily a subjective one. Rather than looking at the actual uttered problem, I was looking more at the conversational fabric around the utterance, both before the utterance occurred (what did the intent of the utterance seem to be?) and afterwards (namely, how did the group respond to the utterance?), indications of surfacing differences that the group appeared to be exploring.

The metaphor that I use to document the patterns of collective problem posing, and reduce the transcript to its "visual essence," is that of the "tapestry." Composed of strands of fabric and color, a tapestry reveals different faces depending on its physical distance from the observer. From afar, which would be the equivalent of summarizing a group conversation and then considering it from both a temporal and contextual distance, the tapestry shows a panoramic scene – a whole composed of a number of parts. Closer, the landscape of the tapestry might still be evident, but now the individual strands are more visible. Move closer still, and now the individual strands are the focus and the overall scene is no longer clear – much in the same way in which it may be easy to follow the individual turns of a conversation but difficult to summarize the gist of the discussion as a whole while it is taking place. At this level, an overall pattern is invisible, but individual contributions and ideas stand out. These strands of individual utterances are ones that weave together into a tapestry as the conversation proceeds.

The production of the tapestry involved a data blurring process, which started with the transcript itself. After multiple iterations of reading and comparing transcripts from the four groups' sessions, I identified the posed problem categories I color coded the utterances in the transcripts according to the problem posing category they best fit. The color-coded transcripts were then shrunk in size, using computer screenshots, to the point where the words of the transcript were no longer visible and the lines of color coding appeared as a visual pattern. The resulting tapestry provides an overall image of the problems posed during the course of the group's session.

RESULTS AND DISCUSSION

At first glance, the structured nature of the Bill Nye task would not appear to allow for many creative possibilities for mathematics students. To solve it, one must understand what the range of time is for opening the gifts, determine the number of time intervals that exist within that time frame, and then find the pair of factors of the number such that one factor is one greater than the other (i.e. 8 and 7). Yet, in working through this apparently straightforward task, these four groups take very different paths to eventually arrive at the same correct solution.

Tapestries

A striking aspect of group work that a tapestry helps to illustrate is how posed problems weave in and out of conversations. A color may appear briefly early in a session – for instance, medium blue in NIJM ("What are the factors of x?") – and not appear again until over halfway through when it begins to occur quite frequently. A problem may be posed and seemingly ignored, only to be reposed later in the discussion, while other problems that seem to have been discussed and resolved may also reappear for more discussion. This suggests that the mention of a posed problem early on in a session may help to seed a later discussion. It also seems to highlight the idea of all ideas being part of the tapestry, visible or not – no utterance truly disappears.

The width of the color bands indicates the approximate length of time a problem is being discussed, and how many connections are made with other posed problems. For example, the chunky[1] pattern displayed in the first third of JJKK's tapestry (Figure 1) is quite distinctive from the tapestries of the other three groups. The chunkiness

[1]Thick bands of color in the tapestry

reflects how a problem is posed, discussed at some length until some kind of agreement is reached, and then disappears, presumably either having been resolved or dropped completely. This pattern also reflects how JJKK poses and reposes far fewer problems than the other groups do (Figure 2).

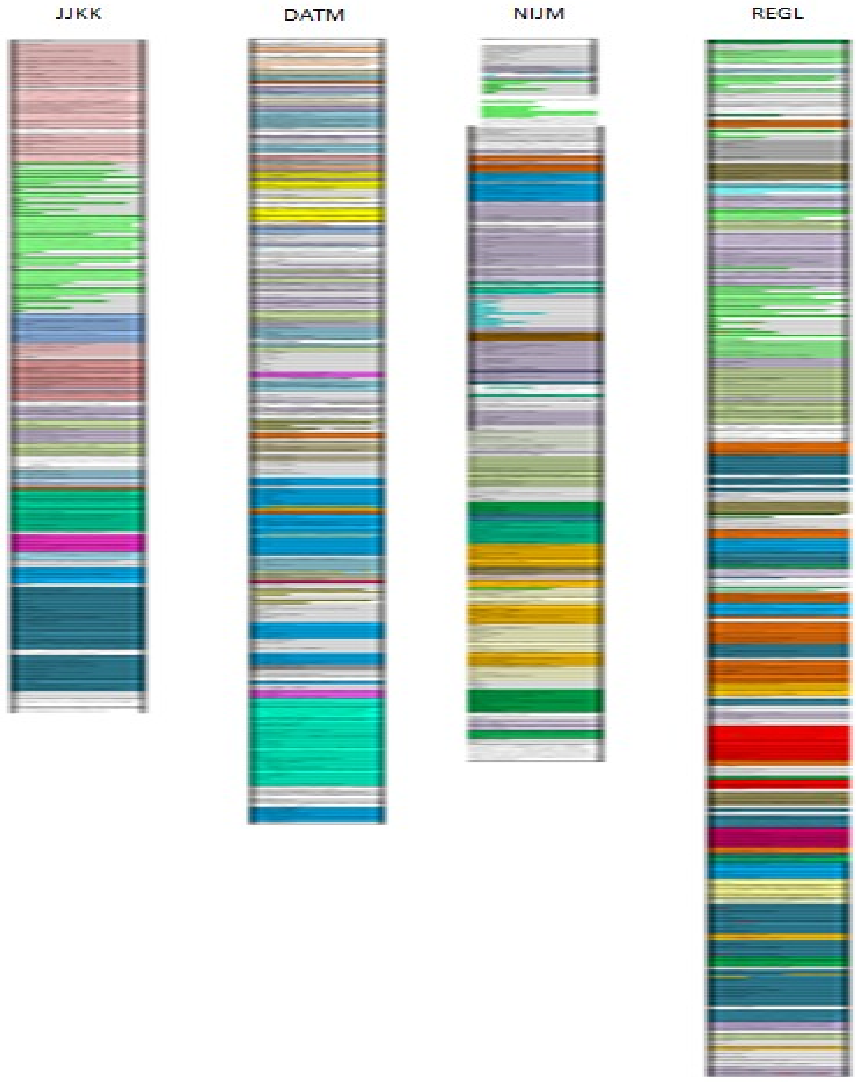

Figure 1: Tapestries

Group	# of different problems posed	Total # of problems posed and reposed
JJKK	13	23
DATM	16	61
NIJM	17	45
REGL	16	66

Figure 2: Comparison of # of problems posed and reposed.

In comparison, “thready”[2] patterns found in the tapestries of DATM, NIJM and REGL tend to show that a number of different problems are being posed and put “on the table,” so to speak. For all three groups there tends to be a thready pattern of different colors at the beginning of their tapestries when they are first considering the task. Finally, the thready pattern also tends to occur late in the sessions when the three groups have come up with a tentative answer, when earlier problems are reposed as a way of checking their thinking.

While lavender and a few other colors appear in all of the tapestries, there are many other colors which do not. For instance, there is a shade of teal (“Is it a square root?”) that only appears in NIJM and JJKK. And still other colors are unique to certain groups, like the light green (“How can we use the 24 hour clock?”) that occurs at the end of DATM’s tapestry. It might be expected that unique problems might be due to experiences/knowledge that is unique to the group, but this is not necessarily the case. For instance, the topic of square roots was one that the groups were all studying in their regular mathematics class, yet only two of the four groups reference it.

Characteristics of problem posing

A notable trend across the sessions is how the role a posed problem plays in a discussion changes each time it is posed even if, on the surface, the wording of the problem appears to be much the same. On the surface, problems like “Do we use time and divide by 5?” which is featured predominantly in at least three of the group’s discussions, may seem to be a clarification problem. For example, consider it functions during NIJM’s session. Posed and reposed eleven times, this problem functions in order to: propose a method of entry into the task; discuss what method would be easiest; discuss how it might eventually lead to solving the entire task; estimate/predict possible answers; narrate ongoing calculations; check possible answers. Most of the other posed problems in the study also show evidence of their roles evolving as the group discussion develops. The only time that a problem does not appear to evolve is when a group does not repose it.

The number of different individual problems posed (Figure 2) is fairly consistent between the groups but there is a large range in the total number of problems posed. One might posit that the difference is due to each group’s “personality.” For example, REGL, who tends to explore concepts more deeply and connect ideas more frequently than the other groups, poses more problems than JJKK who tends to argue about one problem at a time until a consensus appears to be reached. While some problem posing studies in the literature have focused on the number of problems posed, or the quality of problems posed, my findings suggest that the pattern in which problems are both posed and reposed may ultimately tell us more about students’ mathematical behavior and understanding.

[2] Slim bands of color that alternate with slim bands of other colors.

. CONCLUSIONS

This study offers a description of problem posing as collective behavior at the level of the group as an agent. It also provides evidence of the groups' ability to problem pose collectively without having been directed to do so, and without having received any formal instructions about how to do so. It is noteworthy that problems do not emerge in the same order for each of the groups. The varied ways in which groups in this study approach the Bill Nye task may suggest that educators need to be careful of presenting problem solving heuristics as lock-step procedures to be followed in a specific order. Even though the four groups have some common experiences with which to work, the fact that certain groups do not necessarily draw on these experiences, or if they do, do not do so in the same way as other groups, suggests that the process of problem posing is more than simply sitting down and "working with what you have." Perhaps the strength of problem posing is not the generation of a list of problems at the end of the task, but the emerging patterns of problems as the discussion continues and how these problems in turn structure pathways to a solution.

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EMBODIMENT IN TEACHING AND LEARNING EARLY COUNTING: GROUNDING METAPHORS

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This paper contributes to the theory and evidence that mathematical cognition is embodied. Drawing on the practices of primary teachers in South Africa engaged in a longitudinal research and development project – Wits Maths Connect–Primary – we report on aspects of lessons aimed at developing number sense through whole-class teacher-learner interaction. Two episodes are analysed from an embodied cognition perspective. The episodes focus on helping Grade 1 (6-year-olds) learners become fluent in counting forward and back or ordering numbers. Analysis reveals different embodied metaphors underlie the teachers' actions, the nature of which are likely to lead to different learning opportunities. We conclude that our analysis supports a theory of embodied cognition, and demonstrates its usefulness as an analytical tool.

INTRODUCTION

Authors are increasingly arguing that cognitive understandings in general and mathematics in particular are embodied – rooted in perceptual and physical interactions between the body and the world (Barsalou, 2008, Alibali & Nathan, 2012). In this paper we examine two assertions that flow from this theoretical position. First, that teachers, as mathematical knowers, will themselves have embodied understandings and will, often intuitively, draw on these in their teaching. Second, that any such use of embodied metaphors must take the learners' embodiments into consideration if the metaphors are to be supportive of learning.

RESEARCH CONTEXT

National standardized and international comparative test results consistently present a bleak view of mathematical performance in South Africa. For example, the 2012 Annual National Assessments (ANAs) results indicate 27% as the national mean mark at Grade 6 (predominantly 11- to 12-year-olds) (Department of Basic Education (DoBE), 2013-a), down three percentage points from 2011 and well below the target of 60%. In this context, a longitudinal research and development project – Wits Maths Connect–Primary (WMC–P) – is developing and investigating interventions to improve the teaching and learning of mathematics in ten government primary schools. One particular intervention is the Lesson Starters Project (LSP) focusing on improving number teaching in the Foundation Phase so that the students' develop better number sense.

The focus of the LSP is linked to the national South African Curriculum and Assessment Policy Statement (Department of Basic Education, 2011) and the district

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Gauteng Primary Language and Mathematics Strategy (http://gplms.co.za/) that together prescribe content, sequencing and teaching timeframes. Our partner schools are under pressure to follow these policy drivers so we are focusing on the policy mandated ‘mental mathematics’ within ‘whole class activity’ lesson sections.

The CAPS documents provide brief guidance on approaches and tasks expected to figure within this mental mathematics teaching, for example:

> Mental mathematics will include brisk mental starters such as “the number after/before 8 is; 2 more/less than 8 is; 4+2; 5+2, 6+2” etc. (DBE, 2011, p. 11)

Despite such exemplification a diagnostic report on Grade 3 learners’ performance on the 2012 ANAs identified mental skills as a particular weakness that included poor understanding of ‘number concept as demonstrated in being able to count forwards and backwards’ (DBE, 2013-b, p. 6).

Collecting baseline data in 2011, the project team observed and videotaped a numeracy lesson from each of the Grade 2 classes in the ten project schools, to gain insights about the nature of teaching and learning, and the classroom contexts. Analysing this data revealed teachers’ random selection and sequencing of tasks led to a lack of coherence in and across tasks, and in task enactment. The resulting weak coherence within teaching exhibited ‘extreme localization’ and ‘ahistoricity’ (Venkat & Naidoo 2012). Such practices, it is argued, severely impair possibilities for learners to understand number as a connected network of ideas.

Two years later, video data from Grade 1 classes (in the same ten schools) show improvements in coherence and pacing, so our analysis is now examining nuances in how teachers bring coherence to the lesson starters. In doing so, we find embodied cognition a helpful theoretical tool.

THEORETICAL BACKGROUND

As yet there is no unified theory of embodied cognition. Wilson (2002) suggests that there are at least six different views of embodied cognition, one of which is that ‘off-line’ cognition is body based – this is broadly the view taken here. Dehaene (1999) argues that there are spatial aspects to developing number understanding, such as the representation of integers spaced along a line, while Lakoff and Núñez (2000) argue for an embodied view of mathematical understanding, suggesting that developing understanding that goes beyond subitizing small quantities draws on learners’ metaphorising capacities – making sense of numbers (as concepts) through various bodily experiences, such as associating number with distance, movement and location, as well through handling collections of objects.

Such metaphorising capacity is linked, Lakoff and Núñez argue, to two types of metaphors: grounding and linking. ‘Basic’ grounding metaphors ‘allow you to project from everyday experiences (like putting things into piles) onto abstract concepts (like addition)’ (p. 53) while linking metaphors lead to ‘sophisticated’ or ‘abstract' ideas and, in contrast to grounding metaphors 'require a significant amount of explicit instruction' (p. 53). Other theorists support this notion of *grounding* as setting up a mapping between the familiar and concrete and the abstract (Nathan, 2008).

Our videos of the more recent lessons reveal teachers intuitively making use of embodied metaphors. Below we analyse two instances to explore how such metaphors play out in practice and whether this likely to help learners better understand number.

DATA SOURCES

Our data are drawn from the 2013 videotaped classroom lessons of Grade 1 teachers who had also been filmed teaching Grade 2 in 2011 (n = 7). The two lessons focused on here represent the broader dataset in having extended instances of whole class talk around typical tasks. But they also provide 'telling cases' (Sheridan, Street, & Bloome 2000) as both teachers drew on different bodily metaphors.

The first teacher, M is an experienced teacher in a disadvantaged school, whose medium of instruction is Tsonga. The second teacher, R is another experienced teacher at the same school but whose medium of instruction is Tshivenda.

DATA ANALYSIS

Analysis comprised 4 phases: (a) creating a transcript, (b) fleshing out the evidence (c) interpreting (d) producing a 'thick description' and analysis of selected episodes.

Creating a transcript

Bilingual speakers transcribed the video recording, following instruction to capture all the teacher's talk within the lesson and any objects/representations referred to.

Fleshing out the evidence

A narrative account of the unfolding of the lesson was created, using the video to include detail not captured in the transcribing. This account was then parsed into episodes, usually identified by the introduction of a new task, but sometimes marked by shifts of attention within tasks. To improve accuracy and detail, the project team viewed the video recordings several times to clarify the interaction between the enactment of the episode (teacher talk and actions and pupil responses), the choice and sequencing of examples, and the use artifacts to support the teaching.

Interpreting

The team examined and discussed each episode to reach consensus on the likely (a) teaching intentions, and (b) learning opportunities. These interpretations were warranted through reference to the data with the teaching intentions imputed through the enactment of the episode, and not necessarily as explicitly articulated by the teacher. Similarly we interpreted learning opportunities through how the episode played out and the likely consequences for learning.

Producing a 'thick description' and analysis of selected episodes

Many of the episodes identified were, empirically and theoretically, of limited interest as they focused on learners practicing what they already knew. Lessons were then examined for 'critical incidents' – particular episodes where it was clear that learners were not already confident in content being addressed. The team discussed, analysed and wrote up these incidents. We report on two such critical incidents here.

LESSON STARTER CRITICAL INCIDENTS

Incident 1: Teacher M – Forward and backward counting between 1 and 20

This episode was towards the beginning of the starter activity. The teacher settled the learners down and then asked them to count forward from 1 to 20 as a whole class. Some learners were seen using their fingers putting out 1, 2, 3, … while counting. M asked learners to count backwards from 20 to 1. Several learners were heard to say 'twenty, ninety, eighty, seventy,...' and many learners were observed not saying anything. The teacher stopped the class count, saying, 'when counting backward you should say twenty, nineteen, eighteen as you are reversing'. She demonstrated this by taking three steps backwards and gesturing in the direction of her movement by pointing both thumbs back over her shoulders.

> M: If we are counting forward we say one two three up to twenty. In the backward counting we say twenty, nineteen, eighteen. Now let's count backwards again.

Learners started counting: again many could be heard saying 'twenty, ninety, eighty, seventy'. M stopped the counting and shook her head.

> M: We are counting within the range of twenty. You should say, twenty, nineteen not ninety.

M moved from the front of the class to take up position at the back of the room. Stepping towards the front, M counted her steps 'one, two, three, ...' At twenty she stopped counting and stepping, and pointed forward (to the front the class, the direction she had been walking in) with both hands. She then made a backward gesture by pointing her thumbs over her shoulders, saying, 'We are reversing, reversing'. Without turning round, M retraced her steps from the front of the class to the rear, simultaneously counting backwards from twenty to one. As she did this she emphatically enunciated 'nineteen, eighteen' and so on. M asked learners to again count forward from one to twenty. She coordinated this with pointing on her fingers 'one' (thumb), two (forefinger) and so forth as everyone counted, clapping on 'ten' and 'twenty'. Learners were seen to follow the teacher and use their fingers similarly. M asked learners to count backward from 20 to 1. Most learners were observed to count correctly: twenty, nineteen, eighteen down to one.

> M: That is good. So next time don't say ninety. It's nineteen, eighteen.

Analysis

The task enactment showed that learners were fluent in the forward counting sequence but struggled with the backward number word sequence, confusing, for example, ninety with nineteen. M stopping the backward counting and commenting on the errors reveals her awareness of this difficulty. Her actions of stepping forward and backward, language of 'reversal' and gestures of forwards and backwards coheres with the task and teaching intent.

Establishing a sense of numbers as points on line draws on what Lakoff and Núñez (op. cit.) refer to as the 'source-path-goal' schema based in metaphors of a moving trajectory, from a source to a goal. This schema has an internal spatial logic with implications such as having followed a trajectory to a goal, then all prior places on that trajectory must have been passed through. Learning to count backwards could therefore be metaphorically linked to retracing one's path along the trajectory, revisiting all the previous locations in reverse order.

M's actions explicitly embody this metaphor of a moving trajectory. By physically moving to the back of the room, her stepping forward and back and accompanying gestures all were coordinated with the perspectives of the learners: forward in the direction to which the learners faced, backwards being in the same direction over everyone's shoulders. Stepping forward M physically laid out a trajectory, orally indexing locations along her path through counting her steps out loud. Arriving at, and still facing the front of the room, she gestured to indicate the forward direction of the trajectory and then the reverse of this by pointing back over her shoulders and stressing 'we are reversing'. The use of 'we' can be taken as an invitation to the learners to imagine themselves moving, even though only the teacher was actually moving. Without turning round, she retraced her steps, orally indexing these with the backward counting sequence. Thus M clearly enacted a 'source-path-goal' and reverse trajectory metaphor in ways that fitted with the learner's embodied positions and how they would experience the trajectory were they to travel it themselves. Although Lakoff and Núñez take the trajectory metaphor to be a linking metaphor, the teacher's treatment here suggests to us that it can be used as a grounding metaphor as it is set up and used with little explicit explanation and learners appeared to relate to it.

There is also evidence of M's awareness that the counting back errors might arise from the difficulty in hearing the distinction between 'nineteen' and 'ninety' and confusion with the other counting sequence of 'ninety, eighty, seventy, ...' that is frequently practiced. Here again, the teacher addressed this in an embodied way through over-emphasising the enunciation of the counting words. While elsewhere in our data we have found teachers using enunciation to address isolated difficulties, the teacher here incorporates her handling of enunciating words within a coherent and connected moving trajectory metaphor that emphasizes a traversing back through the same path that has just been travelled in the forward direction. Thus 'ninety' is not just an error of enunciation; it is treated as a spatial error in that this indexing of position did not feature in the forward direction.

Incident 2: Teacher R – Numbers ‘behind’ or ‘in front of’ in the range 1 to 10

Up to this point in the lesson R had taken the class through counting forward and backwards to ten and combining two numbers with a sum less than ten (using fingers as artifacts). R turned to a partially completed number line on the chalkboard (0, 4 and 6 labelled, ending at an unmarked 8) to work on finding the missing numbers.

R walked towards the board, which had taped to it a column of numeral flashcards from 0 to 10. Taking down the numeral ‘5’ R asked ‘what number comes behind this number?’ She spoke facing the class and simultaneously gestured by raising her right hand and pointing over her right shoulder towards the board. Some learners said ‘four’, others ‘six’. R restated ‘which number comes behind this number?’ More learners were heard to state ‘four’.

R: Isn’t the number four coming in front?

Now most learners said ‘six’. R took down the numeral ‘8’ flashcard and asked ‘which number comes behind this one?’ As she spoke she again accompanied this with her gesture of raising her hand towards and over her right shoulder. Some learners said ‘nine’, others ‘seven’.

R: Seven? Is this not a number that comes in front?

Learners said ‘nine’. R took numeral 3 down.

R: What number comes in front of this one?

Learners: ‘Three’, ‘two’, ‘four’

R: Four? That is the number that comes behind.

Immediately some learners called out ‘two’. R responded with ‘that is the number that comes in front’. However, at the same time, other learners were still heard to say ‘four’. The teacher did not respond and moved on to the next task.

Analysis

Within a moving trajectory metaphor, four could be considered to be behind five in the sense that having travelled past point four to point five, the former is left behind. The language of behind can also suggest a metaphor of following not leading – if numbers (represented here by numerals on flashcards) are likened to being ‘strung out’ along a line, then the numbers closer to the starting point are, in a sense, behind those coming later: such a metaphor could account for learners giving (from the teacher’s perspective) incorrect answers.

To describe four as ‘coming in front’ of five suggests a different embodiment – perhaps a staircase metaphor of ‘steps’ going up from one to ten, in order of height, the lowest step to the front (such a model can be created using one to ten number rods, although we have not seen it used by our teachers). The teacher’s talk and gestures suggests a positioning within the metaphor, like an ordered set of Russian nesting dolls – shortest in front and others lined up behind in height order. Four would thus be in front of five. If she (the teacher) were 'occupying' the position in height order, of, say, five, then six would be behind her, consistent with gesturing over her shoulder.

Whatever the case, the metaphor here has a more 'static' aspect than in the other episode – R's gesturing indexes a stationery positional metaphor rather than a momentary position on a trajectory. In contrast to the way that teacher M took up position in the room so that her perspective corresponded to that of the learners, R's orientation was such that what was 'in front' or 'behind' *her* occupied a different space relative to that of the learners.

R's response to learners' incorrect answers was to use a rhetorical question to point out that they were wrong (from her frame of reference) – 'Isn't the number 4 coming in front?' While some learners then produce the 'correct' response, any learning provoked is likely to based in association – 'behind' associated with producing the next number name in the counting sequence, rather that being explicitly connected to some grounding metaphor.

DISCUSSION

We make no suggestion here that either teacher was working deliberately with any metaphors, but in keeping with Lakoff and Nunes' theoretical position, the internal consistency of language, gestures and positions strongly suggests a metaphorical origin to their cognition.

Teacher M unambiguously modelled a moving trajectory, embodying a metaphor of numbers along a path and her actions, gestures and talk are coherent and consistent, together with alignment between her spatial perspective and that of the learners. Doing so maximised the chances of learners engaging in '*simulated action*' whereby witnessing actions and imaging actually doing the actions activates appropriate brain areas (Alibali & Nathan, 2012, original emphasis).

In teacher R's case the consistent treatment of 'coming in front' or 'behind' across the examples suggests a different metaphor. Putting things in order of height is a possible grounding metaphor here in that the actions of such ordering (without measuring) require little direct instruction. However, both the more implicit nature of this metaphor in R's episode and the lack of coordination between her position and that of the learners makes it likely that the learners would find difficultly in engaging with this as a grounding metaphor.

CONCLUSION

We commenced with two assertions. First, that teachers, as mathematical knowers, will themselves have an embodied understanding and thus will, intuitively, draw on this in their teaching. In both cases here we are argue that the teachers' talk and actions exhibit evidence of being grounded in bodily metaphors, lending support, albeit limited, for the claims to mathematical cognition being embodied.

Second was the assertion that for embodied metaphors to be helpful, working with them must take the learners' embodiments into consideration. We see here one teacher successfully doing this and another acting more directly from only her position and perspective and thus limiting the potential grounding for the learners.

The theoretical position of teaching and learning number as grounded in embodied metaphors is validated by such examples and, moreover, provides a useful framework for analysis. Given the increased coherence and consistency that we are seeing in lessons, this inferring of metaphors provides us with a useful next step in working with teachers to elaborate, broaden and extend metaphors within their pedagogic repertoire for teaching number sense.

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ANALOGIZING DURING MATHEMATICAL PROBLEM SOLVING – THEORETICAL AND EMPIRICAL CONSIDERATIONS

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The aim of the paper is to provide a process model to evaluate mathematical problem solving by analogy, in order to better determine at which point and under what conditions a learner is prompted to use analogies. The model is a theoretical construct. Qualitative results of an empirical study are used to underline and illustrate core aspects of the model.

INTRODUCTION

The ability to recognize and use analogies (Gick & Holyoak, 1983) is a key aspect of human cognition. If two situations are analogous, which means that there are the same relations between corresponding elements, knowledge transfer from the known situation (source) can help tackling the new situation (target). Thereby, analogical thinking can be used to assist us in understanding certain characteristics, relationships and mechanisms of unknown situations, or to construct plausible hypotheses. It can also play an important role in problem solving "when the solution to one problem suggests a solution to a similar one" (Holyoak & Thagard, 1989, p. 318).

Analogical reasoning is of particular importance in mathematics as the science of patterns, structures, and structure types: "Noticing higher order similarity relationships between such instances of structural similarity is at the core of complex mathematical thinking" (Richland et al., 2004, p. 38.). A closer look at the history of mathematics confirms that analogical reasoning has long played an important heuristic role in this field (Reed, 1985; Zimmermann, 2003).

Solving mathematical problems by analogy is a multi-step process involving higher-order cognitive skills, the first of which requires the identification of a source problem that can be retrieved from memory. A learner's ability to reason analogically is therefore very much dependent on their existing knowledge base (English, 2004). It also involves the mapping between the elements and relational structure of the source problem or known situation (source) and the new one (target). This requires the ability to change representations and therefore to abstract from concrete surface characteristics of the situations worked on (Novick, 1988). Finally, the modus operandi which is considered as appropriate has to transfer onto the new situation and therefore often adjust to its concrete requirements (Novick & Holyoak, 1991).

Although a number of studies have been conducted on using analogies during problem solving (e. g. on analogical transfer), not a great deal of research has been conducted in

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mathematics education examining analogizing from a *qualitative* perspective. Rarely have mathematical problem-solving processes been examined from the point at which analogical reasoning occurs in learners and the conditions that can either facilitate, hinder or prevent this process. This paper has been written with the intent to broaden discussion in this area.

The aim of the paper is to provide a process model to evaluate mathematical problem solving by analogy, in order to better determine at which point and under what conditions a learner is prompted to use analogies. The model is a theoretical construct. Qualitative results of an empirical study are used to underline and illustrate core aspects of the model.

PROBLEM SOLVING AS COGNITIVE MODELING

There is an extensive amount of research literature on mathematical problem solving and modeling, and more recently, on approaches that attempt to combine both in order to describe complex mathematical activities (e. g. Förster, 2000). This approach, derived and justified from the perspective of cognitive psychology and mathematics education, has been most recently taken also by Zawojeski and Lesh (2003). They argue that when students struggle with mathematical problem solving, this cannot always be attributed to a lack of heuristic tools and strategies in Pólya's sense alone. Rather, it is also due to the currently insufficient interpretation or modelling of the given situation. In 1998 Lesh accordingly defined the rather ambiguous term "problem" as follows: "the most important criteria that distinguishes 'non- routine problems' from 'exercises' is that the students must refine / transform / extend initially inadequate (but dynamically evolving) conceptual models in order to create 'successful' problem interpretations" (Zawojewski & Lesh, 2003, p. 318). The issue of "understanding the situation" is, of course, also addressed in classic problem solving models and heurisms a la Pólya can also be used to achieve an appropriate situation model. Moreover, the combination of elements of problem solving and modeling cycles to describe mathematical activity appear to be particularly fruitful when the construction and use of analogies in problem solving is to be analyzed, because analogizing bases upon mental models of mathematical situations.

EMPIRICAL BASIS

In order to examine the construction and use of analogies in mathematical problem solving, we conducted semi-structured clinical interviews (e.g. Beck & Maier, 1993) with 86 primary school pupils. 39 pupils came from regular primary classes of grade 3 to 6, the other 47 pupils participated in fostering projects for mathematically gifted students at university. Every pupil consecutively worked on two problems which were analogous to each other (party intermitted by a disturbing non-analogous exercise), and he was asked to describe what he is doing as far as possible (thinking aloud).

Also if the students failed to show appropriate signs of analogical thinking during the problem solving process, they were afterwards prompted by the interviewer to compare the problems and asked to describe any similarities they found (initiated review).

By now, we used altogether 12 different problems in our study. The most problem pairs tackled by the pupils were not only structurally analogous but also include the same numbers (see also P1 and P2 in Figure 1). This should make it possible to use the analogy for transferring also the results from one problem to the other. For more details of the whole study see Aßmus and Förster (2013a).

For explaining and empirically underpinning the process model presented in Figure 2 we only use two of these problems (cf. Figure 1). The problem combination P1+P2 was tackled by 12 fourth-graders. To ensure that the sample group was as heterogenic as possible, we had asked teachers from different schools to select an above average, average, below average, and if possible a mathematically gifted pupil from each of their classes to take part in the interviews.

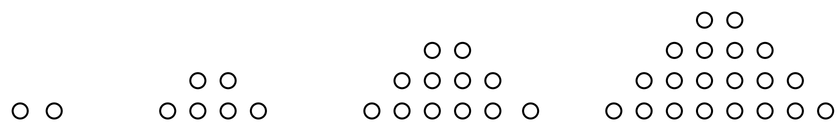
P1	 Paul makes groups of counters on the table. Each new group contains more counters than the last group in a certain way. How many counters do you think he will put in the 20th group?
P2	Anna starts to read a book. She reads two pages on the first day. She continues to read the book, reading 2 pages more than the day before each day. How many pages will she have read after 20 days in total?

Figure 1: Analogical problems (from an expert's view) used in the study.

Both problems were used in varying sequences to ensure that both the source and target points of the analogies to the real problems were empirically accessible.

MATHEMATICAL PROBLEM SOLVING AND CONSTRUCTION OF ANALOGIES

A Process Model

The process model in Figure 2 demonstrates possibilities for analogizing during mathematical problem solving. It attempts to combine the classical models of Pólya (1945) or Mason, Burton and Stacey (1982) to cognitive-modeling approaches while focusing simultaneously on the use of analogies in dealing with challenging mathematical situations.

The model demonstrates possible points in which analogies can be used. A distinction is made between how the situation is dealt with by the learners; whether or not the situation can be defined as the *source problem*, meaning that the ideas, approach or results of which can be transferred to another situation (*outgoing analogy*), or whether or not the situation can be defined as a *target problem*, meaning that a given learner's prior experience with a similar situation can be used (*targeting analogy*). The "reviewing" phase is marked by a dashed box, because in our setting it was, if necessary, initiated by the interviewer to stimulate analogizing processes.

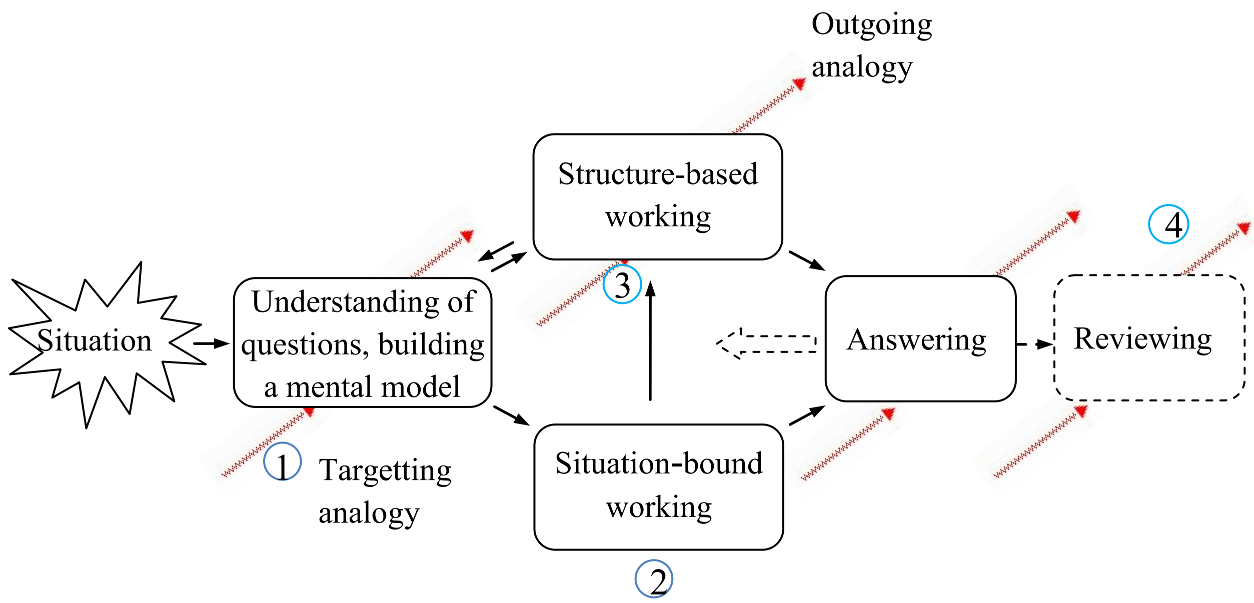


Figure 2: Process model of problem solving focusing possible uses of analogies.

The model also attempts to demonstrate the conditions necessary to facilitate analogizing. They to a large part justify the high cognitive demands of analogizing as a heuristic strategy: The pupil is thus required to understand the problem first and develop an appropriate cognitive model of the situation at hand. As long as the following steps only involve situative characteristics and elements, at most *pseudo-analogies* can be constructed, the level of which is constrained to the situation's surface – such as the same numbers. If the learner is however able to construct mathematical structures that fit the situation, i.e. a mathematical model, it also becomes possible to construct and use *structural analogies*. However, similar surface characteristics, approaches or (partial) results can also present triggers of structural analogies. This is the case particularly in the "Answering" and (initiated) "Reviewing" phases.

Where a learner indeed uses a (targeting) analogy, he may not pass though all phases shown in Figure 2, but other loops and setbacks are possible. In this sense they can be understood as descriptive modules which specifically take shape according to the individual learner's case not only in terms of their type but also in terms of their sequence.

Supportive and exemplifying examples

One the one hand, we are able to show empirical evidence for all possibilities of analogizing in the model depicted in Figure 2. On the other hand, it seems possible to classify all empirically found instances of constructing and using analogies according to this model. For the sake of brevity, we can only present few cases whose classification is also visualized by the numbers in Figure 2.

This first example shows, as referred to in the introduction, the "normal case" of analogizing during problem solving. While working on a problem (P2) analogies to a known problem (source problem, P1) are constructed and then used for solving the target problem P2 (targeting analogy).

<table>
<tr><td colspan="2">① Jenny (10y5m) completes P1 successfully. Upon completion, she reads the instructions for P2 and writes down her answer (420) within 20 seconds.</td></tr>
<tr><td>J: "The same as that one." [Jenny points to the answer sheet for P1.]
I: "Why?"
J: "20 rows there, it's 20 days here, and she always reads 2 pages more, which is why the answer is 420 pages."
I: … "How did you work that out?"
J: "…because she reads two pages more than the day before and the 20."</td><td>After Jenny had read the instructions for P2, she connect the corresponding elements of the two situations ("mapping": 20 rows --> 20 days; always two counters more --> always two pages more), thereby transferring the results from P1.</td></tr>
</table>

Figure 3: Targeting analogy during building a mental model.

The following example shows that a purely situation-bound working may lead to a false answer. Without using any mathematical structures there is no basis for analogizing.

<table>
<tr><td colspan="2">② Ian (10y5m) makes a sketch for P1.</td></tr>
<tr><td></td><td>Ian manages to transfer the predetermined fourth figure, and succeeds to enlarge it correctly across two lines. He then lets the figure become wider and wider somehow, by increasing size, numbers of or distances between counters. Geometric or arithmetic patterns play no role.</td></tr>
</table>

Figure 4: Situation-bound working.

Even a wrong approach can lead to the construction and use of analogies if it is based on a revised mental model and corresponding structural-based working.

<table>
<tr><td>3</td><td colspan="2">Marc (10y2m) works on P1 successfully. To achieve his result, he adds up in parts the even numbers from 2 to 40. While working on P2 he expresses, that 40 pages would be read on the 20th day.
Thereafter he guesses a total number of 80 pages. He hesitates while explaining his answer. So the interviewer explains the problem.</td></tr>
<tr><td colspan="2">I: “The question is how much she reads on all these days, taken together.”
M. writes down in lines 20 + 18 + 16. He leans back and mumbles “count down from 40”.
M: “It’s kind of just the same like this task [points at P1].”
I: “What is the same?”
M: “Well, I have to calculate, down from 40.”
I: “What do you have to calculate down?”
M: “I calculate 40 (…) plus 38 plus 36 plus 34”
I: “Okay, und what is the result?”
M: “420.”</td><td>Marc at first doesn’t understand the question completely. After clarifying the problem he begins to work and writes down some of the even numbers. During structural working he recognizes the one-to-one-correspondence between the summands of P1 and P2 and transfers the result of P1 onto P2.</td></tr>
</table>

Figure 5: Targeting analogy during structural-based working.

The final example shows that an analogy can be achieved in the phase of reviewing, even if the source and the target problem were tackled in different ways and with wrong results.

<table>
<tr><td>4</td><td colspan="2">Michael works on P2 first. He tries to simplify his summation and checks his results by building patterns and subtotals, but doesn’t succeed. Due to some minor calculation errors he finally achieves 400 as a result. Subsequently he deals with P1 and describes several explicit and recursive connections. In order to determine the total number of counters of the 20th group efficiently, he has the following idea: “Within the 20th group one line always adds up to 40 with another. That is to say that all lines except of the 20th can be completed to 40 by another, 2 to 38, 4 to 36 and so on.” Michael supposes to get 19 pairs in this way. Finally he gets his result by calculating: $19 \cdot 40 + 20$.</td></tr>
<tr><td colspan="2">After the computation the interviewer places the worksheets P1 and P2 in front of Michael and initiates the reviewing process:
I: “Do the two problems have something in common?”
Michael thinks for a few seconds.
M: “Somehow it’s nearly the same problem. Because, here she reads a book and … , and here it’s, each group increases by two counters. So only one of the results can be correct. Because both is up to 20. Mmmh, it’s difficult. I think I better check this result [points at P2] … there [points at P1] I may be wrong, too,”</td><td>Recognizing the analogy, Michael controls his results in the source problem (here P2). After achieving the third (now correct) result, he transfers this result to his current target problem (P1). Of course, the question, if and how far he would review his problem solving process self-contained or independently, remains unanswered.</td></tr>
</table>

While recalculating, Michael discovers his computation errors and finally gets the correct result. M: "I think, this should be the result of both [*points at 420*] … Because, I think, it's the same problem, just expressed differently. Because here it's also [*points at P1*], each time, it increases by two, just the same like here [*points at P2*]."	

Figure 6: Outgoing analogy in the phase of reviewing.

WHAT'S ALL ABOUT THIS MODEL?

The process model specified in this paper seems to be suitable for analyzing the construction and use of analogies during problem solving in many respects. On the one hand it can be used as an analysis tool for all problems investigated in this study. Differences regarding points of analogizing in problem-solving processes become comprehensible and describable. Moreover, based on the model phases, conditions that can facilitate or hinder the process of analogizing during mathematical problem solving can be carved out. Such conditions are in some extent already published (Aßmus & Förster, 2013b), but further research is required. This knowledge about constructing and using analogies would constitute an important basis for evaluating pupils abilities in problem solving and analogizing.

On the other hand the model enables us to classify and compare other studies on using analogies more precisely. This includes the arrangement of the studies as well as an assessment of their results. Studies can be systematically compared by determining the specific model phase investigated and by the perspective from which the construction and use of analogies is viewed.

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JUGGLING REFERENCE FRAMES IN THE MICROWORLD MAK-TRACE: THE CASE OF A STUDENT WITH MLD

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This study gives insights into how Logo-like microworlds can affect cognitive development related to mathematics education of students with math learning difficulties. In particular, we analyse the case of a 15-year-old student with dyslexia and severe dyscalculia. Among the various cognitive aspects involved, here we delve into the development of his perspective-taking ability, seen in terms of becoming aware of and juggling two different allocentric frames of reference.

INTRODUCTION AND THEORETICAL BACKGROUND

The ability of perspective-taking (Piaget & Inhelder, 1967; Clements 1999), or being able to embrace different frames of reference based on one's self or on external points of reference, is fundamental both in everyday life and in instruction. The importance of such ability is declared in the Italian National Curriculum Indications (MIUR, 2012) relative to mathematical learning about "Space and Figures". These state that by the end of third grade a student should be able to "…follow simple paths described verbally or graphically, describe a path that he is following, and give instructions to someone so that they follow a given path." (MIUR, 2012, p.50, translated by the authors). Developing the perspective-taking ability may not be straightforward: it involves a transition from "perceptual space" to "representational space" (Piaget & Inhelder, 1967), as well as "connecting different viewpoints" (Clements, 1999, p.3).

While children with a typical development can be assumed to have acquired such ability by the end of primary school, in some children with mathematical learning disabilities (MLD) – including developmental dyscalculia (e.g., Mazzocco & Räsänen, 2013) or more in general mathematical difficulties (as discussed in Karagiannakis et al., in press) – the development of perspective-taking, among other abilities, may be delayed and/or deficient. Although students with MLD may present different mathematical profiles, frequently characterized by the presence of multiple deficits including those of a visual-spatial nature (Andersson & Östergren, 2012; Karagiannakis et al., in press), some remedial interventions that involve microworlds, such as Logo, have been successfully carried out (e.g., Ratcliff & Anderson, 2011).

Our study is part of a larger project investigating qualitative effects of different kinds of remedial interventions for students with MLD. In particular, in this paper, we analyse a student with MLD's cognitive processes involved in juggling different frames of reference, while working in *Mak-Trace*, a Logo-like microworld.

LOGO-LIKE MICROWORLDS AND STUDENTS WITH MLD

As described in the extensive literature on the topic, the idea of microworld involves considering particular computer software as tools providing informal learning environments that have specific knowledge domains embedded (Hoyles et al., 2002). Logo was the first microworld to be developed and to become popular (Papert, 1980). The early plea to study effects of Logo learning on cognitive skills is still a topic of research today (Ratcliff & Anderson, 2011), especially since the original Logo language has been simplified in various ways and adopted to program real or virtual robots used at different school levels (e.g., Highfield & Mulligan, 2008).

The potential of Logo-like microworlds for fostering learning in students with MLD is documented in the literature. In particular, Vasu and Tyler found that Logo may foster the development of spatial abilities and of critical thinking skills (Vasu & Tyler, 1997), and various other researchers have reported several potential benefits of using Logo with students who have learning difficulties (Atkinson, 1984; Maddux, 1984; Michayluk & Saklofske, 1988; Miller, 2009; Russell, 1986), especially using a more structured, mediated approach (Ratcliff & Anderson, 2011).

The Microworld Mak-Trace

Mak-Trace (Giorgi & Baccaglini-Frank, 2011) is a free application for the iPad and the iPhone in which a character (by default a snail) can be programmed to move and draw on a grid. The character can only be programmed to go forwards (F) or backwards (B) of the distance of one side of the grid-square at a time, or to turn 90° clockwise (R, standing for “turn right” in the snail’s perspective) or counterclockwise (analogously L, standing for “turn left”). Therefore, the frame of reference for giving directions is *relative to the character*, not to a North-South-East-West frame relative to the grid. For example, holding the iPad “right side up” if the character’s head is pointing downwards – that is the snail is *oppositely oriented* with respect to the programmer – F (the icon with the arrow pointing upwards relative to the grid) will make it move in a vertical line that the programmer will perceive as “descending”.

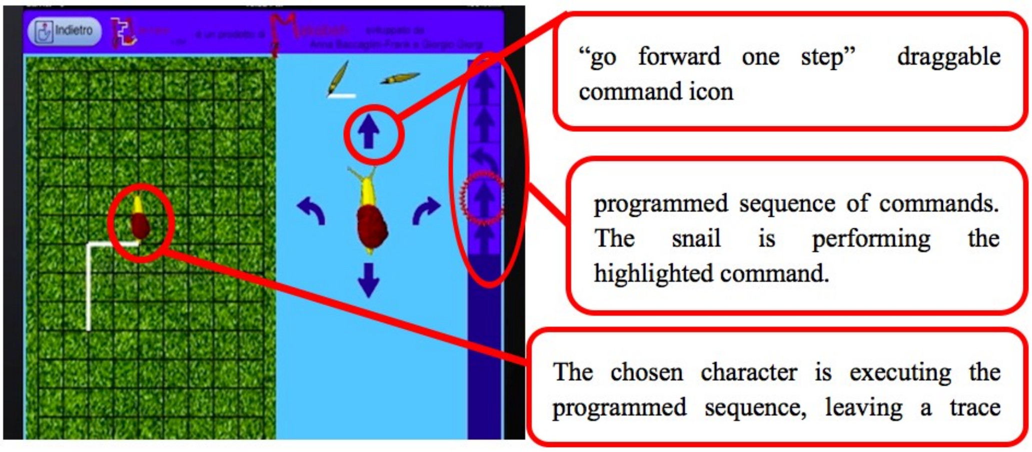

Figure 1: Screenshot of Mak-Trace, where the character is executing a sequence.

Mak-Trace can be seen as a simplified version of Logo; it was designed by one of the authors with the aim of creating an environment accessible to young children, or to students with learning difficulties or disabilities, by trying to offer a more intuitive iconic programming language, with the potential to foster mental planning, visualization and perspective-taking abilities. This aim led to some characterizing design choices. A main difference between Mak-Trace and Logo is the fact that the commands are icons that can be dragged and dropped to build a sequence. Another difference is that Mak-Trace gives no feedback in terms of movements of the character until the student touches “GO”. At this point the character executes the whole list of commands in the constructed sequence. Moreover, to make a variation in the constructed sequence, the student has to go back to the “programming mode”: automatically the character goes back to its original position and all trace marks are cleaned off the screen.

EGOCENTRIC AND ALLOCENTRIC REFERENCE FRAMES

When we represent the location of objects in the environment we can use different frames of reference (Carlson-Radvansky & Irwin, 1994). It is largely accepted that two main frames are the *egocentric* and the *allocentric* ones. In the former, the representation of objects is referred to the self and to the observer's body; in the latter, spatial relations are represented independently of the self. Grush (2000) refined such distinction, identifying four different uses of the term “allocentric”: (A) egocentric space with a non-ego object reference point (*decentred egocentric*); (B) object-centred reference frames; (C) virtual points of view (i.e., maps); (D) “nemocentric” maps. We found this distinction to be quite relevant to our study. In particular, we will take into consideration the first two allocentric frames (A and B in Grush’s distinction), because they apply to programming in Mak-Trace. Let’s describe the reference frames with an example. Let’s say Giovanni and Lucia are in a room, in front of our body looking at us, and Giovanni’s left hand is holding Lucia’s right hand. In our egocentric frame, we can say that Lucia is in front of us. Using an allocentric A-type frame we might say that Lucia is at the right of Giovanni. So, while the left-right axis is referred to our body, we are using Giovanni as the reference to locate Lucia. In the allocentric B-type frame, we might say that Lucia is to Giovanni’s left. In this case, we represent the space as Giovanni might represent it according to his egocentric frame of reference, so “left” is referred to Giovanni’s point of view. In other words, the frame’s origin is centred on Giovanni and its axes are Giovanni-fixed.

In Mak-Trace the perspective-taking ability consists in embracing the character’s moving frame of reference and this requires to coordinate two frames: the decentred egocentric (type A) frame, the character-centred frame (type B). Instead, using an egocentric frame turns out not to be effective, because the iPad usually sits in front of the programmer and terms like “left” and “right” cannot be meaningfully used. In this paper we will analyse processes of juggling of type A and type B frames.

A CASE STUDY: STUDENT WITH MLD JUGGLING FRAMES

The subject in this study, named here Filippo, had been diagnosed with various MLD including dyscalculia and severe dyslexia. From the accounts of his special education teacher (one of the authors), he also was not able to read maps or to give directions (however he did not have difficulty recognizing or naming his left and right hands), he had a short attention span and little – if any – interest in the activities proposed during math class. We planned and carried out a 5-week intervention (globally, 10 hours outside his regular classroom) using Mak-Trace. The tasks he was assigned, that we analyse here, were the following: 1) program the snail to draw a given path; 2) program the snail to draw a square; 3) complete the mazes. Each activity was audio and video-taped, and analyzed according to the frame above.

Task 1: Program the snail to draw a given path

Filippo initially thinks that the commands F, B, R, L make the snail go forwards, backwards, right, and left, where these directions are in the decentred egocentric (A) frame: the forwards-backwards and right-left axes are Filippo-fixed, while the centre of the reference frame is the snail. Therefore Filippo is not able to construct a sequence of commands to make the snail draw a given path. It takes him about 10 minutes, dragging command icons more or less randomly and watching the snail in a confused state, before realizing something is wrong:

Filippo: it is a bit hard. It’s never what it… […] I am not understanding anything […] wait…I didn’t tell him to go right and he went right. […] These two [R and L] are inverted […] I am not understanding anything […] if this arrow [L] makes it turn right, this one [R] makes it turn left.

Teacher: why does an arrow pointing to the left make it go to the right?

Filippo: Ask the person who designed the game!

For over half an hour Filippo tries to understand how each command could be associated to a snail’s movement (translation) on the grid and does not seem to be aware of any reference frames other than the egocentric decentred one that he keeps working in. Clearly, when he realizes that the natural correspondence (R → translation to the East on the grid) does not work (after a long time) he tries to set other correspondences between command icons and movements, but he seems to be overwhelmed and unable to come up with a meaningful correspondence. Filippo comes to an important realization when the teacher helps him analyse the sequence of commands with respect to the trace mark left after the snail executed it:

Filippo: it went backwards, not upwards […]

Teacher: so what do the little arrows refer to?

Filippo: it depends on how the snail is oriented.

We see this as the decisive moment which poses the foundations for the conception of a type B allocentric frame of reference. We note that to embrace a type B frame it is necessary to consider the *snail* as the reference point and the *snail-fixed* axes. Here

Filippo is still only considering his egocentric frame: the snail's rotation can only be perceived from such a frame of reference. However, he states that the type of movement determined by the commands *depends on* the snail, and in particular on its orientation. Although a fundamental step in terms of awareness has been made, Filippo still has trouble embracing the snail's perspective, so when the teacher asks for further explanations on the effects of a command, he appears confused:

Teacher: right for whom?

Filippo: for me, [mutters something], no, for the snail, for both… I don't know! I don't understand…

Filippo refuses to talk any more and uses trial and error to write a sequence of commands (FFRFLFLFRFFLFF) that represents a given path made up of horizontal and vertical adjacent segments. To do this he seems to be embracing the snail's perspective. However along this path the snail is never *oppositely oriented*, which is the situation that creates the greatest difficulties for Filippo.

Task 2: Program the snail to draw a square

The first time Filippo tries to program a sequence to make the snail draw a square starting with the snail pointing upwards, he programs: FFFFLFFFFL [brief pause, he says: "Yes"] B [brief pause] BBBR [long pause] FFFF (Figure 2a). Even though he has hesitations, Filippo is able to program the sequence for the first two sides of the square. He seems to be able to embrace a snail-fixed frame (type B), as shown by the two uses of the command L to make the snail turn, and by the use of the command F to make the snail move along a horizontal segment. From here on, Filippo manifests difficulties: he seems to be programming the third side of the square in a decentred egocentric frame (type A), as shown by his (incorrect) use of the commands B and R; while the fourth side, horizontal in Filippo's frame, is correct again. It is interesting that he uses opposite commands for the first and third sides (F and B, respectively), while for the second and fourth he uses the same command (F). This strengthens our hypothesis that the two pairs of opposite sides were programmed using different frames of reference. In summary, Filippo seems to be mixing the two types of allocentric frames in the same situation, using the snail-fixed frame when it is oriented the same way his frame is or when it is rotated by 90°, and the decentred egocentric frame when his frame and the snail's are *oppositely oriented*.

The second time he tries to program the sequence he composes: FFFFLFFFFL [hesitates, inserts L, erases it, and with the index of his right hand makes the gesture of a counter clockwise turn] FFFF [he says: "I have to always keep the" and does another counter clockwise turn gesture with his right hand] RFFFF (Figure 2b).

So Filippo has now corrected the third side but makes a mistake again on the rotation when the snail is *oppositely oriented*. He re-writes the sequence: FFFFLFFFF [he makes the gesture of a counter clockwise turn with his right hand] LFFFF…[he rotates the iPad so that his frame coincides with the snail's, observing the screen he rotates his right hand counter clockwise]. Now he completes the last turn and side.

Filippo: Done, I found it […] no, I got…lost […] when it is turned around…it goes opposite [*clockwise* rotation gesture with the right hand] so…if I want it to go here [horizontal gesture from left to right with the left hand] … oh, I don't know, I'll try this [RFFFF]… no wait, because this otherwise is like before [he substitutes R with L].

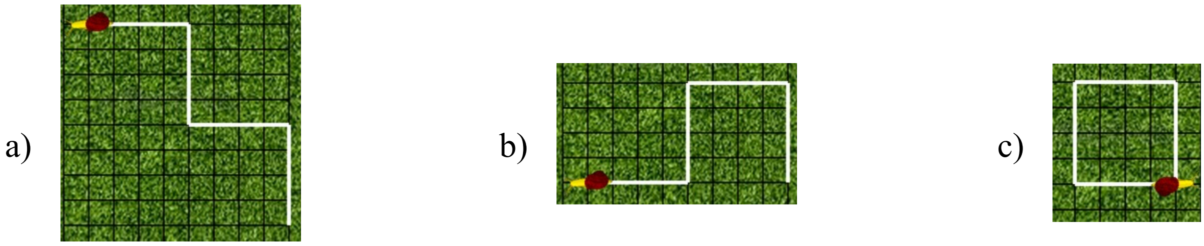

Figure 2: The first (a), second (b) and third (c) traced paths in task 2.

Now the sequence is correct (Figure 2c). We note that rotating the iPad is a gesture that reveals how Filippo is now aware that he should consider the snail's frame of reference, and that this frame is oppositely oriented with respect to his (at the moment of the rotation). This is also testified by the hand gestures, opposite with respect to the previous ones, but even with the rotated iPad he keeps on making mistakes. The way he turns out to solve the task is by remembering the previous sequence he had programmed (second try) and choosing the opposite turn arrow, proceeding by trial and error to successfully compensate his spatial difficulties.

Task 3: Complete the mazes

Filippo is asked to program the snail to get it through a maze. He appears to be convinced of being able to accomplish the task and begins to build a sequence. As in task 2, he stops and hesitates when he needs to program the snail to turn when it is *oppositely oriented*. Filippo grabs a pencil and swivels it around pointing its tip towards himself and making a small rotation in the direction he wants the snail to go:

Filippo: I am doing the snail upside down because otherwise I was getting too stuck.

Whenever the snail is oriented like himself or it is horizontally oriented, Filippo programs more than one segment at a time, but when the frame of reference of the snail is opposite to his (it happens 3 times) he acts as follows. The first time, he uses the word "straight" instead of "forward" to describe the "forward icon". When he has to choose a turn command at the end of the vertical segment he stops and uses the pencil again, as he did the first time. Then he picks up the iPad and rotates it by 180°, thus making the snail's frame coincide with his own, and then he adds R to his sequence. This was the correct choice, but he tests it right away to be sure. The second time, he programs the snail correctly when it has to turn, and the third time he also succeeds in doing this, but he uses the swivelling of the pencil, again. After he successfully concludes the task, Filippo describes how he was able to succeed:

Teacher: When you had to turn, how did you understand when to go left and right? […]

Filippo: I turned the iPad.

Teacher: But weren't you using the pencil, too? Didn't you turn it?

Filippo: Yes […]. You have to imagine [being] the snail.

So Filippo has surely become aware of a new perspective that initially he didn't seem to perceive at all. However, to embrace this type-B perspective, it seems like Filippo is aware that he needs to use some strategy to compensate for his cognitive difficulties.

CONCLUDING REMARKS

The study shows the enhancement of the ability to coordinate different frames of reference in a student with MLD working in a Logo-like microworld. Such enhancement occurred thanks to the specific tasks proposed, the interventions of the teacher (e.g., the use of expressions like "the snail turns", "right for whom?", "refer to…") and the functionalities of Mak-Trace (e.g., the commands are icons that can be spontaneously interpreted in the egocentric frame, but that refer to the snail-fixed frame; the fact that immediate feedback of a programmed sequence is *not* given, etc.), which required continuous juggling between two reference frames. Although this juggling was not spontaneous for the student, due to his disabilities, there was a positive development of his perspective-taking ability. In particular, we observed a transition from not being able to perceive the snail's perspective, and trying to find a way of making sense of the command icons, to recognizing this perspective and trying to embrace it, after a period when the student's confusion seemed to depend on his simultaneous use of the two incompatible frames. Some difficulties persisted but they were partially overcome through different compensatory strategies: trial and error (since there are two choices for the turns), trying to define a rule without embracing the snail's perspective (when the snail is upside down everything is opposite), changing his own perspective (by rotating the iPad or swivelling a pencil pretending it was the snail), and resorting to gestures that bridge one reference frame to the other. In the end the student is aware that he can change frames of reference by mentally trying to "be the snail". Last but not least, similarly to what has been described for Logo, Mak-Trace appeared to help the student to "remain absorbed in a task for a period of time; … tolerate a period of confusion (with appropriate support);… use errors as a source of information about what to try next" (Russell, 1986, p. 103).

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PRIMARY SCHOOL TEACHERS LEARN MODELING: HOW DEEP SHOULD THEIR MATHEMATICS KNOWLEDGE BE?

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We taught a group of experienced in-service primary school mathematics teachers the notion of mathematical model, in order to foster the interdisciplinary mathematics teaching in primary school. In particular, we developed an exercise in which they were supposed to construct a mathematical model on the basis of primary school mathematics. We found out that the formal mathematical knowledge needed to perform the exercise was not sufficient to successfully cope with it. The main factor that influences the ability of the teachers to cope with this type of activity is the depth of their mathematical knowledge which we identify with a person's mathematical insight.

THEORETICAL BACKGROUND

The Israeli primary school curriculum explicitly necessitates the linkage between mathematical curriculum and two other components: other disciplines studied simultaneously, and everyday life experiences. This declaration is realized in several paragraphs of the curriculum, such as data organization and analysis and integrative problems. Nevertheless, the mathematics teaching in Israeli primary schools is usually confined to purely mathematical (mostly arithmetic) contents, with no intentional connections made to the world surrounding the pupils (Arcavi & Friedlander, 2007). We regard this to be an essential drawback and seek ways to cope with it.

Numerous studies indicate the insufficient matching between the mathematical knowledge and skills the schoolchildren are expected to acquire at school, and what they need to be able to do with this knowledge outside the school (English, 2009; Gainsburg, 2006; Pollak, 1979; Zawojewzki & McCarthy, 2007). Hence, the mathematical education specialists face the challenge of finding the ways to cope with authentic and related to them interdisciplinary problems, sometimes rather complicated ones. One of the ways to do so is to embed mathematical model construction in mathematics lessons (English, 2009; Gainsburg, 2008; Kaiser & Schwarz, 2006). In order to embed the interdisciplinary teaching into the mathematics class, several components are needed, such as handbooks, time allocation in the mathematics lessons, and the teachers' competence in the issue. This competence is critical for implementation of interdisciplinary teaching at school. Doerr (2007) claims that teachers refrain from dealing with interdisciplinary problems because their knowledge in mathematical modeling is not sufficient. Hence, it is one of the key research issues – to determine what teachers' knowledge is needed in order to implement modeling at school (Garcia & Ruiz-Higueras, 2011).

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In the present research we focus on the process of building-up the teachers' knowledge of the concept of mathematical model which is pivotal for the interdisciplinary approach (Ng, 2010). In particular, we are interested in the connection between this process and the depth of a teacher's knowledge in other issues of the primary school mathematics curriculum.

Mathematical model is a mathematical object – a graph, a sequence, a diagram, an equation etc., reflecting to a certain extent an outer-mathematical phenomenon. The model construction is a kind of a loop-like process which can be schematically represented in the following way:

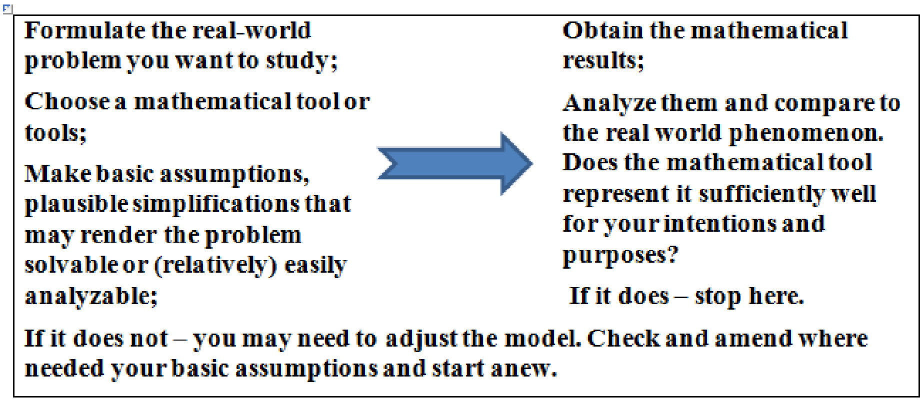

Figure 1: A schematic representation of modeling process

In order for teachers to be able to teach this (as well as any other) approach at school, they must be competent to cope with it at an appropriate level. Speaking of mathematical modeling, we agree with Maab and Gurlitt (2011) who claim that teachers need "modeling competency": the ability to carry out modeling processes independently". Following Cherniak (2007), the empirical research basis in interdisciplinary teaching on which it would be possible to build up practical approaches and curricula, is still lacking, especially in what concerns the teachers' education in these topics. Our research presented here is a part of a bigger research project aimed at the interdisciplinary teaching by expert mathematics teachers in primary school as a part of their professional development.

RESEARCH FRAMEWORK

In this research we follow the process of acquiring the concept of mathematical model by primary school teachers during a one-semester course in mathematical modeling as a part of their M.Ed. program. The notion of a mathematical model was equally new for the audience; nobody has been previously familiar with it.

Our research sought the answer to the following questions:

- Is deep mathematical knowledge of primary school mathematics a necessary basis for the understanding of the concept of mathematical model?
- Is the knowledge of a formal corpus of primary school mathematics a sufficient basis for such understanding?

We studied performance of 14 M.Ed. students who are active and experienced primary school mathematics teachers. In what follows we call them "the teachers".

Tools and methods

More-or-less in the middle of the course on mathematical modeling the students received an exercise in which they were asked to propose a model for evaluating the paper usage at primary school. The exercise was assessed in two different ways; we looked for possible links between the outcomes of these analyses.

Firstly, we analyzed it by the five parameters included in the assignment formulation. The exercise and the assessment parameters appear in Appendix 1. Each of the parameters was assessed using four-level grading, from the lowest (1) to the highest (4) grade. Table 1 represents the assessment criteria. The abridged notations are explained in Appendix 1.

	DC	BA	MR	MA	EM
1	Non-adequate; data not used in the model	Not formulated	Non-structured tables	Not presented	No evaluation
2	Meticulous sheet-by-sheet data collection on paper usage	Not clearly formulated or not relevant for the model.	Structured tables alone.	Inadequate list; tools not used in the model.	Evaluation in terms of the paper usage outcome.
3	A justified sampling coherent with the model assumptions	Plausibly formulated assumptions but not related to the data collection.	Tables and diagrams of a single type (rod diagrams).	The list of tools is adequate but not complete	Evaluation partly refers to the paper usage outcome
4	A sampling method aimed at the specified model and its purposes.	Coherent, clear and justified assumptions	Adequate variety of tables and diagrams.	A complete and adequate list of tools.	Effective evaluation; clear and motivated amendments.

Table 1: Assessment criteria for the exercise.

Secondly, we assessed it from the viewpoint of the depth of teachers' understanding of the mathematical corpus of knowledge. Formally speaking, all the teachers master the mathematical apparatus needed for the exercise, which does not exceed the 6th grade level requirements: data representation, arithmetic of multi-digit numbers, zero in arithmetic operations etc. Should this formal corpus be sufficient, we might expect the more-or-less homogenous results all over the group. What we are interested to appraise is how the depth of the teachers' understanding of this corpus showed itself in the exercise performance; for this we use the notion of mathematical insight. The concise necessary description of the insight parameters appears in Appendix 2.

FINDINGS

Table 2 represents the assessment of the teachers' performance by assignment requirements and by insight components. The teachers are represented in the first column by their numbers. The assessment was validated by three experts

.	Assignment requirements					Insight components		
Nr.	DC	BA	MR	MA	EM	IA	S	ML
1.	1	1	2	1	1	1	3	1
2.	3	4	4	4	4	3	4	4
3.	4	2	3	2	1	2	2	2
4.	4	4	4	4	4	4	4	4
5.	2	3	2	2	1	2	3	2
6.	4	4	3	4	4	4	4	4
7.	2	3	3	3	3	2	4	3
8.	2	2	1	1	1	1	2	2
9.	4	4	4	4	4	4	4	4
10.	2	3	2	2	3	2	2	2
11.	4	3	3	4	4	4	3	3
12.	3	2	1	3	1	1	2	3
13.	4	4	2	4	3	4	3	4
14.	4	4	4	4	3	4	4	4

Table 2: The results of the assessment of the teachers' performance based on the assignment requirements and on the parameters of the mathematical insight.

As one may observe, the results of the group are far from homogenous. In addition, the rows of the Table indicate that the students who performed well according to the assignment requirements also demonstrated higher insight, and vice versa. Having observed that, we decided to "zoom" on the performances of some of the students in order to elucidate the differences. Table 3 enables comparison between the performances in two cases: one of a distinctly low-assessed teacher (Nr.1); one of a distinctly highly-assessed (Nr. 4).

	Nr.1	Nr.4
DC	"…I computed the average paper usage per pupil during the year…". It is unclear how the averaging was performed; the data were not used further in the model.	Based on the school working style using mainly copying: "I asked several teachers how often they hand out copied sheets to their pupils, and evaluated the total."
BA	Mutually incompatible assumptions: "during the academic year a pupil uses about 4 pages a day" (which has nothing to do with the estimate); "A pupil uses in the average at least 500 pages during an academic year" (215 days long).	All the classes are of the same size; the teachers of the same grade work similarly; only writing paper is included; the paper usage for each discipline is similar; most paper usage follows from copying.
MR	Plots a rod diagram for the monthly paper usage for each grade; then plots pie diagrams for the monthly paper usage; Does **not** plot pie or other diagram for the relative paper usage analysis.	A structured table in which the input data are presented in rounded numbers; rod diagrams; pie diagrams representing relative usage by the six grades; comparative rod diagrams
MA	A list of tools most of which were not used or used in a wrong way.	A full list of tools and notions used in the model, such as "ratio", "estimate", "negligible".
EM	"It was difficult to account for all the variables in this problem: the number of pages in a notebook, of copy sheets, etc."	Relates the diagrams to the real school life, e.g. finds real explanations for occasional increases in paper usage;
IA	The teacher clearly did not grasp the idea of mathematical model; she tries in the earnest to gain as precise and extensive information as possible on paper usage.	The newly learned concept of mathematical model is well understood; this can be observed from all the components of the exercise.
S	The teacher's skills are relatively high; e.g., she organizes the data in tables and plots diagrams using the Excel tool; but the skills usage is purely instrumental.	The teacher's skills are well developed and appropriately used; all computations and diagram are well motivated.
ML	Mathematically meaningless usage of such terms as "average", "estimation"; the reasoning comprises logically disconnected statements; uses phrases like "approximately 543 pupils", … "in the average at least".	All the terms are properly used; the reasoning in the work is coherent and consistent;

Table 3

CONCLUSIONS

One of the objectives of this research is to explore whether formal mathematical corpus of knowledge is sufficient for the successful acquiring of the concept of mathematical model. From the preliminary results it can be seen that teachers who succeeded in the exercise demonstrated also higher levels of mathematical insight in this issue. Hence, our preliminary conclusion is that in order to construct and use a mathematical model the teachers should have deeper understanding of the mathematical knowledge they possess. This conclusion is very important: if our goal as educators is developing our students' ability to work with mathematical models, we must find ways to deepen their understanding of the formal corpus of mathematical knowledge they possess.

Appendix 1: the exercise outline

In order to build the model, the students were instructed to perform the following steps (the abridged notation in the parenthesis is used in the text):

<u>**D**ata **c**ollection (DC)</u> – suggest a method of obtaining the information on the paper usage at school needed, in your opinion, to provide a plausible model for evaluation of the school paper usage, such as students' writing habits, teachers' practices, sheets copying policy, etc.

On the basis of their data collection, the students were expected to propose the method for the evaluation of the paper usage at school during the calendric year.

<u>Formulate the **b**asic **a**ssumptions used in the model construction (BA)</u> – e.g., the paper usage in all the classes of the same grade is more or less similar; there are time periods during the calendric year when the paper usage essentially differs from average, for example, during the vacations when it is close to zero. The assumptions of the model are naturally expected to be related to the data collection method proposed by the same student.

<u>**M**athematical **r**epresentations (MR)</u> – use the mathematical representations at the primary school level appropriate, in your opinion, for the model, e.g. structured tables; diagrams of various types, etc.

<u>Correct identification of **m**athematical **a**pparatus (MA)</u> – identify the mathematical tools from the primary school curriculum relevant for the assignment, such as working with big numbers (tens of thousands to millions); zero in multiplication and in addition; estimation methods; ratio and proportion; multiplication of multi-digit numbers etc.

<u>**E**valuate the **m**odel you have constructed (EM)</u> – having constructed your model and obtained the overall results of the paper usage, indicate to what extent it adequately represents the real school situation; which of your assumptions seem to need re-adjustment; is the model really helpful in estimation of paper usage? Etc.

Appendix 2: the mathematical insight

We present here the components of mathematical insight in a form relevant for the present contents:

The **implementation ability** (IA), by which we mean the ability of a person to apply the recently acquired piece of mathematical knowledge, provided this piece is in his or her mathematical ZPD (Vygotsky, 1978). The implementation is expected to occur in the "neighborhood" of the learned issue, obviously "the farther the better".

In the present setting, the recently acquired mathematical knowledge is the concept of a mathematical model.

Skills (S); by which we mean both the variety of mathematical skills at a person's disposal, and his or her autonomy, flexibility, appropriateness and inventiveness in using them.

Extension / generalization ability (EG) by which we mean the ability to extend the acquired knowledge and / or to generalize it.

The **mathematical language** (ML) which includes the ability of a student of take in new terminology and use it appropriately, the competence in using the mathematical notation, the ability to adequately reason mathematically, etc.

We found it next to impossible to plausibly estimate the EG parameter on the basis of the exercise; hence, we did not include it in the general outline of the results.

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A NEW APPROACH TO MEASURING PRE-SERVICE TEACHERS' ATTITUDES TOWARDS MATHEMATICS

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Research (for example Ball, 1988; Philippou & Christou, 1998) have linked teachers' attitudes with classroom practice in teaching mathematics. Previous studies have identified and examined the relationships between different components of teachers' attitudes (Nisbet, 1991). However, a particular criticism of these studies is the lack of content validity of the measures used. In the present study, in line with the conference theme for PME 38, we developed an innovative approach to examining the attitudes of pre-service elementary teachers. The study utilised a mixed methods approach, firstly eliciting qualitative statements from teachers, then using these statements in Likert-scale questionnaire items. We argue that this provides a more valid assessment of attitudes, and a method that can be applied across differing contexts for teachers.

FOCUS OF THE STUDY

Research has highlighted the importance of teachers' attitudes to mathematics. Aiken (1970) stated that teachers' attitudes were particularly important for students' attitudes towards the subject. Ernest (1989) also emphasised the importance of teachers' attitudes as being important for student achievement. Elsewhere, Ball (1988), Philippou & Christou (1998) and Wilkins (2008) have linked teachers' attitudes with classroom practice in teaching mathematics. In the UK context, school inspection evidence shows that teachers' lack of subject knowledge and confidence in mathematics contributes to low standards of mathematics attainment of pupils (Rowland et al., 2000). Despite this importance, researchers have also stated that many pre-service teachers come into the profession with negative feelings towards the subject (Ball, 1988; Nisbet, 1991; Philippou & Christou, 1998). It is therefore important that we use valid measures of pre-service teachers' attitudes to identify any concerns. In this study, we developed an innovative approach to examining and measuring pre-service elementary teachers' attitudes towards mathematics which we describe in this report.

THEORETICAL FRAMEWORK

Oppenheim (1992) defined 'attitude' as a "state of readiness, a tendency to respond in a certain manner when confronted with certain stimuli" (p.174). More specifically, there has been general agreement in the literature that attitudes consist of cognitive, affective and behavioural components (Bagozzi & Burnkrant, 1979; Ajzen, 2001; Crano & Prislin, 2006). According to McGuire (1969), the cognitive component "refers to how the attitude object is perceived, its conceptual connotations – it is the "stereotype the person has of the attitude object"" (p. 155). The affective component "measures the

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degree of emotional attraction towards an attitude object" (Bagozzi & Burnkrant, 1979, p. 915). There are then the "person's gross behavioural tendencies regarding the object" (McGuire, 1969, p. 156). We used this 'tripartite' view of attitude as the starting point for this study.

PRE-SERVICE TEACHERS' ATTITUDES TOWARDS MATHEMATICS

Studies have identified and examined the relationships between different components of teachers' attitudes' towards mathematics (Nisbet, 1991). Schofield (1981) measured two aspects of teacher attitude, namely attitude towards mathematics and attitude towards teaching mathematics. Likewise, Ernest (1989) highlighted these two aspects, identifying within attitude towards mathematics the components of teachers' liking, enjoyment, interest, self-concept and valuing of the subject. Others studies on teachers' attitudes have tried to measure these different components. Nisbet (1991) developed attitude measures to teaching mathematics, consisting of the four separate dimensions of anxiety, confidence and enjoyment, desire for recognition, and pressure to conform in teaching mathematics. Relich, Way and Martin (1994) criticised Nisbet's instruments, and emphasised the inclusion of teachers' self-concept in the subject, alongside anxiety, enjoyment, and belief in the usefulness or value of mathematics. Similarly, Wilkins (2008) used a measure looking at enjoyment, importance and the teaching of the subject, as well as feelings of success within mathematics. Ludlow and Bell (1996) developed an instrument based on existing items on self-concept, teaching of maths and doing or performing mathematics. Finally, more recently, Evans (2011) used an existing questionnaire developed by Tapia (1996, cited in Evans, 2011, p.228) including confidence, value, enjoyment and motivation. It is seen that there are components that frequently occur, such as enjoyment, self-concept, confidence, usefulness and teaching of mathematics.

The above studies used measures of attitudes, mostly based on Likert-scale responses to items related to particular components of attitude, to achieve reliable instruments required for larger scale studies of attitudes of pre-service teachers. However, a criticism that can be levelled at all these studies is the lack of content validity of the measures used. The question raised by Oppenheim (1992) is whether "the items or questions are a well-balanced sample of the content domain to be measures" (p.162). Although there is generally good theoretical agreement regarding the important components of pre-service teachers' attitudes, these are still theoretical assumptions, and the differences between the above studies illustrate the possible problems involved in identifying the 'valid' components. A solution to the problem of construct validity is to derive attitude questionnaire items from students' responses to more open-ended items (Oppenheim, 1992). Therefore, the present study adopted an innovative approach to identifying different components to pre-service teachers' attitudes to mathematics, incorporating both free responses to open-ended items and Likert-scale measures of attitudes.

METHODOLOGY

Methodology and methods

Leading on from the literature, the aim of the study was to develop an approach to identifying components of attitudes for a particular group of pre-service elementary teachers, and in turn develop valid, reliable measures for these components, and to then examine the relationships between these components (in line with Nisbet, 1991). The specific research questions to be answered were:

- Using both qualitative and quantitative approaches, what different components of attitudes towards mathematics emerge from the analyses for a particular group of pre-service elementary teachers?
- Using the resulting quantitative measures of attitudes, what relationships exist between measures of these components of attitude?

In balancing the requirements of identifying both the valid components of attitudes with the requirements of developing reliable measures, a *critical realist* methodological perspective was taken. This perspective balanced the positivist approach of measuring attitudes whilst "taking note of the perspectives of participants" (Robson, 2002, p. 30). Within this perspective, the study used a mixed methods approach, "combining qualitative and quantitative approaches within different phases of the research process" (Tashakkori & Teddlie, 1998, p. 19). In the first phase of the study, a questionnaire was given to pre-service elementary teachers which asked them to give a short written response to three statements: (a) What I perceive/think of with maths; (b) How I feel about maths; and (c) How I behave towards maths. The statements were designed to elicit open responses regarding teachers' cognitive, affective and behavioural components of attitude and no other guidance was given. The resulting statements were then analysed and coded to categorise the statements. In doing so, the analysis was guided by Tesch's (1990) (cited in Creswell, 1994, pp. 154-155) systematic steps to analysing qualitative data. At this stage, the statements from the three areas of attitudes were coded separately. In the second stage of the study, from the twelve most frequently occurring categories, six statements from each category were randomly chosen (if repetition of content occurred within statements, the second statement was discarded and another statement randomly chosen). The resulting statements were then used in a 72-item Likert-scale attitude questionnaire, with the items randomly ordered. Slight modifications of wording within statements were made for clarification if deemed necessary. A response from five possible options to each item was asked for: strongly agree; agree; neither agree nor disagree; disagree; strongly disagree. Having compiled the questionnaire, the pre-service teachers were asked to complete this. The obtained results were coded (5 = 'strongly agree to 1 = 'strongly disagree'), with negative items reverse coded. These quantitative results were analysed in SPSS using exploratory factor analysis to confirm the dimensions of attitude, and reliability analyses were carried out on the resulting groups of items to confirm the quality of the measures. Linear regression analysis was also subsequently carried on the resulting measures of attitude.

Sample

The sample of pre-service teachers involved in this study was comprised of students studying on a one-year Postgraduate Certificate in Education (PGCE) course at Durham University in the UK. The course qualifies students to become elementary teachers. All these pre-service teachers had already obtained an undergraduate degree, although different teachers had studied very different disciplines. In terms of their mathematical qualifications, these ranged from teachers with a minimum of GCSE qualifications in mathematics from examinations at the end of compulsory education in the UK, to teachers with top grades in Advanced-level mathematics from examinations prior to commencing university studies. In the first phase of the study, 78 students completed the open-responses questionnaire. For the second phase of the study, 90 students completed the Likert-scale questionnaire. This difference in numbers was due to the initial questionnaire being given at a pre-course training day to which some students were unable to attend.

RESULTS

Qualitative results

Beginning with the qualitative statements obtained from the pre-service teachers, the statements were categorised into the following groups (Table 1). From the cognitive, affective and behaviour statements, the pre-service teachers could view mathematics positively (i.e. enjoyable, important, confidently, committed) or negatively (difficult, avoiding). Clearly, there were some overlaps between the categories identified for different types of statements, but for the purposes of further analysis, these categories were kept separate for the next stage of the study.

From cognitive statements	From affective statements	From behaviour statements
Maths as difficult (42%)*	Enjoyable/fun (35%)*	Behave positively (36%)*
Maths as important (29%)*	Challenging (29%)*	Committed to maths (35%)*
Maths as enjoyable (27%)*	Confidence or self-concept (28%)*	Behave negatively (29%)*
Involving number (14%)	Very negative (24%)*	Specifically avoid (27%)*
As problem solving (12%)	Useful (15%)	Doing maths (19%)*
As right or wrong (10%)	Prepared to work on (10%)	Do mental maths (9%)
Other (9%)	Teaching of maths (6%)	Other (4%)
	Other (4%)	

Table 1: Categories of statements emerging from the analysis of qualitative statements

Quantitative results

Based on the above categories, the twelve most commonly identified categories (indicated with * in Table 1) were used to compile the Likert-scale attitude questionnaire. The choice of twelve categories were based on gaining a balance between covering as many categories as possible, but not having too many so that the questionnaire became unwieldy. Twelve categories with six items for each category resulted in a 72-item questionnaire which was viewed as reasonable in terms of length. Four subsequent dimensions were identified in the factor analysis, with items grouped as positive attitudes, negative attitudes, commitment to maths and usefulness/importance of the subject (these dimensions tended to be mixed in terms of items related to cognitive, affective and behavioral components). Subsequently, reliability analysis was also carried out on each of these group of items identified, and the Cronbach α values calculated (Table 2). Very high values of Cronbach α were obtained for three of the dimensions, with all the measures having reliability values greater than the benchmark of 0.7.

Dimension identified	Number of items	Exemplar items	Cronbach α of resulting measure
Negative attitude	23	I feel a lack of confidence in maths; I am nervous and anxious about maths	0.97
Positive attitude	16	I am positive towards and about maths; I like maths	0.96
Commitment to maths	9	I try hard in maths; I am keen and willing to learn maths	0.85
Usefulness/importance of maths	6	Maths is a very useful tool; Maths is useful in everyday function.	0.76

Table 2: Dimensions of attitudes emerging from the quantitative data

The above quantitative analysis therefore refined the dimensions of attitude identified in the qualitative stage of the study, and in turn led to the development of reliable and valid quantitative measures for these dimensions. These measures could then be used further to examining the relationships between the different dimensions of attitudes. For example, linear regression analysis was used to find out which other dimensions were significant predictors of larger values on the positive attitude measure, this being deemed to be a desired outcome for pre-service teachers. We found that the negative attitude measure and the commitment measure were both found to be significant correlated to the positive attitude measure. Plotting the position of each of the pre-service teachers on the positive and negative measures of attitude (Figure 1), we found unsurprisingly that there was a strong relationship; however, we also found a triangular relationship which showed that having a high score on the negative attitude measure (and since negative items were reversed, this means not agreeing with negative statements) was a sufficient, but not necessary condition for a high score on the positive attitude measure.

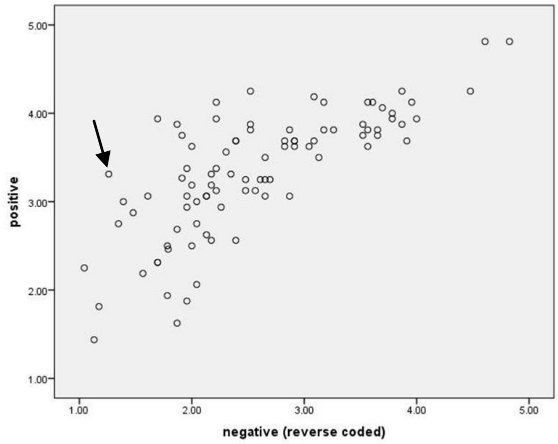

Figure 1: Plot of the positive and negative attitude measures

In fact, from the linear regression results, they showed that a commitment to mathematics also contributed to a positive attitude to the subject. We further illustrate this qualitatively by choosing one of the pre-service teachers who had quite a high score on the positive attitude measure, despite scoring very low on the negative attitude measure (shown in Figure 1 with the arrow). Her qualitative statements included: "A subject that does not come naturally to me. When I was at school I was not inspired by maths ... With maths I feel the least confident out of the core subjects ... Since deciding I wanted to be a teacher I have a very positive attitude towards learning maths. I am/will do everything I can to improve my subject knowledge." What we highlight here is that due to the approach to identifying and measuring dimensions of attitudes where the dimensions emerge from the analysis, we did not exclude important dimensions such as the commitment to mathematics which in turn were related to other important, desirable dimensions of attitude.

DISCUSSION

The methodological approach taken in the study identified a number of components of attitude held by the pre-service elementary teachers involved. An advantage of looking first at the qualitative statements from teachers was that we could identify straightforwardly which were the more significant components of attitude (Table 1). Choosing the twelve most frequently occurring categories identified through the analysis, these significant components involved difficulty of mathematics, importance, enjoyment, challenge, confidence or self-concept, positive and negative views, commitment to the subject and attitude towards doing or avoiding mathematics. There is a great deal of agreement between these identified categories and the literature, for example with Ernest's (1989) components of teachers' liking, enjoyment, interest, self-concept and valuing of the subject. Having identified these categories qualitatively, an added advantage of the current approach was that quantitatively and statistically, through exploratory factor analysis, we could further validate these categories. In fact, from the exploratory factor analysis (Table 2), the analysis refined these dimensions to more general positive and negative components of attitudes towards mathematics, as well as the importance of the subject and a commitment to mathematics. Relating these components to those identified in previous studies, the component which we termed commitment relates to the component of 'motivation' examined by Evans (2011).

Having obtained valid, reliable measures of attitudes of the pre-service teachers, in examining the potential relationships between the different components, although there was an unsurprising inverse relationship between positive and negative attitudes to mathematics, the triangular distribution in Figure 1 emphasised the importance of the commitment component of attitude. Indeed, an extension to this study will be to identify pre-service teachers who score highly on this commitment measure, and to examine further what factors support this commitment, particularly for teachers who may additionally have quite negative attitudes to mathematics.

One component of attitude that did not emerge from the current study, in disagreement to the previous research, was pre-service teachers' attitude towards teaching mathematics. A possible explanation for this is that the teachers in the study were at the very beginning of their training, and therefore had not yet been in schools to teach mathematics as part of their course. Therefore, teaching the subject may not have been a significant component of attitude for the teachers at that particular stage of their careers. In fact, this issue highlights a further advantage of the method used to examine attitudes of teachers. Because of the focus on content validity (Oppenheim, 1992) and the use of qualitative statements to draw out the relevant components of attitude, the particular context of the teachers was taken into account. This means that this approach to examining attitudes can be transferred between quite different contexts, for example teachers at different stages of their careers or in different countries, without assuming the same components of attitude. In addition, the flexibility of the approach allows for an examination of specific aspects of attitude. For example, the study could be extended to specifically examine pre-service teachers' attitudes to teaching mathematics by changing the focus of the initial open statements. Or, we could focus on areas within the subject such as attitudes towards mental calculations or attitudes towards problem solving, two aspects that emerged to a degree from the qualitative statements of teachers. We therefore propose that the approach used in this study can be a powerful method for examining teachers' attitudes towards mathematics (or indeed for other groups or for other topics).

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LINKING CHILDREN'S KNOWLEDGE OF LENGTH MEASUREMENT TO THEIR USE OF DOUBLE NUMBER LINES

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In this paper we report on five Grade 6 students' responses to a proportional reasoning task. We conducted pair interviews within a longitudinal study focused on extending a hypothetical learning trajectory for length measurement. Results suggest that there exists a link between children's level of conceptual and procedural knowledge for length measurement and their ways of using the double number line representation when solving problems involving proportional reasoning.

INTRODUCTION

Researchers have recommended the use of double number lines in the teaching of various content domains (e.g., Kuchemann, Hodgen, & Brown, 2011; Orrill & Brown, 2012; Van den Heuvel-Panhuizen, 2003). In the United States, the Common Core State Standards (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010) specifically recommends using double number lines in the teaching and learning of ratio and proportional reasoning.

Van den Heuvel-Panhuizen (2003) explored the didactical use of a form of a double number line, the bar model. In her work she found that this form of a double number line "can function on different levels of understanding, and that it can keep pace with the long-term learning process that students have to pass through" (p. 30). Kuchemann, Hodgen, and Brown (2011) argued that an understanding of the double number line model is important for helping students make a shift in understanding multiplication as scaling. They also noted that, much of the work relating to the double number line model has been focused on its use as a support for teaching.

In their work, Orrill and Brown (2012) identified conceptual foundations, coordinating units and partitioning, as critical pieces of knowledge for using the double number line representation to support proportional reasoning. Aside from this work, little is known about what concepts and processes are needed to develop fluency with the double number line model. The purpose of this report is to address this gap in the literature.

RESEARCH QUESTION

How does children's knowledge of measurement relate to their ability to use double number lines when solving problems involving proportional reasoning?

THEORETICAL FRAMEWORK

The purpose of this study was to explore children's knowledge of length and how it relates to their use of double number lines while solving proportional reasoning problems. Thus, we needed a theoretical tool that allowed us to describe and differentiate children's knowledge. A hypothetical learning trajectory (LT) for length measurement served this purpose. An LT has three parts: (a) an instructional goal, (b) a likely path for learning, and (c) the instructional tasks that support children's growth through those levels (Clements et al., accepted under review).

LTs are a central feature of hierarchic interactionalism (HI), a theory of cognitive development that integrates empiricism, (neo)nativism, and interactionalism (Clements et al., accepted under review). LTs originate from HI, which postulates that children progress through domain-specific levels in ways that can be characterized by specific mental objects and actions (i.e., both concept and process) that build hierarchically on previous levels (Clements et al., accepted under review).

The following length LT levels (Clements et al., accepted under review) are relevant to the present study.

Length Unit Relater and Repeater (LURR): Children at this level measure by repeating, or iterating, a unit, and understand the relationship between the size and number of units.

Consistent length Measurer (CLM): Children at this level see length as a ratio comparison between a unit and an object. They use equal-length units, understand the zero point on the ruler, and can partition units to make use of units and subunits.

Conceptual Ruler Measurer (CR): Children develop schemes for mentally iterating, partitioning, and unitizing in tandem with a coordinating space and number scheme.

Integrated Conceptual Path Measurer (ICPM): Children incorporate multiple units and collections of units and operate on sub- and super-ordinate units. They have the ability to compensate within a single scale; however, they do not coordinate a series of changes in a systematic way across multiple scales to formulate and justify a valid argument.

Coordinated, Integrated Abstract Measurer with Derived Units (CIAM): At this level, children coordinate multiplicative and additive reasoning in fluent ways and engage in proportional reasoning about repeated or coordinated cases. In addition, they are able to reflect on derived units as an attribute.

METHODOLOGY

The design of the present study was informed by previous work for extending LTs for measurement (Clements et al., accepted under review; Kara, 2013). This organizing methodological structure includes a) posing tasks that reveal children's thinking about a concept outside the LT, b) presenting the tasks to children in the same and adjacent LT levels, c) differentiating children's responses, and d) comparing strategies of children within and across levels to inform extensions to the existing LT.

We focused on five sixth grade children from a public school in the USA. Data were collected over a two-month period as part of a longitudinal teaching experiment (Steffe

& Thompson, 2000). We collected data midway through a four-week unit focused on ratios and proportional reasoning. In the two class periods preceding the data collection, instruction focused on transitioning from using tables of values to double number lines. The following illustrates the teacher's instructional sequence of transitioning from a table of values (Figure 1) to a double number line (Figure 2) and zooming in to find a target value on a double number line (Figure 3).

A 1-liter bottle of cola contains approximately 34 fluid ounces. How many grams of sugar would be in a 1-liter bottle of the cola?

Cola (ounces)	6	12	18	24	30	36
Sugar (grams)	20	40	60	80	100	120

Figure 1: Table Representation

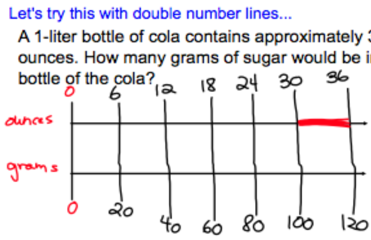

Figure 2: Transition to Double Number Line

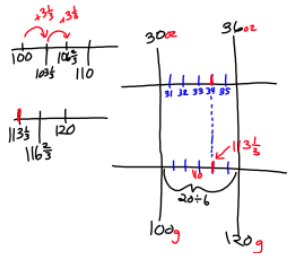

Figure 3: Zooming in to Find a Target Value

The data sources for this report included three 30-minute semi-structured pair interviews and one written assessment. We coded the assessments by LT levels and generated predictions based on these codes. The interviews were videotaped and transcribed. We compared children's responses from the interviews to the predictions to map double number line strategies into the LT.

RESULTS AND DISCUSSION

Predictions Based on the Written Assessment

Based on the written assessment, we identified students at the levels LURR and CLM. On these initial assessment items, two students (Chris and Martha) exhibited LURR level thinking, and three students (Mia, Karen, and Carrie) showed they were operating at least at the CLM level of the length LT. During the set of precursory interviews, Carrie often made use of LURR level strategies; therefore, the research team determined that she was predominately operating at the LURR level. Similar interactions with Chris, Martha, Mia, and Karen provided further evidence that their level placements based on their initial assessments were accurate.

In our prior work, we saw LURR and CLM level thinking predominantly in Grades 2 and 3 (Clements et al., accepted under review). We hypothesize that the Grade 6 students in the present study exhibited LURR and CLM level thinking because the tasks required students to integrate number knowledge and measurement knowledge with ratio reasoning. We think this introduced a level of complexity to the task that might have prompted students to revert back to lower level strategies (Siegler, 1986).

Different LT levels are characterized by specific mental objects and actions (Clements et al., accepted under review); therefore, our research team predicted that students at adjacent levels would use double number lines in different ways. According to the length LT (Clements et al., accepted under review) students at the LURR level measure by repeating, or iterating, a unit; therefore, we expected students at this level to rely on an iterative strategy. Students that are at least the CLM level see a measurement as a ratio between a unit and a length to measure, and can partition units to make use of units and subunits. Hence, we expected to see students who are at least at the CLM level correctly attend to units along one scale, and apply a partitioning strategy. Furthermore, we looked for evidence that they could coordinate units along two scales simultaneously as evidence of concepts and processes of higher LT levels (ICPM or CIAM).

At the beginning of the interview, each student was given the following problem printed on a worksheet:

> While shopping, Kyla found a dress that she would really like, but it costs $52.25 more than she has. Kyla charges $5.50 an hour for babysitting. She wants to figure out how many hours she must babysit to earn $52.25 to buy the dress. Use a double number line to support your answer.
>
> (http://commoncore.org/maps/images/math_documents/G6-M1-Student_Materials_(Eureka_Version).pdf)

The following sections present pairs of students' responses to this task.

LURR Level Pair

Carrie and Martha initially created a table of values, ranging from 1 to 5 for hours and $5.50 to $27.50 for dollars earned (see Figure 4). This suggests that both Carrie and Martha could correctly apply the unit rate of $5.50 per 1 hour to create a table by iteration of units.

Carrie then asked, "Where are we going to?" They settled on a target value of $26.00 as Carrie explained "she wants to buy a dress that's fifty-two dollars and twenty-five cents, so we figured half of fifty-two is twenty-six dollars and so we'd have to find someplace in between twenty-two dollars and twenty-seven is twenty-six and then when we find our answer, then we'll just double our answer because that's half of fifty-two." Carrie and Martha then both drew a double number line, labeling one line as hours and the other as dollars earned. At this point in their solution process, Carrie and Martha were attending to units along only one scale, dollars earned.

When asked what they would do next, Martha explained that they usually make markings in between the tick marks. Carrie said, "Since no numbers are between four and five, we can't put any markings up here (pointing to the hours line)." Martha then said, "so, we'll do this one" (pointing to the dollars line) and told Carrie that they needed to find a number that "goes equally" in the interval between $22.00 and $27.50. Because of Carrie and Martha's discussion of both number lines, the interviewer suspected a transition in their thinking from attending to units along only one scale to coordinating units along two scales simultaneously. Therefore, the interviewer asked how many hours Kyla would need to work to earn the total amount needed for the dress, so the students returned to their tables and extended them as shown in Figure 4.

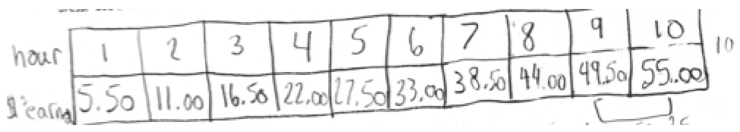

Figure 4: Carrie's Table of Values

The interviewer then asked about the location of $52.25. Carrie explained it was between $55.00 and $49.50. She extended her double number line and created two tick marks on the hours line, and labeled them 9 and 10. Next, Carrie made corresponding tick marks on the dollars line, and labeled them $49.50 and $55.00. Carrie said she would have to make tick marks between these two values. Next, Carrie and Martha applied an iterative strategy. They tried counting by various dollar amounts ($1.00, $1.50, $1.25, $0.50, and finally $0.75). Each time they rejected the value because they could not reach their target value and the $55.00 tick mark. Due to time constraints, the interview ended before Martha and Carrie reached a solution.

LURR and CLM Level Pair

Chris and Karen began solving the problem by creating a table. Using this representation, they were able to correctly apply the unit rate of $5.50 per 1 hour to create a table. When Karen had extended her table beyond 5 hours, she was asked whether she needed to go by one hour or if she could put a 10 in the next box. She explained that she could go from 1 to 5 hours and then double the value for the dollars earned for working 5 hours to get the value for 10 hours. She then subtracted $5.50 to determine the dollar amount that would correspond to 9 hours.

Next, Karen and Chris created a double number line representation to zoom in on a target value (see Figures 5 and 6). Karen then applied a partitioning strategy to this region of the double number line as she drew a tick mark between her tick marks labeled as 9 and 10 on the hours line and connected it to a tick mark on the dollars line. This suggests that, as Karen applied this partitioning strategy, she was able to coordinate units along two scales simultaneously.

Karen then said, "If she worked for 9 hours and 30 minutes, how much will she get?" She labeled the tick mark on the hours line as 9 hours and 30 minutes and recalled that each interval on the dollars line represented $5.50. With computational help from the interviewer, she divided $5.50 by 2 to get $2.75. Next, she asked the interviewer how she could find out the value of the tick mark on the dollars line that corresponded to the tick mark labeled as 9 hours and 30 minutes on the hours line. The interviewer told her it meant that she needed to go $2.75 more than the dollar amount that corresponded to 9 hours, and she added $2.75 to the $49.50 and got $52.25, which she realized was her target value.

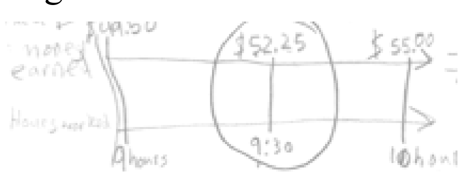

Figure 5: Karen's Partitioning Strategy

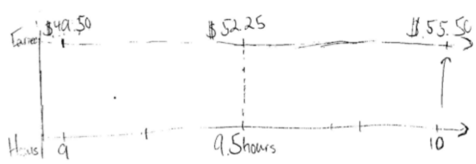

Figure 6: Chris' Partitioning Strategy

Chris followed Karen's partitioning strategy. However, he did not immediately recognize that he had reached the target value, and he continued partitioning the two regions to the left and right of the tick mark labeled as 9 hours and 30 minutes on the hours number line. This suggests that Chris, who had been placed at the LURR level was not able to maintain the coordination of units along two scales simultaneously when applying the partitioning strategy.

CLM Pair

As Mia initially engaged in the task, she drew a double number line and created tick marks on the dollars line with intervals of $5.50 and tick marks on the hours line with intervals of 1 hour. However, she did not maintain even spacing as she drew tick marks along both number lines. This became problematic for her, when she applied the zoom in strategy. She drew a second zoomed in number line, with tick marks labeled as $49.50 and $55.00 on the dollars line. At this point, Mia paused. To prompt her to think about labeling the corresponding tick marks on the hours line, the interviewer asked, "What matches on the bottom of your other number line?" Mia returned to her original number line and labeled more of the tick marks on the dollars line. She then said, "I got a 7" (pointing to the tick mark on the hours line corresponded to the tick mark labeled as $49.50 on the dollars line). Therefore, Mia showed that she could attend to units along one scale.

To help her shift to thinking about coordinating units along two scales simultaneously, the interviewer suggested that Mia draw segments connecting each labeled tick mark on the dollars line to a labeled tick mark on the hours line on her original double number line. The interviewer again asked how many hours corresponded to the value of $49.50. Mia then indicated on her zoomed in number line that the $49.50 tick mark corresponds 9 hours, and the $55.00 tick mark corresponds 10 hours. Next, Mia set out to "find in between of $49.50 and $55.00."

The interviewer then suggested that she show where her target value of $52.25 would be, but Mia said, "I don't know." When asked how much more $55.00 was than $49.50 Mia said, "Five and a half." Next, the interviewer suggested they break this piece of the number line into pieces. Mia initially suggested that they create five pieces. Mia's partner then drew in five tick marks (and later corrected to four) between the tick marks labeled as nine and 10 hours. Mia and her partner labeled the tick marks as nine and one fifth to nine and four fifths. They assigned a value of one fifth of an hour to each interval they created on the hours line; however, they did not apply a partitioning into fifths on the dollars number line. Instead, they reverted back to an iterative strategy, trying to pick a unit that would allow them to span from $49.50 to $55.00 In other words, Mia and her partner could track units (1/5 of an hour) along one scale, but they did not coordinate units along two scales simultaneously. We are not sure if this is because dividing $5.50 is difficult or because they were unable to coordinate.

Mia's partner suggested splitting the interval in half. At first, Mia said she could not split the interval in half because there were five "things." However, when the interviewer asked how much was in the interval from $49.50 to $55.00, Mia said “five and a half.” When the interviewer again asked if she could split it in half, Mia said “yeah,” stating it would be $2.75. Mia explained that the $2.75 represents the halfway point between $49.50 and $55.00. Mia added $2.75 to $49.50 to get $52.25. The interviewer then asked how many hours it would be, and Mia correctly said nine and a half hours. Mia was able to coordinate units along two scales simultaneously with support from the interviewer and only when operating on halves.

CONCLUSIONS AND IMPLICATIONS

Findings suggest a link between length LT level and children’s use of double number lines when solving proportional reasoning tasks. The LURR pair, Carrie and Martha, predominantly relied on an iterative strategy, which is consistent with our prediction. That is, they applied a unit rate by iteration of units to the table representation, and an iterative strategy, of counting by various dollar amounts, to the double line. They also exhibited a lack of understanding of the density of the number line when they noted that there were no numbers between four and five. We conjecture that this is why they did not partition the double number line, which is a CLM level strategy.

The CLM pair, which included Mia, was able to attend to units along one scale and apply a partitioning strategy. However, they could not coordinate units along two scales simultaneously without the interviewer’s expert scaffolding. Chris, who was part of the LURR and CLM pair, followed along with his CLM-level partner’s (Karen’s) partitioning strategy. However, his willingness to continue partitioning the hours line, without checking to see that he had reached the target value on the dollar line, suggests that he was unable to coordinate units along two scales simultaneously. Mia and Chris’ strategies were consistent with our prediction for students at the CLM level of the length LT.

Although not initially placed at the CIAM level, Karen exhibited concepts and processes consistent with this level as she engaged with the double number line representation. For example, she applied a partitioning strategy while maintaining the coordination of units along two scales simultaneously without prompting or support from the interviewer. We take this as evidence that Karen may be operating higher than the CLM level of the length LT. In particular, we think Karen’s simultaneous coordination of units along two scales exemplifies proportional reasoning about repeated or coordinated cases, which is consistent with the CIAM level of the length LT. Although we did not see her exhibit a reflection on a derived unit as an attribute, we conjecture that the task did not require this reflection.

Parallel to prior research, this study established the importance of an understanding partitioning and coordinating units (Orrill & Brown, 2012) for understanding the double number line representation. However, in the present study we established a link between the levels of an LT for length measurement and students’ ability to use the double number line representation when solving proportional reasoning tasks. In particular, our prediction that students at the LURR level would rely on iterative strategies, and children at the CLM level would partition and correctly attend to units along one scale, but not yet coordinate units along two scales simultaneously were correct. Future research is needed to explore ways to support children at LURR and CLM levels in developing these concepts and processes.

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VARIABLE PARTS: A NEW PERSPECTIVE ON PROPORTIONAL RELATIONSHIPS AND LINEAR FUNCTIONS

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We present a mathematical analysis that distinguishes two quantitative perspectives on ratios and proportional relationships: Multiple Batches and Variable Parts. We argue that (a) existing research on proportional relationships has addressed Multiple Batches but has largely overlooked Variable Parts, (b) Multiple Batches makes the co-variation aspect of proportional relationships more explicit, while Variable Parts makes the fixed multiplicative relationship between two quantities more explicit, (c) the distinction between Multiple Batches and Variable Parts is orthogonal to the within-measure-space versus between-measure-space ratio distinction, and (d) Variable Parts affords promising new approaches for addressing linear relationships.

PAST RESEARCH ON PROPORTIONAL RELATIONSHIPS

Ratios and proportional relationships are critical mathematics in elementary and secondary grades (e.g., Kilpatrick, Swafford, & Findell, 2001; National Council of Teachers of Mathematics, 1989, 2000). Although traditional instruction has emphasized applying rote procedures like cross multiplication to solve missing-value and comparison problems, *a robust understanding of proportional relationships involves (a) attending to co-variation between two quantities and (b) forming multiplicative relationships between those quantities*. Despite a significant body of empirical and theoretical research on proportional relationships, understanding how to support students' and teachers' understandings of *both* aspects of proportional relationships remains a significant challenge for the field.

Empirical research has documented numerous difficulties that students, and sometimes teachers, experience with proportional relationships. One line of research has analyzed factors that influence the difficulty of proportion problems for students—including whether students are familiar with problem contexts (e.g., Tourniaire, 1986), whether quantities are discrete or continuous (e.g., Behr, Lesh, Post, & Silver, 1983), and whether ratios are integral, nonintegral, or unit ratios (e.g., Hart, 1981, 1988; Karplus, Pulos, & Stage, 1983; Noelting, 1980a, 1980b). A second line of research has examined students' and teachers' capacities to distinguish missing-value problems that describe proportional relationships from ones that do not (e.g., Cramer, Post, & Currier, 1993; Fisher, 1988; Freudenthal, 1983; Van Dooren, De Bock, Vleugels, & Verschaffel, 2010.) A third line of research has examined difficulties that students and teachers have conceiving of a ratio as a measure of a physical attribute, such as steepness or speed (Simon & Blume, 1994; Thompson & Thompson, 1994). A fourth line of research has examined strategies that students use to solve problems about

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proportions successfully, often before any substantial instruction in these topics. These include forming progressively elaborate unit structures (e.g., Lamon 1993a, 1994; Lobato & Ellis, 2010) and double counting strategies (e.g., Hart 1981, 1988; Lamon, 1993b).

Theoretical research has identified various ways to think about multiplicative relationships in terms of quantities (see Greer, 1992, for a review). There is widespread agreement among mathematics education researchers that ratios and proportional relationships are part of the *multiplicative conceptual field*—a web of interrelated ideas that includes multiplication and division, fractions, linear functions, and more (Vergnaud, 1983, 1988). Furthermore, much of the theoretical work on proportional relationships has been informed by Vergnaud's (1983) identification of isomorphism of measures as one of three fundamental multiplicative structures. Isomorphism of measures covers direct proportions between two measure spaces, and Vergnaud distinguished forming multiplicative relationships within measure spaces from forming such relationships between measure spaces (e.g., Freudenthal, 1973; Lamon, 2007; Noelting, 1980b).

We present an analysis that contributes to the theory of proportional relationships, identifying an overlooked perspective that promises new avenues for reasoning about proportional relationships and foundations for understanding slope and rate of change, among other subsequent topics.

THE TWO PERSPECTIVES ON PROPORTIONAL RELATIONSHIPS

Beckmann and Izsák (2013) identified two distinct, complementary perspectives on how quantities vary together in a proportional relationship. The two perspectives follow from consistently distinguishing the multiplier, M, from the multiplicand, N, in the equation $M \bullet N = P$ (M denotes number of groups, N denotes the number of units in each/whole group, and P denotes the number of units in M groups).

Figure 1 uses ***Punch Problem 1*** to illustrate the two perspectives, which conceptualize and depict covariation and fixed multiplicative relationships in complementary ways. *Multiple Batches* has been widely studied among children—for instance, Lamon, (1993b) and Lobato and Ellis (2010) have referred to it as composed unit reasoning. In this perspective (Figure 1a), a mixture of 3 cups peach juice and 2 cups grape juice is fixed as 1 batch. One *varies the number of batches* to produce different amounts in the same ratio, which corresponds to operating on M. Vertical alignment on the double number line indicates amounts in the same 3-to-2 ratio. Covariation is made visually explicit as movement of that vertical alignment up and down the double number line, but the fixed multiplicative relationship between the quantities—the amount of peach juice is always 3/2 times the amount of grape juice—remains implicit. *Variable Parts* has been largely overlooked in past research and teaching on proportional relationships. In this perspective (Figure 1b), one fixes numbers of parts of peach juice (3) and grape juice (2), and all parts are the same size. One *varies the size of the parts* to produce different amounts in the same ratio (throughout, any one part remains equal to all the other parts), which corresponds to operating on *N*. The numbers of parts show explicitly that the amount of peach juice is always 3/2 times the amount of grape juice, but variation within parts remains implicit.

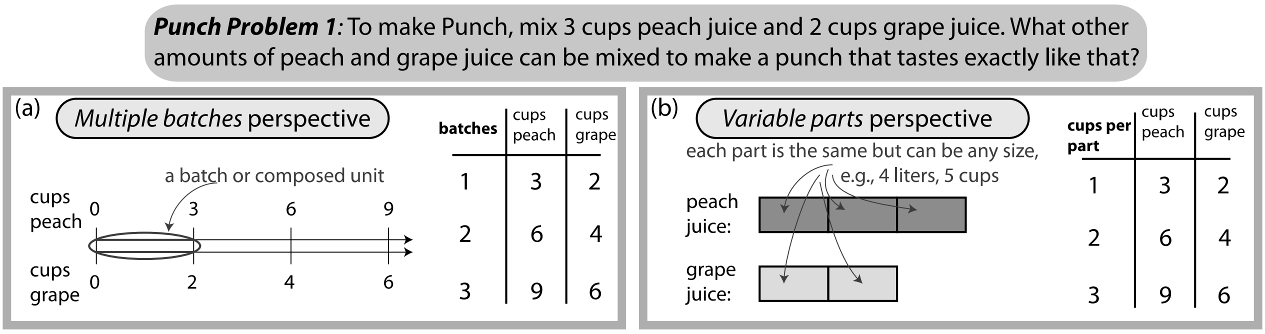

Figure 1: (a) The multiple batches perspective. (b) The variable parts perspective.

The two perspectives on proportional relationships are orthogonal to the within-measure-space versus between-measure-space ratio distinction mentioned previously (Vergnaud, 1983): One can use each perspective to relate quantities within measure spaces or between measure spaces. To illustrate within-measure-space reasoning from the two perspectives, consider the following problem that continues to use the punch scenario. *You made a mixture of 3 cups peach juice and 2 cups grape juice. Now you want to make a mixture in the same ratio using 1/4 as much peach juice. How much grape juice should you use?* Using Multiple Batches, one might view the 1/4 as operating on 1 batch and therefore reason that 1/4 should operate on the cups of peach and grape juice (1/4 batch•3 cups of peach juice in each batch; 1/4 batch•2 cups grape juice in each batch). In this case, multiplying by 1/4 *changes the number of batches* (*M*). Using Variable Parts, one might start with 1 cup of juice in each of the 5 parts and view 1/4 as operating on the size of each part. Here, one needs 3 parts peach juice•1/4 cup in each part and 2 parts grape juice•1/4 cup in each part. In this case, multiplying by 1/4 *changes the size of all 5 parts* (*N*). Beckmann and Izsák (2013) explain how both Multiple Batches and Variable Parts can support between-measure-space reasoning.

Research on proportional relationships has emphasized Multiple Batches, which facilitates within-measure-space reasoning using a variety of strategies, including a “building up” strategy and iterating and partitioning a composed unit (e.g., Kaput & West, 1994; Lamon, 1994, 2007; Lobato & Ellis, 2010; Vergnaud, 1988). Although Lobato and Ellis showed how iterating and partitioning a composed unit can be used to derive a fixed multiplicative relationship between measure spaces, Kaput and West noted that some multiplicative relationships are not well handled by iterating and partitioning within measure spaces:

> A major question not addressed in this chapter is how to deal with multiplicative change situations that are not well modeled [sic] by build-up patterns, change situations that are not inherently replicative. These include the geometric similarity problems handled poorly by our students. The larger, rescaled figure is not the join of several smaller ones. Rather,

> each of the infinitely subdivisible parts of the smaller figure "grows" by the same amount to produce the larger as discussed by Confrey (this volume). This form of multiplicative growth likely has different primitive conceptual roots and is likely to require a different curriculum strand and different types of concrete representations. (p. 284)

We hypothesize that Variable Parts and strip diagrams can provide the needed complementary perspective on multiplicative relationships. In particular, in the next section, we argue that adding Variable Parts to the study of proportional relationships may provide a more robust foundation for the study of linear functions than Multiple Batches alone. Thus, Variable Parts deserves consideration in research on cognition around proportional relationships.

TWO PERSPECTIVES AS A FOUNDATION FOR LINEAR FUNCTIONS

An important theme in the extensive literature on students' and, to a lesser extent, teachers' understandings of algebra is the role of prior experience with arithmetic, including with rational numbers, in supporting and constraining reasoning about linear relationships (e.g., Carraher & Schliemann, 2007; Hackenberg, 2010, 2013; Kieran, 1992). For instance, Kieran (p. 394) argued one source of difficulty is that using algebraic notation to model problem situations requires students to modify their interpretations of symbols like the equal sign and to use arithmetic operations that invert operations they have learned to use almost automatically, while Hackenberg has argued that experience reasoning with fractions in terms of quantities provides a critical foundation for interpreting equations that relate quantities. We focus on the persistent challenge of forming fixed multiplicative relationships between quantities, including slope.

Confusion about meanings of slope, rate of change, and steepness have been found among students using either reform or more traditional curricula (Lobato, Ellis, & Munoz, 2003; Teuscher, Reys, Evitts, & Heinz, 2010), as well as among future teachers (e.g., Simon & Blume, 1994). As one example, Lobato et al. (2003) reported on U.S. high school students' understandings of slope after instruction using a reform curriculum that emphasized slope as a rate of change between covarying quantities in multiple real-world settings and that used multiple representations. The researchers' reported examples of students' persistent difficulties understanding slope as a multiplicative relationship between changes in values of x and y, even when students reasoned about partitioning and iterating Multiple Batches. Such results raise as a question whether other perspectives on covarying quantities might better support appropriate multiplicative conceptualizations of slope (or constants of proportionality). Next, we return to the ***Punch Problem 1*** (Figure 1) and examine how Multiple Batches and Variable Parts can support such conceptualizations.

In a Multiple Batches approach to slope, one thinks of having 3 cups peach juice for every 2 cups grape juice. The value 3/2 specifies the number of cups of peach juice needed for every 1 cup of grape juice (a unit rate) (Figure 2a). This view foregrounds slope as the coordinated variation within the grape juice and peach juice measure spaces: For every new cup of grape juice, the amount of peach juice increases by 3/2 cups. This perspective evokes repeatedly moving to the right 1 unit and up 3/2 units, but the general multiplicative relationship, $y = (3/2)\,x$, is less evident. In a Variable Parts approach to slope, the value 3/2 is a direct multiplicative comparison between the numbers of parts of grape and peach juice: The number of parts peach juice is 3/2 the number of parts grape juice (Figure 2b). Put another way, the value 3/2 is the factor that multiplies the number of parts of grape juice to produce the number of parts of peach juice, regardless of amounts. Figure 2b shows how strip diagrams can be coordinated with Cartesian graphs to support such an interpretation of slope. This view foregrounds slope as a multiplicative relationship: The y-coordinate is 3/2 of the x-coordinate, so $y = (3/2)\,x$.

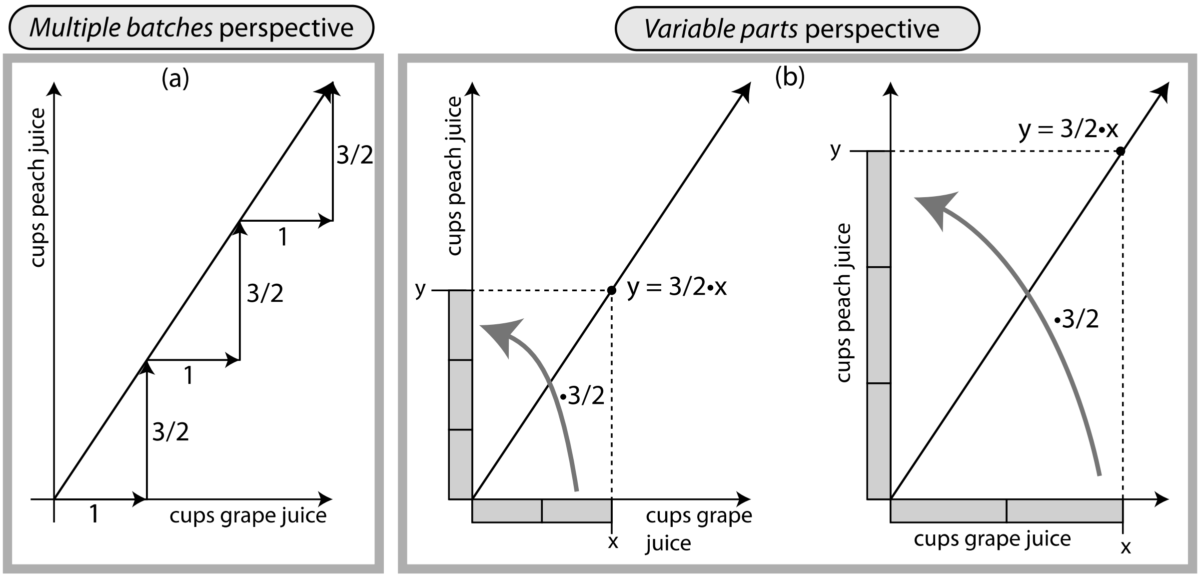

Figure 2: Two perspectives on slope. (a) Multiple batches. (b) Variable parts.

CONCLUSION AND DISCUSSION

An important question for future empirical research is whether introducing Variable Parts as a complementary perspective to Multiple Batches might help both students and teachers develop both key features of proportional relationships between two quantities and help them apply what they learn about proportional relationships to subsequent, central topics, such as slope. Our presentation of the two perspectives on ratios and proportional relationships suggests that adding a Variable Parts perspective may benefit students and teachers.

One benefit is that Variable Parts may support forming direct multiplicative comparisons of two quantities. Past research has shown that children and adults can have difficulty making such comparisons when using Multiple Batches (e.g., Vergnaud, 1980; Schliemann & Nunes, 1990).

A second benefit is that Variable Parts may support understanding not just slope and rate of change as multiplicative relationships but also equations that relate variables.

Numerous studies have demonstrated students' difficulties forming equations (e.g., Clement, 1982; Koedinger & Nathan, 2004). The process of deriving equations from strip diagrams by relating quantities may highlight a relational rather than computational interpretation of the equal sign (e.g., Kieran 1992) and support generating linear equations. Investigating this possibility would be consistent with Kieran's (2007) recommendation for additional research on how students could be assisted to make connections between verbal problem solving activity and generating equations (p. 729).

Finally, an important question for future research is how students and teachers might develop understandings of the two perspectives. It might be that Multiple Batches better supports initial coordination of two varying quantities but that Variable Parts better supports subsequent applications, such as applications to linear functions. Furthermore, it might that students and teachers understandings of the two perspectives could support one another: Seeing covariation explicitly in Multiple Batches might support seeing covariation in Variable Parts and seeing multiplicative comparisons explicitly in Variable Parts might support seeing such comparisons in Multiple Batches. Thus, in combination, the two perspectives on proportional relationships are promising for supporting students' understandings of a central mathematical domain. Therefore, the two perspectives deserve further investigation.

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THE ROLE OF PICTURES IN READING MATHEMATICAL PROOFS: AN EYE MOVEMENT STUDY

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To support university students' understanding of mathematical proofs, pictures accompanying text are frequently used in textbooks as well as in lectures. However, it is unclear if such pictures influence the individual's reading behaviour. By recording the eye movements of eight mathematicians, we investigated whether and how adults with high expertise in mathematics pay attention to additional pictures when reading a written mathematical proof. We found that all participants paid attention to the pictures. As expected, in two out of three items, the text was fixated upon significantly longer than the picture. The data suggest that the participants tried to integrate information from text and picture by alternating between these representations.

THEORETICAL FRAMEWORK

The transition from secondary school to university mathematics and the first semesters of studies in mathematics are considered challenging for many students. Mathematics at the university level makes use of axioms, definitions and theorems that are not easy to understand for novices. In particular, mathematical proof is a central obstacle for students of mathematics. In order to improve students' understanding, pictorial information might be a useful supplement to the written text. Accordingly, the present study focuses on the role of pictures in reading mathematical proofs.

Mathematical proof

Mathematics is a science of proof (Hilbert, Renkl, Kessler, & Reiss, 2008). The deductive structure of (university) mathematics demands dealing with mathematical proofs. In a mathematics lecture, the lecturer typically writes a sequence of definitions, theorems and proofs on a blackboard. Similarly, textbooks typically provide such a definition-theorem-proof structure as well. Accordingly, reading and comprehending proofs is a central activity of studying mathematics (Mejia-Ramos & Inglis, 2009). Dealing with mathematics in this way at the university level differs greatly from the secondary school level, where mathematics is typically presented in a much more concrete and for the most part non-deductive way. For this reason, many students struggle especially with the working with proofs.

In spite of its high relevance for university mathematics, there is only little research on how individuals read proofs (Mejia-Ramos & Inglis, 2009). Inglis and Alcock (2012) asked first-year undergraduate students and academic mathematicians to evaluate mathematical proofs on a computer screen while their eye movements were recorded. The results revealed that the students spent proportionately more time on the formulas

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(compared with the non-formula parts of the proof) than did the mathematicians. Furthermore, the mathematicians shifted their attention back and forth between the lines of the proof more often than the students, suggesting that the mathematicians spent more effort on searching for between-line warrants than the students.

While the study by Inglis and Alcock (2012) investigated how experts and learners read proofs to *evaluate* the proof, there is almost no existing research on how individuals read proofs to *comprehend* them (Mejia-Ramos & Inglis, 2009). Reading proofs for comprehension plays the dominant role in early university studies, and is thus the topic of our study. As a first step, we involved adults with high expertise in mathematics in the current study. In a next step we plan to examine the behaviour of novices (that is students at the beginning of their studies) and to compare the findings of these two groups. This way we want to determine ideal reading strategies to adapt our teaching to student needs.

The combination of text and pictures

In university lectures as well as in textbooks, written mathematical proofs are frequently accompanied by pictures to visualize the main information provided in the text with the intention of facilitating the learning process. Cognitive psychological theories on multimedia learning (e.g. Mayer, 2001; Schnotz, 2005) support this idea and there is empirical evidence that students generally learn better from a combination of text and pictures than from text alone (for a review see e.g. Levie & Lentz, 1982).

The combination of text and pictures seems to be particularly beneficial if the representations are semantically related to each other or presented closely together (Schnotz, 2005). However, there is also evidence that this effect occurs only under specific conditions. For instance, according to Schnotz (2005) pictures can be not beneficial when they visualize the text in a task-inappropriate way, as the form of visualization influences the structure of the mental model which is built from the picture.

Yet, in the field of mathematics, there is little empirically based research about the effect of the combination of text and pictures. In a recent study by Dewolf, Van Dooren, Hermens, and Verschaffel (2013), pictures had seemingly no effect *at all* on higher-education students' behaviour when solving mathematical word problems. The authors showed the students word problems on a computer screen and recorded their eye movements. In the experimental group, the text of every task was accompanied with a picture. The students' answers to the word problems did not differ between the groups with or without the pictures. One possible reason for that was that the students barely looked at the pictures. Only around 1% of all fixations were on the area where the illustrations were presented. In view of these results, it is no matter of course that individuals pay any attention to the pictures presented as part of a proof.

RESEARCH QUESTIONS AND HYPOTHESES

The aim of our study was to find out whether and how experts look at a picture given with a mathematical proof while reading the proof to comprehend it. Following the findings of Schnotz (2005) mentioned above, we used pictures which visualize part of the information given in the text and complete the text without providing any other information than given in the text, so that the text and the picture are semantically related to each other. Such pictures have been referred to as "representational pictures" (Elia & Philippou, 2004).

Measuring individuals' reading behaviour is a methodological challenge. One way is to show the items to participants and to ask them afterwards if they looked at the picture (retrospective reporting). Another way is to ask the participants to think aloud while working on the items (concurrent reporting). A drawback of both methods is, however, that they are highly subjective and do not reliably assess the actual behaviour.

A more objective way to examine reading behaviour is eye tracking. Eye tracking is a technique with which the eye movements of a person, consisting of fixations and saccades, can be made visible. A fixation is the status when the eyes remain still (for example on a word during reading). A saccade is the very fast movement between two fixations where no information is perceived. The underlying idea of analysing eye movement data is that people are mainly processing the information they are looking at (Just & Carpenter, 1980). Although this assumption may not hold true in general, it is arguably reasonable to use eye fixations as a proxy for information processing during reading.

The specific research questions and hypotheses of this study were: 1) Do academic mathematicians look at the representational picture provided along with a written proof at all? 2) If so, how long do they fixate the picture compared to the text? We assumed that the participants would indeed look at the picture (hypothesis 1), because this would help them understand the information provided in the text (see Schnotz, 2005). We further expected that the participants would fixate the text longer than the picture (hypothesis 2), because although the picture would help them understand the text, it was not essential to understand it and it did not reflect the whole content of the text. 3) Furthermore, we were interested if the participants look at the text and the picture in a specific sequence. We assumed that the participants would alternate between text and picture (hypothesis 3), in order to integrate information from the text and the picture. This behaviour was shown by experts in a study by Inglis and Alcock (2012). Here, the experts tried to integrate the information given in consecutive lines.

METHODOLOGY

The participants were six staff members of a German university who had an academic degree in mathematics and two university students majoring in mathematics. The mean age of these eight participants (five female) was 26 years (*SD* = 3.9).

The participants sat in front of a computer screen, which was connected to a binocular remote contact free eye tracking device (SensoMotoric Instruments) with a sampling rate of 500 Hz. The eye tracking device was placed underneath the screen. The participants were asked to avoid head and body movements as far as possible, so that their eye movements could be recorded reliably. First, calibration was performed through fixations of nine small dots on the screen. After that, the participants were instructed that they should try to comprehend the information provided by the items shown on the screen, so that they would be able to answer subsequent questions on the content. They were also informed that there was a time limit of five minutes per item, but that they could proceed earlier by pressing the space bar. In fact, on average the participants spent only 2.5 min on each item.

Then the experiment started and the participants saw the first out of three items. After reading the item, they had to answer two multiple-choice-questions related to the content of the given item by clicking on the correct answer on the screen. The same procedure occurred for the second and third item.

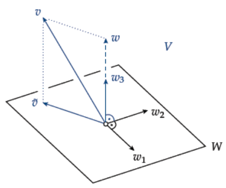
The three items were chosen from German mathematical textbooks that are commonly used for undergraduates. Thus, the items represented highly valid learning materials for students. Every item consisted of a theorem, its proof and a representational picture (see figure 1). The picture was arranged between the text and visualized the written proof without presenting any other information than the text. The contents were selected so that the participants were expected to understand them easily, but did at the same time not include mathematical knowledge that is typically learned by heart.

Orthonormalisierungs-Satz von GRAM und SCHMIDT *Gegeben sei ein euklidischer bzw. unitärer Vektorraum V mit $\dim V < \infty$, und ein Untervektorraum $W \subset V$ mit einer Orthonormalbasis $(w_1, \ldots, w_m)$. Dann gibt es eine Ergänzung zu einer Orthonormalbasis*

$$(w_1, \ldots, w_m, w_{m+1}, \ldots, w_n) \quad \textit{von} \quad V.$$

Beweis des Orthonormalisierungs-Satzes Ist $W = V$, so ist nichts mehr zu ergänzen. Andernfalls führen wird die Ergänzung schrittweise durch. Ist $\mathbf{o} \neq v \in V \setminus W$, so betrachten wir die senkrechte Projektion von v auf W:

$$\tilde{v} := \langle v, w_1 \rangle \cdot w_1 + \ldots + \langle v, w_m \rangle \cdot w_m \in W.$$



Dann steht $w := v - \tilde{v}$ senkrecht auf W:

$$\langle w, w_i \rangle = \langle v - \tilde{v}, w_i \rangle = \langle v, w_i \rangle - \langle \tilde{v}, w_i \rangle = \langle v, w_i \rangle - \langle v, w_i \rangle = 0.$$

Um den nächsten Vektor w_{m+1} für die Orthonormalbasis zu erhalten, setzt man

$$w_{m+1} := \frac{1}{\|w\|} \cdot w.$$

Durch Fortsetzung dieses Verfahrens erhält man eine Orthonormalbasis $(w_1, \ldots, w_n)$. ■

Figure 1: Sample item containing a theorem (shaded paragraph) and its proof, including a representational picture.

RESULTS

Data from one participant had to be excluded from the analysis due to low calibration quality. To analyse the eye movements, we defined three areas of interest (AOIs) for each item. These were fitted around the text above the picture, the picture itself and the text below the picture. As we were only interested in the proof part of the items, the gazes on the theorem itself were not considered. In the following, the values for the text above and below the picture are summarized as values for “text”.

As can be seen from table 1, the fixation times for the pictures were always larger than zero, which is in line with hypothesis 1 and indicates that the participants paid attention to the picture.

To compare fixation times on text and picture, we decided to divide the fixation times (in ms) for the AOI “text” and the AOI “picture” by the size (in pixel; px) of the respective AOI to account for the different areas of the AOIs (AOI sizes: $text_{item\_1}$ = 190498 px, $picture_{item\_1}$ = 78001 px; $text_{item\_2}$ = 158096 px, $picture_{item\_2}$ = 66674 px; $text_{item\_3}$ = 204824 px, $picture_{item\_3}$ = 68214 px). Table 1 displays the fixation times per pixel for each participant and the two AOIs of each item.

	Item_1		Item_2		Item_3	
Participants	Text ms | ms/px	Picture ms | ms/px	Text ms | ms/px	Picture ms | ms/px	Text ms | ms/px	Picture ms | ms/px
P01	120133 | .63	86807 | 1.11	86144 | .54	30660 | .46	141536 | .69	15518 | .23
P02	83385 | .44	49659 | .64	42013 | .27	9786 | .15	55931 | .27	5363 | .08
P03	58428 | .31	48133 | .62	26885 | .17	11104 | .17	43197 | .21	15824 | .23
P04	71020 | .37	16888 | .22	62035 | .39	21261 | .32	67596 | .33	20402 | .30
P06	53572 | .28	28619 | .37	38102 | .24	16449 | .25	63172 | .31	9110 | .13
P07	64469 | .34	23324 | .30	46359 | .29	10862 | .16	71429 | .35	8502 | .12
P08	19869 | .10	60 | .00	14148 | .09	701 | .01	79554 | .39	7797 | .11
M (SD)	67268 (30520) | .35 (.16)	36213 (28246) | .46 (.36)	45098 (23528) | .29 (.15)	14403 (9557) | .22 (.14)	74631 (31700) | .36 (.15)	11788 (5471) | .17 (.08)

Table 1: Fixation times in ms and ms/px for each participant and the two AOIs of each item. Note: M = mean, SD = standard deviation.

We conducted a paired t-test for each item to compare the fixation times of text and picture. There was no significant difference between the fixation times of text and picture in item 1, $t(6)_{item\_1} = -1.26$, $p = .25$. For item 2 and item 3 there were significant differences, $t(6)_{item\_2} = 3.49$, $p = .013$, $t(6)_{item\_3} = 3.17$, $p = .019$. In both cases, the text was fixated upon longer than the picture. Even if there were comparatively large inter-individual differences in the fixation times, this tendency is found within almost all participants and items (see Table 1). This result supports hypothesis 2 for items 2 and 3, but not for item 1.

To illustrate the fixation times, Figure 2 shows the heat map for fixation times of item 2. From blue to green via yellow to red a heat map shows the least and most fixated areas. As can be seen from this figure, participants looked at the text as well as at the picture, but focused longer on the text than on the picture.

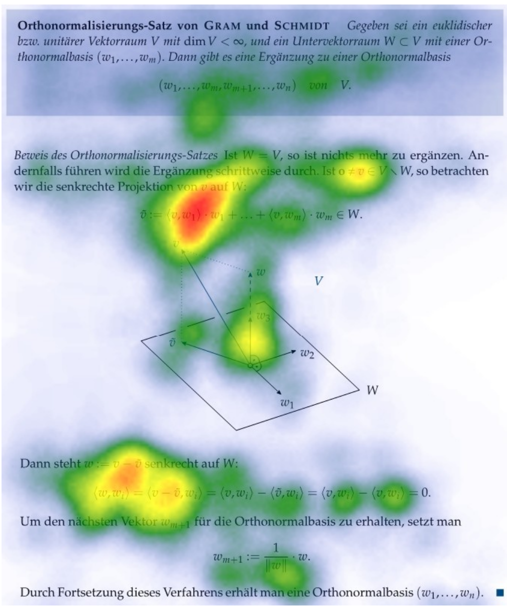


Figure 2: Heat map for fixation times of item 2.

To answer the question if participants looked at the text and picture in a specific order, we analysed the sequence charts which show the order and the duration of fixations of both AOIs for each participant. Figure 3 exemplarily shows the sequence chart for item 2. The gaps result for example from the gaze to the theorem or to regions of the page where no AOIs were defined or from the loss of the eye contact.

Figure 3: Sequence chart for item 2;
fixations of text are coloured green, fixations of the picture red.

The sequence chart for item 2 shows that every participant switched back and forth between the text and the picture frequently. The only exception was participant P08 who merely had short glances at the picture at the end. The sequence charts of item 1 and item 3 looked similar. All in all we can state that the participants alternated between the text and the picture in each item, which supports hypothesis 3.

DISCUSSION

The aim of our study was to find out whether and how academic mathematicians look at a representational picture given with a mathematical proof while reading the proof in order to understand it. We recorded eye movements of eight participants with high expertise in mathematics and analysed fixation times on text and picture, as well as the sequence charts.

We found that all participants paid attention at the pictorial information. This is not in line with the study by Dewolf et al. (2013), who found that students barely looked at pictures presented with word problems, no matter if these pictures were representational or only decorative. One reason for that might be that the picture in our study was positioned in the middle of the text so that it was unlikely to overlook it. However, as the participants switched back and forth between the text and the picture during reading the proof, it is not likely that they looked at the picture just because of its position.

As expected, in two out of three items the relative fixation times for the text were significantly longer than for the picture. As the picture visualized the written proof without presenting any other information than the text, looking at the picture was not essential for understanding the proof. Furthermore, most of the participants were academic mathematicians who were certainly familiar with proofs in general and with the presented topics (chosen from undergraduates' textbooks) in particular. This could explain why shorter fixations at the picture might have been enough to comprehend the proof. For the first item, there was no difference in the fixation times of text and picture. One reason for this unexpected result could be that only after working on the first item, the participants learned that the subsequent questions would not explicitly refer to the picture, so that they payed less attention to the picture in the following two items.

Furthermore we could show that the participants alternated between the text and the picture during reading the proof, which suggests that they tried to integrate the information given in the text and in the picture. This is plausible as the text and the picture were semantically related. It might also be an indicator of mathematical expertise (see Inglis & Alcock, 2012). An interesting question for a follow-up study will be to see if individuals who are not familiar with reading proofs, such as first-year students of mathematics, show the same fixation pattern.

We used eye movements as a relatively new method to assess mathematical tasks. We could show that this method is feasible to analyse whether and how participants look at a picture while reading a proof in order to understand it. Based on eye movement data, we could draw reliable conclusions, which would not have been possible through verbal reports (retrospective or concurrent).

A limitation of the study is the small number of participants, which restricts the generalizability of our findings. Certainly, further studies with a larger sample size are necessary to replicate the present results. Moreover, we aim to examine the reading behaviour of novices (that is students in their first year at university) and compare these data to our present findings, to trace students' problems. On the long run, these studies could help developing learning materials tailored to student needs when learning university mathematics.

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THE INFLUENCE OF TEACHER-TRAININGS ON IN-SERVICE TEACHERS' EXPERTISE: A TEACHER-TRAINING-STUDY ON FORMATIVE ASSESSMENT IN COMPETENCY-ORIENTED MATHEMATICS

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The research-project Co²CA investigates the influence of teacher-trainings on in-service teachers' expertise. Within a teacher-training-study 27 in-service teachers have been trained in selected ideas about teaching, having an exemplary focus on central aspects of formative assessment in competency-oriented mathematics. At the end of the teacher-trainings the teachers' pedagogical content knowledge as crucial aspect of teachers' expertise has been evaluated by using tests being sensitive to the teacher-trainings. Results of this evaluation point out: Within teacher-trainings pedagogical content knowledge can be conveyed to in-service teachers successfully.

INTRODUCTION

As part of fundamental debates about effective teaching in general and successful competency-oriented teaching of mathematics in detail the importance of (mathematics) teachers' expertise for students' learning has been pointed out by several studies within the last decade (Ball, Hill & Bass, 2005; Baumert et al., 2010). Next to open questions about how to train pre-service teachers to build up expertise it's furthermore not clear by now how in-service teachers can best be supported building up their expertise. Therefore the interdisciplinary research-project Co²CA [1] investigates the influence of teacher-trainings on teachers' expertise, focusing on selected ideas of teachers' pedagogical content knowledge about assessing and reporting students' performances if dealing with modelling tasks. Within this article, (1) an overview about theoretical and empirical discussions about the role of teachers' knowledge as part of teachers' expertise is given. Furthermore (2) some central ideas about mathematical modelling as part of competency-oriented mathematics as well as (3) empirical findings on assessing and reporting students' performances are pointed out. Based on these considerations (4) a teacher-training-study as part of the research-project Co²CA is presented: research-question, design and test-instruments are illustrated as well as results pointing out the effectiveness of teacher-trainings for building up in-service teachers' expertise.

[1] *Conditions and Consequences of Classroom Assessment*. Research project supported by the German Research Society (DFG); principal researchers: E. Klieme, K. Rakoczy (both Frankfurt), W. Blum (Kassel), D. Leiss (Lueneburg).

TEACHERS' KNOWLEDGE AS CORE DIMENSION FOR THE QUALITY OF TEACHING AND STUDENTS' LEARNING

Nearly the whole 20th century researchers have tried to explain students' learning by investigating the teacher's role for the quality of teaching – once by describing the teachers' personality, once by analysing learning processes and products in classroom, once by assessing teachers' expertise. Especially since the work of Shulman (1986) the idea of harking back to teachers' content knowledge (CK), teachers' pedagogical content knowledge (PCK) and teachers' general pedagogical knowledge (PK) as central aspects of teachers' expertise to explain the quality of teaching and of students' learning is a crucial one. These "three core dimensions of teacher knowledge" (Baumert et al., 2010, p. 135) help to understand the teachers' role in the classroom. By assessing, describing and analysing teachers' CK, PCK and PK, several studies resort to these dimensions of teachers' expertise – and with a special focus on mathematics teachers the COACTIV-project (Baumert et al., 2010), the Michigan Group (Ball, Hill & Bass, 2005) and the TEDS-project (Döhrmann, Kaiser & Blömeke, 2012) point out the importance of CK, PCK and PK for the quality of teaching and for students' learning. And having especially a closer look at teachers' pedagogical content knowledge, Baumert (2010) stresses:

> PCK – the area of knowledge relating specifically to the main activity of teachers, namely, communicating subject matter to students – makes the greatest contribution to explaining student progress. This knowledge cannot be picked up incidentally, but as our finding on different teacher-training programs show, it can be acquired in structured learning environments. One of the next great challenges for teacher research will be to determine how this knowledge can best be conveyed to both preservice and inservice teachers. (Baumert et al., 2010, p. 168)

MATHEMATICAL MODELLING AS ONE ASPECT OF COMPETENCY-ORIENTED MATHEMATICS

Based on general ideas about competency-oriented mathematics (Niss, 2003) several countries implemented national standards for the teaching and learning of competency-oriented mathematics within the last years (see besides others: NCTM, 2000). One of the main ideas of these standards is to not only telling teachers any longer which mathematical content should be taught and learnt at school but to describe which mathematical competencies students should possess at the end of a course. Besides other competencies (e. g. problem solving, reasoning, communicating – see Blomhoj & Jensen, 2007), mathematical modelling is one of these competencies students should acquire if dealing with mathematical topics at school. The main idea of being able to do mathematical modelling is: One should not only be able to solve pure mathematical problems but to work on (complex) real world problems which can be solved by using mathematics. In detail, the competence of mathematical modelling includes (see also Blum, 2011; Maaß, 2010):

- Being able to understand, structure and simplify a complex real world problem and being able to transfer the reduced real world problem into a so called mathematical problem which can be worked on mathematically.
- Being able to work on the mathematical problem mathematically, to interpret and validate the mathematical result by transferring it back to reality and finally being able to give an answer to the initial, proper real world problem.

ASSESSING AND REPORTING STUDENTS' PERFORMANCES WITHOUT GIVING MARKS – THE IDEA OF FORMATIVE ASSESSMENT

Next to fundamental discussions about competency-oriented mathematics the question of how to assess and report students' performances to support students' learning as good as possible is a central question of improving the quality of teaching in general and the quality of teaching mathematics in detail. While in school students' performances are quite often assessed only once at the end of a course and the students is given a mark summarizing their performances which "does not normally have immediate impact in learning" (Sadler, 1989, p. 120), theoretical and empirical studies hint at the importance of a more formative assessment at school (Baker, 2007; Black & William, 2009; Hattie, 2008; Shepard, 2000): Students' performances should be assessed in short intervals and more than once during learning processes, diagnoses of students' performances should immediately be used to support students' learning. As a central element of such a formative assessment, feedback should be given to the students whenever assessing performances which mainly answers three questions: Where am I going? How am I going? and Where to next?" (Hattie & Timperley, 2007, p. 88). Furthermore meta-analyses point out the following ideas of how feedback "with which a learner can confirm, add to, overwrite, tune, or restructure information in memory" (Butler & Wine, 1995, p. 275) as part of formative assessment should look like to support students' progress as good as possible:

- Kluger & DeNisi (1996) stress that feedback should first of all be close to the tasks students are working on: "effects on performance are augmented by (a) cues that direct attention to task-motivation processes and (b) cues that direct attention to task-learning processes" (Kluger & DeNisi, 1996, p. 268).
- Deci, Koestner & Ryan (1999) emphasize that feedback should inform students concerning their learning processes without any kind of pressure. Furthermore the information provided to the students should not only tell the student whether he is right or wrong but offering additional information about how to improve (see also Bangert-Drowns, Kulik, Kulik & Morgan, 1991; Pittman, Davey, Alafat, Wetherill & Kramer, 1980).

A TEACHER-TRAINING-STUDY FOR IN-SERVICE TEACHERS

Based on theoretical and empirical discussions (1) about the importance of teachers' expertise for the quality of teaching and for students' learning, (2) about competency-oriented mathematics in general and mathematical modelling in detail and

(3) about the importance of feedback as central element of formative assessing and reporting students' learning, within a teacher-training-study the research-project Co^{2}CA aims at answering the following research-questions:

Research-question I

Is it possible to develop tests on teachers' expertise being sensitive to the topics of a teacher-training? Or more specifically: Is it possible to develop tests on teachers' pedagogical content knowledge concerning formative assessment if dealing with modelling tasks in competency-oriented mathematics which can be used to evaluate a teacher-training reliably?

Research-question II:

Is it possible to support teachers building up their expertise if attending in teacher-trainings? Or more specifically: Is it possible to foster teachers' pedagogical content knowledge concerning formative assessment if dealing with modelling tasks in competency-oriented mathematics if teachers are trained in these topics?

Design and content of the teacher-training-study

For being able to answer the main research-questions stated above, the following teacher-training-study as one part of the research-program of Co^{2}CA looks like as follows (see also figure 1): Overall 27 mathematics teachers participate in teacher-trainings taking place from September 2013 to December 2013. Before starting the trainings every single teacher is assigned to one out of two experimental groups (EG A and EG B). Over a period of 10 weeks there are three-day teacher-trainings twice for these two experimental groups, once at the beginning of the teacher-training-study, once at the end of the teacher-training-study.

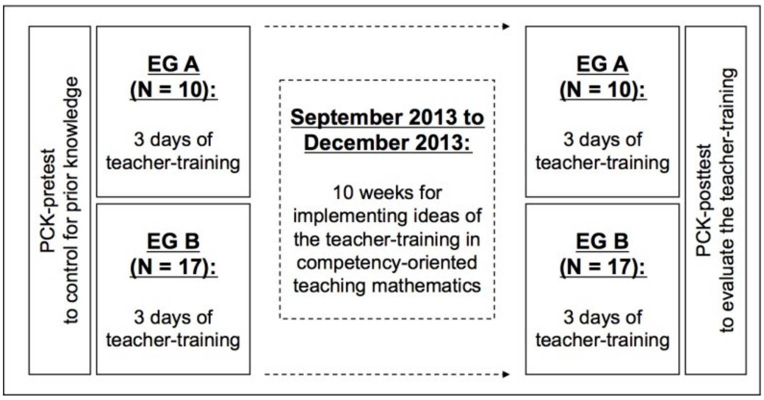


Figure 1: Design of the teacher-training-study

The contents of the teacher-trainings differ between the two experimental groups: Teachers of EG A are trained in central ideas of formative assessment if dealing with modelling tasks in competency-oriented teaching of mathematics, teachers of EG B are trained in selected aspects of competency-oriented mathematics in general (see table 1 for details). Next to taking part in the teacher-trainings itself the participating teachers have to implement central ideas of the teacher-trainings within their teaching mathematics at school, that is to assess and report students' modelling performances regularly (EG A) respectively to make use of problem-solving-tasks and modelling-tasks (EG B) if teaching mathematics.

EG A	EG B
(1) Formative assessment and feedback as central element of formative assessment: A general psychological and pedagogical point of view.	(1) Mathematical problem solving as a central element of competency- oriented mathematics: General didactical ideas and task-analyses.
(2) Mathematical modelling as a central element of competency- oriented mathematics: Analysing students' solution processes and giving feedback to the students.	(2) Mathematical modelling as a central element of competency- oriented mathematics: General didactical ideas and task-analyses.
(3) Implementing formative assessment in teaching mathematical modelling.	(3) Implementing mathematical problem solving and mathematical modelling in every-day teaching.

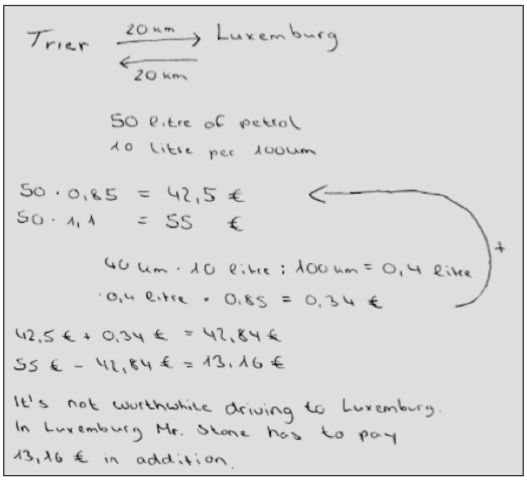
Table 1: Contents of the teacher-training-study

Test-instruments to evaluate the teacher-training-study

For being able to evaluate the teacher-training-study and to evaluate the effectiveness of the trainings, the following test-instruments are used within the study to compare the teachers' pedagogical content knowledge of the two experimental groups:

- Firstly a pretest on teachers' mathematical pedagogical content knowledge is used at the beginning of the training to control for teachers' PCK. This test is taken from the COACTIV-project (see e. g. Krauss et al., 2008) and asks for general didactical ideas if teaching mathematics.
- Secondly there is a newly developed posttest on teachers' PCK which is used to compare the teachers' expertise between the two experimental groups at the end of the teacher-training-study and which is sensitive to the contents of the teacher-trainings in EG A. In detail this PCK-posttest consists of overall 10 items dealing with (1) ideas about mathematical modelling processes in general as well as with (2) ideas about how to analyse students' solution processes to modelling tasks. Furthermore it is asked (3) for how to give feedback to students working on modelling tasks and (4) for concepts of how to implement formative assessment in teaching mathematical modelling (an example of an item of this PCK-posttest is given in figure 2).

There is given the following task on the left and a students' solution to the task on the right:

Task:	**Students' Solution:**
Mister Stone lives in Trier close to the border of Luxemburg. For filling up his VW Golf he drives to Luxemburg. Immediately behind the border, which is 20 km away from Trier, there is a petrol station. There you have to pay 0.85 Euro for one litre petrol in contrast to Trier where you have to pay 1.1 Euro. Is it worthwhile for Mister Stone to drive to Luxemburg?	

Give a written feedback to the student which has a great potential to support further learning of the student.

Figure 2: An example of a PCK-posttest-item

Results of the teacher-training-study

By January 2014, the PCK-posttest has been coded completely (whereas data of the PCK-pretest still has to be analysed): For every single item, teachers' are given – depending on the item – up to 3 score-points, by theory a minimum of 0 score-points and a maximum of 21 score-points is possible. Based on this coding the following answers to the research questions stated above can be given (see also table 2):

- Research Question I: The PCK-posttest is reliable with Cronbach's alpha = 0.68 regarding the whole sample of 27 teachers within EG A and EG B.
- Research Question II: Having a closer look at the teachers' performances to the PCK-posttest, it can be seen that teachers of EG A outperform their counterparts of EG B. In detail, teachers of EG A do not only score a higher empirical maximum but do also have a significantly higher mean-score in the PCK-posttest ($t(25) = 4.90$; $p < .001$). The effect-size of this difference is a medium one, that is Cohen's d for independent samples with a differing sample size is $d = 0.7$.

PCK-posttest: 10 items; alpha = 0.68 (N = 27)

	N	m	SD	emp. min.	emp. max.		
EG 1	10	15.10	1.91	12	18	p < .001	d = 0.7
EG 2	17	9.65	3.18	2	14		

Table 2: Results of the PCK-posttest

SUMMARY AND OUTLOOK

Teachers' expertise has been pointed out to be central for the quality of teaching. However it is still an open question how to support teachers building up their expertise. Within the research project Co²CA a test on teachers' expertise has successfully been developed which is not only sensitive to the content of teacher-trainings but which can furthermore reliably be used to evaluate teachers' expertise at the end of trainings. Results using this instrument illustrate that teachers' expertise is significantly higher if they are specifically trained within the topics being tested comparing to teachers not being trained. So it is not only general knowledge about competency-oriented mathematics which is needed to answer the posttest-items (EG B) but special knowledge about formative assessment if dealing with modelling tasks (EG A). Within the next steps the PCK-pretest has to be analysed to control for teachers' general didactical knowledge. Furthermore results presented here have to be discussed within the broader context of the research-program of Co²CA to investigate the influence of teacher-trainings on teachers' expertise in depth.

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THE BELIEFS OF PRE-SERVICE PRIMARY AND SECONDARY MATHEMATICS TEACHERS, IN-SERVICE MATHEMATICS TEACHERS, AND MATHEMATICS TEACHER EDUCATORS

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This paper presents a comparison of the responses to nine beliefs items and one confidence item of samples of Australian mathematics teachers, pre-service primary teachers, pre-service secondary mathematics teachers, and mathematics teacher educators. Significant differences were found between each pair of groups. The implications of these for the effectiveness of mathematics teacher education and professional learning for mathematics teachers are discussed.

TEACHER BELIEFS

Teachers' beliefs influence practice in subtle but powerful ways (Bray, 2011). Beliefs about the nature of mathematics, mathematics teaching and mathematics learning have been considered most relevant to practice and have been the focus of much research in the area. In relation to the nature of mathematics, Ernest (1989) identified three philosophies comprising systems of beliefs. The first, an Instrumentalist view, regards mathematics as a set of useful skills for practical purposes. The second, the Platonist view, sees the discipline as a structured body of pre-existent knowledge. According to the third view, Problem-Solving, mathematics is a creative human activity and product.

Ernest and others have considered beliefs about mathematics teaching and learning that follow from these views of the discipline. Van Zoest, Jones and Thornton (1994), for example, conceptualised three views of mathematics teaching. These were Content-focused with an emphasis on performance, Content-focused with an emphasis on understanding, and Learner-focused. Beswick (2005) used a modification of Ernest's work to identify three views of mathematics learning that align with each of his philosophies. These were, Skill mastery, passive reception of knowledge; Active construction of understanding; and Autonomous exploration of own interests.

There has been a trend towards incorporating teachers' beliefs research into studies of teacher knowledge because of the practical difficulty of distinguishing them and/or theoretical arguments about their equivalence (e.g., Kuntze, 2012). This paper arose from a study of teacher knowledge, conceptualised to include their beliefs. It focuses on broad beliefs about the nature of mathematics, and mathematics teaching and learning and draws upon seminal work of Ernest (1989) and others, on conceptualising mathematics teachers' beliefs, to allow data on mathematics teacher educators' (MTEs) beliefs to be considered in the context of established understandings about pre-service teachers (PSTs) and in-service teachers' (MTs) beliefs, and for comparisons among various groups to be made. The research question that guided the study was: What differences are there between the beliefs of primary and secondary PSTs, MTs and MTEs?

Pre-service teachers' beliefs

PSTs commence their university study with beliefs based on their own experiences of learning mathematics at school (Van Es & Conroy, 2009). PSTs' beliefs about mathematics have been characterised as fixed and hence aligned with Ernest's Instrumentalist or Platonist views. Similarly, Philipp, et al., (2007) reported that many PSTs see mathematics as a collection of rules and procedures. This is problematic for MTEs conducting teacher education programs underpinned by constructivist views of learning, and student-centred teaching that emphasises conceptual understanding. Beswick and Goos (2012) reported that primary PSTs responded positively to beliefs items reflecting a student-centred approach to teaching but that their responses to items about the nature of mathematics were more ambiguous suggesting that the nature of the discipline may have received too little attention in their teacher education programs.

Many studies have reported favourably on the effectiveness of teacher education programs in influencing PSTs' beliefs to be more compatible with student-centred teaching but almost always accompanied by caution about depth and longevity of observed changes (e.g., Conner, Edenfield, Gleason, & Ersoz, 2011). Studies of primary PSTs appear more common than those involving secondary PSTs but the research that has been conducted with the secondary group has contributed important insights to conceptualisations of the belief structures of secondary MTs and their development (e.g., Cooney & Shealy, 1997). Implicit in this is an assumption of consistency between the beliefs of secondary PSTs and MTs.

In one of few comparative studies of the attitudes and beliefs of primary and secondary PSTs, Kalder and Lesik (2011) found that primary PSTs who had not chosen to specialise in mathematics teaching were more likely than secondary PSTs to have negative attitudes to and beliefs about mathematics.

In-service mathematics teachers' beliefs

Studies of MTs' beliefs have been in the context of professional learning (PL) initiatives aimed at influencing them in similar ways as described in relation to PSTs (e.g., Kuntze, 2012); describing and categorising them (Kalder & Lesik, 2011), or exploring their connection with practice (e.g., Beswick, 2005). Archer (1999) interviewed 17 primary teachers and 10 secondary MTs in order to compare their beliefs. She found that primary teachers were more inclined than secondary MTs to see mathematics as linked to everyday life and to other areas of the curriculum. In contrast, secondary teachers tended to see it as a self-contained, orderly and logical. It seems that primary teachers are inclined to hold Instrumental views of mathematics (Ernest, 1989) whereas secondary MTs are more likely to have a Platonist view.

Mathematics educators' beliefs

The beliefs of MTEs have received little attention, however, Callingham, Beswick, Clark, Kissane, Serow, & Thornton (2012) reported on the knowledge (including beliefs) of MTE members of the Mathematics Education Research Group of Australasia (MERGA) using the same instrument as used with primary and secondary PSTs. They reported that the MTEs found the beliefs items more difficult to endorse than did PSTs. There were no differences for different employment types (continuing, fixed term, and casual) or length of tertiary teaching experience other than for those with more than 16 years of experience who were less inclined to endorse the items.

Differences among the beliefs of various groups

Ashman and McBain (2011) investigated the beliefs about mathematics teacher education of primary MTs and PSTs. They found a tendency among both groups to value classroom experience over university study, but that both groups shifted to a more balanced view of the relative value of learning in the two contexts following a semester long intervention that involved substantial interaction between MTs and PSTs as well as liaison with MTEs.

The effectiveness of MTEs' work with PSTs, and with MTs in the context of PL, depends upon their capacity to influence. If there are important differences between the ways in which MTEs and PSTs or MTs view mathematics and its teaching and learning, and these are neither acknowledged nor addressed, MTEs' ability to influence may be compromised. Differing beliefs of experienced MTs and/or MTEs and newly graduated MTs may contribute to both the importance that PSTs place on learning during practicums (Korthagen, 2010) and to the fact that many beginning teachers perpetuate the teaching that they experienced in school (Ball, 1990). Differing beliefs may also contribute to the perceived theory-practice gap that has concerned researchers, teacher educators, and teachers (Korthargen, Loughran, & Russell, 2006). Studies, such as that reported here, that examine the nature and extent of belief differences between PSTs, MTs, and MTEs are thus important and timely.

THE STUDY

The PST data reported and discussed in this paper were part of a larger Australian study of the knowledge required to teach mathematics. Aspects of the study related to primary PSTs and the use of the survey with MTEs have been reported elsewhere (e.g., Beswick & Goos, 2012; Callingham et al., 2012). This paper focuses on the responses to beliefs items for these cohorts and for MTs and secondary mathematics PSTs.

Instrument and procedure

Data about participants' beliefs were collected as part of an online survey that also included questions designed to examine their mathematical content and pedagogical content knowledge. Due to constraints on the overall length of the survey the beliefs items were limited to the nine Likert type items listed in Figure 1. They required responses on 5-point scales from Strongly Disagree to Strongly Agree. The three items concerning each of beliefs about the nature of mathematics (Items 1, 4 and 7), mathematics teaching (Items 2, 5 and 8), and mathematics learning (Items, 3, 6 and 9), were modified from existing sources (e.g., Van Zoest et al., 1994). A tenth item asked respondents to rate on a similar scale their confidence to teach the mathematics at the level they were or would be qualified to teach.

1. Mathematics is a beautiful and creative human endeavour.
2. Periods of uncertainty and confusion are important for mathematics learning.
3. Acknowledging multiple ways of mathematical thinking may confuse children.
4. Mathematical ideas exist independently of human ability to discover them.
5. Students learn by practicing procedures and methods for performing mathematical tasks.
6. Teachers must be able to represent mathematical ideas in a variety of ways.
7. The procedures and methods used in mathematics guarantee right answers.
8. Justifying mathematical thinking is an important part of learning mathematics.
9. The teacher must be receptive to the students' suggestions and ideas.

Figure 1: The nine beliefs items

The survey was made available to PSTs at seven Australian universities, to MTs through the website of the Australian Association of Mathematics Teachers (AAMT), and to MTEs through the MERGA website. Respondents accessed the survey via an anonymous link. An analysis of variance using SPSS, was used to examine differences in mean responses among the groups to the beliefs and confidence items.

Participants

Participants comprised 294 primary PSTs and 86 secondary mathematics PSTs. Most (81.6%) of the primary PSTs had not studied mathematics or statistics beyond secondary school. Of these 58.3% (47.6% of the whole PST sample) had studied a Year 12 mathematics subject that could contribute to university entrance, and 18.3% (15% of whole sample) reported Year 10 as the highest level of mathematics studied. A similar percentage (13.6%) had studied some university mathematics or statistics. Nearly half (48.2%) of the secondary PSTs had studied mathematics or statistics as part of a bachelor degree and a further 5.8% reported postgraduate study of these subjects. Some of the secondary PSTs (12.9%) had studied no mathematics beyond Year 10 or had studied a Year 12 subject that did not count for university entrance.

The 57 MTEs and 65 MTs were drawn from every Australian state and territory. Most of the MTEs (77.2%) had post-graduate qualifications but 29.8% had not studied tertiary level mathematics. Almost half had been working in universities for 5 years or less and 38.4% had taught in schools for at least 15 years. Of the MTs, 35% reported having postgraduate degrees and 51.2% were more than 50 years old. Three quarters (75.4%) had studied mathematics or statistics at tertiary level. Almost two thirds (63.5%) taught secondary school mathematics. This profile is consistent with the MTs having been recruited through the AAMT website and hence likely to members of that association and to consider themselves to be specialist MTs.

RESULTS

Differences were found for all beliefs items except Items 6 and 9, and for the confidence item. For Item 1 MTs, MTEs and secondary PSTs all agreed more strongly than primary PSTs, $F(3, 470) = 37.767$, $p = .000$ in each case. For Item 2, MTs and MTEs agreed more strongly than Primary PSTs, $F(3, 470) = 4.468$, $p < .05$. In relation to Item 3 both groups of PSTs agreed more strongly than each of MTs and MTEs, $F(3, 469) = 13.152$, $p < .01$ in each case. The only difference for Item 4 was between MTs and MTEs with the former group agreeing more strongly, $F(3, 468) = 3.340$, $p = .015$. For Item 5, MTEs agreed less strongly than all other groups, $F(3, 463) = 12.201$, $p = .000$ in each case. Both groups of PSTs agreed more strongly with Item 7 than did MTEs, $F(3, 461) = 6.806$, $p = .000$ in each case. For Item 8 both MTs and MTEs agreed more strongly than Primary PSTs, $F(3, 462) = 6.772$, $p = .001$ for primary PSTS and $p = .024$ for secondary PSTs. MTs and MTEs were more confident than each of the PST groups, $F(3, 459) = 29.622$, $p < .01$ in each case.

There were significant differences between MTEs and primary PSTs for Items 1, 2, 3, 5, 7, and 8 with MTEs more likely to view mathematics as a “beautiful and creative human endeavour”, and to agree that periods of confusion and uncertainty, and justifying mathematical thinking are important to mathematics learning. PSTs were more likely than MTEs to believe that students learn by practicing procedures, that procedures guarantee right answers, and that acknowledging multiple ways of thinking mathematically could confuse students.

MTs and primary PSTs responded significantly differently to Items 1, 2, 3, and 8, with MTs, more likely than primary PSTs to view mathematics as a “beautiful and creative human endeavour”, and to agree that periods of confusion and uncertainty, and justifying mathematical thinking, are important to mathematics learning. Primary PSTs were more likely to believe that acknowledging multiple ways of thinking mathematically could be confusing.

Significant differences between MTEs and secondary PSTs were found for Items 3, 5, and 7, with secondary PSTs more likely to believe that students learn by practicing procedures, that these procedures guarantee right answers, and that acknowledging multiple ways of thinking mathematically could confuse students.

MTEs and MTs differed for Items 4 and 5, with MTs more likely to believe mathematical ideas are pre-existing and that students learn by practising procedures. The only significant difference between the PST groups was in relation to Item 1, with secondary PSTs more likely to see mathematics as a “beautiful and creative human endeavour”. MTEs and MTs were more confident than both PST groups.

DISCUSSION

Primary PSTs were less likely than other groups to agree that “Mathematics is a beautiful and creative human endeavour” (Item 1), and less likely than both MTs and MTEs to agree that “Periods of uncertainty and confusion are important for mathematics learning" (Item 2) and that "Justifying mathematical thinking is an important part of learning mathematics" (Item 8). Both of these results are consistent with the well-documented unease that this group have with the discipline (Kalder & Lesik, 2011). Many regard mathematics with fear and dislike and many of their own experiences of uncertainty and confusion in learning mathematics may not have resulted in eventual understanding. Similarly, they may not have had positive experiences of having to justify their mathematical thinking.

Both groups of PSTs were more likely than MTs or MTEs to agree that "Acknowledging multiple ways of mathematical thinking may confuse children" (Item 3), and more likely than MTEs to agree that "the procedures and methods used in mathematics guarantee right answers" (Item 7). Acknowledging multiple ways of mathematical thinking is broadly consistent with progressive views of mathematics teaching as described by Beswick (2005) and so this result suggests that the PSTs had views less aligned with reform teaching than did MTEs and MTs. The results for this item are, however, difficult to interpret because some may have agreed because they regarded confusion as a negative experience to be avoided whereas others may have agreed but regarded confusion as a necessary to achieving greater understanding. Item 7 is consistent with an Instrumentalist view of mathematics (Ernest, 1989). The results for this item are thus consistent with the stronger Problem solving view of the discipline of MTs and MTEs than primary PSTs evident from the data for Item 1.

MTEs were less likely than all other groups to agree that "Students learn by practicing procedures and methods for performing mathematical tasks" (Item 5). This statement is broadly consistent with a Skill mastery, passive reception of knowledge view of mathematics teaching (Beswick, 2005) and contrary to an emphasis on teaching for understanding that is prevalent in mathematics education literature. It is also consistent with the apparent counteractive effect of the practicum on the changes in beliefs that MTEs strive to instil in PSTs (e.g., Conner et al., 2011). Given that the MTs in this study were well qualified, largely experienced, and engaged with their profession, this result should not be dismissed lightly. It could be that distance from the reality of mathematics classrooms causes MTEs to adopt less nuanced rhetoric in regard to this and possibly other pedagogical issues.

MTs were more likely than MTEs to agree that, "Mathematical ideas exist independently of human ability to discover them". This item expresses a broadly Platonist view of the discipline (Ernest, 1989) and so the result is consistent with that for Item 1. The greater confidence of MTs and MTEs than PSTs is consistent with their relative experience.

Consideration of Items 1, 4 and 7 together suggests that the MTEs tended to hold views of the nature of mathematics most closely aligned with a Problem Solving view of the discipline, MTs were more likely to be Platonists, and PSTs, particularly primary PSTs, more inclined to hold Instrumentalist views. In relation to mathematics learning as reflected in Items 2, 5, and 8, a similar ordering of perspectives is evident. For mathematics teaching there was, however, a difference between the groups only for

Item 3. That result is consistent with MTs and MTEs having more learner-focussed views of mathematics teaching than PSTs but given the interpretative difficulties associated with Item 3 and the absence of differences for other teaching related items it is not clear that beliefs about teaching fit the same pattern. Rather, the teaching practices endorsed by the four groups are little different in spite of differences in their beliefs about the nature of mathematics and how it is learned. This accords with evidence that beliefs about mathematics and its teaching and learning manifest in practice in subtle ways (Bray, 2011) and that some MTs confound the discipline with the mathematics of the school curriculum (Beswick, 2012).

CONCLUSION

This study represents an initial attempt to compare the beliefs of MTs, MTEs and primary and secondary PSTs. Beliefs of MTEs are particularly under-researched. The extent to which MTEs appear to hold different beliefs from either MTs or PSTs points to a need for further exploration of the bases of these differences. It could be that MTEs, a relatively small community in Australia, are susceptible to adopting accepted rhetoric without appropriate critique. Although few, the differences between MTEs and experienced, professionally engaged MTs are relevant considerations in PL work with teachers; to what extent might MTEs be perceived as, or actually be, out of touch with classroom realties? The even greater differences between MTEs and the PSTs, especially Primary PSTs, with whom they work have implicitly been acknowledged but these data should prompt reflection on the extent to which MTEs are able to communicate with PSTs credibly. The beliefs measure used in this study was necessarily crude and so there is scope for far more detailed studies of the issues raised.

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TURNING DISINTEREST INTO INTEREST IN CLASS: AN INTERVENTION STUDY

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This paper presents an intervention study in three 11th grade classes on calculus with the aim to overcome the students' amotivation by boosting their situational interest. This was successfully done by teachers implementing interest-dense situations. Data analyses further revealed two different principles of how the teachers transferred the theory on interest-dense situations into practice when arranging the epistemic processes, and two directions of dissemination that the teachers undertook additionally.

INTRODUCTION

Though the contribution of interest to performance in mathematics is not so clear (OECD 2004), implementing learning mathematics with interest in school is relevant for at least two reasons: (1) learning mathematics with interest prevents students becoming amotivated, hence, overcoming a significant obstacle to learning mathematics, and (2) interest can be regarded as the driving force for learning mathematics in self-determined and in-depth ways, since "interest and enjoyment of particular subjects […] affects both the degree and continuity of engagement in learning and the depth of understanding reached" (OECD 2004, p. 117). But what are the didactic tools that teachers can use to plan and implement math lessons that support learning mathematics with interest? This question will be answered by presenting a qualitative intervention study as a case study on how a group of teachers solved motivational problems in their 11th grade classes by the use of the theory of interest-dense situations (briefly expressed as IDS) (Bikner-Ahsbahs & Halverscheid, 2014). Since IDS has not yet proven its applicability as a teaching tool, this intervention study had two aims: to solve the practical problem of increasing learning with interest in math classes and, through that, to prove the applicability of IDS in practice.

THEORETICAL BACKGROUND

Interest in mathematics is a relationship between a person and mathematics (Krapp, 2002) that can appear in two forms. (1) Situational interest is triggered by situational conditions, and may vanish if these conditions dry down. (2) Personal interest is a long lasting kind of interest that, independent of situational conditions, students bring with them into the class. Mitchell (1993) has worked out a concept of situational interest that later Hidi and Renninger (2006) have identified as a step towards the development of personal interest. Personal interest in mathematics is shown by epistemic actions leading to increased knowledge, being accompanied by positive emotions and placing high value on mathematics (cf. Krapp, 2002). Situational interest (Mitchell, 1993) is triggered by the situational conditions. It is relatively easy to catch situational interest in class, but difficult to hold it. Interest can be held if students experience themselves as being *involved in the mathematical activity* which is *meaningful* to them (Mitchell, 1993). Research on Self-Determination Theory (Deci, 1998) has shown that the experience of three basic psychological needs of competence, autonomy and social relatedness support interest development. Thus, fostering learning with interest in class means arranging lessons that support fulfilling these needs. But what do these arrangements look like? Psychological research on interest does not answer this question.

IDS (Bikner-Ahsbahs & Halverscheid, 2014) is a theory of learning mathematics that results from a paradigm shift, merging the concepts of situational and personal interest and turning them into the social-situational concept of interest-dense situations. An interest-dense situation is shaped by social interactions in class. It may appear within an epistemic process of solving a mathematical problem exhibiting three features: The students are deeply involved in the mathematical activity (*involvement*), they construct mathematical meanings in an in-depth way leading to deepened insight (*positive dynamic of the epistemic process*), and they value highly the mathematics at hand (*value attribution*). Research on interest-dense situations has disclosed how these situations may be arranged (Bikner-Ahsbahs & Halverscheid, 2014; Bikner-Ahsbahs, 2004a; 2004b; Stefan, 2009). Some of these conditions are now briefly described:

- *Involvement*: The teacher follows the students' line of thought, the students are focused on their own train of thought.
- *Dynamic of the epistemic process*: The epistemic process comprises three epistemic actions: gathering and connecting mathematical meanings may - if adequately arranged - lead to structure-seeing; for example, this may happen if students first collect examples or ideas (gathering), then relate them to each other (connecting), and finally search for patterns of these relationships (structure-seeing) where a structure is regarded as an entity made of the relations among pieces of knowledge. In every interest-dense situation the epistemic process leads to structure-seeing, i.e. perceiving a new structure or a familiar structure in a new context.
- *Value attribution*: The teacher's and the students' behaviours are based on a didactic contract: the students act as authors producing valuable mathematical ideas, and the teacher acknowledges the students' authorship and fosters such processes, for example assists in finding suitable words, naming mathematical products concerning the original author such "Emma's rule".

Stefan (2009) has researched interest-dense situations concerning how grade 2 students investigate a dice of 1 million cubes. She has identified specific kinds of participation patterns indicating forms of situational interest, such as being interested in theoretical considerations, or being interested in accurately working with dice to build a dice of 1 million.

METHODOLOGICAL CONSIDERATIONS

Applicability, hence, the relevance of IDS, is proven by an intervention study in the math classes of the SINUS-Set teachers in Nordrhein-Westfalen (Germany), who wanted to solve motivational problems in their 11th grade calculus courses. The study was conducted to answer the following questions: a) Do the intervention lessons exhibit interest-dense situations (as described above)? b) How did the teachers organize and conduct the arrangement in class to implement interest-dense situations (i.e. concerning the three features: involvement, dynamic of the epistemic process, value attribution)? c) Did the teachers solve their motivation problem? What kind of indicators can be found? d) Was the experience with IDS disseminated? If yes, how?

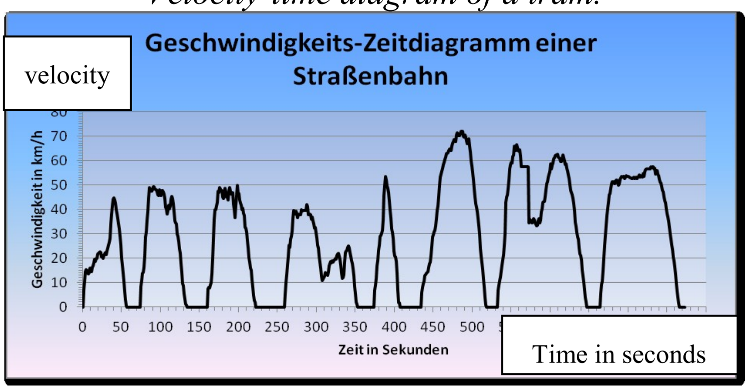
For that, the teachers first were trained in the use of IDS. That was done by a workshop dealing with transcripts of typical interest-dense situations, their specific kinds of social interaction, teacher behaviour and student involvement, the specific epistemic processes and different ways of arranging the epistemic process by means of the epistemic actions model. Moreover, examples of value attribution were presented and discussed (Bikner-Ahsbahs & Halverscheid, 2014). Finally, some mathematical tasks were presented that have the potential to promote a progressive dynamic of epistemic processes. In addition, the SINUS-Set was provided with a summary of the theory of interest-dense situations with some practical advice, such as "interest in the students' learning supports students' interest in learning".

The intervention study comprised three steps:

Preparation: The SINUS-Set commonly planned two lessons for implementing IDS in their 11th grade classes. Two cycles of teacher-planning and researcher-revision were conducted before the teachers implemented the lessons: introduction of the definite integral and finding sufficient conditions on extreme and inflection points.

a) *Introducing the integral concept by application situations*:

Velocity-time diagram of a tram.



1) Find information from the graph and exchange them [in the group].

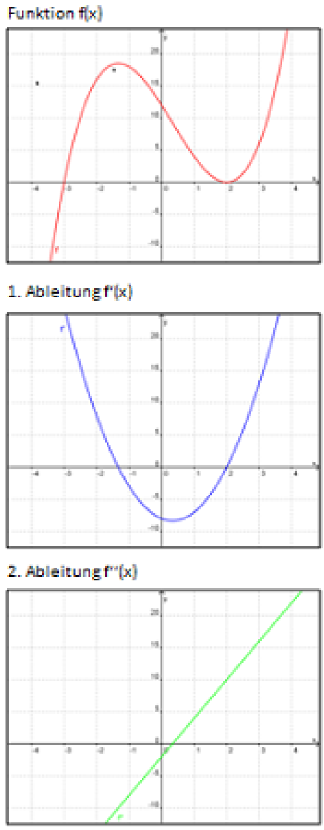
2) How far does the tram approximately drive between 435 sec and 520 sec and between 665 sec and 765 sec? Explain.

3) Prepare a poster presentation with your results. Every member of the group must be able to explain them.

Figure 1: Task example of introducing an integral.

The teachers prepared similar tasks (cf. Figure 1) of seven different application situations which all had the same idea in common: Through converting the area under the graph within an interval into a rectangle, a quantity can be estimated which later is called definite integral.

b) *Particular points of functions*: The teachers prepared the same tasks for seven different functions f (Figure 2), hence, for seven groups of students. Their planning encompassed three phases: Phase 1: Gathering information with the help of graphical representations to create conjectures and some first connections (Figure 2, (1), (2)). Phase 2: connecting mathematical meanings by comparing and contrasting the findings of the groups of students to revise hypotheses (Figure 2, (3)). Phase 3: structure-seeing towards general features of the particular points based on the hypotheses gained.



Task:

(1) Transfer these graphs onto your poster. Use the same colours as the ones in the pictures.

(2) Find correlations among the three graphs. Be aware of the particular points. Formulate these correlations in brief sentences as hypotheses. Find logical substantiations for your hypotheses.

(3) Check your hypotheses by looking at the posters of the other groups. Change and make your sentences more precise if necessary.

(4) Write those sentences that after your revisions seem right and valuable on DIN A4 paper sheets, so that it is well readable by the others.

Figure 2: Example of the tasks about particular points of function graphs.
Legend: f (red), f' (blue), f'' (green)

Implementation and data collection: Two teachers implemented the same lesson on particular points (nulls, extreme and inflection points) of functions (Figure 2) in their calculus ground level courses, and one teacher implemented the lesson of introducing the concept of integral (Figure 1). In order to answer the questions a), b), c), the three lessons were videotaped recording the group work and the whole class situation. To answer the questions c), d), a written interview with the communicator of the SINUS-Set took place.

Analyses of the video data: First, the video was reorganized by episodes that were rated according to the features of interest-dense situations. Second, data analyses were carried out observing and interpreting the video episodes several times ac-cording to the tools of IDS. Each time, the researcher decided whether and then reconstructed how the episode shows *deep student involvement, positive dynamic of the epistemic process* and *value attribution*. Third, the epistemic process was analysed according to the epistemic actions model to obtain phase diagrams (Bikner-Ahsbahs, 2004b; Figure 5). Finally, the communicator of the SINUS-Set was interviewed to validate whether the intervention had boosted interest and how the experiences were disseminated.

DATA ANALYSES AND SOME RESULTS

I will now reduce the description of the analyses to the lessons on the particular points of the function graphs and their first and second derivative. The two lessons were arranged in the same way: In groups of four, the students gathered mathematical meanings on their tasks, connected them in order to prepare initial substantiated conjectures and poster presentations thereof (Figure 2. (1), (2)), second revised and improved them (Figure 2, (3)) and third, a class discussion led to structure-seeing.

Involvement: Through the whole process, the students followed their own way of thinking. Within the group work they gathered ideas concerning the given particular points, shared them among the four students and created conjectures about the features of these points. When they compared their group results with those of the other groups written on posters, they systematically contrasted, checked, and revised their hypotheses: The students went to the board, showed their group mates their observations, and improved their findings. After that, the teachers asked the students to present and substantiate their final hypotheses and to hang the paper sheets with their hypotheses on the blackboard. The student audience critically checked and discussed the validity of hypotheses, valuing them as *generally valid, sometimes valid, invalid,* or *just in one case valid.* The teachers' behaviour was steered by the situation, in that they organized the discussion and visualized results on the blackboard but did not intervene.

Dynamic of the epistemic process (Figure 5): During the group work (phase1), initially gathering actions took place leading to building conjectures (connecting). The teachers assisted the groups in organizing and expressing their conjectures. Phase 2 was a connecting phase: The conjectures were checked by comparing them with results shown on the other groups' posters (Figure 3). During this phase the teachers did not intervene and left space for revisions. Each conjecture was finally written on a paper sheet. Phase 3 was that of structure-seeing: The students presented and justified their conjectures to the whole class. The other students checked them. While grouping, the conjectures were hung on the blackboard, though not all the groups found substantial propositions. Figure 3 represents one substantial proposition (4a), a valid but irrelevant one (4b) and a special one (4c). Through a process of collectively reasoning and rearranging the hypotheses, a process of structure-seeing was organized resulting in sufficient conditions for inflection and extreme points of differentiable functions (4a).

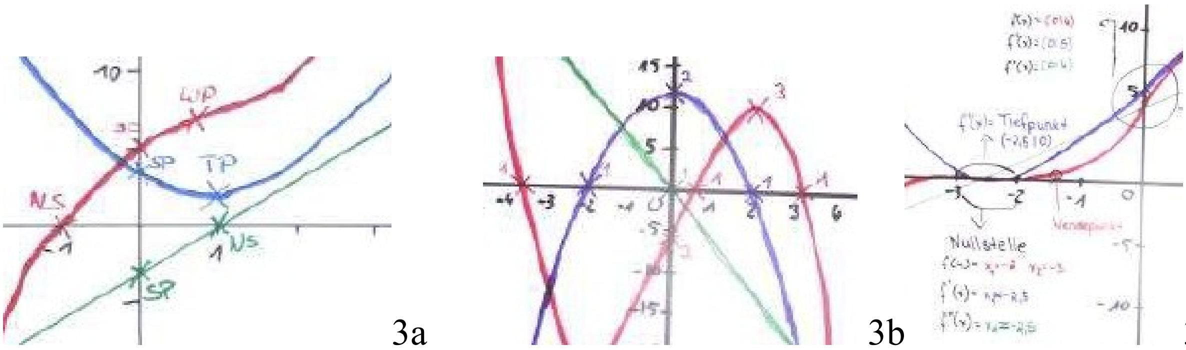


Figure 3: Poster presentations of three working groups (WP, Wendepunkt (inflection point), TP, Tiefpunkt (minimum point), Ns, Nullstelle (null))

"the maximum point of f(x) is above the null point of f'(x)" Der Hochpunkt von f(x) liegt über der Nullstelle f'(x) 4a	"all graphs have an intersection point with the y axis" Alle Graphen haben ein Schnittpunkt mit der der y-Achse 4b	"nulls are at the same time the inflection points" Nullstellen sind gleichzeitig die Wendepunkte 4c

Figure 4: Conjectures: 4a: generally valid, 4b: valid, but irrelevant, 4c: only sometimes valid

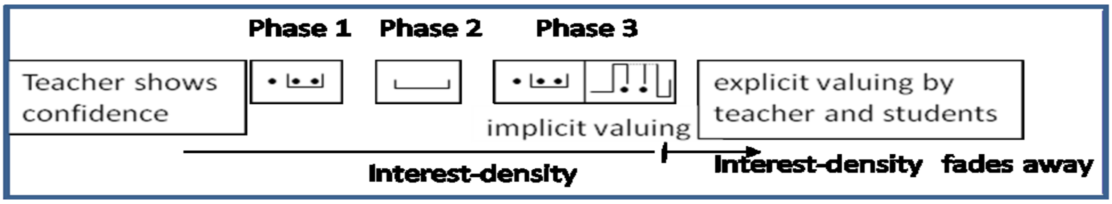


Figure 5: Phase diagram of the epistemic process expressed by pictographs (cf. Bikner-Ahsbahs 2004a; 2004b). Phase1: gathering-connecting, phase 2: connecting, phase 3: gathering-connecting, then structure-seeing

Value attribution (Figure 5): According to value attribution, the two teachers behaved differently towards the creation of valuable ideas. One of them started the lesson showing confidence in the students' capacity to solve the task: "I am curious about the up-and-coming lesson" she said. Her lesson took place in the late afternoon and lasted longer than usual. The students maintained their interest although the bell already had rung. In the end the teacher apologized and thanked them for their engagement. The students reacted by applauding and rapping on the table, hence, they highly valued learning mathematics this way. The other teacher acknowledged the students' engagement in a different way. He clarified the status of all the conjectures by arranging them according to the degree of validity; this way he valued every hypothesis by giving space for it and keeping it on the blackboard. In addition, he acknowledged authorship of valuable ideas. For instance, he asked a student to put his justified conjecture in the right place on the blackboard and finished the sentence by saying: "since you are the discoverer". Figure 5 represents the phase diagram of the epistemic processes including value attribution.

Answers to the three questions

All the three lessons show interest-dense situations. The teachers linked the features of IDS to the topic of their lessons in a substantial way. In the derivative lesson, the core principle was *finding hypotheses and working on them to obtain propositions*. This was done by preparing, revising and collectively checking the validity of conjectures as a three-step-design by means of taking gathering, collecting and structure-seeing as planning tools. The arrangement of the integral lesson was built around the core idea of a definite integral according to the principle of *framing it by different application problems in similar ways*: the estimation of a quantity by calculating an area under the graph of a function. This way, structure-seeing was arranged on two occasions: first, during the group work within each single application context, and second through seeing the same structure underlying other application contexts. (See also Bikner-Ahsbahs & Halverscheid, 2014).

The third question was answered by the interview with the teacher group's communicator. Main points are expressed by the answer to the subsequent questions: Did the concept of interest-dense situations help to solve motivational problems? How? Did you and your colleagues integrate this kind of teaching into other lessons, too? How?

> In all courses in which we tried out our teaching scenarios, all, really all the students be-came actively and very intensively involved in the mathematical activities according to their capacity. We had not expected this. For some students – especially the weak ones – the experience of discovering mathematics themselves within an atmosphere free from fear was an initial ignition. They took this positive experience as motivation with them into the subsequent lessons. In some schools, the developed teaching scenarios have been implemented as a permanent feature of the respective lesson series, slightly adapting them to the specific students, whereby they are thus repeated every year. (Own translation)

This answer indicates the teachers' surprise that they were able to turn amotivation in class into situational interest being held over time. Dissemination of the teachers' experience does not only take place within the teachers' schools, but also by in-service teacher training workshops that one of the teacher is offering: As he wrote:

> [...] That on the base of the concept of interest-dense situation it is possible to develop instruction scenarios leading to an active and markedly motivating lesson for the students independent of the mathematical content and its application, that was a helpful information for the participants of the workshops. Thus, during these workshops such scenarios were developed for further mathematical topics. (Own translation)

CONCLUSIONS

This implementation study has resulted in the new insight that amotivation in a class can successfully turn into learning mathematics with interest by tools extracted from IDS, and that teachers can successfully work with these tools to plan and implement interest supporting lessons. The teachers even have developed suitable transferable principles for the two types of structures to be seen: a concept and a proposition. Since applicability of a theory is an important criterion of the theory's relevance, this implementation study has shown the theory of interest-dense situation is of relevance for practice. However, although the teachers have already practiced some dissemination of IDS and Stefan has applied IDS at primary level (2009), it is not yet clear whether this theory also offers fruitful tools for other school levels or other countries. Larger implementation studies are needed to investigate to what extent the theory of interest-dense situation can be disseminated in practice.

Acknowledgement

I thank the teachers of the SINUS-Set of Nordrhein-Westfalen for their engagement, especially Ulrich Hoffert [hallo@herrhoffert.de]. A book about the teachers' products and experiences is about to be published, too.

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SEMIOTIC AND THEORETIC CONTROL WITHIN AND ACROSS CONCEPTUAL FRAMES

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This paper refers to the concept of semiotic and theoretic control describing resources to conduct decisions in epistemic processes. We consider an argumentation process from a complex problem-solving activity involving different conceptual frames related to parabolas. Using a micro-analytical interpretative lens, we will show that, in order to carry out the argumentation activity, semiotic and theoretic control within conceptual frames (local control) needs to be co-ordinated with control across different conceptual frames (global control).

INTRODUCTION AND THEORETIC BACKGROUND

In the context of argumentation and proof activities, Arzarello & Sabena (2011) show how students' processes are managed and guided according to intertwined modalities of control, namely *semiotic* and *theoretic control*. As introduced by Schoenfeld (1985), control in problem solving activities deals with "global decisions regarding the selection and implementation of resources and strategies" (p. 15). It entails actions such as: planning, monitoring, assessment, decision-making, and conscious meta-cognitive acts. Arzarello & Sabena (2011) speak of *semiotic control* "when the decisions concern mainly the selection and implementation of semiotic resources" (p. 191), and of *theoretic* control

> when the decisions concern mainly the selection and implementation of a more or less explicit theory or parts of it […]. For example, a semiotic control is necessary to choose a suitable semiotic representation for solving a task (e.g. an algebraic formula vs a Cartesian graph), while a theoretic control intervenes when a subject decides to use a theorem of Calculus or of Euclidean Geometry for supporting an argument. (ibid.)

Although these kinds of control seem relevant for epistemic processes, little is known about how they play together and how they relate to the respective content area.

In this paper we will consider the dialectic between semiotic and theoretic control to give account for students' progresses and standstills during a complex problem-solving activity in geometry context, in which the elaboration of an argument is required. Our analysis will be based on the model of epistemic actions (Bikner-Ahsbahs & Halverscheid, 2014), and on the notion of conceptual frame developed by Arzarello, Bazzini and Chiappini (1995). The former provides a tool to focus on epistemic processes in groups of students working together, the latter allows specifying semiotic and theoretic control by mathematical content.

The *epistemic actions model* (Bikner-Ahsbahs & Halverscheid, 2014) comprises three collective actions: gathering, connecting and structure-seeing. In order to solve a mathematical problem, students may *gather* mathematical meanings, i.e. collecting similar pieces of knowledge (such as ideas, examples or counter examples). Through *connecting* actions students may link some of these collected mathematics meanings, for instance by checking whether a number of collected coordinates fulfil a specific equation. *Gathering* and *connecting* actions disclose an amount of mathematical meanings that shape the base for *structure-seeing*. Structure-seeing is an epistemic action of perceiving (1) a new entity of relationships built by gathering and connecting actions, condensing a possible infinite number of examples, or (2) a familiar structure in an unfamiliar/new context.

Students' epistemic processes in mathematical activities are usually organized around specific *conceptual frames*, which are related to their knowledge and expectations. For example, the coordinates of a point of a function graph can be framed as a pair of numbers, but also as lengths in the coordinate system. Arzarello, Bazzini, and Chiappini (1995) introduced the conceptual frame in algebraic context, as "an organized set of notions (i.e. mathematical objects, their properties, typical algorithms to use with them, usual arguing strategies in such a field of knowledge, etc.), which suggests them [the students] how to reason, manipulate formulas, anticipate results" (p. 122). The term *frame* is taken from artificial intelligence studies (Minsky, 1975), where it indicates a knowledge structure that contains fixed structural information. In mathematics education, the idea of conceptual frame can be related to the notion of *cadre* (setting) discussed in Douady (1986), for its strong mathematical dimension. It is also akin to *framing* by Krummheuer (1992), i.e. a stabilized and conventionalized way of seeing things based on previous experiences (Krummheuer, 1992).

In the outlined background, our research seeks to answer the following question: What role does semiotic and theoretic control play in the students' epistemic pro-cesses when developing an argumentation? How is it related to different conceptual frames?

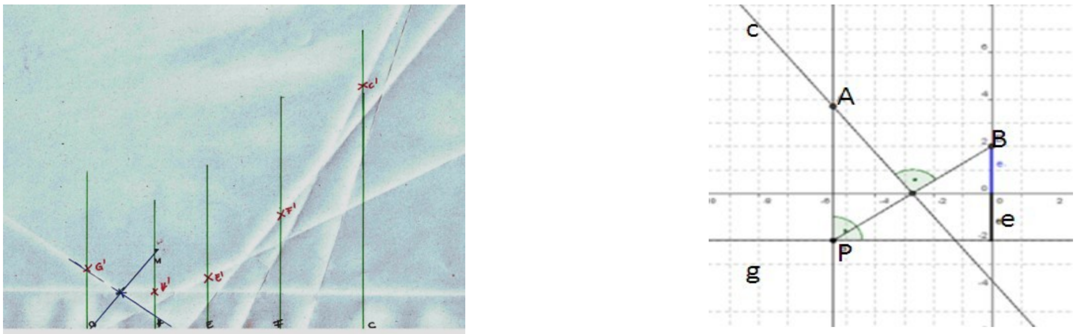
METHODOLOGICAL AND METHODICAL CONSIDERATIONS

We investigate this research problem by observing couples of grade 10 students solving the "parabola task". These students are indicated as high achieving by their teachers. The task, adapted from Gilboa, Dreyfus & Kidron (2011), has been designed to investigate epistemic processes with micro lenses of analysis (Krause & Bikner-Ahsbahs, 2012).[1]

Students are firstly given a paper sheet to construct a curve by a folding process (Figure 1a): *(1) Take any point C on the bottom edge of the given sheet of paper. (2) Bend the sheet such that the chosen point touches the given point M. (3) Through point C, draw*

[1] Data is taken from the project "Effective knowledge construction in interest-dense situations" promoted by the German-Israeli-Foundation, grant 946-357.4/2006.

the line perpendicular to the bottom edge. (4) Mark the point of intersection with the folding line. Keep on doing that until you recognize a curve.



(1a) (1b)

Figure 1: Folded sheet and GeoGebra worksheet of the parabola task.

The marked points indicate a parabola and the folds represent the tangents touching the curve at the corresponding marked intersection points, as well as reflecting axes.

The folding process is translated then into the dynamic geometry software GeoGebra (Figure 1b), with *g* representing the edge of the paper sheet, and B the given point (which was M in paper folding). By dragging P (which corresponds to C in fig 1a), a curve is traced. The distance 2*e* between B and the fixed line *g* can be varied by a scroll bar. The task consists now of three steps: *(a) identifying the curve as a parabola, (b) justifying this conjecture (c) providing a definition for the parabola as a locus of points (in analogy with a given definition of a circle)*. During the whole session, an interviewer is sitting at the students' desk: his role concerns organising the process of working on the task and assisting when the students get stuck.

In our view, the parabola task is especially apt to our research question, since it includes complex problem-solving and argumentation activities, fostering the articulation of a variety of conceptual frames (arithmetic, algebra, geometry, etc.).

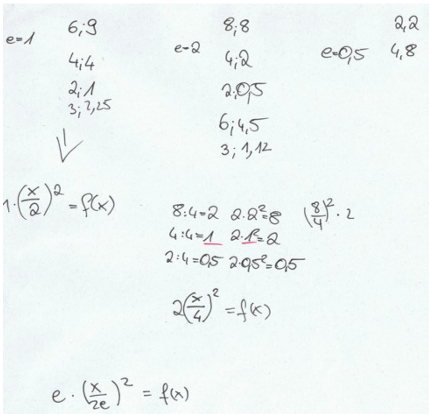
The data collection is organized in order to grasp all the kinds of signs that the students are using (language and gestures, inscriptions, dynamic signs of GeoGebra, etc.). For each couple of students, three video cameras allow us to synchronically observe the gestures, the faces, the computer screen, and the writing processes. Out of this record, a detailed transcript of the language exchanges is obtained, keeping record of verbal and non-verbal modes of expression (Table 1). This allows conducting analyses of social interaction respecting two levels of meaning, the locutionary and the non-locutionary level. Referring to Austin (1975), the former entails what is explicitly expressed, the latter concerns implicit ways of meaning making, for instance through pauses, voice intonation and non-linguistic signs such as gestures.

DATA ANALYSIS

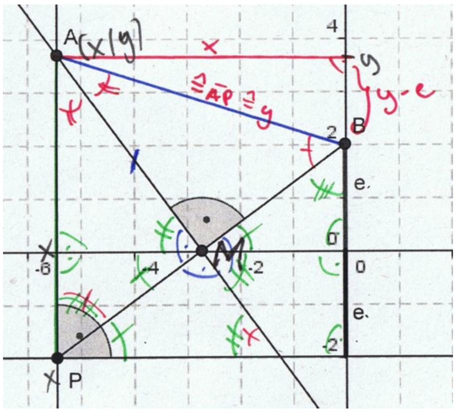
In this paper we consider the epistemic processes of two German students, Rosa and Lisa, who are facing the second part of the parabola task. Rosa and Lisa have done the folding process, clarified how this process has been translated into the dynamic geometry worksheet, and just conjectured that the curve might be a parabola. Now they are asked to justify their conjecture: *How can you convince somebody that it is actually this curve? Use everything you found so far.*

There are several ways to conduct such a justification. The students' knowledge on parabolas is reduced to parabolas as graphs of quadratic equations; thus, a justification of the conjecture must draw on this knowledge. Considering the German curriculum, we expected that the students used the Pythagoras Theorem with subsequent algebraic manipulations. In this argumentation process, it is necessary to consider the geometric conditions and translate them into an algebraic equation: it requires both semiotic and theoretic control encompassing graphical and algebraic frames as well as theoretic knowledge thereof.

We consider the transcript (and video) starting from line 618. In order to prove that the function is a parabola, Rosa and Lisa look for some quadratic equation representing it. The interviewer shows them how to vary *e* with the scroll bar and how to make the coordinates of the point A visible (Figure 1b). The students use the scroll bar several times to create curves, systematically gathering specific coordinates *x* and *y* for *e*=1 and *e*=2 and writing them down in a structured way (Figure 2a, top-left part).



(2a)



(2b)

Figure 2: Students' written sheets.

The top part of Figure 2a shows that the girls have structured the writing process, systematically deciding to begin the gathering process with *e*=1, followed by *e*=2. Experiencing that *e*=3 is not provided by the scroll bar, they decide to add some coordinates for *e*=0.5. This phase of gathering signs (lines 618-665) is led by semiotic control of the arithmetic signs, intertwined with theoretic control within a conceptual frame considering parabolas as quadratic (functional) relationships between numbers. From line 666 to 753, the students leave the computer aside and look at the inscriptions gained. The process of interpreting the number pairs is now conducted by the aim of gaining a function equation with squares of *x*. The following transcript shows the last steps of this process:

740 R: (*laughs*) Yeah (*briefly looks up at I, L writes "=f(x)" after "2(x/4)^2"*) (4sec)

741	R:	That means (*first points at the brackets in "2(x/4)^2", then at the brackets in "(x/2)^2"*) this here ,so this two can […]
744	/L:	yes, yes- (*slides the note sheet to herself, writes "1" in front of the equation for e=1*) times one. (*laughs*)
745	/R:	yes that is ,no that is (*points at the just written "1"*) e times'
746	L:	Yeah yeah. but here (*points at the "1" in the equation for e=1*) times one' (*points at the "2" in the equation for e=2*) there times two.
747	/R:	that means if we say that more general now that would be-
748	/L:	(*writes at the bottom of the note sheet "e·"*) *e* times
749	/R:	(*synchronic*) times, *x* divided by, two *e*'
750	/L:	(*synchronic*) (writes "(*x*/(2*e*))") two *e*'
751	R:	Square
752	/L:	(*nearly synchronic*) (*writes "2" as an exponent next to the bracket and then "=f(x)"*) square.
753	R:	Thus we would have our' (*R looks up at I, L as well, but only briefly*) (...), our (*lays her hand on the note sheet, briefly looking at it, then looks at I again*) square. (*R looks at the note sheet, L sits leant back and looks towards the note sheet*) (5sec).

I: interviewer *L: Lisa* *R: Rosa*
EXACT: with a loud voice
Exact: emphasized
e-x-a-c-t: prolonged
exact.: dropping the voice
exact´: raising the voice *,exact: with a new onset*
(.),(..)...: 1, 2 ... sec pause
(....): more than 3sec pause
(gets up): nonverbal activity
/S: interrupts the previous speaker
[...] some lines are missing
Table 1: Transcription key

By comparing the structure of the arithmetic terms and keeping the square of *x* as an invariant according to the idea of quadratic functions, Rosa and Lisa gain the equation: $2\left(\frac{x}{4}\right)^2 = f(x)$ (line 741, Figure 2a). By comparing the terms for *e*=1 (744-753, arrow in Figure 2a), they obtain the next equation: $1 \cdot \left(\frac{x}{2}\right)^2 = f(x)$. Comparing both equations they finally get: $e \cdot \left(\frac{x}{2e}\right)^2 = f(x)$ (Figure 2a, bottom). In this first phase, the students select and implement theoretic knowledge about arithmetic structures and quadratic functions, which finally are generalized. Indeed, the final equation can be transformed into $(y =)f(x) = \frac{x^2}{4e}$, the correct equation the students will reveal in the end. To carry out this process, the students turn from the numeric to the algebraic system of representation. They start from a gathering process led by semiotic control over the arithmetic-algebraic representation, and then turn into a process of generalisation through connecting actions that lead to structure-seeing. Here, semiotic control intertwines with theoretic control based on arithmetic, algebra, and quadratic functions knowledge.

However, the students have created the equation of the parabola without questioning general validity. A pause of 5 seconds (line 753) appears to the interviewer as an indication that they need some help in this direction:

754	I:	Okay. (..) ,is that universally valid now?
755	R:	No that is a conjecture (..) ,which we found with- (*points at the notes in the upper part of the note sheet, fig. 2a) (.)* eight points or so- (.)
756	I:	Y-e-s- ,so with these two examples (*points at the two lists for e=1 and e=2, fig. 2a*) you have-
757	/R:	yes
758	/L:	(*synchronic*) yes
759	/I:	seen that now already right' (*takes a deep breath*) ,the problem now is that- (..) well- (*points at the screen*) ,here alone you have just seen *e* equals three- ,so you would have to try out all.
760	R:	mhm'
761	I:	It's about general- convincing now
762	R:	(*looks at the note sheets*) Yes. (*looks at the screen*) (…)
763	I:	How can you do that (*both students look at the note sheet*) (9sec), you have now just read out (*points at the screen*), the coordinates right'
764	R:	Mhm' (*nods*)
765	/L:	Mhm- (*I touches some of the sheets on the table*) (...)
766	I:	My proposal would be now that you take another look, at one of these diagrams' (*puts the printed sheet to the top*) (..), and now (*points at A in fig. 2b, slightly shaking his finger*), assume- generally here ,that this- (.) point. A' has the coordinates x y.
767	R:	Yes' (.)
768	I:	(*takes back his hand*) And then proceed from there
		L writes "(x|y)" next to "A" on the printed sheet, then after a short break marks the y-coordinate of A on the y-axis and labels it "y", then labels P "x" (23sec).

The interviewer's intervention shifts the focus on the general validity of the obtained formula, beyond the specific considered three cases of *e*=1; 2; 0,5 (lines 754-759). On the non-locutionary level, he invites the students to justify their conjecture, i.e. to build an argument proving that their formula is generally valid ("it's about general-convincing", line 761). Under this prompt, the students express awareness about the status of their equation (755: "that is an conjecture"), but they are stuck in the arithmetic-algebraic frame. From voice intonation, pauses and broken language, a certain reluctance is observable. The interviewer reacts by pointing (both verbally and with gesture) to the coordinates of A on the diagram of the work sheet (Figure 2b), and asks explicitly for a change of view ("take another look", line 766). With his speech-gesture intervention, the interviewer is also supporting the students at a semiotic level. In order to shift to a geometric frame he shows that this "other look" must consider the coordinates *x* and *y* of the points. Following the interviewer's suggestion, Lisa and Rosa include the coordinates in their considerations while manipulating the graphs on the computer screen. However looking at the value and size of the coordinates, their investigations are still kept in the arithmetic frame (e.g. "the larger *e* gets the larger *y* too", line 811): the students are still framing the parabola as a structure of arithmetic relationships. Thus, the interviewer plays the role of a teacher to initiate a change of frame and draws an auxiliary line from A perpendicular to the *y*-axis (fig. 2b) (naming the intersection point Y):

835 I: I'll do in red then, there's not so much of that in there yet' (4 sec) so I now draw (*draw the red line in fig. 2b*), another- line- here. (..) (*slides the print to the students again*) ,and now- you can- (*circles with his finger around the new triangle AYB*) take a look at this triangle here ,and look what of that is known to you (6 sec) (*R, pointing at the right angle AYB, says: The angle*)

The interviewer first acts at a semiotic level, adding a new line to the figure, and then prompts the students to use their theoretic knowledge ("look what of that is known to you", 834). In this way, he is bringing into the scene a new conceptual frame, the geometric one, and is inviting the students to work in it. Once entered into the new frame, the students suitably connect geometric features with algebraic terms revealing *y-e* and *x* for two sides of the triangle AYB. By that, they show a semiotic control on connecting geometric with algebraic signs. In line 857 Rosa sees a structure in the diagram and suggests applying the Pythagoras Theorem on triangle AYB. As a consequence, they get the equation $x^2+(y-e)^2=(y+e)^2$. Finally they transform it into $y=\frac{x^2}{4e}$ and identify its equality to $f(x)=e\cdot\left(\frac{x}{2e}\right)^2$.

CONCLUSION

The analysis of Lisa and Rosa's epistemic processes in the case study shows that *gathering* and *connecting actions* are supported by the *intertwining of semiotic and theoretic control within a given conceptual frame*. The students in fact are successful in gathering and connecting signs first within the arithmetic frame (parabolas as quadratic relationships between numbers, lines 618-665), then within an algebraic frame leading to an equation as a generalization of the arithmetic structures gained (lines 666-753), and finally within a geometric-analytic frame (836 onwards).

However, although the students seem to act according to well-developed semiotic and theoretic control *within* each conceptual frame (we may call it *local* control), they get stuck. Acting as a teacher, the interviewer exploits a synergy between different semiotic resources (written signs, speech and gestures: in particular, lines 766, 835), and supports the students in considering a new frame (geometric), which enriches the previous one (algebraic). With this “new look” at the problem, the students become able to *see a new structure* (Pythagoras Theorem) and to quickly complete the task, exploiting again their semiotic and theoretic control within the new enriched frame.

The interviewer’s intervention was not expected by the research plan, but it helped us to seize the importance of theoretic and semiotic control at a global level, encompassing different conceptual frames on the same situation. Our analysis suggests that in order to carry out complex problem-solving or argumentation processes, semiotic/theoretic control within each single conceptual frame (*local control*) needs to be grounded in a higher order kind of control, which allows flexibly looking for other conceptual frames, and suitably connecting them (*global control*).

The next step in our research will consist in validating our results for a wider range of contexts, and in studying what didactical interventions in the mathematics classroom can develop students’ semiotic and theoretic control at a global level.

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LEARNING WITH FACEBOOK: PREPARING FOR THE MATHEMATICS BAGRUT - A CASE STUDY

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To help students prepare for the resit exam of the mathematics Bagrut (Israeli matriculation) of 2013, the Center for Educational Technology established a virtual review session using Facebook, for four days before the exam. 614 students and 16 teachers participated. We examined three central questions, each about using Facebook to prepare for the mathematics Bagrut: What opportunities for learning were created? What are the students' opinions? What are the teachers' opinions? Analysis of the posts on Facebook revealed five types of situations with potential for learning. Answers to on-line questionnaires show that both students and teachers hold positive opinions towards the solution for learning provided by Facebook. We recommend researching the opportunities for learning afforded by the social networks.

INTRODUCTION

Online social networking sites like Facebook have developed in recent years and have become the most popular meeting places for youth and adults (Boyd, 2010). Many studies have investigated the potential of using these networks to promote learning (e.g. Forkosh-Baruch & Hershkovitz, 2013; Neman, Lev, & Amit, 2013). In some of these study teacher-student interactions the student has the status of the teacher's friend (Madge et al., 2009) and hierarchies are formed as a result of this friendship status (Steinfield, Elison & Lampe, 2008). Asterhan et al. (2013) discuss whether and how teachers may use Facebook for innovative, collaborative forms of online learning that extend beyond the traditional classroom, and whether this is at all recommendable or feasible.

Recently many researchers have studied the Facebook option of creating a group where teacher and students belong but do not need to be "friends". Students perceived learning in this environment as very intensive and collaborative in nature (Meshar-Tal, Kurtz, & Pitwerse, 2012). In the learning of mathematics social network sites have been found to invite student collaboration and encourage learning (Baya'a & Daher, 2013).

In this present research the learners were members of a group on Facebook opened specially for preparation for the mathematics Bagrut (Israeli matriculation) exam. We investigated what opportunities for learning were created as a result of the interactions that formed within the group and examined the viewpoints of students and teachers who took part in the study group. We present the results of a pilot study, in preparation for a wider research on this subject.

OUTLINE OF THE RESEARCH

Eight groups – four in Hebrew and four in Arabic – were opened on Facebook for four days, twelve hours a day, before the resit of the mathematics Bagrut exam. Teachers were on call to respond to students (three shifts of four hours). The Hebrew speakers' group comprised 513 students, and the Arabic group 101 students. The groups were divided according to the questionnaires in the Bagrut exams at intermediate and advanced levels. The teachers were trained in online teaching, in the principles of a forum, and in the Facebook tools, and were given technical instructions on how to provide responses in the forum.

During the activity the students raised questions in whatever subject they wish. The questions were uploaded to the forum as photographs or as details of book, page, and exercise number (the teachers were provided with all the relevant textbooks). On receipt of a question the teacher sent a reply, "I will upload an answer soon" and after several minutes (on average 10 minutes) he uploaded a response to the forum in a similar manner: as a photo of the page on which he wrote the solution, or hints on how to reach it. At the end of the study session online questionnaires were sent to the students and teachers who took part in the forums. 105 students and 15 teachers completed the questionnaires.

RESEARCH METHODS AND TOOLS

We used a mixed methods research model (Johnston & Onwuegbuzie, 2004) which combines qualitative and quantitative data analysis. The research tools were two online questionnaires, one for students and one for teachers, comprising open and closed questions. The open questions for the student included those on his background, which exam paper he was taking, how he heard about the study group, and his suggestions for what he would like to preserve in the study group and what he would like to improve. The open questions for the teacher included those on his seniority, his online teaching experience and the classes he usually teaches. Teachers were also asked to write down their feelings about teaching through Facebook, to describe interactions they remember favourably, etc. The closed questions in both questionnaires comprised statements on a Likert scale from 1 (disagree) to 4 (strongly agree). These statements included issues such as the use of technology, peer learning, motivation to continue learning/teaching in a similar manner in the future, interactions with students, etc.

The participants answered the questionnaire at the end of the Facebook review session. The answers to the open questions were analysed by three mathematics education experts to improve validity reliability by triangulation (Denzin & Lincoln, 2000). The analysis was carried out in four stages: first the answers were collected; in the second stage all the answers were divided into short sentences; subsequently each sentence was classified according to general subject matter; and finally the sentences in the same subject matter group were collected together and arranged according to categories. After much discussion 100% agreement was achieved between the judges about the categorisation of the data.

In order to learn what opportunities for learning were created as a result of revising for the Bagrut exam in mathematics through the medium of Facebook an analysis was made of the content appearing in Facebook throughout the review session. First we mapped the participators in the interactions: teacher-student as opposed to student-student. In the second stage we analysed all the interactions and learning opportunities that arose.

FINDINGS

Figures 1 shows the number of students in each group according to the levels of the exam papers (804 and 805 - intermediate, 806 and 807 - advanced) and the number of questions or discussions raised (the posts). Figure 2 presents a map of the opportunities for learning observed throughout the review session.

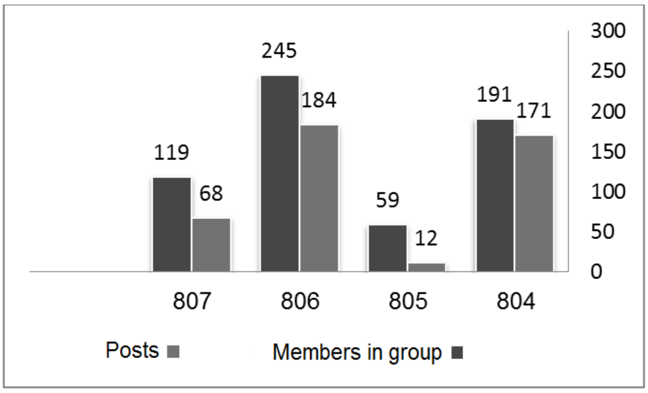


Figure 1: Number of participants and number of posts in each study group.

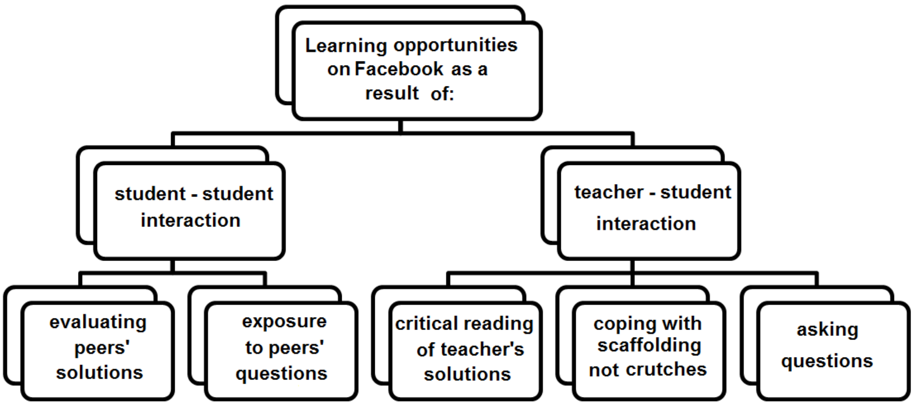
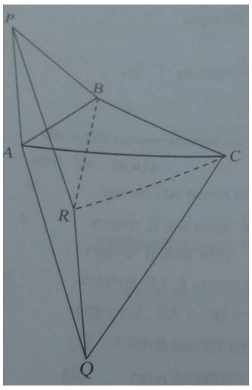


Figure 2: Learning opportunities on Facebook resulting from teacher-student interactions and student-student interactions.

We now provide a short description of each opportunity and some episodes from the forum.

Evaluating peers' solutions

During the review session students asked for help in pinpointing the mistakes they had apparently made in their solution, intending that the teacher would evaluate their work and find the mistake. We observed that during the time that passed between a student uploading his solution and receiving a reply from the teacher (perceived as the source of authority in the forum) other students responded and tried by themselves to pinpoint the source of their peer's error. The students' attempts created a cognitive appeal to the correctness or incorrectness of the evaluation and thus started a chain of responses until a final response was given by the teacher. Similarly we noticed that throughout the review session students had considerable success in taking the teacher's role by attempting to provide explanations through the forum. This finding is strengthened by the students' answers to the questionnaire at the end of the review session. 72% (N=104) stated that they learned from responses given by other students. Figure 3 presents a solution uploaded by a student (Shiran) and is followed by an excerpt from the forum where another student (Achinoam) evaluates the solution before a reply is received from the teacher.



Triangles APB, ACQ, BCR are similar isosceles triangles whose bases are the sides of triangle ABC.

Prove: ∡ACB = ∡QCR

AB = AC given
PA = PB given
AQ = QC given
∡AQC = α
⇒ ∡QAC = ∡QCA = 90 − α
⇒ ∡BRC = α similar triangles so angles are equal
⇒ ∡RBC = 90 − α = ∡BCR
∡RCA = β
⇒ QCR = 90 − α − β subtracting angles
⇒ ∡ACB = 90 − α − β " "
⇒ ∡QCR = ∡ACB

Figure 3: Geometric problem and student's solution uploaded on the forum.

Achinoam: I think you made a mistake with the angle QAC. Shouldn't it be 90 minus half alpha?

Shiran: Yes you're right. And that changes them all to 90 minus half alpha… So it's the same proof, I simply need to change the alpha to half alpha… right?

Achinoam: Yes, got it, great, thanks ☺ But I think that generally you can't say that AC=AB is given. Right? Shouldn't it be 90 minus half alpha?

Shiran: I meant br=rc

Shiran: I've got too many mistakes ☹

Achinoam: Aaah. Now it all makes sense! It's really not so bad. It's a mini mistake! ☺

Exposure to peers' questions

Throughout the review session students were exposed to questions raised by other students and tried to answer these questions themselves. This finding is based on the number of observers of each post in the forum, on the students' reports in the questionnaire, and on the responses of the students in the forum itself. Exposure to peers' questions expanded the available pool of exercises and presented the additional challenge of dealing with questions that were difficult for their peers to solve. This finding is supported by the students' questionnaires where 78% reported that they learned from questions raised by other students.

Critical reading of teachers' solutions

The most significant learning opportunities that occurred during the review session were the chance to read, to analyse, and to understand the teachers' solutions on the forum. On some of the posts, after reading the teachers' solution the student returned to his own solution to compare the two methods. In this excerpt we can see the comparison one student made after receiving the teacher's answer to his question. At the end of this post an error was found in the book, thanks to the student's "stubbornness".

> Thanks. But somehow in the answers they put 3/4 instead of 3 root 3 divided by 2. And according to the volume of the prism that you found I got the correct t but the maximum volume is different. Maybe they made a mistake? I'd like you to solve the rest because I didn't get the same answer …

Coping with scaffolding not crutches

In not a few cases the teacher's response was advice for continuing the solution and the student had to deal with the problem on his own. 87% of the students claimed that the teachers' tips helped them learn. In this excerpt we see a hint given by the teacher and the student's satisfied response that it helped him to solve the problem.

Teacher:	I recommend you to try to finish this by yourself. If not, let me know and I'll post the solution. Tip: the lateral area is the sum of the areas of the rectangular faces without the bases.
Student:	Thank you very much for the help. I got it right! ☺

Asking questions

Throughout the review session, in addition to the problems the students posted as photos or text, they asked concrete questions on particular parts of a solution, and expressed doubts that arose during a solution. In contrast to questions asked face to face, here asking questions requires another skill – the ability to formulate the question in writing, with suitable emphasis for the teacher who is supposed to answer.

The following excerpt show a student's questions after a solution has been posted by the teacher. It includes a search for explanation/proof, indicating critical reading of the solution.

> It's not clear to me why you can deduce from the sketch of the graph alone that there are no maximum or minimum points? Who says there isn't one before the asymptote? And how can you tell without a table if the function is increasing or decreasing from the asymptote? Thanks!!

Instructional interactions on the social network

The answers to the questionnaires were analyzed as described in the section on research methods. In Table 1 we show examples of students' and teachers' remarks in each of the categories: motivation for continued learning, peer learning, technology utilisation, and supportive learning climate.

Categories	Student questionnaire	Teacher questionnaire
Motivation for continued learning	I'm glad I got the chance of the Facebook forum. It gave me the option with exercises that I couldn't solve, not to give up like I usually do, but to get the solutions from a teacher – that really helped me.	I really liked the fact that the students asked relevant questions, related to the answers, and didn't give up until they understood.
Peer learning	The forum was a very good idea. We could learn from other students' questions and answers.	A student posted a question after a lesson, and I noticed that students started to help each other in the forum, and succeeded in solving some parts of it.
Technology utilisation	I would recommend improving the method of posting pictures on Facebook.	The idea of photographing the problem or the solution and posting is brilliant and effective in making best use of the time and for presenting the solution.
Supportive learning climate	I would be very happy to get this kind of help throughout the year. It is all over and above what a student can expect for success. Thank you so much for all the help.	The students' appreciation was heart-warming.

Table 1: Students' and teachers' remarks about the integration of Facebook in preparing for the Bagrut exam in mathematics.

As can be seen in the table, students' and teachers' responses were mainly positive, and in general the participants' responses indicate great satisfaction with the use of Facebook in preparing for the exam. 75% of the students (N=105) stated that it was easy for them to ask questions and receive replies through Facebook, 79% stated that they would like to use Facebook in this way also for learning other subjects, and 87% stated that they would like to continue learning in a similar manner throughout the year. 93% of the teachers (N = 15) stated that the environment encourages meaningful learning and that the project justifies the investment of resources. There was 100%

agreement among the teachers on willingness to continue in a similar manner next year. 93% stated that they would be interested in opening similar learning environments for their own students during the year.

A little criticism on the use of technology was heard from both students and teachers, relating to the uploading of pictures that were sometimes not clear, thus making it difficult to understand and respond to the problem. In addition, teachers in charge of forums where there was a lot of activity indicated the need for extra staff to help manage the responses where necessary.

DISCUSSION AND CONCLUSION

The findings in this research indicate students' great satisfaction with the opportunity given them to study for the mathematics Bagrut exam through the medium of Facebook. The Facebook forum encouraged different interactions between teachers and students and among the students themselves. These interactions provided the learners with learning opportunities which included: asking questions, peer learning, different methods of problem solving, and critical reading of solutions. They were motivated to deal with questions their peers found difficult, and were exposed questions from different textbooks and to solution methods of different teachers. These learning opportunities carry extra value and are important in the learning process leading up to the Bagrut exam and in general. Individual study without interactions with peers or with a teacher is unlikely to afford any of these opportunities.

The findings relating to the students' positive opinions of learning in a Facebook environment strengthen findings of earlier studies about learning on social networks (Meshar-Tal, Kurtz, & Pitwerse, 2012). The teachers also expressed great satisfaction with the Facebook environment for learning and declared their intention to adopt a similar environment preparing for Bagrut exams in the following years and for teaching during the school year.

This research was an initial testing of teacher-student and student-student interactions on Facebook in a four-day review session in preparation for the Bagrut mathematics exam. The results encourage continuation and further research into these and other related aspects, on wider groups of teachers and students, and for longer time periods. A wide based research in the subject would be likely to lead to peer learning also among the teachers themselves – on how to characterize students' questions leading up to the exam, and in general. In addition, we recommend that continued research on these issues could provide educational policy makers with an understanding of the value of investing in similar projects in the future.

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LEARNING NEGATIVE INTEGER CONCEPTS: BENEFITS OF PLAYING LINEAR BOARD GAMES

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Linear board games have shown great promise as tools to teach whole number concepts (Ramani & Siegler, 2008), but little is known about their utility for supporting negative integer concepts. This study sought to extend the use of linear board games to teach integer concepts. Forty-eight first graders (ages 6-7) counted along an integer board game with negative numbers or did control activities with integers. Students who played the board game made significant gains on several measures related to identifying and ordering integers. Findings suggest that even young children can benefit from games with negative integers, and we provide implications for instruction.

Young children are often exposed to negative numbers in contexts, such as negative temperatures, negative points in golf or video games, or when selecting floors below zero in elevators in some countries, before they are formally introduced in school. An understanding of negative numbers can help students better understand that zero is not the smallest number, a conception they have difficulty overcoming (Bofferding, 2014). Further, having experiences counting backward through zero can help them understand that expressions such as 3-5 are meaningful (Bofferding, 2011), making it unnecessary to teach the misleading rule that you cannot subtract a larger number from a smaller one. However, there is little data on how children's informal experiences with negative numbers might influence how they make sense of these numbers. This study provides initial data on the benefits that playing linear board games can have on first graders' (6-7 years-old) understanding of negative numbers.

THEORETICAL FRAMEWORK

According to Case's (1996) Central Conceptual Structures for Number Theory, by the age of about 6, children have coordinated their understanding of number concepts involving symbols, number order, number values, and the relations among them. Therefore, they can say the counting sequence, know that numbers further in the counting sequence correspond to larger values, and know that the larger values correspond to larger sets of objects, all of which can be represented with numerals. Further, they understand that saying the next number in the sequence corresponds to getting 1 more (or adding 1) and that saying the previous number in the sequence corresponds to getting 1 less (or subtracting 1). Researchers describe this understanding as using a mental number line. These concepts form students' initial mental models of number that they must change to accommodate new numbers like negative integers (Vosniadou, Vamvakoussi, & Skopeliti, 2008).

In order to extend their mental number line to include negative integers, they must extend the number sequence to the left of zero (or less than zero), with numerals symmetric to the positive ones, but marked with negative signs. They also must extend the idea that numbers further to the left on their mental number lines correspond to smaller values, e.g., -5 is less than -3. The purpose of this research is to determine the extent to which informal experiences with negative integers could help students extend their mental number lines in this way.

RELATED LITERATURE

Drawing on Case's (1996) theory, Ramani and Siegler (2008) posited that young children could learn whole number concepts if they played linear board games which supported the development of concepts important to establishing a mental number line. Through a series of experimental studies, they found that pre-schoolers (3 to 5 year-olds) who "counted on" from 1 to 10 while playing linear board games made significant gains on a variety of number concepts compared to children who did numerical, control activities. The children who played the game improved in counting to 10, determining which of two numbers is larger, correctly identifying numbers 1-10, and estimating the positions of numbers 1-10 on an empty number line (Ramani & Siegler, 2008).

The literature on young student's understanding of negative integers is fairly sparse but suggests that they are capable of learning about them if given purposeful experiences with them. Before learning about negative numbers, first graders order negative integers next to their positive counterparts (Peled, Mukhopadhyay, & Resnick, 1989) or treat them as numbers that have been taken away and order them before or after zero (Bofferding, 2014; Schwarz, Kohn, & Resnick, 1993). Similarly, they treat negative numbers as positive and consider -5 > 0 and -5 > -3 (Bofferding, 2014; Peled et al., 1989). However, after instruction on the order and value of negative integers and working with these concepts through a series of card games utilizing number line contexts, first graders at the end of the school year (7 and 8 year-olds) improved significantly on such tasks compared to students who had not received this instruction (Bofferding, 2014). Further, students who know the order of the negative numbers can begin to reason about integer addition and subtraction problems and successfully solve some of these problems (Bofferding, 2010; Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2013). Because integer concepts rely on the same order and value relations underlying the whole number mental number line, it is likely that the board game experiences that help students develop their whole number mental number line could also help students develop an integer mental number line.

RESEARCH AIMS

The primary aim of this research is to test whether students who play linear, numerical board games develop a deeper understanding of negative integer concepts than peers who participate in integer-related control activities. We hypothesized that the board game group would make significant gains on negative integer tasks compared to students in the control group because the board game provides students with the opportunity to experience an extension of the positive number line into negatives.

METHODS

Setting and Participants

The study was conducted at an elementary school located in a low-income area where approximately half of the students were Hispanic and a third spoke a language other than English at home. The study was conducted during the first three months of the school year. Across six classrooms, students were randomly selected for participation until we had a sample size of 50 (26 female; 24 male).

General Design

To test our hypothesis, we employed an experimental design involving a pre-test, random assignment to control or treatment group, intervention, and post-test. The design extended the methods and measures used by Siegler and Ramani (2009) to include negative integers. For the intervention, each participant worked with a researcher for three, 15-minute sessions. However, two children in the treatment group withdrew from the school before the post-test measure was given, resulting in data for 23 children who played the integer board game and 25 children who did the control activities. A professor and graduate research assistant collected data in corners of the classrooms, replacing a portion of the students' whole-class mathematics instruction. Most participants worked with both researchers during the course of the study.

Pre-test and Post-test Measures

The pre-test and post-test were identical so that we could measure gains in the participants' mathematical knowledge. We conducted both tests as individual interviews with the students, and we did not provide specific feedback on their performance, just general encouragement.

The first section of the tests involved students *counting* forward to ten and backward from ten as far as they could go. If they stopped at 0, we asked if they could keep counting down any further. The second section involved verbal *integer identification* of the numerals from -10 to 10, presented on isolated pages in random order. The third section concerned *integer order*. Students were asked questions such as, "What number comes two numbers after 7?" They also ordered a set of eleven cards labelled -5 to 5 and indicated which were the least and greatest.

The fourth section on *integer values* asked students to identify which of two integers was closer to 10 (n=10) and further from 10 (n=10). The two numbers were a mix of positive integers, negative integers, and zero. For each question, children were shown an equilateral triangle with 10 inside the peak and the pair in question equidistance away, placed in the left and right corners. Further, each integer was named by the researcher. If students were confused on the use of the words "closer" and "further," the researcher would make a statement such as, "If 10 is the largest, which of these [pointing] -9 or 5 is less or further away from 10?"

The fifth section dealt with *operations*. Students were first given additive expressions involving strictly positive integers, and then negatives were introduced into the expressions. Second, the researchers presented expressions with subtraction, initially with positive differences, and then negative differences. The section ended with a contextualized addition problem and two, two-digit arithmetic problems.

The final section of the tests involved students placing integers on *number lines*. Students completed a packet involving positive integers followed by one involving negative integers. Each page of the packet contained an empty number line 25.5cm long with two integers marked. On the first page of both packets, students were asked to put a pen mark where 0 would go, given the locations of -5 and 5. For the positive packet, the remaining pages contained empty number lines marked with 0 and 10. The placement of zero in the middle, i.e., leaving space for the negative numbers to the left, was an important feature. Students were asked to make a mark where a given integer should go a total of 18 times (1 through 9 in random order, twice). The researchers gave instructions such as, "If here is 0 [point to the middle] and here is 10 [point to the right], then make a mark anywhere on this line [motions to whole 25.5 cm line] where 6 should go." The negative number packet worked similarly only with -10 marked on the left and 0 marked in the middle. Students were told to place the negative integers -1 through -9 on the respective pages (see Figure 1 for examples).

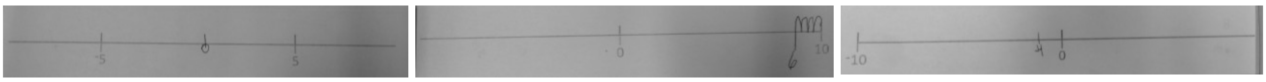

Figure 1: Three number line tasks: mark where zero goes (left), where positive integers 1 to 9 go (middle), and where negative integers -1 to -9 go (right).

Control Group

For their three sessions, the control group students cycled through three types of activities with the researcher. The first activity involved counting a collection of 1-10 poker chips and counting backward as far as they could. They did not receive feedback on correctness.

For the second activity, students put six integer cards in order from least to greatest. The researchers rotated between three sets of cards; for example, one set they ordered included the following integers: 6, -9, -4, 0, 3, and -1. After the students ordered the set, they were asked to point to the least and the greatest. Students were not given any feedback on the ordering or the identification of the cards.

The last activity in the cycle was a game of memory where the goal was to match integers. Students were given corrective feedback if they attempted to collect an incorrect match but were not told the names of the numbers. Both the positive and negative versions of integers appeared, so if students tried to match *n* and *–n*, the researcher would interject that the cards did not look exactly the same.

Treatment Group (Game)

During each 15-minute intervention session, the treatment group played a board game against the researcher using a board labelled with the integers -10 to 10 (see Figure 2).

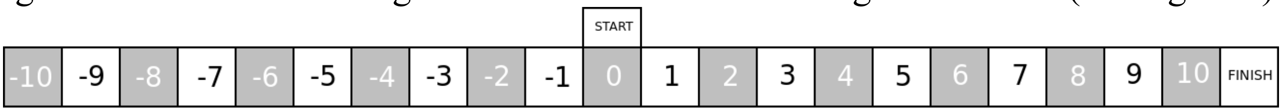

Figure 2: An illustration of the linear, numbered game board.

Players started by placing their tokens at zero, and the first player drew a card from a card deck. In the first version, all but one of the cards was labelled with a 1, 2, or 3. The remaining card contained the text, "All players go back to -10." When this card was drawn, the student had to count backward while moving their tokens back to -10. The researcher always stacked the deck so that this card would come up in the first few turns of the game, assuring players would proceed from -10 to 10 in each round.

If players drew a 1, 2, or 3, they moved their tokens that number of spaces and named the numbers on the spaces they passed through. For example, if a player on -4 drew a "2", then she would move her token to -3 and say "negative three", then move her piece to -2 and say "negative two." The game ended when a player crossed 10.

During the third session, the card sending players back to -10 was removed and a new stack of cards was introduced. Cards in the new stack were labelled either -2 or -4. Players began the game by drawing from this stack and counting backwards as they moved to -10. Once a player reached -10, on her next turn she would begin drawing with the deck containing positive numbers. From this point, play continued as normal, with the game ending once a player crossed over 10.

The researchers gave enough feedback to ensure legal turns by the students. Sometimes this involved correcting the name of the integer that they landed on, other times it involved correcting the number of spaces the game piece was moved. The game would not proceed until the student had said the correct number names aloud. Students played an average of 4 games in 15 minutes.

ANALYSIS

The focus of the analysis and results will be primarily on the negative integer items. For the *counting* backward to -10 task, students were given 1 point per correct number named below zero until their first error. Therefore, a student who counted "-1, -2, -4," would receive a score of 2 out of a possible score of 10. Similarly, on the negative *integer identification* task, students received a point per negative integer identified (for a possible total of 10). On the *integer values* task, students received a point per correct problem (for a possible total of 20). Finally, for the *number line* tasks, we calculated students' percent absolute error (with small numbers being better), comparing where they marked a number on the empty number line to its proper location.

RESULTS

Based on our preliminary analyses, students in the treatment group made larger gains across item types even after playing the board game for only 45 minutes (see Table 1).

	Pre-test		Post-test		Gain	
Item Type	Control	Game	Control	Game	Control	Game
Counting to -10	.36	.43	1.32	4.35	.96	3.91
Negative Integer Identification	1.08	1.61	3.00	9.22	1.92	7.61
Number Values	7.32	6.70	7.5	9.30	.18	2.61
Number Line Percent Abs. Error	41.60	49.81	50.24	39.61	8.64	-10.20

Table 1: Average scores on the pre-test and post-test and average gain for the Control group (n=25) and Treatment or Game group (n=23) across item types.

We examined the multivariate effects of condition (game treatment versus control) across the four tasks using a MANOVA on the average gain scores for the four item types. There was a significant effect of condition on the average gains for all items, $F(4, 43)=5.821$, $p=.001$, partial eta squared= .351. The game group significantly improved across all measures. Separate univariate ANOVAs on the outcome variables revealed significant treatment effects on identifying negative integers, $F(1,46)=17.059$, $p=.000$ and counting to -10, $F(1,46)=6.540$, $p=.014$, with smaller effects on identifying integer values closer to or further from 10, $F(1,46)=4.521$, $p=.039$, and in their percent absolute error when marking numbers on a number line, $F(1,46)=4.097$, $p=.049$.

Counting back to -10

Other than either stopping at 0 or counting back all the way to -10, one student stopped counting at -3 and another student named negative numbers in random order.

Identifying negative numbers

Generally, students in the control group ignored the negative signs and called negative integers by positive integer names. A few students made up new names for the numbers, e.g., "infinity one" for -1 or "equals three" for -3. Only one student in the game (treatment) group was unable to identify any of the negative integers.

Integer Values closer to/further from 10

The smaller gains that students made on these items reflect the difficulty that some of them had in overcoming their desire to treat negative numbers as equivalent to positive numbers in value. Students in the control group learned that -5 could not be matched to 5 in the matching game, but they had no reason to think that the values were different. When choosing whether -7 or -10 would be further from 10, one student in the control grouped explained that -7 is further because -10 is the same as 10. More nuanced, when choosing whether -8 or -5 would be closer to 10, a student in the game group said -8, but then justified it by clarifying that -8 is closer to -10. Therefore, she may have interpreted -10 as something different than 10 but with equivalent values. This is true in some sense as both numbers are equal distances away from 0.

Placement of numbers on an empty number line

Across both groups, students displayed a strong tendency to mark 1 toward the left edge of the number line, even when 0 was marked in the middle of the line. They explained that 0 always starts at the beginning of the line and then 1 would come after that. Most students continued to use this logic even after being reminded that zero was already marked for them on the page. By the post-test, on average, students in the game group made significant progress placing integers closer to their actual locations; however, several continued to place positive numbers to the left of zero on some trials.

CONCLUSIONS AND IMPLICATIONS

As demonstrated by the large gain in the game group, once students knew that the negative sign was an important feature for designating a new number, students were quick to learn the negative integer names. Further, a significant number of students in the game group successfully counted back to -10, suggesting that saying the backward number sequence as part of regular instruction can help students extend the whole number sequence to the negative integers. Interestingly, some of these students did not count backward to -10 even though they could do this while playing the board game. Students are generally expected to stop counting once they reach zero, so continuing into the negatives may not have seemed an appropriate response to our prompting. Stopping at zero could also contribute to students' resistance in subtracting a larger number from a smaller one.

Although the integer value result was significant, even students who could count backward to -10 had difficulty determining which of two integers was closer to 10. This is striking because students in the game group often commented about how they did not want to be in the negatives because their goal was to get to positive ten. These results provide further evidence that students rely on the absolute value meaning of integers, using zero as a reference point, and that this reasoning trumps their inclination to consider numbers further to the right on the number line as larger. Students need explicit experiences to help them negotiate the differences between absolute value and ordered value; including language into the game about integer values could provide needed scaffolding. Finally, the number line results suggest that teachers should talk about and use number lines that do not always start at zero to help students from overgeneralizing ideas about zero. Students already had strong conceptions about zero at the beginning of first grade, suggesting targeted instruction about zero and its placement relative to all integers could be beneficial before this point.

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REVISITING TRANSFER OF LEARNING IN MATHEMATICS: INSIGHTS FROM AN URBAN LOW-INCOME SETTLEMENT

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Debates on transfer of learning in mathematics are not new. Claims of situatedness of learning within tasks and non-transferability of knowledge between tasks are widely contested. Direct Application (DA) of learning is a common paradigm for characterising transfer which led to many instances of transfer failure (Bransford & Schwartz, 2001). The alternative framework presented here shows that some of the transfer failures in mathematics can be considered as partial transfer by broadening the DA paradigm which can scaffold classroom pedagogy by drawing upon everyday mathematics. Claims are supported by data drawn from an economically active low income settlement where sample middle graders are engaged in house-hold based micro-enterprise, possess diverse opportunities for gaining mathematical knowledge.

THE CONTEXT

Debates on situated perspective and transfer of learning in mathematics are not new. There have been several arguments in favour of and against learning transfer. Beginning with the pioneering work of Lave (1991), Lave and Wenger (1991) and Greeno, Smith and Moore (1992) situated learning in the domain of mathematics education broadly claimed that learning is situated within tasks at hand and that knowledge is non-transferable between different tasks. But, Anderson, Reder and Simon (1996) contested by arguing that such claims are “sometimes inaccurate and exaggerated”, and the “implications drawn are mistaken” (p. 5, 6). Anderson *et al.*'s contention (*ibid*) was further challenged by Greeno's (1997) objections of the generality and presuppositions about the levels of analysis that Anderson *et al.* had adopted. Greeno argued that the counterclaims of Anderson *et al.* addressed different questions by focusing on “knowledge and contexts of performance” and not the “activities and situations in which activities occurred and learned” (p. 6) and therefore they answered wrong questions. Earlier debates on transfer of learning indicate a growing feeling within cognition researchers about too many instances of transfer failure and lack of evidence that can challenge Thorndike's assertion that transfer is rare and occurs only between two similar situations (Bransford & Schwartz, 2001). Transfer literature is thus based on claims and counterclaims that looked at different notions of transfer devoid of unanimity over any concrete outcome. Most debates came within the paradigm of “direct application” (DA) of learning to new problem situations often following “sequestered problem solving” (SPS) method (Bransford & Schwartz, 2001). In this paper, it is argued that sticking to the rigid boundaries of DA paradigm could be one reason for many instances of transfer failure and that we need a broader perspective to incorporate the doer's goal structure while using algorithms and learning

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the underlying principles in order to look at the transfer phenomenon holistically. The broader perspective on transfer can then help us address educators' concern for ensuring effective learning among students and increase their ability to carry forward such learning. DA and SPS characterisations detect transfers with a "yes-no", or "either-or" result and fail to indicate occurrences of partial transfer which can actually prepare ground for future learning. Transfer in everyday settings seldom leads to black and white conclusions. In this paper, an alternative framework is employed to revisit the transfer problem and reject the previously held characterisation of transfer as only direct application (DA) and also Lave's claim of non-transferability of knowledge. Extending and partially revisiting Bransford and Schwartz' notions of transfer, the alternative framework (*Algorithm Goal Structure*) addresses the transfer problem by considering comprehension and use of arithmetical algorithms as the central goal followed by learning to apply or relate to the underlying principles implicitly or explicitly.

Drawing from data collected through interviews and discussions with working middle graders from an economically active urban low income settlement, it is observed that children's work-contexts are diverse and consequently, the extent and type of mathematical knowledge that students acquire outside school can be expected to show diversity. Such diverse engagement with contexts help children develop effective context specific problem solving ability that could be used for effective mathematics learning in the classrooms (Bose & Subramaniam, 2013). In this paper, transfer of learning is explored among the sample middle graders while they solve mathematical problems reflecting everyday contextual situations in the school set up as well as in the situations that emerge in the work contexts. It is claimed that the occurrence of partial transfer works as scaffolds for better learning of different components of the algorithms and principles, unlike Bransford and Schwartz's relatively vague and indefinite notion of "preparation for future learning" (p. 69).

TRANSFER OF LEARNING (IN MATHEMATICS)

Different paradigms looked at transfer of learning and common among them was the direct application (DA) paradigm based on the notion of "initial learning followed by problem solving". Bransford and Schwartz (2001) however moved from direct application of knowledge to the "perspective of preparation for future learning". This notion is in opposition to those that Lave, Anderson *et al*. or Greeno had adopted. Table 1 below highlights the transfer notions that prominent researchers adopted.

Claims for both successes and failures in achieving learning transfer came up due to inconsistencies prevalent in the way transfer was defined. It is pertinent therefore for the educational researchers to revisit the definition and make generalisable claims that can be used for drawing larger pedagogic pointers for effective mathematics learning. Lave and Wenger (1991) looked at learning in the processes of co-participation as a situated activity, focusing on skill acquisition through engagement in tasks and claimed that situated perspective demonstrated that skills (or action) grounded in tasks often did not "generalise to school situations". In contrast, Anderson *et al*. argued that closer analyses of the tasks were required to make tenable claims of non-transferability of learning (1996, p. 6) and demonstrated situations where learning transfer occurred across contexts by showing transfer of mathematical competence from classroom situations to laboratory situations.

Thorndike's definition	Whether people can apply their knowledge to new a problem or situation (1901, as cited in Bransford & Schwartz, 2001)
Lave's definition	Transferring one's knowledge and skills from one problem-solving situation to another (1988)
Anderson *et al*.'s & Greeno's definition	Not explicitly defined, used prevalent notion of direct application
Bransford & Schwartz's definition	Moved from Direct application of knowledge (DA) to Preparation for future learning (PFL) (2001)

Table 1: Definition of transfer used by prominent researchers

Everyday mathematics vs School mathematics

"Everyday mathematics" is considered here as a form of mathematics used in out-of-school settings while engaging in contextually embedded practices. Everyday and school mathematics describe two forms of activities based on the same mathematical principles but on different cultural practices (Nunes *et al*., 1993). While the former offers freedom of using alternate techniques than those learnt at school, the latter is more of symbol based often detached from meaningful contexts. Distinction between everyday and school math can be summed up by contrasting their *basic features* (group-work, division of labour vs individual, independent work; tool manipulation vs pure mentation), *goals* (situation specific competencies vs generalised learning), *difference in procedures* (orality vs written, use of multiple units, contextualised reasoning vs use of symbols and formal reasoning), *difference in mechanisms of knowledge acquisition* (sharing, communication vs textbook based knowledge), and *metacognitive awareness* (meaningfulness, continuous monitoring vs algorithm-use, lack of meaningfulness, relevance) (Nunes *et al*., 1993; Lave and Wenger, 1991; Resnick, 1987; Saxe, 1988). We make use of the everyday mathematics framework to locate the transfers of learning as they emerge in the classroom practices or in the everyday settings, in routine work-contexts, or during economic transactions.

THEORETICAL FRAMEWORK: ALTERNATIVE VIEW OF TRANSFER

The whole debate depends upon what counts as transfer. In the alternative framework of *Algorithm Goal Structure* (AGS), "direct application" is used as the first filter and Bransford and Schwartz' definition of "*preparation for future learning*" as the second filter. The first filter considers definition of transfer on the following criterion:

- Have students learned the algorithms?
- If they have learned the algorithms, whether they can apply the underlying principles implicitly or explicitly?

In the second filter, it is checked whether students are able to transfer elements from their everyday or school knowledge in terms of some components of the underlying principles or the algorithms. In the present analysis, Bransford and Schwartz' notion of "*preparation for future learning*" is restricted to the possibility of learning the algorithms and elements of everyday knowledge that contribute to the components.

SAMPLE & METHODS

The sample for the larger ethnographic study done over two years was drawn from grade 6 of the municipality-run English and Urdu medium schools located in a low income settlement in Mumbai. This area has a vibrant economy in the form of house-hold based micro enterprise, which provide employment to the dense population living there. A total of 31 students were chosen randomly (every third student from the attendance register) to form the sample. Data was collected in three separate parts: the first part was semi-structured interviews of all 31 students to understand their family-background, socio-economic conditions, parental occupation, productive work done at home/elsewhere and student's involvement in them. The second part was interviews based on a structured questionnaire to understand students' basic arithmetical knowledge while the third part focused on students' knowledge about their work-contexts. All the interviews were audio recorded with prior permission from the respondents, the school authorities and also from the parents. The present data source is the second part of the interviews (arithmetical knowledge).

Location of the study

The large low-income settlement is located in central Mumbai where practically every house-hold is involved in income-generating work in which children take part from a young age. Being an old and established settlement, this low-income area attracts skilled and unskilled workers from all parts of India who come to the financial hub Mumbai in search of livelihood. The settlement is multi religious (Muslim majority) and multilingual (different language groups: Hindi/Urdu, Gujarati, Marathi, Tamil, Telugu). Common house-hold occupations include embroidery, *zari* (needle work with sequins), garment stitching, making plastic bags, leather goods (bags, wallets, purses, shoes), recycling work, etc. The goods produced here are sold not only in Mumbai but even exported, mainly to the Middle East countries.

INSIGHTS FROM THE FIELD

Transfer from everyday setting to school setting: Contextual problem-solving

Students' strategies while solving contextual problems presented in the school setting involved use of halving methods and convenient groupings that are commonly encountered in everyday setting. For example, while solving a school-type proportion problem of finding the price of 25 *burfi* when 20 *burfi* cost 42 rupees, some students found the prices of 10 and 5 *burfi* by halving 42 and 21, *"bees ka bayalees, dus ka ikkis aur paanch ka gyarah*" [*forty two for twenty, twenty one for ten and eleven for five*]. However, not many students could do this way and opted for "unitary" method of finding the price of one *burfi* first and then that of 25, but eventually got stuck in the middle. Some students arrived at 53 as the answer and justified that sellers often do not return small change of Re 0.50. Most students (barring a few who used unitary method) could not figure out how to proceed and struggled to choose an arithmetical operation for solving the problem. Under the first filter of DA, transfer does not occur in the sense of using formal algorithm (unitary method) as a generalised technique. But, using the second filter, we notice that some students were able to use their everyday contexts and reality perspective in using halving technique that allowed them to find the price of the "difference" in the number of *burfis*. From pedagogic viewpoint, this is a scaffold for teaching the generalised requirement of finding the price of one *burfi*. Under AGS, it fits as a case of transfer through second filter.

In another problem (*finding the number of days 16 kerosene oil cans can last if one can lasts for 7 days*), most students used their everyday mathematical knowledge. For example, one student while computing orally, grouped 15 days for 2 cans and arrived at 4 months for 16 cans (considering 30 days per month) and then compensated the extra counts of 1 day per 2 cans, by subtracting 8 days and arrived at 112 days. Under the DA filter, one can claim that transfer is not happening if the algorithmic goal was to use formal multiplication. But, upon relaxing this goal, it fits as a case of transfer since the student could draw upon her everyday mathematical knowledge to solve the problem. Interestingly, only one student used the multiplication table of 16 while some students used the tables of 10 and 6 and added the partial products.

Many students found the formal algorithm for division difficult to use and they often use convenient strategies. For example, one student actually divided 315 by 5 presented symbolically on the paper and obtained 13 as the answer. He soon realised the error that the actual result cannot be that less. Subsequently, he aborted the formal division and did mental computation. Under DA filter this example does not show transfer but under the second filter, student's use of everyday experience emerges in realising the error and learning of repeated distribution as the underlying principle.

Use of fractions

Binary fractions like *aadha* (half), *paav* (quarter), *aadha-paav* (half-quarter, i.e., one-eighth) etc. are part of the everyday discourse that most students were exposed to and comfortable in using. The common contexts where binary fractions are used and which students regularly come across, are while buying provisions, vegetables, milk, etc. Non-routine fractions remain difficult for most students to comprehend and remains poorly developed, whereas binary fractions are easy for them to visualise and concrete visuals of whole numbers are easy to come by. Beyond these imagining and correlating divisions with numbers become difficult. The transfer of learning in case of binary fractions are only partial which fails at the DA filter since transfer does not reach formalism but students are able to transfer some components of their everyday knowledge (fraction knowledge) which can scaffold learning of non-routine fractions.

Classroom activities[1]

During a classroom activity of shirt measurement which aimed at drawing upon students' everyday mathematical knowledge for informing classroom teaching, almost all the students preferred using *inch* tape (a popular measurement mode) for taking measurements although textbooks deal only with the international standard units. Most students however, did not know the relations between old British units (inch, foot) and standard international units (centimetre, metre). Students commonly used other indigenous measuring units like *bitta* (arm length) and *futta* (template used in tailoring work) during classroom discussions. AGS framework does not consider this example as transfer under DA since students could not convert two systems of units, but this example qualifies for partial transfer under the second filter on account of bringing in elements of everyday knowledge which can scaffold further learning.

Transfer from school setting to everyday setting: Problem-solving

In the work-contexts

Work-contexts of some students require doing quick calculations and use of approximation and estimation skills. Garment recycling work, for example, involves many children and requires weight measurement of the collected cloth pieces of varying size, colour and texture. The collected pieces are then sold off and the price is negotiated which requires children to make quick decisions and calculations. Children use convenient strategies and develop situation specific competencies, some of them reported use of multiplication tables that they learned at school. One student said that he does the multiplication "up in the air" by visualising the whole operation. He claimed that he does multiplication to cross-check the money he received.

Everyday shopping

School taught formal algorithms are often part of the daily routine, for example, at general stationary stores, sellers use paper and pen to arrive at the total cost. Oral computations are preferred while dealing with small amount of goods. Some students claimed that they cross-check the calculations on a paper using the school learned algorithms. Such examples indicate direct application of school taught methods and show transfer of learning to a different context. Number approximations during computations however qualify the second filter and shows partial transfer of learning.

Transfer failure

There are occasions where learning transfer did not seem to occur. It could be due to poor mathematical learning and lack of preparedness to handle complex calculations.

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[1] The classroom activities are drawn from the vacation camp classes for the grades 6 and 7 students of the Urdu school that the researcher and his colleagues conducted. The activities do not reflect actual classroom teaching.

For example, when the researcher discussed with a student whether she was satisfied with the *Rakhi* making (decorative wrist-bands) wage, she answered in affirmation. Upon asking she could only tell the market price of one dozen *Rakhi* – at least Rs 60 (one *rakhi* is sold for Rs 5). She was unable to find the price at which one gross (twelve dozen) of *rakhi* is sold – Rs 720 for making of which she only earns Rs 15 or less. She could not calculate the amount she earns for making a single *rakhi* – which is about 10 paise (one-tenth of a rupee). She could neither use the school taught multiplication algorithm nor any other computation strategy. There was no reflection of the use of any form of mathematical knowledge – school or everyday. Transfer of learning from either context was not visible. One can also argue here that poor learning of school mathematics impedes workers like her from checking the fairness of a deal or the wage and entitlements that are distributed among the workers.

DISCUSSION & IMPLICATIONS

Interactions with students showed instances where transfer of learning occurred and where it did not occur. Transfer or non-transfer both emerged as instances of computation strategies and such instances are often connected with knowledge and skill acquisition that are valorised in the community. Economic, social and cultural practices of the households often influence children's learning of strategies to meet different needs, like optimal use of limited resources, management of house-hold chores, routine purchase of provisions, and so on. Gaining such traits are seen as essential in the community. Arguably therefore, low socio-economic conditions affect diverse skill acquisition and induce transfer achievement or partial transfer as seen among the sample students. Diverse work-contexts and everyday settings create affordances that support mathematics learning and they are important sites of learning transfer between school and everyday mathematics. From pedagogic viewpoint such potentially rich contexts offer strong foundation for effective mathematical learning.

The examples discussed above showed how rigid boundaries of DA would term many of them as transfer failures whereas many of these examples carried elements of everyday mathematical knowledge and some components of the underlying principles. Transfer failure often occurs not just due to the lack of exposure to everyday contexts but also on account of less conceptual mathematical reasoning and cognitive preparedness. Encumbrance of using formal algorithms is another possible reason. However, transfer failure occurring due to partly applying algorithms can scaffold learning of the underlying principles and achieve the algorithm goal, i.e., complete understanding of the procedure and the rationale. Thus, partial transfer emerging from the use of small components of the underlying principles has strong pedagogic relevance for better learning. From a pedagogic viewpoint these are pointers that can help the educators connect everyday mathematics with school mathematics. Hence, it was rather essential to look for possible instances of partial transfer and that required broadening of the DA filter. The proposed “Algorithm Goal Structure” (AGS) model looks at the instances that are potentially strong to work as scaffolds for effective learning. AGS matches with the growing consensus and with several educational policy documents on bringing children's experience and prior knowledge to the classrooms and treating them as good starting points for building new knowledge. The arguments and claims made here are however evolving and calls for deeper exploration of their systemic and pedagogic underpinnings.

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EFFECTS OF A HOLISTIC VERSUS AN ATOMISTIC MODELLING APPROACH ON STUDENTS' MATHEMATICAL MODELLING COMPETENCIES

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The paper deals with the question of the practicability and the effectiveness of different approaches to foster students' mathematical modelling competencies. Within the modelling project ERMO (Acquirement of modelling competencies) a holistic and an atomistic approach of mathematical modelling were compared in order to find out which approach is more effective in fostering the students' modelling competencies. The results of modelling tests with three measurement points show that both approaches foster students' modelling competencies, but both approaches have strengths and weaknesses. The data indicates that the holistic approach is more effectively for students with weaker performance in mathematics.

INTRODUCTION

For several years, there was an intense national and international didactic discussion and research in mathematical modelling (see Blum et al., 2007; Kaiser et al., 2011; Stillman et al., 2013). Furthermore, the development of students' mathematical modelling competencies is a central goal of German mathematics lessons, since the competency of mathematical modelling has been described as one of the central competencies in German educational standards in mathematics. Projects to foster students' mathematical modelling competencies can each be assigned to one of two approaches: either a holistic or an atomistic approach (Blomhøj & Jensen, 2003). The main goal of the presented study is a comparison of the effectiveness of these two approaches in terms of the development of modelling competencies of students.

In the first part of the paper the theoretical framework will be documented. Then, the design of the modelling project will be presented as well as the methods of data collection and evaluation. Finally, selected results of the study will be described.

THEORETICAL FRAMEWORK

In recent years, mathematical modelling was an internationally highly discussed topic of didactics of mathematics. From the discussion resulted various perspectives of mathematical modelling that include different representation of the modelling process as a cycle as well as goals and modelling competencies. An overview is given for instance in Kaiser and Sriraman (2006).

However, the various definitions have in common that mathematical modelling is described as a process of solving real world problems by using mathematical methods (Niss, Blum, & Galbraith, 2007). In addition, an ideal-typical process of mathematical

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modelling is usually illustrated in the form of a cycle (Kaiser, Blomhøj, & Sriraman 2006), while in reality such processes are characterized by frequent switching between the various stages of modelling cycles (Borromeo Ferri, 2011; Martinez & Brizuela, 2009). Corresponding to the different perspectives of mathematical modelling there are various modelling cycles, which are either more useful for application in mathematics lessons or in science (Borromeo Ferri, & Kaiser, 2008). The project ERMO refers to a didactical modelling cycle developed amongst others by Kaiser and Stender (2013; see Figure 1).

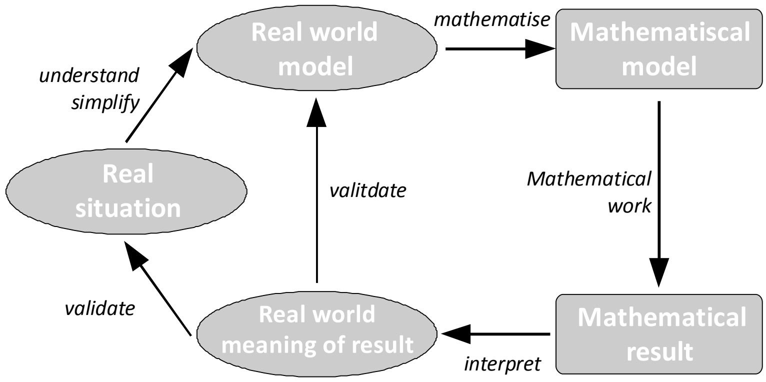

Figure 1: Modelling cycle (Kaiser & Stender, 2013)

The specific definition of modelling competencies depends on the particular underlying concept of mathematical modelling (Zöttl, Ufer, & Reiss, 2010). Widely accepted is that modelling competencies include abilities and a willingness to solve real-world problems by using mathematical modelling (Maaß, 2006; Blomhøj & Jensen, 2003). The concept of mathematical modelling competencies contains different components, namely sub-competencies of mathematical modelling, metacognitive modelling competencies, competencies of structuring given problems appropriately and goal-oriented, competencies of argumentation and documentation and competencies of realising the possibilities of mathematics as well as positively valuing these (see for example Maaß, 2006).

The sub-competencies are based on the underlying modelling cycle and include the abilities needed to perform the different steps of the cycle. Based on the modelling cycle from Kaiser and Stender (2013) different sub-competencies of mathematical modelling are distinguishable which can be assigned to three sub-processes of mathematical modelling (referring to Zöttl, Ufer, & Reiss, 2010):

- *Simplifying / Mathematising* (including all competencies needed for the transition between real world and mathematics)
- *Working mathematically* within the mathematical model
- *Interpreting / Validating* (including all competencies needed for the transition between mathematics and real world)

The sub-competencies of mathematical modelling are seen a necessary part of the modelling competencies, as they enable the modeller to perform the different steps of the modelling process adequately. However, the presence of the sub-competencies does not automatically include the existence of the *overall modelling competence* (Zöttl, Ufer, & Reiss, 2010). According to Maaß (2006) or Stillman (2011) the *metacognitive competencies* play a significant role for the modelling competencies. A non-existent or low meta-knowledge about the modelling process as a result may lead to considerable problems while working on modelling tasks, for example at the transitions between the different phases of the modelling process.

According to the survey by Blomhøj and Jensen (2003) projects to foster mathematical modelling competencies can mainly be assigned either to a holistic or an atomistic approach. The holistic approach is based on the assumption that the fostering of modelling competencies will be the most effectively by tackling whole modelling tasks. The complexity and difficulty of the modelling tasks should correspond to the competencies of the students. The atomistic approach is based on the assumption that particularly at the beginning of the work with modelling problems the tackling of whole modelling tasks would be too time-consuming and not be effectively referring to the fostering of sub-competencies of mathematical modelling. Propagated is separated fostering of the sub-competencies by tackling only sub-processes of a whole modelling process (Blomhøj & Jensen, 2003).

DESIGN OF THE STUDY

The central goal of the project ERMO (*Er*werb von *Mo*dellierungskompetenzen: Acquirement of modelling competencies)[1] was to foster the students' modelling competencies. Furthermore, the design of the single modelling activities was oriented towards the promotion of the students' ability to reflect about their own working processes and results.

The modelling project was carried out in 2012 in Hamburg (Germany) and started with a teacher training course conducted in cooperation with Dr. Katrin Vorhölter in February 2012. The participating classes integrated six 90 minutes modelling activities, including the tackling of different authentic modelling problems in co-operative, self-directed learning environments, as well as a modelling test in a pre-, post- and follow-up-design into their mathematics lessons (for an overview see Figure 2). The classes were divided into two groups: The modelling activities of group A were assigned to the holistic approach, while the modelling activities of group B were assigned to the atomistic approach. The students of the holistic group dealt with complete modelling tasks with an increasing complexity, the students of the atomistic group dealt with sub-processes of mathematical modelling separately, especially the transitions real world → mathematics and mathematics → real word. The tasks of the atomistic group contained active parts, i.e. tasks that require for example to develop own real and mathematical models, as well as passive parts, i.e. given models or

[1] The project benefited from experiences of the Hamburg working group on mathematics education with carrying out and evaluating modelling projects (see for example Kaiser, 2007).

solution that were to assess and validate. As hypothesis it was formulated that modelling activities of the holistic approach is more effectively concerning the fostering of the *overall modelling competency* while the atomistic approach might be more effectively regarding the sub-processes of mathematical modelling (*Simplifying / Mathematising, Working mathematically* and *Interpreting / Validating*).

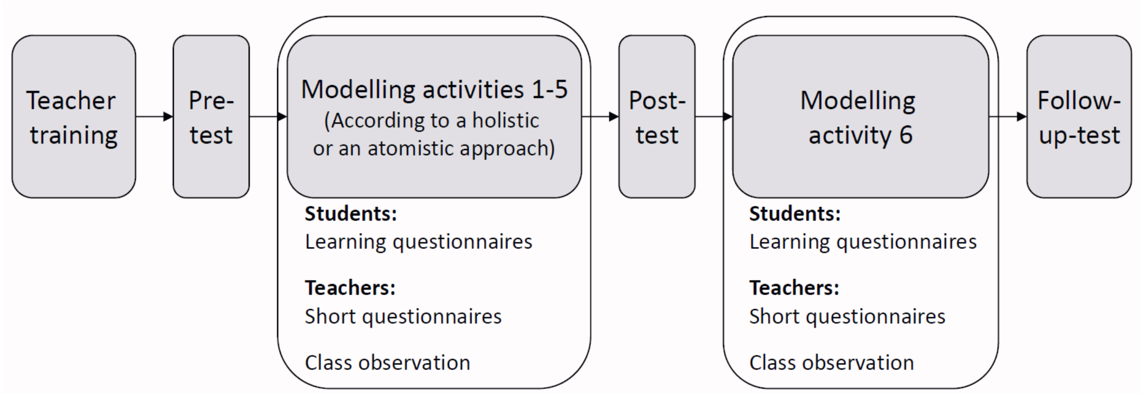

Figure 2: Design of the study

Altogether, N=377 students from 15 classes of 9th grade of four secondary higher-track schools and two comprehensive schools took part in the project, while only 204 students of 13 classes participated in all three measurement points (MP), 132 students of the holistic group and 72 students of the atomistic group (see Table 1). The presented results are based on this panel.

	MP 1	MP 2	MP 3	Panel
Holistic approach	168	164	169	132
Atomistic approach	159	152	97	72
Total	327	316	266	204

Table 1: Sample – number of participating students

The modelling test was designed in a pre-, post- and follow-up-design and conducted to evaluate the students' progress of their modelling competencies. The design of the modelling test refers to work by Haines, Crouch and Davis (2001) and Zöttl, Ufer and Reiss (2011) and others, who developed items that tested different sub-dimensions of the modelling competencies. Because of this structure, it is possible to measure different dimensions of students' modelling competency independently from potential weaknesses in single phases of the modelling process. The developed modelling test covered the three sub-processes of mathematical modelling (*Simplifying / Mathematising, Working mathematically* and *Interpreting / Validating*) as well as an *overall modelling competency* including the competence of carrying out a whole modelling process and matching different parts of a solution of a modelling task to the right phases of the modelling cycle. Per measurement point, the number of used items per dimension of the modelling competency varied between 15 and 24.

The data were scaled by using methods of multidimensional item response theory and with an approach of so-called virtual persons for all items of the three measurement points (Rost, 2004). The various dimensions of the modelling competency are considered as being the latent variables that can be estimated as a multivariate function of the items solved. The scaling was carried out with Conquest (Wu et al., 2007). In a first step, different psychometrical models of the structure of the modelling competency were scaled, a one-dimensional model as well as a four-dimensional between-item model and two multidimensional within-item models. To select the best model for the data, the psychometrical measures Akaike Information Criterion (AIC), Bayes Information Criterion (BIC) and Consistent AIC (CAIC) were used (Rost, 2004). After the model selection, weighted likelihood estimates (WLE) were estimated as individual ability parameters and converted to an average value of M=50 and a standard deviation of SD=10. To analyse the progress of the modelling competencies within the two groups, amongst other evaluations, the average test performances of the students were tested for significance that were corrected by the Bonferroni method. In addition, the effect sizes of the performance differences were calculated.

RESULTS

The comparison of the four psychometrical models points to the four-dimensional between-item model (see Figure 3). Considering the psychometrical measures AIC, BIC and CAIC, which are the lowest for this model, the four-dimensional between-item model describes the collected data the best compared to the others (Rost, 2004). In addition, the reliabilities of the four dimensions are acceptable and vary between 0.767 and 0.821.

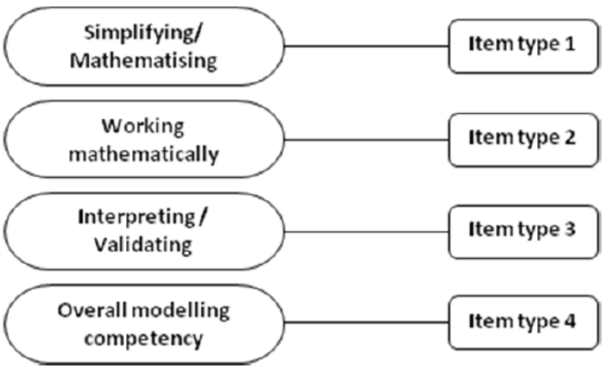

Figure 3: Four-dimensional between-item model

Regarding the development of the four dimensions of modelling competencies the data show for all groups of students highly significant increases between the first and the second as well as between the first and the third measurement points (see Table 2). In the first dimension *simplifying / mathematising* there is a higher effect size in increase of the holistic group between the pre- and the post-test (0.88) compared to the atomistic group (0.72). Between the pre- and the follow-up-test the atomistic group shows a larger effect size (0.68) than the holistic group (0.59). The effect sizes in the dimension of *working mathematically* are larger in the atomistic group between measurement point one and measurement point two (0.57 versus 0.47) as well as between measurement point one and measurement point three (0.46 compared to 0.32). The effect sizes in increase in the dimension of *interpreting / validating* are higher in the holistic group between the pre- and the post-test (0.77 compared to 0.69) as well as between the post- and the follow-up-test (0.65 instead of 0.57). In the fourth dimension, the overall modelling competency, there are larger effect sizes in increase in the holistic group as well (0.90 versus 0.68 between the first two and 0.61 instead of 0.35 between the first and the third measurement point.

	Mean MP 1 (SD)	Mean MP 2 (SD)	Mean MP 3 (SD)	MP1→MP2 (Cohen‘s d)	MP1→MP3 (Cohen‘s d)	MP2→MP3 (Cohen‘s d)
Simplifying / mathematising						
Holistic	48.26	57.60	54.90	+9.33***	+6.64***	-2.69*
group	(11.29)	(9.87)	(11.14)	(0.88)	(0.59)	(-0.26)
Atomistic	51.21	57.62	57.08	+6.41***	+5.87***	-0.54
group	(7.80)	(9.84)	(9.39)	(0.72)	(0.68)	(-0.06)
Working mathematically						
Holistic	49.94	54.92	53.10	+4.98***	+3.16***	-1.82*
group	(10.18)	(10.83)	(9.29)	(0.47)	(0.32)	(-0.18)
Atomistic	48.85	54.76	53.81	+5.91***	+4.95***	-0.96
group	(9.16)	(11.35)	(12.33)	(0.57)	(0.46)	(-0.08)
Interpreting / validating						
Holistic	47.93	55.83	54.38	+7.90***	+6.45***	-1.45
group	(9.42)	(11.07)	(10.50)	(0.77)	(0.65)	(-0.13)
Atomistic	50.55	56.73	55.79	+6.19***	+5.24***	-0.95
group	(8.86)	(8.95)	(9.59)	(0.69)	(0.57)	(-0.10)
Overall modelling competency						
Holistic	49.78	58.08	55.55	+8.30***	+5.78***	-2.53**
group	(9.34)	(9.20)	(9.62)	(0.90)	(0.61)	(-0.27)
Atomistic	50.75	57.28	54.21	+6.52***	+3.46*	-3.07*
group	(9.74)	(9.46)	(9.81)	(0.68)	(0.35)	(-0.32)

***p<0.000, **p<0.01, *p<0.05

Table 2: Means and performance increases of the different dimensions of the modelling competency

To receive detailed information about the differences in the performance increase between different groups of students, two-way ANOVAs with repeated measures were used. The results of the two-way ANOVAs show that there can be seen only significant effects of the modelling approach (in favour of the holistic approach) in the dimension *simplifying / mathematising* and *the overall modelling competency* and only between the first two measurement points. Differentiated between the two school types, the two-way ANOVAs reveal that there is no significant effect of the modelling approach for the students of the secondary higher-track schools while there were significant effects of the modelling approach for all dimensions of the modelling competencies (in favour of the holistic approach) for the students of the two comprehensive schools.

DISCUSSION OF THE RESULTS

The results of the study are on the one hand related to the structure of modelling competencies and on the other hand on the development of the modelling competencies of the students.

The results of the model comparison confirm the possibility to distinguish different facets of modelling competencies. The multidimensionality of the construct modelling competencies was also shown by Zöttl, Ufer, and Reiss (2010), while in their study the *overall modelling competency* was not seen as separated from the sub-processes of mathematical modelling as it could be shown in this study. This fact may be explicable by various aspects, mainly a different definition of the dimension of *overall modelling competency* including meta-cognitive aspects (see above).

The evaluation of the modelling tests shows that the effectiveness of the two approaches towards fostering students' modelling competencies has to be considered in a differentiated way. On the one hand the data showed that, despite the limitations of the reliability of the results particularly in field studies, in the project ERMO both the holistic and atomistic approach fostered the development of the different dimensions of the students' modelling competencies under real teaching conditions successfully. On the other hand, differences between different groups of students and between the four dimensions of modelling competencies became apparent. A general superiority of one approach could not be stated. The results indicate that the approach for high-performance students plays a minor role, since no effect of the approach was found for the dimensions of modelling competencies for students of higher track schools (so-called *Gymnasien*). In contrary, especially for relatively less powerful or for heterogeneous classes the holistic approach seems to be superior to the atomistic approach, because for the students of the comprehensive schools (the so-called *Stadtteilschulen*) there higher performance increases were found in the holistic group.

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FOSTERING THE ARGUMENTATIVE COMPETENCE BY MEANS OF A STRUCTURED TRAINING

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We report on a long-term study which was executed in a German secondary school with 128 eights graders (ages 14 to 15) in four different classes. Two of these classes served as control groups. The mathematics lessons of the other two classes (treatment groups) were frequently enriched by distinguished phases in which structured argumentation and the use of heuristics was trained. The study aimed at investigating the development of the argumentation competence of the students over that period. For this report, the products of four different geometry tasks of 15 students from one of the treatment groups and 15 from one of the control groups respectively were evaluated.

INTRODUCTION

Both "reasoning and proof" and "problem solving" are important parts of mathematics curricula all around the world (e.g., NCTM 2000). Though both deal with aspects of producing mathematical argumentation, mathematics educators tend to compartmentalize those two domains (Mamona-Downs & Downs 2013). Problem solving is being perceived as focusing on progressing work, whereas the proof tradition highlights evaluating the soundness of the product of reasoning (cf. ibid.).

We report on a 1.5-year study covering two experimental and two control classes emphasizing reasoning and proof as well as problem solving. In this paper, we confine ourselves to the "reasoning and proof" part of this study with a focus on the methodology of rating the students' products. Additionally, we present initial results by highlighting quantitative (scores) as well as qualitative (ways of reasoning) analyses of the students' products at the beginning and at the end of the study.

THEORETICAL BACKGROUND

Reasoning and proof is a significant aspect of mathematics and therefore also important for mathematics at school. It is, however, very difficult for students of all grades up to university level to generate or even read proofs on their own. Reid and Knipping (2010, p. 68 ff.) summarize several studies regarding the construction of proofs, which all agree on the fact that most students cannot write a correct proof.

There is a need for good teaching concepts regarding reasoning and proof as well as for studies that accompany related teaching experiments. An important part of such studies are methods to measure the argumentational competencies of the participating students. These methods need to be able to account for the (partially) complex structures of proofs, to appropriately compare different approaches and levels of elaboration of proofs, and consequently to show progress in the generation of proofs.

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Many researchers studying reasoning and proof use the Toulmin (1958) model which has been developed to reconstruct arguments in different fields (cf. Knipping 2008). According to Toulmin, the basic structure of rational arguments can be described as consisting of the pair of *datum* and *conclusion*. As this step might be challenged, a *warrant* can be added to justify it. Toulmin adds additional elements to his model (like *qualifiers* that can restrict the conclusion or *backing* for warrants) as do other researchers that use it. For example, Ubuz et al. (2012) add elements to describe statements and actions of teachers in classroom situations (like *guide-redirecting*) and specifications of existing elements (like *deductive warrant* and *reference warrant*).

However, the Toulmin model has its limitations. For example, it "is not adequate for more complex argumentation structures [in classrooms]" (Ubuz et al. 2012, p. 168) and it "de-emphasises the times" (Knipping 2008, p. 439) and thus is not able to outline the development of argumentations. Most notably, the Toulmin model is not designed to analyze written argumentations such as students' solutions of proof tasks. Analyses of students' solutions with this model would mostly contain of data and conclusions, missing rebuttals of dialogue partners and according backings.

As an alternative method to reconstruct argumentation steps and streams in written work of students, we propose in this article an adapted version of the multigraph representation by König (1992). He uses different graphical elements to denote elements like "starting quantities", "solution state" and "intermediate states" as well as logical derivations between states and heuristics elements that might help proceeding from one state to another (see the Methodology part for an example of such a graph).

König had designed his method which he refers to as a "solution plan" to compare written solutions of proof tasks – be it different solutions of the same task or solutions of different tasks. The standardized way of depicting an argumentation allows for a mostly objective analysis of students' work in different states of elaboration.

Our **research intention** is to adapt the solution plan sensu König to our study and to apply it onto the written argumentations (the *products*) of students that worked on mathematical problems and proof tasks. A secondary research question deals with detecting differences between and improvements of the argumentative competence of the students that underwent our training compared to those from the control group.

DESIGN OF THE STUDY

The HeuRekAP[1] study was launched at the beginning of the 2011/2012 school term (August 2011) in a German secondary school and lasted for one and a half years (until the end of January 2013, see Figure 1). It covered the whole eighth grade consisting of four parallel classes. Altogether there were 128 students initially aged 14 to 15. Two of these classes were continuously taught by the first author (treatment groups T_1 and T_2),

[1] ***Heu**ristisch **Rek**onstruiertes **A**rbeiten und **P**roblemlösen* means Heuristically Reconstructed Working and Problem Solving, for details of the concept of Heuristical Reconstruction see Gawlick (2013).

the two others served as control groups (C_1, C_2). Treatment group T_1 and control group C_1 were both mathematical profile classes, which implies an additional mathematics lesson per week in grades seven and nine.

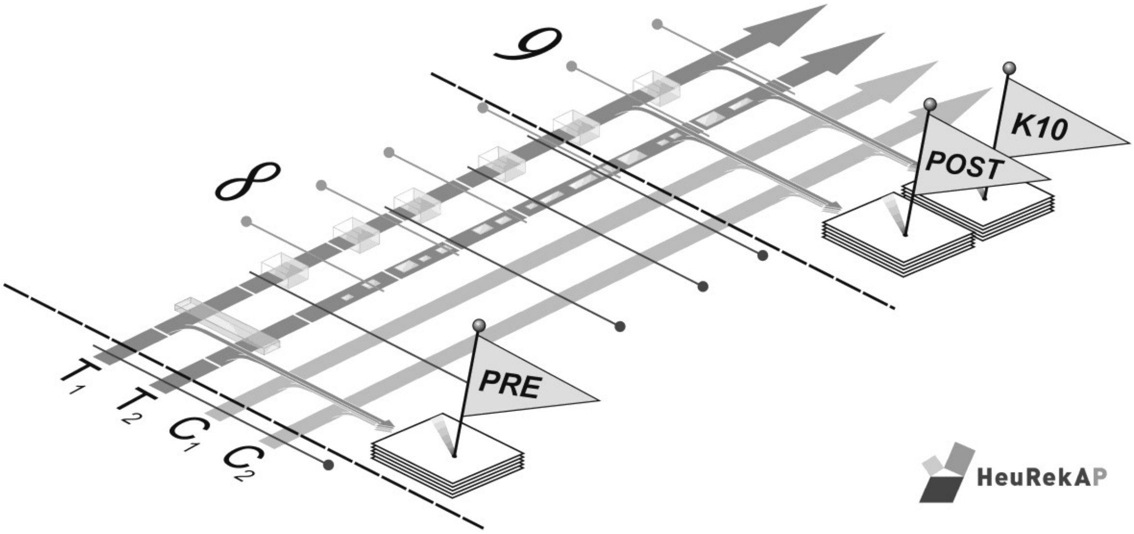

Figure 1: Overview of the study and the ascertainments relevant for this paper

For this paper, 15 students from each of the profile classes (T_1 and C_1) have been chosen by two criteria: (a) The selected students from both classes were supposed to be comparable with each other referring to their initial performance (parallelized samples). This was measured by the average school marks in Mathematics and German over the past four years before the study started. (b) The selected students should have had an above average motivation to participate in the ascertainments of the study. Therefore a survey on motivation was conducted at the beginning of the study.

The mathematics lessons of treatment of group T_1 included distinguished phases in which structured argumentation and proving as well as the use of heuristics were trained. The students were involved into the whole process of proving according to Boero (1999) and learned to write down their proofs in the Two-Column-Format (cf. Herbst 2002). Amongst the heuristics they became familiar with are the use of auxiliary elements, principles like analogy and strategies like working backwards. See Brockmann-Behnsen (2013) for an example of a typical educational unit.

At regular intervals, sets of reasoning problems have been given to the students. Relevant for this paper are two items of the pretest, which was handed out before the treatment started, and three items of the posttest. The problems Rhombus 1, given in the pretest, and Rhombus 2, given in posttest are similar, Angle was given to the students both in the pretest and the posttest. With these pairs of problems the development of the quality of argumentation can be examined. Additionally, K10 was given to the students at the end of the study – with no matching pretest item because it is more complex than the other problems (see Table 1 for the problems).

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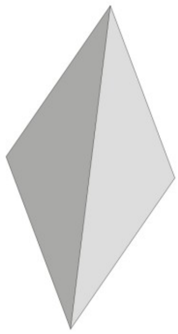
Rhombus 1 (Pretest)

A rhombus is divided into two triangles by its diagonal.

Demonstrate that these two triangles are congruent.

Write down all your considerations and arguments step by step.



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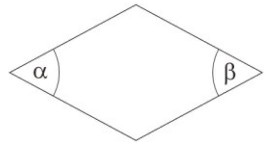
Rhombus 2 (Posttest)

A rhombus is defined as a quadrilateral with four sides of equal length.



Given: A rhombus with opposite interior angles α and β.

Prove: $|\alpha| = |\beta|$

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<td>Source: Griesel et. al. 2006, p. 27 f.</td>
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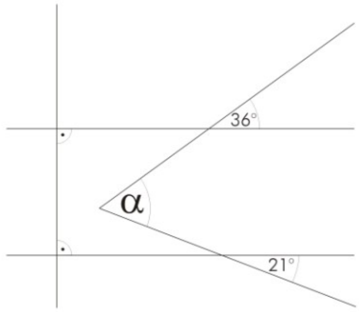
Angle (Pretest/ Posttest)



Determine the value of angle α.

Write down all your considerations.

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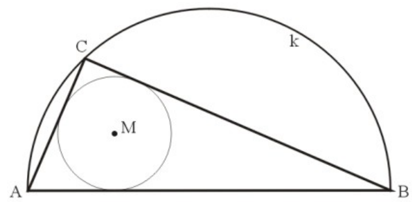
K10

AB is the diameter of a semicircle k, C is an arbitrary point on the semicircle (other than A or B) and M is the center of the circle inscribed into ΔABC. Determine the value of ∠AMB



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<td>Source: Lergenmüller et. al. 2006, p. 64</td>
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Table 1: The four tasks selected for the analyses in this paper

METHODOLOGY

The research questions stated in this paper demand an instrument which is suitable to analyze and categorize the quality of argumentation in the students' products. These products often differ strongly in their form and structure. The spectrum ranges from disjointedly noted statements – partly written in mathematical symbols – over prosaic texts up to highly structured Two-Column-notations.

Therefore in a first step it is necessary to transform this variety of forms into one standardized format to facilitate comparability of the products. Orientated multigraph representations sensu König (1992, p. 25) serve as a basis for this standardized format. The vertices of these multigraphs comprise of the given magnitudes framed by circles, operators like Thales Theorem (TT) or the Angle Sum Theorem (AST) framed by rhombuses, intermediate aims surrounded by a mixture of rectangles and circles and the target magnitude enclosed into a rectangle.

[2] In contrast to the TIMSS III format in this study no solution alternatives were given to the students.

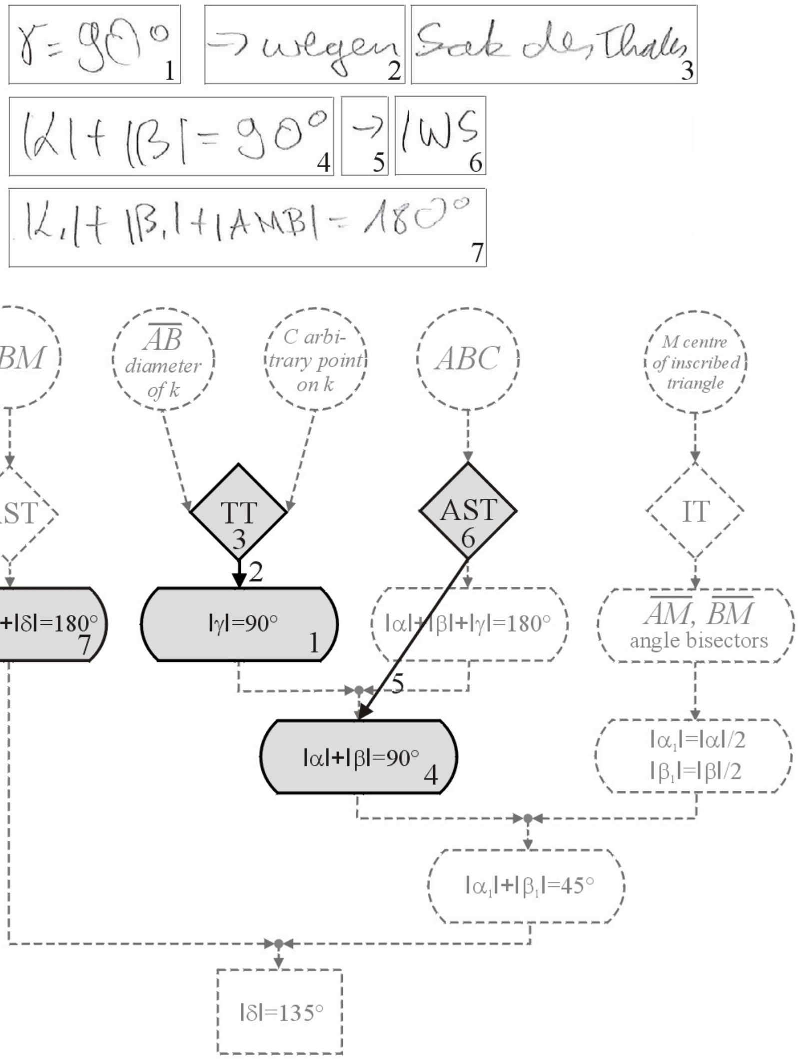


Figure 2: T_1-04-K10 original notations and standardized representation

The orientated multigraph representation depicts a survey of a complete solution of the given problem and highlights all the details reached by the student and their relation to each other. Figure 2 gives an example of such a representation. Shown are the original notations of student T_1-04 who worked on problem K10. The notations have been parsed into units that correspond to intermediate aims, identified operators or phrases that indicate connections between them. Beneath the original notations the appendant multigraph representation can be seen. The units of the original notations have been registered within this standard solution.

In a second step the quality of the students' argumentations were graded into six categories (Cat. 0 to Cat. 5) based upon the multigraph representation (see Table 2).

The notations of student T_1-04-K10 as stated in Figure 2 consist of some intermediate aims and a logical connection between the operator Thales Theorem (TT) with its conclusion $|\gamma| = 90°$. The required premises for the application of that operator are not stated. Therefore these notations were categories into Cat. 2 (Molecules).

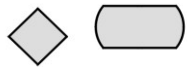
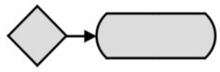
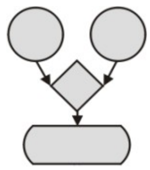
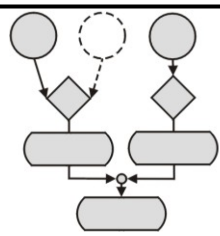
Cat. 0	**No access:** No detail of the students' notations is relevant for the solution.	Blank
Cat. 1	**Atoms:** At least one detail of the students' notations (operator, intermediate aim etc.) is *relevant* for the solution.	
Cat. 2	**Molecules:** At least two details of the students' notations are *logically connected* with each other.	
Cat. 3	**Deductive Cells:** At least *one correct and complete elementary deduction* that is relevant for the solution can be found. This is called a *Deductive Cell*. It includes the notation of the premises required for the application of an operator, the operator itself and the correct conclusion derived by the application of that operator on the stated premises.	
Cat. 4	**Deductive Torso:** At least *two deductions* relevant for the solution are *logically connected* with each other. Either one of the connected deductions or the connection itself is correct and complete (existence of at least one Deductive Cell).	
Cat. 5	**Deductive Body:** A complete solution *without any logically deficits* is being given.	Complete solution graph

Table 2: Categories for grading the students' products

RESULTS

For all tasks presented in this paper as well as additional ones within this study, the coding of students' written argumentations by representing it with an oriented multigraph and grading it into one of the six categories (Cat. 0 to Cat. 5) proved to be highly objective and reliable. Interrater correlations for 5 randomly selected students' products per task have been calculated. The percentage of agreement scores for researchers who have coded the products individually range between 65% and 100% with the median interrater correlation being 83%.

The coding of the students' products into categories via the multigraph representations allows us to compare their argumentative performances. For this report, we examined a parallelized sample of 15 students each from the treatment group T_1 and the control group C_1. Because of the fact that the category coding yields only ordinal data and because of the small sample size, in the following we use non-parametric statistical methods like interquartile ranges and chi-square-tests instead of parametric methods like standard deviation and *t*-tests.

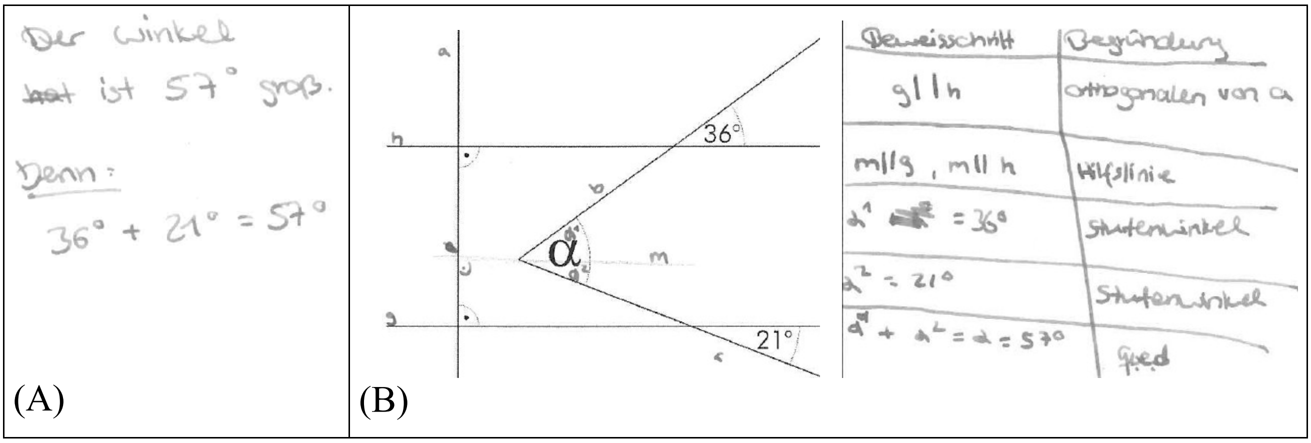
Comparing the two groups shows that they scored equally at both pretest items as it was expected because of the parallelization with regard to previous achievement. The three posttest items, however, show a significant difference in favor of the treatment group (see Table 3). This was proven by chi-square-test ($\chi 2 = 19{,}72$, $p < 0{,}0001$).

	Rhombus 1	Rhombus 2	Angle (pre)	Angle (post)	K10
T_1: median (interquartile range)	2 (1)	4 (1)	2 (3)	4 (0.5)	2 (0.5)
C_1: median (interquartile range)	2 (1)	1 (2)	2 (2)	2 (2)	1 (1)

Table 3: The mean results of the students for each task

This result can be supported by an analysis of the individual development of the students between the two matching pairs of pre-posttest items (Rhombus 1/2 and Angle pre/post). From the tasks of the pretest to the tasks of the posttest only 5 out of 30 products of the treatment group had no change or even a decline in their categories, whereas 21 products ascended by two or more categories. In the control group, 18 out of 30 products had no change or even a decline in category from the pretest tasks to the posttest tasks and only 5 products increased by two or more categories.

We like to illustrate the development of the students argumentative competence exemplarily by the elaborations of student T1-15 working on the Angle Problem in the pretest (A) and in the posttest (B). In the pretest the student merely states the correct result with the argument: “Denn: (Because:) 36°+21°=57°”. No mathematical connections between the given and the demanded angles are being drawn. In the posttest the solution is structured by a Two-Column-System and heuristic elements such as auxiliary lines and notations can be found.



(A) (B)

DISCUSSION

We introduced a study to foster the argumentative competencies of eighth graders. To examine such competencies and possible advancements, we developed a method based upon multigraph representations that enabled us to categorize and thereby compare written products of students working on mathematical problems and proof tasks. We challenged the objectivity of this method by measuring its interrater reliability and gained very satisfactory results.

With the help of this method, we were able to grade the students’ argumentations before and after the 1.5-year period of our study. In accordance with the literature, most of the students scored quite bad results in proof tasks previous to the study. The control group (with no special training in heuristics and argumentational strategies) showed equally poor results at the posttest. The treatment group, on the other hand, reached significantly better results after the training. Ongoing research has to further demonstrate the effectiveness of the teaching method elaborated in this study.

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STUDENT STRATEGIES IN ENACTING AFFORDANCES

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This paper reports an instrumental case study of the strategies employed by Year 11 students engaged in solving a functions population task. The task was implemented as part of a study of students studying functions in a Technology-Rich Teaching and Learning Environment (TRTLE). Student strategies related to the perception and enactment of affordances of the TRTLE that would be useful during task solution. The number and nature of strategies used and the combinations of affordances perceived and enacted were diverse. This was true even when students had the same function related intention, for example, find a model to represent data.

THEORETICAL FRAMEWORK AND RELATED LITERATURE

A *Technology-Rich Teaching and Learning Environment* (TRTLE) is a classroom environment where both teachers and students have access to, and teachers' professional development support for, a range of electronic technologies. To qualify as 'rich' the environment includes unrestricted access to electronic technologies that enable mathematical explorations. See Brown (2005) for further details.

The term *affordance*, prominent in educational literature, has a proliferation of different uses and meanings. In the research reported, following Gibson (1979) who invented the term, "*affordances of a TRTLE*…are the offerings of the environment for facilitating and impeding teaching and learning. Affordance bearers are those specific objects within the environment that enable an affordance to be enacted" (Brown, 2006, p. 241). Being opportunities, affordances need to be perceived and acted upon if the opportunity is to be taken up. In this study affordances were described in the same linguistic form used by Gibson (e.g., *Communicate-ability, Represent-ability*). Gibson saw affordances as a precondition for activity defining allowable actions between the object and actor; however, the existence of an affordance does not necessarily imply that activity will occur. The language form '-ability' is intended to convey this potentiality. *Affordance bearers*, a term coined by Scarantino (2003), can be described in general terms or more particularly as specific features of the particular technology being used as is the case here (e.g., ZOOM).

The number of studies focussing on function appears to have declined over the last two decades. Those that are reported tend not to involve real world contexts, situations where students are required to determine what function to use, nor often consider student actions in a normal classroom environment. For example, Kouropatov and Dreyfus (2012) in a study of advanced-level mathematics student volunteers learning integral calculus report on the accumulation function concept as core to developing "a flexible proceptual understanding of the integral and integration" (p. 11). An

2014. In Nicol, C., Liljedahl, P., Oesterle, S., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 2, pp. 201-208. Vancouver, Canada: PME.

interesting study is reported by le Roux and Adler (2012) with a group of four "first-year undergraduate students solving a function problem" (p. 51). Although this function task is situated in the real world (chemical reaction) the function type (quadratic) is specified. Analysis focuses on "the interplay between students' ways of talking about and looking operationally and structurally at the quadratic function" (p. 57). Watson and Harel (2013) follow up Harel's earlier analysis of the weak treatment in US textbooks of functions to investigate the impact of teacher mathematical knowledge on their teaching. With respect to TRTLE's, Brown (2007) reported on Year 9 students' early conceptions of function whilst Minh (2012) reported on Year 11 and 12 students in a French technology-rich geometrical and symbolic environment learning about functions through modelling geometric dependent situations. Minh found "joint development of knowledge about the artefact's capabilities together with mathematical knowledge about functions during the instrumental genesis" (p. 217) takes time.

METHODOLOGY

An instrumental approach to collective case study (Stake, 1995, pp. 3-4) was used in this study. The research question that is the focus of this paper is: In a TRTLE, where myriad affordances are present and would be useful, what strategies do students employ in solving real world functions tasks? The strategies relate to the perception and enactment of affordances that would be useful during task solving. Affordances that would be particularly useful in determining and subsequently using models for the task included: *Data Display-ability, Function View-ability, Represent-ability*, and *Check-ability*. These affordances are briefly illustrated from the lesson sequence on functions prior to task implementation; the focus then becomes the strategies employed during task solving.

The data reported[1] here involved students in two Year 11 classes of 16-17 years olds solving a function task, *The Platypus Task* (see Figure 1), as part of a larger study (Brown, 2006, 2013). Platypus are found in the Yarra River close to the school where the study took place. Teachers and students had access to both TI-graphing calculators and laptops with a selection of mathematical software. The initial units of analysis were TRTLE's: P11 - 17 students taught by Peter and J11 – 20 students taught by James (all names are pseudonyms).

The task was implemented during term 2, after the classes had completed course work related to the area of study of functions (including linear, quadratic, and cubic functions and relations). The task was introduced by the researcher and students worked on the task independently with little interaction with other students or the teacher. Data sources included task scripts, a post task record sheet - seeking to ascertain information that may not have been recorded (i.e., consideration of multiple

[1] Data were collected in RITEMATHS, an ARC funded Linkage Project – LP0453701 at The University of Melbourne.

function types, checking), and audio and video recording of at least one student pair in each class. In addition, key recordings were made (using a *Key Recorder App* that recorded user button presses). Sixteen students were interviewed post-task.

The platypus is an endangered species that may become extinct unless action is taken to save it. An annual survey held in a nearby national park showed an alarming decrease in the number of platypus over the years 1993-1998. The Save the Platypus Project commenced in 1999. Two sets of data representing the platypus population, before and after the intervention project, were provided.

Part A: Use any method you feel is appropriate to determine a model to represent platypus numbers over time (a) before and (b) after the beginning of the project.

Part B: Analysis included: did the intervention improve the situation, what was the predicted population a decade later, when would the population return to the initial value, and if successful when will the population return to the 1993 level?

Figure 1: Details of The Platypus Task.

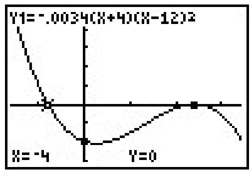
Analysis of the data followed a grounded theory approach (Strauss & Corbin, 1998). After data collection, audio recordings were transcribed. These transcripts and video and key screen recordings were re-read, re-listened to and re-watched to immerse the researcher in the data. Screen shots were used to re-create student actions. Open and axial coding followed. The former identified categories such as affordance perceived, affordance enacted, and action promoting uptake of an affordance (Brown, 2013). Axial coding (Strauss & Corbin, 1998, p. 127) focused on discovering relationships amongst categories by answering questions such as: Who used the technology, how, for what purpose, and what was the consequence of use? Thus sets of combinations of affordance enacted, strategies employed, and affordance bearers used emerged.

AFFORDANCES ILLUSTRATED

Given the varied use of the term affordances in the literature, an understanding of the author’s use of this term is critical. This section illustrates what the term meant in practice in this study. Classroom situations (e.g., attempting tasks, discourse, and other interactions) often involved perceiving and enacting multiple affordances, as was the case here.

One affordance identified in all TRTLE’s in the study *was Communicate-ability*. The affordance *Communicate-ability* is defined as Affordances of a TRTLE involving support of/for communication between humans through electronic technologies. Each affordance can be manifest in a variety of ways, for different purposes within the broad purpose enabled by the particular affordance. *Communicate-ability* was manifest through *display of screens, lesson flow, program sharing, and vicarious experiences. Lesson flow* and *vicarious experiences* were manifestations of this affordance for teaching, always initiated or enacted by the teacher. The others were both teaching and learning manifestations initiated by both teacher and students at different points in time. *Display of screens*, for example, involved sharing electronic displays for the purposes of communication in TRTLE’s supporting teaching and learning. The display to be shared could belong to teacher or student. Sharing displays could be deliberately planned or occur spontaneously as the lesson unfolded. Figure 2 presents a short dialogue between two students and a recreation of their graphing calculator screens. This occurred when Hugh and Tony were trying to find a function in the form $y = \mathrm{A}(x + \mathrm{B})^2(x + \mathrm{C})$, to model curves in a wooden strip. However, Tony did not correctly match the repeated factor to the correct root, and hence found an incorrect function.

Tony:	Check it [the function graph] hits the points.
Hugh:	y = -.0034(x + 4)(x - 12)^2.
Hugh:	Mine looks right. [Comparing his plot to the correct function, he observes the function graph passing though his plot of the data, and having the required features.]
Hugh:	[Looks at screen of Tony.] You had a square on the wrong line. [Indicating the wrong factor has been squared rather than the wrong 'line']
	Lesson 32 Lines 169-175



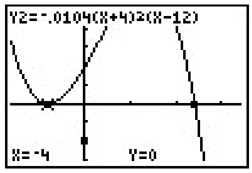


Figure 2: Illustration of *Communicate-ability*.

Figure 2 shows an example of *display of screens* where each student shared their screen with the other for the purpose of *Communicate-ability*. The dialogue illustrates that both students were expecting the correctly identified function to pass through their plot of the data. Following their sharing of Hugh's screen showing the plot and function graph, Hugh was able to identify the source of Tony's error. Thus screen sharing facilitated collaborative work, which enabled informed error correction.

Multiple affordances were often perceived and enacted in the same instance. Three other critical affordances are apparent in Figure 2, for example. These are *Data Display-ability* (affordances of a TRTLE to provide a graphical display of data, i.e., plot of numerical data), *Function View-ability* (affordances of a TRTLE to identify type of function to fit given data or identify a specific function) and *Check-ability* (affordances of a TRTLE allowing local or global checking or verification).

ANALYSIS AND RESULTS

Analysis of the data identified sets of combinations of affordance enacted, strategies employed, and affordance bearers used. A particular affordance could be used in employing a range of strategies and/or with a selection of affordance bearers. Equally many strategies could be employed using different selections of affordances. A student may consciously perceive an affordance first or a strategy first, or this may occur simultaneously, however it is generally impossible to determine (see Figure 3).

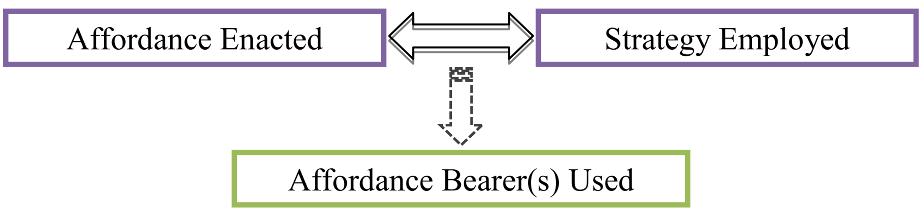


Figure 3: Relationship between categories.

The specification of affordances is based on the critical ones being perceived and enacted for the particular strategy. One affordance may be enacted individually or concurrently with others. In some cases, enactment of an affordance implies enactment of previously perceived affordances. One affordance can be associated with several strategies, and vice versa, and in enacting any particular affordance, one usually has a choice of affordance bearers. For example, in enacting *Data Display-ability* several strategies were identified. These included: View plot of data to consider appropriate function type given shape; View plot of data after function graph has been viewed (i.e., in already set up window); View plot of data after function graph has been viewed, set up new window; and View plot of data simultaneously with function graph. However, a strategy involved both a purpose and the use of particular affordance bearers. For example in Viewing a plot of data to consider appropriate function type given shape, three different choices of affordance bearers were identified (a) LIST, STAT Plot, ZOOM Stat; (b) LIST, STAT Plot, WINDOW; and (c) LIST, STAT Plot, WINDOW + ZOOM Out.

Model Finding

Students took quite different approaches as they began the task. For example Len (P11) began by entering the population data before the intervention project in Lists 1 and 2. He found a linear regression model and pasted it into the function window as shown in Figure 4 (first 3 screens). Pressing GRAPH, no part of the function graph was visible. He edited the Window Settings, clearly informed by the data, and immediately saw the function graph (Figure 4, final screens). Hence, Len began the task by first enacting *Function Identify-ability* followed by *Function View-ability*. His strategy in the former was Identify functions using linear regression (using affordance bearers LIST and STAT CALC LinReg), and for the latter View the graph of model by editing window settings directly (using affordance bearers function window y=, WINDOW, and GRAPH).

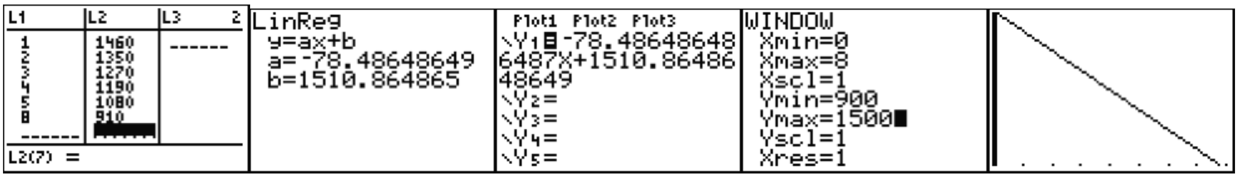


Figure 4: Evidence of enacting *Function Identify-ability* and *Function View-ability*.

In contrast, Cam (J11) entered all the data provided into Lists 1 and 2. He correctly set up the plot, pressed GRAPH and saw the standard viewing window with no part of the plot visible. Selecting ZOOM Stat the data were displayed in the viewing window. Thus he had begun the task by enacting *Data Display-ability* as shown in Figure 5. His strategy was to View the plot of data to consider appropriate function type given shape using affordance bearers LIST, STAT Plot, and ZOOM Stat.

The number of strategies in each TRTLE for these affordance combinations and the total number of times the affordance(s) for each part of the tasks are shown in Table 1. Row 1 shows that in P11, in Part A of the task seven affordances were perceived and enacted with eight different combinations. In enacting these, 16 different strategies were used. Some students enacted multiple affordance combinations and strategies and hence the number of instances, 52, is greater than the number of students (17). In addition, models were found for two sets of data. In Part A, students in J11 enacted the same affordances although not necessarily the same combinations. Values in the final columns for Part B of the task less than 17 and 20, respectively, indicate that not all students completed all parts of the task.

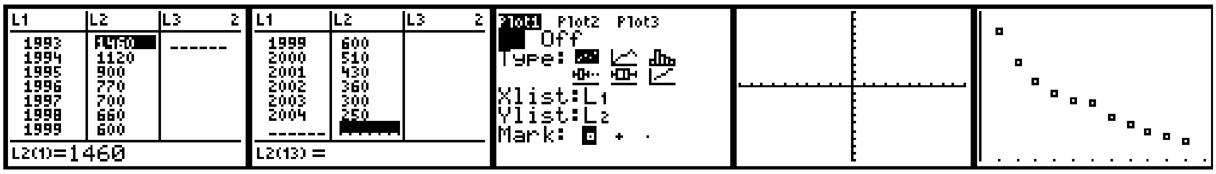


Figure 5: Evidence of Cam enacting Data Display-ability.

Part of Task	Affordance(s) Perceived and Enacted: Number		Affordance(s) Perceived and Enacted: No. of Combinations		Number of strategies		Number of Instances	
	P11	J11	P11	J11	P11	J11	P11	J11
A: Model Finding	7	7	8	9	16	24	52	103
B: Q1	4	4	3	6	5	9	11	15
B: Q2	3	4	3	7	8	13	16	17
B: Q3	3	3	2	2	4	6	10	9
B: Q4	4	3	2	5	2	6	6	9

Table 1: Number of affordances enacted and strategies used during task solving.

As Part A of the task allowed greater diversity of approaches, as evidenced in row 1 of Table 1, more detail is provided with respect to affordance combinations and strategies. There were 10 combinations of affordances in total, eight in P11 and nine in J11. Those combinations of affordances with number of strategies used (s) and instances occurring (N) are shown in Table 2, which indicates that some affordances were distinguished at a more specific level, that is, *Check-ability* was specified to be either local (shaded cells) or global. For *Data Display-ability* where the focus was multiple plots rather than a single plot and similarly for *Function View-ability* where the focus was multiple function graphs. All three relate to strategies being employed. *Check-ability* (local) was evident when Tabya employed the strategy: Perform a local check of function value(s) and compare with corresponding data value(s) using affordance bearer CALC Value. *Check-ability* (global) occurred, in conjunction with *Data Display-ability* and *Function View-ability* when Cam and others employed the strategy: View plot and graph simultaneously to see if the model matched the data using affordance bearers, ZOOM Stat and GRAPH. Multiple plots occurred when students viewed all the data at the same time. Multiple graphs occurred in two situations, either when students considered different function models for one set of data or when they compared a model for the data before intervention and a model for the data after intervention.

Affordance Combination	*s*		*N*	
	P11	J11	P11	J11
Data Display-ability	3	5	8	20
Function Identify-ability	4	6	16	17
Function View-ability	3	3	16	16
Check-ability (global) with *Data Display-ability* and *Function View-ability*	2	2	8	15
Degree of Fit-ability with *Represent-ability* and *Function View-ability*	0	1	0	12
Check-ability	1	2	1	8
Represent-ability with *Data Display-ability* (multiple plots)	1	2	1	7
Represent-ability with *Function View-ability* (multiple graphs)	1	2	1	7
Degree of Fit-ability with *Calculate-ability*	1	1	1	1

Table 2: Additional details for the model finding phase of task solution.

DISCUSSION AND IMPLICATIONS

A greater number of affordance combinations were identified in J11 and a greater number of strategies used in enacting the various affordance combinations. Whilst this is in part a result of a greater number of students in J11, this is not the only reason. Students in J11 made more diverse use of function types and had an increased tendency to consider multiple function types when identifying models for the data, or a combination of these factors.

Approximately half of the students in P11 compared to most students in J11 (i) enacted *Data Display-ability*, (ii) viewed at least one model graphed simultaneously with a plot of the data, enacting *Data Display-ability* in conjunction with *Function View-ability*, and (iii) did so for at least one model before intervention and one model after intervention thus perceiving the usefulness of plotting the data and graphing a model of the function together. In addition, in P11, one student found multiple models for both sets of data and six did so for only one set (1 before, 5 after). In contrast in J11, eight students found multiple models for both sets of data and eight for one set only (7 before, 1 after). This was a contributing factor in the larger number of affordances enacted and strategies employed in this TRTLE.

Whilst it is interesting to compare students in the two TRTLE's, this study is using an instrumental approach rather than the cases themselves being of primary interest (Stake, 1995, p. 171). For teachers and researchers, several implications arise. Firstly, the great diversity of approaches taken by students may be an eye opener to teachers of senior secondary mathematics students. These approaches relate to both affordances perceived and enacted and to strategies employed by students. Secondly, the number of students who found only one possible model for a data set that clearly could not be perfectly modelled by a polynomial or exponential model was surprising. Thirdly, related to this, was the number of students who failed to either perceive affordances of the TRTLE for, or the need to, either compare a function model to data, to compare multiple models for the one set of data – to each other and the data – or to compare the two sets of data or the models for these. This researcher wonders if these students are simply assuming their model must be a perfect fit and hence there is no need to view the plot? Further research is needed to consider this.

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COMPETENCIES DEVELOPED BY UNIVERSITY STUDENTS IN MICROWORLD-TYPE CORE MATHEMATICS COURSES

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We report on an empirical study grounded in our sustained implementation over ten years of a sequence of three-term undergraduate core mathematics courses centred on microworlds. The survey study investigates students' views on 15 competencies potentially developed as they, individually or in pairs, create 12 Exploratory Objects, i.e., microworld-type environments, on diverse mathematical topics as part of their workload. Results suggest that students develop further the competencies as they repeat designing, programming, and using microworlds to learn and do mathematics, and that original projects in which students start by selecting their own topic, is key to the development of these competencies. No gender differences were found.

INTRODUCTION

Mathematics microworlds have long been acknowledged as providing a rich mathematics learning experience for students (Healy & Kynigos, 2010). There is abundant literature, mainly at the research level, on the topic. Most involve a few student participants (e.g. Wilensky, 1995) or a class for a one-time project (e.g., Jiménez, Gutiérrez, & Sacristán, 2009). There seems to be relatively little sustained classroom implementation of microworlds, probably for the reason that "[t]he ideas behind the microworld culture have not yet been presented in a form readily acceptable not only to school systems, but also to other stakeholders in education" (Healy & Kynigos, 2010, p. 68). In this paper we report on an empirical study on competencies grounded in our sustained implementation over ten years of a sequence of undergraduate core mathematics courses centred on microworld-type activities.

CONTEXT

Mathematics Integrated with Computers and Applications (MICA) program is a unique core undergraduate mathematics program offered at Brock University since 2001 (Ralph, 2001). As a central component of the program are the innovative first-year MICA I and second-year MICA II courses, two core project-based courses for mathematics majors and future mathematics teachers. We can describe these courses by their common activity repeated throughout the two courses (at least four times/term), though each time in a more complex situation and on a different mathematics topic: to design, program, and use an interactive and dynamic computer-based tool, called an *Exploratory Object* (EO), for systematically investigating a mathematics concept, theorem, self-stated conjecture or a real-world situation (Muller, Buteau, Ralph, & Mgombelo, 2009). These microworld-type environments are either assigned to them, i.e. the topic and exploration questions are provided to students through guidelines, or are original projects in which students start by selecting their own topic. For example in 2011-12, 471 assigned EOs and approximately 98 original EOs were created. Examples of original students' EOs can be found in (MICA, n.d.).

To date we have conducted diverse studies based on our insightful reflections about our MICA students' experiences. For example, we have examined the students' instrumental genesis of programming technology to create their own EOs for their mathematical investigations (Buteau & Muller, 2014). Based on a task analysis (Buteau & Muller, 2010), we have recently conducted a literature review aiming at contextualizing the EO learning activity. As a result, we describe that students engaged in an EO activity "experience, in a context of experimental mathematics, inquiry-based learning and mathematics learning through programming and simulation" (Marshall & Buteau, forthcoming, [p. 17]). The review included literature about microworlds (in the area of 'learning university mathematics through simulation'). The literature study also aimed at theoretically identifying competencies that could be attained through the EO learning activity, resulting in a list of 15 competencies (Marshall, Buteau, & Muller, forthcoming); see Table 1.

	Competencies
a	To self-motivate to learn/do mathematics
b	To engage in divergent thinking
c	To research mathematical topics
d	To develop mathematical intuition
e	To understand mathematical models
f	To closely reflect on problems
g	To program mathematics (simulations, mathematical experimentation etc.)
h	To get a feel for inappropriate answers
i	To work with abstraction
j	To visualize mathematics
k	To connect different representations of concepts
l	To interpret mathematical results
m	To communicate one's mathematical results
n	To engage in the process of mathematics research
o	To learn/do mathematics independently

Table 1: Theoretically identified competencies developed through the EO activity.

These competencies were thereby identified in the literature in areas with common activity elements to the EO learning activity. Many of these competencies were discussed in the context of microworlds (Marshall et al., forthcoming).

Aiming now at empirical research about the students' experiences through the repetitive microworld-type activities implemented in our department, we first conducted a study to gather preliminary empirical evidence of our theoretical results. Our guiding research questions were: what are the students' views on i) the nature of the MICA I & II courses, and ii) the competencies developed in these courses? In the following we report on the results found in relation to the second question.

METHODOLOGY

To provide some insight into the research questions a student survey was undertaken. The voluntary and anonymous on-line survey was run, during laboratory sessions, of the MICA I and MICA II courses of the 2012/13 academic year. The questionnaire contained three sections. Section 1 focused on the demography of the respondents; Section 2 inquired about students' views on the nature of the MICA courses; and Section 3 questioned their views on competencies developed in these courses. Questions in the latter two sections were based on our theoretical results (Marshall & Buteau, forthcoming; Marshall et al., forthcoming). In the third section students were asked questions that focused on: each of the 15 competencies listed in Table 1; two competencies related to more traditional mathematics courses, namely y-'to write mathematics proofs' and z-'to perform complex calculations by hand'; and an optional open-ended question to comment on any of the competencies. The students' responses were recorded on a five point Likert scale, for example, the question related to competency a) in Table 1 was, "The activities in the [MICA I (and MICA II)] course[s] prompt me to self-motivate to learn and do mathematics" with answer options: "[4]-Very much; [3]-Much; [2]-Some; [1]-Not at all; or No opinion". In this paper, we only report on the survey results related to competencies (third section).

The results from the 17 Likert-scale questions were analysed using simple descriptive statistical methods. Mann Whitney non-parametric tests were used to identify significant statistical differences between groups of participants. Qualitative data from the open-ended question was coded by competencies, followed by a frequency analysis. This data was also used to help interpret the statistical results.

RESULTS AND DISCUSSION

In total 55 MICA students participated in the study (57% participation rate), with 27 MICA I and 28 MICA II students. In terms of gender, 24 female and 31 male students participated. Of the respondents 38% were mathematics majors, 40% future mathematics teachers, and 22% were enrolled in other programs.

Answers to the survey questions provide students' estimations of their acquisition and/or improvement of competencies potentially developed in the MICA courses centred on the EO activity. Although the data is ordinal we decided to calculate and plot means (removing the 'No opinion' responses) in order to provide a visual overview and pointers to possible areas where differences may be found. In the exploratory graphs created we have joined points to offer more visual distinction between the results by different groups – the lines joining points have no meaning.

Overall the means for each 15 competencies from Table 1 ranged from 2.46 to 3.34 (see Figure 1). It suggests that students view that they've developed, to a certain extent, these competencies through the repetitive EO activity. The two additional competencies (y and z in Figure 1) received the lowest means, i.e., 1.98 and 2.3. In fact, this is in line with the learning objectives of the EO activity, and suggests that the implementation aligns with the activity expectations, namely, to design, program, and use an interactive and dynamic computer based tool to learn and do mathematics. For example, a participant commented that through programming mathematics, "[i]t was interesting to actually visualize the topics studied of mathematical conjectures and projects." A respondent stressed the self-motivation to learn and do mathematics: "I have been very interested in the exploration of real world applications and creating [EOs] to test real life examples!!!" Another participant also commented that,

> [s]ince we code the projects ourselves (mostly) there isn't a nice likes help option to tell us what the results are telling us, so it does require really thinking about what the numbers actually mean.

This aligns with Wilensky's (1995) study involving university mathematics students using microworlds: "It was not until [the student] programmed a simulation of the problem that she began to resolve the paradox" (p. 272).

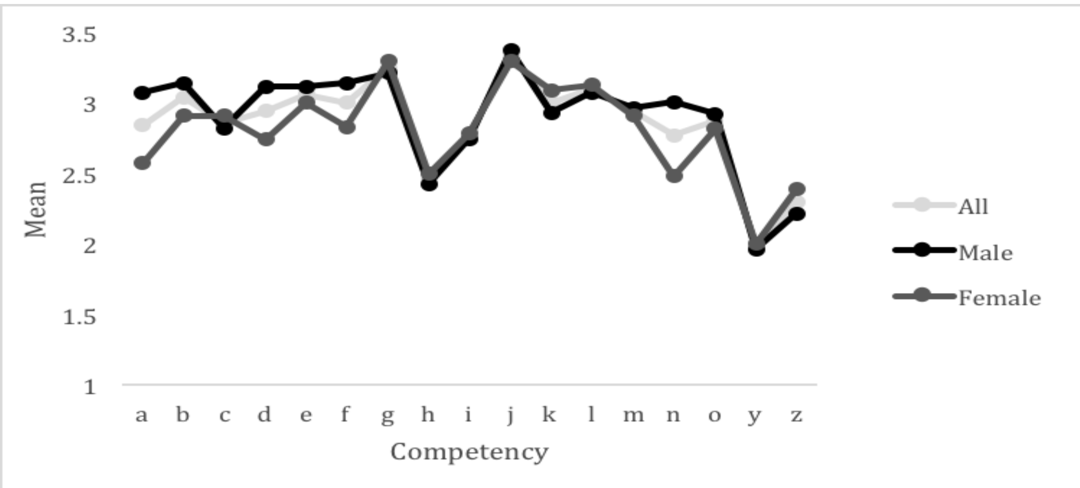

Figure 1: Means of students' views, by gender, on competencies developed through EOs (N=55; with scale: 4-very much; 3-much; 2-some; 1-not at all).

The results by gender in the graph (see Figure 1) seem to be relatively the same. Indeed, when Mann Whitney U tests were performed on data for each competency categorised by gender, no statistical differences (α=0.05, two tailed) were found for any of them. This indicates that students, independent of gender, demonstrate a similar awareness that they are acquiring and/or improving the competencies while engaged in the EO activities. This could be contrasted with Barkatsas, Kasimatis, and Gialamas (2009) study which found gender differences about school mathematics students' achievement and views towards the use of technology in mathematics.

A similar comparison was done for responses of MICA I and MICA II students. Figure 2 visually summarizes the results. It is worth noting that for each competency, the mean is greater for MICA II students than the paired mean for MICA I students.

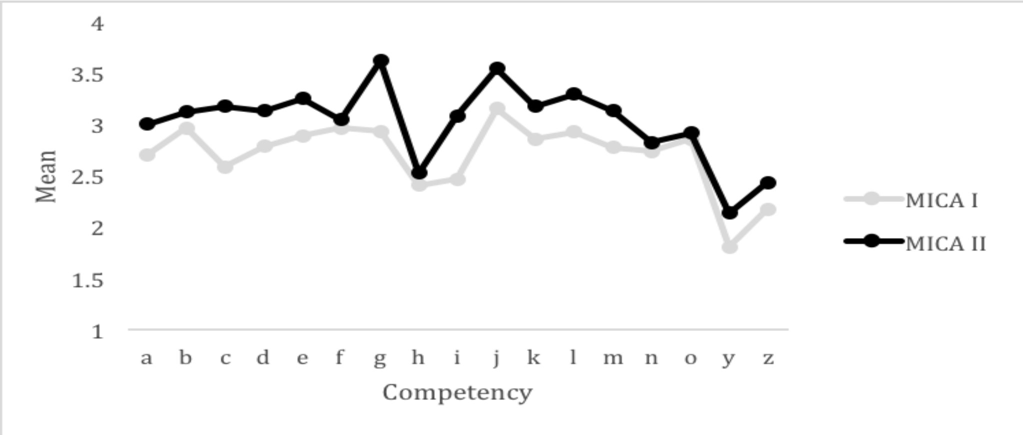

Figure 2: Means of students' views, by course year, on competencies developed through EOs (N=55; with scale: 4-very much; 3-much; 2-some; 1-not at all).

Mann Whitney U tests were performed on the data for each competency categorised by MICA I and MICA II students. Significant differences were found (α=0.05, two tailed) for seven of the competencies, namely: c 'to research mathematical topics' (p= 0.032); e 'to understand mathematical models' (p= 0.029); g 'to (computer) code mathematics' (p=0.003); i 'to work with mathematical abstraction' (p=0.012); j 'to visualize mathematical concepts' (p=0.045); l 'to interpret mathematical results' (p=0.021); and m 'to communicate mathematical results' (p= 0.046).

The surveys were undertaken close to the end of the academic year, so one would expect that students at the end of their second year of a mathematics program would be more mature, both mathematically and in their ability to work with a microworld-type environment, than their counterparts at the end of their first year. Furthermore students completing MICA II would have realised 11 EOs, including two self-directed Objects and would be working on their third. On the other hand MICA I students would have completed only three EOs and would be engaged in their first major self-directed Object. The differences in Figure 2 point to the possibility that the competencies theoretically identified need repetition and maturity that requires more than one MICA course to become established.

For the fifteen competencies the views of students in the MICA I and MICA II courses were statistically significantly different for seven of them. Because of space limitations we will comment on only two of the seven cases. We have selected one case in which we anticipated a difference, and another one that was a surprise. MICA I students will have experienced by the end of the semester (i.e., after they filled out the survey questionnaire) their first independent 'research of a mathematical topic'. The MICA II (two semesters) course has important research components arising from both the complex questions generated by the instructor through the EO assignments, that require on line research and computer based experimentation, and also from the mathematical conjectures or real world problems chosen by the students to be studied in their original final EO projects. It was therefore not surprising to find a significant difference between the views of MICA I and MICA II students in regards to the ‘research of a mathematical topic’ competence.

However we had not expected much of a difference of views for the competence ‘to (computer) code mathematics’. In the MICA I course students learn to code through a well-defined sequence of mathematical problems that require an increasing number of different programming concepts (Buteau & Muller, forthcoming). It is therefore possible that students in MICA I are so focused on acquiring the procedures of programming that is new to them that they lose sight of the mathematics. In contrast students in the MICA II course are sufficiently familiar with programming that they may now become more aware that they are ‘coding mathematics’ in their EOs.

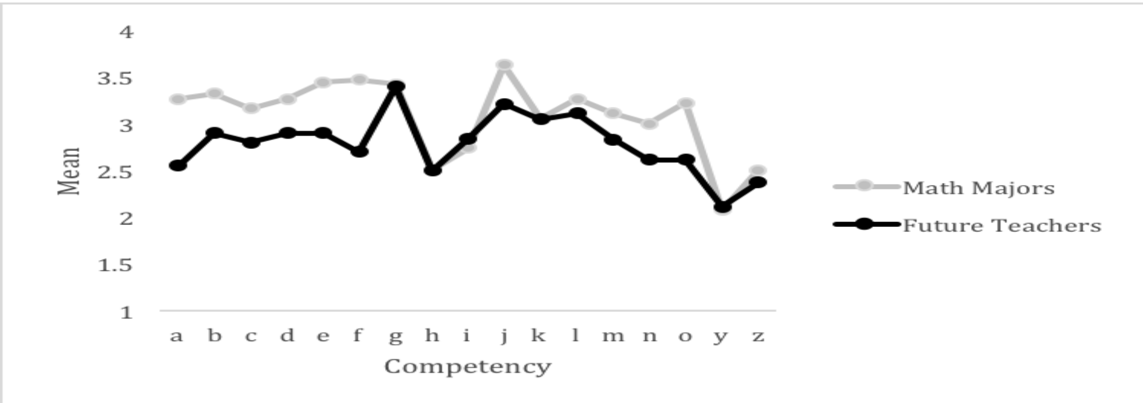

Figure 3: Means of students’ views, by program, on competencies developed through EOs (N=55; with scale: 4-very much; 3-much; 2-some; 1-not at all).

Finally we also compared the responses of the students according to their programs, namely the math majors and the future math teachers (Figure 3). We didn’t consider students enrolled in other programs. The trends for the responses from the two groups are generally similar, with the math majors mostly providing the greatest agreement (largest means) on their development of the competencies. The Mann-Whitney tests identify significant differences ($\alpha=0.05$, two tailed) between the two groups for only three competencies, namely: e ‘to understand mathematical models’ ($p=0.031$); f ‘to carefully reflect (think over carefully) on mathematical problems’ ($p=0.008$); and o =‘to learn and do mathematics independently’ ($p= 0.038$).

Both in MICA I and MICA II courses, there is an over-riding importance placed on the original projects. In both these courses future teachers are allowed to substitute the original EO projects by *Learning Objects*, i.e., interactive, dynamic computer-based environments designed to “engage a learner through a game or activity and that guide him/her in a stepwise development towards an understanding of a mathematical concept” (Muller et al., 2009, p. 64). Thus mathematics majors may experience their original EO project as a means for themselves ‘to learn and do mathematics independently’ (as well as competencies e and f), while the future teacher may experience the Learning Object project as a means for themselves to design a well-defined sequence of teacher-defined mathematical activities for someone else to learn mathematics. As such, these three competencies (e, f, and o) do not seem to relate to the LO activity, and this could explain why future teachers didn't view developing as much these competencies. We stress that the theoretical list of competencies was generated on the basis of EOs, and for future teachers the list could be modified to include didactic competencies. Overall, this could suggest that original individual EO projects in which students start by selecting a topic of their choice, is key to the development of these three, or many of the 15, competencies.

At the end of the survey, participants were invited to comment on some of the competencies. Figure 4 shows a summary created using *Wordle* (Feinberg, n.d.):

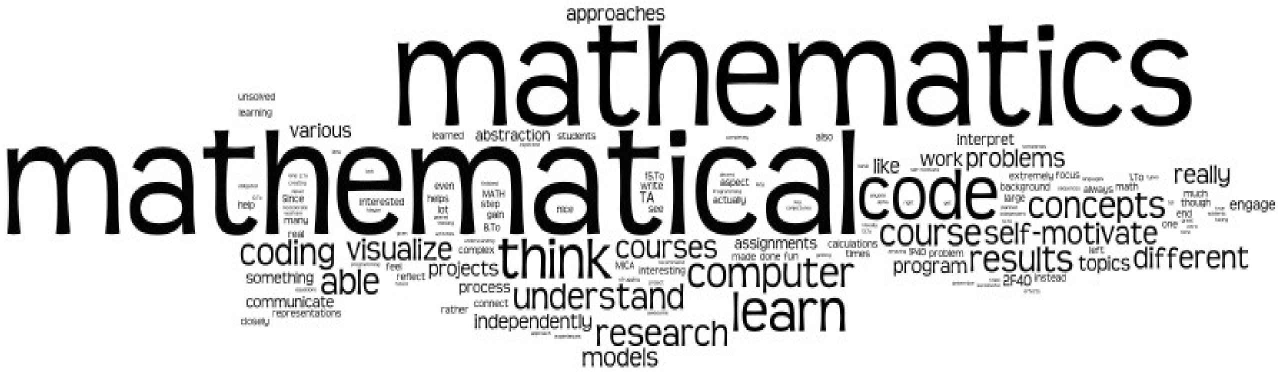

Figure 4: Word cloud of MICA students' comments on competencies.

Clearly the participants identified mathematics as the main focus within the competencies in the MICA courses, followed by coding, thinking, learning, understanding, researching, computer, concepts, and able. When analysing how often each competency appears in the comments (see Figure 5), we find that the most often selected competency for comments was 'to program mathematics', followed by 'to self-motivate to learn/do mathematics', and 'to engage in divergent thinking. Whereas g is the competency, or skill, likely to be most easily identified in relation to the EO MICA courses, the other two (a & b) are deep competencies normally beyond first and second-year university mathematics students.

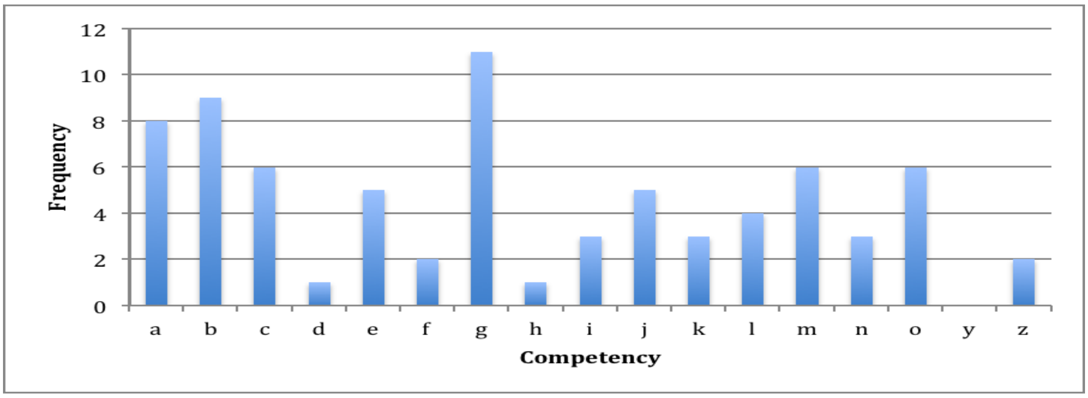

Figure 5: List of competencies selected by students for further comments.

FUTURE RESEARCH

Results of our preliminary empirical study suggest that the 15 theoretically identified competencies (Table 1) may be further developed through a process of *repetitive* microworld-type activities. In addition, the original EO projects, where typically mathematics students independently carry out an investigation of their choice, may be key in developing these competencies. We now aim to conduct a comprehensive empirical study to investigate the evolution of these students' competencies throughout their 12 individual EO activities in the three-term MICA core courses.

Finally, results of the survey study also suggested no gender difference in students' competency development. We postulate this may be linked to the creativity aspect of the EO activity (Buteau & Muller, 2014), which could be a topic of further research.

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SECONDARY TEACHERS' RELATIVE SIZE SCHEMES[1]

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This paper explores the usefulness of understanding quotients as measures of relative size in mathematics. The paper characterizes the types of thinking displayed by high school mathematics teachers on two novel tasks designed to reveal teachers' meanings in contexts where making comparisons of relative size is productive.

INTRODUCTION

Comparing the relative size of two quantities is an important mental operation that can be employed productively to reason about topics that span the grades 2-12 curriculum. These topics include measurement, fractions, rates, slope, trigonometry, and derivatives. After discussing topics where conceiving of the relative size of two quantities is useful, we will describe results from a study designed to reveal meanings held by 100 high school teachers in regard to two items where conceptions of relative size are useful.

Comparisons of relative size are critical in conceiving of a quantity's measure. The measure of some quantity tells us how many times as large the quantity is as the unit by which it is measured. Second grade students are supposed to measure an object using two different sized units and describe how measurements are related to the size of the unit chosen (CCSS.Math.Content.2.MD.A.1, 2010). Third grade students are asked to understand the quotient 32/8 as telling us that 32 is some number of times as large as 8 (CCSS.Math.Content.OA.B.6). This meaning for quotient helps students make sense of situations where division is used. For example, 25 inches divided by 12 inches per foot tells us that 25 inches is 25/12 times as large as the standard measure of one foot. Thus 25/12 feet is the same magnitude as 25 inches. Fourth grade students are asked to know the relative sizes of units within one measurement system (CCSS.Math.Content.2.MD.A.1) as well as express measurements given in a larger unit in terms of a smaller unit (CCSS.Math.Content.4.MD.A.2). Fifth grade students are asked to convert among different sized measurement units within a given measurement system (CCSS.Math.Content.5.MD.A.1).

Fractions are a critical part of the Common Core curriculum in grades three through five. Thompson and Saldanha (2003) discuss the utility of conceiving of fractions as reciprocal relationships of relative size. The fraction *p/q* tells us how many times as large *p* is as *q*. Reciprocally, *q* is *q/p* times as large as *p*. Middle school students continue to study multiplicative relationships between two covarying quantities. For example, a rate can be considered to be a measure of the relative sizes of changes in

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two quantities. Constant speed measured in miles per hour tells us that the number of miles traveled is so many times as large as the number of hours elapsed. Two quantities change proportionally if, as each quantity changes, the relative size of the two quantities' changes is constant. If an object is traveling at a constant speed, any change in the measure of distance is always the same number of times as large as the associated change in the measure of time (Thompson, 1994).

In algebra, conceiving of slope as a measure of how many times as large a change in y is as a change in x is useful in modelling and writing equations of lines. In trigonometry, radian measure can be thought of as a measure of how many times as large an arc length is as the radius of the given circle. In calculus, the difference quotient $(f(x+h) - f(x))/h$ can be understood as comparing the relative size of $f(x+h) - f(x)$ and h. We believe this meaning for difference quotient is more useful than thinking of "$f(x+h) - f(x)$ out of h" or "go up $f(x+h) - f(x)$ for every time we go over h" because these meanings do not work well when h is small. Additionally, because h becomes increasingly small and is typically not equal to one, thinking of slope as how much y increases for a one-unit change in x is not productive in calculus. This list represents only a subset of mathematical topics where considerations of relative size are productive.

THEORETICAL PERSPECTIVE: QUANTITATIVE REASONING AND MEANING

The theoretical perspective guiding the creation and scoring of the items reported evolved from the work of Piaget and von Glasersfeld. A project team designs items to reveal secondary teacher's mathematical meanings. The intent of our items is find out what meanings teachers have with regard to various mathematical concepts; notice, this is not equivalent to classifying teachers into categories according to those who can solve a problem and those who cannot. This theoretical perspective, and what we mean by "meaning" is addressed in depth in Thompson, et al.(2013). For any mathematical idea, there are a variety of potential meanings, some of which are more useful than others because of the coherence they provide a teachers' thinking and instruction. For example, the meaning of quotient as a measure of relative size would allow a teacher to explain why division is used in the slope formula.

It is possible to have multiple meanings for one topic, and each meaning can be either quantitative or computational. For instance, a computational meaning for quotient held by some calculus students is that quotient is the answer that results from performing long division (Byerley, Hatfield, & Thompson, 2012). We attempt to determine whether a teacher's meaning is computational or is based on reasoning about the quantities in the item.

Much has been written on student and teacher understandings of the curricular topics connected to conceptions of relative size such as fractions, rates of change and derivatives (Armstrong & Bezuk, 1995; Bowers & Doerr, 2001; Harel & Behr, 1995; Izsák, Jacobson, de Araujo, & Orrill, 2012; Orton, 1983; Steffe & Olive, 2010). See

Sowder et al. (1998) for a good overview of the literature related to teachers' understandings of multiplicative structures. In short, there is much evidence that both teachers and students struggle with topics that have a comparison of relative size at the heart of the idea.

METHODOLOGY

The two items discussed in this paper are part of the assessment project *Mathematical Meanings for Teaching secondary mathematics* (MMTsm). Items in the MMTsm were developed based on conceptual models of thinking that arose from prior research, our teaching or from interviewing teachers and students. For example, prior research on quotient (Ball, 1990; Ma, 1999; Simon, 1993) shows that both elementary and secondary mathematics teachers have stronger computational meanings than quantitative meanings. In items where teachers were asked to create a story problem for division by a fraction, most did not demonstrate a meaning for quotient as the relative size of two quantities. Coe (2007), Castillo-Garsow (2010) and Johnson (2010) found that often secondary students' and teachers' meanings for rate of change did not entail the idea of relative size of changes.

Items went through a process with multiple revisions as a result of doing item interviews with teachers, showing items to mathematicians and math educators, and analysing data from approximately 150 teachers collected in summer 2012. Further details of the methodology were described in a methodology paper submitted to PME 38 (P. W Thompson & Draney, under review).

RESULTS

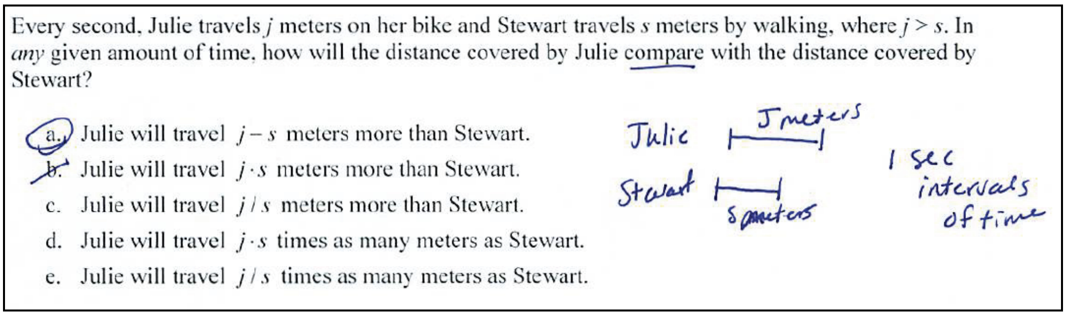
In the results section we will present two items, the rationale behind the items, and the teachers' results on the items. The first item, shown in Figure 1, was created to reveal teacher's meanings for constant speed.

Every second, Julie travels j meters on her bike and Stewart travels s meters by walking, where $j > s$. In *any* given amount of time, how will the distance covered by Julie compare with the distance covered by Stewart?

a. Julie will travel $j - s$ meters more than Stewart.
b. Julie will travel $j \cdot s$ meters more than Stewart.
c. Julie will travel j / s meters more than Stewart.
d. Julie will travel $j \cdot s$ times as many meters as Stewart.
e. Julie will travel j / s times as many meters as Stewart.

Figure 1: An item on relative rates.

Based on prior research we hypothesized that some teachers' meanings for speed were "chunky" (Castillo-Garsow, 2010). For those with a chunky meaning, speed is the distance travelled in a 1-unit interval (i.e. chunk) of time as opposed to a measure of how many times as large the measure of distance travelled is as the measure of elapsed time. We suspected that teachers with chunky meanings for speed might choose *j-s*, an answer that is only true for the first one-second interval. There is some evidence in the written work and interview data to support this hypothesis, an example of which is provided in Figure 2.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e. Julie will travel j / s times as many meters as Stewart.

Julie J meters
Stewart s meters
1 sec intervals of time

Figure 2: A teacher's response.

In teacher responses to other items, interviews, and in the literature, we also noticed teachers using the formula $d = rt$ inappropriately and thought that some teachers may expect to see a product as part of the answer (Bowers & Doerr, 2001). For example, some teachers used the formula $d = rt$ to find the total distance travelled on a trip with a non-constant rate of change by simply selecting the rate of change at the end of the trip. We have not yet done interviews to see why teachers selected *j*s* in (b), (d). The only difference between response (c) and (e) is that the response (e) uses multiplicative language and (c) uses additive language. When scoring the items we do not think of teachers' answer in terms of correct and incorrect, but in terms of how productive those meanings are for teaching. In this case we believe (e) is the most productive way to think about speed because it generalizes to situations where the change in time is not one-unit. Some teachers who substituted values for *j* and *s* were able to determine that the quotient *j/s* was important but did not select the phrasing "times as many" and instead chose (c).

The teachers' responses to "Relative Rates" are shown in Table 1.

Response	Math Majors	Math Ed Majors	Other Majors	Total
j-s	15	22	10	47
*j*s* more	0	0	4	4
j/s more	4	5	3	12
*j*s* times	0	1	5	6
j/s times	5	10	15	30
"no time"	1	0	0	1
Total	25	38	37	100

Table 1: High school teacher's responses to "Relative Rates."

The majority of responses (70%) do not reflect multiplicative comparisons of the relative size of distance travelled for any amount of elapsed time. Teachers with "Other" degrees (e.g., Art History, Biology or Religion) were more likely to choose the highest-level response (40%) than teacher with math (20%) or math education degrees (26%), but we did not find this relationship to be statistically significant.

We designed the second item, shown in Figure 3, to see whether teachers' thinking about a relative size situation would be constrained by the quantitative relationships or would be primarily algorithmic. Although the item is most closely aligned with elementary measurement standards, this foundational understanding is important in secondary mathematics standards as well. The first quantitative relationship is that when the magnitude of the unit is increased, the measure of the container will decrease. The second relationship is that if the new unit is 189/50 times as large as the old unit, the measure of the container is 50/189 times as large in the new unit.

> A container has a volume of *m* liters. One gallon is $\frac{189}{50}$ times as large as one liter. What is the container's volume in gallons? Explain.

Figure 3: The second item, "Liters to Gallons."

We scored responses to Liters to Gallons from 100 high school math teachers using a rubric that was negotiated by the project team. Responses that omitted *m* or did not somehow indicate the idea of "number of liters" at level zero (e.g. the teacher only wrote *189/50*). Uninterpretable responses, responses that cubed part of the expression to find volume, and responses of "I don't know" were also scored at level zero. For example, Figure 4 shows a level zero response from a math major who has taught forty high school math courses.[2]

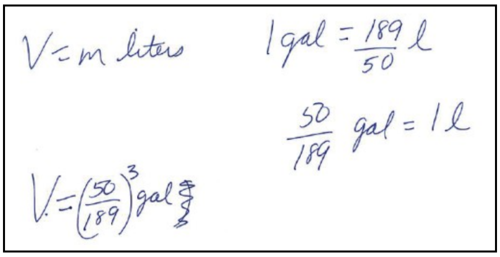

Figure 4: Level zero response to liters to gallons.

This was not the only response that used a cubic term in the answer and some stated explicitly that there must be three variables or that something must be cubed in volume problems. These responses do not reflect an awareness of how the relative size of the two units influences the relative size of the two measures.

The response in Figure 5 from a teacher with a math education degree who has taught seven high school math courses is an additional example of a level zero response. The teacher used the letter *G* to refer to both the magnitude of one gallon in the first line, and the number of gallons in the second line. The teacher did not demonstrate awareness of the reciprocal relationship between the measure of a quantity and the size of the unit measuring it.

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[2] For example, if a teacher taught one Algebra class, four geometry classes, and one study skills class in a school year we would say they taught five math courses that year.

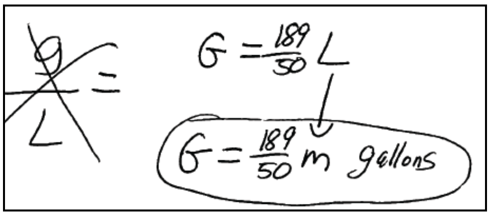

Figure 5: Example of level zero response to “Liters to Gallons”.

The response *G = 189/50 m* is level one if the teacher never used the same letter to represent two different quantities. The teacher in Figure 5 would have been scored at level one if the response had not used the letter “*G*” to represent two different quantities.

Level two responses demonstrate the correct relationship of relative size between the volume in gallons and the number of liters, by using the reciprocal 50/189. However they are not scored at the highest level because teachers wrote that a volume in gallons is a number of liters. If the response in Figure 6 had omitted the word “liters” in the final line or wrote that 50/189 had units of gallons per liter, the response would have been considered highest level.

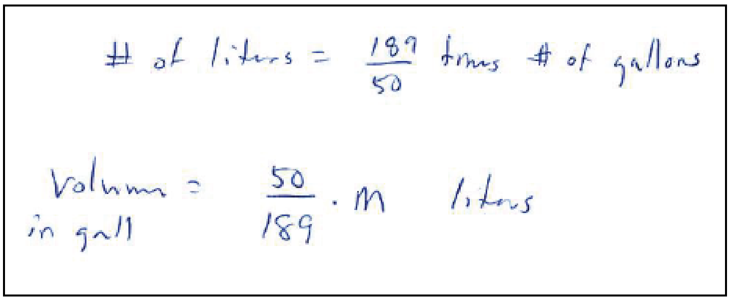

Figure 6: Example of a level two response.

Correct answers with explanations and correct answers without explanations were both scored at level three. Sometimes the response was only written symbolically such as *(50/189)m*. Level three responses may have incorrect work crossed out, but the teacher settled on a response of “the number of gallons equals *(50/189)* times *m*.”

Response	Math Majors	Math Ed Majors	Other Majors	Total
Level 0	5	8	12	25
189/50 m	14	20	14	48
50/189 m liters	1	1	1	3
50/189 m gallons	5	9	10	24
Total	25	38	37	100

Table 2: Responses and degree type for "Liters to Gallons."

The majority of responses (63%), regardless of teacher degree, demonstrate that the teacher did not consider the quantitative relationships regarding relative size when producing their answer. Although Table 2 shows that teachers with Other Majors have a higher percentage of highest-level responses (27%) than Math Majors (20%) and Math Ed Majors (23%), we found no statistically significant relationship between degree type and level of response.

The MMTsm had one additional item involving the comparison of two measures. The item in Figure 7 included an image of a circle with a highlighted arc.

In Nerdland they measure lengths in Nerds. The highlighted arc measured in Nerds is 12 Nerds. In Rapland they measure lengths in Raps. One Rap is 3/4 the length of one Nerd. What is the measure of the highlighted arc in Raps?

Figure 7: Additional measurement item named "Nerds and Raps."

Most teachers answered either 9 or 16, with 50 out of 100 high school teachers giving a highest-level response of 16. Out of those 50 teachers who had a highest-level response to “Nerds and Raps” only 17 gave a highest-level response to “Liters and Gallons.” We hypothesize using the letter “*m*” to represent an arbitrary number of liters required additional meanings for variables or increased the likelihood of using algebraic computations without considering quantitative relationships of relative size. However, even if the difficulty of Gallons to Liters was primarily caused by the variable “*m*”, Nerds and Raps shows at least 50% of the teachers were not constrained by quantitative relationships of relative size-in this case, the smaller the unit, the larger the measure.

After meeting and interviewing a number of teachers who responded to these items, we suspect their daily work does not require them to consider quantitative relationships of relative size. We believe most teachers are capable of reasoning quantitatively, but they have had few occasions to do so. When we used the Gallons to Liters problem in a workshop for teachers who took the MMTsm, they were able to think about it quantitatively. We believe MMTsm can be used in professional development to help teachers develop more productive meanings. We have conducted one three-day professional development using a number of our items that was positively received by the teachers, but further research is needed to help teachers build meanings related to relative size.

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USING INTERVIEWS TO EXPLORE TEACHER KNOWLEDGE PROFILES IN THE AREA OF PERMUTATIONS

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This paper illustrates consideration of multiple facets of mathematics teacher knowledge through interviews, demonstrates how teachers have different knowledge profiles, and discusses the implications for professional development. This study focuses on interviews with eight teachers, which were analyzed using Ball, Thames, and Phelps' (2008) framework of teacher knowledge, assigning knowledge types to statements made during the interview. All teachers exhibited multifaceted knowledge, but different profiles emerged. The implication is that teacher profiles should be considered in designing professional development. In addition, this analysis supports the use of teacher interviews as a tool to consider professional development needs.

INTRODUCTION

Since the introduction of the idea of pedagogical content knowledge (PCK; Shulman, 1986), studies and theoretical papers have attempted to clarify, specify, or measure Shulman's construct. However, as pointed out by Hill, Ball, and Schilling (2008), there is still little information about how teachers' PCK relates to student-level outcomes, or even about what constitutes PCK. A detailed view of teachers' professional knowledge may help us to enrich it.

This paper considers teacher knowledge to be multifaceted, using the framework of types of teacher knowledge put forth by Ball, Thames, and Phelps (2008), rather than viewing it as unitary. Not all theorized knowledge types have been measured independently or shown to be independent constructs (Hill, Ball, & Schilling, 2008). Thus, they remain, in part, theoretical distinctions. However, even if a different model of teacher knowledge emerges at a later date and is empirically validated, the premise of the analysis in this paper still stands as it points to the importance of recognizing teacher profiles, rather than defining specific profiles a priori.

Here, we are assigning knowledge types to teacher statements made during an interview. In doing so, two themes emerge: (1) teachers' knowledge profiles vary as seen in an interview setting, and (2) these profiles are useful when considering professional development. The study summarizes the results of analyzing eight teacher interviews. In the interviews, the teachers were presented with a mathematical problem and were asked to solve the problem, and also to reflect upon it, explain their solutions, and provide alternative strategies. The mathematical problem asked about how many ways there would be to arrange four distinct objects. The mathematical content, a permutation of all n of n objects [$P(n,n) = n!$] and its influence on responses cannot be disregarded; this study does not make claims about what these teachers would do when faced with a different type of problem.

In contrast to written assessments, interviews may provide teachers with more freedom in their responses, afford a more detailed look at their thinking, and serve as a more practical tool for educational administrators or professional development providers seeking to provide targeted development opportunities for teachers.

Defining Teacher Knowledge Types

Shulman (1986) introduced PCK in response to research and standards that heavily emphasized pedagogical procedures divorced from specific content areas. Rejecting this dichotomy, Shulman proposed that teachers needed not content-free pedagogy, nor pedagogy-free content, but a particular kind of professional expertise that went "beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (p. 9). Inside PCK, Shulman included representations, examples, and explanations, as well as common difficulties, common student preconceptions, and ways of changing incorrect student conceptions.

Hill, Ball, and Schilling (2008) give the most comprehensive look at PCK for mathematics. They propose that PCK is part of a larger construct, mathematical knowledge for teaching (MKT) and separate the universe of MKT into (a) subject matter knowledge and (b) pedagogical content knowledge. In this framework, subject matter knowledge includes both common content knowledge (CCK) and specialized content knowledge (SCK). Common content knowledge (or "'common' knowledge of content" in Hill et al., 2005, p. 387) includes what we might consider to be pure mathematical content; this is the knowledge of mathematics apart from the need to teach it. For example, knowing the solution for x in the expression $10^x = 1$. Specialized content knowledge is content knowledge that would be useful only to a teacher; this SCK is still mathematical knowledge, not pedagogy. One example is knowing how to evaluate three methods for multiplying two digit numbers and determine which of the methods are always mathematically valid. This SCK sits next to PCK but does not contain it; neither is it contained by it (Hill et al., 2008). It is knowledge that would be useful while engaged in teaching, but does not require one to know anything about students or about teaching.

PCK includes knowledge of content and students (KCS) and knowledge of content and teaching (KCT). KCS includes "knowledge of how students think about, know, or learn this particular content" (Hill et al., 2008, p. 375). KCT "combines knowing about teaching and knowing about mathematics" (Ball, Thames, & Phelps, 2008, p. 401), that is, knowledge of instructional strategies, choosing examples, and other elements that link the mathematics to the practice of classroom teaching.

Ball et al. (2008) do not limit the types of teacher mathematical knowledge to those described above and leave room in their model for future discovery and definition of knowledge types, particularly as relates to knowledge of the mathematical horizon and knowledge of curriculum. In addition, Ball et al. (2008) and Hill et al. (2008)

acknowledge the difficulty and subtlety in these distinctions, even at a theoretical level. However, the analysis presented here is restricted to these four relatively well-defined knowledge types. To clarify the theory behind these distinctions, Ball et al. (2008) describe examples of tasks in which teachers may engage that would be manifestations of a particular type of knowledge. Their examples paved the way for the analysis carried out in this study.

Accessing Teacher Knowledge Types through Interviews

While teacher knowledge types are theoretical distinctions, they have been described through tasks and measured through written assessment questions (Hill et al., 2005) that attempt to engage respondents in the same types of activities they would be doing as teaching professionals. While teachers may shift fluidly between knowledge types during teaching, and may hold knowledge in complexes (Sherin, 2002), examining the tasks connected to each knowledge type elaborated by Ball et al. (2008) enables us to disentangle the knowledge used.

This paper shifts examination of knowledge types to the interview setting. While Hill et al. (2008) describe conducting interviews as follow up to a written assessment to confirm that teachers were using specific knowledge types, since the interviews in this study will be analyzed statement by statement, the approach, analysis, and results presented here are novel. The interviews elicit teacher knowledge of all types, and the question is how much teachers use each type, and how different profiles emerge.

METHOD

Participants were eight secondary school teachers in a U.S. city or nearby urban rim community participating in a summer professional development workshop who accepted the invitation to be part of this study, which was independent of the workshop. Data were collected through flexible, open-ended individual interviews. The teachers were given problems to solve and, after solving each problem, they were asked for an explanation of their work and then to show a different way to solve the same problem and a different explanation. They were also asked what they believed their students would do when working on the same problem. The analysis presented here focuses on teacher responses to the first problem presented to the teachers. In this problem, teachers were asked how many ways they could arrange four objects. The four objects were presented as characters in boxes. In half of the interviews, the characters were numbers and in half the characters were letters. However, no teacher gave a response specific to either of the two formats, so the answers by all eight teachers were analyzed together.

ANALYSIS

The eight interviews were fully transcribed and statements in each interview were linked to items from the lists of teaching tasks in Ball et al. (2008) for each knowledge type. In each teacher statement, particular tasks were carried out, described, or referred to. A statement was defined as the full length of what a teacher said without response or interruption from the interviewer. For example, the statement “Yes – after we do a few, most of them see the pattern is that you multiply. So after we do the tree diagram, typically I go into the fundamental counting principle” included four different tasks: Sequence particular content for instruction (KCT); Linking representations to underlying ideas and to other representations (SCK); Anticipate what students are likely to think (KCS); and Anticipate whether students will find a task easy or hard (KCS). Multiple codes were allowed, so more than one knowledge type may have been applied to a single statement, as was the case in this example. The eight teachers produced a total of 168 statements.

RESULTS

In the interviews, all teachers made statements classified under all four knowledge types, exhibiting multifaceted mathematical knowledge for teaching. Considering the eight teachers together, SCK appeared in 55% of statements (see Table 1). The other knowledge types (CCK, KCS, and KCT) appeared in 33% to 35% of statements.

Knowledge type	# of statements linked to this knowledge type (N = 168)	% of statements linked to this knowledge type
CCK	59	35%
SCK	92	55%
KCS	55	33%
KCT	58	35%

Table 1: Relative frequency of statements linked to each knowledge type.

As shown in Table 2, the dominance of SCK varies across individual teachers. SCK, more frequent in the aggregate data, was the most commonly used knowledge type for only four of the eight teachers. In addition, none of the teachers mirrored exactly the profile of the study population as a whole.

Name	**# CCK**	**CCK %**	**# SCK**	**SCK %**	**# KCS**	**KCS %**	**# KCT**	**KCT %**	**HIGHEST %**
Jessica	8 of 14	57%	7 of 14	50%	2 of 14	14%	2 of 14	14%	CCK
Anna	7 of 14	50%	6 of 14	43%	4 of 14	29%	5 of 14	36%	CCK
Sarah	8 of 23	35%	16 of 23	70%	4 of 23	17%	9 of 23	39%	SCK
Whitney	8 of 17	47%	9 of 17	53%	4 of 17	24%	3 of 17	18%	SCK
Annie	6 of 20	30%	16 of 20	80%	7 of 20	35%	3 of 20	15%	SCK
Laura	10 of 35	29%	22 of 35	63%	15 of 35	43%	18 of 35	51%	SCK
Betsy	3 of 18	17%	6 of 18	33%	10 of 18	56%	4 of 18	22%	KCS
Shana	9 of 27	33%	10 of 27	37%	9 of 27	33%	14 of 27	52%	KCT
Total	**59 of 168**	**35%**	**92 of 168**	**55%**	**55 of 168**	**33%**	**58 of 168**	**35%**	**SCK**

Table 2: Percentage of use of knowledge types, by teacher.

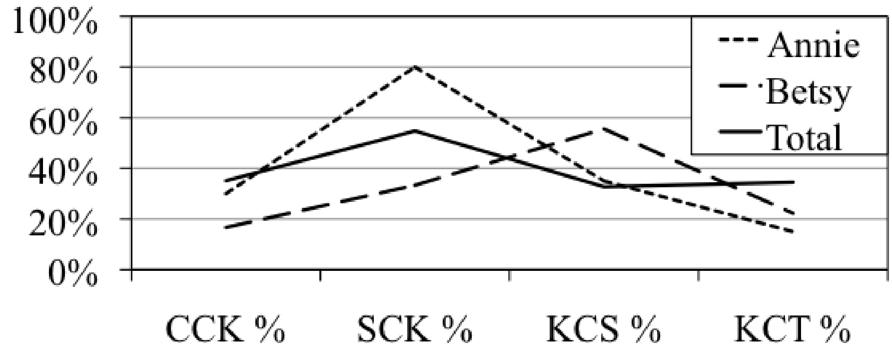

Figure 1: Percentage of statements exhibiting each knowledge type.

Figure 1 provides an image of the percentages for the total and two of the teachers (the data points are connected to easily show all the points corresponding to each teacher). Data on eight teachers are not sufficient to generalize profiles; however, a detailed description may help us begin to construct ideas about the different profiles that could emerge from the teacher knowledge types revealed in interviews. To this goal, two cases will be described here, those of Annie and Betsy. Annie was chosen because she exhibited SCK more than any other knowledge type, mirroring the aggregate data, and because she had the greatest percentage difference between any two knowledge types, with SCK used in 80% of statements and KCT used in only 15%. Betsy was chosen because her profile is at the opposite extreme: she was the only teacher to exhibit KCS more frequently than any other knowledge type, and the only teacher to use KCS more frequently than SCK. These two teacher profiles will enable a more detailed discussion below about the potential for this type of analysis.

Annie, whose work is shown in Figure 2, was a teacher with a strong background in mathematics. She had been teaching for less than five years, but her teaching had always been in secondary school mathematics. When presented with the problem, she quickly found the (correct) answer and explained the procedure she had used. She was able to also talk about two different methods for finding the solution and discuss which one she preferred and why. The use and critique of different representations was a major factor in her high percentage of statements exhibiting SCK, as two of the tasks linked to SCK are "Recognizing what is involved in using a particular representation," and "Linking representations to underlying ideas and to other representations." Annie referred to one (or both) of these tasks in 8 of the 20 statements she made about this interview question, as was the case in this statement:

> We start off with listing them all out, and then do the tree diagram, we can do the tree diagram for it, and then we came up with the formula, so they can see how many choices do they have. And I eventually show them the slots. Like think of 4 chairs that you have and then one person sits here there's only 3 people left, so you take one out.

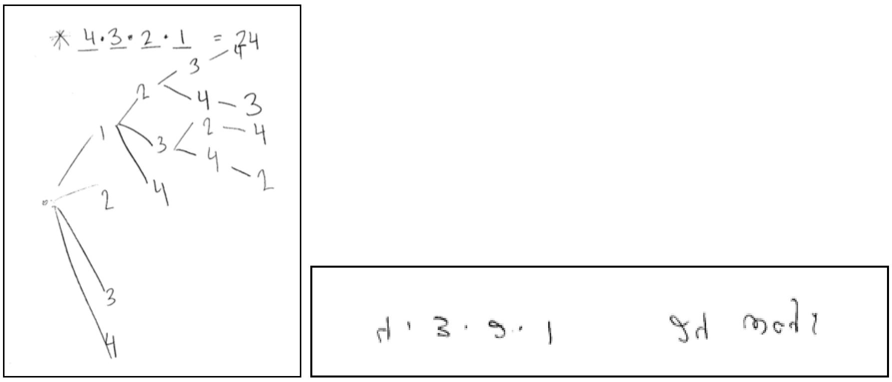

Figure 2: Work on the interview problem. Left: Annie’s; Right: Betsy's.

Other than discussion and use of different representations, the other major factor in Annie’s high percentage of statements exhibiting SCK referred to the task, “Using mathematical notation and language and critiquing its use,” which was identified in 8 of her 20 statements. All of the instances sprung from discussion of how her students would struggle with knowing to multiply, rather than add, the numbers, because of the use of the word “and.” She clarified that students would have trouble “just with the ‘and’ and the ‘or.’ Because doing this stuff [permutations], it means different things.” She elaborated cases when this would occur and how her students would react. Note that these instances of SCK occurred *only* because she was engaged in a task associated with KCS, namely “anticipate what students are likely to think.”

Betsy had been teaching for much longer than Annie had, more than 20 years, but she had not always been a mathematics teacher. She had started by working with special needs students in different subject areas, and then had begun to focus on teaching mathematics with the same population of students at the secondary school level. When Betsy was given the interview problem, she was able to solve it quickly and correctly, as shown in Figure 2, but she was more tentative in her work than Annie was, saying, “Okay, this is the factorial. And granted, I don’t do that too much, but what I understand is you go 4, 3, 2, 1?” When asked about other methods, Betsy was not able to spontaneously think of an alternative, so she did not refer to the same tasks in SCK that Annie had, but she had no difficulty describing how her students would react to the problem and what they would do with similar problems, referring to the tasks “anticipate what students will find confusing” and “anticipate whether students will find a task easy or hard.” For example, when talking about what students would do, Betsy said, “and the other [kind of problem] that they have trouble with too is replacement and without replacement. I mean, some of the kids got it but others just really struggled with it.” Note that by “replacement” and “without replacement,” Betsy meant whether an item could be used again in a permutation once it had already been used once. These terms are common in secondary school classrooms, where they often talk about pulling items from a bag and either replacing or not replacing the selected item before choosing the next.

DISCUSSION

As mentioned above, the analysis of teachers' statements reveals that they all demonstrated all types of knowledge during the part of the interview analyzed in this paper. No teacher had an individual profile that matched the profile of the combined data from all eight participants. What can we make of these wide variations? While making decisions based on these differences now would be unwise, if we were to apply this technique to a larger sample, we might see a set of teacher profiles emerge. In connection with classroom data, we could begin to understand what these different profiles suggest about the teacher's work of teaching.

This is illustrated by the two profiles, Betsy and Annie, described above. We are not naming one profile as superior to the other or preferable for helping students to learn. However, the differences between these two cases illuminate the breadth of experience in mathematics and the variety of perspectives that exist in the teaching force. The analysis of different knowledge types highlights and clarifies the differences between the profiles, and could ultimately help to provide professional support to the teachers. For example, Betsy made relatively few statements showing evidence of SCK. This might lead us to infer that *for this particular mathematical area* (permutations), Betsy could benefit from working in professional development activities related to SCK, such as working with and connecting a variety of representations. Conversely, Annie made few statements that showed evidence of KCT. She might be better supported, then, by professional development that focused on the teaching aspect, such as choosing examples or deciding how to respond to student contributions. Another advantage of examining these teacher profiles is that we begin to see that different profiles may complement each other. That is, perhaps Betsy and Annie would be able to each take the lead in turn in sharing teaching knowledge with each other in a mutually beneficial way.

This particular interview analysis is different from previous work on distinguishing teacher knowledge types. The analysis of individual statements in interviews is based upon Ball et al. (2008), but not recommended or endorsed by them. While not as easy to code, the interview allows for a more descriptive view of a teacher's varied knowledge. This may help us not only to understand the different teacher profiles, but also to begin to see how they complement each other, as in the cases described above. In addition, written assessments that can claim to measure a particular type of teacher knowledge, like that described by Hill et al. (2005), need to be developed and tested extensively. By necessity, they can only cover a finite number of mathematical topics. If we want to know more about teacher knowledge about something specific, like the permutation question analyzed above, a coded interview allows this targeted examination. Interested researchers and those who work on professional development could look at teacher knowledge in their particular mathematical domain, even when they do not possess the resources that would be required to develop a written assessment.

It is important to consider that the freedom of an interview may make it more likely that teachers will elaborate on the elements that interest them. In doing so, they move back and forth quite fluidly between knowledge types. This is supported by the findings above that all eight teachers exhibited all four knowledge types. In fact, Sherin (2002) suggests that teachers may access “content knowledge complexes” (p. 124), where the teachers’ past experience creates a link between the content and the pedagogy that results in accessing these types of knowledge together. The way that the teachers in this study moved easily between knowledge types lends support to Sherin’s theory. However, using the terms put forth by Shulman (1986), she says, “I claim that there are larger elements of teacher knowledge that cannot be categorized either as subject matter knowledge or as pedagogical content knowledge” (Sherin, 2002, p. 124-125.) We would suggest instead that it is not that a complex exhibited by a teacher can be classified as *neither* type of knowledge, but rather that it can be classified as *more than one* type of knowledge. The idea of content knowledge complexes gives us a view of how different knowledge is called forth by a teacher, but it does not preclude us from categorizing teacher statements more specifically.

While mapping and coding knowledge types may begin as a theoretical exercise, it is one with a practical goal. A unitary approach to professional development for teachers ignores their varying knowledge profiles. While it may seem obvious that teachers differ, we know little about how to determine which tasks they need support with and how to provide this support. Careful examination of their profiles through interviews about the complex tasks of teaching can only help.

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PROBLEM-SOLVING STRATEGIES AS A MEASURE OF LONGITUDINAL CURRICULAR EFFECTS ON STUDENT LEARNING

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This study examined the longitudinal effects of a middle school reform mathematics curriculum on students' open-ended problem solving in high school. Using assessment data from a large, longitudinal project, we compared the open-ended problem-solving performance and strategy use of high school students who had used the Connected Mathematics Program (CMP) in middle school with that of students who had used more traditional mathematics curricula. When controlling for sixth-grade state mathematics test performance, high school students who had used CMP in middle school had significantly higher scores on a multipart open-ended problem. In addition, high school students who had used CMP appeared to have greater success algebraically abstracting the relationship in the task.

INTRODUCTION

Problem solving is an integral focus of the school mathematics curriculum. Studies of problem solving in mathematics education have already moved from a focus only on the product (i.e., the actual solution) to a focus on the process (i.e., the set of planning and executing activities that direct the search for solution). Individual differences in solving mathematical problems can sometimes be understood in terms of differences in the uses of different strategies. Proficiency in solving mathematical problems is dependent on the acquisition, selection, and application of both domain-specific strategies and general cognitive strategies (Schoenfeld, 1992; Simon, 1979). Thus, competence in using appropriate problem-solving strategies reflects students' degrees of performance proficiency in mathematics. This implies that assessment tasks should reveal the various strategies that students employ. In addition, students' problem-solving strategies become more effective over time. In fact, researchers have long used the examination of problem-solving strategies to assess and evaluate instructional programs and education systems (Cai, 1995; Fennema et al., 1998). Therefore, both the examination of the strategies that students apply and the success of those applications can provide information regarding the developmental status of students' mathematical thinking and reasoning.

The purpose of this study is to use problem solving strategies to investigate how the use of different types of middle school curricula affects the learning of high school mathematics for a large sample of students from ten high schools in an urban school district. This paper reports findings from a large project, *Longitudinal Investigation of the Effect of Curriculum on Algebra Learning* (LieCal).

BACKGROUND AND RATIONALE OF THE STUDY

The LieCal Project began with an investigation of the differential effects of a reform middle school mathematics curriculum called the Connected Mathematics Program (CMP) and more traditional (called non-CMP) curricula on middle school students' learning of algebra. The CMP and non-CMP curricula are very different. In particular, they make use of strikingly different conceptions about algebra – a functional approach in the CMP curriculum and a structural approach in the non-CMP curricula. For example, the CMP curriculum defines a variable as a quantity that changes or varies. The variable idea is needed to describe relationships in the problem situations that the CMP curriculum uses. In contrast, the non-CMP curricula define a variable as a symbol (or letter) used to represent a number. Variables are treated predominantly as placeholders and are used to represent unknowns in expressions and equations. By introducing the concept of variables in this fashion, the non-CMP curricula support a structural approach to algebra. In the non-CMP curricula, similarly, equation solving is introduced symbolically by using the additive and multiplicative properties of equality (equality is maintained if the same quantity is added to, subtracted from, multiplied by, or divided into both sides of an equation). On the other hand, in the CMP curriculum, equation solving is introduced using real-life contexts that are incorporated into contextually based justifications of the equation-solving steps.

In the LieCal Project, we found that on open-ended tasks assessing conceptual understanding and problem solving, the growth rate for CMP students over the three middle school years was significantly greater than that for non-CMP students (Cai et al., 2011). At the same time, CMP and non-CMP students showed similar growth over the three middle school years on the multiple-choice tasks assessing computation and equation-solving skills. These findings suggest that the use of the CMP curriculum is associated with a significantly greater gain in conceptual understanding and problem solving than is associated with the use of the non-CMP curricula. However, those relatively greater conceptual gains do not come at the cost of lower basic skills, as evidenced by the comparable results attained by CMP and non-CMP students on the computation and equation solving tasks.

The LieCal Project has subsequently followed the students into their high school years. All high schools in the district are required to use the same district-adopted mathematics curriculum. CMP and non-CMP students were mixed into each class in each of ten high schools in the same district. Thus, all of the former CMP and non-CMP students used the same curriculum and were taught by the same teachers in their high schools. We have been examining whether the superior problem-solving abilities gained by the CMP students in middle school result in better performance on a delayed assessment of mathematical problem solving in high school.

In a previous study, we used problem posing as a measure of middle school curricular effect on students' learning in high school (Cai et al., 2013). Using problem posing as a measure, we found that in high school, students who had used the CMP curriculum in middle school performed equally well or better than students who had used more traditional curricula. The findings from this previous study not only showed evidence of the strengths one might expect of students who used the CMP curriculum, but also demonstrated the usefulness of employing a qualitative rubric to assess different characteristics of students' responses to the posing tasks. In the same vein, the present study uses open-ended problem-solving strategies as a measure to examine longitudinal curricular effect on students' learning.

METHOD

Participants

In the LieCal Project, we followed more than 1,300 students (650 using CMP and 650 using non-CMP curricula) from a school district in the United States for three years as they progressed through grades 6-8. In the 2008-2009 school year, most of these 1,300 CMP and non-CMP students from the middle school study entered high schools as freshmen. We then followed the students enrolled in the 10 high schools that have the largest numbers of the original 1,300 CMP and non-CMP students.

Assessment Tasks and Analyses

As part of the LieCal Project, we developed and used 13 open-ended tasks to assess students' learning in high school, specifically the 11th and 12th grades. Students' responses were analyzed in two ways. The first was to quantitatively score each student response using a prior-developed holistic scoring rubric. The second was to qualitatively analyze students' responses with a focus on their solution strategies. In this paper, we mainly draw on results from an analysis of solution strategies to a pattern problem called the doorbell problem (see Appendix). This five-part task assesses students' ability to find regularities of a pattern and make generalizations. We chose to report the results from this task as it is a representative task that assesses students' generalization skills.

Data Collection and Coding

As part of the larger longitudinal study, we assessed students in the fall of 11^{th} grade (Fall, 2010), spring of 11^{th} grade (Spring 2011), and spring of 12^{th} grade (Spring 2012). The data for the analyses of students' strategies came mainly from the 12th grade spring assessment. In a small number of cases, if a student did not participate in the Spring 2012 assessment but did participate in the Spring 2011 assessment, we used the data from the Spring 2011 assessment. If a student did not participate in either the Spring 2012 or Spring 2011 assessments, but had participated in the Fall 2010 assessment, we used the data from the Fall 2010 assessment. This allowed us to look at the students' most recent attempt at each task.

As noted above, students' responses to the doorbell problem were first scored using a holistic scoring rubric that took into account the students' numerical answers and their explanations of their strategies. The responses were then also qualitatively coded for the types of strategies used. We coded students' solution strategies for parts A, B, C, and E as an abstract strategy, a concrete strategy, an unidentifiable strategy, or no strategy. Students who used an abstract strategy were able to recognize that the number of guests entering for each ring was equal to either two times the ring number minus one (i.e., $y = 2n - 1$) or the ring number plus the ring number minus one (i.e., $y = n + (n - 1)$). Students who used a concrete strategy were able to identify that the number of guests who enter increases by two for each doorbell ring and then sequentially adding two until they reached the desired number of rings, but did not abstract an algebraic formula. An unidentified solution strategy was a strategy that did not particularly make sense for the problem (e.g., $y = [r(100) + 2] - 1$). Lastly, a student was said to have used no strategy if the student did not show work for his or her answer, or if he or she did not attempt to answer the question at all.

Students' strategies for part D were coded in one of five ways. First, the student could have completely abstracted the algebraic formulas $2n - 1$ or $n + (n - 1)$. Secondly, they could have completely abstracted the pattern in a verbal description (e.g. "The number of guests who entered on a particular ring of the doorbell equalled two times that ring number minus one."). Third was an incomplete abstraction that only captured a recursive relationship, such as, "When the bell rings, two more people come." Fourth was an unidentified strategy, which either represented the strategies for students who incorrectly answered the question or had a provided a strategy that did not make sense. Finally, a strategy was coded as "no strategy" if no attempt was made to solve the problem.

RESULTS

Overall Performance on the Doorbell Problem

We first conducted two ANCOVA analyses based on the quantitative scoring to student responses to the doorbell problem. The ANCOVA analyses indicated significant curriculum effects under two covariates for the doorbell problem. When controlling for overall state math test exam scores for 6th grade, CMP students scored significantly higher than non-CMP students on the doorbell problem ($t = 2.09$, $p = 0.0371$). When controlling for scores on the algebra subtest on the overall state math test for 6th grade, CMP students still scored significantly higher than non-CMP students ($t = 2.47$, $p = 0.0141$).

Performance on Individual Parts of the Doorbell Problem

Chi-squared tests were performed to look for relationships between curriculum and correctness of answers on each part of the doorbell problem. For part A, there was a significant relationship between curriculum and correct answers ($\chi^2 = 6.5363$, $p < 0.040$). That is, a significantly larger percentage of the CMP students had correct answers than the non-CMP students. For parts B, C, D, and E, there were no significant relationships between curriculum and correct answers. For each of the five parts of the problem, Table 1 provides the percentage of students with correct answers in that part. Note that Table 1 shows a considerable decreasing trend in the number of students who found a correct solution from part A to part E.

	Doorbell Problem Part				
Curriculum	A	B	C	D	E
CMP (n = 321)	80.1	38.6	27.7	18.1	7.5
Non-CMP (n = 212)	74.5	35.4	27.4	16.0	5.2

Table 1: Percentages of CMP and non-CMP students who correctly solved each part of the Doorbell Problem

Concrete and Abstract Solution Strategies

Focusing specifically on the solution strategies of those students who provided correct solutions for parts of the Doorbell problem, some differences in strategy use arose between the two groups. For part B (see Table 2), 73.4% of CMP students (n=124) and 60% of non-CMP students (n=75) abstracted the problem to an algebraic formula to find the correct solution, whereas 17.7% of CMP students and 24.0% of non-CMP students used a concrete strategy. A significantly greater proportion of CMP students used the abstract strategy than did the non-CMP students (z = 1.97, p < 0.050), but there was no significant difference in proportion between CMP and non-CMP students for the concrete strategy.

For part C (see Table 2), 71.9% of CMP students (n=89) and 67.2% of non-CMP students (n=58) abstracted the problem to an algebraic formula, whereas 7.9% of CMP students and 19.0% of non-CMP students used concrete strategies to find a correct solution. A significantly greater proportion of non-CMP students used the concrete strategy than did the CMP students (z = -2.27, p < 0.025), but there was no significant difference in proportion between CMP and non-CMP students for the abstract strategy.

For part A (see Table 2), 67.3% of CMP students (n = 257) and 63.9% of non-CMP students (n = 158) used a concrete strategy to find the correct answer, whereas 26.1% of CMP students and 27.8% of non-CMP students abstracted the problem to an algebraic formula. There were no significant differences in proportion between CMP and non-CMP students for each strategy.

For part D, almost every student who provided a correct solution responded in nearly the same way. All of the 34 non-CMP students and 54 out of 58 CMP students who correctly answered this part generated an algebraic abstraction and provided a mathematical formula. The remaining four CMP students wrote out a verbal description of the mathematical formula, which would still require them to have first abstracted the relationships before translating those relationships into written form.

		Type of strategy			
Problem part	*n*	Abstract	Concrete	Unidentified	None
A					
CMP	257	26.1	67.3	3.5	3.1
Non-CMP	158	27.8	63.9	1.9	6.3
B					
CMP	124	73.4	17.7	3.2	5.6
Non-CMP	75	60.0	24.0	4.0	12.0
C					
CMP	58	71.9	7.9	9.0	11.2
Non-CMP	34	67.2	19.0	5.2	8.6
D					
CMP	58	100.0	0.0	0.0	0.0
Non-CMP	34	100.0	0.0	0.0	0.0
E					
CMP	24	62.5	29.2	4.2	4.2
Non-CMP	11	45.5	36.4	0.0	18.2

Table 2: Percentages of CMP and non-CMP students who used each type of strategy to correctly answer parts of the doorbell problem

Part E seemed to be a challenging question for both the CMP and non-CMP students. Only 24 CMP students and 11 non-CMP students provided a correct solution to this part of the doorbell problem. Given these small sample sizes, although there were noticeable group differences in raw percentages of students using algebraic and concrete strategies, with a greater proportion of CMP students than of non-CMP students using algebraic strategies, these differences were not statistically significant.

DISCUSSION

As part of a larger longitudinal study of curricular effect on mathematics learning, the results we have presented above provide a useful perspective on the potential long-term impacts of reform mathematics curricula on students' mathematical thinking and problem solving. Although we have presented data from only one open-ended task, the results suggest that high school students who used the CMP curriculum in middle school were more successful than their peers who used more traditional middle-school curricula at solving the doorbell problem and explaining their solution strategies. This result accords with those obtained when these students were still in middle school (Cai, et al., 2011). The result is also consistent with our previous findings using problem posing as measure of curricular effect (Cai et al., 2013). Thus, it would appear that the CMP students' problem-solving gains persist well into high school.

The retention of these gains over longer time intervals also parallels the findings from research on the effectiveness of problem-based learning (PBL) in medical education (Hmelo-Silver, 2004). In that context, medical students trained using a PBL approach performed better than non-PBL students on conceptual understanding and problem-solving ability even when assessed at a later time. In a similar fashion, the CMP students in the LieCal project experienced problem-based instruction that focused on developing students' conceptual understanding and problem solving abilities.

In addition, our analysis of the strategies used by the students in this study suggests that the CMP students who correctly solved the parts of the doorbell problem were somewhat more likely to make generalizations. This appears to reflect the emphasis in the CMP curriculum on relationships between quantities (i.e., the functional approach). The ability to abstract algebraic relationships from real-world situations appears to also have persisted in the CMP students.

Note that for this analysis, we focused on the strategies of students who correctly answered one or more parts of the doorbell problem. We did not consider the strategies of students who failed to provide correct answers. Additional analyses that will further probe the strategies of students who provided incorrect answers are in progress at the time of this proposal. Also, we are analysing data from other open-ended problems.

APPENDIX

Sally is having a party.

The first time the doorbell rings, 1 guest enters.

The second time the doorbell rings, 3 guests enter.

The third time the doorbell rings, 5 guests enter.

The fourth time the doorbell rings, 7 guests enter.

Keep going in the same way. On the next ring a group enters that has 2 more persons than the group that entered on the previous ring.

A. How many guests will enter on the 10^{th} ring? Explain or show how you found your answer.

B. How many guests will enter on the 100^{th} ring? Explain or show how you found your answer.

C. 299 guests entered on one of the rings. What ring was it? Explain or show how you found your answer.

D. How many guests will enter on the n^{th} ring? Show or explain how you found your answer.

E. If we count all of the guests who entered on the first 100 rings, how many would we get in total? Show or explain how you found your answer.

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ALGEBRA FOR ALL: THE HIDDEN COST

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PISA 2012 results indicate that school systems that group students based on ability levels tend to have lower performance than those that do not divide students by ability. One way some in the United States have sought to increase equity of opportunity is to mandate enrollment of students in college-preparatory mathematics, i.e., Algebra 1 in eighth or ninth grade. This paper is based on a study conducted on one such curricular change. It uses a multiple linear regression model to compare two graduating class cohorts—one from before the initiative and one after—on test scores, courses completed, grades, and drop-out rates. There were positive gains for select groups of males and negative results for most females with the highest losses found for White females, especially those qualifying for special education services.

BACKGROUND

Countries around the world vary in their approach to mathematics education. Some have a highly stratified system, sorting students from an early age, while others delay sorting students until their last two years of schooling, if at all. PISA 2012 results indicate that countries with systems that group students according to their ability tend to have lower performance than those that do not. Across countries, students in schools that do not use ability grouping on average outperform students in schools that do. Furthermore, ability grouping or "tracking" has a disproportionate impact on students of lower socio-economic status (SES), and that impact is greater the earlier the age at which students are divided according to ability (OECD, 2013).

In the United States, ability grouping in mathematics is often firmly established by grade eight, around age 13, with results similar to those found in PISA 2012: lower levels of mathematical achievement—particularly for students from historically marginalized or economically disadvantaged groups. Many scholars in the U.S. have called for increased access to college-preparatory mathematics curricula at grade eight or nine (age 13 or 14), especially for historically marginalized or disadvantaged groups, as a way to increase equity of opportunity (e.g. Pelavin & Kane, 1990; Silva, Moses, Rivers, & Johnson, 1990; Smith, 1996; U.S. Department of Education, 1997). Initiatives aimed at this grade level are often called *Algebra for All* initiatives because Algebra 1 is the course students generally enroll in at age 13 or 14 if they are to complete a college-preparatory mathematics course sequence by age 18.

This study examines one such initiative in an economically and ethnically diverse school district in the Midwestern United States: the Madison (WI) Metropolitan School District's (MMSD) *Algebra for Everyone* initiative. Analogous to results seen internationally, students of color were under-represented in the district's higher-level mathematics classes and over-represented in basic and vocational mathematics classes, thus denying many students of color the opportunity to apply to and attend college due to inadequate high school mathematics courses. The MMSD identified institutional and systemic racism as a large contributor to this situation and decided to discontinue using staff recommendations for students' mathematics class placements and instead place all students in a college-preparatory mathematics track.

In 2003 non-college-preparatory mathematics classes at the high schools such as Pre-Algebra and Consumer Math were discontinued district-wide and all students were required to enroll in an Algebra 1 or higher-level mathematics class by grade nine. The only students who had an option for enrollment in non-college-preparatory mathematics were Special Education students who planned to apply for an exception based on their diagnosed disabilities (graduating via Individualized Education Plan (IEP)).

By 2004 the *Algebra for Everyone* initiative was in full swing and a disturbing new trend was appearing in the Algebra 1 classes: higher and higher failure rates were observed across all sections and for all teachers. By 2007, failure rates in Algebra 1 had skyrocketed to 40% (from an average of just 10% in 2000) with some Algebra 1 classes having 65% of students failing.

Anecdotally, students of color seemed to be more likely to fail than White students, Special Education students (those with diagnosed cognitive and/or emotional disabilities) seemed to be more likely to fail than those not qualifying for Special Education, and the students who struggled the most in Algebra 1 seemed to have very low middle-school mathematics achievement. It was extremely disheartening for classroom teachers to both literally and figuratively fail so many students.

When examining results for the school district as a whole, the policy of largely eliminating ability grouping in ninth grade seemed to be a success. More students were completing a college-preparatory mathematics course sequence than had ever before, and yet there was this seemingly contradictory anecdotal evidence that the policy was actually *lowering* achievement for many students. Was it just that these students were struggling at first but were able to recover and catch up, or was it that the positive effects on some students masked the negative effects on others when outcomes were aggregated? This study was conceived in order to explore the effect of eliminating ability grouping via the *Algebra for Everyone* initiative on students in the MMSD and whether that effect differed at all depending on a student's demographic group.

METHODS

Data Source

I used transcript and demographic data from two graduating class cohorts: the last cohort in the MMSD whose students were able to enroll in classes below Algebra 1 and the cohort entering high school soon after the implementation of the *Algebra for Everyone* initiative. The first cohort (Cohort A) entered ninth grade in 2000 and was the last cohort for whom Pre-Algebra was still an option at all four high schools in the district. Cohort B entered high school in 2004, which was the second year in which Algebra 1 was the lowest-level math class offered at all four high schools. I chose the second year of the *Algebra for Everyone* initiative to avoid as much as possible any effects of the adjustment period on student achievement.

The raw data I received contained transcript data for grades 8 through 12 for 4,440 students in the MMSD who were either enrolled in ninth grade in the fall of 2000 or the fall of 2004. Students who were not first-time ninth-graders in either 2000 or 2004 were excluded from the study. After these exclusions, there remained 2,019 students in Cohort A and 2,006 students in Cohort B.

Eighth-Grade (Incoming Ninth-Grade) Achievement

Using independent-samples *T* tests to compare the means of Cohort A and Cohort B, I discovered that there were not significant differences ($p \leq 0.05$) in mean eighth-grade achievement between the two groups either overall, or when divided into each of the eight main demographic subgroups (Asian males and females, Black males and females, Hispanic males and females, White males and females), in terms of the number of eighth-grade mathematics credits earned or the eighth-grade mathematics grade point average (GPA).

I also compared scores from the state-wide standardized test given in eighth grade: the Wisconsin Knowledge and Concepts Examination (WKCE). Unfortunately, I was unable to conclusively compare eighth-grade WKCE scores from Cohort A to Cohort B because of changes to the WKCE test which occurred in 2002 (WI DPI, 2003), but the change in the MMSD's 8^{th} grade scores from Cohort A to Cohort B closely resembles the changes seen across those years state-wide. When this result is paired with the favorable comparison of the measures of 8^{th} grade mathematics GPA and 8th grade mathematics credits earned, it gives confidence that students in the two cohorts entered high school with essentially the same prior achievement.

Criteria on Which Cohorts Were Compared

I then set out to measure whether any of the positive effects desired by proponents of *Algebra for All* initiatives, as well as any possible negative effects, were realized during the implementation in the MMSD. I compared the two cohorts on measures of student achievement chosen to address specific claims found in the literature (see Table 1).

Measure(s) of student achievement	Notes on how data were recorded
Level of initial high school mathematics class enrollment. Level of highest mathematics class taken in high school for which credit was received.	1 = Special Ed or Pre-Algebra (non-college prep mathematics) 2 = Algebra 1 3 = Geometry 4 = Algebra 2 5 = Algebra 3, Pre-Calc, or AP Stats 6 = Calc AB or higher.
GPA in high school mathematics classes and overall high school GPA.	GPA was unweighted and on a four-point scale.
Overall ACT scores and ACT mathematics sub-scores.	If there was more than one score, the highest one was used.
Number of mathematics credits earned in high school.	Programming classes were not included in the total.
Drop-out rate.	

Table 1: Measures of student achievement.

Statistical Methods and Justification

I used a standard multiple linear regression model because it allowed me to better isolate the effects of the *Algebra for Everyone* initiative from other known variables, such as gender or socio-economic status. For example, if a particular subgroup had an increase in drop-out rates from Cohort A to Cohort B, it may be due to the initiative, but it could also be due to an increase in the proportion of students in that group with low socio-economic status. Multiple linear regression calculates the magnitude of change we can expect to see in the dependent variable due to each predictor (independent variable) and create a model which quantifies this change.

Regression models for all variables have coefficients for the following predictors where possible: Cohort, Gender, each of the races/ethnicities except for White, Special Education status, English Language Learner status, and Socio-Economic status. I translated the demographic data into dummy codes of 0 and 1 so as to be able to use them as predictors in linear regression models for each measure in Table 1. Because the drop-out variable took only values of yes (1) or no (0), I used a binary logistic regression model to analyze this change.

The focus of this study was the coefficient for Cohort, which represents the amount of change from Cohort A to Cohort B for a given variable that may be attributed to the *Algebra for Everyone* initiative.

This study was conceived primarily out of concern that the *Algebra for Everyone* initiative was having a differing effect on certain demographic subgroups versus others. This interest necessitated that the analysis not stop at simply calculating results for the MMSD as a whole, males vs. females, or even the eight main demographic subgroups. In all, I calculated regression equations for approximately 150 different demographic subgroups: for example, one of the subgroups was the group of White Female Low-SES Special-Ed students. At first glance, this seems like a great deal of unnecessary calculations, but the fine grain size proved to be pivotal in terms of attaining useful results. Many variables did not show significant differences for the larger demographic group but differences became significant as the group was subdivided.

The fine grain size allowed this study to answer, in a way that would not have been possible otherwise, the question of whether the *Algebra for Everyone* initiative had divergent effects on different demographic groups.

RESULTS

Positive Results: Increased Achievement for Select Groups of Males

As hoped, the *Algebra for Everyone* initiative did increase the mathematics achievement in the MMSD of some historically marginalized and/or disadvantaged groups, including Asian and White males of low socio-economic status, Black males who were not of low socio-economic status and were not receiving Special Education services, and Hispanic males who were not classified as English language learners. For these groups, the initiative yielded:

- An increase in the number of credits earned in mathematics classes.
- An increase in the level of the highest mathematics class.
- An increase in the mathematics GPA and the overall GPA.
- Higher college entrance examination scores (measured here by ACT test scores) and more students taking college entrance examinations.
- A decreased or stable drop-out rate.

These are encouraging results because they show that the theory behind an *Algebra for All* initiative is sound: many more students than previously thought are ready for college-preparatory mathematics and when given the opportunity to enroll they will rise to meet the challenge.

Negative Results: Decreased Achievement for Females and Vulnerable Males

Unfortunately, other demographic groups in the MMSD did not fare as well, suffering large losses in academic achievement after implementation. Sadly, these were some of the very groups the initiative was designed to empower, including Black male and female Special Education students, Black males who were of low-socio-economic status, Hispanic females, Hispanic males who were English language learners, White females (especially those eligible for Special Education services), and Special

Education students of all races and genders. For these groups, the consequences of the initiative included:

- Fewer credits earned in mathematics classes.
- A reduction in the level of the highest mathematics class.
- Lower mathematics grade point average (GPA) and lower overall GPA.
- Lower college entrance examination scores (measured here by ACT test scores) scores and fewer students taking the college entrance examination.
- An increased drop-out rate.

Any groups of students not named here showed mixed results, with the exception of Asian females for whom there was inconclusive evidence of either a positive or negative overall effect.

DISCUSSION

These results would show that *Algebra for Everyone* had positive effects on many students in the district, opening up the doors to college to many who would not otherwise have considered it. However, it did this while closing the doors to a traditional high school diploma for many others and leading still others to elect minimal mathematics preparation—the opposite of what was intended.

Teacher Expectations and Student Achievement

When examining the list of students for whom the *Algebra for Everyone* initiative met its goals, the salient feature is the gender they all have in common: male. These results could be an example of what Rosenthal and Jacobsen (1968) termed the "Pygmalion effect" in which teacher expectations of student learning become reality. Males are traditionally viewed as being better at mathematics and, given that their SES may not be readily apparent, for those not receiving Special Education or English Language Learner services there would have been no reason for a teacher or their classmates to expect them not to do well.

Correspondingly, the second list contains students from demographic groups society has historically deemed more likely to struggle or fail in mathematics classes: females, students of color with low socio-economic status, English language learners, and students with diagnosed cognitive or emotional difficulties that qualify them for Special Education services. In these students' cases, disliking mathematics or struggling to do well in it might be seen as common and/or not unexpected and therefore would not be cause for alarm.

Mathematical Identity

Ma's (2003) research on the acceleration of regular students also may apply here. Ma found that when regular students are accelerated (defined as students who score at the 65th percentile or lower, taking Algebra I in seventh or eighth grade), their attitude toward mathematics declines more quickly than their peers who were not accelerated and their anxiety increases at a higher rate than their regular peers who were not accelerated. Ma was unable to find any student-level or school-level factors that could reliably predict this attitude decrease or increase in anxiety level. Using previous research on attitudes and how they relate to learning, Ma came to the conclusion that the negative effects are due to regular students being overwhelmed by the demands of the higher-level class.

Students who were enrolled in a grade-level class when they would otherwise have enrolled in a below-grade-level class may have an experience similar to a regular student who was accelerated to an above-grade-level class. Students who had lower prior academic achievement may have been more susceptible to feeling discouraged and overwhelmed, leading to the increased dropout rates and a loss of the lower-achieving students from the group of students taking the ACT.

Another influence on how students experience mathematics classes is how they perceive themselves to perform as compared to their peers. Correll (2001) determined that students' self-assessment of their mathematical ability is done in reference simply to others in their daily classes, not in reference to the entire grade-level or student body. Prior to *Algebra for Everyone*, lower-achieving students would have been placed in a Pre-Algebra or lower class where they could have excelled relative to others in their class. Post *Algebra for Everyone*, these same students were placed in a more difficult Algebra 1 class with students with stronger prior achievement. The lower grades achieved in Algebra 1 vs. Pre-Algebra and their lower performance relative to their classmates may have affected students' views of their mathematical abilities correspondingly.

Individual Agency

A third possible explanation is that the *Algebra for Everyone* initiative inadvertently changed the cost/benefit ratio of pursuing higher mathematics and/or a high school diploma. Correll (2001) found that girls who were strong in both English and mathematics were less likely to elect to enroll in Calculus (the most advanced mathematics course offered at a typical U.S. high school) than girls who were also strong in mathematics but not in English. In a sense, many girls who stayed with mathematics may have done so not because they loved mathematics but because they had no other viable alternatives.

The groups with the greatest negative effects from the *Algebra for Everyone* initiative could perhaps be those for whom another option besides continuing with mathematics was readily available. This may have taken the form of enrolling in more history or English classes or, for those students who also struggle in the other disciplines such as many of the Special Ed students, it could have meant dropping out.

Increasing equity of opportunity without harming vulnerable students

Of course, the theories posited above are simplifications of the complex reality which influences students' choices, but all seem to point to *Algebra for Everyone* not as the cause of the results we see here, but rather as a trigger for amplification of already-existing trends and dynamics.

It would appear that school systems that seek to eliminate ability grouping may unknowingly wield a double-edged sword, and further research is needed to paint a clearer picture of the dynamics involved and the optimal solutions. In principle, a policy designed to increase equity of opportunity, such as an *Algebra for All* initiative, would function only to place underestimated students in classes that were more appropriate, thereby unlocking their heretofore untapped potential. However, this study suggests that this result was achieved for only a fraction of students and that the success of these students was attained only at the cost of their peers' achievement.

This study would suggest that eliminating formal ability grouping is but one factor in increasing student achievement. Another important factor in student achievement is how students incorporate cultural beliefs about mathematics into their identities, and it is one that will be much more challenging for schools to address.

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STUDENT ACADEMIC SELF-CONCEPT AND PERCEPTION OF CLASSROOM ENVIRONMENT IN SINGLE-SEX AND COEDUCATIONAL MIDDLE GRADES MATHEMATICS CLASSES

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In this paper we present findings from a study investigating the relationship between all girls' classes, all boys' classes and coeducational classes on student mathematics self-concept and student perception of classroom environment. Further, we compared responses of girls in all girls' classes to girls in coeducational classes and responses of boys in all boys' classes to boys in coeducational classes. Using the Mathematics Attitude Scale *and the* What is Happening in This Class *questionnaire, we found no significant differences in student responses on any of the subscales or domains for any of the subgroups, except for Math as a Male Domain. Our findings indicate that student mathematics self-concept and student perception of the classroom environment are similar regardless of whether students are in a single-sex or a coeducational class.*

FOCUS OF THE PAPER

In U.S. public schools, academic classes consisting of only girl students or only boy students became permissible in certain circumstances in October, 2006. Thus, in contrast to many other countries, single-sex classes in U.S. public schools are recent phenomena. Many schools and districts in the U.S. are implementing single-sex classrooms within coeducational schools, rather than separating boys and girls into different schools. This provides scholars with an opportunity to investigate the efficacy of single-sex classrooms in public schools. The authors are engaged in studies that seek to contribute to our understandings of to what extent, in what ways, by what means, and for which students, single-sex mathematics and science middle grades classrooms influence learning environments, classroom discourse, student academic self-concept, and student performance. In this paper, we present findings on student perception of classroom environment and student academic self-concept in single-sex and coeducational mathematics and science classrooms at the middle level. In particular, we focus on the following questions: To what extent and in what ways are student academic self-concept and student perception of classroom environment related to class type (all girls, all boys, or coeducational)? How do girls in all girls' mathematics classes compare to girls in coeducational (coed) classes in their academic self-concept and perception of classroom environment? How do boys in all boys' mathematics classes compare to boys in coed classes in their academic self-concept and perception of classroom environment?

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THEORETICAL FRAME AND RELATED LITERATURE

Marsh and Yeung (1997) discuss the importance of distinguishing between academic and non-academic components of self-concept. They also emphasize that, even within a notion of academic self-concept, domain-specific distinctions of academic self-concept make sense because, for instance, one's mathematics self-concept may not necessarily be correlated with one's English self-concept (Marsh & Yeung, 1997). Bong and Skaalvik (2003) concur with the utility of domain-specific self-concept constructs, as they discuss how "academic self-concept reflects an aggregated judgment or overall impression of one's competence in given academic domains" (p. 29). For this study, we consider mathematics self-concept to represent one's perspective of one's competence within the domain of mathematics. Our focus on middle school students is driven by our understanding of the middle grades—spanning approximately ages 10 to 15—as a critical juncture in the development of students' knowledge and attitudes towards mathematics. Ma & Kishor (1997) identify the middle grades as a crucial period in which students shape their attitudes toward mathematics.

While it can be illuminating to understand more about student academic self-concept in a variety of classroom settings, it is also meaningful to inquire about student perceptions of the learning environment, particularly when those learning environments are novel to the typical schooling contexts. The importance of student perception of classroom environment has become so clear that an entire field devoted to the study of learning environments is now well established. Dorman, Adams, and Ferguson (2003) report that several studies spanning three decades have linked the quality of the classroom environment to learning outcomes in mathematics. In addition, drawing on Fraser's (1998) study, they note the possibility that classroom environment could vary by school type (coeducational, boys' and girls' schools). In this study, we investigate whether and to what extent student perception of the mathematics classroom environment is related to classroom type (coeducational, all boys', and all girls') within coeducational public middle schools.

METHODS

Context of the Study

A total of 215 students enrolled in one of the three class types (all boys, all girls and coeducational classrooms) in two rural middle schools (grades 6-8) from one school district in the southeastern region of the United States participated in the study. Specifically, 85 participants were enrolled in all-boys classes, 66 in all-girls classes, and 64 in coeducational classes (40 boys and 24 girls). Thus, there were a total of 125 boys and 90 girls participating in the survey. The students completed an electronic survey and responded to subscales from two survey instruments – the *Fenemma-Sherman Mathematics Attitudes Scales* and the *What Is Happening In this Classroom* (WIHIC) questionnaire. The former scale addresses the research questions related to student mathematics self-concept and the latter scale addresses the research questions related to student perceptions of the classroom learning environment. Both instruments are discussed in more detail below.

Instruments and Analysis

The *Fenemma-Sherman Mathematics Attitudes Scales* (MAS) (Fenemma-Sherman, 1976) have long been used to investigate students' attitudes and beliefs towards mathematics across all levels of schooling. For the purposes of this study, we focus on four of the nine domains of the MAS; the Math as a Male Domain Scale, the Confidence in Learning Mathematics Scale, the Mathematics Usefulness Scale, and the Teacher Scale. The MAS is organized as a 5-point Likert scale from strongly disagree to strongly agree. Prior to analysis, we reverse coded negatively-worded items from the subscales. For the subscale Mathematics as a Male Domain, we coded items so that a high rating reflected rejection of the notion that mathematics is a male domain. Thus, a score higher than neutral (higher than 3 on the 5-point scale) represents disagreement with the idea that mathematics is a male domain, whereas scores lower than neutral represent agreement with the idea that mathematics is a male domain. Fennema and Sherman (1976) obtained split-half reliabilities ranging from 0.87 to 0.93 for these scales.

The *What is Happening in this Classroom* (WIHIC) questionnaire was developed by Fraser, Fisher, and McRobbie (1996) as an instrument to assess student perceptions of their classroom learning environments. By incorporating scales that have been shown to be important predictors of learning outcomes, this instrument reflects recent cognitive views of learning in mathematics and science (Kim, Fisher, & Fraser, 2000). The WIHIC contains seven scales or subsets, each consisting of ten items on a Likert scale: (1) Student Cohesiveness, (2) Teacher Support, (3) Involvement, (4) Investigation, (5) Task Orientation, (6) Cooperation, and (7) Equity. Fraser (1998) notes that it is important to separate variations of a survey that asks about students' perceptions of the classroom environment as a whole from variations of that survey that ask about that particular student's experiences in the classroom; he advocates for extricating these perspectives into separate class and personal forms. In this study, we use the personal form because we are interested in sub-group analysis (Fraser, 1998). Fraser (1998) reports alpha reliabilities of more than .80 for each subscale for the WIHIC instrument.

A non-experimental one-way analysis of variance (ANOVA) of student responses was conducted for each research question. For the ANOVA, the dependent variables were the student responses to items on each scale. The independent variables were class type, more specifically an all-girl, an all-boy, and a coeducational class setting, and students' sex. For ANOVA in which a significant difference ($\alpha = .05$) among the means was concluded, Tukey's Pairwise Comparison post hoc test was utilized. When significant differences were found for subscales, Bonferroni adjustments were made for subsequent ANOVA analyses of individual items in that subscale. All statistical calculations were performed using the software program JMP Pro 10.

RESULTS

The research questions for this study are: To what extent and in what ways are student academic self-concept and student perception of classroom environment related to class type (all girls, all boys, or coeducational)? How do girls in all girls' mathematics classes compare to girls in coeducational (coed) classes in their academic self-concept and perception of classroom environment? How do boys in all boys' mathematics classes compare to boys in coed classes in their academic self-concept and perception of classroom environment? In presenting our findings, we address the research questions relating to mathematics self-concept first, followed by our findings addressing student perception of classroom environment.

Findings from the Fenemma-Sherman Mathematics Attitudes Scales

We began our analysis of mathematics self-concept by investigating student responses across the three class types (all girls, all boys, and coed). We found no significant differences in the responses of students in all-boys, all-girls and coed classrooms for three of the four MAS scales: Mathematics Usefulness, Confidence in Learning Mathematics, and Teacher scales. The Mathematics as a Male Domain scale, however, indicated significant differences, with all-girls' and coed classes scoring differently from all-boys' classes. Further analysis indicated that responses from students in all girls' classes differed significantly from responses from students in all boys' classes on four items of the scale. Table 1 shows the results of the analysis of mathematics self-concept by classroom type.

	All Girls		All Boys		Coed		p
Subscales	Mean	SD	Mean	SD	Mean	SD	
Confidence in Learning Math	3.60	0.92	3.63	0.87	3.52	0.86	.7580
Mathematics Usefulness	3.93	0.71	3.87	0.69	3.74	0.75	.3223
Teacher Perceptions	3.48	0.68	3.46	0.72	3.51	0.73	.9096
Math as a Male Domain	4.08^{a}	0.52	3.63^{b}	0.61	3.85^{a}	0.54	$<.0001^{*}$
When a woman has to solve a math problem, she should ask a man for help.	4.13^{a}	1.19	3.41^{b}	1.18	3.82	1.02	.0012
Women who enjoy studying math are a little strange.	4.22^{a}	1.14	3.58^{b}	1.13	3.94	1.05	.0023
Women certainly are smart enough to do well in math.	4.56^{a}	0.76	4.07^{b}	0.87	4.33	0.91	.0036
I would have more faith in the answer for a math problem solved by a man than a woman.	4.13^{a}	1.19	3.41^{b}	1.18	3.82	1.02	.0008
OVERALL	3.76	0.51	3.65	0.56	3.65	0.54	0.3789

Table 1: MAS by class type

Note. * indicates significant difference based on F-test with $p < .05$. Item means with a different letter superscript indicate significant difference based on F-test with $p<.0046$.

Our second layer of analysis of mathematics self-concept was to investigate whether girls in all girls' classes responded differently from girls in coed classes, and how responses from boys in all boys' classes compared with those from coed classes. There were no statistically significant differences on any of the four subscales for girls in single-sex classes and girls in coed classes, and the same situation holds for boys in single-sex classes and boys in coed classes (see Tables 2 and 3). There were two individual items on which girls in single-sex classes and girls in coed classes differed significantly; those items are included in Table 2. Likewise, there was one item on which boys in single-sex classes differed significantly from boys in coed classes; this item is included in Table 3.

	Female Coed		Female Single-Sex		
Subscales	Mean	SD	Mean	SD	p
Confidence in Learning Math	3.54	0.77	3.62	0.92	0.7186
Mathematics Usefulness	3.71	0.79	3.93	0.72	0.2270
I will use mathematics in many ways as an adult.	3.50	1.02	4.05	0.93	0.0192
Teacher Perceptions	3.63	0.58	3.50	0.67	0.3924
Math as a Male Domain	4.01	0.45	4.12	0.44	0.2960
Studying math is just as good for women as for men.	4.17	0.82	4.53	0.67	0.0346
OVERALL	3.76	0.49	3.63	0.57	0.0708

Table 2: MAS Female Coed by Female Single-Sex comparison

	Male Coed		Male Single-Sex		
Subscales	Mean	SD	Mean	SD	p
Confidence in Learning Math	3.52	0.92	3.62	0.88	0.5429
Mathematics Usefulness	3.76	0.74	3.87	0.69	0.4181
Math is not important for my life.	3.40	1.43	3.89	1.06	0.0340
Teacher Perceptions	3.43	0.80	3.44	0.73	0.9551
Math as a Male Domain	3.76	0.57	3.61	0.64	0.2077
OVERALL	3.61	0.58	3.64	0.57	0.8367

Table 3: MAS Male Coed by Male Single-Sex comparison

Findings from the What Is Happening In this Classroom Questionnaire

To address our research question regarding student perception of classroom environment in single-sex and coed classes, we first compared responses to the WIHIC survey across the three class types (all boys, all girls, and coed). We found no significant differences across the three class types for any of the subscales or individual items on the survey. The results for this analysis at the subscale level are presented in Table 4.

	All Girls		All Boys		Coeducational		
Subscales	Mean	SD	Mean	SD	Mean	SD	*p*
Social Cohesiveness	3.05	0.09	3.02	0.08	3.07	0.09	0.9264
Teacher Support	2.48	0.10	2.55	0.10	2.43	0.10	0.7099
Involvement	2.55	0.10	2.58	0.10	2.51	0.10	0.8725
Investigation	2.34	0.11	2.45	0.10	2.32	0.11	0.6322
Task Orientation	3.14	0.10	2.94	0.09	3.06	0.10	0.3610
Cooperation	2.96	0.10	2.76	0.09	2.90	0.10	0.3160
Equity	2.66	0.11	2.79	0.10	2.85	0.11	0.4812

Table 4: WIHIC by Class Type

Table 5 shows the results of our analysis of girls' responses in coed classes and girls' responses in all girls' classes. None of the subscales indicated significant differences in the responses, although two individual items showed significance. Those items are included in Table 5.

	Female Coed		Female Single-Sex		
Subscales	Mean	SD	Mean	SD	*p*
Social Cohesiveness	3.01	0.15	3.07	0.09	0.7382
Teacher Support	2.46	0.17	2.50	0.1	0.8350
Involvement	2.36	0.17	2.58	0.1	0.2860
Investigation	2.05	0.17	2.37	0.11	0.1174
I solve problems by using information obtained from my own investigations.	1.83	0.21	2.42	0.13	0.0218
Task Orientation	3.16	0.16	3.18	0.1	0.9109
Cooperation	2.91	0.17	2.99	0.11	0.697
Equity	3.00	0.19	2.68	0.11	0.1398
I get the same amount of help from the teacher as do other students.	3.04	0.21	2.56	0.13	0.0503
OVERALL	2.71	0.13	2.76	0.08	0.7525

Table 5: WIHIC Female Coed by Female Single-Sex comparison

Similarly, results of our analysis of boys' responses in coed classes compared to boys' responses in all boys' classes are presented in Table 6. None of the subscales indicated significant differences, although one item showed significant differences. That item is included in Table 6.

Subscales	Male Coed Mean	Male Coed SD	Male Single-Sex Mean	Male Single-Sex SD	*p*
Social Cohesiveness	3.11	0.11	3.01	0.08	0.4595
Teacher Support	2.41	0.13	2.53	0.10	0.4548
Involvement	2.60	0.12	2.55	0.09	0.7591
Investigation	2.49	0.13	2.42	0.10	0.6508
Task Orientation	3.00	0.13	2.92	0.10	0.629
Cooperation	2.90	0.12	2.73	0.09	0.2624
When I work in groups in this class, there is teamwork.	3.05	0.15	2.67	0.11	0.0463
Equity	2.75	0.14	2.78	0.10	0.8810
OVERALL	2.75	0.11	2.70	0.08	0.6902

Table 6: WIHIC Male Coed by Male Single-Sex comparison

DISCUSSION

Our findings at the subscale level of the WIHIC survey suggest that class type, whether coeducational, all boys, or all girls, did not influence student perception of the classroom environment. Student self-concept, assessed through the Mathematics Attitude Scales, was not significantly different for the subscales Confidence in Learning Math, Mathematics Usefulness, or Teacher Perceptions. The only subscale that showed statistically significant differences between single-sex classes was Math as a Male Domain. Our findings indicate that, while both boys and girls rejected the notion that mathematics is a male domain, girls tended to do so more strongly.

Our comparisons of girls in coed to girls in single-sex classes and boys in coed to boys in single-sex classes indicate that, on the subscale or domain level, single-sex education does not significantly influence student mathematics self-concept or student perception of the classroom environment. That is to say, we have not found that girls in all girls' classes (or boys in all boys' classes) have significantly different views of their classrooms or themselves as mathematics learners than girls and boys in coeducational classes. However, we realize that the presence or absence of a relationship between class type, academic self-concept, and student perception of classroom environment is not the sole rationale for instituting single-sex education. For this reason, our research team continues to investigate classroom discourse, student performance, and student engagement in single-sex and coeducational classrooms in addition to self-concept and perception of classroom environment.

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SHAPING MATHEMATICS CLASSROOM DISCOURSE: RELATING PROFESSIONAL DEVELOPMENT AND CLASSROOM PRACTICE

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Professional development that furthers teachers' understanding of mathematics classroom discourse offers possibilities to improve students' learning of mathematics. It is not clear, however, how teachers relate such professional development experiences to their own classroom practice. In this paper, I discuss the features of mathematics classroom discourse that were most salient for teachers in relation to their classroom practice as they engaged in professional development focused on secondary mathematics classroom discourse.

BACKGROUND

Providing students with opportunities to engage in mathematical argumentation and conceptual explanations improves students' learning (Chapin, O'Connor, & Anderson, 2009). Despite documented benefits of students engaging in such rich discourse, most mathematics classroom discourse follows a pattern in which students take only brief turns in discussion followed by evaluation or feedback from the teacher (Cazden, 2001). Consequently, there is a need for professional development (PD) that supports teachers to become purposeful about engaging students in mathematical explanations, argumentation, and justification. Identifying what teachers learn from any PD, however, is a complex task. The purpose of this paper is to share findings from an investigation into what teachers learned from a particular case of the Mathematics Discourse in Secondary Classrooms (MDISC) PD program (Herbel-Eisenmann, Steele, & Cirillo, 2013). Specifically, I discuss one aspect of the findings which addresses the following question: *What features of mathematics classroom discourse are most salient for teachers related to their classroom practice as they engage in PD focused on secondary mathematics classroom discourse?*

ANALYTICAL FRAMEWORK

Two bodies of literature informed this study, literature that examines: (a) particular features and practices associated with enhancing mathematics classroom discourse for students, and (b) influences on teachers' learning from PD. From this literature, I generated an analytic framework for instructional practices and concepts that teachers might learn from engaging in PD focused on mathematics classroom discourse. This framework is comprised of the following four categories of practices, which have been shown to influence students learning of mathematics, including: (a) shaping classroom

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discourse, (b) shaping classroom social norms, (c) making student thinking visible, and (d) promoting mathematics during classroom discussion. Here I briefly describe the features of these categories and later I outline how this framework forms the basis of analysis for this study and the ways in which the MDISC PD experience addresses these categories.

Shaping Classroom Discourse

Teachers' instructional moves can shape classroom discourse patterns in order to support the mathematical thinking and learning of their students (Chapin et al., 2009; Stein, Engle, Smith & Hughes, 2008; Wood, 1999). Teachers' may purposefully shift classroom discourse for many reasons, including efforts to assess students' understanding, or to help students to more meaningfully engage with each other's reasoning (Cobb et al., 2001; Nathan & Knuth, 2003; Stein et al., 2008; Staples & Truxaw, 2010). Teachers' recognition of the ways in which they shape discourse in their classrooms is an important step towards enacting these types of instructional practices.

Shaping Classroom Social Norms

Students' participation within the classroom is heavily influenced by the social expectations and contexts of that classroom (i.e., Yackel & Cobb, 1996). Based on their prior experiences, students in secondary mathematics classrooms may not be inclined to openly share their in-progress ideas and solution strategies. The moves teachers make to support students to share their solution strategies can establish new social norms in the classroom regarding expectations that students should explain their reasoning (Forman, Larreamendy-Joerns, Stein, & Brown, 1998; Herbel-Eisenmann & Cirillo, 2009; Stein et al., 2008). Similarly, teachers' efforts in close listening, engaging with students' thinking, and pressing students to engage with each other's reasoning indicate to students that relevant mathematics discourse is valued in their classroom. As teachers become more aware of their control over the social norms present in their classrooms, they are able to purposefully shape those norms.

Making Student Thinking Visible

Classroom discourse can provide a mechanism by which individual student's thinking and reasoning can be made visible to both the teacher and to other students. Therefore, classroom discourse can provide a source of data for formative assessment that teachers can use to monitor students' understanding of mathematical concepts. Teachers who are learning about mathematics classroom discourse will likely engage in practices that help make student thinking visible. These include, (a) making students' reasoning a part of classroom discourse (Stein et al., 2008), (b) sharing ideas students generated independently as a part of whole class discussion, and (c) pressing students to clarify and justify their reasoning (Cobb et al., 2001; Staples & Truxaw, 2010).

Promoting Mathematics During Discussion

Teachers can support student learning by foregrounding the mathematics in classroom discourse, such that mathematical ideas at the heart of teachers' lessons remain prominent throughout the instruction (Stein et al., 2008). This can be accomplished through practices such as (a) revoicing (Forman et al., 1998) or highlighting a particular aspect of a student's contribution in order to connect to more advanced mathematical ideas (Herbel-Eisenmann, Steele, & Cirillo, 2013; Nathan & Knuth, 2003) and (b) focusing on the mathematical content of the discourse through purposefully developed symbolic records of students' contributions (Cobb, et al., 2001).

Relating the MDISC Professional Development Goals to the Literature

The MDISC PD curriculum is a set of practice-based, case-based materials. The materials are organized around five constellations of activities anchored by a mathematical task and a narrative or video case of a teacher engaging students in work on the task. The materials introduce six Teacher Discourse Moves (TDMs) as tools for teachers in developing their discourse practices (see Herbel-Eisenmann et al., 2013 for more detail). I examined the content of the MDISC PD in light of the aforementioned analytic framework. Each activity within the MDISC materials provides multiple opportunities for teachers to engage with a number of these ideas. For example, *Activity 1.5: Examining Whole-Class Discussion as a Context for Communicating Mathematics* provides teachers the opportunity to examine transcript excerpts of a whole-class mathematics discussion to explore (a) the ways in which students participate in a whole group context, (b) the ways in which students' opportunities to engage in mathematical practices are influenced by their participation in the classroom discourse, and (c) the ways in which classroom discourse can position mathematics. Although I use *Activity 1.5* as an example, all activities in the materials follow a similar pattern of providing teachers with multiple opportunities to engage with practices across the analytic framework.

METHOD

The setting for this study was a yearlong pilot of the MDISC PD materials with four mathematics teachers at a suburban middle school in the Midwest. The group was comprised of two seventh grade teachers, referred to here as Stephanie and John, and two eighth grade teachers, Nick and Brenda. The teaching experience within the group ranged from Stephanie having no prior full-time teaching experience to Brenda and John having taught mathematics for over 20 years. None of the participants had previously engaged in PD focused on mathematics classroom discourse. They became aware of the project through recommendations from their colleagues in the mathematics department at the high school in the same school district. All four participants also expressed a strong learning disposition and desire to improve their practice. Both the facilitator of this pilot and the author worked as developers for the MDISC materials.

Data Collection and Analysis

This paper is informed by data collected from the PD study group sessions, observations of teachers' classroom, and individual interviews. The study group met approximately once each month for six hours each session, with the exception of the second and sixth sessions, which occurred after school and for only two hours each. The distribution of the sessions and data collection is represented in Figure 1.

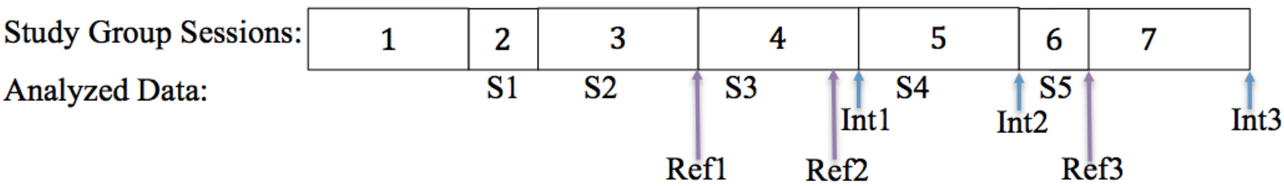

Figure 1: Schedule of study group sessions and data collection

I attended, videorecorded, and took detailed field notes of all study group sessions. Using my field notes from the entire set of study group sessions, I identified any segments of conversation during which the primary focus was on the teachers' own classroom practice (marked S1-S5 in Figure 1). I also observed three lessons selected by participants, during which I video recorded and took field notes. Additionally, I communicated with the teachers prior to each observation to gather data about their goals for the lesson, and immediately following each observation I asked teachers to reflect on their teaching episode. Subsequently, about one week later, I engaged the teachers in a semi-structured follow-up interview. The data used for the analysis presented in this paper comes from the semi-structured interviews, not the classroom observations (marked Int1-Int3 in Figure 1). Additionally, I collected three written reflections from the participants (marked Ref1-Ref2 in Figure 1). The data from these reflections were used for triangulation purposes, rather than as a primary source. The nature of the interview protocol and related methods will be discussed in more detail in the presentation of this paper.

To analyse the data, I first transcribed all study group session segments, written reflections, and teacher interviews and then imported the transcriptions into the qualitative analysis software NVIVO. Then, I used a modified grounded theory approach (Strauss & Corbin, 1998) to identify the ideas related to mathematics classroom discourse most salient to teachers in their discussions of their own classroom practice. Using open-coding, I categorized teachers' statements related to their own classroom practice and to classroom discourse. I then re-examined these data, specifically looking for statements that included references to the four categories of the analytic framework described above. Through this process, I developed the coding scheme in Table 1, with code definitions and subcategories refined through a constant comparative method.

Category	Code Definition
Shaping Classroom Discourse	**General Moves Related to Discourse.** Teacher discusses moves they made in order to shape their classroom discourse.
	Specific TDM Terms. Teacher explicitly referenced one of the six TDMs terms from the materials: Asking, Creating, Inviting, Probing, Revoicing, and Waiting.
	Use of TDMs Without Term. Teacher discussed moves that fit the descriptions of the six TDMs described by the PD materials, without explicitly referencing the terminology specified in the materials.
Shaping Social Norms	**Teacher Shapes Social Norms.** Teacher discussed the ways in which they influence, both purposefully and implicitly, the social norms of their classroom.
	Attention to Social Norms. Teacher described implicit or explicit social norms present in their classroom without acknowledging his/her role in shaping those social norms. This code applies to the teachers' statements describing existing norms or those they wish to change.
Making Student Thinking Visible	**Students' Non-verbal Evidence.** Teacher discussed evidence of students' thinking that were non-verbal. This code applies to statements about students' written work or gestures
	Inference About Student Thinking. Teacher discussed students' thinking without specifically attending to verbal or non-verbal evidence.
	Assessing Via Specific Student Discourse. Teacher referenced assessing students' understanding of mathematical concepts via students' specific statements, written or non-verbal.
Promoting Mathematics	**Promoting Mathematics Content During Discussion.** Teacher explicitly described bringing out mathematical ideas during classroom discussions (i.e. functions, equations).

Table 1: Coding Categories

I synthesized the similarities I observed across the teachers and across the data sources to identify the themes most salient in what the teachers talked about in relation to their classroom practice. Although the teachers discussed a wide range of ideas, I selected representative examples focused on the features of classroom discourse that appeared most consistently across the group and across the data set.

RESULTS

Overall, my findings suggest that as the teachers engaged in the MDISC PD, the themes that were salient regarding their classroom practice represent elements of all four categories of the analytical framework: (a) shaping classroom discourse, (b) shaping social norms, (c) making student thinking visible, and (d) promoting mathematics during discussions. The data revealed that the features of mathematics classroom discourse that teachers discussed most consistently related their role in shaping the discourse in their classrooms. Consequently, this paper specifically focuses on the salient themes from the category *shaping classroom discourse*.

Shaping Classroom Discourse: Seeing the Need to Move Towards More Open Discourse Patterns

As an early step towards purposefully shaping discourse within their classrooms, the teachers acknowledged the ways in which they controlled the discourse within the classrooms. The teachers expressed a desire to allow for more natural interactions between students, in which students communicated productively with each other about mathematics. Building upon the teachers' understanding of their role in shaping discourse, they discussed the impact their interactions had on their students' discourse. Reflecting on the video of the first lesson observation John noted,

> The thing that struck me in the first half…was the amount of very traditional interactions. You know, prompt, response, feedback, prompt, response, feedback, – consistently. [The students gave] very factual answers. [It was] very [teacher] centered…I don't know if I'm dumbing things down, without realizing it even, by trying to put it in little tiny steps for them, because that's the way that I see things… So, by making it so explicit, does that help them? (Int1)

In this instance, John reflected on whether or not his interactions with students allowed them flexibility to share and develop the mathematical concepts. This is characteristic of a theme that I observed throughout the data; the teachers worked towards a goal of more open discourse patterns. As a part of this effort, teachers described their use of questioning practices. Reflecting on the video recording of the second lesson I observed, Nick described his concerted efforts to ask more open-ended questions as follows:

> When I was asking kids to explain something, I wasn't asking them yes or no questions. It was more open-ended. You know, "What did you get for your solution? And talk us through the steps." And I saw more of that, which I was happy about. But I still saw that it was a lot of the teacher-guided questions. (Int2)

In this statement, Nick both identified his own growth in terms of his efforts to ask more open-ended questions and acknowledged that he had further to go before he met his goals. As a group, the teachers' discussion of their classroom practice in both interviews and study group sessions demonstrated a combination of (a) an increased awareness of the ways in which their teaching moves affected the discourse patterns in their classroom and (b) a desire to support more natural student-to-student interactions with less central control attributed to the teacher.

During the third professional development session, teachers were introduced to the IRE pattern of discourse (Mehan, 1979). During subsequent study group discussions and interviews, all of the teachers noted their tendencies to follow the IRE pattern as part of their reflection on video recordings of their instruction. John's quote above illustrates this type of reflection. Later, during the fifth study group session, John described his efforts to limit his evaluation of students' responses, he said, "We were doing this thing yesterday and I hadn't been saying nice job, good work, or whatever. And a kid gave an answer and I said, "Great answer!" [hesitates] "I meant, *another* great answer!" (S4). John recognized that his "nice job" comments affected how his students responded; they waited for him to validate their answers, as is typical in IRE patterns. Acknowledging their tendency to fall into the IRE pattern marked a point of comparison for the teachers between the discourse they wanted to have in their classrooms and the sorts of interactions they were presently experiencing.

Although the teachers struggled to change the discourse patterns in their classrooms, as they developed an understanding of the sorts of interactions they wanted to support they began to catch themselves engaging in unproductive discourse patterns, and thus began to make changes towards their goals. Specifically, the teachers began making a variety of efforts to move the classes towards more open-ended discourse patterns, including modifying mathematical tasks and using the specific Teacher Discourse Moves suggested by the PD.

DISCUSSION

These findings highlight the features of mathematics classroom discourse that were most important to the teachers in relation to the classroom practice as they engaged in the MDISC PD. Additionally, these findings show the ways in which teachers described how they learned from their engagement with the ideas of the professional development in the context of their own classrooms. Reflecting on the PD experience John said,

> [The MDISC professional development experience] is an opportunity to improve what we're trying to do and to look at yourself in a little different light…You see things and you go, 'Oh no!' but we have to confront the image we have of ourselves and what's actually going on in our classrooms and what the reality is. (Int3)

Spurred by his recognition of the contrast between what he encountered in the study group sessions and his classroom experiences, John described his desire for change. John's quote highlights a group commitment to continue learning as they worked to connect the ideas discussed in the PD to their use of those ideas in the reality of their teaching. Throughout the PD, teachers had opportunities to engage with multifaceted theoretical ideas related to mathematics classroom discourse. These findings reinforce the notion that teachers can and will make sense of information from PD in complex and meaningful ways that are connected to their classroom experiences (Herbel-Eisenman, Drake, & Cirillo, 2009). If professional developers are thoughtful about enacting the recommendations from the field for high-quality PD, rather than devoting energy to developing assessments of what teachers learn from PD, these findings suggest research should focus on the ways teachers conceptualize and engage the professional development content through their discussion of their classroom practice as alternative means to assess the impact of PD. By prioritizing teachers' perspectives and valuing what they find most salient, this study offers possibilities for how we can begin to bridge the gap between teachers' learning from professional development and sustained change in classroom practice.

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COGNITION IN THE WORKPLACE: ANALYSES OF HEURISTICS IN ACTION

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Studies on cognition have capitalized on the role of contexts and experience in shaping our cognitive competence. In the past few decades, the mathematical education research field has begun to pay increased attention to the mathematics practices of both adults and children that take place in non-academic settings. As a result, theoretical fields such as ethnomathematics, situated cognition, and workplace mathematics have gained prominence. In this paper, we use Vergnaud's theory of conceptual fields to highlight the workplace mathematical activities of two groups of practitioners-Street vendors in Lebanon, and Bus conductors in India.

INTRODUCTION

In the past two decades, researchers have increasingly emphasized the elicitative role of cultures in impacting mathematical thinking and problem solving. For many researchers, sociocultural settings not only determine how mathematical knowledge is acquired, but also how it is represented, organized and retained (Sanin & Szczerbicki, 2009). Although there are several contexts in which mathematical ideas develop and are discussed, mathematics education has mostly been associated with the institutional context (Mukhopadhyay, Powell, & Frankenstien, 2009). The problem is that usually in the school setting, mathematical knowledge is presented as a "prized body of knowledge" (Millroy, 1992, p. 50), stripped of its rich cultural and historical connotations, and far removed from the " lives and ways of living of the social majorities in the world" (Fasheh, 2000, p. 5). We, alongside prominent mathematics education researchers, take an exception to this view and argue for countering the narrow vision of mathematics that confines it to the school walls.

Investigations that have focused on studying people's use of mathematics outside the classroom is divided into two main groups; namely, those interested in "everyday cognition" where Lave (1988) is a prominent figure; and those interested in "ethnomathematics," where D'Ambrosio (1992) is prominent. Both groups of researchers call for a new conceptualization of mathematics that is rooted in nonacademic practices. The mathematical ideas that are generated and used outside of learning institutions allows people with little or no schooling experience to practice crafts and trades, conduct business transactions and make their livings in a variety of ways.

We, the authors, have been immersed in the field of mathematics education for over two decades in a wide range of settings and in both western and nonwestern countries: from K-12 schools to research universities and graduate schools of education. During this tenure, we have encountered many implicit and explicit questions about mathematical competence and its role in defining and shaping the identity of individuals in the classroom, the workplace, and the society as whole (Naresh & Chahine, 2013). We list some of these questions here: *What are the mental processes underlying an act of labor or service? What is the nature of the problem solving behavior of workers while immersed in everyday work practice? How is experienced knowledge represented and employed as part of the daily decision-making manners undertaken by workers?* Researching in the context of the workplace provided us with just the right context for addressing such questions. It also afforded us the opportunity to make connections between what are seemingly two disparate worlds — the world of mathematics learning and the world of mathematics in the workplace. To further pursue this line of inquiry, we devised a study (Chahine & Naresh, in press) to carry out a meta-analysis of the problem-solving behavior of two groups of workers - street vendors in Beirut, Lebanon and bus conductors in Chennai, India. The research reported in this paper is part of this larger project; in particular, we provide a narrative of workers' problem-solving behaviors using Vergnaud's theory (1988, 2000) of conceptual field.

THEORETICAL BACKGROUND

We situate our work in the broader theoretical fields of ethnomathematics and situated cognition. The foundation of ethnomathematics rests in its "openness to acknowledging as mathematical knowledge and mathematical practices elements of people's lives outside the academy" (Mukhopadhyay, Powell, & Frankenstien, 2009, p. 75). There is strong evidence in the literature on situated cognition that supports the hypothesis that by actively engaging in everyday activities, individuals gradually incorporate culturally constructed artefacts into their repertoire of thinking and further develop context- specific problem solving competencies (Wenger, 2000). Such evidence predictably challenges the conventional definition of what counts as mathematics by reinforcing the claim that mathematical activity can be seen as interwoven with everyday practice outside the academic formal settings.

Analytical Framework: Vergnaud's theory of conceptual field

Vergnaud's (1988) theory of conceptual fields is based on the idea that concepts always involve three facets: invariants, representations, and situations. Invariants refer to the mathematical properties or relations associated with the concept. Vergnaud contends that invariants are expressed through representations and that they are not the only factor affecting performance, but rather the way in which concepts are formed might be important. Also, concepts are always tied to situations which make them meaningful. More importantly, Vergnaud (2000) argues that the existence of these mathematical concepts does not necessarily mean that people are fully aware that they are behaving accordingly, but most often these concepts are only "implicit" in theorems or what he calls "Theorems-in-Action". Vergnaud (1988) has defined theorems-in-action as those "... mathematical relationships that are taken into account by students when they choose an operation or a sequence of operations to solve a problem" (p.144). Vergnaud's theory of conceptual fields brings us to the idea that to understand how mathematical concepts are acquired it is necessary to analyze the situations through which these concepts were made meaningful and useful in the context in which they are invoked. Vergnaud's model provides not only guidelines for coding vendors' and bus conductors' problem solving behaviors, but also an understanding of the underlying properties and relations implicit in these behaviors. Pursuing this model, we conducted comparisons along three major dimensions (representation systems, heuristics-in-action, and situations) to decode and examine the problem solving behaviors of street vendors and bus conductors. In this paper, we will provide an overview of data analyzed along the three dimensions; however, we will present and discuss data specific to one dimension – heuristics-in-action.

METHODOLOGY

The goal of the larger research study (Chahine & Naresh, in press) was to examine cognition at work in order to define and describe practical mathematical knowledge that emerged in the context of work activities. To this end, we chose a methodological approach that was based on an iterative process of data collection, analysis, and hypothesis. Our methods comprised qualitative secondary data analysis (of the ethnographic case studies – case refers to the groups of bus conductors' and street vendors), narrative inquiry of solution schemes, and focused discussions (researchers as participants).

Ethnographic case studies (ECSs) and related data

The overall goals of the ECSs were to unravel, analyze, and describe the mathematical ideas and decisions employed by the participants to solve work-related mathematical tasks. In the street vending context, participants included 10 male vendors randomly selected from two market settings in the southern suburbs of Beirut. Vendors in the sample varied in years of schooling (three to seven years), in age (10 to 16 years), and vending experience (one to eight years). Four of the vendors worked alone while the other six helped their fathers or neighbors. Only three were totally responsible for purchasing the produce at wholesale market and pricing it for selling. In the bus conducting context, five bus conductors selected from two bus depots were included. Four male bus conductors and one female bus conductor participated; the bus conductors varied in their educational qualifications (two had high school diplomas and 3 had Bachelor degrees) and years of experience (9 to 31 years). Data collected from the ECSs on street vending and bus conducting included field observations and notes, transcriptions of interviews, researchers' introspection notes, problem solving narratives, and work sample artifacts.

Data Analysis

Our first approach towards data analysis was to engage in a secondary data analysis (Moore, 2006) using data collected from the ECSs. Examining pre-existing data from the two studies enabled data linkage and afforded powerful insights into the problem-solving behavior of practitioners. Furthermore, revisiting data related to field observations, interviews, and researchers' notes in two workplace contexts allowed transparency within research as we continuously interrogated the quality of qualitative data in terms of coding and completeness. Next we engaged in a narrative inquiry (Coulter & Smith, 2009) to describe the problem-solving heuristics of the street vendors' and bus conductors' workplace activities. Developing a coding system for the problem solving behavior and narratives of vendors and bus conductors involved careful readings of transcriptions taken from practitioners' written solutions as well as interviews, with particular attention to researchers' comments. The first two stages of data analysis required us to engage in focused discussions centered on the secondary data collected through the ECSs. These discussions targeted specific mathematical frames that were captured as the practitioners are immersed in the context of street vending and bus conducting. Such discussions produced nuanced insights that would be less accessible without our intensive face-to-face purposeful interactions. As we listened to each other verbalizing and recollecting our field experiences, memories, ideas, and experiences were stimulated and validated as we discovered a common language to describe our recollections and reveal shared understandings or common views.

A qualitative analysis of the problem solving behavior of vendors and bus conductors was established by comparing, contrasting, and synthesizing these properties across work settings namely, vending and bus conducting , and across cultures i.e. Lebanon and India. We conducted three comparisons to decode and examine the problem solving behaviors of street vendors and bus conductors. Comparisons are carried out along three major dimensions: (a) representation systems; (b) heuristics-in-action; and (c) situations. In this section, we present data analyzed along the second dimension -- heuristics-in-action.

RESULTS

Street Vending and Bus Conducting heuristics-in-action

Vergnaud (1988) maintains that all "mathematical behaviors" are tied to certain mathematical concepts and that the existence of these concepts does not necessarily mean that subjects are fully aware that they are behaving accordingly. We call these mathematical behaviors as heuristics-in-action and define them as the ways in which practitioners utilized the mathematical properties or relationships to resolve a problem or complete a task that emerged in their work settings. Two major heuristics-in-action were employed by the participants to reach a satisfactory solution, namely *building-up* and *multiplicative* which in turn led to scalar and functional solutions. These heuristics are virtually based on the properties of linear functions, specifically isomorphic and functional properties (Vergnaud, 2000). To illustrate, consider the following transactions that were extracted from the two contexts:

Transaction 1: Context: Street vending: The following exchange occurred between the researcher posing as a customer and Masri (pseudonym), a 12- year old vendor:

Researcher: I will take 6 kilos of lemon, how much do these cost?

Masri: 1 kilo for 1250 L.L., then if 1 kilo cost 1000 L.L then 6 kilos will cost 6000 L.L and 6 of 250 LL. Will be 1000... then 7500 L.L

Transaction 2: Context: Street vending: This time, we approached Masri selling onions, 750L.L/ 1 kilo:

Researcher: I want 3 kilos of onions, how much do I owe you?

Masri: 2 kilos for 750 L.L plus 750L.L which gives 1500 L.L, and another 1 kilo for 750L.L then 2250L.L".

Transaction 3: Context: Bus conducting: A passenger approached the conductor requesting 4 tickets for destination A and 2 tickets for destination B.

Passenger: I want 4 tickets (tokens of travel) from to Sayani (exit point) and 2 tickets to the Sanitarium (a different exit point)

Conductor: Unit ticket price to Sayani is 3.75 so 4*4 is 16… take away 4 quarters, so the price is 15; unit ticket price to sanitarium is 4.25… so 4*2 is 8 and add 50 to it to get 8.50. The total fare is 15 +8 is 23 … add another 50 to it. Give me 23.50.

Let us consider the second transaction for analysis. We viewed the problem posed as one of multiplication, precisely 750 * 3. However, Masri did not multiply using the standard algorithm; rather, he solved the problem mentally through a *building-up* heuristic involving repeated additions which could be formalized as follows:

$$\begin{array}{rcl} 1 \text{ kilo} & \longrightarrow & 750 \text{ L.L} \\ 1+1=2 \text{ kilos} & \longrightarrow & 750+750=1500 \text{ L.L} \\ 2+1=3 \text{ kilos} & \longrightarrow & 1500+750=2250\text{L.L} \end{array}$$

Each variable, i.e., weight and price, remains independent of the other and parallel transformations are carried out on both variables, thereby maintaining their values proportional. When selling something at a price X, the vendors were perfectly aware of the fact that when there is an increase in the number of kilos (k) , there is a proportional increase in the price i.e., as many X's are increased in the price as kilos are increased in the purchase. The solution thus obtained has been termed by Vergnaud (2000) as scalar solution. Representing the above solution formally or explicitly:

Cost(3 k) = Cost(1 k+1 k +1 k) = Cost(1 k) + Cost(1k) + Cost(1 k) = 3 * Cost(1 k).

If we propose a relation between weight and price, more precisely a mapping f: to every weight there corresponds a well-defined price, then the above expression can be formalized as $f(1+1+1) = f(1) + f(1) + f(1)$. More generally, $f(X+Y) = f(X) + f(Y)$, which Vergnaud (1988) describes as the isomorphic property of addition, or more specifically, the linear property of function f.

The first transaction, on the other hand, represents another heuristic employed by the same vendor, namely multiplicative heuristic which can be also formalized as follows:

$$\begin{aligned}
&1 \text{ kilo} \longrightarrow && 1250 \text{ L.L} \\
&6 \text{ kilos} \longrightarrow && 6(1250) = [6(1000 + 250)] \text{ L.L} \\
&&& = [6(1000) + 6(250)] \text{ L.L} \\
&&& = [6000 + (4+2)\ (250)] \text{ L.L} \\
&&& = [6000 + 4(250) + 2(250)] \text{ L.L} \\
&&& = [6000 + 1000 + 500] \text{ L.L} \\
&&& = 7500 \text{ L.L.}
\end{aligned}$$

Here, Masri's solution method can be conceived in terms of a variable f(X), the price, as a function of a variable X, the number of kilos, and hence a relation can be formed through multiplying the value of X by a constant, unit price, in order to find the value of f(X). The solution obtained is called functional for it relates to two different variables, the ratio thus attained is termed "intensive" or "external" ratio (L.L/ kilo). Using the preceding argument:

$$\text{Cost (6 kilos)} = \text{Cost 1 kilo} * \text{6 kilos}$$

$$f(X) = f(1) * X$$

If we assume that the cost of 1 kilo = f (1) = constant a, then the above expression can be replaced by f(X) = a * X, which is the constant function coefficient (Vergnaud, 1988).

In the third transaction, the bus conductor broke the in-situ problem into three smaller problems: Find 3.75 * 4 (b) Find 4.25 * 2 (c) Add the answers from (a) and (b). We can combine the derived facts and related discussions from the first and the second transactions and propose the following (note that the currency denomination is in rupees abbreviated as Rs.):

Cost of four Rs. 3.75 tickets and two Rs. 4.25 tickets

= Cost of four Rs. 3.75 tickets + Cost of two Rs. 4.25 tickets

= Cost of one Rs. 3.75 ticket * 4 + Cost of one Rs. 4.25 ticket* 2

= (3.75*4 + 4.25*2) = (3.75*4) + (4.25 *2) = 3.25 * 4 + 4.25 * 2

This expression can be stated as

g (3.75*4 + 4.25*2) = g (3.75*4) + g (4.25*2) = 3.25 * g (4)+ 4.25 * g (2) or more generally as g (aX + bY) = a g (X) + b g (Y). Here, we can conceive the conductor's solution in terms of g (aX + bY), the price, as a function of the variables X and Y, the number of tickets two different ticket denominations with unit prices a and b respectively. Thus we form a relation by multiplying the values of X and Y by constants a and b.

DISCUSSION

In analyzing the practitioners' problem solving at work, one thing was clear namely, the fact that the participants utilized common heuristics-in-action in their understanding of simple proportional relationships, a model which Vergnaud terms "the isomorphism of measures model of situations". When using *building-up* heuristic, practitioners maintained the proportionality between the values by carrying parallel transformation on the variables without dividing or multiplying values in one variable by values in the other variable. It is worth mentioning here that, the rule-of-three, the algorithm learned in school to solve simple proportional problems differs from the isomorphism schema because it involves the multiplication of values across variables instead of parallel transformation on the variables. Hence, practitioners employed concepts that challenges the rule-of-three algorithm taught in school today clearly preferring the use of multiplicative heuristic due to its strong ties to problem situations, which led to functional solutions.

It seems fair to conclude that street vendors and bus conductors have developed mathematical concepts as a result of immersion experiences in everyday situations. The work setting represented vendors and bus conductors' natural habitat and thus introduced familiar problems. As a result, practitioners systematically and smoothly built up their solutions using intuitive computational strategies and without losing track of the strategy, even if many numbers are involved. In other words, the practitioners kept the meaning of the problem in mind during problem solving. This understanding that the practitioners acquired in the work situation elicited a coherent problem solving behavior that was attained through the following steps: (a) translating the problem from its real life context into an appropriate mathematical calculation problem, (b) performing the mathematical calculations, and (c) translating the result of this calculation back into the context of the problem to see whether it made sense.

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IS A MATHEMATICS TEACHER'S EFFICACY INFLUENTIAL TO THEIR STUDENTS' MATHEMATICS SELF-EFFICACY AND MATHEMATICAL ACHIEVEMENT?

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The main purpose of this study was to examine the effects of fifth-grade mathematics teachers' efficacy (MTE) on their students' mathematics self-efficacy (SMSE) and mathematical achievement (SMA) in the classroom. Two instruments (for MTE and SMSE) were administered to 62 classes (62 teachers and 1283 fifth-graders) for gathering data, associated with SMA scores in school. Corresponding statistical analyses were applied to the obtained data. The findings revealed that mathematics teachers' efficacy beliefs were significantly influential to both SMSE and SMA. It also showed that MTE ratings could effectively predict SMA. Consequently, suggestions derived from findings and discussions were proposed for further improvement of these mathematics teachers' efficacy and, in turn, for enhancing fifth-graders' mathematics self-efficacy and mathematical achievement in the future.

INTRODUCTION

Contemporary educational reforms in many countries focus on advancing the quality of teaching and learning in every classroom (Goddard, Goddard, & Tschannen-Moran, 2007, Moolenaar, Sleegers, & Daly, 2012). Grounded on Bandura's (1977) social cognitive theory and his construct of self-efficacy (SE), teacher efficacy (TE) has been recognized as "a variable accounting for individual differences in teaching effectiveness" (Gibson & Dembo, 1984, p. 569) and has a strong relationship with student learning and achievement (Cantrell, Young, & Moore, 2003; Gibson & Dembo, 1984; Ross, 1998). Tschannen-Moran, Woolfolk Hoy, and Hoy (1998) defined TE as "the teacher's belief in his or her capability to organize and execute courses of action required to successfully accomplish a specific teaching task in a particular context" (p. 223). Actually, from research in 1970s (e.g. Armor et al, 1976), "teacher efficacy was first conceptualized as teachers' general capacity to influence student performance" (Allinder, 1995, p. 247). Further, Ross (1998) indicated that most researchers treated "teacher efficacy as a type of self-efficacy" (p. 50). Since then, TE has been viewed as "self-efficacy beliefs directed toward a teaching context" (Knoblauch & Woolfolk Hoy, 2008, p. 167). That is, teacher efficacy referred to "their belief in their capability to have a positive effect in student learning" (Ashton, 1985, p. 142).

The concept of self-efficacy consists of two kinds of expectation, efficacy expectation and outcome expectancy. A teacher's efficacy expectation influences her/his thoughts and feelings, her/his selection of instructional activities, the amount of effort s/he

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spends in teaching, and the degree of her/his persistence while confronting difficulties (Bandura, 1981). The outcome expectancy refers to her/his own estimate of the likely consequences of teaching performance at the expected level of competence (Bandura, 1981). Applying this construct to the subject of mathematics, "Mathematics Teaching Efficacy Beliefs Instrument (MTEBI)" was originated by Enochs, Smith, and Huinker (2000) in measuring pre-service teachers' efficacy beliefs. Later, the researcher (Chang & Wu, 2004; Chang & Wu, 2009) adapted the MTEBI to assess elementary in-service mathematics teachers in Taiwan; that is, "Elementary Mathematics Teacher Efficacy Instrument (EMTEI)" was established consequently. EMTEI includes two cognitive dimensions: personal mathematics teaching efficacy (PMTE) and mathematics teaching outcome expectancy (MTOE). Accordingly, EMTEI is employed in this study to obtain targeted mathematics teachers' efficacy ratings.

As Bandura (1997) argued, SE, defined as "belief in one's capabilities to organize and execute the courses of action required to produce given attainments" (p. 3), had a great influence on one's task choices, effort, persistence, and achievement. Based on this concept, a student's self-efficacy refers to "belief in her/his capabilities to organize and execute the courses of learning". Thus, students who are self-efficacious in learning are likely to pay more efforts, persist longer while facing obstacles, and eventually attain better achievement. As to the domain of mathematics, students' mathematics self-efficacy (SMSE) beliefs have a powerful impact on the level of academic achievement and performance they may eventually achieve in learning mathematics (Chang, 2012; Kitsantas, Cheema, & Ware, 2011; Pajares & Miller, 1994; Pajares & Kranzler, 1995); that is, SMSE has been evidenced to predict students' mathematical achievement (SMA). In this study, "Elementary Students Mathematics Self-Efficacy Instrument (ESMSEI) is employed to assess targeted students' mathematics self-efficacy ratings, which was developed and validated by the researchers (Chang, 2012) based on Bandura's (1977, 2006) theory and his guidelines. ESMSEI also consists two cognitive constructs, "General Self-Efficacy—Related Mathematics (GSE-M)" and "Self-Efficacy for Mathematical Learning (SEML)".

Since teacher efficacy has a strong impact on student learning and achievement, does teacher efficacy beliefs have a direct influence on the development of students' self-efficacy in the classroom? In fact, several studies, domestically and internationally, indicated that a teacher's efficacy belief and her/his students' self-efficacy were significantly correlated (Bandura, 1982; Janet et al., 1995; Shao, 2005; Liu & Zhou, 2007; Tang & He, 2006). However, little knowledge was attained for the domain of mathematics learning, as well as for elementary students. Further, empirical evidences revealed that self-efficacy began to decline in grade 7 or earlier (Urdan & Midegley, 2003), particularly obvious in mathematics at the transition to middle school (Jacobs, et al., 2002). Thus, for fifth and sixth grades, children are positioned right at the developmental transition period, in which they confront with dramatically psychological, physiological, and social changes. As new challenges await them in this fast-growing stage (Schunk & Meece, 2006), to understand the relationship between teacher efficacy and students' self-efficacy becomes more beneficial while learning mathematics. Consequently, the first intention of this study is to assess the effect of a mathematics teacher's MTE on her/his students' SMSE, who are at the beginning stage of this transitional period (i.e. fifth-graders).

As verified by the researchers' previous study (Chang, 2012), a student's mathematics self-efficacy (SMSE) is predictive to her/his mathematics achievement (SMA). In addition, teacher efficacy is significantly influential to students' learning. However, less empirical evidence existed in supporting the effect of teacher efficacy on students' achievement, especially for mathematics in Taiwan. Therefore, besides assessing the effects of MTE on SMSE, it is also essential to testifying the effects of MTE on students' mathematics achievement (SMA). Altogether, in this study, it is valuable to verify whether the two factors, i.e. MTE and SMSE, are predictive to SMA or not. This effort will help us to clarify the relationship among the three factors, which will be also useful for further improvement for the quality of teaching and learning in mathematics.

Based on the background and motivation stated above, the three purposes of this study are as follows: (a) to investigate the effects of teachers' MTE on their students' SMSE; (b) to examine the effects of teachers' MTE on their students' SMA; and (c) to assess the effects of MTE and SMSE on SMA. Based on foregoing purposes, this study has three research hypotheses as follows:

- H1: MTE has a significant effect on SMSE, and significantly predicts SMSE.
- H2: MTE has a significant effect on SMA, and significantly predicts SMA.
- H3: MTE and SMSE significantly predict SMA.

METHOD

A total of 62 fifth-grade classes, including a classroom teacher (who taught mathematics) and fifth-graders in every targeted classroom, were selected by a stratified random sampling method (by school size) in elementary schools in Taiwan. Thus, a total of 62 mathematics teachers and 1283 students participated in this study. Based on the purposes of this study, data were collected through background sheets (for teachers and students), MTEBI (for teachers), and students' MSEI and mathematics achievement in school.

"Elementary Mathematics Teacher Efficacy Instrument", adapted from Mathematics Teaching Efficacy Beliefs Instrument (MTEBI) for pre-service teachers (Enochs, Smith, & Huinker, 2000), were used in this study in order to explore mathematics teachers' efficacy beliefs (Chang & Wu, 2004; Chang & Wu, 2009). The EMTEI consists of "Personal Mathematics Teaching Efficacy (PMTE, 13 items)" and "Mathematics Teaching Outcome Expectancy (MTOE, 8 items)", rated on a 5-point Likert scale; also, 5 items were written in a positive orientation and 16 items were written negatively. EMTEI has respectable internal consistency of .77, .81, and .71 for the whole scale, PMTE, and MTOE subscales respectively; PMTE and MTOE accounted for 20.82% and 15.86% of variance, respectively. (Chang & Wu, 2004).

In measuring SMSE, Mathematics Self-Efficacy Instrument (MSEI) was developed on the basis of Bandura's (1977, 2006) theory and his guidelines, which includes "General Self-Efficacy—Related Mathematics (GSE-M, 24 items)" and "Self-Efficacy for Mathematical Learning (SEML, 23 items)", rated on a 100-point scale. MSEI has high internal consistency of .96, .93, and .95 for the total scale, GSE-M, and SEML subscales respectively (Chang, 2012). Also, GSE-M and SEML accounted for 27.68% and 20.41% of variance, respectively. Both subscales significantly correlated, $r = .74$, $p < .001$. Also, mathematical achievement in school was represented in terms of their overall mathematics scores at the fifth-grade level. Mathematics scores, named as mathematical achievement T scores (MA-T), were collected at the end of the school year and then transformed into T scores for further analyses.

RESULTS

For teachers, the mean rating of all 62 fifth-grade mathematics teachers on MTE was 78.95 (SD=7.01), which meant that on average they had nearly 75% confidence in their own mathematics teaching capabilities. Also, for students, the mean rating of all 1283 fifth-graders on SMSE was 70.19 (SD=7.25), which meant that on average they had nearly 70% confidence in their own mathematics learning abilities.

The effects of fifth-grade teachers' MTE on SMSE

In order to examine the effects of MTE on SMSE through ANOVA, all teachers' MTE ratings were divided into three levels, i.e. "high (top 27% of them)", "middle", and "low (bottom 27% of them) MTE. Further, regarding the effect of MTE on SMSE, the results showed that there were statistically significant differences in fifth-graders' SMSE ratings among the three levels of MTE, $F\,(2, 59) = 5.13$, $p < .01$. The strength of the relationship between MTE and SMSE, as assessed by η^2, was strong, accounting for 14.8% of the variance for MTE. The post hoc comparison based on Scheffé concluded that fifth-graders taught/led by the teacher with high MTE (M=73.95) scored significantly superior in SMSE than did those taught/led by the teacher with low MTE (M=66.93), while the other two comparisons were not significant (i.e. high MTE and middle MTE [M=69.80], and middle MTE and low MTE). In addition, fifth-graders taught/led by the teacher with medium MTE scored higher in SMSE than did those taught/led by the teacher with low MTE.

To determine whether a mathematics teacher's efficacy belief could predict her/his students' mathematics self-efficacy, a simple regression analysis of MTE regressing on SMSE was conducted. The findings showed that MTE significantly predicted SMSE, $F\,(1, 60) = 17.88$, $p < .001$, suggesting that 21.7% of SMSE variance was explained by MTE. The standardized regression coefficients indicated that MTE (B = .48, t = 4.23, p < .001) had significant effects on SMSE. In brief, these findings indicated that fifth-graders who taught/led by the teacher with higher MTE would influence their students' SMSE. It means that a fifth-grade mathematics teacher with high MTE would be valuable in helping fifth-graders to build up their SMSE in the classroom. Accordingly, H1 was supported in this study.

The effects of fifth-grade teachers' MTE on SMA

Regarding the effect of MTE on SMA, the results showed that there were statistically significant differences in fifth-graders' SMSE ratings among the three levels of MTE, $F\,(2, 59) = 53.44, p < .001$. The strength of the relationship between MTE and SMA, as assessed by η^2, was quite strong, accounting for 64.4% of the variance for MTE. The post hoc comparison based on Scheffé concluded that fifth-graders taught/led by the teacher with high MTE (M=86.84) scored significantly superior in SMA than did those taught/led by the teacher with medium (M=81.46) and low MTE (M=71.42). In addition, fifth-graders taught/led by the teacher with medium MTE scored higher in SMA than did those taught/led by the teacher with low MTE.

To determine whether a mathematics teacher's efficacy belief could predict her/his students' mathematics achievement, a simple regression analysis of MTE regressing on SMA was also conducted. The findings showed that MTE significantly predicted SMA, $F\,(1, 60) = 119.02$, $p < .001$, suggesting that 65.9% of SMA variance was explained by MTE. The standardized regression coefficients indicated that MTE (B = .82, t = 10.91, p < .001) had significant effects on SMA. In short, these findings indicated that fifth-graders who taught/led by the teacher with higher MTE would influence their students' SMA. It indicates that a fifth-grade mathematics teacher with high MTE would be valuable in helping fifth-graders to increase their SMA in the classroom. Accordingly, H2 was supported in this study.

The effects of MTE and SMSE on SMA

To determine whether a mathematics teacher's efficacy belief and a student's mathematics self-efficacy could, together, predict a student's mathematics achievement, a simultaneous regression analysis of MTE and SMSE regressing on SMA was conducted. The findings showed that MTE and SMSE significantly predicted SMA, $F\,(2, 59) = 63.48$, $p < .001$, suggesting that 67.2% of SMA variance was explained by both MTE and SMSE. The standardized regression coefficients indicated that MTE (B = .74, t = 4.23, p < .001) yielded significant effects on SMA, which were greater than non-significant effects of SMSE (B = .15, t = 1.83, p> .05) on SMA. In summary, this finding revealed that fifth-graders who taught by a mathematics teacher with high MTE tended to have better mathematics achievement, with a minor support of her/his own and higher mathematics self-efficacy. Therefore, H3 was patricianly supported in this study.

DISCUSSION

MTE significantly influence fifth-graders' SMSE and SMA

First of all, the findings of regression analyses, paralleling with the result of ANOVA, indicated that MTE significantly predicted fifth-graders' mathematical achievement with 65.9% variance. This finding of significant effects of a mathematics teacher's efficacy belief on her/his students' mathematical achievement in school is corresponding to the previous studies (Ashton & Webb, 1986; Rosenholtz, 1989); even analogous to studies with different subject areas (Bandura, 1982; Denham & Michael, 1981; Janet et al., 1995). It is notable that MTE had great effects on students' mathematical self-efficacy as well. Thus, this result apparently indicate that the more efficacious a mathematics teacher the better her/his students' mathematical achievement in school. As mentioned previously, as teacher efficacy plays an important role on promoting students' learning achievement and their self-efficacy development in the classroom, we as teacher educators must devote extensive efforts to establish a positive and collaborative working and in-service learning environment that promotes mathematics teacher efficacy. In addition, all 62 mathematics teachers, on average, had nearly 75% confidence in their own mathematics teaching capabilities, and around 22 of them were even lower than 70%. This low efficacy and inadequate readiness in teaching elementary mathematics needs to be carefully acknowledged while discussing the future task of teacher professional development. Since teachers with high efficacy tend to put more efforts in preparing and teaching, persist longer while facing students' learning problems, and have more flexible selection of instructional activities, these enthusiastic actions combing with positive thoughts and adaptive expectations will be definitely beneficial for establishing a preferable learning environment, which in turn support students' mathematical learning.

Fifth-graders' SMSE had a effect on their mathematical achievement

In this study, all 1283 fifth-graders had averagely 70% confidence in their own mathematics learning abilities. Since "self-efficacy" was a powerful factor for students' learning performance (Bandura, 1977), which was evident in the researchers' previous study that the higher SMSE the better mathematical achievement (Chang, 2012), "how to increase or maintain the status of their SMSE became more essential to help them be successful in learning mathematics in school both at this transitional period and in the future" (Chang, 2012, p. 524). As a result, effectively providing a positive learning environment in this fast-growing and transitional stage will help to prevent possible declines of their SMSE (Jacobs et al., 2002), which is also helpful for promoting their learning achievement.

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HOW STUDENTS COME TO UNDERSTAND THE DOMAIN AND RANGE FOR THE GRAPHS OF FUNCTIONS

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To understand the mathematical concept of function, students must understand certain subconcepts, such as domain and range. Many researchers have studied students' understanding of functions, but no study has focused on how students come to understand the domain and range for the graphs of functions. In this study, we identified four common strategies, two transitional conceptions, and two representational challenges evidenced by students. In general, determining the range was more difficult than determining the domain for the students.

HOW STUDENTS COME TO UNDERSTAND MATHEMATICS

Functions play a key role throughout the mathematics curriculum. The U.S. Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Office, 2010) states that high school students should be able to: a) create functions that model relationships between two quantities, b) analyze and employ functions using different representations, and c) interpret functions for applications in terms of the context of the situation. However, the concept of function is one of the most difficult for students to understand (Tall & DeMarois, 1996). To understand the concept of a function, students must understand numerous subconcepts, such as input, output, ordered pairs, and correspondence to name a few.

Additional complications can arise when students graph, use graphs to reason about, or try to understand graphs of functions. Prevalent evidence suggests that piecewise functions cause substantial difficulty for students (Norman, 1993). Graphs play a role in how students come to understand and work with functions. Functions and their graphs are of interest in an instructional sense because they tend to focus on relationships as well as entities. While many have studied students' conceptions of functions (Markovits, Eylon & Bruckheimer, 1983) and how they understand domain and range (Arnold, 2004), there has not been a specific focus on how students understand the graphical representation of a function's domain and range.

Mathematics ideas and relationships can be represented using a variety of multimodal resources (e.g., inscriptions, speech, gestures, and artifacts, for more see Moore-Russo and Viglietti (2012)). Representations "help to portray, clarify, or extend a mathematical idea by focusing on its essential features" (National Council of Teacher of Mathematics [NCTM], 2000, p. 206). For example, to express the domain and range of a function, students often use interval or inequality notation. The representations used often come to impact how students make meaning of the concept at hand. During

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the meaning making process, individuals often: a) rely on strategies to help develop their understanding and b) develop individual conceptions regarding the idea under consideration. Chiu, Kessel, Moschkovich, and Muñoz-Nuñez (2001) defined a *strategy* as "a sequence of actions used to achieve a goal, such as accomplishing a particular task or solving a particular problem" (p. 219). They defined a *conception* to be "an idea that is stable over time, the result of a constructive process, connected to other aspects of a student's knowledge system, robust when confronted with other conceptions, and widespread" (p. 219). Following Smith, diSessa, and Roschelle's (1993) recommendations, Moschkovich (1999) defined a *transitional conception* as "a conception that is the result of sense-making, sometimes productive, and has the potential to be refined" (p. 172). To study a particular individual's meaning-making process, it is imperative to consider the transitional conceptions that occur and the strategies employed when students are engaged in tasks. In this study, we explore students' transitional conceptions of the domain and range of a graphical representation of a function.

METHODS

Setting

For this research, a qualitative, multiple-case study was conducted. The research site was a community college adjacent to a large city in eastern region of the United States. The lead researcher administered a pre-test to all students enrolled in two precalculus classes to determine their performance on graphical tasks that involved the concepts of domain and range. Study participants for the study were selected from middle-achieving students whose test scores ranged from 51% to 79% on the initial instrument, since evidence suggested that these students were developing an understanding of domain and range, and were more likely to have transitional conceptions than those with very low or very high results on the initial instrument.

Data Collection

The data sources for the study were students' written answers to domain and range tasks that involved functions' graphs on four test sets as well as videos and transcripts of subsequent student interviews. Five participants were asked to solve short-answer items. Each participant completed items 1-20 first, either in the classroom or researcher's office. Then after completing the test without interruption, the lead researcher immediately interviewed the student asking about how the tasks were completed. After the first interview, the lead researcher administered a second set of tasks, items 21-40, to the participant after a short break. Upon completion a second interview was then conducted. In a later setting, the third and fourth interviews were administered in a similar format. All interviews were videotaped and transcribed within two weeks after the interviews were completed. The four test sets were designed with different purposes. The first set, items 1-20, was designed with basic questions and figures. The second set, items 21-40, contained more advanced problems whose graphs included more turning points, open points and horizontal sections. The third set, items 41-60, included more complicated piecewise function graphs. The fourth set, items 61-108, was designed to check the impact of the small graphical differences on the domain and range. For example, pairs of tasks might contain one item where all the turning points were closed and another related item where the same graph was given except all the turning points were open.

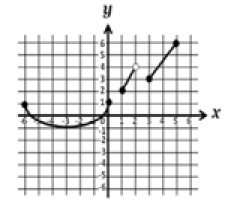
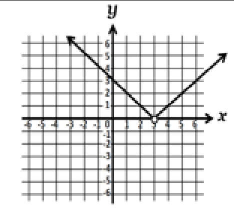
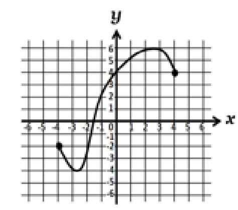
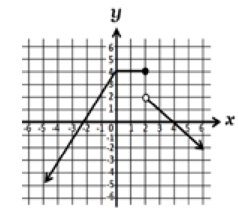
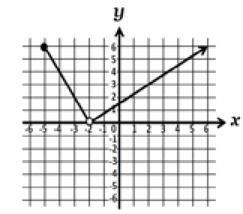
Data Analysis

The data were examined for emerging categories through a general inductive analysis. The research team used theoretical memoing (Glaser, 1998) to record and classify observations during the multiple passes through the data. This process was guided by the use of rich, thick descriptions of participants' activities and their responses. As the research team combed through the data, they began to cluster similar entries to form unifying categories. Upon studying the data, the research team determined it would be best to start with a loose structure of three broad classifications (strategies, conceptions, and representations) allowing more specific categories to emerge from the data under this structure. The research team then used these categories to make sense of observed activity (Thomas, 2006). During the constant comparison of data across participants as well as across interviews and written responses while considering what information might be of greatest benefit to instructors, there was a slight refinement of the overall structure to concentrate primarily on students' common strategies, transitional conceptions, and challenges with representations.

RESULT: STRATEGIES, CONCEPTIONS, AND REPRESENTATIONS

In this study, we considered two research questions: a) Which strategies and transitional conceptions are evident when students consider the domain and range of a graphical representation of a function? b) How do students' use of strategies and their understanding of concepts and representations impact their understanding of the domain and range of a graphical representation of a function? To address the first research question, the research team determined the most prevalent strategies and conceptions that students used when engaged in domain and range tasks for a given graph. To address the second research question, the research team analysed all data sources to see how students were using strategies, concepts, and representations related to the domain and range of a function's graph.

To determine and denote the domain or range of a graph, students need to be able to use appropriate **strategies** that fit the context and the problem at hand; they need to hold particular **conceptions** to understand and work with certain mathematical concepts; and they need to be able to represent their ideas and responses with an appropriate **representational notation**. Consequently, all three are needed to work with the domain and range of a function's graph. Next, we report our findings based on these three classifications: common strategies, transitional conceptions and representational challenges. We use representative examples, displayed in Figure 1, which are a subset of 10 items and 4 participants' responses to these items.

① Item 20 Mary	② Item 74 Mary	③ Item 22 Kara	④ Item 36 Mary	⑤ Item 10 Victor
				
20. Find the range	74. Find the range	22. Find the range	36. Find the range	10. Find the range
[-1, 1]U[2,6]	**(0, ∞)**	**[-2, 4] →** **[-4, 6]**	**[4, 2) →** **[4, 2)U(2, -∞)**	**(-2, ∞) →** **(0, ∞)**

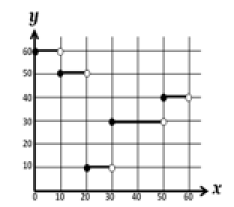
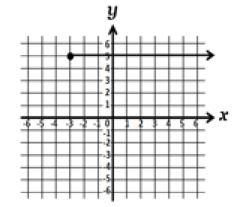
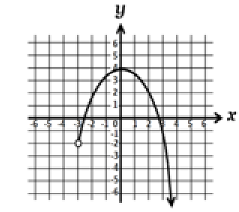
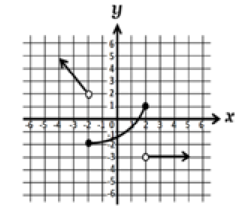
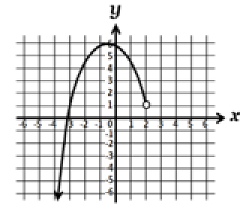
⑥ Item 16 Louis	⑦ Item 64 Louis	⑧ Item 12 Kara	⑨ Item 38 Victor	⑩ Item 24 Mary
				
16. Find the range	64. Find the range	12. Find the range	38. Find the range	24. Find the range
[10, 60]	**ø**	**(-∞, 4) →** **(-∞, -2) → (-∞, 4]**	**[-2, 1] (2, ∞) →** **y=-3U[-2, 1] (2, ∞)**	**(6, -∞) →** **[6, 1)U(1, -∞)**

Figure 1: Representative examples (For each item response, an arrow sign "→" represents a subsequent attempt during the interview process.)

Common Strategies

1. Projecting the graph onto the x-axis (or y-axis) to determine the domain (or range). Projecting the graph onto the x-axis strategy is a glancing-and-imagining method. With this strategy, students glanced at the graph and projected the graph onto the x-axis without any other body motion. They mentally projected the graph onto the x-axis and used the imagined horizontal segment or line to determine the domain. Projecting the graph onto the y-axis strategy is a strategy similar to projecting the graph onto the x-axis strategy. On item 20, see Figure 1-(1), Mary projected the piecewise function onto the y-axis. When determining the range she looked at the graph, staring the longest at the right side of the graph, where the two linear segments' ranges of [2, 4) and [3, 6] overlapped. She reported that she projected the graph to the y-axis with her eyes and merged the two intervals to determine the answer $[-1, 1] \cup [-2, 6]$.

2. Pushing the graph to the x-axis (or y-axis) to determine the domain (or range). Pushing the graph to the x-axis was an embodied strategy that involved student gesturing. To determine the domain of the graph, students used a motion with their hands or fingers as if pushing or pressing down the graph to imagine it as a horizontal segment (or line) on the x-axis. A slight variation of the pushing gesture, students used a clapping gesture to make a noise by actually clapping their hands as they imagined the graph physically being pressed to the x-axis. Pushing the graph onto the y-axis strategy is a strategy related to the strategy of pushing the graph onto the x-axis, with the only distinction being pushing to the *y*-axis rather than the *x*-axis. On item 74, see Figure 1-(2), Mary gestured by "pushing" the piecewise function to the *y*-axis using her both hands.

3. Focusing on the endpoints when tracing the graph from the starting point to the ending point. Tracing the graph is a graph-following strategy that involved eye or finger movement. When students traced a graph with their eyes or fingers, typically they focused on the endpoints then traced the graph from the starting to the ending point. While this strategy would yield a correct response for the domain, it would not necessarily do so for the range. In some instances, students traced from right to left, where the domain was $(-\infty, p)$ or $(-\infty, p]$ for some point p. For example, if a graph was bounded by a closed endpoint on the right yet unbounded to the left, often students reported they traced the graph from the right point to the left arrowhead because the arrowhead's direction caused their eyes to naturally follow the arrowhead's path. On Item 22, see Figure 1-(3), Kara answered [-2, 4] for the range of this graph. She traced the graph from the left boundary point to the right ending point. She initially used the *y*-coordinate values of the both end points, even though they were not the minimum and maximum values of the graph.

4. Not overlapping sections of a graph to determine its range. When a graph is not a one-to-one function, there is at least one portion of the graph were the *y*-coordinate values overlap. However, some students did not notice the overlapped portions. For example, if an open point exists in the overlapped portion, the open point's *y*-coordinate value should not be eliminated since the open point can be overlapped with another portion of the graph. On item 36, see Figure 1-(4), Mary determined the range [4, 2) on her first attempt focusing on the two end points. Upon noticing this in the interview she then responded $[4, 2) \cup (2, -\infty)$ on her second attempt, even though the desired answer was $(-\infty, 4]$. In her second response, she did not include the *y*-coordinate value of 2, even though the point (-1, 2) is on the left part of graph. In addition, she started at the top most part of the graph following the arrowheads down and then reported the interval in a nonstandard descending order.

5. Using the closest axis value; using x-coordinate values instead of y-coordinate values, and vice versa. When students determine the domain of a graph, they should focus on the *x*-coordinate values of the graph. However, if a graph intersected a *y*-axis or had a vertex on the *y*-axis, students often focused on the *y*-coordinate values of the point. This phenomenon seems to suggest that the students' eyes are attracted or drawn to the closest number on the *y*-axis. Similar situations occurred for range when a graph had a critical point on the *x*-axis. On item 10, see Figure 1-(5), Victor used the *x* value of -2 rather than the *y* value of 0 from the open point (-2, 0) when reporting the range of this graph.

6. Measuring the range from the lowest value to the highest value of a piecewise function. Some students used the lowest and highest values to determine the range even though the graph was vertically discontinuous. On item 16, see Figure 1-(6), Louis answered [10, 60] for this step function's range.

Transitional Conceptions

7. Belief that a horizontal line or segment of a line has no range. A horizontal line is a specific, special case since it has no change in its range. Some students believed that a horizontal line or segment had no range at all. Their belief stemmed from the conviction that the range should have some length or distance. On item 64, see Figure 1-(7), Louis' answer was an empty set "Ø" and his reasoning in the interview was "There is no range because it is a flat line."

8. Dealing with marked open or closed points as boundaries. Students felt it was especially difficult to determine the range of a horizontal segment when it included open points on the ends of its graph. They did not seem to recognize that a horizontal segment consists of infinitely many closed points.

Students preferred clearly designated points, either open or closed, when finding the range and when the graph was not horizontal. When a graph had an absolute maximum where the function was concave down, the point's y-coordinate value should be the range's maximum. However, some students hesitated to put the vertex's y-coordinate value as the range's greatest point. The reasoning was that when an absolute maximum is a part of curve, there is no clear point but a curve. Instead of using the absolute maximum (or minimum) on a curve, students preferred to use the open or closed point that was highlighted at the boundary points of intervals.

On a related note, some students would purposely use open parentheses in their responses when turning points where the absolute extrema. On item 12, see Figure 1-(8), Kara's original answer was (-∞, 4) for the range. She hesitated to use the maximum vertex, the point (0, 4), specifically mentioning because there was no closed point specifically marking a definite point.

Representational Challenges

9. Difficulty with the notation in representing the range of horizontal lines. The horizontal line is a specific case of a graph since it has only one point for its range. Even when students realized that a single point was the range, they did not know how this should be represented. In the special case of a single value, using conventional set notation, the degenerated interval is represented by braces { }. Students were at times unfamiliar and more often uncomfortable using this notation. On item 38, see Figure 1-(9), Victor did not include the horizontal part of the graph for the range originally. He thought that the horizontal ray's range did not exist because of an open point. His original answer was [-2, 1] (2, ∞). After he realized that there were many points that had the y-coordinate value -3, he added y = -3 to his response. His final answer was y= -3 ∪ [-2, 1] (2, ∞).

10. Representing an interval in descending order. By convention, the interval notation is written in ascending order with the smaller number located on the left side of the interval and the greater number located on the right side. However, many students used a descending order, especially when the graph decreased or when they traced an increasing graph that was bounded on the right. Often they traced from right to left following the direction of the arrowhead on the left hand. On item 24, see Figure 1-(10), Mary determined the range from the maximum point to the arrowhead. Rather than writing $(-\infty, 6]$, she wrote $(6, -\infty)$. She then rewrote the answer $[6, 1) \cup (1, -\infty)$, still using the descending order, which related to the transitional conception labelled #8 above.

DISCUSSION

Common Challenges

All five students had difficulty with the range of horizontal lines. This was related to the fact that they thought of graph as flat and without any vertical distance or length. They felt it should not have a range. Even when students began to recognize that a horizontal segment would have a range of a single point, they were often only considering the endpoints of the segment and felt that horizontal segments with open endpoints as boundaries would not have a range. Others recognized that the range would be a single point but often struggled with how to represent this. The fact that all five students had difficulty with this suggests that instructors should take this into account in their task selection.

When a graph extends infinitely and its representation includes arrowheads, students impulsively followed the arrowhead direction. When students used this strategy to determine the range (especially for the piecewise functions), they frequently created overlapping intervals. The instructional implication is that students often view parts of graphs that should be considered at the same time as separate entities. Of note was the fact that the students who used the “projecting” or “pushing” strategies rather than the tracing strategies seemed to treat the piecewise graphs as all belonging to a single whole and were less likely to give the overlapping interval responses.

One of the notable findings was when students used open parentheses when boundaries were not endpoints nor where they specifically designated (either open or closed) points. Three of five students did not want to use the y-coordinate value of the absolute maximum point since the turning point had no closed point. Instead, they favoured using only specifically represented closed points serving as boundaries for certain sections of the graph, since they were clearly specified. The underlying source of this transitional conception relates to the fact that either students did not realize that a line consisted of infinitely many closed points or they did not realize that points do not have to be represented by either open or closed circles (i.e., segments and other curves represent a continuous collection of closed points). This finding suggested that students’ challenges related to domain and range may stem from not understanding the meaning of curved sections of graphs but may go back to understanding the fact that continuous curves represent an infinite set of closed points.

More challenges arose when students were determining the range, rather than the domain. Since the functional inputs are not repeated but are unique for the domain, there are no overlapping sections as can be the case for the range. This was a particular challenge for students for graphs of functions that were not one-to-one.

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USING THEMATIC ANALYSIS TO STUDY CURRICULUM ENACTMENTS

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We use thematic analysis to explore how mathematical concepts are developed in four enactments of the same task. Thematic analysis emerges from Systemic Functional Linguistics and provides a means to explore the development of mathematical ideas in the classroom discourse. Thematic analysis was used to explore how themes related to the comparison of like quantities were developed in a task designed to introduce different types of comparisons and different ways to represent comparisons. The thematic analysis showed similarities across the four teachers' enactments, suggesting an influence from the design of the task. There were also differences that pointed to teachers' different emphasis and their own understanding of the thematic pattern.

CURRICULUM ENACTMENT AS AN OBJECT OF STUDY

Teachers' use of curriculum materials has become an active field of study, especially since many districts have used innovative curriculum materials to drive instructional change (Remillard, 2005). The view of the teacher's role has expanded over the last two decades in response to new conceptions of teachers as curriculum developers (Ben-Peretz, 1990) and to the influx of innovative curriculum materials developed in the U.S. The enactment of curriculum materials is not straightforward, as there can be considerable variation in the ways teachers enact materials from the same program (cf. Remillard & Bryans, 2004; Tarr et al., 2008). To account for this variation, researchers have developed a number of perspectives to better understand curriculum enactments. These perspectives have focused on the extent to which curriculum content is covered, the fidelity with which teachers draw on the materials to design instruction, the kinds of instructional practices evident when using particular curriculum materials, and the level of cognitive demand of instructional activities (Chval, Wilson, Ziebarth, Heck, & Weiss, 2012). While these perspectives provide nuanced and detailed accounts of teachers' interactions with curriculum materials and, to a lesser extent, the curriculum received by students, they shed little light on how teachers use curriculum materials to develop mathematical ideas and on how different design features of curriculum materials influence the ways mathematical ideas get developed in curriculum enactments. This paper seeks to address that gap.

In this study, we use *thematic analysis* to explore how mathematical concepts are developed during the enactment of curriculum materials. Thematic analysis has been used to characterize discourse in mathematics classrooms (Herbel-Eisenmann & Otten, 2011) and in science classrooms (Lemke, 1990). Thematic analysis allows a researcher to explain the ways that concepts are developed by looking at the underlying semantic

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relations and the ways they are explicitly or implicitly constructed. Thematic analysis emerges from *systemic functional linguistics* (SFL) (Eggins, 2004; Halliday, 1978; Halliday & Martin, 1993), which looks at language as a *resource for meaning* rather than as a system of rules, so that language use is viewed in terms of "learning how to mean versus learning how to speak" (Halliday, 1978, p. xx).

THEMATIC ANALYSIS AS A FRAMEWORK

Systemic Functional Linguistics

Systemic functional linguistics looks at language as a meaning making resource rather than as a conduit through which thoughts and feelings are expressed (Halliday & Martin, 1993). As such, SFL is built on the supposition that language is not only situated in context but produces context, and moves away from idealized views of language and of speakers (Halliday, 1978). SFL treats grammar as the realization of discourse, as a means of expressing semantic relations that are the heart of meaning making, rather than inherently carrying some unambiguous meaning.

Thematic Analysis

Analysis of thematic patterns allows researchers to see how ideas and concepts are developed in the classroom discourse, where discourse is broadly construed to include language, gesture, and other resources for conveying meaning (Herbel-Eisenmann & Otten, 2011). Thematic analysis focuses on how relationships between discourse objects are expressed, how relationships are made explicit, and how these relationships cohere into themes (Lemke, 1990). Lemke states that a thematic pattern is a way of picturing the network of relationships among the meanings of key terms in the language of a particular subject. Often, students are drawing from one pattern that is based on their everyday experiences while teachers are drawing from a pattern that is based in the conventions of the discipline they are teaching. In order for students to learn disciplinary content, teachers must recognize students' thematic patterns and draw connections to the conventional disciplinary pattern (Lemke, 1990; Schleppregrell, 2007). Learning can thus be construed moving from thematic patterns based in everyday language use to those found in disciplines.

Thematic patterns involve the construction of lexical relations and lexical chains. Lexical relations (Eggins, 2004), or semantic relations (Lemke, 1990) express relationships between various discourse objects. Lexical relationships include taxonomic relations such as hierarchy, similarity, or contrast, and nuclear relations, such as agent-process-medium and activity sequences (Martin & Rose, 1993). Taxonomic relations include the ways terms are similar (synonyms) or contrast (antonyms) in addition to hierarchical relations such as hyponyms (member – class relationship), co-hyponyms (two members of the same class), meronym (part of a whole) and so forth (Lemke, 1990). Lemke states that semantic relations tend to be variants of a relatively small number of basic ones, and which form thematic patterns that are highly standardized in each field of science.

Lexical chains or strings (Eggins, 2004) or *semantic chains* (Lemke, 1990) provide insights into how thematic patterns are developed. Eggins defines a lexical string as a list of "all of the lexical items that occur sequentially in a text that can be related to an immediate prior word" (p. 44) either taxonomically or through an expectancy relation (word or phrases that we expect to see used in proximity to each other). These strings contribute to the *cohesion* of a text. Eggins (2004) defines cohesion as "how what we're saying hangs together and relates to what was said before and to the context around us" (p. 12). Text cohesion builds within a clause via expectancy relations or via the ways that clauses are connected by conjunctions.

METHODS

Thematic analysis was conducted on transcripts of four teachers' enactments of the same task. The *Bolda Cola* problem introduces the *Comparing and Scaling* unit in the Connected Mathematics Project (CMP) curriculum (Lappan et al., 2006). The unit explores different ways to compare like and unlike quantities, eventually leading into unit rate and algebraic representations of unit rate. The Bolda Cola problem asks students to explore four claims around a fictional taste test of two brands of cola, Cola Nola, and Bolda Cola. The purpose of the problem is to introduce different kinds of comparisons (part to part and part to whole) and ways to represent comparisons (e.g., fraction, percent), ideas that will be explored for several more tasks over the span of a week or more.

The thematic analysis involved the construction of lexical chains and maps of the lexical relations based on the transcripts from each enactment, to focus on similarities and differences in those thematic patterns. The transcripts were first parsed into Topically Related Sets (TSRs)(Mehan, 1979), which consist of a series of exchanges around a single topic, such as discussion around a strategy or a specific question.

Each TRS was parsed into two separate themes, one related to mathematical concepts and the other related to the context of the Bolda Cola problem. This was done by creating a column for mathematical terms and language and a column for references to the taste test context. A third column was used to track which part of the Bolda Cola task was being addressed during the TRS. A column was created to track the conjunctions and prepositions to help map the relationships being constructed between the lexical items in the mathematics and context columns. Another column was created as well to keep track of the verbs being used, in order to consider the ways that mathematics was being construed in each class (Herbel-Eisenman & Otten, 2011). For each topically related set, we summarized the mathematics and context themes, which now became the lexical items used in subsequent parts of the analysis.

To construct the lexical chains, we used the summaries of the mathematics and contextual themes from each TRS and created a new spreadsheet which tracked these lexical items across the TRSs, with the mathematical themes and contextual themes grouped separately, with a third grouping for the task part being addressed in that particular TRS. This spreadsheet showed how lexical items were used over time, whether they appeared in multiple TRSs, and how they mapped on to the task parts.

The different lexical items and the relationships expressed between each of the items from the TRS summaries were then used to create a transcript-based map. Lemke (1990) stated that thematic patterns are best expressed in the form of diagrams that can show the interconnected semantic relationships among several terms or thematic items. These transcript-based maps were adapted from the ‘clean map’ (Herbel-Eisenmann & Otten, 2011), which was constructed based on an analysis of the lexical relations expressed in the textbook, on discussions with one of the textbook authors, and on our own understanding of mathematics. The clean maps were used to identify key concepts that were the intended focus of the Bolda Cola task and the Comparing and Scaling unit in general.

A final spreadsheet was created, using the primary concepts in the unit as column titles and the TRS number as row title. Then, the lexical items identified in the first spreadsheet were placed under the column or columns that represented the concepts referenced by the lexical item. We were looking for instances in which a lexical item appeared multiple times in one column, which formed a lexical chain for that concept. These lexical chains not only exhibit the development of the thematic pattern, but also how the mathematical and contextual themes worked in tandem to express the underlying mathematical concept.

We also looked for when one TRS contained multiple concepts, which constituted a *thematic nexus*. Lemke (1990) explains that an important point in the development of a thematic pattern is when there is a *thematic nexus*, which multiple thematic relations are brought in contact with each other at one point in time.

RESULTS

The results show similarities in the transcript maps that potentially show the role of task design in eliciting and developing a thematic pattern related to comparing quantities. Conversely, there are subtle differences in the transcript maps and lexical chains that demonstrate different. First we discuss the lexical chain tables.

Lexical Chain Tables

The lexical chain tables provide an indication of how the lexical items varied across the enactment and how they mapped onto the part of the task being addressed. In Figure 1, for the teacher named Allen, the first section is the mathematical lexical items (e.g., comparing two quantities using a 'for every' statement; dividing the quantities by the same number gets a scaled down ratio), the second the contextual items (e.g., For every 17139 who liked Bolda Cola, 11426 liked Cola Nola; Dividing 17139 and 11426 by 5713 gives you 3 and 2), and the third section the task part (e.g., Do the four statements from the same data?). In figure 1, for the teacher named Allen, as the task part changes, one can see a related sequence of mathematical and contextual themes.

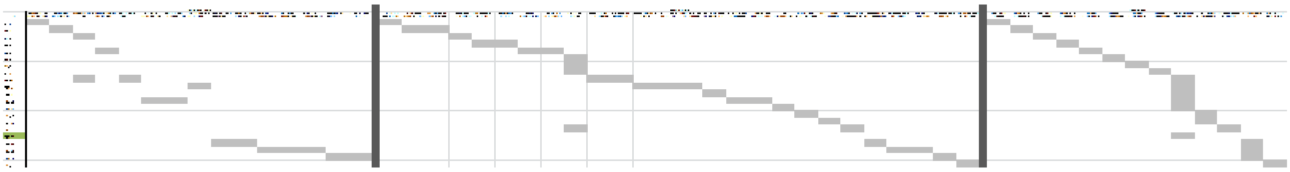

Figure 1: Lexical chain table of Allen.

The differences across the lexical chain tables of the four teachers indicate the extent to which the teachers focused on particular questions to develop the mathematical or contextual themes. Granville, for example, focused much of her discussion on whether all four statements came from the same data, during which nearly half of the mathematical themes were developed, as seen in Figure 2, while Sadosky focused a good portion of her discussion on the question most related to the context (which statement would make the best advertisement), which focused relatively more of her discussion on developing contextual themes than mathematical themes.

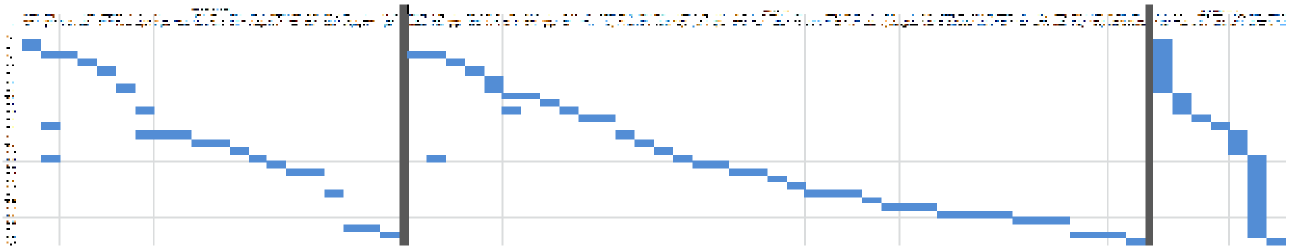

Figure 2: Lexical chain table of Granville.

Transcript Maps

The transcript maps showed which lexical items were discussed and how they were related in each enactment. Figure 3 shows the transcript map for Allen. The four transcript maps had roughly similar sets of lexical items and lexical relationships, which speaks to the design of the Bolda Cola problem. However, there were subtle differences between the maps that indicate teachers' intended focus for the task and perhaps their own understanding of the thematic pattern. Allen's map, for example, was not as clearly connected as the others and there was more ambiguity in the classification and composition of comparisons, particularly the ratio comparisons, as can be seen, for example, in the characterizing of most ratios as part to part ratios. Sadosky's map showed a relatively greater emphasis on processes and different ways of expressing comparisons but the taxonomic relations were not as coherent as those of Pless and Granville. Pless's map was parsimonious but coherent, showing a clear taxonomy of terms. Granville's map was the most elaborate and well-connected, reflecting the greater amount of time spent on the task and the greater explicitness in discussing the lexical relations as seen in Figure 4.

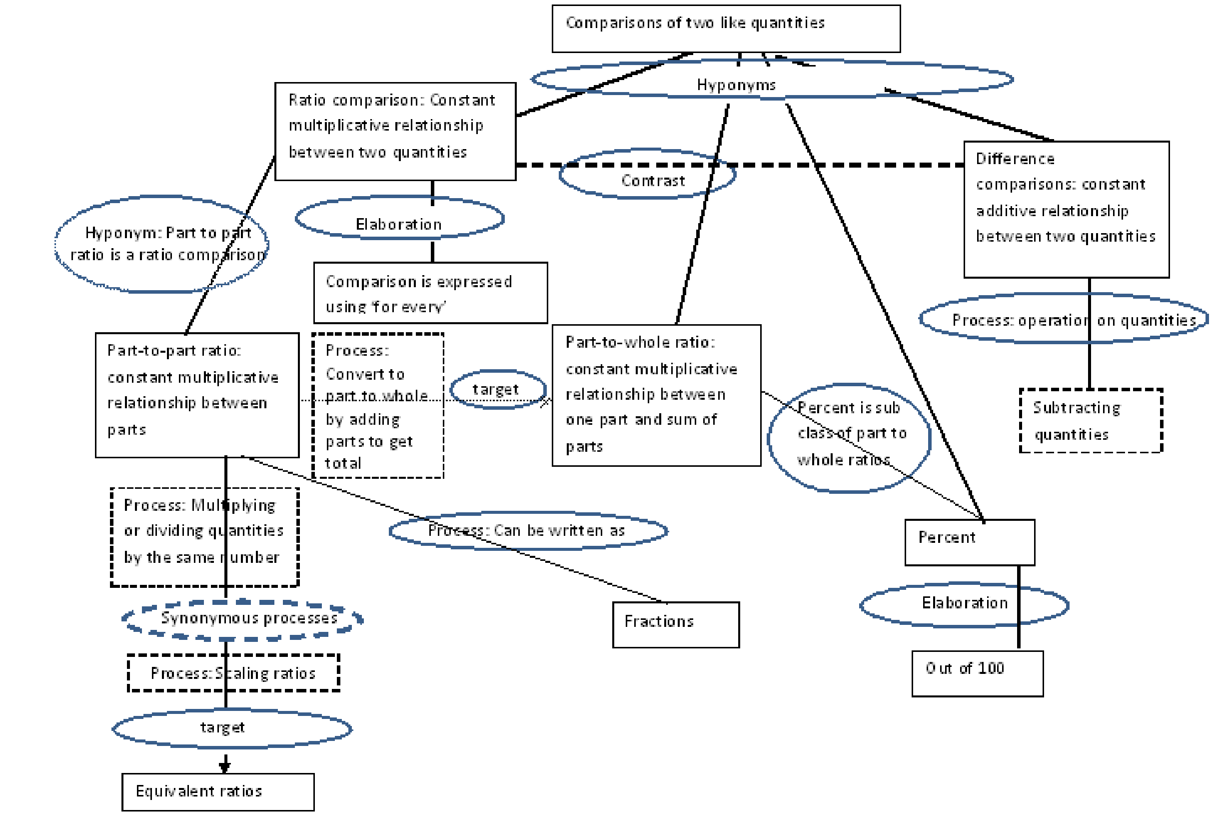

Figure 3: Transcript map for Allen.

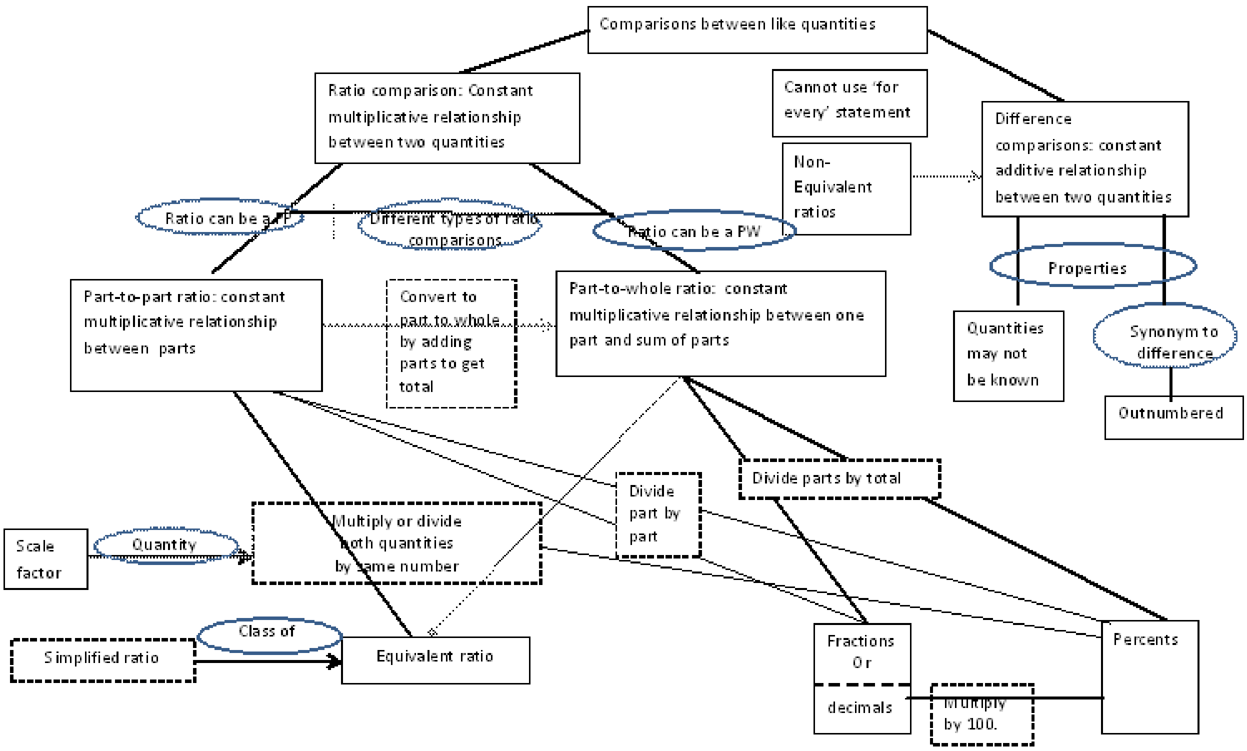

Figure 4: Transcript map for Granville.

The final spreadsheets, reflecting lexical strings for the main concepts and any occurrences of a thematic nexus, primarily showed the distribution between the mathematical themes and contextual themes for developing the thematic pattern. The tables showed how couplings of math/context themes occurred relatively frequently and how each contributed to the various lexical strings and the overall thematic pattern. Figure 5 shows how Pless explicitly describing the lexical relations, which contributed to her coherent transcript map.

Part to part comparison		Part to whole comparison		
Ratios can be simplified or scaled up	Ratios can be written as fractions or percents	To get the total you add the parts	Ratios can be simplified or scaled up	Ratios can be written as fractions or percents
A big number is unlikely to be a simplified ratio				
A big number means it's the actual number of people surveyed				
The bigger numbers means the majority of people preferred BC				
The quantities can be written in the ratio 17139 to 14126				
There are different types of ratios				
BC to NC ratio and BC to total ratios are different				
There are part to part and part to whole ratios				
BC to NC ratio and BC to total ratios are different				
		One type of ratio compares people who like one brand of coke to the total population surveyed.		

Figure 5: Partial Second Lexical Chain Table for Pless.

DISCUSSION

The paper set out to show how mathematical ideas were developed in four separate enactments of the same task. A goal was to introduce a perspective on studying curriculum enactments that allowed researchers to better understand how teachers' uses of curriculum materials provides opportunities for students to develop understanding of mathematical concepts. Thematic analysis allowed for a fine-grained and multi-tiered analysis of the development of mathematical ideas across the four classrooms. The results also show how thematic patterns related to problem contexts can be used to develop mathematical thematic patterns.

The thematic analysis showed similarities across the four teachers' enactments, suggesting an influence from the design of the task. There were also differences that pointed to teachers' different emphasis and their own understanding of the thematic pattern. Future research needs to look at enactments of tasks from different curriculum materials and at how thematic pattern related to comparison of like quantities develops over multiple tasks from the same instructional sequence.

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TEACHERS' PRODUCTIVE MATHEMATICAL NOTICING DURING LESSON PREPARATION

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This study uses lesson study to investigate what mathematics teachers notice about students' mathematical reasoning during the planning of a lesson on fractions. Most research examines teaching noticing during or after a lesson, and focuses on the specificity of what teachers notice as a characteristic of noticing expertise. In this paper I propose a new notion of productive noticing, and apply it to analyse two vignettes of teachers' mathematical noticing during lesson preparation. Findings suggest that teachers' noticing is most productive when it goes beyond the specificity of what teachers notice to include justification based on what they have noticed about students' thinking. The study also demonstrates the usefulness of this construct in analysing what mathematics teachers notice when planning lessons.

INTRODUCTION

Mathematics teacher noticing—what mathematics teachers see and how they understand instructional events or details they see in classrooms (Mason, 2002; Sherin, Jacobs, & Philipp, 2011)—is central to mathematics teaching practices, and is needed for improving teaching (Mason, 2002). Most researchers who study mathematics teacher noticing do so by examining what teachers observe from video clips of lessons (Star, Lynch, & Perova, 2011; van Es, 2011); while others (Sherin, Russ, & Colestock, 2011) capture what teachers notice in-the-moment during lessons. In this paper, I extend the notion of productive noticing to enable investigation of what mathematics teachers notice during the planning of mathematics lessons. The key research questions addressed in this paper are: What do mathematics teachers notice about students' mathematical thinking during lesson preparation? More importantly, what distinguishes teachers' productive noticing from less productive noticing?

THEORETICAL CONSIDERATIONS

Mathematics teacher noticing

According to Mason (2002), noticing is a set of practices that work together to enhance teachers' awareness to new responses in classroom situations. These practices include "reflecting systematically; recognising choices and alternatives; preparing and noticing possibilities; and validating with others" (Mason, 2002, p. 95). Many researchers view noticing as consisting of two main processes: "attending to particular events and making sense of events in an instructional setting" (Sherin, Jacobs, et al., 2011, p. 5), but Jacobs, Lamb, and Philipp (2010) also include how teachers decide to respond to instructional events in order to link the intended responses to the two main processes of noticing. This triad view of noticing—attending to; making sense of; and deciding to respond—ties in with Mason's (2002) idea that noticing should bring to the mind of teachers a different way to respond.

However, it can be very challenging to notice salient mathematical details in a classroom setting. Marking and discerning instructional events that are critical and useful can be difficult for teachers. In a video-club study involving 30 pre-service teachers, Star et al. (2011) found that they had problems attending to specific mathematical details of lesson tasks. Vondrová and Žalská (2013) also found that the pre-service teachers in their study did not notice mathematics-specific details, even when they were shown short video clips with prominent mathematical incidents.

Developing teachers' ability to notice

Approaches to develop teachers' noticing often centre around the use of video clips of teaching—where teachers are shown clips of classroom teaching and asked to notice certain features of the instruction (Sherin, Russ, et al., 2011; Star et al., 2011; van Es, 2011). These approaches tend to focus largely on noticing instructional details after lessons are conducted. In order to examine teachers' in-the-moment noticing, Sherin, Russ, et al. (2011) asked teachers to record short segments of video clips of what they noticed during lessons, using a wearable camera, before they discussed these recorded segments. Even though this approach gave researchers improved access to teachers' in-the-moment noticing by triangulation with teachers' reflections on the recorded segments, the researchers acknowledged that the sense-making and decision-making processes may not be fully captured (Sherin, Russ, et al., 2011).

One issue with this approach of developing teachers' ability to notice is the lack of focus on preparation to notice. As Mason (2002) put it, "noticing is an act of attention, and as such is not something you can decide to do all of a sudden. It has to happen to you, through the exercise of some internal or external impulse or trigger" (p. 61). More specifically, Mason (2002) highlights advanced preparation to notice, and the use of prior experience to enhance noticing in order to have a different act in mind. In this paper, I propose a development of teachers' noticing ability through explicit preparation during the planning of a mathematics lesson.

Productive mathematical noticing—focusing on the 'Three Points'

Most research focuses on the specificity of what teachers notice, but specificity is not sufficient for noticing to be productive. In a study involving seven prospective secondary school mathematics teachers, Fernandez, Llinares, and Valls (2012) found that most were unable to relate the strategies used by students to the characteristics of the problem, even though they were all able to describe the specific strategies at the beginning of the study. In the context of lesson planning, one possible approach is to support teachers' ability to notice mathematical features by directing their attention to key mathematical ideas and students' learning difficulties related to these concepts.

In a previous paper (Choy, 2013), I proposed a characterisation of productive noticing using Yang and Ricks' (2013) Three-point framework—key point; difficult point; and critical point. According to Yang and Ricks (2013), the *key point* refers to key mathematical concepts or ideas of the lesson; the *difficult point* refers to cognitive obstacles encountered by students when they attempt to learn the key point; while the *critical point* refers to the approach taken by teachers to help students overcome the difficult point. I propose that teachers' productive mathematical noticing occurs when they are able to:

- attend to specific details related to the key point, difficult point or critical point that could potentially lead to new responses;
- relate these details to prior knowledge and experiences to gain new understanding for instruction (key point and difficult point);
- combine this new understanding to decide how to respond (critical point) to instructional events.

This characterization of productive mathematical noticing uses the 'three points' not only to direct teachers' attention to specific details of what they notice, but also to highlight the need to connect the critical point to the key point and difficult point.

METHODOLOGY

This paper uses data from a seven-week lesson study cycle situated in a Singapore primary school. Lesson study, as a collaborative inquiry approach, provides a means to make teachers' thinking during lesson planning "more visible" (Lewis, Friedkin, Baker, & Perry, 2011, p. 171). There are five key tasks in lesson study—developing a research theme; working, discussing and anticipating student thinking through mathematics tasks; developing a shared lesson plan; collecting data during observation of research lesson and conducting a post-lesson discussion (Lewis et al., 2011). In this paper, I report results drawn from the first three tasks corresponding to the lesson preparation phase of the lesson study.

Six mathematics teachers formed the lesson study group that explored the teaching of 'fraction of a set' for Primary Four students (aged 10). Five of the teachers have more than 10 years of teaching experience and the other has at least five years.

To facilitate productive mathematical noticing during lesson preparation, I introduced Yang and Ricks' (2013) Three-Point Framework to teachers and encouraged them to focus their discussion for each lesson study task on the specifics of these three points. The teachers discussed explicitly the key mathematical ideas they wanted to teach, and the associated "difficult points" from their readings, prior experience or observations of their own students. Next they focussed their discussion on possible approaches (critical points) that could help students overcome the difficulties and learn the key ideas, before they agreed on a teaching approach. Teachers then designed the main task and anticipated students' possible responses to the task in relation to the points raised.

Finally, the team prepared a shared lesson plan containing the lesson sequence, key tasks, anticipated students' responses and planned teachers' responses to students.

The researcher primarily took on the role of observer during the seven lesson study sessions, and served as a resource person for the mathematical knowledge for teaching, while Ms Kirsty (a pseudonym), the team leader, was facilitator. Data were collected and generated through voice recordings of the lesson study sessions and video recording of the lesson. The recordings were parsed and segmented into episodes, as determined by the goal of the conversation. The findings were developed through identifying categories, codes and themes related to what teachers noticed in the episodes. The episodes were then classified as more or less productive using the framework above. Noteworthy episodes were further developed into vignettes to highlight the characteristics of more and less productive mathematical noticing.

RESULTS AND DISCUSSION

Focussing on the data drawn from the first four sessions on lesson preparation, the language of the 'Three-Point Framework' seemed to have helped teachers attend to specific key points, difficult points, and critical points related to the topic.

Less productive noticing

During the second session, Mr Anthony went through how the textbooks present a diagrammatic representation of 2/3 + 1/4 by showing two diagrams with 12 equal parts each. Mr Anthony then highlighted that the reason for the 12 parts was not obvious to the students.

> Mr Anthony: So the children will ask, why do you give me 12 equal parts? Why didn't you give me 6 or 18 equal parts? So, Ah… we look at the multiples of 3, 6, 9, and so on… at the end, we have 4, 8, 12… Coincidentally, we find just the lowest common multiple, so we have to use 12.

Here, Mr Anthony attended to a specific mathematical detail (key point) that might present new possibilities in the approach. He was also very specific with regard to students' difficulties—that they did not understand why 12 parts were used in the fractional representation (difficult point).

When asked how he would helped them to bridge this gap, Mr Anthony recounted:

> Mr Anthony: No choice… Because they are not in the same family, we want them to do some transaction, or you want to mix them together, we need to do something alike.

Furthermore, he highlighted that students often just latch on to the procedure:

> Mr Anthony: They will tell me this: My teacher tells me this… you multiply me and I multiply you. [Laughter] So, if the question is not that big, some times they are given 5/6 and then 4/9. They start to multiply 9 with 6 and 6 with 9…Yeah! That's right! And the numbers get bigger and bigger… Then they don't know how to do.

The rest of the teachers in the team also agreed with Mr Anthony that the problem was common among students. However, the teachers did not explore this difficulty further and attributed the difficulty to a lack of procedural competency in finding the lowest common multiple:

Ms Kirsty: Because they fail to understand the factors and multiples well. They don't know the least common multiple.

Ms Regina: They don't know how to list and find the lowest.

Mr Jeff: This is like the easy way out.

The teachers thought that students could not find the lowest common multiple, but did not suggest why this was the main issue. It seemed that the crux of the problem was the reason behind the 12 equal parts instead of finding the multiple 12. However, they attended to specific key and difficult points, even though they did not reason and make sense of the difficult point to arrive at a possible approach (critical point).

Productive noticing with reasoning and justification

When discussing students' difficulties in learning about a fraction of a set, Mr Jeff highlighted that students' difficulty in understanding fraction of a set could be due to a 'met-before' (Tall, 2004) of the notion of fraction as 'part of a whole':

> I think the objective for fraction of a set is for students to see, to interpret fraction as part of a set of objects. Previously, the fraction [concept] they learnt is more of part of a whole. They are very used to thinking about part out of a whole. Now that we give them a lot of whole things, they cannot link that actually these fractional parts can refer to a set of whole things also. So I think, to me, I feel that the connection that is missing, is that, how this fraction concept—which is part of one whole, which they have learnt so far—can be linked to whole things. For example, previously we used to teach fractions as parts of a cake or pizza. From that, how can it be that we have many pizzas, we don't cut out the pizza, there is a fraction of the pizzas. I think they cannot make a link there.

Mr Jeff elaborated further what he meant:

> For me, the main difficulty is to relate part of a whole into items that are "whole" but you take a fraction out of it. So, I think that's where the confusion comes.

He went on further to give a more concrete example:

> For example, if you say ¾ of the cats are… [Imitating the students] Ah… you cut the cat into three quarters? [Laughter] Cut each cat into four parts. So, yeah, but based on what they learnt so far, that may be the first thought they might have. To them, fraction could still be cutting up into parts. Whereas, fractions of a set, we leave the things as a whole entity but we look it as a collection of things. So out of these five things, how many are blue etc… For me, that would be the main difficulty.

In this short exchange, Mr Jeff clearly identified the need to extend the notion of fraction to a set of items (key point). He was also able to attend to the expected difficult point with a good level of specificity. Mr Jeff linked students' difficulty with a met-before of fraction—'part of a whole'—and suggested how students' image of fractions as ‘parts’ might conflict with the concept of fractions referring to a subset of ‘whole’ objects. However, unlike the first vignette, Mr Jeff used two examples—pizza cutting and cat cutting—to illustrate students’ difficulties. His use of specific examples strengthened what he attended to, and how he made sense of his prior experiences with students. Therefore, the link between the key point and difficult point was made explicit for the other teachers, and this provided an impetus for other teachers to notice students’ thinking. Hence, Mr Jeff’s noticing of students’ possible difficulty had productive potential for enhancing students’ thinking because it helped other teachers to focus their attention during the design of the task.

Besides directing teachers’ attention to the three points, Mr Jeff’s productive noticing also heightened other teachers’ sensitivities to students’ thinking when they were teaching. For example, Ms Kirsty became more cognisant of her students’ difficulties in grasping the concept and related what she attended to during another planning session.

Ms Kirsty:	And I think what we said is very right. They are not equating this concept of fraction as being the relationship between the part and its whole.
Mr Jeff:	As in, the object being the whole, right?
Ms Kirsty:	Not the fraction… part… and… what.
Researcher:	Part of a whole?
Mr Jeff:	… the number of whole things?
Ms Kirsty:	Part of a whole… not as relationship between a part and its whole… but as part of a whole. They are still with the impression of ‘part of a whole’.
Mr Jeff:	Actually the item that we use must be something that we cannot cut out one… like cars… tables… chairs

Ms Kirsty’s observations resonated with Mr Jeff’s noticing of student thinking about fraction of a set, and this later advanced the design of the task. Noticing is “validated” when others recognise that what is being noticed corresponds to their own experience (Mason, 2002, p.93). This validation heightens one’s sensitivity to notice, and promotes the possibility of improving practice (Mason, 2002). Mr Jeff’s reasoning based on his noticing also seemed to provide some justification for the proposed approach or response to students’ difficulties: Mr Jeff suggested using items that cannot be “cut” to help students get over the ‘part of a whole’ image of fractions. Moreover, when Mr Jeff was asked about a possible approach to help students understand the concept, he suggested an approach that made explicit links between the key point, difficult point and critical point:

> I think the confusion part also comes when… for example… this example here… we tell that … ¼ of the cups are yellow and then the answer is 4 cups. Huh… ¼ and then why got 4 in the 1/4? They cannot link between the… the ¼ in their mind is still ¼ of a whole… and then there is this four cups, four whole things… and so they cannot link… I was thinking whether we can put it into… something more familiar because… eh… they have learnt

> models, how to represent questions in model also, so, I was just looking at this… instead of just doing this, could we box the whole thing up instead. And to them, they are familiar with the part-whole model… a whole box is a whole… so while keeping the items inside and we draw the box… and… and… yes… we tell them that this looks familiar, and it looks like the model as a whole, right? These lines can be the partitioning of the whole model. While doing that… they can still see that the 4 items are still inside the parts. I don't know whether that can help them to make the connection that if this one box [partition] is ¼ of the whole, inside that box, I have four things. And this is where the 4 came from?

Mr Jeff's suggested approach (critical point) was directly linked to students' image of ¼ as 'part of a whole' (difficult point). Mr Jeff attempted to use the part-whole model, which the students were familiar with, as a scaffold to help students see that there could be 'whole items' inside a 'part'. This provided a bridge for students to extend their notion of fractions by emphasising fraction as a means to express the relationship between a part and its whole (key point).

What distinguished Mr Jeff's noticing as more productive was not the workability of the approach suggested, but rather the justification that reinforces the alignment between the three points. Justifying based on what was noticed not only helped the teachers maintain their attention on specific key and difficult points, but also lessened the likelihood of generating a critical point that does not provide opportunities to enhance students' reasoning.

CONCLUSION AND IMPLICATIONS

Productive mathematical noticing brings to the minds of teachers different ways to respond during teaching, and this can potentially improve the teaching of mathematics. This study highlights how processes of noticing can be incorporated into lesson planning. The findings suggest that the construct of productive noticing can be used to analyse teachers' noticing during lesson preparation. Moreover, teachers' noticing seems to be more productive when it goes beyond the specificity of what teachers notice about the three points, to include justification as a means to strengthen the linkages between the three points. The ability to notice productively during lesson preparation is important because it sensitises teachers to think about what to teach, students' possible misconceptions, and ways to deal with these problems. Further research is needed to characterise productive noticing more rigorously, and more work is needed to show how this construct can be applied to teacher noticing during and after instruction. Nevertheless, this study brings out the value and potential of productive noticing to improve teachers' practice.

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MODALITIES OF RULES AND GENERALISING STRATEGIES OF YEAR 8 STUDENTS FOR A QUADRATIC PATTERN

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This paper reports on the performance of 167 eighth graders in Singapore making generalisations for a quadratic figural pattern presented in a non-successive format. Data were collected through administering a written test in which the students had to establish the functional rule underpinning the pattern. The findings revealed that the students constructed a variety of functional rules, expressed prevalently in symbols using a range of generalising strategies, some of which were novel in the literature.

BACKGROUND

Most generalising tasks used in pattern generalisation research involve linear rather than quadratic patterns. The quadratic patterns typically depict the widely-recognised square and triangle numbers (see Steele, 2008). Moreover, the patterns are all too often presented in the form of a successive sequence of numerical terms or configurations. The generalising strategies that students employ to formulate a rule for predicting any term of a linear pattern are well established. However, if the rule were to change from a linear to a quadratic relationship, would the strategies that students engaged in the former case change to suit the latter? What types of rules would the students then establish for the latter? To gain more insights, an empirical study was conducted on a group of Year 8 students in Singapore to examine how they construct the rule underpinning a quadratic pattern presented in figural form. Specifically, this paper addresses these research questions: What are the different forms of rules that the Singapore students formulate for a figural quadratic pattern? What is the modality of the rules that the Singapore students formulated? What are the generalising strategies employed by the Singapore students in formulating the quadratic rule?

THEORETICAL FRAMEWORK

Students are often asked to construct a rule to describe the pattern structure that they see in a generalising task. Their rules take on mainly two forms: *recursive* and *functional*. The *recursive* rule allows the computation of the next term of a sequence using the immediate term preceding it whereas the more powerful *functional* rule refers to the rule expressed as a function that computes the term directly using its position in the sequence. Consider the linear task comprising a square made of four matchsticks in Figure 1, a row of two squares made of seven matchsticks in Figure 2, and a row of three squares made of 10 matchsticks in Figure 3. A recursive rule for this matchstick task could be expressed as "*add three to get the next term*" and its functional rule in closed form is $3n + 1$.

The functional rules are often represented in three different modes: purely symbolic (S), purely in words (W), and in alphanumeric form (SW). These different modes of representation are referred to as the *modality* of the rules. The functional rule for the matchstick task above, $3n + 1$, is expressed entirely in symbols. This rule can be stated wholly in words as: *add one to three times the number of squares*. Written alphanumerically, it can take the form: $3 \times number\ of\ squares + 1$. Stacey and MacGregor (2001) reported that nearly half of their sample of 2000 Australian students in Years 7 to 10 described the functional rule underpinning a pattern in words. Mavrikis, Noss, Hoyles and Geraniou (2012) noted a student using the alphanumeric form in their study.

The wealth of research on students' generalising strategies suggests that students use a variety of strategies to derive the rule connecting the term and its position in a pattern. Bezuszka and Kenney (2008) identified three numerical strategies: (1) *comparison*, where the terms in a given number sequence are compared with corresponding terms of another sequence whose rule is already known, (2) *repeated substitution*, where each subsequent term in a number sequence is expressed in terms of the immediate term preceding it, and (3) *the method of differences*, which is an algorithm for finding an explicit formula when the pattern is derived from a polynomial.

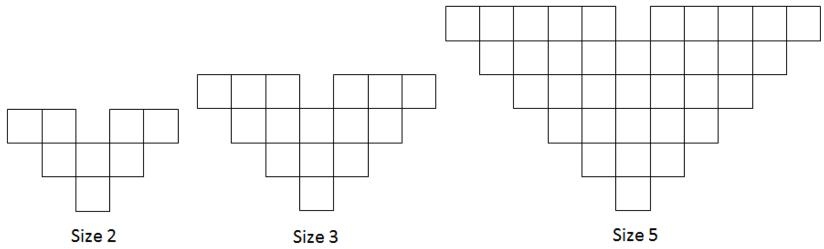
Different categories of *figural* strategy have also been identified. Rivera and Becker (2008) distinguished between (1) *constructive generalisation*, which occurs when the diagram given in a generalising task is viewed as a composite diagram made up of non-overlapping components and the rule is directly expressed as a sum of the various sub-components, and (2) *deconstructive generalisation*, which happens when the diagram is visualised as being made up of components that overlap, and the rule is expressed by separately counting each component of the diagram and then subtracting any overlapping parts. Chua and Hoyles (2010) introduced two further strategies into Rivera and Becker's (2008) classification scheme: *reconstructive*, which involves rearranging one or more components of the original diagram to form something more familiar, and *figure-ground reversal*, which entails augmenting the original configurations to become part of a larger composite configuration.

METHODS

167 Year 8 students (89 girls, 78 boys) of average learning abilities from three secondary schools participated in the study. The students had to complete two linear and two quadratic generalising tasks in 45 minutes and were asked to produce the functional rule in each task. Only one of the quadratic tasks, *Tulips*, in Figure 1 below is discussed here. Prior to participating in this study, the students had learnt the concept of variables and the topic of number patterns in the Singapore mathematics curriculum. They should also be far more familiar in dealing with linear patterns than with quadratic ones, which are less commonly featured in their mathematics textbooks.

Tony used identical square tiles to create flower designs of different sizes for his art project.

The diagrams below show two flower designs he made.



As the size number became larger, more square tiles were used.
Tony wanted to find the number of square tiles he had to use to make any size.
He used a rule to find this number.

Figure 1: Tulips

The *Tulips* task was deliberately designed to depict the pattern with three non-successive configurations starting with Size 2 and made less structured without any part questions that gradually led students to detect and construct the general rule. This was to allow the students a greater scope for exploring the pattern structure so that we could then see how they recognised and perceived the pattern without any scaffolding.

All the student responses for the *Tulips* task were analysed comprehensively to identify the types of rules produced and the generalizing strategies used. Several types of equivalent functional rules were observed and those with similar structure were collapsed into the same category after further examination, thereby developing the coding scheme for the types of rules. When two or more equivalent expressions of the functional rule were seen in a student response, the initial one, albeit simplified to another form subsequently, was coded. The rules were also coded for their modalities. The coding scheme for generalising strategies relied on *a priori* ideas drawn from different sources, including, mainly the research literature and our observations made during the analysis of the student responses. The generalizing strategy of every student was matched with the available codes and when it was not found to match any, a new code was created. Some student responses were subsequently selected and passed to a mathematics teacher for coding. After the inter-rater reliability was established, the frequencies of each type of rule and each type of generalising strategy were then determined.

RESULTS

93 students (56%) produced a correct functional rule for *Tulips*. Another nine students identified the first differences between consecutive terms correctly but only six of them articulated the recursive rule successfully.

Types of functional rules

Nine categories of different but equivalent expressions of quadratic functional rules were constructed, as shown in Table 1. The rules display variation in the mathematical operations used to join different terms together, involving both addition and subtraction. For instance, $n^2 + 2n$ illustrates the sum of two terms whereas $(n+1)^2 - 1$ exemplifies the difference of two terms.

	Rule Modality				Rule Modality		
Rule type	S	W	SW	Rule type	S	W	SW
$n(n+2)$	40	5	5	$n(2n+1) - n(n-1)$	1		1
$n^2 + 2n$	22	1	2	$(2n+1)(n+1) - n(n+1) - 1$	1		
$n + n(n+1)$, $n + (n^2 + n)$, $n + \frac{2n(n+1)}{2}$	8		1	$(n^2 - 1) + (n+1) + n$	1		
$3n + n(n-1)$	2	1		$n + 2[n + (n-1) + (n-2) + \cdots + 3 + 2 + 1]$			1
$(n+1)^2 - 1$	1						

Table 1: Rules and their modalities

The two most common functional rules are $n(n+2)$ and $n^2 + 2n$. Figure 2 below illustrates how Student A established $n(n+2)$ by means of producing the missing Size-4 configuration and rearranging it into a 4 by $(4+2)$ rectangle, followed by recognising the link between the dimensions and the size number.

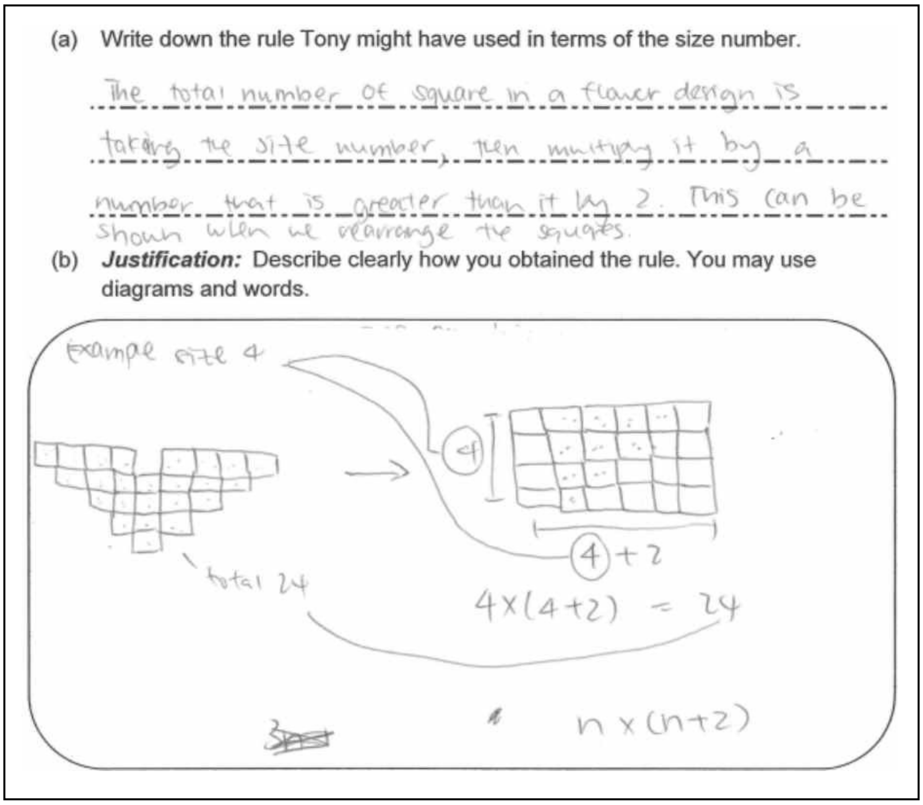


Figure 2: Functional rule $n(n+2)$

In Figure 3 below, Student B generated the rule, $n(2n+1)-n(n-1)$, by first shifting the bottom-most single tile to fill the gap in the top-most row, then imagining the resulting configuration as being formed by removing staircase-shaped tiles from each corner at the bottom left and bottom right of a "perfect" rectangle with dimensions $(2n+1)$ by n. The two sets of staircase-shaped tiles that are removed can be joined to form a rectangle of dimensions n by $(n-1)$, hence the rule.

The rule, $n+2[n+(n-1)+(n-2)+\cdots+3+2+1]$, is worth highlighting even though it occurred only once in this study. Although it describes the structure underpinning the pattern, it is not *algebraically useful* in Lee's (1996) language. This is because it does not allow the direct computation of the output when given an input.

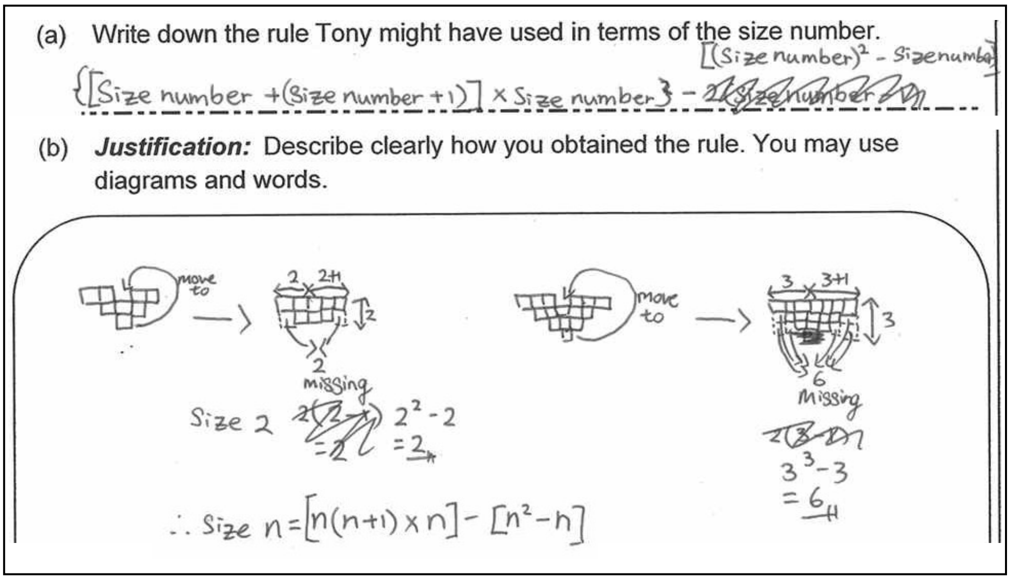


Figure 3: Functional rule $n(2n+1)-n(n-1)$

Modalities of rules

Three categories of modalities were identified, as indicated in Table 1. The functional rules were articulated predominantly in symbols, whilst the word and alphanumeric modes of representation were seldom used. Student A expressed the rule correctly in words and in symbols, thus the more sophisticated symbolic form was considered. Similarly, Student B also articulated the rule in two different forms: symbolic and alphanumeric, but the latter was considered because the former was incorrect (Note: $n(n+1)\times n$ should have been $[n+(n+1)]\times n$).

Generalising strategies

Eight different strategies were used, the most common being what we call a *combo* strategy involving both the *constructive* and the *comparison* strategies (see (d) below). Descriptions of the various strategies, excluding guess-and-check, now follow.

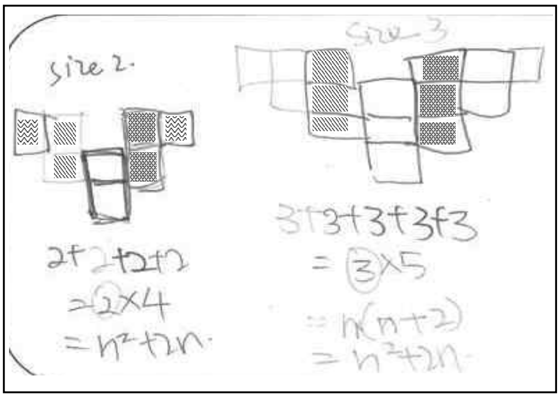
a. ***Grouping***. In Figure 4(a), the size number is used to generate the number of groups of tiles in each configuration: for instance, there were four groups of two tiles in Size 2, and five groups of three tiles in Size 3. Hence, there are $(n+2)$ groups of n tiles in Size n, or $n(n+2)$ tiles.

b. ***Reconstructive***. Figure 2 exemplifies this strategy where the original configuration is rearranged into a rectangle of dimensions $(n+2)$ by n.

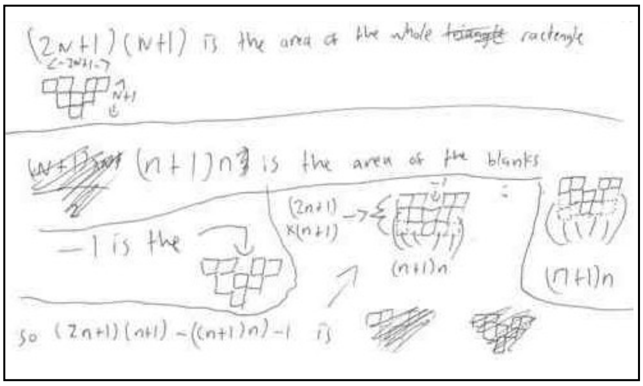
c. ***Figure-ground reversal***. The original configuration is visualized as being formed from a $(2n+1)$ by $(n+1)$ rectangle with two step-shaped components removed from its bottom-left and bottom-right corners alongside a tile in the top-most row. Given that the two step-shaped components can be repositioned to form a n by $(n+1)$ rectangle, the rule is thus $(2n+1)(n+1)-n(n+1)-1$ (see Figure 4(b)).

d. ***Constructive-comparison combo***. In Figure 4(c), each configuration is first viewed as comprising two non-overlapping parts: the top-most part made up of two rows, and the "step pyramid" (i.e., the *constructive* strategy first). The number of tiles in each "step pyramid" is then worked out and compared with the square numbers (i.e., the *comparison* strategy next).

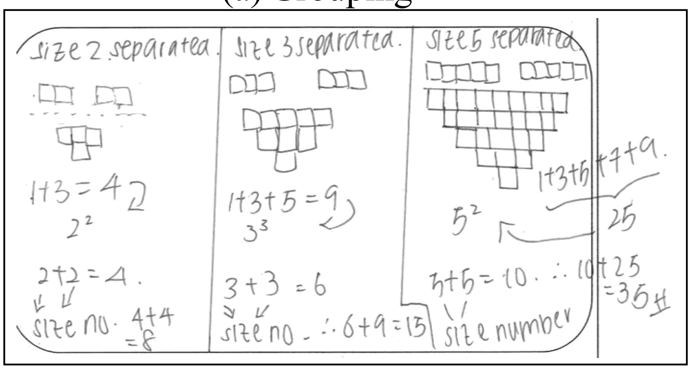
e. ***Constructive-reconstructive combo***. As Figure 4(d) shows, the discernment of the pattern begins with separating the original configuration into the "stalk" and "petals" (i.e., *constructive* first), then rearranging the "petals" into a rectangle before combining it with the "stalk" to form a larger rectangle (i.e., *reconstructive* next).



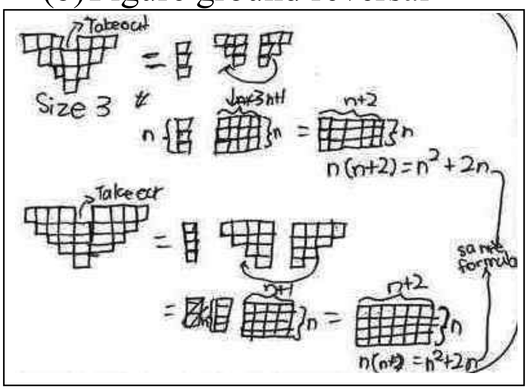
(a) Grouping



(b) Figure ground reversal



(c) Constructive -comparison



(d) Constructive-reconstructive

Figure 4: Generalising strategies

f. ***Reconstructive-constructive combo***. This strategy is similar to (e) except the order of applying the strategies is switched around.
g. ***Reconstructive-figure-ground reversal combo***. Figure 3 illustrates an example involving the repositioning of a tile (i.e., *reconstructive* first) followed by envisioning the resulting configuration being cut out from a larger rectangle (i.e., *figure-ground reversal* next).

DISCUSSION AND CONCLUSION

It is a fact that making generalisations for a quadratic pattern challenges secondary school students (see Jurdak & El Mouhayar, 2014; Steele, 2008). In Singapore, quadratic patterns are rarely used in mathematics textbooks. Moreover, with the *Tulips* pattern presented in a non-successive format, the task of finding a general rule might be even more testing. It is therefore surprising, yet encouraging, to see the students achieving moderate success in *Tulips*. A key to their success in detecting the inherent pattern structure lies in their recognising the need to use the size number as a generator of the term-to-position relationship.

The prevalence of functional rules expressed in symbols in *Tulips* stands in contrast to previous results by Stacey and MacGregor (2001). The fact that many Singapore students could develop the rule as an algebraic expression indicates that the concept of variables is generally well understood, a result of their prior experience with algebra where the teaching of number patterns follow the introduction of variables.

A marked observation to emerge from the analysis of the generalising strategies used in *Tulips* is the lack of *repeated substitution*, a common strategy for linear tasks. Using this strategy to generate the quadratic rule is not as straightforward as one might expect and students favouring it might have faltered and did not know how to employ it when the first differences of the pattern were not a constant, like in *Tulips*. Another remarkable finding is the use of certain strategies that are hardly described in the literature: *grouping* and the *combo* strategies such as the *constructive–reconstructive* and *constructive–figural-ground reversal* strategies.

To conclude, most studies on pattern generalisation have been undertaken in the west, offering a vast knowledge of students' generalising abilities and strategies. We hope this paper provides new insight into how Asian students visualise, think and reason about patterns, and opens the door for comparisons and future research.

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THE DEVELOPMENT OF THE VISUAL PERCEPTION OF THE CARTESIAN COORDINATE SYSTEM: AN EYE TRACKING STUDY

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The aim of the research is to investigate the transformation of the perception process through mathematics education, by an example of scanning the Cartesian coordinate system in order to locate a target point. We compared participants with different competence in mathematics. Historically, motion along axes appeared as a specific "theoretical" action that constituted the Cartesian coordinate system. We detected this specificity as a dominance of vertical and horizontal saccades of eye movements in perception processes of all groups of participants. Experts-novices differences dealt with an ability of experts to use additional essential information and to discard unnecessary data. Furthermore, a lot of evidences of shortening of the perception actions from novices to experts are presented.

THEORETICAL FRAMEWORK

The theoretical background of this work is based on the culture-historical tradition and an activity approach in its application to the development of perception, which are based on the dialectical-materialistic philosophy. From this point of view the process of perception should be constituted in accordance with scientific theoretical understanding of a represented object. Davydov supposes that the correct perception of visual models should appear by developing "special *object-related actions* by which they [students] can disclose in the instructional material and reproduce in models the essential connection in an entity" (Davydov, 1972/1990, p. 174). These special actions have been elaborated through cultural-historical development of the represented object and they, according to Davydov, need to be approached by a child through specially constructed educational activity.

Radford, following Marx, also assumes that an eye as a receptive organ should be converted to a "theoretician" and then it would be able to perceive scientifically essential features of figures. Radford (2010) supposes that this transformation occurs due to participation of a student in spontaneous but cultural forms of activity in classroom, which includes gestures, voice intonations and other embodied aspects of learning: "the senses … *become* shaped in certain historically formed ways as we engage in sociocultural practices" (Radford, 2010, p.2). So, by one or another way, education transforms the perception process of a student into historically elaborated system of actions, which allows detecting essential features of a visual model.

Let us now turn to some historical information in order to trace the main steps of the formation of the Cartesian coordinate system and to reveal the transformations of

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perception that are needed to approach this mathematical visual model. The history of the Cartesian coordinate system had started long before R. Descartes in ancient time from the practical usage of a rectangular grid on the plane in astronomy and in geography independently (Jushkevich, 1970, p. 98). The Cartesian system of coordinates appeared in mathematics due to attempts to find analytical descriptions of geometrical curves by Fermat and Descartes (Jushkevich, 1970). An idea was that the distance from one given line to a point of a curve could be counted by measuring the distance *along* another selected direction. Both Fermat and Descartes drew only one axis and the direction of another one, which usually wasn't perpendicular to the first one. Gradually, a rectangular system of coordinates became more popular.

Another important difference from the modern Cartesian coordinate system was that Descartes used only positive numbers; directions of axes could differ from one illustration to another. Fluent usage of both axes and negative coordinates appeared in 18 century (Jushkevich, 1970; Burton, 2011).

So, we suppose that the specific action, which is needed to perceive the Cartesian coordinate system, is a motion along one of the axis in order to find a distance from zero-point to a projection of a target point. Another important step towards correct perception of coordinate system is an ability to find a correspondence between a positive or negative value of a coordinate and an axis orientation.

This study claims that a child needs to acquire the specific ways of perception that were elaborated in the history of mathematics, during his educational practice. There are *three stages* of perception development distinguished by investigations framed in activity theory (e.g. Zaporozhets, 1986/2002). The first stage includes external, material actions with objects; for example a child could run along the axes by an index finger. The second stage reflects deployed sensory processes in which perceptual actions "are performed with the aid of motions of receptor apparatuses and anticipate subsequent practical actions" (Zaporozhets, p. 41). At this stage we should find a movement of eyes by the same route as fingers run at first stage. The third stage is a stage of most mature perception, the stage of shortening and automation. "Orienting-research action transforms into ideal action, into the movement of attention across the perception field" (p. 42), writes Zaporozhets.

Our *research question* dealt with the transformation of the perception process by mathematics education: whether indeed matured perception of Cartesian plane includes specific "theoretical" actions, revealed in this research in our historical analysis, and whether these kinds of actions become dominant in perception of highly mathematically educated respondents. We also investigated a shortening of perception actions proposed by psychologists of activity approach, as a way of transformation of external perceptual actions into mental ideal actions.

METHODOLOGY OF RESEARCH

Eye-tracking methodology appears more and more often in educational researches especially in area of multimedia learning, including multi-representational materials (van Gog & Scheiter, 2010). Only a few papers are devoted to mathematics education. Most of these papers are framed by a semiotics paradigm and analyze perception of representations as signs of mathematical objects. They calculate numbers and describe kinds of saccades between representations and compare number and duration of fixations in different area by participants with different level of mathematics competence. As Andra et al. write (2013), all kinds of analyses could be distinguished to macrolevel (analysis of shifting between representations, e.g. Andra et al., 2009; Andra et al., 2013), mesolevel (analysis of attendance of each representation by a gaze) and microlevel (analysis of activity inside of one representation, e.g. Epelboim and Suppes, 2001; Peter, 2010).

Our research is focused on microlevel analysis since a development of perception, which doesn't supposed by classical semiotics perspective, could be understood only through deep analysis of interaction of a subject with a visual representation. The most well founded fact is that experts are able to detect significant parts of representations (thus more fixations by experts were found in these areas in comparison with novices' fixations). It was shown for such knowledge domains as sports, medicine, transport (Gegenfurtner, 2012), zoology (Jarodzka, Scheiter, Gerjets & van Gog, 2010), meteorology (Canham & Hegarty, 2010) and others. In mathematics there are evidences that experts are able to focus on a blank area, which is essential for additional constructions in geometry (Epelboim & Suppes, 2001).

From cultural-historical point of view, all evidences of magnetism by essential parts of a picture or a text for experts could be interpreted as a reorganization of the perception process in accordance with their deeper theoretical knowledge. In our research we investigated if experts are able to choose an appropriate quadrant of the Cartesian plane faster than novices. We supposed that theoretical knowledge about negative or positive coordinate of a target point would influence on the perception process of experts to create an ability to use this information in a search for a point.

Applying ideas of the activity approach we were focused on a procedural aspect of perception trying to understand perceptual actions. We supposed that directions of saccades reflect cultural way of approaching the Cartesian coordinates system: saccades should be performed along axes. So vertical and horizontal saccades should prevail on any other directions if perception is reorganized in a theoretical way.

Participants

In our research we compared eye movements of participants of 3 levels of mathematics competence. There were 11 participants with higher mathematics education, 23 students of a first year of non-mathematical departments at University (they have passed school mathematics exam), 10 students of 9-11 grade of high school (14-16 years old). We will refer to these groups as experts, intermediates and novices.

Apparatus

For data collection we used SMI RED eye tracker with a sampling frequency of 120 Hz, participants seated at approximately 40-50 cm distance from a monitor. IViewX was used for tracking eyes' activity. Stimuli were presented by Experiment Center 3.0. Data analysis was conducted by Begaze 3.1 and SPSS 20.0. Before the main experimental procedure, eye tracker was calibrated for an each participant by 9-points procedure with validation. Only those participants who showed better than 0,5 degree accuracy at the calibration stage were accepted for main experiment.

Materials and procedure

Each participant had to solve 9 tasks on detection a point on the Cartesian plane with determined coordinates. There was an instruction at the beginning of experiment: "Now you will receive tasks on the Cartesian coordinate system. Try to solve these tasks as accurate and as fast as you can." Each task consisted of three slides: 1) a task with the coordinates of a point, 2) the Cartesian plane with two axis and four labeled points, 3) labels of the points to choose a correct answer. All tasks had an equivalent wording, for instance: "Choose a point with coordinates (3, -4)". There were one or two points (of four) in target quadrant of the Cartesian plane. Participants switched from one slide to the next by pressing Space bar. There were no time limits either for reading of a task or for searching for a point.

Hypotheses

1. Vertically and horizontally directed saccades are prevailed on saccades with other directions. This ratio is more pronounced for experts than for novices.
2. A number of fixations in irrelevant quadrants of Cartesian plane decrease with growing of participant competence.
3. Perceptual actions lessen with growing of competence: the better participants are educated, the shorter are their gaze paths, and the more the number of their fixations is reduced, and the durations of their tasks solving become shorter.

DATA ANALYSIS AND RESULTS

First part of analysis dealt with directions of saccades. The problem is that the standard algorithm of saccades detection implemented in Begaze 3.0 defines a saccade as a vector from the center of an initial fixation to the center of an ending fixation. Thus our observations of raw eye-movements showed that directions of many saccades were calculated incorrectly due to significant drifts during fixations (Figure 1 gives an example of raw data). We elaborated our own software that detected saccades by simple Velocity Threshold algorithm (Salvucci & Goldberg, 2000). Eye movements were considered

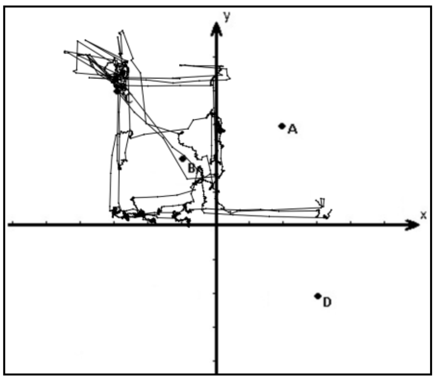

Figure 1: An example of raw eye movements by experts

saccades when velocity exceeded 120°/sec. Therefore, in respect to fixations, a saccade appeared as a vector from the point where the previous fixation is completed to the first point of the next fixation (instead of fixation centers). Saccade direction was computed as an angle from 0° to 90°. All saccade directions were divided into 6 sectors by 15 degrees: from a sector 0°-15° to a sector 75°-90°. Saccades of the first and the last sector were considered as horizontal and vertical respectively. The mean numbers of saccades of different directions were compared by repeated measures ANOVA with mathematics competence as a between group factor and saccade direction sector as within subject factor.

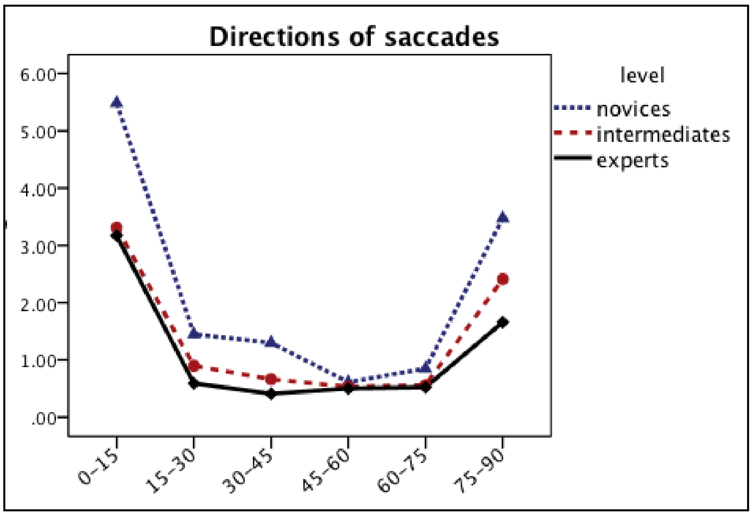

Figure 2: Directions of saccades in groups with difference competence

Saccades with vertical or horizontal orientation appeared approximately 4 times more often than those with directions from other sectors ($F=31.554$, $p<0.001$), see Figure 2. This ratio is stable across groups. In spite of this fact we found a significant interaction between factors ($F=4.225$, $p=0.021$): a dominance of vertical and horizontal saccades is most noticeable for novices, and intermediates also use vertical and horizontal saccades 4 times more often than all other saccades. But experts use only vertical, but not horizontal saccades as often as intermediates. Also it was shown that a total number of saccades decreases with mathematics competence ($F=5.446$, $p=0.008$); the result confirmed our third hypothesis about the shortening of the perception process.

Next part of analysis was dedicated to participants' ability to be focused on essential parts of a diagram. We defined six AOI (Aries of Interest): four quadrants of the plane and two axes. A target point belonged to one of the quadrants and this target quadrant could be figured out only by taking into account the sign (positive or negative) of the both coordinates. Three other (non-target) quadrants are irrelevant to the task. Number of fixations in irrelevant AOIs decreases with mathematics competence (Kruskal Wallis Test, $\chi^2=11.065$, $p= 0.004$). There were large individual differences but at the average expert did only 3.5 fixations (6% of all fixations) in irrelevant AOIs for the whole session of 9 tasks, intermediates and novices did 10.2 (14.8%) and 14.1 (15.3%) irrelevant fixations correspondently.

To investigate shortening of perceptual actions we compared 1) number of fixations, 2) length of the gaze paths, 3) total time to solve each task in different groups using repeated measures ANOVA with mathematics competence as between group factor and task as within subject factor. Means and statistics for all parameters for each group are presented in Table 1. All parameters significantly indicated the reduction of explicit perceptual actions from novices to experts. Also specific tasks significantly influenced the process of problem solving ($p<0.001$). Significant interactions between tasks and competence were found for all parameters ($p<0.03$). Below we'll consider only the number of fixations.

	Level of competence			Results of ANOVA	
Parameter	novices	intermediates	experts	F	p (sig.)
Time of solution (sec)	4.638	3.285	2.681	4.916	0.013
Number of fixation	14.02	9.8	7.54	5.794	0.006
Gaze path (px)	1810.2	1250.1	814.5	5.744	0.007

Table 1: Parameters of the shortening of perceptual actions (between group analysis)

Figure 3 represents mean number of fixations for each task in different groups. First task provoked the most explicit search by novices and intermediates, which was reduced in next tree tasks. Experts solved first task with the number of fixations comparable to all other tasks. In contrast to the previous tasks, tasks 5, 6, 7, 8 had two points in the target quadrant (one correct and one wrong). Figure 3 shows that the presence of an additional point influenced the search process in novices, while perception of intermediates and experts is kept as short as it was in tasks with the only one point in the target quadrant. Altogether, the results provide evidence for shortening of perception from one group to another and from the first task to the following tasks.

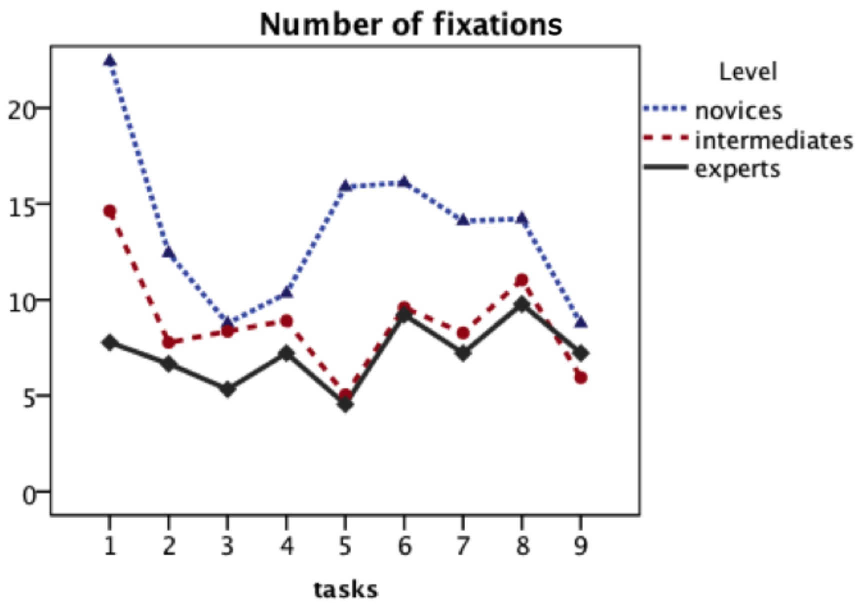

Figure 3: Reduction of fixations from group to group

DISCUSSION AND SOME CONCLUSIONS

The main result is that we have found an evidence of "theoretical" perceptual actions: vertical and horizontal saccades appeared much more often than saccades of other directions. We observed these specific actions in perception of participants with all levels of mathematics competence. It means that their perception had been transformed in a cultural way already. And that special system of tasks (as Davydov (1972/1990) claims) is not needed for it; but it could be acquired through classroom practices as Radford (2010) supposed.

Another possible interpretation is that this way of perception is a natural one for these tasks. In order to trace "non-theoretical" perception, where the main principle of motion along the axes is not approached yet, we intend to collect data from less experienced participants in our future work, and check if their perception is not vertically and horizontally organized. Only then will we be able to claim that perception was transformed to be "theoretical" as Radford anticipates (2010).

Davydov supposed that the transformation of the perception process is a result of theoretical understanding. Our results show that perception, which is structured by special actions (vertical and horizontal motions), could still be enriched by additional knowledge about negative or positive values of coordinates. Our Hypothesis 2 was confirmed: experts were almost never focused in irrelevant parts in comparison with other groups (6% for experts vs. about 15 % for other groups). Empirically the result repeats the evidences that experts are able to distinguish essential parts of visual representations (e.g. Gegenfurtner, 2012; Jarodzka et. al, 2010; Canham & Hegarty, 2010). But what is more interesting is that this result is similar to observations by Andra et. al (2009), that novices more often revisit different alternatives of answer than experts. From the activity theoretical point of view it means that experts conduct only executive actions, which lead to almost algorithmic solution, while novices need to perform orient-research activity to construct an image of a representation of the task and an algorithm how to perceive it (e.g. Zaporozhets, 1986/2002).

We also observed that the experts were able to solve the tasks using only necessary information, while missing additional data: there was a reduction of vertical saccades from intermediates to experts when horizontal saccades were performed with the same frequency (Figure 2). Indeed, the second coordinate wasn't necessary to choose a correct answer.

As it was expected (Hypothesis 2) we have shown that orient-research parts of actions are reduced in perception of experts (they had the less amounts of fixations, the shorter gaze paths and the faster solutions (Table 1). It is interesting that this difference was especially strong for the first task (see Figure 3). We can explain it as follows: in the first task the orient-research activity of novices and intermediates was unfolded and it allowed them to construct an appropriate algorithm of perceptual actions. This algorithm was applied in further tasks. But new elements in tasks 5-8 (see Figure 3) broke the perception process of novices and returned it to the stage two of orient-research actions (see above from Zaporozhets, 1986/2002), while perception of experts and intermediates kept its maturity.

In summary, an inclusion of special "theoretical" actions in perceptual process appears as only a first stage in the transformation of perception by education. The difference between experts and novices deals with the ability of experts to use additional essential information and to discard unnecessary data. Apart from this other evidences related to the shortening of perceptual actions from novices to experts was found. So, being culturally organised, perception continues its development in order to find the shortest and simplest way and at the same time to include theoretical information. Future investigation of less experienced participants is also necessary.

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SUPPORTING THE INTRODUCTION TO FORMAL PROOF

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In this study, a tool that worked to support teachers with the introduction to formal proof in geometry is discussed. The tool helped teachers navigate the "shallow end" of proof. More specifically, the tool was shown to support teachers with introducing and scaffolding proof. Findings from this study suggest that the tool may be useful for supporting formal reasoning in geometry as well as other areas.

INTRODUCTION

Considering the teachers' role in navigating the proof terrain, Herbst (2002) conducted an analysis of what is involved when teachers attempt to engage students in the production of a proof. He argued that alternative ways of engaging students in proving must be found if proving is to play, in the classroom, the same instrumental role for knowing mathematics that it plays in the discipline. Thinking about possible instructional alternatives for the reform-oriented classroom as an opportunity, Herbst (2002) stated: "The mandate to involve students in proving is likely to be met with the development of tools and norms that teachers can use to enable students to prove and to demonstrate that they are indeed proving" (p. 200). A primary goal of this paper is to describe and discuss the reported benefits of a tool that was developed in a research study whose aim was to better understand the challenges teachers faced when teaching proof in geometry. Following Smith and Southerland (2007), here "tool" references a teaching tool or guide that was used to help teachers envision a new way of teaching, in this case, mathematical proof. The research question addressed in this paper is: How can the mathematical proof tool (MPT) serve as a guide to support teachers' work of introducing proof in secondary geometry?

THEORETICAL PERSPECTIVE

Past research has shown that students have difficulty with proof at various levels in many parts of the world (Knipping, 2004). The finding that most U.S. students are not developing through the van Hiele levels at all (Fuys, Geddes, & Tischler, 1988), is problematic because it implies that students enter high school unprepared for the formal deduction required in many geometry courses (Clements, 2003). This is important because students must understand geometric ideas in the middle grades in order to be successful in subsequent mathematics experiences (Sinclair, Pimm, & Skelin, 2012), including secondary level geometry. Thus, there is an obvious need for this curricular gap to be bridged. However, some secondary teachers have claimed that they do not have strategies for teaching proof and even expressed the belief that you cannot teach someone how to develop a proof (Cirillo, 2011). This belief may be the

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reason that geometry is often thought of as the most difficult portion of school mathematics (Knuth, 2002).

Much of Herbst and colleagues' work has focused on classroom interactions and proving in geometry at the secondary level (Herbst & Brach, 2006; Herbst et al., 2009). For example, Herbst et al. (2009) described instances of student engagement with proof in various geometry courses in a high school. Through this work they unearthed a system of norms that appear to regulate the activity of "doing proofs" in geometry class. The authors contended that a collection of actions related to filling in the two-column form are regulated by norms that express how labor is divided between teacher and students and how time is organized as far as sequence and duration of events. They argued that despite the superficially different episodes in which doing proofs were observed, there were deep similarities among those events. The first 5 of 25 norms reported by Herbst et al. (2009) are listed below:

> …producing a proof, consists of (1) writing a sequence of steps (each of which consists of a "statement" and "reason"), where (2) the first statement is the assertion of one or more "given" properties of a geometric figure, (3) each other statement asserts a fact about a specific figure using a diagrammatic register and (4) the last step is the assertion of a property identified earlier as the "prove"; during which (5) each of those asserted statements are tracked on a diagram by way of standard marks …(pp. 254-255)

This model of the instructional situation of doing proofs in terms of a system of norms is helpful to those who wish to investigate what it might mean to create a different place for proof in geometry classrooms (Herbst et al., 2009).

The documentation of classroom norms is relevant here because it provides a frame for examining the alternative practices supported by the tool used by the teachers in this study. This study builds on the work of Herbst and colleagues by examining possibilities outside of these normative practices. It also takes seriously the call to bridge the curriculum gap by supporting students' development through the use of a teaching tool that has the potential to lead to new norms in geometry classrooms.

METHODS

To learn more about the challenges that teachers face when cultivating formal proof in their classrooms, a three-year study that made use of qualitative methods of inquiry, was designed. For the larger study, five teachers who had between one and ten years of experience with teaching proof in geometry were recruited. Baseline data, collected in Fall, 2010, included two non-consecutive weeks of classroom observations in one target classroom of each teacher. Beginning Spring, 2011, 20 professional development (PD) sessions were designed and implemented to attend to and reconsider the ways in which the study teachers taught proof. These sessions took place over the course of a year. In Fall, 2011 and 2012, additional data were collected to observe and understand changes made to the introduction and teaching of proof in geometry. Interviews designed to help the researcher better understand the data and the teachers' evolving beliefs about teaching proof were also conducted.

Data and Analysis

This paper draws on a subset of the teachers and the data described above. For this study, interview transcripts from two teachers' data sets were transcribed and analyzed. This includes a total of 4-5 interviews with each teacher, comprising a total of 3-3.5 hours per teacher spread across the three years of the project. Interviews were coded for instances where the teachers discussed how engagement with the PD and the tool influenced their practice. Data from two classroom episodes are also presented. The teaching episodes were purposefully selected because it was from these two classroom lessons that the idea to develop the tool grew. Last, a written curriculum developed by the two teachers over the second and third years of the project was analyzed. Together, this collection of data allows me to describe how the tool was developed and used over time, given the limited space provided here.

Setting

Participants for this study include Mike and Seth (pseudonyms) who, at the onset of the study, had eight and five years of mathematics teaching experience, respectively. Mike had previously taught a high school geometry course every year since he began teaching, while Seth had only taught the geometry course once. Mike and Seth taught in a private, all-boys school with a racially diverse population and small class sizes (14-17 students). During Year 1, they taught from a conventional geometry textbook, teaching Euclidean geometry proof primarily over the course of the first semester.

FINDINGS AND DISCUSSION

The findings in this study are explored through three data sources. Two excerpts from Mike's Year 1 baseline classroom data are presented. I then describe the Mathematical Proof Tool and explore its use in the classrooms through examples from the curriculum developed by Mike and Seth. Interview data is also included.

The "Shallow End" of the Proof Pool

In the first year of the study, project teachers were asked to invite the research team in when they first introduced formal proof. Before beginning the first proof, Mike said the following to the students:

> Here we go. So proofs are tough. You know one thing about proofs is, there's no easy way. There's no way to do it. There's no shallow end. You can't like wade into the proof pool. You gotta kind of jump right in the deep end with these tough ones. (11/2/10)

To this introduction, a student responded, "I would drown." The next day, Mike began the lesson by explaining how difficult proofs are:

> These proofs are really hard and I think I said last time a couple things. One, there's no real easy way to start proofs. It's not like algebra where you could start with easy problems and work to more difficult problems and then do really challenging problems. The proofs start, and they are immediately difficult and they are immediately unlike anything that you have ever seen before and that's okay. Alright, so you'll learn how to do 'em by sort of trying them. (11/2/10)

These two examples suggest that Mike seemed to hold similar beliefs to those of the teacher in Cirillo's (2011) study of a beginning teacher learning to teach proof in geometry. Like, Matt, despite his eight years of experience teaching geometry, Mike's introduction to proof provides evidence that he was at a loss when it came to scaffolding the introduction of proof.

The Development of the Mathematical Proof Tool

There were four important findings from the first year of data collection across all five project teachers. First, teachers did not understand all that was involved in teaching students how to develop and write proofs. Second, as was demonstrated, teachers did not know how to scaffold the introduction to proof. Third, teachers thought that the only way to teach proof was through show-and-tell. Last, a set of ideas that were implicitly taught during these show-and-tell presentations were found in the analysis of classroom observations. In particular, students must learn the following simultaneously: (a) postulates, definitions, and theorems; (b) how to *use* definitions to draw conclusions (c) how to work with diagrams (i.e., what can and cannot be assumed); (d) a variety of sub-arguments and negotiated classroom norms for writing them up; and (e) how sub-arguments come together to construct the larger argument. It was through these observations in conjunction with the consideration of Mike's claim that there is no shallow end to proof that the Mathematical Proof Tool (MPT) was developed. Based on the shallow end proof pool metaphor suggested by Mike, I hypothesized that perhaps there was a set of competencies that students needed in order to develop proofs that could be ramped up over time. The PD sessions and subsequent observations gave me a way to test that hypothesis.

In Spring, 2011, the group of teachers and the research team began meeting for PD sessions. In these sessions, the teachers participated in the following activities: discussing research and practitioner articles on proof and geometry, reflecting on practice through writing and watching teaching videos, participating in PD on classroom discourse, and considering alternative teaching approaches. After learning about van Hiele levels and coming to believe that their students were not ready to engage in proof, the MPT became a major focus of alternative teaching approaches.

The tool began as sample alternative tasks and evolved into the tool that is shown in Table 1. The MPT works as an instructional guide to support teachers by offering pedagogical content knowledge that breaks down the practice of proving. It unpacks the sub-goals of proof and identifies competencies that occur frequently and are necessary to make a lesson focused on proof go well. By the third year of the study, project teachers were using the MPT as a planning guide to make sure that they were addressing each of the sub-goals, and providing learning activities that would foster the competencies in their students. In the paragraphs that follow, I briefly describe the sub-goals and include examples of each. Examples come from the written curriculum that Mike and Seth developed around the sub-goals.

Sub-Goals	Description	Competencies
Understanding Mathematical Objects and Mathematical Notation	This sub-goal connects a definition and notation to a particular instance of that object.	1) Communicating a mathematical object by making use of spoken or written text
		2) Communicating or reading a mathematical object by making use of diagrams. Sometimes notation is used to mark these diagrams
		3) Communicating or reading a mathematical object by making use of symbolic notation
		4) Determining examples and non-examples
Understanding the Nature of Definitions	This sub-goal highlights the nature of definitions, their logical structure, how they are written, and how they are used.	1) Writing a "good" definition (includes necessary and sufficient properties)
		2) Knowing definitions are not unique
		3) Understanding how to write definitions as biconditionals
		4) Knowing you cannot prove a definition
Drawing Conclusions and Developing Conjectures	This sub-goal presents the idea of an open-ended task that leads to conclusions that can be drawn from given statements and/or a diagram.	1) Understanding what can and cannot be assumed from a diagram and recognizing that sometimes diagrams can be misleading
		2) Knowing when and how definitions can be used to draw a conclusion from a statement about a mathematical object
		3) Using combinations of postulates, definitions, and theorems to draw valid conclusions from some given information
		4) Developing conjectures that could be used to prove or disprove a mathematical statement where part of the process is making, testing, and refining conjectures as one works
Sub-arguments	This sub-goal presents the idea that there are common short sequences of statements and reasons that are used frequently in proofs and that these pieces may appear relatively unchanged from one proof to the next.	1) Recognizing a sub-argument as a branch of proof and how it fits into the proof
		2) Understanding what valid conclusions can be drawn from a given statement and how those make a sub-argument (e.g., knowing some commonly occurring sub-arguments)
		3) Understanding how to write a sub-argument using acceptabl notation and language (often negotiated with the teacher)
Understanding Theorems	This sub-goal highlights the nature of theorems, their structure, and how they are used.	1) If applicable, marking a diagram that satisfies a hypothesis
		2) Interpreting a theorem statement to determine the hypotheses and conclusion
		3) Rewriting a theorem written in words into symbols and vice versa
		4) Understanding that a theorem is not a theorem until it has been proved (using definitions, postulates, or previously proved theorems, lemmas, and propositions) and that one cannot use the conclusions of the theorem itself to prove the conclusions of that theorem (i.e., avoiding circular reasoning)
		5) Understanding that theorems are mathematical statements that are only sometimes biconditionals
		6) Determining the theorem proved when presented with a proof
		7) Understanding the connection between logic and a theorem, for example, how to write the contrapositive of a conditional statement and the connection between laws of logic and the hypothesis and conclusion of a mathematical statement

Table 1: Mathematical Proof Tool (MPT)

The first sub-goal, *Understanding Mathematical Objects and Mathematical Notation*, supports students in working with commonly used terms in geometry, for example, angle bisectors. Students need to know particular definitions since these (along with theorems and postulates) are what make up the substance of a proof. Understanding Mathematical Objects connects a definition and notation to a particular instance of that object. Mike and Seth made use of this sub-goal early and often in their first unit on definitions and constructions. For example, students were asked if it is possible to draw a picture in which $\overline{DF}$ bisects $\overline{PO}$ but $\overline{PO}$ does not bisect $\overline{DF}$. Students were expected to explain their answers. Students also worked with compasses, constructing medians and perpendicular bisectors, for example.

The second sub-goal, *Understanding the Nature of Definitions*, highlights the nature of definitions, their logical structure, how they are written, and how they are used. An example from the curriculum was: "Write the two conditional statements that comprise the biconditional: *Two angles are complementary if and only if their measures sum to 90 [degrees]*." Similarly, another problem asked the students to write out the complete statement in words: "Isosceles triangle ↔ 2 ≅ sides."

The *Drawing Conclusions and Developing Conjectures* sub-goal presents the idea of an open-ended task that leads to conclusions that can be drawn from given statements and/or a given diagram. This sub-goal is useful, for instance, in helping students understand what you can and cannot assume from a diagram. For example, you can assume vertical angles, but you cannot assume perpendicular lines. A benefit of explicitly attending to this sub-goal is that it helps teachers correct common errors students tend to make regarding the conclusions they draw from the given information before they begin developing formal proofs. An example of this sub-goal is provided in Figure 1.

The *Sub-arguments* sub-goal presents the idea that there are common short sequences of statements and reasons that are frequently used in proofs and that these pieces may appear relatively unchanged from one proof to the next. An example of a common sub-argument is a proof of the proposition: If lines are perpendicular, then congruent angles are formed. In the teachers' curriculum, after reviewing some common sub-arguments, students were asked to complete sub-arguments such as the one in Figure 2, justifying each claim with a reason.

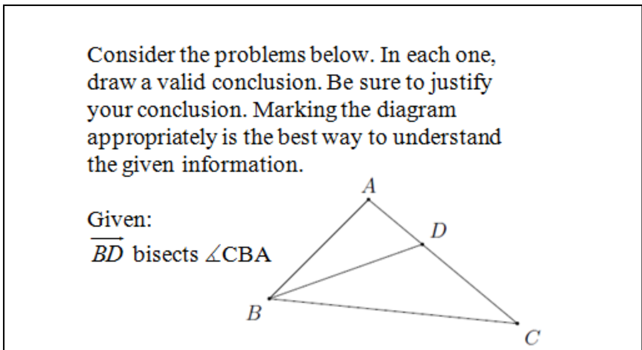

Figure 1: Sub-arguments Example

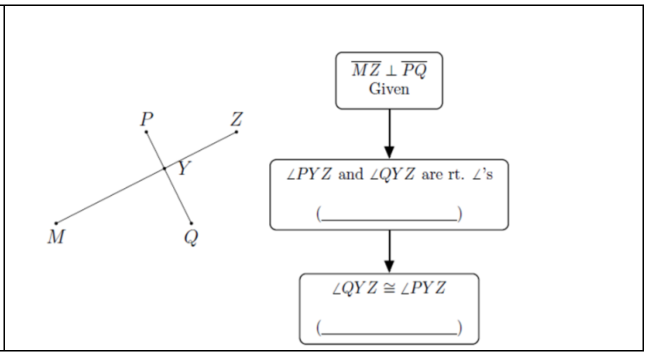

Figure 2: Drawing Conclusions Example

Last, the *Understanding Theorems* sub-goal highlights the nature of theorems, their structure, and how they are used. For example, rather than always providing students with a diagram, a given statement, and a conclusion to prove, students are asked to set up the proofs themselves. A sample problem from Mike and Seth's curriculum was as follows: "Set up the following statement to be proved: *If a figure is a parallelogram, then its opposite sides are congruent*."

Teachers' Reactions to the MPT Implementation

After the spring and summer PD that followed the baseline data collection in Year 1, Mike explained how one of the readings (see Cirillo, 2009) influenced his thinking about how he taught proof in geometry:

> One of the readings…was Ten Things I Wish I Knew, and I was like van Hiele levels, give me a break. I don't wish I knew that. But, I actually wish I knew that [laughing]. So one of my 'aha' moments is that we have to adjust the curriculum, adjust our approach so that we're communicating with our students. (Mike, 8/23/11)

In an interview during Year 2, Mike discussed the types of tasks he engaged his students with through the new curriculum that he started developing that semester (Seth later partnered with Mike in teaching with the new curriculum). Mike described the *Understanding Mathematical Objects* example provided above and said:

> I never would've asked this before. But just getting at the idea, you gotta look at what's bisecting what. There's a subject and an object there. Here was bisects but is not perpendicular. Perpendicular but does not bisect. Perpendicular and bisects. So does such a thing even exist? Oh perpendicular bisector. So now you come back here and construct a perpendicular bisector. (Mike, 10/6/11)

During an interview at the conclusion of Year 2, the first year of using the MPT, Seth explained the impact that the tool had on him and his students:

> The really big change was all that scaffolding that we built up to the proofs…which provided some of these comments [from students] like proofs were easy, you know, that was fun….I think back to my first year teaching proof. Straight agony….I probably, like when I took geometry, I sort of understood it myself…but I certainly didn't have a great grasp of how to teach it. I mean, as we said…I just threw it up one day, like here we go, we're gonna do a bunch of these and you have two options – you can either understand what's going on or you're gonna recognize that there's only about ten of them, like in different forms and you can probably, if you're good enough, you can memorize basically what's going on and survive. But…there's no takeaway from that. So I think the way we built it this year was remarkable in terms of their retention. (Seth, 6/5/12)

Like Seth, Mike also reported that he found that the tool supported him in teaching proof and supported his students in learning proof.

DISCUSSION AND SUMMARY

The tool described in this paper was developed in response to some of the findings related to the challenges of teaching proof in high school geometry. The tool was intended to scaffold the introduction to proof for the students. In contrast to the traditional teaching methods reviewed in the literature, the tool was intended to assist the project teachers with introducing proof to their students in a manner that did not feel like such an "abrupt transition" (Moore, 1994) into the deep end of the proof pool. The five sub-goals of the tool were intended to provide teachers with a support for teaching proof. Although this study only presents findings from two teachers using the MPT, these findings are promising because the teachers did more than just use the tool in a casual way. Rather, they saw enough potential in the tool use to develop a new curriculum around them, and they reported strong effects from their use. Additional research that explores the use of the Mathematical Proof Tool with additional teachers in varying contexts are warranted to determine if this tool can be used by teachers to improve the teaching and learning of proof, even potentially in other sub-areas of mathematics.

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ORDINALITY, NEUROSCIENCE AND THE EARLY LEARNING OF NUMBER

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Throughout the twentieth century there was debate as to the primacy of ordinality or cardinality in the development of the concept of number. Psychological experiments have largely given way to neuro-science in deciding this issue. There are results suggesting students' awareness of symbol-symbol relations is the best predictor of future mathematical attainment, which could be interpreted as meaning ordinality is the key awareness. This report draws on evidence from a recent project in primary schools in the UK that took an ordinal approach to learning number. One suggestion arising from this work is the potential educational power of a pedagogy based on developing an awareness of mathematical structure.

INTRODUCTION

In this report, I first set up the theoretical notion of an ordinal approach to learning number, partly drawing on neuro-scientific evidence. I then discuss possible educational implications, before reporting on an empirical study in Primary classrooms in the UK where, although the focus was not on ordinality, it is clear the approach taken to number was ordinal. I conclude with implications for further study.

ORDINALITY VERSUS CARDINALITY

Ordinality refers to the capacity to place numbers in sequence, for example, to know that 4 comes before 5 and after 3 in the sequence of natural numbers. Cardinality refers to the capacity to link numbers to collections, e.g., to know that "4" is the correct representation to denote a group of four objects. A significant question dealt with in the twentieth century, was which aspect of number was most primitive. On the assumption that ordinality and cardinality are the only two dimensions to developing a concept of number, there are three possible views and each one had its proponents. It could be that cardinality is primary (Russell, 1903), it could be that ordinality is primary (Gattegno, 1974), and it could be that both are equally primary (Piaget, 1952). I will briefly summarise each perspective.

Russell (1903) based his analysis of number on the concept of cardinality. For Russell, a number was what was common to sets containing members that could be placed in one-to-one correspondence. Lest there be doubt that questions of mathematical philosophy have relevance, it is only necessary to look at the prevalence of one-to-one mapping tasks in the first years of schooling in the UK, or the fact that work on number is limited to the integers 1-20 (the ones we can grasp), in the early years, to see the influence of Russell's thinking.

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An opposing view is that ordinality is the more primary. This view was used to inspire at least one mathematics curriculum (Gattegno, 1970) and the use of Cuisenaire rods. In Gattegno's curriculum, students' first experiences are to play with the Cuisenaire rods (wooden blocks with 1cm square faces and different lengths – each length associated with a unique colour) and work on relations (bigger than, smaller than). The first number to be introduced is "2", to represent the action of placing two rods of the same size to match the length of a single rod. Numbers are introduced as relations, rather than denoting objects.

A third perspective is that both ordinality and cardinality are equally primitive, and such a view was advocated by Piaget (1952). Piaget believed that the development of ordination and cardination was characterised by the same three stages, which occurred at the same age, hence his conclusion that they are acquired simultaneously.

Experiments in the 1970s appeared to suggest that ordinality occurred in young children at a much earlier age than cardinality (Brainerd, 1979). Recently, the kind of ingenious psychological experiment conducted in the twentieth century, has given way to brain research. One of the findings of broad agreement from neuro-science is that humans share an early (in evolutionary terms) Approximate Number System (ANS), our 'number sense' which we use to judge the relative size of groups of objects (Neider and Dehaene, 2009), i.e., the ANS is a non-symbolic form of numerical reasoning. Research is currently being undertaken to try and map out how the ANS links to our symbolic use of number, since there is evidence that ANS acuity is correlated with later mathematical achievement (e.g., Gilmore et al., 2010).

Some studies suggest a link between our symbolic and non-symbolic awareness of number, which could be taken to imply that cardinality is the key to learning early number. However, the situation may not be as simple as that. Lyons and Beilock (2011) suggest that many experiments related to ANS share an *assumption* that cardinality is the primary aspect of the number concept. Lyons and Beilock (2011, 2013) conducted experiments that test this assumption and their conclusion was:

> a key aspect of transitioning from ANS to symbolic representations of number involves extraction of ordinal information from the ANS and codification of these ordinal relations in terms of direct associations between symbolically represented quantities (2011, p. 257).

In other words, ANS acuity may not be a simple case of awareness of cardinality. Instead, codifying relations between symbols for numbers (characteristic of ordinality) may be key. Furthermore, Lyons and Beilock (2013) found that qualitatively distinct areas of the brain are active during ordinal tasks with number symbols, compared to tasks involving collections of objects (with or without the link to number symbols). There is evidence, then, that in the development of our concept of number, distinct processes are occurring in relation to our awareness of relations between number symbols (in an ordinal sense) and our awareness of how to link objects to numbers. Furthermore, there is evidence (again, from brain imaging) that when working with number in more complex contexts, areas of the brain significant for linking numbers to objects are not activated (Lyons &Beilock, 2011).

There are different interpretations of the neuro-scientific evidence. But one clear hypothesis to emerge is that students' awareness of ordinality may be distinct from awareness of cardinality and, in terms of developing skills needed for success in mathematics, that ordinality is the more significant. If such a conclusion were accepted, it would represent a huge challenge to current practice in the UK where, as stated above, the emphasis in the first years of schooling is firmly on linking number symbols to collections of objects.

In the next section of this report, I draw out educational implications of taking an ordinal approach to number, before then reporting on the results of an empirical study conducted in the UK where such an approach was adopted.

EDUCATIONAL IMPLICATIONS OF AN ORDINAL APPROACH

To take an ordinal approach to number, the focus shifts from linking numbers with the concrete (collections of objects) onto linking numbers with each other. Such an approach was developed by Gattegno (1974) where number is introduced as a relation. Rather than an appeal to collections of objects, number skills and awarenesses can be developed from a structure. As well as the structured Cuisenaire rods, mentioned above, Gattegno devised a chart (see Figure 1) that offers one powerful view of our number system.

0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	2	3	4	5	6	7	8	9
10	20	30	40	50	60	70	80	90
100	200	300	400	500	600	700	800	900
1000	2000	3000	4000	5000	6000	7000	8000	9000

Figure 1: Gattegno's tens chart

There is a choice of what rows to display and early work may leave the decimal rows hidden, perhaps with larger numbers added below. When introducing the Gattegno chart to a group for the first time, students need to see how numbers are named on the chart. Rather than concern about the meaning or place value of numbers, the focus is on how to say and write numbers and to gain awareness of how they are ordered. The teacher might tap on a number in the units row and get the class to chant back in unison the number name. This can extends to numbers in the tens row. For example, the teacher taps on "4" (class chant FOUR) and then "40" (class chant FOUR-TY); tap on "6" and then "60"; tap on "8" and then "80". Attention can be focused on how the number name changes (i.e., adding '-ty'), the task for students is to say and read the numbers. In contrast to limiting students to 1-20, on such an approach, the single awareness of how to move from the units to tens row allows access to 1-99 (students can enjoy saying the structurally correct "three-ty" for thirty, "two-ty" for twenty and "one-ty" for ten).

Gattegno's ordinal approach to number was the background to a research project in the UK that aimed to develop creativity in the Primary mathematics curriculum as a way of tackling underachievement. In the next section I report on this project, drawing out the links to an ordinal approach (that were not made explicit at the time), before giving some results and offering implications.

AN EMPIRICAL STUDY IN THE UK

During 2010-2013, I worked in collaboration with the charity "5x5x5=creativity" (5x5x5) with 5 different Primary schools (and one teacher in each school) to develop creative approaches to teaching mathematics. Projects with 5x5x5 often involve an artist working with a group of students in a school, to develop and document their learning in relation to a provocation. In 2010-11, 2011-12 and 2012-13, I acted as a mathematician-artist with the project schools as well as co-ordinating meetings (six a year) between teachers from project classrooms. As the mathematician-artist, I would go in to schools to take lessons. In all schools, we agreed the project lessons would centre around the notion of students 'becoming a mathematician'. We emphasised that mathematicians look for pattern and ask questions. The content of the lessons I taught was always discussed and agreed with the classroom teachers and we would de-brief afterwards. Teachers continued to work on developing activities that would allow students to notice and develop patterns, when I was not there, and at the meetings would share their ideas and activities (see Coles, Fernandez and Brown, 2013). Some teachers devoted one lesson a week to activities linked to 'becoming a mathematician', in a few cases, teachers shifted their entire approach to teaching mathematics and every lesson had a focus on students' noticing and emerging ideas.

The tool that was used more than any other in schools (in the context of the project) was the Gattegno chart (Figure 1). A common activity with year 1 (age 5-6) students was to tap on a number of the chart and get the class to chant back (in unison) the number one higher (or one lower). After working on this and taking different starting points, the students might be invited to choose their own starting number and to keep on either adding or subtracting 1 and to see what they noticed.

Another activity tried in several schools, usually with year 3 or 4 students (age 7-9), involved tapping on the chart and getting students to chant back the number ten times bigger. This can be done on the chart with a simple movement down a row. After practising in unison, the class do the same for division by 10, then for multiplication and division by 100. For this activity, the class were then invited to choose a 'starting number' somewhere on the chart, to go on a 'journey' of multiplying and dividing by powers of 10, with the challenge to get back where they started from.

To give a sense of what students might do in the course of these activities, a typical example from a students' book is copied below (see Figure 2). This student was in year 3 (age 8) and was working on multiplication and division journeys. She had decided to challenge herself to go on a journey and get back in one go, "I went back in one" is her comment on the right of the page (Figure 2).

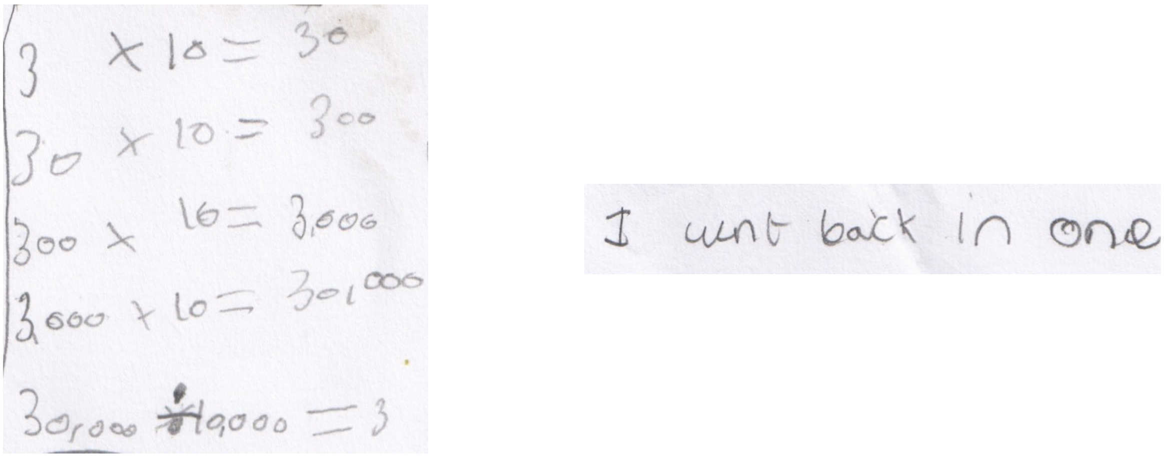

Figure 2: One student's 'journey' and comment

Division by 10,000 is many years in advance of what a year 3 students would normally be expected to compute. There is tentative evidence in Figure 2 that this student has become aware of a relation between successive multiplications by 10 and their inverse. She is making connections between the symbols themselves and seems to be gaining some confidence in working with symbols in their own right (something closely linked to an ordinal view of number).

While ordinality was not an explicit focus of the project, the description above, of activities on the Gattegno chart, demonstrates that the approach to number was one of linking symbols to symbols and moving away from concrete representations.

METHODOLOGY

The original focus of the project was on teacher development, hence audio recordings were taken of all meetings with teachers and these have been analysed (e.g., Coles, Fernandez and Brown, 2013). For this report, I have re-analysed the audio recordings of teacher meetings from 2012-13, using the theoretical framing of ordinality/cardinality, i.e., looking out for instances where ordinality/cardinality was being discussed as an issue. The taking of multiple views of data is in keeping with the enactivist methodology (Reid, 1996) that underpinned the study. Rough transcripts had already been created for the project meetings. I re-read these transcripts and returned to the audio data to confirm and make accurate the transcription of any sequence of talk that touched on issues of symbol use or the connection between symbols and objects. I also report briefly on the statistical progress data that was collected across the 5 schools. All the schools routinely monitored student progress (in relation to a system of National Curriculum levels) and schools, at points throughout the year, assessed students from project classrooms. Assessments were made by teachers, informed by written tests and moderated by a local authority. For the purposes of the project, progress was judged from the end of the year before the work began, to the end of the year in which the project took place.

RESULTS

The issue of ordinality/cardinality is raised at three meetings during 2012-13, and always by Teacher G. These three meetings are reported briefly below and analysed.

In November 2012, Teacher G (who had a year 2, age 6-7, class) reflected on the work of a student who is attaining well below government expectations for his age.

G: He's loved doing the number journeys, loved exploring what's happening when dividing by ten and dividing by a hundred. He didn't always know what the numbers were. He might know it has two zeros at the end but not know it's six hundred. He's used the pattern in terms of how it looks without being able to say the number. That makes me a bit uneasy.

The student in question appears to have been able to write out some journeys successfully, but G expresses concern that he is working with numbers he cannot say. A similar discomfort was expressed again when Teacher G reflected (February 2013) on further work he was doing using the Gattegno chart and a group of students who had been working on writing out multiples of 21 (students had chosen what multiples to work on):

G: They were doing 21 and then 42 and 63 and 84 and they were looking to see what was happening with the digits. So they could see what was happening and they could see could see the pattern, they could predict next one … I'm not sure if it's a danger but I'm aware some children see the patterns and can write a sequence of digits but maybe not know how to read those digits as a number … it just makes me aware you can't just leave it there because they just see it as patterns of numbers and they don't get to feel the truth underneath it, the place value underneath it.

I interpret Teacher G here as grappling with precisely the ordinal/cardinal issue. He reports his students being successful writing multiples of 21 (beyond what would be expected of students at that age in the UK) and yet being concerned whether students got the 'truth underneath it', the place value sense of the number – which may be a wish for a more cardinal awareness of the link to objects.

Another teacher responded directly after G's turn above:

E: you mean the symbols representing numbers have become disconnected from what they represent … the thing we're always trained not to do is to take children beyond those numbers they can grapple and handle. It's almost the whole thing is, what happens when we do do that, and is it empowering or is it actually quite shocking, quite weird, I don't know.

Teacher E here interprets the whole purpose of the 5x5x5 project: 'the whole thing is, what happens when we do' take children beyond those numbers they can 'grapple and handle. E is a headteacher and he gives an interesting insight into the orthodoxy of Primary teacher training in the UK: 'the thing we're always trained *not* to do', is move beyond students' cardinal sense of number.

By July 2013, students in G's class made, on *average*, 18 months progress over the academic year. The headteacher at G's school described the impact of the project as 'transformational' and in 2012 and 2013 (the years the school was involved) the school achieved its best ever results for the end of year 2 (the project class in 2012 and 2013). The progress by students in this school was higher than in the other 4 project schools (although in all schools, student progress matched or exceeded government expectations). Factors that were different at G's school compared to the others included: the teacher involved in 2011-12 having responsibility for developing numeracy across the school; the teachers at this school in 2011-12 and 2012-13 adopting a 'project' approach more consistently throughout their teaching than in other schools; students in the school having lower prior attainment than other schools and coming from areas of higher deprivation (as judged by the UK school inspectorate, Ofsted).

Teacher G and E's concerns and questions are significant and also give an insight into the challenge of creating new ways of working. Teacher G was subject to an Ofsted inspection during 2013, which he discussed at the meeting in June 2013. Whilst being impressed by what they saw, the inspector picked up on the issue of students working with numbers they could not read and raised this as a concern. The issue of reading numbers is an intriguing one. The Gattegno chart (Figure 1) can be used to support number reading and can be powerful in this respect. I interpret, in the concerns expressed by G, E or the inspector the exact issues discussed at the start of this report – what is a number? and, what does it mean to know a number? At what point is it okay to work with numbers we cannot 'grapple with and handle'?

DISCUSSION

In this report, I have presented neuro-scientific evidence and results from an empirical study that both suggest the idea of an ordinal approach to early number should at least be taken seriously as a possible focus in Primary school. Experimental brain studies have suggested that awareness of ordinality may be the key attribute determining the chance of success in later mathematics. In the empirical study, we certainly witnessed students becoming excited, interested and successful in mathematics, through a focus on the structure of the number system and through giving students permission to explore larger numbers than they would normally be allowed. There is clearly a need for further work on the neuro-scientific basis of early number acquisition and this is on going. There is also a need for further work in the classroom, and with teachers, to develop and trial materials, activities and ways of working that support students' awareness of ordinality. Not only that, we need to know more about effective way of working with teachers to support the development of an ordinal approach to number and to address the real concerns expressed about place value.

There are also possible implications for mathematics teaching in higher years. One interpretation of the success of an ordinal approach to early number is that it stems from having a focus on developing awareness of mathematical structure in an almost game-like manner. Once the structure (the rules of the game) is established (for example, through choral response with the Gattegno chart, Figure 1) there is space for creativity as students enter into a dialogue with the challenge of learning mathematics. There is nothing to stop such an approach being used at any level of mathematics (see Coles & Brown, 2013).

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MATHEMATICAL DISCOURSE FOR TEACHING: A DISCURSIVE FRAMEWORK FOR ANALYZING PROFESSIONAL DEVELOPMENT

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The framework of mathematical knowledge for teaching (MKT) is brought under the discursive framework of Commognition in order to track learning in professional development (PD). I follow MKT in differentiating between subject matter discourse and pedagogical discourse. The framework, which I call Mathematical Discourse for Teaching (MDT) permits a combined view on mathematical and meta-mathematical issues as constituted in discourse. Such meta-issues are found to be a significant part of what is taught and learned in a particular PD, where mathematics Ph.D. students teach elementary school teachers. Through the analysis of a lesson on parity I show how "knowing" has different meanings in mathematical and pedagogical discourses, and find evidence of learning in the evolving ways in which the parties use this term.

INTRODUCTION

What are teachers learning? This is an important question for any professional development (herein PD) program. Yet it is not clear how we should go about answering it. Though the ultimate goal of PD is a sustainable change in teaching practices, it is important to track learning as it occurs or fails to occur. In this paper I present a discursive framework for conceptualizing and analyzing knowledge and learning in mathematics PD, and demonstrate how this framework helps make sense of a particular session on parity, in which the participants were 1st and 2nd grade teachers and the instructor was a mathematics Ph.D. student. This unusual PD setting highlights the strength of the discursive approach; the instructor and the teachers are shown to have had very different ideas about what it means to know, learn and do mathematics, ideas that are constituted in their discursive practices. The crossover of these meta-mathematical ideas, as mathematical content is being discussed, is shown to be a significant aspect of the learning that is taking place.

THEORETICAL FRAMEWORK

The framework of MKT – Mathematical Knowledge for Teaching (Ball, Thames, & Phelps, 2008) has been influential in conceptualizing what mathematics teachers need to know for effective teaching, differentiating between subject matter and pedagogical content knowledge (PCK). However, to track learning as it occurs in PD, we must find indications of learning in the parties' discourse. For this I propose to embed MKT in an overarching discursive framework. The discursive approach I adopt is *commognition*

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(Sfard, 2008), whose basic tenet is that fields of human knowledge (such as mathematics) are nothing more than well defined forms of communication, and thus communication and cognition are aspects of a single entity termed *discourse*[2] . Ball et. al. lament that "after two decades of work, the nature of this bridge [PCK bridging content knowledge and the practice of teaching] remains inadequately understood" (p. 3). The commognitive view, seeing PCK and teaching as aspects of a single entity – discourse – may be exactly what is needed.

To understand how MKT may be embedded (and extended) in a commognitive framework, I first present a short exposition of commognitive assumptions and methods. Discourses are types of communication common to particular communities. They are identifiable through four interrelated characteristic features: *keywords*, *visual mediators*, distinctive *routines*, and generally endorsed *narratives*. Most commognitive research to date has focused on mathematical discourses; however in PD we are interested in the discourse of *teaching mathematics*. This discourse makes use of keywords, mediators, routines and narratives of mathematics, but also of *teaching* mathematics, much in the same way as MKT consists of content knowledge and PCK. Thus, each of the MKT categories of knowledge may be redefined as a discourse, calling their union *Mathematical Discourse for Teaching* (MDT). For the purpose of this paper it will be sufficient to distinguish between a mathematical discourse within MDT (paralleling subject matter) and a discourse of teaching mathematics, which I will call Pedagogical Content Discourse (PCD, paralleling PCK). The keywords, mediators, routines and narratives of PCD will be those that are related to teaching, students and curriculum, for example: words such as *difficult, prior knowledge*, *understand*, *misconception*; visual mediators such as manipulatives, routines of teaching, and narratives about how to teach particular content. The notion of *discourse* goes far beyond the cognitivist notion of *knowledge*. To demonstrate this point, the empirical part of this paper analyzes discursive aspects of the notion of *knowing* that some mathematical claim is true. Following Wittgenstein (1958, p. 20), the meaning of a word is taken to be the ways in which it is used, which in our framework means: what are the endorsed narratives in which the word *knowing* features, what are the routines that are invoked by this word, and what are the visual mediators and other keywords associated with it.

METHOD

The PD under investigation was the initiative of a university professor of mathematics, and was taught by mathematics graduate students. Approximately 90 teachers enrolled in the 2011-12 program, which consisted of ten 3-hour sessions taught in six groups spread over the year. The data collected consists of audio recordings of all the sessions, interviews with the instructors before and after the lessons, and teacher questionnaires

[2] The present analysis does not rely strongly on this assumption, and is valid under the weaker assumption that ways of talking do not neutrally reflect social practices such as teaching but rather play an active role in forming them.

– expectations at the outset and feedback after each session. In this paper I analyze part of a lesson on parity in which approximately 15 1st and 2nd grade teachers participated.

The decision to focus on meanings of *knowing* is not arbitrary; rules and routines by which knowledge is endorsed are a central characteristic of mathematical discourse.

The instructors' stated goal for the PD was mathematical – to broaden and deepen the teachers' understanding of the mathematical content they teach. The teachers' expectations, based on questionnaires, were pedagogical – classroom-ready activities and teaching tips. These conflicting goals are the backdrop for my discursive analysis.

DATA ANALYSIS

A comprehensive analysis of the transcript is beyond the scope of this report. I limit my analysis to utterances that reflect meanings of the word know for various participants. I omit utterances that are not relevant for the analysis.

Turns	Duration	What's going on
1-84	4:30	Teachers suggest 5 definitions for even number
85-280	15:00	Discussion: Do we want to give this as a definition?
281-321	3:00	Comparing definitions – which are similar?
322-401	6:30	Even + even = even. How do we know this?
402-480	4:00	How to define an odd number
481-665	11:00	Sign of parity (even ones digit) – why does it work?

Table 1: Overview of transcript data

Segment 1: Do we want to give this as a definition?

85 I[3]: Do we want to give 0, 2, 4, 6, 8, etc as a definition of even number?

91 I: If we tell a child that 0, 2, 4, 6, 8, etc are even, will he know to say if 1024 is an even number?

96 T1: Of course he'll know, according to the ones digit.

98 T2: If we only explain it to him this way.

Here are two different meanings of *knowing*. T1, drawing on the teaching routines of her Pedagogical Content Discourse (PCD), says that children *know* 1024 is even based on a rote endorsement routine (checking ones digit). In contrast, T2 understood the instructor's intention – that the imaginary child only *knows* what he was told explicitly – the *definition* – and that this knowing should be the basis of endorsement.

114 I: What's bothering *me* is that I can continue differently. 0, 2, 4, 6, 8, then 12.

119 T3: But we learned skip counting; he knows it's by 2, he won't pull a 12 on you.

A real child *knows* that 10 follows 8, thus in a pedagogical discourse skipping by 2 does not need to be made explicit. However the instructor's endorsement routine is mathematical in spite of his pedagogical phrasing (*will he know*), where *knowing* is based on what is explicit in the definition.

[3] 'I' indicates *Instructor*. 'T2' (capital T) indicates a particular teacher. t165 (small t) indicates turn number 165.

124 I: So when I skip by 2 from 0 I reach 2, 4, 6, and I can continue ... eventually I'll arrive, it's not very efficient to say if 1024 is even, but it's something.

The instructor "fixes" the definition – skipping by 2 is made explicit; yet it is being judged by a rather basic routine – determining evenness – and is deemed inefficient.

They proceed to discuss the definition: *a number that divides into two identical parts*.

158 T4: But... for example divide 5 into two and a half and two and a half.

163 I: Alright, that's important. In grade 2 and 1, I'm not sure the kids know...

164 T4: They know only halves.

165 I: Ok, if they know then we must be precise.

The instructor seems to have adopted a pedagogical discourse. Mathematically speaking, the precision of a definition does not depend on what any particular audience does or does not know, but an imprecise definition may be endorsed in a pedagogical discourse if the imprecision is unlikely to create a problem for students.

166 T5: They say 5 is divisible, they take the concrete, break the stick...

167 I: Ok... even numbers are only in the context of integers. We don't even know fractions. A number will be called even if I can divide, if I can take that quantity of objects and divide them into two equally large sets.

168 I: This is one way. I'll write another: a number is even if one can take such a quantity of objects and divide them into equally large sets without applying violence, without breaking things along the way. We don't permit breaking.

In the context of integers are the words of a mathematician, who has alternative contexts (natural, integer, rational, real, or complex numbers). "*We don't even know fractions*" is a code, having little to do with what real people know. In retrospect t165 appears less pedagogical. It is not a question of whether children know that a 5-foot stick can be divided equally into 2, but rather are rational numbers part of the children's world? The instructor is now aware of two different discourses. In the pedagogical (t168) we specify *without violence*, since halves are in the child's discourse; in the mathematical (t167) this is not necessary; everything is in the context of integers.

Segment 2: Proving even plus even is even

322 I: Let's say I gave you some oranges, and the number of oranges is divisible by two, that is even. And I also gave you oranges, and you checked, and this number is also even. Now we take the oranges that you both received and put them in a crate. Do I need to check all over if the number is even or not?

329 T6: No. It's even.

330 I: Why?

331 T6: Because it's divisible by two. Even plus even is even.

338 T7: If mine is divisible by 2 and hers is divisible by 2, the definition didn't change... mine remains even and hers remains even, why should it change?

The instructor chooses to ask about the sum of even numbers realized by *quantities*. The teachers return the discussion to abstract numbers. T6 knows that *even+even=even*, but this doesn't answer the instructor's *why*. In contrast, T7 accepts the need to prove the claim based on a definition – *divisible by two* – but does not yet see what exactly needs to be proven.

347 I: You checked, and [each of the quantities] can be arranged in pairs.

350 T8: You transfer them in pairs, you don't change [the pairing].

The instructor takes them back to *quantities* and T8 completes the mathematical proof.

Segment 3: Definition of odd number

397 T9: Every odd [number], if I go with a division into pairs, you have one left.

398 I: Why?... So what's an odd number?

Odd number has not yet been defined. To endorse the narrative in t397 the instructor explicitly asks for a definition , which will become the basis for an endorsement.

437 I: Suppose I tell you that a number is odd if it's not even. How do we show that the remainder, when we try to divide into pairs, I'll have one left over?

445 T9: I'd ask them to arrange in pairs... I'd like them to experience it themselves... Because if the remainder is 3, they need to check if this is really the remainder... so they see the two that can be arranged in [another] pair.

The instructor gives his definition for *odd*. Knowing in t437 relies on *showing*, but what does showing mean? T9 suggests a demonstration, using children as a visual mediator. This routine is clearly pedagogical, but it is also mathematical – in t445 this demonstration becomes the foundation for a generic proof by contradiction – if you have 3 left over, you can form another pair.

Segment 4: Sign of parity

481 I: Let's try to understand now from the definitions we have, why if a number's ones digit is 0, 2, 4, 6, or 8 - it's even. Here's a number...

500 T11: The ones digit is the end of your pairing. After you've paired them, what you have at the beginning doesn't matter; it's only the bottom line that matters... You bring down the ones digit.

512 T8: All the numbers before the ones digit are even.

T11 is proving based on a definition (pairs), but has not provided an acceptable argument. She appears to be influenced by the long division routine. T8 provides the missing link – we have already shown that the sum of two even numbers is even. She will show that all numbers are the sum of even numbers and the ones digit. The routine here is proving a property based on previously proven properties; we no longer need to refer all the way back to the definition.

554 T12: When she says 90 and 500 are even, she's basing it on the ones digit. You must! How else can you know that 1000 is even?

568 T13: I know it's a multiple of 2. 500 times 2 is 1000.

T12's rote endorsement routine is so entrenched that she can't imagine any other. T13's proof draws on the more abstract definition of even number – multiple of 2. Later, the instructor helps generalize – numbers are shown to be a sum of an even number (in fact divisible by 10) and the ones digit. Thus, every number is either even+even=even or even+odd=odd.

DISCUSSION

The transcript has shown various *meanings* (in the commognitive sense) of knowing, that is: *narratives* of knowing, *routines* invoked for knowing (e.g. proof), *visual mediators* used to support knowing (e.g. demonstration), and *words* associated with knowing (e.g. *showing*). In Pedagogical Content Discourse (PCD) we see meanings that are concerned with learners and the rules by which they endorse their narratives (t96, t331). In this discourse knowing is not linear – learners may know halves before they have officially learned fractions. Conversely, in the instructor's mathematical discourse knowing is structured. Its endorsement routines begin with definitions, and proceed through theorems that are proven based on these definitions. Furthermore, it is reflective – at any point we know what we "know" and what has yet to be shown. With this in mind, I now ask about the learning that took place, where learning is conceptualized as discursive change. The limited scope of this report cannot show that a learning trajectory was completed. *Opportunities* for learning, where interlocutors meet new discourses and engage in them, will be the focus of this discussion.

Participation in mathematical discourse may be ritualized or explorative (Sfard & Lavie, 2005). The goal of exploration is endorsing new narratives, thus explorative discourse will focus on the autonomous derivation of new narratives and their deductive endorsement. The PD episode can be seen as modeling explorative participation in mathematical discourse, where progressively sophisticated endorsement rules are introduced. This is seen twice. First in the mathematical content where the topic is parity (what are even numbers, prove that even+even=even). In this context, virtually all of the mathematical narratives came from the teachers. The instructor's contribution was in organizing well known narratives into a structure, where endorsement begins with definitions and proceeds, by means of deductive proof, to more sophisticated properties and theorems. The second exploration was meta-mathematical, where the implicit topic was *definition* ("do we want to take this as a definition?"). In both contexts the rules of endorsement evolved. Evenness was first endorsed based on a "rote" property (ones' digit), later it was based on checking a definition, and finally on proven properties. At the meta-mathematical level, definitions were at first endorsed for efficiently deciding if a number is even, later for their productiveness in routines of proving properties and theorems. For the teachers, engaging in these explorative routines is not only a model for classroom teaching, it is also an opportunity to "forget" the rote endorsement routines they have adopted as adults, which for many have become automatic, and recall what there is to learn in such a seemingly straightforward topic as parity.

What in the instructor's pedagogical discourse enabled learning? Modeling, as described above, is not the only tool the instructor used. When an expert is teaching novices, the expert's discourse may be incomprehensible to the learner, and it is up to the expert to adopt a discourse that bridges the discursive gap[4] . I have shown instances where this is achieved by means of a discursive move called interdiscursivity – "the use of elements in one discourse and social practice which carry… meanings from other discourses and social practices" (Candlin & Maley, 1997). A common instructor move was carrying mathematical meanings of words (e.g. *knowing*, *checking*) into pedagogical narratives (t91 t322), all in the context of mathematical routines of proving. The instructor chose to mediate one proof by means of quantities (t347) – a pedagogical realization of *number* – after teachers failed to find a proof using abstract numbers (t331).

Much of the commognitive research to date has focused on the asymmetrical situation of children learning. In PD, adult learners are accomplished teachers, and thus the situation is more symmetrical. It is not only the teachers who learned - the instructor came to appreciate the significance of PCD (e.g. t168). Furthermore, the teachers did not blindly adopt the instructor's patterns of participation in mathematical discourse; the discourse that emerged is an interdiscursive synthesis: t554 prefers a decimal decomposition (1000+500+90+2) over the instructor's decomposition, recognizing place value as a critical topic, and in t445 T9 added a pedagogical mediator – children pairing up – to achieve a mathematical proof. This interdiscursivity on the part of teachers shows that they are appropriating[5] a new mathematical discourse – an indication that learning is taking place. For this to happen, the teachers and the instructor need opportunities to reflect on the mathematics in the context of teaching, thus bridging the gap between their different goals for the PD. The instructor's interdiscursive routines support this. This is also supported in the open nature of the questions he asks, e.g. *do we want to give this as a definition of even number?* Who is meant by *we*? What are the considerations to *want* a particular definition? *Give* to whom? How do we endorse a statement as a *definition*? The fact that all these are left open permits the discussion to draw on multiple discourses. The pedagogical discourse is concerned with learners, for whom numbers are realized as quantities (*a number that can be divided into 2 equal sets*). It addresses classroom routines such as determining efficiently if a number is even. In the mathematical discourse the abstract concept of number is disassociated from *quantity*, precision is crucial, and the routines that involve definitions, such as proving properties, are more sophisticated. The instructor was careful not to let the teachers' pedagogical concerns derail his mathematical goals, but he delayed voicing his own ideas until after the teachers had had their say (t114). It is clear that the instructor was uncomfortable with imprecise phrasing *number that can be divided into quantities*, but he merely revoiced it more precisely – *I can take that quantity of objects*...(t167) – perhaps recognizing that the less precise wording is

[4] This is a discursive paraphrasing of Wertsch's notion of intersubjectivity (1984).

[5] *Appropriation* as used by Moschkovich (2004), in the sense of actively transforming goals and meaning.

productive in a pedagogical context. Even when he adopted parts of a pedagogical discourse (t168), it was alongside the mathematical discourse he is aiming for (t167).

SUMMARY

In this paper I have argued for a discursive approach in analyzing learning in PD, and have shown how a commognitive embedding and extension of mathematical knowledge for teaching, which I call Mathematical Discourse for Teaching (MDT), provides both theoretical framework and methods for such an analysis. This framework highlights discursive aspects of knowing, which may be difficult to conceptualize in a more cognitivist approach. Through focusing on a discursive analysis of meanings of *knowing*, I have shown the kind of learning, conceptualized as discursive change, that is taking place alongside the learning of mathematics. The instructor adopted elements of the teachers' PCD, and the teachers participated in an explorative mathematical discussion, which drew on the instructor's university routines and narratives and on the teachers' pedagogical discourse. In this discussion, the concept of *definition* took on new meanings, as it was used in increasingly sophisticated mathematical endorsement routines. This explorative experience may eventually serve as a model for the teachers' classroom teaching. They did not blindly adopt the instructor's discourse, but rather transformed the mathematical discourse into a discourse *for teaching*, appropriating it for their pedagogical purposes.

In this paper I too have tried to model an explorative discursive practice, enriching the commognitive framework with new words, routines and narratives, interdiscursively drawing on other theories (i.e. discourses) such as MKT.

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USING HABERMAS TO EXPLAIN WHY LOGICAL GAMES FOSTER ARGUMENTATION

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Argumentation and proof have been in the focus of attention in mathematics education research for several decades. While it has often been pointed out that it is important to give argumentation a prominent position in the mathematics classroom, it is far from clear how to teach argumentation, particularly to students from non-privileged backgrounds. In this paper, I show how Habermas' theory of communicative action gives a valuable perspective on what makes argumentation likely to occur. The context of a logical game situation in which argumentation happened is analysed to support the following result: the exclusion of force and a cooperative mode of communication are helpful elements in understanding the fostering of argumentation.

LEARNING ARGUMENTATION IS DOING ARGUMENTATION

Research in the past decades has looked at argumentation from various angles, often connected to mathematical proof. In this paper, Knipping's (2003, p. 34) view is adopted: argumentation is seen as a sequence of utterances in which a claim is made and reasons are brought forth to rationally support this claim; so proof is one form of argumentation. Different approaches have been made to promote proving in the mathematics classroom[1]. Boero (2011, p. 120, italics in original) claims that to teach the rules of argumentation and proving: "the best didactical choice is to exploit *suitable mathematical activities* of argumentation and proof". Douek (1999) however pointed out that having proof as the goal of the activity can be a restraint for argumentation. I decided to look purely at argumentation and the question of how to involve students with a non-academic family background in reasoning.

The only way of learning argumentation is engaging in argumentation, and as Ernest (1986, p. 3) pointed out, "playing games demands involvement". In this theoretical paper I show how Habermas' threefold approach to argumentation from his theory of communicative action provides a useful perspective for looking at classroom situations, and how logical games used in the context of mathematics teaching can provide a fruitful environment for argumentation. I support this approach with an example from my research in which I use a game to involve my students in argumentation. After playing two rounds of the logical game "Da Vinci Code" in a competitive mode, the students were faced with a hypothetical situation based on the game, whose solution required deductive reasoning. A part of the students'

[1] Knipping (2012) presents a concise overview on different approaches to the teaching and learning of argumentation and proof, including graphical representations, the debate approach and the concept of cognitive unity.

sophisticated argumentation is presented in this paper. I conclude by illuminating how mathematics education can benefit from Habermas' view on argumentation, and how logical games can provide a context to promote argumentation.

"IN THE MIDDLE, THERE IS TWO AND FIVE"

In my doctoral research, I worked for an entire year as a teacher-researcher in a group of five 15-year-old girls from different schools in Bremen, whose mother tongue is not German. In different learning situations, some purely mathematical and some including elements of logical games, I tried to evoke argumentation. The transcript given in this paper is an excerpt from a lesson in March 2013. The girls and I had known each other for 6 months. For the last lesson prior to the spring holidays I decided to pose a task related to a logical game called "Da Vinci Code", also known by the name "Coda" by Eiji Wakasugi. "Da Vinci Code" is about correctly guessing the numbers of your opponents. The game consists of black and white stones, 12 each, numbered from 0 to 11. At the beginning, each player takes a certain amount of stones and puts them up in front of him or herself, so that the other players cannot see the stones. They have to be put up in ascending order, and if a player has both stones of one number, the black number must stand left of the white number. In the course of the game, stones are taken up from the middle and wrong guesses lead to stones being tipped over, thereby revealing the number. At a certain point it can become possible from a player's perspective to correctly deduce all of the remaining stones.

I introduced the girls to the game in that lesson. There were only three girls present on that day, one of them does not make a contribution in the transcript; the others are labelled as S1 and S2. My contributions are labelled as "I". I translated the transcript from German to English as thoroughly as possible. The girls were allowed to play two rounds of the game before I took away the material and presented them with a fictional situation (cf. Figure 1) based on the game. The task for the girls was to find out all of the missing numbers; the only information they had was that all stones were arranged according to the rules of the game. I decided to work with a fictional situation so that the students could collaborate in finding a solution, in contrast to the game situation in which they were opponents. The transcript covers a time span of approximately 2 minutes, which took place directly after distributing the worksheets. In the situation, one of the girls (S2) argues that the two black stones in the middle need to be 2 and 5.

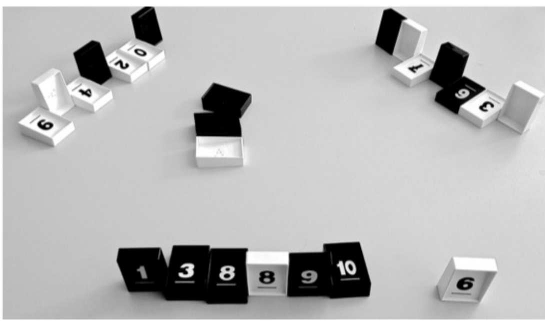

Figure 1: The fictional game situation

S1: (points to the stones in the middle) Do we have to guess these, too?
I: Yes, but of course there you cannot assign which one's which. But you will guess them in the end, so to say. Actually they are only those, which are left over.
S2: Hey, the two is IN THERE, in that, in that hole there.
S1: Mhm (agreeing; 27s silence)
I: (gets up) Feel free to tell the others when you found out a number. (11s)
S2: Well, in the middle there is five and two.
I: Mh (questioning), how do you think you know? (walks over)
S2: BECAUSE (1s), eh, the two, here (points to right opponent) it would not fit because there is a white one.
I: Mhm (agreeing).
S2: (4s) And here it would not fit (points to left opponent) because the black one isn't in front.
I: Yes (1s), that's true.
S2: Oh, and this is the three (points to the middle), isn't it?
I: No, the three is lying in front of you. I'm //not saying it's wrong//
S2: //No, I mean five, I mean five// and the five can't fit here (points to right opponent) because there is the six. And here, the five can't fit here (points to left opponent) because there is a four.
I: Yes (1s). Very nice. So the two and the five black are already set in the middle. (4s). Well considered. I actually thought they'd just be left over in the end (laughs).

After this situation, the girls found all other missing numbers with hardly any guidance and arrived at a correct solution for the overall situation in less than 10 minutes.

Analysis of the argumentation structure

In my analysis of the situation, I reconstructed the argumentation using the Toulmin scheme in the way Knipping (2008) introduced. The analysis is based on the transcript; the numbers in the boxes indicate the referenced lines. Implicit data and warrants are added for clarification, marked by dashed lines. Roman numbers indicate the three different warrants which occurred:

- I. All 24 stones (0 to 11, each once in white and once in black) are on the table, and there are no more stones than these.
- II. The stones are arranged in ascending order in front of the players.
- III. If a player has one number in both colours, the black stone stands left of the white stone.

In the scheme, data are represented as ellipses, both final and intermediate conclusions as rectangles, and warrants as diamonds. The implicit data is that the black 2 and the black 5 do not stand in front of the player; the blackened box stands for the false intermediate conclusion that the black 3 is in the middle.

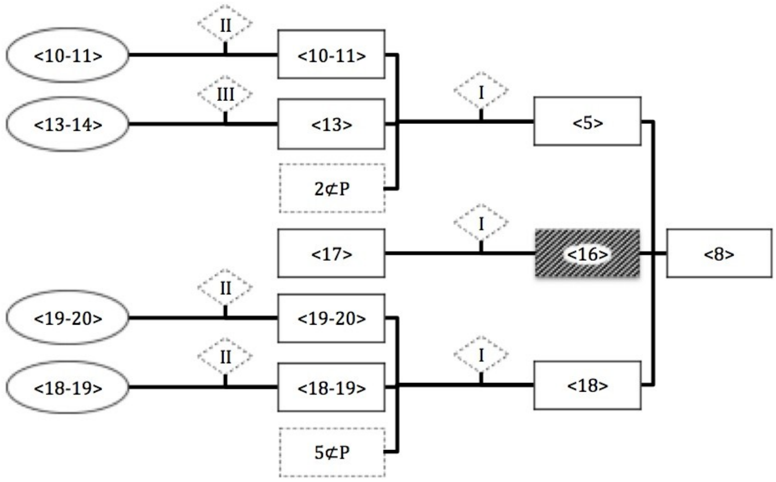

Figure 2: Logical analysis of the presented argument

The warrants used, not only in this transcript excerpt but also in all other arguments in that lesson, are equivalent to the rules of the game. In this argument, all warrants were left implicit which is common according to Toulmin (1958/2008). The structure of the argument is highly complex, and the only implicit parts are the warrants and the pieces of data referring to the immediately visible situation in front of the player.

Obviously, S2 was capable of using the rules learnt in the game to create a sophisticated argument. The deductions she makes to show that the black stones in the middle need to be 2 and 5 are similar to those used in mathematical proving. She comprehensibly establishes that both black 2 and black 5 have to be in the middle, for they cannot be in front of any of the players. In many other much less complex classroom situations, this particular student was not capable of creating arguments. This leads to the question, how argumentation was facilitated in the presented situation. In the following, I will elaborate a theoretical framework that can explain why logical games are likely to evoke argumentation and reasoning.

HABERMAS' THEORY OF COMMUNICATIVE ACTION

Boero (2006) introduced Habermas' concept of rationality into the analysis of argumentation and proving in mathematics education. This concept of rationality provides a fruitful tool for the analysis of argumentation and proving processes and their products. In this paper I use another concept from Habermas' theory of communicative action: the three-layered view on argumentation as a process, procedure and product. While Habermas' theory of rationality provides a tool for analysing the epistemic and cognitive aspects of actual argumentation and proving processes and products, the view on argumentation presented in this paper can provide an explanation why students do or do not engage in argumentation. In his theory of communicative action, Habermas (1983) elaborates on how the sciences of rhetoric, dialectic and logic differ in their approaches to argumentation as processes, procedures and products.

Argumentation as a process

Rhetoric analyses focus on the process character of argumentation. From this perspective, Habermas (1983) describes argumentation as an act of structured communication that follows almost ideal preconditions. Characteristic for argumentation processes is the exclusion of force from outside and the reliance on nothing but the best arguments. The rules which, according to Habermas, any argumentation process needs to fulfil are: Every subject capable of speech and action may participate in the discourse; every participant may problematize and introduce any statement and utter his or her wishes, attitudes and needs; and no forces from within or without the discourse may hinder any participant to use these rights. While hardly any communicative situation objectively fulfils these criteria, Habermas clarifies that the subjective impression that these criteria are met is sufficient. The subjective feeling that there is no force from outside the situation is a prerequisite for engaging in argumentation.

School situations are usually marked by an imbalance in the distribution of power between teacher and students. The teacher controls and defines topics, suitable arguments, relevant background information and data that can be regarded as shared knowledge. For the students, it is often far from obvious which inference rules and data can be seen as common knowledge and where further clarification is required. Control remains with the teacher. Logical game situations, on the other hand, are shaped by clear instructions and equal positions of the participants. Although players may have a different level of experience, the game treats them as equal. Superiority can only arise from a better understanding of the instructions. The possibility of eye-level communication, the shared knowledge of rules and premises and the absence of force from outside create ideal preconditions for argumentation. In the situation presented in this paper, S2 self-confidently supports her claims with arguments. She clearly feels encouraged to engage in reasoning and to bring forth arguments to support her claims, and no force prevents her from doing so.

Argumentation as a procedure

Dialectic is the science concerned with argumentation as a procedure. Habermas (1981) characterizes argumentation procedures as cooperative communication situations in which proponents and opponents hypothetically check claims and their appropriateness by reasons, acting without pressure arising from experience or from a call to action. The rules for argumentation procedures (1983) are the following: Speakers are only allowed to claim what they believe, and if they attack statements or norms outside of the initial discussion matter, they need to give a reason. Arguments are the only way of reaching agreement, and cooperative communication of all participants is necessary to reach a decision.

In the mathematics classroom, students sometimes make claims without being convinced of their truth, looking for their teacher's evaluation of validity. In game situations, on the other hand, claims are made according to the players' best hypothesis, intrinsically motivated by the desire to win. If the rules of the game do not allow for any activity but hypothetically considering logical implications, reasons are the only available way of dealing with the situation. Topics which are outside of the game content are unlikely to be introduced into the discussion while playing a game, because the validity of the rules is strictly limited to the game setting. In the two rounds preceding the task, the opponents were responsible for checking the validity of claims. The teacher did not play a role; true and false was exclusively defined by the students. This independence transferred to the task situation: The students trusted their argumentation and did not require feedback from the teacher once they were convinced they had found the right number. Durand-Guerrier et al. (2012) pointed out how conjecturing can motivate students to look deeper into logical structures. In a game situation, the desire to win can motivate students to find good arguments and make conjectures. In the competitive mode the finding of arguments is practiced, whereas the fictional task promotes the movement from a strategic desire to win towards an internal motivation to cooperate. This cooperative communication situation creates ideal preconditions for argumentation as a procedure.

Argumentation as a product

Arguments are the products of argumentation processes and can be examined from a logical point of view. Habermas (1983) states rules for the logical structure of arguments: No speaker may contradict himself, every speaker who uses a warrant for an inference in one case needs to be willing to use this warrant in analogous cases, and different speakers may not use the same expression with varying meanings.

In most argumentations, the warrants used remain implicit. In everyday interaction we usually assume that our conversation partners share the knowledge from which the warrants arise. In mathematical argumentation, it is common to leave out inferential steps if the reader can easily fill the gaps. For students, however, it is not always obvious which knowledge counts as shared and how to find arguments. In logical games, there is not only a fixed set of rules but also a limited number of outcome possibilities. Analogous cases are easily identified and contradictions are easy to see. Context complexity as described by Douek (2002) is reduced: time and space are irrelevant, the sources of arguments are clearly defined by the game's rules and structure, and frame changes between the abstract rules and the concrete situation are easily undertaken. In the task, a further reduction of complexity is achieved by giving the same situation to all students. This way, communication is facilitated.

Although the game is not directly connected to any mathematical content, the mode of reasoning used is essential for the learning of mathematics. General inference rules are used to deduce hypotheses from the data given on the worksheet. The conclusions the students arrive at are certain as long as we assume that all players act according to the instructions. In his work about proving, Jahnke (2007) has established the dependence of statements on hypotheses as characteristic for mathematical argument. The analysis of the situation at hand has shown that the warrants applied by the students to create arguments were equivalent to the rules of the game. Furthermore, the available data to arrive at conclusions was limited by the game setting presented to all students. This creates an ideal situation for the development of argumentation.

WHAT IS THERE TO LEARN FROM LOGICAL GAMES?

Proof is an essential component of academic mathematics, and so the products of argumentation have often been in the focus of mathematics education research. However, if we want to take a closer look at the products, we might have to look more closely at what Habermas calls ‘processes’ and ‘procedures’ as well. Habermas’ theory of communicative action does not specifically focus on the mathematics classroom but on how argumentation spontaneously develops in society. If we want to include more students in argumentation, taking a closer look at Habermas’ criteria for when individuals engage in argumentation can be a helpful means.

Logical games may help to establish a situation where force from outside is excluded and a cooperative mode of communication is predominant. In this environment, argumentation can be practiced in a meaningful and motivational way. Especially for students who are not used to argumentation, this presents a good opportunity to develop and practice their reasoning skills. In a game situation, all participants have equal power, rights and duties, and the same limitations seem true for everyone. In the light of social imbalances whose high impact on mathematical argumentation Knipping (2012) has pointed out, games could present one way of overcoming problems.

Another clear advantage of game situations is their clarity about applicable warrants and about the scope of data that can be used as a reference. The concrete and the abstract are tightly linked in the game situation, because the abstract rules guide the argumentation in a concrete situation. The steps from data to conclusion in a logical game, which one student makes, are easily comprehensible for the other participants in the situation. Despite the easy construction of arguments in this structured game context, the conclusions are not obvious. Logical games are often designed so that logical thinking and arguments with several intermediate steps are necessary to arrive at a conclusion. The products arising from these situations are likely to be sophisticated arguments.

Last but not least, the motivation to win a game by producing the cleverest argument creates a positive atmosphere in the classroom. Children are fascinated by games in general, and if these games contain argumentation they may become even more interested in the search for the best argument, which is so typical for the science of mathematics.

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TOWARD BUILDING A THEORY OF MATHEMATICAL MODELLING

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This study utilized an innovative data analysis approach to examine how engineering undergraduates engaged in mathematical modelling. Individual modelling routes were constructed via modelling activity diagrams and were used to critically examine the theoretical framework. Implications for the theoretical model are offered along with implications for future research.

The purpose of this study is to contribute to the growing body of work indicating that the mathematical thinking which supports mathematical modelling is not regular and cyclic, but is instead idiosyncratic and context-dependent. Theories of how individuals engage in mathematical modelling – the practice of combining mathematical and nonmathematical knowledge to develop mathematical explanations for natural phenomena – assert regularities in the construction of mathematical models. In particular, claims have been made that the process is cyclic and iterative, involving a sequence of stages of model construction and mathematical activities that transform them (Blum & Leiß, 2007). Others suggest that this may not be the case (Ärelebäck, 2009; Borromeo-Ferri, 2007). As the next step in developing a model of individuals' mathematical modelling activity, the existing theory must be evaluated in light of a broader observational base and analytic techniques. Thus, the theory of model construction was adopted both as a theoretical framework to guide data collection and analysis and as a research framework.

This study is a close, systematic inspection of the mathematical thinking that constitutes the activities involved in mathematical modelling. Two questions guided task selection and data analysis: (i) Is mathematical modelling a regular, quasiperiodic process? (ii) How do individuals engage in mathematical modelling tasks?

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

This study uses the theory of model construction as a research framework and as a theoretical framework. Mathematical modelling has been theorized as an iterative, cyclic process that renders a real world problem as a mathematically well-posed problem that is then analysed mathematically and its solution interpreted in terms of real world constraints. The model is then validated against real-world observations and rejected or revised. Typically, models begin as crude representations or explanations and become more detailed and sophisticated after multiple iterations of this process. A schematic describing the process is given in Figure 1 (Blum & Leiß, 2007). The mathematical modelling cycle (MMC) is a series of six stages of model construction (stages [a] – [f]) sequentially linked by a series of six transitions (transitions [1] – [6]).

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Tables 1 and 2 give brief descriptions of each of the stages and transitions among them. The MMC was adopted as theoretical framework for this study.

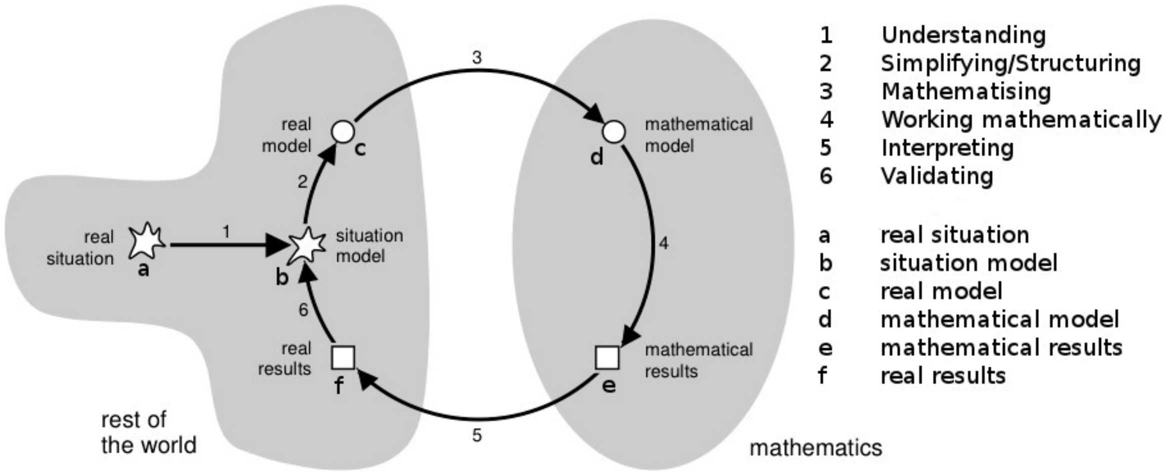

Figure 1: The mathematical modelling cycle (Blum & Leiß, 2007)

Stage of Model	Definition
[a] real situation	situation, as observed in the world
[b] situation model	conceptual model of problem
[c] real model	idealized version of the problem (serves as basis for mathematization)
[d] mathematical model	model in mathematical terms
[e] mathematical results	answer to mathematical problem
[f] real results	answer to real problem

Table 1: Stages of Model Construction

Transition	Captures	Sample Indicator
[1] understanding	forming an idea about what the problem is asking for	reading the task
[2] simplifying & structuring	identify critical components of the problem situation	making assumptions to "simplify" the problem
[3] mathematizing	represent the idealized real model mathematically	writing mathematical representations of ideas
[4] working mathematically	mathematical analysis	explicit algebraic or arithmetic manipulations
[5] interpreting	recontextualizing the mathematical result	speaking about results in context of the problem
[6] validating	verifying results against the real world	implicit or explicit statements about the reasonableness of the answer

Table 2: Transitions among stages in the modelling process

Using a similar theoretical framework, Borromeo-Ferri (2007) suggested individual modelling routes as a means for documenting individuals' cognition during mathematical modelling. An individual modelling route is "the individual modelling process on an internal and external level," (p. 265) though only *visible modelling routes* (verbal utterances or external representations) can be observed. Borromeo-Ferri's individual modelling routes took the form of arrow diagrams which traced the modeller's work through the modelling cycle. Findings suggested that modelling routes are idiosyncratic rather than cyclic as predicted by theory. Additionally, it was suggested that a change in representation might aid in understanding how individuals combined their mathematical and nonmathematical knowledge.

Ärelebäck (2009) adapted problem solving activity diagrams (Schoenfeld, 1985) to create Modeling Activity Diagrams (MADs) in order to study groups' modelling activity. MADs map a modelling event to a staff where each line is colour-coded to the modelling transition that leads the line. The result is a concise graphical representation of model construction with the advantage of providing chronological structure to the modelling activity.

The MADs track the length of time that the solver(s) were engaged in each activity. There are two drawbacks to this approach. First, the researcher cannot precisely determine when a particular transition begins or ends. Second, it is unclear how the time unit is meaningful because duration of the transition may not correspond to its meaningfulness mathematically or to its import to modelling progress. For example, if an individual spends a long time working mathematically, it may indicate a task with many steps to analysis; it may indicate an individual's difficulty in carrying out that analysis; or it could indicate that the individual paused to think about something else though outwardly he appeared to be on task. To further complicate matters, an individual may be engaged in more than one activity simultaneously or not visibly engaged in any activity. Both issues are important to consider because interpretations of the MADs are highly sensitive to the grain size of analysis and to whether verbal and written externalizations of the model are treated equivalently.

This study responds to Borromeo-Ferri's call for examining modelling routes and it uses MADs to do so. By reducing the grain size of the analytic unit and treating verbal and written externalizations of the mathematical model with equal weight, analysis of individual modelling routes and MADs can strengthen theoretical models of individuals' mathematical thinking during mathematical modelling.

METHODOLOGY

Participants were four engineering majors enrolled in a course on differential equations at a large US Midwestern university. A calculus screening test based on the Calculus Concept Inventory (Epstein, 2006) was administered to volunteers and four participants were selected such that two had high performance and two had low performance. The individuals were purposefully selected to maximize variation in their backgrounds and ability levels. All participants were male: Mance (sophomore, environmental engineering, low performance), Trystane (sophomore, mechanical engineering, low performance), Orys (freshman, chemical engineering, high performance), Torrhen (freshman, electrical engineering, high performance).

Seven one-on-one, semi-structured, task-based interviews and one follow up member check interview were conducted. The goal of each primary interview was to elicit modelling activity. Interview techniques were drawn from experiment principles such as cross-fertilization and thought experiments (Brown, 1992). Nineteen tasks were designed to elicit the stages and transitions of the MMC and were developed through an iterative process starting with gathering modelling tasks from textbooks and research papers, mapping expected student responses against the MMC, and then review by a panel of mathematics educators and mathematicians. Many were solvable through multiple methods ranging from arithmetic to differential equations. Fourteen tasks were administered and 7 eliciting all transitions (some of the 14 focused on only one) were used for analysis.

Interviews were video recorded, transcribed, and reduced to MADs in the following way. The unit of analysis was one student working on one task, termed an event. The transcript of each event was parsed into a series of mathematically complete verbalized or written ideas. Using the method of constant comparison, a rubric of indicators for each transition activity in the MMC was developed and these indicators were applied to each unit. Sample indicators are given in Table 2.

The MADs were constructed in MATLAB as two dimensional graphs. Time (in seconds) is along the horizontal axis and transitions from the MMC along the vertical axis. Each transition was assigned a colour and vertical position. Each analytic unit was assigned the ordered pair (timestamp, transition). In this way, interview protocols were reduced to individual modelling routes represented as MADs (Figures 2 – 4). Each coloured mark represents when that particular transition between two stages of model building began. Elongated marks are artefacts of the scale do not indicate the length of time an individual was engaged in an activity. This serves to emphasize sequencing of transitions through the MMC, when the MADs are read left-to-right, rather than relative lengths of time spent executing each activity.

Each event was regarded as a product of some configuration of personal experiences, mathematical knowledge, and nonmathematical knowledge. These configurations were then examined for regularities across events and for divergences from predictions of the MMC. To accomplish the latter, an "ideal" MAD (Figure 2) was generated from the idealized MMC.

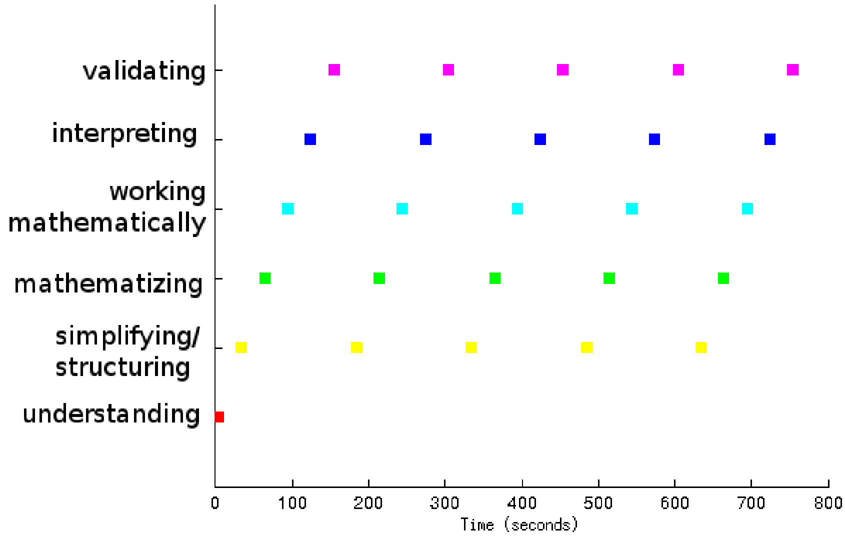

Figure 2: MAD corresponding to the MMC

RESULTS

This section presents the findings of a cross-event analysis of all MADs, but due to space constraints, the MADs for only one task, the Falling Body Problem (Figure 3 and 4), are displayed. This task was chosen because its MADs most clearly show each of the five deviations of the data from theoretical predictions. The task was: *On November 20, 2011, Willie Harris, 42, a man living on the west side of Austin, TX died from injuries sustained after jumping from a second floor window to escape a fire at his home. What was his impact speed?*

This was a standard dynamics problem (a critical variable is *time*) from physics and calculus solvable using kinematics, energy, or first-order differential equations. Mance used kinematics and made only one pass through the modelling cycle. In his MAD (Figure 3), each of the transitions fades in and out over time. That is, simplifying/structuring ceased as mathematizing took over and mathematizing faded out as working mathematically dominated. Torrhen, Trystane, and Orys made multiple passes through the modelling cycle as they changed their approaches by considering the effect of air resistance. Trystane refined his model multiple times, changing his conceptual model from *energy* to *kinematics* to *differential equations*, ultimately considering variables such as force-due-to-drag and surface area of the falling object. For all students except Mance, understanding, simplifying/structuring, and validating were exhibited frequently and consistently throughout the MADs.

The MADs provide an overview of an individual's modelling activity. The ideal MAD (Figure 2) exhibits a sawtooth pattern corresponding to the individual traversing the MMC over and over again as he adjusts the model to make it more accurate. Considering the MAD as encoding information about the individual's mathematical thinking during modelling, then this pattern is a signal and deviations from it are noise. Analysis revealed five deviations from theoretical prediction and possible reasons for the noise are discussed below.

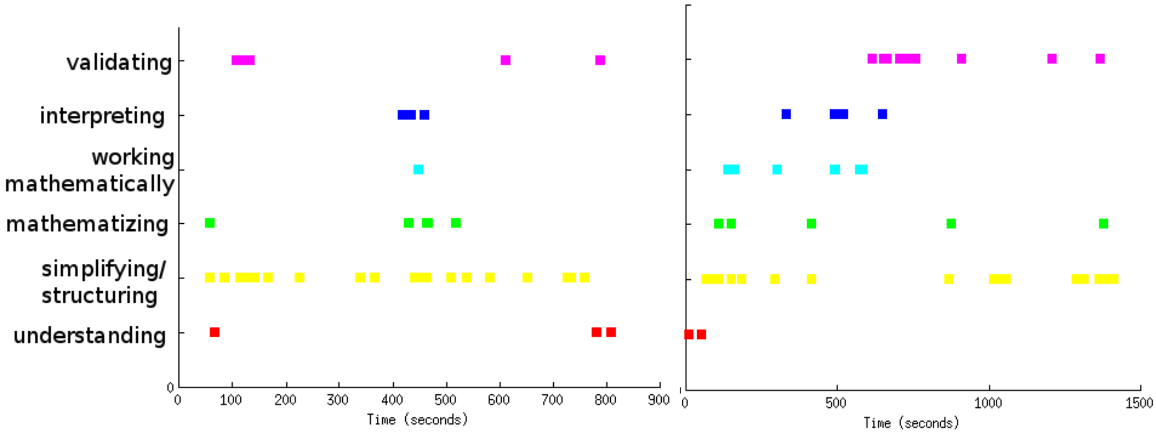

Figure 3: MADs for Mance (left) and Torrhen (right)

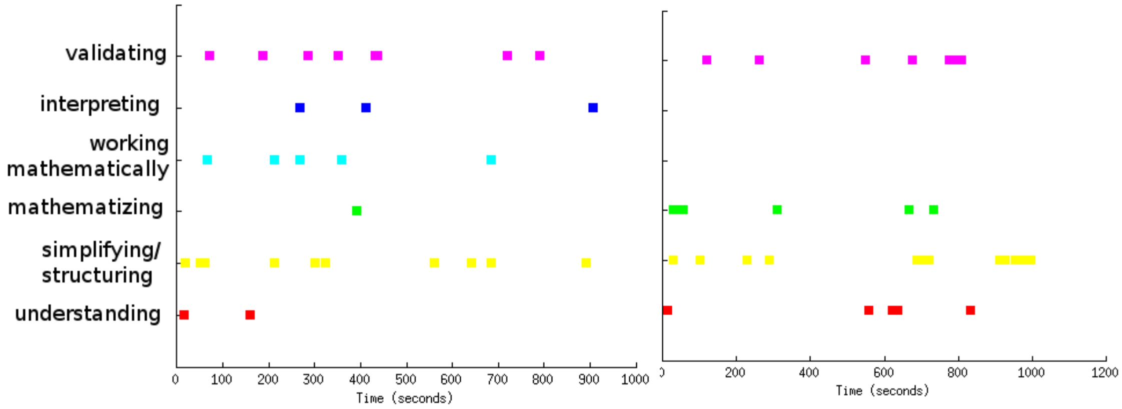

Figure 4: MADs for Orys (left) and Trystane (right)

First, the MADs show that an individual's movements are not solely "forward" in the modelling cycle. The individual may go back to consider previous stages of the model, may consider multiple stages simultaneously, or may skip transitions altogether. For example, at 600s, Torrhen was considering important variables and relationships while he is mathematizing them and at around 1000s, he rereads the problem statement (understanding) but returned directly to working mathematically without exhibiting the transitions in between.

Second, the sawtooth pattern is present, but noisy and spread out over time. For example, Mance's MAD progresses through the transitions in the MMC over the 900s, but is neither linear nor cyclic. Mance made corrections to his mathematical work, but did not revise his model. When revisions occur, they appear as bands of activity rather than neat, linear, sequential steps of a sawtooth pattern. The macroscopic banding structure is most clear in Trystane's MAD. Trystane's MAD exhibits three bands (0 – 300s, 400 – 900s, and 1000 – 1400s), but they are difficult to distinguish because understanding, simplifying/structuring, and validating occur throughout the MAD.

Third, understanding activity is present throughout the MADs; its appearance does not correspond to the start of a cycle. This is visible in Torrhen's, Orys's, and Trsytane's MADs. The most common source of this noise was the student returning to read the problem statement. Some instances could be considered monitoring because the individual compared his goal or subgoal to the task. In other instances, the problem statement was used to find more information for the simplifying/structuring phase.

Fourth and fifth, there is increased presence of simplifying/structuring and validating activities. These features are evident in all of the MADs. Both transitions typically occurred throughout the MADs. In MADs presented here, with the exception of Mance, they occurred throughout each pass through the MMC. Taken together, this suggests that the individuals were consistently checking throughout modeling whether the variables and relationships assumed to be important in the model were necessary and sufficient. That is, it was not an activity that occurred only at the end of a cycle.

Validating often occurred at sites where there were no real results to verify. For example, Torrhen checked the accuracy of a computation at 200s prior to obtaining a result to evaluate in terms of the real world. At 100s, Orys engaged in validating activity immediately after reading the task when he questioned the legitimacy of the task itself asserting “most people would survive from jumping from a second floor window.” The individuals were indeed validating other aspects of their models and how real world information might relate to their models. A focused investigation is necessary to determine the nature of the role of validating in mathematical modelling and in particular its relationship to simplifying/structuring.

DISCUSSION AND CONCLUSIONS

Analysis shows that the mathematical thinking involved in mathematical model construction is not sequential nor quasi periodic. The macroscopic structure of the MADs echo the idealized MMC. However the kind and quantity of deviations of the observed individual modelling routes from the model’s predictions suggest that there are critical phenomena which are unaccounted for by the theoretical framework. These findings confirm prior conjectures that “the view presented on modelling as a cyclic process is highly idealised, artificial, and simplified” (Ärelebäck , 2009, p. 353). This is expected, since models are representations of simplified versions of reality. These discrepancies should lead to revision of the MMC.

There is tension between a desire for an accurate, predictive model and a model that is too complex or situation-specific to be of general use. Zbiek and Connor (2006) responded to the irregularities within students’ work by introducing more stages and transitions which may collapse when an individual is facing a routine task. Collapsing would be consistent with the appearance of Mance’s MAD for the Falling Body Problem. The MMC accurately describes the practice of modelling, but requires additional consideration to account for factors like individuals’ prior knowledge, experiences modelling, and the purpose of the model.

The MADs and their subsequent analysis are a product of how the list of indicators operationalized the transitions in the MMC and grain-size of the unit of analysis. These modifications were necessary to capture the students’ mathematical work and thinking, especially in the advanced mathematical settings not yet explored with the MMC. In particular, working mathematically was defined broadly to include observations like *using deductive reasoning*; validating was redefined in terms of indicators instead of by the MMC. One avenue for future research is to use similar analytic techniques to examine the nature and role of validating activity and how it interacts with other mathematical activities. Another is to use the MADs to investigate where validating and simplifying/structuring occur in the modelling sequence as a means to examine how individuals combine mathematical and nonmathematical knowledge.

The goal of this line of research is to model individuals' mathematical thinking as they conduct mathematical modelling. The MMC provides an overview of its macroscopic structure. There is enough variation across tasks and individuals that we cannot claim that a cyclic, quasiperiodic description provides the only theoretical view of how individuals combine mathematical and nonmathematical knowledge. Mathematical modelling is a complex process and there is much work to be done to build a comprehensive theory of mathematical modelling.

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"THAT SOUNDS GREEK TO ME!" PRIMARY CHILDREN'S ADDITIVE AND PROPORTIONAL RESPONSES TO UNREADABLE WORD PROBLEMS

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Both additive and proportional reasoning are types of quantitative analogical (QA) reasoning. We investigated the development and nature of primary school children's QA reasoning by offering two missing-value word problems to 3rd to 6th graders. In one problem, ratios between given numbers were integer, in the other ratios were non-integer. These word problems were written in the Greek alphabet, and thus totally incomprehensible to the children. QA answers considerably increased with age. Younger children more frequently chose additive relations, whereas older children chose more proportional relations. The nature of the ratios between the given numbers also affected the answers, particularly in 5th grade.

THEORETICAL AND EMPIRICAL BACKGROUND

Solving proportional missing-value problems

In primary school, children frequently encounter proportion problems, mainly with a missing-value structure (Cramer & Post, 1993), in which three magnitudes are given and a fourth one has to be found by identifying the multiplicative relation between two given magnitudes and applying this relation to the third given magnitude (Kaput & West, 1994; Vergnaud, 1997). To illustrate the structure of missing-value word problems, and the two main approaches to solve them, we use the 'placemat problem' of Kaput and West (1994): "A restaurant sets tables by putting seven pieces of silverware and four pieces of china on each placemat. If it used thirty-five pieces of silverware in its table settings last night, how many pieces of china did it use?" (p. 254). Proportional reasoners using the external ratio assume a proportional relationship between silverware and china pieces (i.e. $7 \cdot 4/7 = 4$), and apply this relationship to the third magnitude (i.e. $35 \cdot 4/7 = 20$). Proportional reasoners using the internal ratio assume a proportional relationship between the first and second number of silverware pieces (i.e. $7 \cdot 5 = 35$), and apply this relationship to the third magnitude (i.e. $4 \cdot 5 = 20$).

From 4th grade on, children get ample instruction in, and practice with, the solution of proportional missing-value problems in a diversity of contexts (such as equal sharing, constant price, or uniform speed) (Vergnaud, 1983, 1988). However, previous research (e.g., Hart, 1988, Kaput & West, 1994; Karplus, Pulos & Stage, 1983) has shown that in the beginning children frequently give additive solutions instead of proportional ones. In the aforementioned 'placemat problem', those children would assume an additive relationship between pieces of silverware and pieces of china (i.e. 'the external difference', $7 – 3 = 4$), and apply it to the third known magnitude (i.e. $35 – 3 =$

32). Additive reasoners could also assume an additive relationship between the two numbers of silverware pieces (i.e. 'the internal difference', 7 + 28 = 35), and apply this to the third magnitude (i.e. 4 + 28 = 32). Studies have pointed out that children use additive solution methods in proportional problems more frequently when the numbers in the problem form non-integer ratios (Kaput & West, 1994; Karplus et al., 1983; Vergnaud, 1983, 1988).

Solving additive missing-value problems

Of course, not every missing-value problem should be solved by means of proportional reasoning. In some missing-value word problems, another type of reasoning (e.g. quadratic, or exponential) is required. In this paper, missing-value problems where additive reasoning is required are of specific interest. An example is the one that Cramer, Post and Currier (1993) gave to pre-service elementary education teachers: "Sue and Julie were running equally fast around a track. Sue started first. When Julie had run 3 laps, Sue had run 9 laps. When Julie completed 15 laps, how many laps had Sue run?" (p. 159). Here, the relation is an additive one (i.e. a relation of difference). Sue is 6 laps ahead of Julie, so when Julie ran 15 laps, Sue ran 15 + 6 = 21 laps.

We are not aware of any mathematics curriculum where attention is spent to solving additive missing-value problems. Still, this could be valuable, given that (analogously to our overview of the incorrect use of additive reasoning to proportional missing-value problems as given above) many children erroneously use proportional solution methods to additive missing-value word problems. For instance, the most frequent erroneous answer to the aforementioned runner problem of Cramer et al. (1993) is "15 · 3 = 45". Previous research pointed out that the improper use of proportional reasoning is also strongly determined by task and subject characteristics, similar to those for the improper use of additive strategies (Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005; Van Dooren, De Bock & Verschaffel, 2010): First, the application of proportional methods occurs more frequently when the numbers in the word problem form integer ratios, and, second, the overuse of proportional methods to additive problems tends to increase with age during elementary school and the first years of secondary school. Moreover, between the stage where children overuse additive methods on proportional problems (as described in the previous paragraph) and the stage where they overuse proportional methods on additive problems, there is a stage of simultaneous overuse of additive and proportional methods. Children in this intermediate stage give additive answers to word problems with non-integer ratios and proportional answers to problems with integer ratios, independent of their actual mathematical structure. In Flanders (Belgium), this intermediate stage typically occurs in 5th grade of primary school.

Similar despite differences: quantitative analogical reasoning

Most research on the development of proportional reasoning considered children's additive reasoning as an indicator of not having reached the stage of proportional reasoning yet (or at least not yet completely). While we agree with this conclusion, a basic tenet of the present paper is that children who reason additively in those proportional word problems have already taken a valuable step in their development towards proportional reasoning, as compared, for instance, to children who just add all the given numbers. Kaput and West (1994) already emphasized that children who improperly use the additive approach for proportional reasoning problems of the missing-value type, still "distinguish the quantities, construct units, and correctly identify the unknown quantity" (p. 251). In other words, improper additive reasoners demonstrate insight into the different known and unknown magnitudes and the fact that these are analogously related. They focus on the quantitative relation between two magnitudes that are given in the word problem, and apply this relation to a third given magnitude in order to calculate the missing one. So, regardless of the correctness for a given problem, additive and proportional missing-value reasoning have in common that children focus on the analogical relations between the four magnitudes in the word problem. Thus, both additive and proportional missing-value reasoning are types of quantitative analogical reasoning (hereafter abbreviated as QA reasoning).

RATIONALE

In this study, we applied a novel approach to investigate the development of QA reasoning, namely by giving children word problems that were unreadable to them. We will explain the rationale for this – at first sight indeed strange – methodological choice. In all aforementioned previous studies into children's choice for an additive or proportional solution method, word problems with an underlying mathematical model that could be determined clearly and unquestionably by carefully reading and processing the word problem, were used. In the current study, besides the development of children's quantitative analogical reasoning per se, we also wanted to investigate children's choice for an additive or proportional approach in situations where they were not directed whatsoever by the mathematical structure of the word problem. This allowed us to get a view on children's general and spontaneous inclination towards QA reasoning, and, in case such reasoning occurred, which type of QA reasoning then would be used (additive or proportional). For this reason, we used an atypical kind of items, namely mathematically neutral word problems. We designed such neutral problems by posing them in Greek literal symbols which were completely inaccessible to the (Flemish) children involved in our study. The numbers were of course accessible as they were presented in their usual Arabic form. Still, children were asked to try to solve these 'incomprehensible' word problems. Our intention was thus to find out to what extent they would look for a quantitative analogical relation between the given numbers, and if so, if they would opt for an additive or a proportional one.

RESEARCH QUESTIONS AND HYPOTHESES

Our first research question was: To what extent do children apply quantitative analogical reasoning in neutral word problems, and how is this affected by age? Because of elementary school children's increasing classroom experiences with solving missing-value word problems, we expected that even those neutral word problems would elicit a substantial amount of QA reasoning (*hypothesis 1*), and that this amount would increase with age (*hypothesis 2*).

Our second research question was: What is the nature of children's QA reasoning, and how is it affected by age and by number characteristics of the neutral word problem? Given that both additive and proportional types of answers to missing-value problems were observed in previous research, we hypothesized that we would observe both types of QA answers to our neutral word problems (*hypothesis 3*). Furthermore, based on the aforementioned previous research results about clearly additive and proportional word problems, we anticipated that among the QA answers, there would be a development with age, from a dominance of additive answers towards a dominance of proportional answers for neutral word problems too (*hypothesis 4*). We also expected a reliance on the characteristics of the numbers in the word problem. More specifically, we predicted that problems containing non-integer ratios would lead to a higher number of additive answers than problems with integer ratios, and that the latter problems would lead to a higher number of proportional answers than problems with non-integer ratios (*hypothesis 5*). Finally, we anticipated that the sensitivity to the numbers in the problem would be the strongest in the intermediate stage of children's development, between the initial stage, with mainly additive answers, and the final stage, wherein mainly proportional answers were expected (*hypothesis 6*).

METHOD

Participants were 325 children from 3^{rd} to 6^{th} grade from two primary schools in Flanders (88 3^{rd} graders, 78 4^{th} graders, 81 5^{th} graders and 78 6^{th} graders). The number of boys and girls was approximately equal in the sample. The children solved two neutral word problems, that will be the focus of the current paper. These neutral word problems were part of two larger paper-and-pencil tests. Each of these tests contained one neutral word problem, along with some buffer items (related to various parts of the children's curriculum). Both neutral word problems were stated in Greek literal symbols, but the numbers were given in the usual Arabic form as shown in Figure 1. Flemish children could absolutely not read nor understand the text of these problems, so neither the proportional nor the additive solution method − nor any other solution method − could be considered as correct or incorrect. The two word problems only differed with respect to the numbers used in the problem: the given numbers formed integer (internal and external) ratios (e.g., 4, 16 and 8 as given magnitudes) for one problem, and non-integer (e.g., 4, 14 and 6 as given magnitudes) for the other one. To minimize the influence of the specific numbers in both problems, several sets of numbers forming integer and non-integer ratios were used.

The two tests were administered on two separate moments, with one week in between. The researcher told children that the test was aimed at assessing general mathematics achievement. For the neutral problems the test merely mentioned that the problems were in Greek but that children were nevertheless invited to try to fill them in.

This word problem is a Greek one. Try to fill in a number on the dotted line.

Αδα καλκα πορελαντορα λικτουν κοττορ.
Νοϖεργανιχα τινεσταρι 4 ποσστορ ιο χηιον ανπερα τον ϖορχον 16 στατον εστανο τυϖ μαγχανετο.
Προβαλεντι μογρονατεσ 8 ογροντ ο γνοστον καλκονο τοτ λινδεναν, ναγ κιφ νισϖορ κ σχκρινον λοπεναδο μαορν εωεινστ?

Answer:

Γελομαλ λοπανδορα ριτ .... νιφφ τοτο.

Figure 1: ‘Greek’ word problem.

RESULTS

Quantitative analogical reasoning

In a first step of the analysis, the responses to the two neutral word problems were classified as ‘QA answers’ when either proportional or additive operations were executed on given numbers (i.e. calculating x in $b / a = x / c$ or in $b – a = x – c$), or as ‘other answers’ when the given numbers were combined in another way than specified above, or when the problem was left unanswered.

While coding the responses, a third category, namely ‘sum of three’ answers was added for coding cases wherein the three given numbers were added (i.e. calculating x as $x = a + b + c$). This solution method is not of specific interest for the present study (as it is not a QA answer in the sense explained above), but was still included because a large number of children had used it.

Table 1 gives an overview of the percentage of all QA, other and sum-of-three answers in different grades. This table reveals that 20.5% of all answers were QA answers. Another 42.6% was of the sum-of-three type, and the remaining 36.9% were other answers. So, in line with hypothesis 1, we found a substantial number of QA answers, especially given that the two neutral word problems were completely incomprehensible to these children. However, even more interesting is the effect of age on the percentage of QA answers.

Grade	QA	Other	Sum-of-three
3	9.1	42.0	48.9
4	7.7	35.9	56.4
5	25.3	33.4	41.4
6	41.1	35.9	23.1
Total	20.5	36.9	42.6

Table 1: Percentages of quantitative analogical (QA), other and sum-of-three answers in different grades.

A generalized estimating equations analysis revealed that children’s age affected their answers. The percentage of QA answers significantly increased from 9.1% in 3rd grade to 41.1% in 6th grade ($\chi^2(3)$= 43.858, p < .001), which was in line with our second hypothesis. As shown in Table 1, the initially low percentage of QA answers was due to the remarkably large percentage of sum-of-three answers. Almost half of the answers (48.9%) was characterized as such in 3rd grade, and still almost a quarter in 6th grade ($\chi^2(3)$= 24.579, p < .001). The percentage of other answers also decreased with age, from 42.0% in 3rd grade to 35.9% 6th grade, but this decrease was much smaller and non-significant.

Proportional or additive quantitative analogical reasoning

In a second step, we focused on the subset of answers being coded as QA answers (20.5% of all answers, i.e. 133 out of 650), to answer our second research question about the precise nature of QA reasoning. All QA answers were further categorized as 'proportional answers' (when multiplicative operations were executed on given numbers, i.e. calculating x in the expression $b / a = x / c$) or 'additive answers' (when additive operations were executed on given numbers, i.e. finding x in $b – a = x – c$).

Table 2 gives an overview of the percentage of additive and proportional answers. As expected (hypothesis 3), the neutral word problems elicited both proportional and additive answers. Of all QA answers, half were additive (49.6%), whereas the other half were proportional (50.4%). Moreover, the percentage of additive and proportional answers differed depending on children's grade and on the nature of the numbers. The results of a generalized estimating equations analysis indicated, first, that the percentage of proportional answers significantly increased with age, from 25.0% in 3rd grade, to 64.1% in 6th grade ($\chi^2(3)$= 884.927, p < .001, see Table 2). Accordingly, the percentage of additive answers significantly decreased from 75.0% in 3rd grade to 35.9% in 6th grade. These findings were consistent with hypothesis 4. Second, the nature of the numbers affected the kind of QA answers, as expected in hypothesis 5. The integer problem evoked significantly more proportional answers than the non-integer problem (69.4% vs. 27.9%, $\chi^2(1)$= 1349.979, p < .001). Third, the number effect interacted significantly with the effect of grade ($\chi^2(2)$= 452.825, p < .001), which was in

Type of numbers	Grade	*n*	Additive	Proportional
Integer	3	12	60.0	40.0
	4	11	80.0	20.0
	5	20	26.1	73.9
	6	23	17.6	82.4
	Total	66	30.6	69.4
Non-integer	3	4	100.0	0.0
	4	1	100.0	0.0
	5	21	77.8	22.2
	6	41	56.7	43.3
	Total	67	72.1	27.9
All problems	3	16	75.0	25.0
	4	12	91.7	8.3
	5	41	48.8	51.2
	6	64	35.9	64.1
	Total	133	49.6	50.4

Table 2: Percentages of additive and proportional answers by grade and type of numbers.

line with hypothesis 6. The number effect was the largest in 5^{th} grade (leading to a difference of 51.7% between the percentage of proportional answers to the integer and non-integer variant), and decreased towards 6^{th} grade (39.1%). However, the difference in 3^{rd} grade (40.0%) and 4^{th} grade (20.0%) was not reliable, due to the very low absolute number of QA answers.

CONCLUSION AND DISCUSSION

This study focused on children's quantitative analogical (QA) reasoning in word problems that could be considered neutral in terms of their underlying mathematical model, given the completely unknown alphabet and language in which they were posed. In a first step, we analyzed children's tendency to give answers based on QA reasoning. This kind of analysis is rather unique, because previous research into this topic has mainly focused on either additive reasoning or proportional reasoning, without explicitly recognizing the common nature of these two types of reasoning. Our study indicated that the neutral word problems did elicit answers based on QA reasoning, in approximately one out of five cases. This percentage considerably increased with age. Consciously or not, older children more frequently looked for a relation between two given numbers in the word problem and applied this to the third number, in order to calculate a fourth one.

The finding that children became more focused on quantitative relations relates to the notion of 'spontaneous focus on relations' (SFOR) introduced by McMullen, Hannula-Sormunen and Lehtinen (2013). However, they studied this SFOR tendency by means of non-explicitly mathematical tasks, whereas we conceptualized QA reasoning in the context of missing-value word problems which are clearly mathematical. Future research should study the relation between these two notions.

In a second step, we investigated on which kind of quantitative relation the quantitative analogical reasoners relied. The same overall percentage of answers was additive or proportional, but the percentage of additive answers decreased with age, while that of proportional answers increased. Furthermore, problems with integer ratios evoked more proportional than additive answers, whereas there reverse was true for problems with non-integer ratios. This number effect was most prominent in 5^{th} grade.

The explanation for our findings is still open for discussion, but it may at least partly be found in the current elementary mathematics curriculum. Children encounter in their elementary mathematics lessons a restricted and stereotyped diet of word problems, and are taught to solve them by recognizing the problem type and activating the arithmetic solution method that is associated with it (e.g., Verschaffel, Greer, & De Corte, 2000). The majority of word problems with a missing-value structure with which children are confronted must be solved by focusing on the proportional relations. Moreover, when proportional reasoning is introduced, problems typically first involve numbers forming integer ratios (Van Dooren et al., 2010). This way, it is not surprising that older children increasingly reason proportionally, and that children connect superficial cues in the word problem (i.e. number characteristics) with concrete solution methods.

Regardless of the fact that additive analogical reasoning often inappropriately occurs in proportional missing-value problems, it is still an important and valuable step in children's development towards proportional reasoning. Additive reasoning is after all already a way of QA reasoning. Therefore, we suggest that both additive and proportional missing-value problems should be included in the elementary school curriculum, and that children repeatedly should be stimulated and helped to distinguish between them.

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ALGEBRA-RELATED TASKS IN PRIMARY SCHOOL TEXTBOOKS

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Even though there is growing consensus for engaging primary students with early algebraic ideas, there is limited research knowledge about the relevant learning opportunities designed in textbooks. Textbooks are considered to play an important role in what is happening in classrooms, especially in educational contexts where classroom instruction relies heavily on textbooks. An analytic framework was developed to identify the opportunities designed in textbooks for engaging students with algebra-related tasks and to examine the respective guidance (or lack thereof) in the accompanying teacher guidebooks. The framework was used to analyse a primary textbook series for grades 4-6 and relevant findings are presented. Implications for textbook design, research, and practice are discussed in light of these findings.

INTRODUCTION

Algebra is seen as a gateway to higher mathematics, and both researchers and curriculum frameworks recommend that primary students should be offered learning opportunities that can prepare them for formal algebra learning at secondary school (e.g. Carpenter, Franke & Levi, 2003; NCTM, 2000; Stacey, Chick & Kendal, 2004). Even though the issue of which specific algebra-related topics are appropriate for primary students is not settled in the educational community, there is extensive reference to generalised arithmetic (e.g. Kaput, 2008), patterns and functions (e.g. NCTM, 2000), and problem solving and modelling (e.g. Kieran, 2004).

While intervention studies showed that it is possible for primary students to successfully engage with early algebraic ideas (e.g. Carpenter et al., 2003), little is known about what is happening in ordinary primary classrooms in terms of algebra teaching and learning. Textbooks offer a way to investigate this issue, as they can influence what and how mathematics is taught and thus students' learning experiences (Tarr, Chávez, Reys, & Reys, 2006). Indeed, the TIMSS 2007 study showed that, on average, 65% of the fourth grade teachers from the participating countries used textbooks as a primary basis for their lessons while 30% used textbooks as a supplementary resource (Mullis et al., 2008).

Only few studies have investigated how textbooks promote algebra-related topics. A comparative study analysed how algebraic concepts are introduced and developed in five primary curricula: the Chinese, South Korean, and Singaporean curricula, and selected Russian and U.S. curricula (Cai, Lew, Morris, Moyer, Ng & Schmittau, 2005). They found that the main goal in all curricula was to deepen students' understanding of quantitative relationships but the emphasis and approaches in achieving this goal

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differed across curricula. Also, Hodgen, Küchemann and Brown (2010) analysed two textbooks in widespread use in lower secondary schools in England (one currently used and one from the 1970s) focusing on linear relations. They found that none of the textbooks exploited available research knowledge on the teaching and learning of algebra. Both of these studies examined the selected textbooks across some general dimensions such as their goals for algebra-related learning and their content organisation and coverage. Another approach would be to use a systematic unit of analysis such as the textbook task (Stylianides, 2009), and to examine all the textbook tasks in regard to their algebra-related content (or lack thereof). Such an examination could offer a good basis for inferences about the kind of learning opportunities offered to students in classrooms, especially in educational contexts whose teachers tend to follow closely the textbooks.

Primary school teachers tend to recognise algebra-related tasks by the existence of letter symbolism or symbol manipulation (Stephens, 2008). Yet, this conception of algebra does not reflect the breadth of algebra-related topics currently mentioned in the literature. Considering that teachers' use of textbooks depends not only on the opportunities designed in textbooks but also on teachers' interpretations of these opportunities, it is important to explore also whether the accompanying teacher guidebooks offer some support to teachers to understand or appreciate the learning potential of algebra-related tasks in the textbooks. Indeed, the value of curriculum materials that aim to promote teacher learning alongside student learning is well elaborated in the literature (e.g. Ball & Cohen, 1996; Davis & Krajcik, 2005).

This paper presents an analytic framework for investigating algebra-related tasks in primary school textbooks and the respective guidance (or lack thereof) in the accompanying teacher guidebooks. The framework was used to analyse the textbooks for the fourth, fifth and sixth grades in the Cypriot educational context. In this context there is a unique textbook series that is used in all state schools and teachers rely heavily on textbooks to plan and enact their teaching (Kyriakides, 1996). These two characteristics of the Cypriot educational context elevate the importance of a textbook analysis, as such an analysis can offer a good insight into the learning opportunities offered to students in Cypriot primary classrooms.

ANALYTIC FRAMEWORK

The development of the framework involved four stages. First, it was decided that the unit of analysis would be the textbook task, which is taken to be the smallest unit identified by a separate marker in a textbook page (Stylianides, 2009). Second, it was decided that algebra-related tasks would not be limited to tasks that involved the use of letters; this is because letter symbolism is considered to be neither a necessary nor a sufficient condition for algebraic thinking (Radford, 2010). Third, three categories of algebra-related tasks were originally identified by synthesising key definitions of algebra from the literature (Bednarz, Kieran & Lee, 1996; Kaput, 2008; Kieran, 2004;

NCTM, 2000) and were then refined after piloting. Finally, two codes were selected that would provide evidence about the respective guidance in teachers' guidebooks.

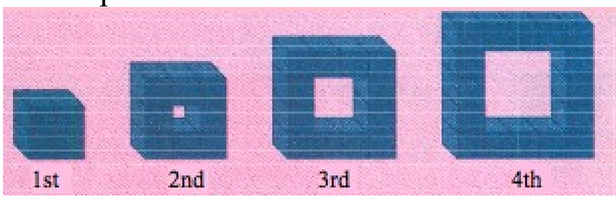
Algebra-related tasks were grouped into the following three categories according to the relations between numbers and quantities in the tasks: arithmetically-situated relations, rule-based relations and known-unknown relations. *Arithmetically-situated relations* tasks focus on the structure of arithmetic by attending to the behaviour of arithmetic operations and properties as mathematical objects and why they work. Also, these tasks could engage students in generalising these relations. This category of tasks corresponds to what is referred to in the literature as generalised arithmetic (Carpenter et al., 2003; Kaput, 2008). An example is a task that asks students to form a general expression for the commutative property of addition.

Rule-based relations tasks focus on the relations within a dataset or between datasets. These tasks could engage students in forming a rule that applies for all the elements of the datasets, testing plausible rules, extending a rule to nearby and far away cases and generalising a rule. Also, these tasks could provide opportunities for working with equivalent representations of the same rule (e.g. verbal and algebraic expressions). An example is a task that asks students to generalise verbally the functional rule of a growing geometric pattern. This category of tasks relates with the study of patterns, functions, change and variation (Kaput, 2008; NCTM, 2000). The generalisation perspective on the introduction to algebra (Bednarz et al., 1996) includes topics that engage students in generalising activities such as numeric or geometric patterns which would belong to rule-based relations tasks according to this categorisation of tasks, and laws governing numbers which would belong to arithmetically-situated relations tasks.

Known-unknown relations tasks focus on the relations between known and unknown quantities and numbers, and treat unknowns as objects (entities that stand on their own) rather than as processes. The nature of the relations range from simple direct relations to complex non-direct relations (i.e., relations for which there is no direct bridge between known and unknown). An example is the following story problem: 'A farm has chickens and rabbits. We counted the heads and we found 27. We counted the feet and we found 78. How many are the chickens and how many are the rabbits?' This category of tasks draws on the description of algebra as a cluster of modelling languages (Kaput, 2008) and the problem solving approach on the introduction to algebra (Bednarz et al., 1996). The potential of forming expressions and equations during engagement with the three categories of algebra-related tasks aligns with the purpose of generational activities as defined by Kieran (2004) (i.e., forming general expressions that arise from patterns and numerical relationships, and equations that represent problem situations).

In investigating the guidance provided in the teachers' guidebooks regarding the role of algebra-related tasks, it was examined whether these tasks were explicitly or non-explicitly identified. The code *explicitly identified* algebra-related task was used when there was an explicit reference to the task's relationship with an algebraic idea as signified by the presence of at least one of the following key words in the commentary for the task: algebraic symbols/thought/representations/equations, verbal/symbolic/algebraic generalisation, finding the rule/formula, general numbers, investigating relations between numbers/quantities, patterns, functions, arithmetic properties and relations, forming and solving equations, finding the unknown, problem-solving. The code *non-explicitly identified* algebra-related task was used when there was no such relevant key word or commentary in the teachers' guidebook. For example, the two algebra-related tasks below (Figure 1) were both coded as rule-based relations tasks since they can engage students in extending the pattern to a far away case, but only Task 1 was coded as explicitly identified due to the presence of the key word 'identifying pattern' in the commentary for the task in the teachers' guidebook.

1. Observe the figures and find the number of cubes that will compose the 25th figure of this sequence.



(MEC Grade 5 Volume D, 1999, p.43)

2. Draw the fourth figure and complete the table.

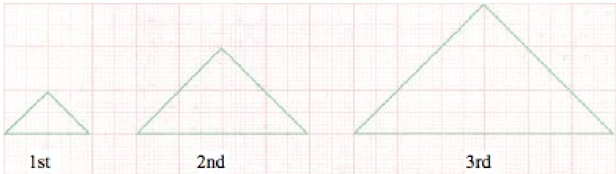


Figure Series	1st	2nd	3rd	4th	5th	10th	20th
Length of base (cm)							
Area (cm^2)							

(MEC Grade 4 Volume C, 1998, p.80)

Figure 1: Algebra-related tasks in students' textbooks

Inter-rater reliability was tested by comparing the coding of the primary rater (first author) with the codes of a second rater, who coded a subsample of 25% of the tasks in the textbooks of grades 4, 5 and 6. Two reliability values were calculated. The first reliability value concerned the decisions on whether or not a task in the subsample was algebra-related. The inter-rater agreement was kappa=0.82. The second reliability value concerned the decisions on assigning algebra-related tasks to the three categories described earlier. The second inter-rater agreement was kappa=0.84.

FINDINGS AND DISCUSSION

The framework was applied to the textbook series used in the Cypriot educational context in the three upper grades of state primary schools. It was found that 10.7% of the total number of tasks in the textbooks for the fourth, fifth and sixth grades (N=2814) were algebra-related. The specific percentages for grades 4, 5 and 6 were 10.7%, 9.2%, and 16.7%, respectively.

These findings suggest that algebra-related tasks seem to become more frequent in the sixth grade textbooks. Also, 43.4% of the total identified algebra-related tasks were found in this grade (see Table 1). This is possibly because grade 6 is the last one before secondary school and students' preparation for algebra gets more priority then than in the previous grades, which may focus more on the development of students' fluency with arithmetic calculations. Another hypothesis is that sixth grade students might be considered more developmentally ready to engage with algebra-related tasks than younger students. However, this hypothesis is inconsistent with the observed decrease of algebra-related tasks from fourth to fifth grade (as shown in Table 1). Of course this inconsistency may be a byproduct of the specific definition used in this study to identify algebra-related tasks, which may differ from the textbook authors' (working) definition of these tasks.

Categories of algebra-related tasks	**Fourth grade (n=90, 29.8%)**	**Fifth grade (n=81, 26.8%)**	**Sixth grade (n=131, 43.4%)**
Arithmetically-situated relations (n=36, 11.9%)	15.6	7.4	12.2
Rule-based relations (n=128, 42.4%)	28.9	45.7	49.6
Known-unknown relations (n=138, 45.7%)	55.5	46.9	38.2
Total	100	100	100

Table 1: Distribution by percent of the three categories of algebra-related tasks across grades

Opportunities for engaging with rule-based relations (42.4%) and known-unknown relations (45.7%) prevail in this textbook series. This possibly reflects a systemic view on mathematical development that focuses on these two categories of tasks while fewer opportunities seem to be designed for arithmetically-situated relations (11.9%). Table 1 shows further that opportunities for engaging with arithmetically-situated relations are more frequent in the fourth and sixth grades than in the fifth grade. Tasks that involve rule-based relations are increasingly more frequent while those that involve known-unknown relations decrease in frequency from grade four to grade six. This indicates that students have more opportunities to construct first notions of algebra relevant to known-unknown relations while they have more opportunities to develop ideas relevant to rule-based relations in the last two grades of primary school. Opportunities for students to engage with known-unknown relations and rule-based relations are desirable because they are likely to help students recognise the limitations of arithmetic problem-solving approaches and start familiarising themselves with making generalisations.

The low percentage of arithmetically-situated relations tasks indicates that there are relatively limited opportunities for students to attend to the structure of arithmetic and how that relates with algebra. Linchevski and Livneh (1999) mentioned that students' difficulties with the structural properties of the algebraic system originate in a limited understanding of the number system. This difficulty is partly attributed to the lack of attention given to students' awareness of the mathematical structure and of arithmetic operations as general processes during the learning of arithmetic (Booth, 1984).

Table 2 shows that 84.1% of the tasks categorised as 'algebra-related' in this investigation were explicitly identified as such in the teachers' guidebooks while the remaining 15.9% were not. For the non-explicitly identified tasks, there were no relevant key words or commentary in the teachers' guidebooks. It is unknown if these tasks were actually intended by textbook authors to engage students with algebra-related topics, but in any case this lack of clarity leaves space for teachers to interpret in different ways the role of these tasks in the textbooks.

Guidance in the teachers' guidebooks for algebra-related tasks	**Fourth grade**	**Fifth grade**	**Sixth grade**
Explicitly identified (n=254, 84.1%)	66.7	88.9	93.1
Non-explicitly identified (n=48, 15.9%)	33.3	11.1	6.9
Total	100	100	100

Table 2: Distribution by percent of explicitly and non-explicitly identified algebra-related tasks across grades

The number of explicitly identified algebra-related tasks increases from fourth to sixth grade. This indicates that the available guidance in teachers' guidebooks differs across grades. Also, the opportunities designed for students seem to become more explicit for the fifth and sixth grade teachers than for the fourth grade teachers. For the non-explicitly identified algebra-related tasks, the lack of key words in the teachers' guidebooks seems to hinder the role that these tasks can serve in the curriculum. Referring back to the two textbook tasks presented above, the tasks seem to promote similar algebra-related learning goals since they can engage students in extending the geometric pattern to far cases. Yet, the algebra-related goal of Task 2 was not explicitly stated in the teachers' guidebook for grade 4 and this may obscure the relationship between this task and other similar tasks in the curriculum, such as Task 1, which students would encounter in grade 5.

CONCLUDING REMARKS

The distribution of tasks across the three categories of algebra-related tasks raises questions about the possible implications of designing limited opportunities for students to engage with one kind of tasks, in this case the arithmetically-situated relations tasks. Fewer opportunities are designed for students to understand that the same underlying properties of arithmetic are applied to algebra and more opportunities are designed for them to solve algebraic problems, identify and generalise quantitative relations.

The findings raise also questions about what might be the essential support in teachers' guidebooks. Given the fact that algebra has traditionally been considered a mathematical topic for secondary school, in cases where algebra-related tasks in primary textbooks are non-explicitly identified, there is a danger that the potential of these tasks to engage students with early algebraic ideas will not be fulfilled. Teachers' approaches to tasks are underlain by the different ways they read the textbooks, which in turn are influenced by their beliefs about teaching and their expectations of students' learning (Remillard, 1999). Therefore, by not providing explicit information about the role of these tasks, textbooks allow further space for disparate interpretations among teachers and thus more variability in the opportunities that teachers offer to students to engage with algebra-related topics.

One could argue that it does not matter whether or not these tasks are explicitly identified in the teachers' guidebooks as long as teachers encourage students' engagement with algebraic ideas. However, research suggests that primary teachers have rather narrow conceptions about algebra-related tasks (Stephens, 2008), and this raises concerns about the implementation of non-explicitly identified algebra-related tasks in primary school classrooms. Further research is needed to explore primary teachers' interpretations and enactment of different kinds of algebra-related tasks.

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MATHEMATICS TEXTS: WORKSHEETS AND GENRE-BENDING

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This paper reports an in-depth study that explores the nature and use of mathematics worksheets using a genre analysis approach. Nine secondary level teachers with collective experience from five different countries participated. Through individual online and focus group interviews teachers shared their own worksheets and their understandings and use of worksheets for teaching and learning math. Results indicate that mathematics worksheets have culturally recognizable features and characteristics, they are used to emphasize procedural over conceptual aspects of mathematics learning, and can structure the way mathematics is taught. This study highlights the potential of genre-bending as an approach to extend and re-imagine the structure and use of mathematical texts such as worksheets.

INTRODUCTION

In many countries around the world teaching and learning mathematics involves the use of curriculum materials such as mathematics textbooks (Schmidt et al., 1997). In addition to the textbook, or sometimes in place of it, mathematics worksheets distributed by teachers to their students also play a role in mathematics education. As with other curriculum materials mathematics worksheets can impact the ways teachers teach and interact with their students as well as influence their own and students' interaction with mathematics (Mousley, 2012). Problems selected by teachers and posed to students can communicate implicit understandings of what it means to do mathematics and what is involved in getting better at it (Schoenfeld, 1992). Understanding better the kinds of mathematics problems teachers select or design and offer their students can provide insight into how materials are used, what is taught and learned, and how teacher education can better support learning to teach (Nicol & Bragg, 2009). Although there is increasing research on the nature and use of mathematics textbooks (Haggarty & Pepin, 2002), we know little about the nature and use of mathematics worksheets including their textual features and how they are used to teach mathematics (Kaymakci, 2012).

In this paper we explore the textual and contextual features of mathematics worksheets. We use genre analysis (Gerofsky, 2012; Kearsey, 1997) as a dynamic and holistic method to examine how worksheets, as text, are shaped and used, and thereby how they might provide opportunities to be imagined differently. With this paper our purpose is twofold: 1) to provide insight into the nature and use of mathematics worksheets through a small empirical study; and 2) explore the potential of genre analysis as an approach to understand the use and impact of curriculum materials such as mathematics worksheets.

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THEORETICAL CONSIDERATIONS

Studies on the analysis of curriculum materials reveal their impact and role in mathematics teaching and learning. They explore what texts are (e.g. Love & Pimm, 1996), how they are accessible to students (e.g. Van Dormolen, 1986), how teachers and students conceptualize and use texts (e.g. Remillard, 2000), and how the power of texts can become a “surrogate curriculum” (O’Keeffe, 2013). International studies of textbooks such as the Third International Mathematics and Science Study [TIMSS] found that student achievement is impacted by curricular and pedagogical intentions presented in textbooks (Schmidt et al., 1997) and that textbooks are “important mediators between policy and pedagogy” (Valverde et al., 2002, p. 171). These studies typically analysed textbooks for their structural features and organizational characteristics through examination of the intended, implemented and attained curriculum.

More recent studies employ a linguistic approach to analyse mathematics curriculum materials in order to better understand how mathematics activities are presented and the kinds of mathematical messages portrayed. Drawing from a systemic functional linguistic approach Morgan (1996) proposes a method to examine mathematical texts through the analysis of language and considers “the ways in which reasoning is expressed” (p. 7). Herbal-Eisenmann and Wagner (2007) build on Morgan’s work and examine the use of imperatives, pronouns and modality in mathematics textbooks and found how language choices within a text not only influences how readers make sense of it but also how the text might position students in relation to other students and their teachers.

Although mathematics worksheets are used at the elementary and secondary school levels there are few studies that focus explicitly on worksheets. An exception is Mousley (2003) who examined how a particular worksheet on the topic of percentages was used by two Grade 6 teachers and found that the worksheet shaped how the teacher and students interacted with each other. However, we know little about the nature of mathematics worksheets, how they are conceptualized, their language, and how they are used. Genre analysis provides an approach to understand both the features of text and the relationship between the participants (producers, consumers, and content itself) in that text (Kearsey, 1997). Whereas most previous studies of curriculum materials in mathematics education focus on either the features of or the use of the materials, genre analysis brings the study of these two areas together for a more holistic approach to text analysis.

Genre analysis conceptualizes genre as “a culturally-recognizable form” and involves asking questions about the presence and nature of the particular generic form, such as mathematics worksheets, from many different disciplinary perspectives. Genre can be defined as a category, a kind, or a type of artistic, musical, or literary composition that is characterized by a certain style, form, or content. More recently genre is considered to be a “form of cultural knowledge that conceptually frame[s] and mediate[s] how we understand and typically act within various situations” (Bawarshi & Reiff, 2010, p. 3).

From this view, a genre analysis of a worksheet not only provides insights regarding the features of the text, but also the producers' intentions, how the readers are addressed, and the kind of motives portrayed by the genre itself. Genre analysis can help to answer the question of what a worksheet is. It is also an approach that can provide opportunities to consider or re-imagine what a worksheet could be; to explore what Gerofsky (2012) refers to as the genre-benders of mathematics texts. Thus the focus of our study is on mathematics worksheets: What kind of genre are mathematics worksheets? How do teachers report on their use? And in what other ways might worksheets be imagined?

DATA COLLECTION AND ANALYSIS

This report draws upon an in-depth study that included both conceptual and empirical data phases. The conceptual phase involved examining worksheets as a cultural and pedagogic genre. The empirical phase worked with 8 experienced and 1 novice secondary school level educators who, at the time of this study, were also graduate students at a major university (8 participants) or retired (1 participant). Four participants earned their undergraduate degrees from Canada, two from China, one from Belize, one from India, and one from the United States of America (US). Participants were asked to provide samples of mathematics worksheets from their own teaching resources and participated in two individual on-line interviews along with a 1 hour focus group interview.

The first individual interview focused on gathering participant background information and how participants say they used worksheets in their teaching. This was followed by a focus group interview in which participants shared their experiences and understandings of mathematics worksheets. In order to better understand what counted as a worksheet and what didn't participants were provided with a range of worksheets and asked to examine them for their features, similarities and differences. They were also asked to provide samples of worksheets they had used from their own teaching resources and examples they thought could lie on the boundary of counting as a worksheet. When needed participants translated their worksheets to English. The focus group was followed by a second individual interview to further pursue comments and ideas shared during the group interview. Questions asked during the interviews included: What are your memories of worksheets as a student? How would you describe what a worksheet is and what it is used for? In what situations would you use mathematics worksheets? Are worksheets for all students? What did using worksheets accomplish for you? How do they differ from mathematics textbooks? All interviews were audio recorded. Data therefore included copies of participant's mathematics worksheets (22 worksheets; 295 questions in total), online interview text, and transcriptions of the audio-recorded interviews.

Data analysis drew upon genre analysis approaches that included a study of the existence of worksheets, their historical development, and their defining features and characteristics. Language analysis included coding worksheets at a word level by identifying imperatives, pronouns and modality (Herbel-Eisenmann & Wagner, 2007) and in terms of the placement of mathematics in the worksheets (Morgan, 1996). Analysis focused not only on the features of worksheets but also on the contexts in which they were used as well as the stated relationships by teachers on the interactions between the teacher, student and the mathematics.

RESULTS

Our findings are presented in two parts: 1) a focus on the existence, features and characteristics of mathematics worksheets; and 2) a focus on their use and purpose. For this paper we briefly highlight results on worksheet features and characteristics (part 1) in order to provide more depth on their use (part 2).

Existence, features, and language of mathematics worksheets

The conceptual phase included a general search to verify the cultural recognition of mathematics worksheets as a genre. Entering "mathematics worksheets" as a key phrase in the Google search engine revealed a result of 4,410,000 documents (May 14 2013). A search was conducted in other languages as well: Turkish "matematik calisma kagitlari" revealed 445,000 documents; German "mathematic arbeitsblatt" revealed 823,000 documents, and Chinese "数学随堂小试卷" revealed 2,110,000. This abundance of documents provides evidence that internationally mathematics worksheets are a culturally recognizable form.

Analysis of participants' submitted mathematics worksheets found common features among the worksheets on form, content, graphics and language as shown in Table 1.

Features	Detailed Characteristics
Form	Page full of a series of math questions; organized into columns and rows; includes a math topic as title; requires completion by students
Content	Developed by teachers; required by students to complete; questions listed from easy to more difficult; focuses on one particular math topic; emphasizes certain skills
Graphics	No or few graphics; few words; no or few variations of representation such as graphs, numerical, illustrations
Linguistic	Use of symbolic statements; use of imperatives (e.g. find, calculate, divide); use of sentence phrases (e.g. fill in the blanks)

Table 1: Common features of teachers' mathematics worksheets

Participants self-identified their mathematics worksheets as central to, peripheral to and contrasting with what might commonly be referred to as mathematics worksheets, or in other words with the genre of mathematics worksheets. According to participants, worksheets that are central to the genre provide repetitive practice questions for mathematical fluency and accuracy. Participants identified worksheets that were peripheral as focusing on skill practice but also including different activities such as sketching, using a number line, or communicating mathematical reasoning or including questions that were more open-ended or personalized to student interests. All participants agreed that worksheets that could be considered as on the boundary of counting as a worksheet were less focused on drilled practice and included more use of graphics (pictures or graphs), complex language structures, and required more critical reasoning.

A language analysis of participants' worksheets focused on the frequency of pronouns, imperatives, and modality of the text indicated that the use of imperatives was the most frequent occurring linguist form in the data set. There were 139 imperative statements within the worksheets. Of these 17 were categorized as inclusive imperatives (e.g., show, explain, describe) while 121 were exclusive imperatives (e.g., find, calculate, express, determine, write) [I'm not sure of the difference between inclusive and exclusive – can you add something to this?]. There were 25 pronouns found across all the samples: 2 first person plural pronouns (we), 17 second person pronouns, and 8 third person pronouns. No first person singular pronouns were found. Modality, an aspect of text that reveals how human agency is constructed within the text, was also analyzed. Modality examined through the use of modal auxiliary verbs such as must, will, could, or might revealed that 11 of the 22 worksheet samples used modal verbs. In total 22 modal verbs were found across all the samples with the verb "can" being the most frequent (11 times).

How teachers describe, use, critique and imagine worksheets

Analysis of individual and focus group interviews reveals that all participants distinguished worksheets as being quite different from class handouts. For example, worksheets were for "practice, practice, practice" (Anton) or "a series of questions with single right answers" (Rambo). Four of the nine participants associated worksheets with acquiring fluency and accuracy while handouts were used to engage students in critical thinking or conceptual understanding. In this way participating teachers stated they found worksheets limiting, with a focus on repetitive, drill type questions, worksheets tended to lack challenging questions or prompt critical thinking. Mohna's comment reflected others: "teachers don't use [worksheets] as a tool for critical thinking or conceptual thinking" worksheets generally focus on "what you've already learned."

Although participants' descriptions of worksheets were similar, their pedagogical use of worksheets varied. Some said that they used worksheets as preparation for tests (Lizzy), homework and practice to sharpen students' memory (Anton), for practice following a lesson (Chloe), to develop students' independent studying habits (Serena), to provide time for teachers to circulate around the class and provide individual help to students (Pascal), or for evaluation (Mohna). For some, such as Mohna, it was the way teachers used worksheets that made the difference: "worksheets should be created and utilized as formative evaluations and also for encouraging hands-on, minds-on learning not just as copied text-based list of problems."

All participants reported using mathematics worksheets but they were also critical of them. Worksheets were described as "drill and kill exercises," offering few opportunities to challenge students (Rambo), hindering creativity in favour of the teacher's activities (Lizzie), "getting the answer at the expense of in-depth understanding" (Anton), or closed problems that didn't provide teachers access to their students' thinking (Serena). Some, such as Rambo, reported his experience that students sometimes preferred worksheets to other more challenging work that required creativity or critical engagement.

In order to better understand mathematics worksheets as a genre, participants were asked to imagine possibilities for other ways in which mathematics worksheets might be structured or used. Participants suggested playing with the form of the worksheet as well as the content. They identified a sheet with the single division problem:

$$\frac{(-1)^{998} + (-1)^{895} + (+1)^{1000}}{(-1)^{901}}$$

accompanied with prompts: "I think the answer is ….because … and… therefore …." as an example of pushing the boundaries of what counts as a mathematics worksheet. All participants agreed with Rambo and considered this sheet to be more of a handout than a worksheet because it contains one question and "it asks for an explanation… it's asking students to think." Participants also discussed possible worksheets that challenged the typical structure of worksheets as progressing from easy to more difficult questions. In addition, it was suggested that worksheets could be designed to engage students in working with others in order to challenge the individual nature of worksheet engagement. All participants acknowledged Rambo's suggestion of challenging the idea of worksheets as consisting of multiple questions offered to students all at once by instead developing "one problem at a time worksheets." Such a worksheet could allow teachers to differentiate the questions to individual student interests or needs.

CONCLUSIONS

The results of this study suggest that mathematics worksheets can be considered a genre. Findings revealed that mathematics worksheets have typified regularities (Miller, 1984) that conform to a certain consensus and mediate interactions (Bawarshi & Reiff, 2010). Analysis of the form, content, graphic and linguist features of worksheets provided characteristics that help define worksheets as culturally recognizable forms. Comparing the typical mathematics worksheet with contrasting or marginal examples provided further clarification in terms of the nature and generic structure of the worksheet genre. For instance, our results indicate that worksheets central to the genre were composed of questions that emphasized the calculational and procedural aspects of mathematics and reinforced drill-type skill development by highlighting accuracy and speed. Marginal forms, on the other hand, included questions that involved mathematical reasoning, pattern recognition and critical thinking. Results indicate that although teachers are familiar with worksheets as a genre they are also able to consider possibilities for bending the genre, that is, thinking of possibilities for worksheets that extend its features and use. Worksheets could be created that allow for one question at a time, include questions that are more open-ended or questions which require students to raise their own questions.

A genre analysis of worksheets has helped clarify the form of worksheets that can shape activities in the mathematics classrooms. Genre-bending provided opportunities to think about worksheets and their use in a different way, and therefore opened possibilities for improving the structure and use of mathematics worksheets.

This study revealed that worksheets emphasized the procedural aspects of mathematics but not conjecturing, relating, or testing activities. The language analysis revealed that the authoritative language of worksheets positions students outside the mathematics community. If worksheets are used to force students to learn in a specific way, and treat them as a cohort rather than individuals without addressing individual abilities, interests and needs, worksheets become hegemonic and homogenizing force. This study contributes to our understanding of mathematics worksheets and provides a strategy, genre analysis, to engage in a critical analysis of worksheets as a genre. This study is an example of how educators can come together to reconsider and re-invent their use of worksheets in mathematics teaching and learning. As mathematics worksheets are recognized internationally as playing a role in mathematics teaching and learning, it is important that we gain a better understanding of how they are conceptualized, how they are used, and how they might be re-imagined. This study adds to the beginning research in this area and contributes to the analysis of mathematics texts by focusing on one kind of text, the mathematics worksheet.

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TEACHERS' BELIEFS ABOUT STUDENTS' GENERALIZATION OF LEARNING

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Researchers in psychology and mathematics education have been conducting systematic investigations of students' generalization (or transfer) of learning since the beginning of the 20th century. However, we do not know how teachers, the people typically associated with student learning, think about this phenomenon. This study, thus, identified teachers' beliefs about students' generalization of learning. Five categories of teacher beliefs were identified, highlighting the importance of bringing teachers into the ongoing transfer conversation as the categories identified both extend current conceptualizations of transfer into the domain of mathematics education and identify new beliefs regarding students' transfer of learning.

INTRODUCTION

The idea that students generalize classroom learning to novel situations serves as the foundation for our educational system (Bassok & Holyoak, 1989; McKeough, Lupart, & Marini, 1995; National Research Council, 2000). One body of research that has examined students' generalization of learning is the research on *transfer* (e.g., Bereiter, 1995; Engle, 2006; Gick & Holyoak, 1983; Lobato, Rhodehamel, & Hohensee, 2012; Markman & Gentner, 2000; Singley & Anderson, 1989; Thorndike & Woodworth, 1901). Traditionally, transfer has been characterized as "how knowledge acquired from one task or situation can be applied to a different one" (Nokes, 2009, p. 2). Transfer has had a rich and varied history dating back to the turn of the 20th century and has evolved to include a multitude of perspectives regarding what transfer is, how it occurs, and how it might be supported. One might assume that since teachers are the people typically associated with student learning, some of the transfer literature would have identified teachers' beliefs about their students' generalization of learning. However, such studies do not appear to exist. Thus, I sought to determine whether teachers think about students' generalization of learning as part their typical practice and, more specifically, to answer the following question: What are teachers' beliefs regarding students' generalization of learning?

FRAMEWORK

As noted above, *transfer* has traditionally been conceived of in terms of the application of one's previously acquired knowledge. Heeding critiques of such acquisition-application views of transfer (e.g., Lave, 1988), researchers have reconceived of and redefined the phenomenon in many different ways (e.g., Bereiter, 1995; Lobato et al., 2012). Here, I use the term *transfer* to reference the phenomenon in which students generalize, extend, or in some way make use of their learning when

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engaging with novel situations (rather than in reference to a particular conception or definition of transfer).

This study identified teachers' *beliefs* about students' generalization (or transfer) of learning. Drawing upon Philipp's (2007) definition, I define *a belief* regarding transfer as a conception (i.e., a general notion or view) regarding transfer that I, the observer, can respect as intelligent and reasonable even when it differs from my own conceptions regarding transfer. Here, beliefs are distinguished from knowledge (i.e., conceptions that I can not respect as intelligent and reasonable when they differ from my own). This decision indicates my own orientation towards conceptions of transfer. The fact that transfer is one of the most researched topics in psychology coupled with the fact that many different conceptions of transfer are documented in the transfer literature leads me to believe that transfer is a complex phenomenon, best studied as a belief wherein I am supported in making sense of differing conceptions of transfer rather than casting them aside as unintelligent and/or unreasonable.

METHODS

Participants

I recruited eight practicing teachers from multiple urban school districts in Southern California to participate in this study. Teachers were selected on the basis of several criteria. First, *practicing* teachers were selected to help ensure the selection of teachers who, at the time of the study, naturally thought about the phenomenon of interest to this study. Second, participants were recruited on the basis of the nature of the mathematics courses they taught and whether they had the opportunity to develop students' understanding of slope during the 2011-2012 or 2012-2013 school year. (Slope provided the mathematical context in which teachers' beliefs were examined.) Finally, teachers were recruited so there was variation across the following: the forms of practice enacted in their classrooms, the number of years teaching experience, the amounts of training and professional development received, and the type of school where employed (charter school vs. non-charter school). The rationale for seeking such variation was to increase the chance of selecting a group of teachers who held different beliefs regarding students' generalization of learning.

Data collection

I engaged the eight practicing-teacher participants in two 2-hour semi-structured clinical interviews (Clement, 2000; Ginsburg, 1997). During these interviews, teachers were asked questions and posed tasks that were designed to elicit their beliefs regarding students' generalization of learning. The same major questions and tasks were posed to each teacher, but follow-up probes were tailored to individuals. The interviews were recorded with a video camera and a table microphone. The video camera was aimed to capture teachers' gestures, written inscriptions, and verbal reports. All written work and materials were collected.

Instruments

The major questions and tasks designed to get at teachers' beliefs about students' generalization of learning were separated into three sets: (a) questions related to an instructional item teachers selected and brought to the first interview, (b) questions and tasks designed to provide teachers with opportunities to explicitly *espouse* their beliefs regarding students' generalization of learning and me data from which to *infer* such beliefs, and (c) questions related to a lesson plan teachers constructed to support their students' generalization of learning. Each set is briefly discussed. (Note that because the word *transfer* is a researcher construct, it was not used with teachers; rather, phrases like "generalization of learning" were used.)

Prior to engaging in the first interview, I asked teachers to select an item or items (e.g., an activity, lesson plan, test, or homework) they had used during a unit on slope and linear functions that they believed demonstrated an instance in which they thought about supporting their students in being able to generalize their understanding of slope to a new task, activity, or situation. The discussion of this teaching item took place at the beginning of the first interview and involved questions like: "Describe how your [item] shows you were thinking about helping students to make future use of their learning." All of the teachers brought a task or activity involving slope and were thus asked questions like "As a consequence of your students' engagement with this [item], what types of tasks and activities do you believe your students are (and are not) prepared to successfully engage with?"

The second set of questions and tasks was not associated with either the aforementioned instructional item or the lesson plan still to be discussed. It involved more general questions like "What do you do, or what do you think teachers in general can do, to help enable students to be able to generalize their learning to new situations? Explain how these actions support students' generalization of learning." This set of questions also involved more specific tasks and questions including a task in which teachers were presented with hypothetical student responses to a slope task and a set of novel slope tasks, and asked to discuss which of the novel tasks the hypothetical students would be able to successfully engage with given their work. Teachers were also presented with three hypothetical instructional activities and asked to discuss which activities best supported students in generalizing their understanding of slope.

Between interviews, teachers were asked to develop a lesson plan on slope that implemented some of the ideas they discussed during the first interview regarding students' generalization of learning. They were also asked to design a novel task (not discussed in the lesson) with which their students could successfully engage after participating in the lesson. In the second interview, teachers' were asked questions about their lesson plans, novel tasks, and the relationship between the two.

Data analysis

I transcribed and analyzed all interview data qualitatively, using what Miles and Huberman (1994) describe as "partway between a priori and inductive coding" (p. 61).

Categorizing teachers' beliefs about students' generalization of learning involved drawing upon the transfer literature. For instance, some teachers appeared to believe that students would be able to productively generalize their learning to a novel situation if the novel situation prompts students to make use of a learned association, procedure, or formula—a belief found in the research literature from an associationist view of transfer and mainstream cognitive accounts of transfer (e.g., Singley & Anderson, 1989; Thorndike & Woodworth, 1901). Other categories of teachers' beliefs were induced using *open coding* from grounded theory (Strauss, 1987).

RESULTS

I identified five categories of teacher beliefs about students' generalization of learning. These categories fit within three super-categories: content, students' disposition, and students' affect (see Table 1). Content refers to the mathematically specific knowledge students generalize; students' disposition refers to the general orientation towards problem solving students generalize; students' affect refers to student-held beliefs that support students' generalization of learning. The number found within parentheses indicates the number of teachers holding a particular belief. (Note that teachers held multiple beliefs about students' generalization of learning.)

<table>
<tr><th>Content (7)</th><th>Students' Disposition (3)</th><th>Students' Affect (7)</th></tr>
<tr><td>Category 1:
Associations, Procedures, and Formulas (3)</td><td rowspan="2">Category 3:
Orientation towards Problem Solving (3)</td><td>Category 4:
Students' View of Self (6)</td></tr>
<tr><td>Category 2:
Meaning (4)</td><td>Category 5:
Students' View of Mathematics (3)</td></tr>
</table>

Table 1: Categories of teachers' beliefs about students' generalization of learning.

Content

The first two categories of teachers' beliefs about students' generalization of learning involved the role of mathematical content. Specifically, 3 of the 8 teachers seemed to believe that students productively generalize their learning to a novel situation when the novel situation prompts the use of a learned association, procedure, or formula (Category 1). *Association* refers to students linking a specific word, phrase, or image to a particular mathematical response. *Procedure* refers to the use of a pre-determined set of steps to solve a problem. *Formula* refers to the employment of a conventional rule to solve a problem. For instance, Anne believed that students would productively generalize their learning to a novel activity that asked students to select appropriate graphs for given sentences like "We raced down the hill away from the museum" if they were prompted to make use of the associations she had previously instructed them to copy into their notes (e.g., "away" and an unlabeled graphical image of a diagonal line going up as one looks from left to right). She explained that if students focused on phrases like "down the hill" rather than "away" when confronted with such sentences, they would not be prompted to make use of the learned association and would therefore be unsupported in choosing the correct graph. (Note that gender-preserving pseudonyms are used for all participants.)

In contrast, half of the teachers in the study seemed to believe that students' generalization of learning is based on the ways in which students interpret their mathematical activity and the *meanings* they develop for mathematical topics like slope (Category 2). Moreover, these teachers appeared to believe that students' productively generalize their learning when they develop mathematically-valid interpretations of topics like slope, for example, *slope is a ratio providing a description of the multiplicative relationship between two quantities*. Thus, teachers holding this belief made predictions about students' generalization of learning based on the meanings they thought students might develop for a particular topic rather than on whether they thought a particular task would prompt students to make use of a pre-determined association, procedure, or formula. For example, Patrick believed students would be able to find and explain the meaning of slope in a novel slope task involving a burning candle if, during previous classroom activities, they had developed an interpretation of slope as a ratio, or a multiplicative comparison, of two quantities. However, Patrick predicted that students who had not fully developed such an interpretation of slope would attend primarily to the height of the candle, saying a slope of -2.5 means "the candle is shrinking" or "the candle isn't as tall" rather than "the candle burns 2.5 cm *per hour*."

Students' disposition

Whereas teachers in the first two categories emphasized particular mathematical content in their beliefs about students' generalization of learning, the emphasis in this category was on students' more general dispositions toward problem solving. The term *disposition* is used in the spirit of Gainsburg (2007) to refer to students' personal outlook on or orientation towards problem solving; this includes what problem solving is about. Teachers seemed to believe that students productively generalize their learning to novel situations when they develop and make use of particular dispositions. Moreover, these teachers appeared to believe that the dispositions themselves carry over to novel situations and function to facilitate students' generalization of learning. For instance, Emma shared that students will be better able to "assess where to go" and "find the solution" in novel problem-solving situations if their orientation towards those situations is one of sense-making and visualization of the problem (e.g., by asking questions like "What is *actually* going on here?" rather than "What equation do I use to solve this—what is the formula?").

Students' affect

The last two categories of teachers' beliefs about students' generalization of learning involve the role of students' affect, specifically students' beliefs. To avoid confusion from using the word "belief" twice, I use the word "view" in reference to the students.

Hence, Category 4 involves teachers' beliefs regarding the role *students' view of self* plays in their generalization of learning and Category 5 involves teachers' beliefs regarding the role *students' view of mathematics* plays in their generalization of learning. This follows McLeod (1992) who conceived of both views of self and views of mathematics as components of the affective domain in mathematics.

Six of the 8 teachers in this study seemed to believe that students generalize their learning to novel situations when they develop confidence in their ability to engage in mathematical activity (Category 4). Here, confidence refers to a student's view of his or her "competence in mathematics" (McLeod, 1992, p. 583) or the "belief that one can learn to do that which is expected of one" (Broekmann, 1998, p. 18). These teachers believed that students' generalization of learning is dependent upon how confident a student is that he/she can engage in mathematical activity. For instance, Donna said, "It's hard to get kids to generalize [their learning] … because you have to break down their beliefs of 'I just suck at this; I don't know anything." She went on to say, "For me, it is making that 'Ah-ha' like 'Oh, I *can* do this.' … You have to build self-esteem into those learners like 'No, you're not stupid.'" Similarly, 3 of the 8 teachers in this study seemed to believe that students generalize their learning to novel situations when they view mathematics as relevant and useful outside of the mathematics classroom (Category 5).

These beliefs seemed vague in the sense that they were not well specified as mechanisms for supporting students' generalization of learning. It could be that the teachers in these categories believed students' views acted like a key to unlock the door to their engagement with new situations thereby creating an opportunity to apply particular mathematical understandings. Alternately, it could be that teachers believed students' views acted at a more general level allowing students to productively engage with new situations regardless of the particular mathematical topic. In this way, the limits of these beliefs regarding students' generalization of learning remain unclear.

CONCLUSION

The findings outlined above point to the importance of bringing teachers into the ongoing conversation regarding students' transfer of learning. Using artifacts from their own teaching, teachers were engaged in conversations about transfer using the terminology of students' "generalization of learning." This resulted in the identification of new beliefs about students' generalization of learning. In other words, talking to practicing teachers resulted in the identification of beliefs not found in the transfer literature. Looking across Categories 4 and 5 (see Table 1), the role of students' affect was present in 7 out of 8 of the teachers' beliefs about students' generalization of learning despite the fact that it is absent in the transfer literature. This finding indicates that while researchers have yet to identify affect as an important factor in the generalization of students' learning, teachers have.

This is not to say that overlap did not exist between teachers' and researchers' beliefs regarding students' generalization of learning. For example, Anne (from Category 1) appeared to hold the Thorndikean belief that transfer is mediated by common associations (cf., Thorndike & Woodworth, 1901). However, the teaching items she selected to illustrate her belief were drawn from reform-oriented and constructivist-inspired textbooks suggesting practice-based decision-making that transcended a Thorndikean approach to curricula.

The Category 3 belief is similar to Bereiter's (1995) *dispositional* approach towards problem solving wherein Bereiter argued that students' "way of approaching things" is of primary concern when teaching for transfer (p. 23). Teachers holding this belief emphasized, in the spirit of Bereiter, the ways in which students orient towards problem solving and the roles their dispositions play in the generalization of their learning. However, Bereiter illustrated his ideas with examples from moral education and science education. Thus, the particular dispositions articulated by the teachers in this study (e.g., a visualization and sense-making disposition) are new to the transfer literature. In other words, by talking to teachers of mathematics, I was able to identify specific beliefs about dispositional approaches to transfer relevant to the field of mathematics education.

Lastly, the fact that all but one of the teachers appeared to hold multiple beliefs regarding students' generalization of learning suggests that *in practice* multiple beliefs about students' generalization of learning may actually function together. Together, these findings suggest that investigations into the transfer of student learning may benefit from a shift in point of view—from the eyes of researchers to the eyes of practicing teachers.

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WHAT DETAILS DO GEOMETRY TEACHERS EXPECT IN STUDENTS' PROOFS? A METHOD FOR EXPERIMENTALLY TESTING POSSIBLE CLASSROOM NORMS

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We report on the development and piloting of a method that provides a sufficient condition for confirming that an observed regularity in a classroom is a norm. The method we describe is a refinement of the breaching experiment technique (Garfinkel, 1963; Mehan, 1979) that uses random assignment to experimental conditions as a means to facilitate controlled comparisons between participants' reactions to different episodes of instruction. We use this method to confirm the existence of normative ways that teachers scrutinize the details of proofs in geometry.

INTRODUCTION

International comparisons of teaching have brought attention to the notion of cultural scripts and the claim that regularities are observed across episodes of teaching in a given country (Stigler & Hiebert, 2009; Santagata & Stigler, 2000). The existence of these cultural scripts is warranted by observations of different teachers who share a national culture engaging in stable patterns of classroom activity—patterns that are similar to each other yet distinct from patterns of teachers from other national cultures (Stigler & Hiebert, 2009). On account of the scale of such comparisons, the identified scripts have been largely subject-independent and rather general. Furthermore, the extent to which cultural scripts capture *norms* of classroom action—that is, what is expected to happen, for the absence of which would be seen as a violation of the social order (Garfinkel, 1963)—as opposed to provide descriptions of what is observed to happen in classrooms—is an open question. A social norm is not merely an action that might be frequently observed, but actually an action that participants expect (or expect their coparticipants) to engage in. Developing methods for identifying the classroom regularities that are actually norms is pressing because providing an account of what teachers expect to happen in classrooms—as opposed to just recording those things that do happen—brings us closer to understanding what it might cost to change classroom instruction.

In this paper, we report on the development and piloting of a method that provides a sufficient condition for confirming that an observed regularity in a classroom is a norm. We use for that the classroom activity *doing proofs in geometry* (Herbst & Miyakawa, 2008) and norms that we call *semiotic norms*. By semiotic norm, we mean a norm of the way in which semiotic resources (e.g., written words, diagrams) are used to produce and evaluate mathematical work. The method we describe is a refinement of the breaching experiment technique (Garfinkel, 1963; Mehan, 1979) that uses

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random assignment to experimental conditions as a means to facilitate controlled comparisons between participants' reactions to different episodes of instruction. The method we developed for confirming the existence of classroom norms will help researchers describe more precisely the mathematics that students have an opportunity to learn and will also help identify levers for piecemeal alterations to curriculum and instruction in order to improve the mathematical quality of the work students are involved in.

THEORETICAL FRAMEWORK

Doing proofs in geometry is an example of an *instructional situation*: A stable segment of classroom activity within which students trade (or "cash") completed work for a claim—from the teacher—that they have acquired a particular item of knowledge (Herbst 2006; Herbst & Chazan, 2011). When doing proofs, the work to be produced is a proof of a particular mathematical statement and when a proof is so produced it may be exchanged (i.e., cashed) for a claim that some knowledge exists implicit in that proving work (such as the knowledge of how to produce a specific kind of mathematical argument). Within any instructional situation, the exchange of work for knowledge-claims is made possible through the available semiotic resources (Herbst & Chazan, 2012) that, together, comprise the semiotic currency of the situation. We are concerned with describing the normative ways that semiotic resources are used in such situations, or what we call semiotic norms.

From the perspective of social semiotics (van Leeuwen, 2004), instructional situations may be conceptualized as *genres* of classroom activity that have different realizations (Christie, 1997; Lemke, 1990; Martin & Rose, 2008). From video records of different geometry classrooms doing proofs[1], we identified presenting/checking a proof as a realization of the doing proofs situation in which the teacher presents a completed proof to the class and the students in the class take turns scrutinizing its written steps. In video episodes of different geometry lessons, there were recurring instances of details of the proof being insufficient under such scrutiny. These included instances when conceptual entailments—such as the conclusion that two angles that form a linear pair are supplementary—were not unpacked into more primitive steps (i.e., a statement that identifies such angles as being a linear pair followed by a statement that angles forming a linear pair are supplementary) and instances when distinctions between geometric objects and their measures (such as a segment versus the length of a segment) were not strictly enforced. Since the kinds of details that were scrutinized in the written arguments of proofs recurred in different geometry classrooms, we hypothesized the existence of a *details norm* when checking proofs. To confirm that there are, in fact, normative ways of scrutinizing the details of a proof, we devised a planned comparison study between groups of teachers in treatment and control conditions. The design of the experiment and the results of the data analysis are reported in the next sections.

[1] This data corpus had been gathered with the support of NSF grant 0133619 to P. Herbst.

METHOD

The method we developed to confirm the existence of the details norm combines the technique of a virtual breaching experiment (Herbst & Chazan, 2003; Nachlieli & Herbst, 2009) with a planned comparison study. As a virtual breaching experiment, we developed storyboards consisting of a sequence of classroom images to represent episodes of high school geometry lessons and showed these to participants. There were from 9 to 22 images in each storyboard. These scripted image sequences were adaptations of geometry lessons that were based on video recordings of classrooms doing proofs. As a planned comparison study, participants were randomly assigned to treatment and control conditions in which the teacher in the storyboard episode breaches (treatment) or complies with (control) the details norm. The purpose of randomly assigning participants to conditions was to be able to compare reactions (both within and across conditions) to the different lessons.

We used image sequences, rather than actual video, for two reasons. One, since we wanted to compare reactions to episodes where a norm is breached to reactions to episodes where a norm is not breached, using actual classroom video was not feasible, since in actual geometry classrooms, the norm is not usually breached. Second, using sequences of images allowed the breach and control conditions to feature representations of instruction that were minimally different from each other—that is, for a given instructional episode, its breach and control versions were identical except for those images in the sequence that depicted the breach of (or compliance with) the norm. This principle of minimal variation allowed us to make comparisons across the conditions.

What is described above as the details norm was the subject of four classroom stories (A, B, C, D), and each of these classroom stories had a version (A', B', C', D') in which the norm was breached and a version in which the norm was not breached. In stories A and B, the teacher allows minor omissions[2] in the written argument of a proof to stand without correction (thus breaching the norm), while in stories A' and B'—the control duals of A, B, respectively—the teacher corrects the omissions. In stories C and D, the teacher insists that students explicitly justify claims[3] that are tacitly warranted by a diagram (thus breaching the norm), while in stories C' and D'—the control duals of C and D, respectively—the teacher uses the diagram to elide some steps in the proof.

As a group, these four sets of stories concern the necessary details of the semiotic currency for a valid exchange of proof-for-credit when doing proofs. We hypothesize that the teacher in stories A and B would be seen as breaching the details norm because the teacher accepts less detail in the written argument of a proof than what is usually

[2] Respectively: failing to include an explicit step that establishes the congruence of two segments from the definition of midpoint, and failing to distinguish between angles and their measures.

[3] Respectively: that a point of intersection between two rays exists, and that two angles are collinear.

required, while the teacher in stories C and D would be seen as breaching the details norm because the teacher asks for more details than what is usually required. The instrument we developed thus allowed us to test two different ways in which the details provided in a proof might be seen as breaching the norm. We thought important to test both hypotheses to be able to argue that the norm is not actually a generic one (insisting on detail, no matter what detail), but rather a mathematically specific one—some details are insisted upon, others frowned upon, and the semiotic systems involved are the bearers of the distinction.

The structure of the instrument was the same for all stories: participants were shown one of the classroom stories, then asked a series of questions. These included a general open response question—“What did you see happening in this scenario?”—a general closed-response rating question—“How appropriate was the teacher’s review of the proof?”—and two targeted, closed-response rating questions (described below). All of the rating questions used the same 6-point Likert-style rating scale, with choices from 1 (very inappropriate) to 6 (very appropriate). The rating questions also included a “please explain your rating” follow-up prompt.

For the targeted rating questions, participants were shown a “clip” of the story (that is, a segment of the storyboard) that focused on a particular teaching action. One of these targeted rating questions showed participants the 3 to 5 image clip in which the norm was either breached or not breached, stratified by condition. The purpose of this targeted rating question was to focus participants’ ratings on the part of the story where the teacher complies with or departs from the norm. Participants were also asked to rate a different clip. For this other targeted rating question, participants in the breach/control conditions were shown identical sets of 3-5 images in which the teacher in the story does a routine instructional action unrelated to the target norm. It was possible to identify such clips because each set of breach/control stories were identical except during those parts of the story that represent the breach of (or compliance with) the target norm. The purpose of including the two types of targeted rating questions was to enable comparisons across the breach and control conditions. These comparisons and their results are described below.

DATA

We gathered data from 34 high school teachers (working in different schools and districts) during a pilot study in the fall of 2013. The teachers were randomly assigned to treatment and control conditions. 16 teachers were assigned to a condition where they viewed stories that breached the norm (7 assigned to the “less details” breach, 9 assigned to the “more details” breach), and 18 teachers were assigned to a condition where they viewed stories that complied with the norm (9 assigned to the “less details” control, 9 assigned to the “more details” control). Within each condition, a teacher either viewed two stories that breached the norm or two stories that complied with the norm. No participant viewed the breach and control version of the same story, and the order in which the stories appeared was randomized (to neutralize any effects from the order in which the stories are viewed). Since each participant viewed and rated two stories, there were 32 responses to each question about stories where the target norm was breached and 36 responses to each question about stories where a norm was not breached.

ANALYSIS AND RESULTS

The study was a planned comparison study between participants assigned to treatment and control conditions. Since a norm is not only what is routine but also what is expected, we hypothesized that participants would find the work of the teacher less appropriate in those stories that breached the hypothetical norm. We made three comparisons of answers on closed-response questions both across and also within the different conditions to test this hypothesis. The first was a comparison of the mean scores on the general rating question across the breach and control conditions. The second was a comparison of ratings on the targeted rating questions between breach conditions and control conditions, while the third was a within-condition comparison between ratings on the targeted rating questions—i.e., comparing ratings on the breach/nonbreach storyboard segments to the ratings on the other storyboard segment within a condition. These comparisons and the results of statistical tests are reported below.

Comparison 1: Across condition comparison of mean scores on the general rating questions

The general closed-ended rating question asked participants to rate the appropriateness of the teacher's review of the proof: "how appropriate was the teacher's review of the proof?" There were 32 responses to this question across 4 stories that breached a norm, and 36 responses to this question across 4 stories that complied with a norm. Using the 6-point Likert-syle rating scale for appropriateness described above, the mean rating of the breach responses was 1.14 points lower than the mean rating of the control response (3.47 compared to 4.61, respectively), a statistically significant difference in means at the .05 level (two-tailed, heteroscedastic t-test assuming unequal *Ns*, $p<.01$). Because of the random assignment, this difference in means provides some evidence that any secondary math teacher would notice when the details norm is breached when doing proofs in geometry.

Comparison 2: Across condition comparison of mean scores on the targeted rating questions

The targeted rating questions asked participants to rate the appropriateness of the teacher's actions at a specific place in the story. Each participant answered two types of targeted rating questions: one that targeted the place in the story where the teacher breaches (or complies with) the norm, and one that targeted a moment in the story when the teacher engages in some other action. We refer to the first type of targeted rating question as the "targeted breach/compliance" (TBC) rating and the second as the "targeted distracter" (TD) rating. By design, the TD rating questions targeted an action that appeared in both the breach and compliance versions of a story, so participants across the conditions viewed identical story segments when answering this rating question. The purpose of including these targeted rating questions was to be able to compare ratings both across and within conditions at specific points in the stories.

Across the conditions (32 and 36 respective responses, as above), the mean ratings on the TBC questions for those who viewed breach stories was 1.61 points lower than the mean rating on the TBC questions for those who viewed compliant stories (2.78 to 4.39, respectively), a statistically significant difference at the .05 level (two-tailed, heteroscedastic t-test assuming unequal *Ns*, $p < .001$). This significant difference in means on the rating questions that target the moments in the stories that either breach or comply with the norm is complemented by a non-significant difference in means on the TD rating questions: 4.1 (breach) to 4.6 (control), a .5 difference that is not significant at the .05 level (two-tailed, heteroscedastic t-test assuming unequal *Ns*, $p = .13$). The significant difference in mean TBC ratings together with the non-significant difference in TD ratings suggests that participants' overall lower ratings on the breach stories (compared to the control stories, reported above) are linked to the teacher's breach of the norm, rather than some other action the teacher takes in the story. The experimental design and the deliberate scripting of the stories to be identical in all places except for where the teacher breaches the norm underscores this point.

Comparison 3: Within condition comparison of mean scores on the targeted rating questions

Further evidence that participants were responding to breaches of a norm—as opposed to other aspects of the stories—comes from within condition comparisons of the targeted rating questions (32 and 36 responses, as before). For the breach stories, the mean scores on the TBC ratings was 1.31 points lower than the mean scores on the TD ratings (2.79 to 4.1, respectively), a statistically significant difference at the .05 level (paired, two-sample t-test, $p < .001$). Complementing this, there was no significant difference between TBC and TD ratings for the stories in the control condition (means scores of 4.39 and 4.6, respectively, $p = .15$). The fact that, in the breach condition, participants' ratings on the TBC questions were significantly lower than their ratings on the TD questions—together with the fact that there were no such significant differences between the targeted rating questions for participants in the control condition—indicates that participants noticed the moments in the episodes of instruction when teachers were shown departing from the norm.

Open-response data

The open response data also indicate what participants view as appropriate or inappropriate ways of scrutinizing a proof. For example, a participant who viewed story D—one in which the teacher breaches the details norm by problematizing the existence of a point of intersection for the angle bisectors of a parallelogram—remarked: "The rays [of the parallelogram] intersect by definition. We don't need a theorem to justify it (participant ID 2248)". As a comment on this same story, another participant remarked: “I don't think we need to validate the fact that the two rays intersect here. This is…focusing on *minutia* that will prevent kids from focusing on the important parts of the problem (participant ID 2333, emphasis added).” Yet other open responses indicate that the scrutiny of some aspects of a proof is compulsory. For example, a participant who viewed story B—one in which the teacher allows a student to make statements about the sum of the angles of a triangle as opposed to the sum *of the measures* of the angles—said: “The teacher is down-playing the little things. Sometimes those little things can change the whole outcome (participant ID 2300).” Viewing this same story, a different participant commented: “When you do proofs, you *can't assume anything* (participant ID 2359, emphasis added).” These comments would seem to be directly at odds with those reported above. That both under-scrutiny of the written argument (second example responses) of a proof and hyper-scrutiny of the diagram accompanying a proof (first example responses)—practices that could be seen as equivalent from the perspective of justifying every step in a proof—can draw the concern of secondary teachers provide evidence that the routines for checking the details of a prof are, in fact, norms.

Two-column proof has been criticized for being ritualistic or attentive to excessive detail (e.g., Harel & Sowder, 1998; Schoenfeld, 1988); however, our research shows that such statements are too broad—attention to detail depends on what details are being considered and how those details are being expressed. When it comes to statements—such as the existence of a point of intersection—that are tacitly warranted by a diagram, participants reacted unfavorably to episodes that showed a teacher asking for the explicit justification that would warrant those statements, on the grounds that doing so was focusing on minutia. However, when it comes to statements—such as deducing the congruence of two segments from the definition of midpoint—that are tacitly entailed by definitions, participants reacted unfavorably to episodes that showed a teacher *not* asking for the explicit justification that would warrant those statements, on the grounds that every step in a proof requires an explicit justification. That teachers would hold different views of the appropriate level of detail in a proof is not *a priori* obvious, and the account we have provided highlights the affordances of the method we have developed.

CONCLUSION

The research reported here describes a method for confirming that a routine classroom practice is a norm and uses that method to confirm the existence of semiotic norms when doing proofs in geometry. The articulation of a semiotic norm contributes an elaboration of the theory of instructional exchanges, while its experimental confirmation contributes a method that can be used to identify normative practices in instruction. More generally, we have shown that representations of lessons may be used in an experimentally controlled way to target what teachers notice about instruction.

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AN EXPERT TEACHER'S LOCAL INSTRUCTION THEORY UNDERLYING A LESSON DESIGN STUDY THROUGH SCHOOL-BASED PROFESSIONAL DEVELOPMENT

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Using data from a research project in Shanghai, China, this paper reports on an expert teacher's implicit 'Local Instruction Theories' (LIT) (Gravemeijer, 2004) that underpin his guidance of a junior teacher in lesson design and implementation. Our analysis focuses on the expert teacher's input to the junior teacher to help her understand how and why to redesign a lesson as part of a school-based teacher professional development project. We identified three key points of the expert's implicit LIT: mathematics has its own form of exploration; each student should have their own thinking path at each key point of the learning process; and each student should not only be able to experience use of their own representation, but also learn about other students' representations and the excellence of representations.

INTRODUCTION

At a PME36 Research Forum, Li and Kaiser (2012) examined "the concept and nature of teacher expertise in mathematics instruction valued in selected education systems" (p121). In doing so, they highlighted different approaches, practices and cultural resources that are used to develop teacher expertise in mathematics instruction in different countries. In similar vein to Jaworski (2004), who sees teachers and educators working together in an *inquiry community* and in a "reciprocal relationship of a reflexive nature" (Jaworski 2001, p. 315), the analysis of five nation-wide teacher professional programs (Canada, China, Japan, Norway, and USA) by Kieran, Krainer and Shaughnessay (2013) concludes that teachers should be viewed as key stakeholders in research – "stakeholders who co-produce professional and scientific knowledge" (p. 387).

In Shanghai (SH), China, Gu and Wang (2003) have proposed the 'Action Education' (AE) model ('Xingdong Jiaoyu' in Chinese) to tackle the challenge of improving teaching through inservice teacher professional development (TPD). Three key features are emphasized in the AE model: the use of *Keli* ('exemplary lesson development' in English) (see Huang & Bao, 2006), the collaborative work of teachers with expert teachers and university researchers (mostly local but sometimes foreigners in the case of SH), and teacher follow-up reflection and action in their own class. Paine and Fang (2006, p286) consider that this SH AE as a hybrid model – a means of connecting Chinese educators to foreign ones – that characterizes reform in Chinese TPD. Such a teacher/expert collaboration attempts to develop and promote the

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teacher's expertise by absorbing and building on a combination of Chinese experts' accumulated "wisdom of practice" (Shulman, 1986, p. 9) and international expertise. Given the long tradition of China's own cultures of teaching and learning (Paine, Fang, & Wilson, 2003), it remains under-researched how this combination works out in practice. It is this that is a focus of our research.

In a previous paper, Ding, Jones and Pepin (2013) report how an expert teacher guided a junior teacher to develop what we called a 'hypothetical learning structure' (HLS) in her lesson design. We carefully distinguished this HLS from Simon's (1995) 'hypothetical learning trajectory' (HLT), as the HLS in our study was not based on constructivist theory but rather on the Chinese expert teacher's 'wisdom of practice' in the form of their expertise and experiences with local classroom practice. In this paper, we seek a deeper understanding of the pedagogical principles of this local expert teacher through studying his coaching of a junior teacher during our lesson design study.

In this we refer to Gravemeijer's (2004) '*local instruction theories*' (LIT) of the expert teacher. As pointed out by Gravemeijer (2004), local instruction theories go "beyond the level of an instructional sequence in terms of a series of instructional activities" (p. 108); rather, LIT are a "description of, and *rationale for*, the envisioned learning route" (p. 107; emphasis added). Our research question in this paper is: "what are the expert teacher's implicit LIT that underpin his guidance of a junior teacher in lesson design and implementation, with the particular teaching objective of developing individual children's mathematical reasoning in the class?"

THEORETICAL FRAMEWORK

Simon (1995) suggested the HLT as a way to consider the reflexive relationship between a teacher's design of activities and considerations of students' thinking as the students engage and participate in particular classroom tasks. As pointed out by Simon (1995), the term HLT underscores the importance of having a goal for teaching, some ideas for learning activities, and a sense of the direction of students' learning. The HLT consists of three components: the learning goal; learning activity/ies; and the hypothetical learning process.

Gravemeijer (2004) points out that it is not easy for teachers to design the HLT for reform mathematics in which the aim is to transform of students' current ways of reasoning to more sophisticated ways of mathematical reasoning. The central problem that teachers face involves the tension between the openness toward the students' own constructions and the obligation to work toward certain given endpoints. As Gravemeijer (2004) clarifies:

> I reserve the term *hypothetical learning trajectories* for the planning of instructional activities in a given classroom on a day-to-day basis, and I use the term *local instruction theories* to refer to the description of, and rationale for, the envisioned learning route, as it relates to a set of instructional activities for a specific topic. (p. 107)

That is, the term local instruction theory is coined to "convey the intention of offering more than a description of a learning route, or the corresponding instructional activities. In addition to these two, a local instruction theory also includes a rationale" (Gravemeijer, 2004, p. 100). As such, and akin to Simon's HLT with the addition of a rationale, the conjectured LIT consists of three components: (a) learning goals for students; (b) planned instructional activities and the tools that will be used; and (c) a conjectured learning process in which one anticipates how students' thinking and understanding could evolve when the instructional activities are used in the classroom.

In our study, we use the three components of Gravemeijer's (2004) conjectured LIT (noted above) to analyse both the junior teacher's and the expert teacher's pedagogical thinking and decision-making during the lesson design and implementation, as well as during the lesson redesign.

METHOD

Our school-based TPD study is being conducted in a local laboratory school located in Qingpu district, a western suburb of SH (see also Ding et al., 2013). The overall methodological approach of our TPD study is in the form of the AE model by Gu and Wang (2003) that aims at developing the teacher's professional knowledge – in the nature of absorbing and building on the accumulated "wisdom of practice" (Shulman, 1986) – through the teacher's lesson planning, lesson delivery, post-lesson reflection and lesson re-delivery. Two features highlighted by Huang and Bao (2006) distinguish the SH AE model from other types of TPD used in other countries – such as 'Japanese Lesson Study', case inquiry (Shulman, 1986), and course-based training and workshops: (1) the expert's input to upgrade teacher ideas in the context of peer support; and (2) the whole process of teacher action follow-up and reflection is included. At the present stage of our data analysis, we particularly focus on the expert's input to the junior teacher to help her understand *how* and *why* to redesign the lesson.

The participant groups of the study were: (1) four researchers (the four authors); (2) an expert teacher (Mr Zhang); and (3) three teachers (two in Grade 3 (G3) and one in G4; 4) twelve mathematics teachers from the mathematics teacher group of the school (from G1 to G6, ranging from newly-appointed teachers to teachers with about ten years teaching experience). In this paper we focus on one of the G3 teachers, who we call Peipei (a pseudonym), who, at the time of the research, had four years teaching experience in primary school mathematics.

Our data sources include: Peipei's initial lesson plan and accompanying classroom tasks; the transcript of her video-recorded lesson; the transcript of the video-recorded comments of the expert teacher and his work/documents to redesign the lesson and tasks; and the transcript of the video-recorded re-taught lesson.

The analysis of the development/design research approach (Gravemeijer, 2004) was used to analyse the cumulative interactions between the junior teacher's initial lesson design and implementation, and the expert teacher's comments and lesson re-design.

In so doing, we aim to make the expert teacher's implicit LIT explicit, and explain *how* and *why* the teacher reflected and revised her mathematics teaching across an interactive series of teaching cycles.

FINDINGS

Understanding the learning goal of the lesson

In Peipei's initial lesson plan and implementation, we found that the teacher tried to guide students to achieve the learning goal given in the SH official teacher's textbook reference (TTR). The TTR suggested the teacher to make one point of mathematical knowledge clearly to students in the lesson inquiry activity. In this case, the core of the inquiry was the relationship of the area, length and width of rectangles (including squares) with the constant perimeter as a stepping stone to understanding the relationship of the constant sum of two numbers and the maximum product of them.

After observing Peipei's initial lesson, the first point that Mr Zhang suggested to Peipei was carefully to consider about the learning goal suggested by the TTR. Mr Zhang explained to Peipei the learning goal as follows:

> In primary mathematics, this content is considered as a typical topic to learn how to establish a mathematical proposition. Strictly speaking, it is not about concept learning, but about proposition learning [learning how to find laws and relations in mathematics].

Redesigning the instructional activities and the tools

In the initial lesson, Peipei directly used the task given in the textbook (using 20 matches to form rectangles and to find the largest area). To achieve the learning goal explained in the TTR, Peipei organized three main instructional activities in her lesson plan and implementation: (1) The starting activity: Peipei asked students to use four numbers 1, 3, 4, 5 to combine two two-digital numbers, and then to guess which of the two to multiply to get the largest result. (2) The main activity: Peipei asked students to cooperate in a group of four students and to respectively use 20 and 18 matches to form rectangles and to record the possible length, width and area of rectangles with the constant perimeter on the worksheet. Students were also asked to use mathematical language to represent their findings on the worksheet. (3) The exercise activity: One of the tasks in this activity was to ask students to find the larger product of 94×83 and 93×84.

Mr Zhang considered that Peipei constructed the learning process not from the perspective of students, but from the perspective of the textbook. Mr Zhang said the following:

> From the teaching perspective, the logic of the lesson structure [the three instructional activities] is clear. If the teacher added one more activity to ask students to talk about the conclusion of the lesson, I guess most students could make it. Such a way of teaching is very traditional as it merely concerns on students' learning product, not on their learning process. However, students would gain benefits from the learning process, not merely from the learning product. The application of the learning product is based on students' learning

> experience, method and thinking path. To support individual learning, the teacher should address questions [pertaining to] students' starting points in their own learning and experience and what they can achieve in the lesson.

Accordingly, Mr Zhang suggested to Peipei not to use the activity of four numbers 1, 3, 4, 5 to start the lesson. Instead, Mr Zhang suggested Peipei to start the lesson by using a smaller number of matches so as to enable students with various levels of skills to handle the task within the available lesson time. The instructional activities were redesigned to enable students to experience the whole reasoning process of rediscovering the mathematical proposition (e.g., observation/operation – guesses – plausible reasoning / proving – using proper representations and language to represent the mathematical proposition) as follows: (1) Starting activity: students were asked to use matches to form rectangles and then to record the length, width, perimeter and area of the rectangles in a table; (2) Follow-up activity: students could make guesses and reasoning about their findings and then confirm their own guess. (3) Conclusion of the activity: students should learn to use different representations (e.g., drawing, symbols, their natural language and mathematical language) to characterize and to simplify the mathematical proposition of the relationship of perimeter and area of rectangles.

The expert teacher's implicit LIT

We analysed the complexity of Mr Zhang's implicit LIT according to his perspectives on students' learning methods, students as active learners, and students' mathematical reasoning development.

Students' learning methods: Mr Zhang highlighted the role of the worksheet as an effective tool to develop individual students' independent learning method. For instance, in the redesigned starting activity, students were given opportunities to independently decide the length and width of rectangles and the number of matches. As the worksheet was A4 size, the space was limited for students to draw and put matches on the worksheet. A maximum of 10 matches could be used. In using the worksheet, students would have opportunities to experience the process of reviewing their own previously learned knowledge of perimeter and area of rectangles and squares, drawing and forming rectangles and gradually to develop their reasoning of their observations and guesses.

Students as active learners: Mr Zhang explained to Peipei the complex relationship between the cognitive processes of an individual student and the classroom learning community. Mr Zhang's view is evident in his discussion with Peipei about students' group discussion, as follows:

> Students' group [or class] discussion is based on each individual's own learning experience and the related learning results. It would be too abstract for students if the teacher asked students to discuss their observation during the starting activity. Because students had not yet experienced the cognitive processes such as from sample [of matches] to operations [form rectangles by matches], and from the diagram to language, the group [or class] discussion encouraged by the teacher was from one student's language to another

> student's language. The individual student's cognitive process was interrupted by others' discussion. Sometimes, other students' talk is positive to develop the individual student's thinking development. Yet, other times, it may prevent the individual student's independent thinking. The teacher should reflect on her role of how to enable each student to develop their own learning outcome and then how to help students to correct and revise their learning experience.

Mr Zhang further used an example to explain to Peipei the teaching strategy of how to tackle such complexity of the relationship between individual, group and class during the follow-up activity in the lesson: (1) individual students should be selected by the teacher to report their worksheet data to the class; (2) a group of students would share the similar data (due to the same size of worksheet); (3) the whole class could share all reported data listed on the blackboard.

Students' mathematical reasoning development: Mr Zhang referred to two theoretical ideas to address the teacher's role in students' mathematical reasoning development: (1) the teacher can use *variation* as a means of "Pu dian" (scaffolding in Chinese) (Gu, 2012) to enable different students' reasoning and representations to be shared in the whole dynamic mathematical activity; (2) the teacher should ensure that at each key point of the learning process, each student should have their own thinking path. He said:

> Students should first develop their independent representation of their findings. After that, they can present their representations in the class. They would then learn from their peers in the class which representation is correct or incorrect, which one is a suitable, rigorous or scientific form of representation. The representation of mathematical proposition is complex as it can be represented by multiple languages and reasoning path. The teacher should ensure that each student not only has learning opportunities to demonstrate their representations and to compare with others, but also to learn to appreciate the excellence of the multiple forms of representation.

DISCUSSION AND CONCLUSION

By analysing the cumulative interaction between the junior teacher's initial lesson design and implementation and the expert teacher's comments and lesson redesign, we can identify three key points of Mr Zhang's implicit LIT as follows:

1. Mathematics has its own form of exploration. The teacher should think about how to develop students' ways of mathematical reasoning during their exploration process. The lesson should be designed in such a way that students are able to experience on their own the whole process of plausible reasoning in mathematics.
2. To experience the whole process of mathematical reasoning (plausible reason in this study), the construction of the learning process should focus on each individual student. That is, at each key point of the learning process, each student should have their own thinking path. Each student should enjoy a whole process of their own independent thinking in the learning process.

3. Mathematical proposition is complicated for it can be represented by multiple kinds of languages and various types of thinking. Each student should not only be able to experience to use their own representation, but also to learn others' representations and the excellence of representations.

In our previous studies (Ding et al., 2013; Ding, Jones, Pepin & Sikko, 2014), we focused on the expert teacher's voice. For instance, in Ding et al. (2014) we reported that while guiding the teacher to understand the new teaching norms from the overseas textbooks (e.g., Pepin & Haggarty 2001), the expert teacher simultaneously encouraged our case study teacher to use the traditional Chinese 'two basic' (basic knowledge and skills, briefly named as TB) teaching (e.g., Shen Tou) method carefully to develop students' TB in mathematics. In this paper, the expert teacher highlighted an alternative teaching method (Pu Dian) in the redesigned activities to develop students' mathematical reasoning. The expert teacher's voice on the empirically-grounded teaching approaches echoes Shulman' (1986) influential work on the nature of teachers' professional knowledge development – absorbing and building on the accumulated "wisdom of practice". In our case, it is as a *key stakeholder* (Kieran, et al, 2013) in our *inquiry community* (Jaworski, 2004).

Li, Huang and Yang (2011) show the complexity of the Chinese expert teachers' teaching expertise valued in China. In our study, we showed the complexity of the expert teacher's implicit LIT. As Mr Zhang pointed out, 'at each key point of the learning process, each student should have their own thinking path'. That is, while individual students participate into the group and class-shared thinking process, they should not stop their own thinking path and passively listen and take others' thinking path. Others' thinking path should be considered as an alternative means for individuals to develop and complete their independent thinking path. If we borrow Simon's (1995) metaphor of travel plan, the teachers ought to have a sophisticated 'travel plan' not only for one individual, but for the class of pupils. Our next step in our project is towards understanding the expert teacher's sophisticated 'travel plan' that makes the connection to each individual student's thinking in their mathematics learning journey within the class.

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UNPACKING CHILDREN'S ANGLE "GRUNDVORSTELLUNGEN": THE CASE OF DISTANCE "GRUNDVORTSELLUNG" OF 1° ANGLE

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Results of the last thirty years in mathematics education have shown the importance of an operational concept development. One of the geometrical concepts that has been researched for years already, however, not with the particular focus on its systematic teaching in school mathematics, is the concept of angle. In this paper we focus on children's understanding of the angular size of 1° and its development obtained through a test followed by a task-based interview. The interview results with 9 pupils showed that they have a fragmented understanding of the angle concept, enabling them to fully grasp what 1° angle is. Moreover, many of the children's misconceptions were directly connected to the measuring tool, namely set square, and angle notation. Implications for systematic teaching of the angle concept are given at the end.

INTRODUCTION

The angle concept is a fundamental concept of plane geometry and central to the development of geometric knowledge and thinking. This concept is not only relevant for the entire geometry teaching, but also in everyday situations and in different careers. In Germany, the angles, namely right angle, get introduced at the elementary level, but its systematic learning starts at 5th grade and lasts throughout grade 10. Both the state curriculum and the standards give guidelines as to what ideas, knowledge with respect to the angle concept should be learned. The angle concept, although being an elementary concept of plane geometry, poses problems for many middle-school and high-school students; the students have no sense of angle size, have fragmented knowledge of angle aspects, lack knowledge of angle attributes, do not understand the protractor as a measuring tool, and so on (Dohrmann & Kuzle, 2013, in press; Krainer & Cooper, 1990; Mitchelmore & White 1995, 2000; Van de Walle, 2001). For that reason, project WiKUL (***Wi****nkel* ***k****onstruktiv* ***u****nterrichten und* ***l****ernen*, that is teaching and learning of angle concept under constructivist epistemology) was developed. The goal of the project is to understand which of these ideas and operations about angle are encountered by middle- and high-school students and how the angle concept can be conveyed to students in a meaningful manner by using fundamental ideas of concept learning to prevent the development of angle misconceptions. For the purpose of this paper we focus on one of these aspects, namely students' angle measure understandings past elementary level, which was focus of previous research (Kaur, 2013; Mitchelmore & White, 1995, 2000) and have prevailed in our previous research (Dohrmann & Kuzle, 2013, in press).

THEORETICAL PERSPECTIVE

The nature of the angle concept has been vividly debated for over two thousand years, and the discussion is not close to be over (Krainer & Cooper, 1990). This discussion resulted in three different definitions or aspects as well as different representations being typified in school mathematics. Having this complexity in mind, research has shown that students have serious misconceptions about the concept of angle based on their personal experiences. For that reason a somewhat radical approach is needed to alter preexisting concept structure. With this in mind, the conceptual change theory is becoming more prominent in the mathematics education research to explain student's difficulties in learning mathematical concepts (Posner et al., 1982). According to this this theory, that draws from both Kuhn's sociology of science and Piaget's developmental psychology, learning can occur in two manners: (1) new knowledge is added to the prior knowledge (assimilation) and (2) old knowledge is first reconstructed as a result of disequilibrium or conflict when confronted with new knowledge (accommodation) before the conflict can be resolved or it gets overthrown (rejection) by the learner. Following this process students can then undergo the process of accepting, integrating and using the new concepts.

Though the conceptual change approach has been proven to be a fruitful framework for analyzing student difficulties, it does not exhaustively reflect the complexity of the learning process, student's understanding of a particular concept nor student's learning difficulties. Essential reasons for these problems are due to the fact that mathematical concepts and symbols, which are used in the teaching of mathematics are often understood by students with a totally different meaning from what was intended by the teacher (vom Hofe, 1998). For that reason, different concepts of the generation of "mental models" have been developed to counteract these problems. In Germany these mental models, which bear the meaning of mathematical concepts or procedures are called *Grundvorstellungen* (GVs), which emphasize the constitution of meaning as a central aim of mathematical teaching. They can be interpreted as "elements of connection or as objects of transition between the world of mathematics and the individual world of thinking" (vom Hofe, 1998, p. 320), which show structural and functional aspects of a mathematical subject. GVs are not static mental models, which are valid forever, but its generation is a dynamic process of changes, reinterpretations and modifications as involvement with new mathematical subjects takes place. It is a cognitive net in which single GVs are in correlation to others.

GVs cannot be directly studied but require the need to be aware of three different types of behavior, prescriptive (basic idea), descriptive (individual image) and constructive. In mathematical literature, prescriptive notion of angle GVs are given describing adequate interpretations of the core of the respective mathematical contents which are intended by the teacher in order to combine the level of formal calculating with corresponding real live situations. For instance, from a normative aspect (basic ideas) of 1° angle can be the amount of openness between the two rays of an angle, which corresponds to 360^{th} part of the circle circumference with *degree* as a unit of measure equal to it, which openness is so small (on paper) that one can barely see the difference between the two rays. However, descriptive notion focuses on describing ideas and images, which students actually have and which usually more or less differ from the relevant mathematical thoughts intended by mathematical instruction. Thus, in the teaching-learning context it is important that the teacher specifies an adequate basic idea of 1° angle, so that the students do not generate an image detached from it, such as 1° being understood as a Euclidean distance between the two rays or as measure of the extent of a two-dimensional shape. The third perspective focuses on developing and confronting students with learning situations that would allow them to change, rebuild, and refine their individual images.

In summary, when thinking about the teaching-learning process, the first focuses on ideas that have to be formatted by the students, the second on images, which have been activated by a student and third initiated by the teacher as a result of faulty or not fully developed basic ideas. In this paper we focus on the process of teaching-learning of the angle measure of 1° with the interplay between individual images and basic ideas, and how these can lead to the constitution of basic ideas of the students in a psychological sense.

METHODOLOGY

The study took place in a Montessory comprehensive school in the state of Saxony. We administered a WiKUL test to approximately 300 students in grade 5 to grade 10. The purpose of the test was to grasp and understand their existing ideas and aspects about the angle concept, and to obtain an image for the understanding of the concept and the associated operations. The students had 45 minutes for the test. The test items were aligned with the Saxon curriculum and consisted of two types of items, that focused on the following two aspects: (a) intra-mathematical knowledge on both grade and across-grade tasks, and (b) patterns of thinking in application tasks about the angle concept (Dohrmann & Kuzle, in press). A special test item was used, namely Anna-letter developed by Thomas Jahnke, as a source of data for the pupils' individual images about the angular size of 1°. In this data source a 12-years old bright girl by the name of Anna is introduced asking students for an advice or help. For the purpose of our study, Anna-letter focused on asking the pupils to help Anna understand what 1° angle is:

> Dear …,
>
> Yesterday we repeated angles in math calls. Our teacher wanted to know what 1° is. With the question I was totally overwhelmed. Although I know that we have constantly used this, I cannot exactly explain what 1° means. Can you please help me? Maybe you can also draw a sketch.
>
> Thank you and best regards, Anna.

This item was used for 5-10th graders. By using this data source and through children's communication, representations and arguments, we obtained an insight into children's images about the 1° angle. The analysis of Anna–letters occurred in several steps where both inductive and deductive methods were used as suggested by Patton (2002). The analysis showed that pupils held many different images about 1° angle (Dohrmann & Kuzle, in press) with distance GV about the 1° angle being highly coded and across different grades (ca. 10% of children). This GV was assigned when word distance was used in the verbal explanation, and/or when 1° was equated with a distance measure (e.g., 1°=1mm, 1° equals distance between two dashes on the set square).

To confirm written explanation and to better understand this GV, nine pupils were chosen on the basis of their contrasting responses (GVs and misconceptions) about the angular size of 1° and interviewed; two from grades 5 and 6, one from grades 7, 8 and 9, and two from grade 10. The interviews lasted ca. 15-20 minutes and focused on student's elaboration of Anna-letter and how this GV developed. In addition, another instrument was used, namely Anna-video. In it a girl Anna measures the angle as described by each pupil in the Anna-letter; she measured the angle by measuring the distance between the two rays, concluding that since the distance between the two rays equals 1mm, the angular size corresponded to 1°.

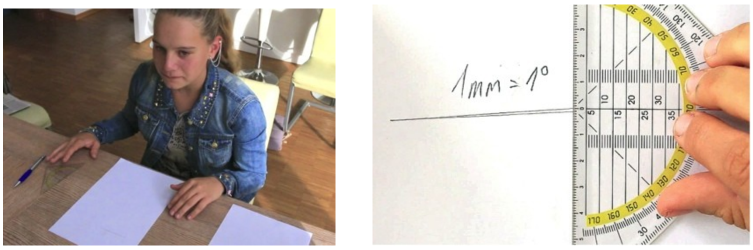

Figure 1: Anna-video.

The children were supposed to comment on Anna's solution and give us a better understanding of their image by explaining their notion, refining, rejecting or rethinking their distance GV. In other words, we were interested how the children deal when confronted with new experiences and challenges as explainer earlier.

This data was again analyzed using content with contrasting comparative methods (Patton, 2002). To increase the reliability of the study, both authors coded the data separately and meet to discuss the codes. When agreement was meet, the code was assigned.

CHILDRENS' DISTANCE GRUNDVORSTELLUNGEN OF 1° ANGLE

A summary of findings is presented in terms of children's distance image(s) in angle context, their understanding of 1° angle given through elaboration and arguments on the basis of their Anna-letter and reaction to Anna-video, and relationship among their identification of 1° angle and its representation on paper and set square. Brief descriptions are provided for the categories with quotes from participants.

Individual image of "distance" in angle context – in mathematics, from a normative perspective distance is a function that describes how apart objects are. In the angle context the normative aspect of distance is described as the length of the unit circle arc enclosed by a particular angle. However, the descriptive perspective exhibited by the participants was different. They held three different images of distance GV with respect to 1° angle: (1) the distance between the two dashes on the set square which was equal to 1mm (N=3), (2) the 1 mm distance between the two half-rays (N=3), and (3) the length of the arc closed by the 1° (N=3). First image was observed by Lynn (5th grade), Elli (5th grade), and Toni (7th grade). These children showed the 1° angle on the set square; it was identified as a plane between zero and two dashes on the set square. However, the distance between the dashes was then estimated to 1mm and equated with the 1° angle.

Lynn: Well, I just thought that 1mm …So when I have here the set square, that here between the two lines maybe 1mm is …

Toni (7th grade) argumented similarly, but by using the half-circle scale.

Tony: So, I'd say that for instance here on the set square 1° is 1mm here on the circular edge marking.

Joanna (6th grade), Toni (7th grade), and Mike (8th) similarly identified 1° degree as 1mm "distance" but between the two half-rays, whereas Elaine (10th grade) as 1cm distance. Ally (6th grade), Jess (9th grade) and Layla (10th grade) associated the "distance" with the arc length. For instance, Jess viewed it as 360th part of circle, which had the length of 1mm. Based on the sample we can conclude that independent of the grade level, pupils held these different misconceptions about the 1° angle. Hence, these different sub-misconceptions were stabile throughout grades 5 to 10. Moreover, the source for some was traced to the tool itself, namely the set square, used in all grades when teaching and learning the angle concepts.

Individual image of 1° angle on the basis of Anna-instruments given through elaboration and arguments – children differently reacted to Anna-video. Four pupils tried to make sense of the newly acquired information by trying to make connections to their own learning. At the end they assigned it to another way of measuring an angle. Hence, they accommodated the new technique into their existing scheme.

Elaine: I'm not sure, maybe one can measure an angle like that. I think, when she would have measured the angle a bit further, then it would have become bigger, the distance… However, I guess that it does not make a difference how much one extends the dashes.

Five pupils after trying to make sense of it, consciously rejected the new technique.

Jess: I would tell her [Anna], that one has to differently lay the set square. So one lays it onto the angle where zero is und that one reads it off like that, but yeah I write it myself incorrectly.

Interviewer: At what point would you say that you wrote it down incorrectly?

Jess: Well I assumed the same ideas… I also measured it from the top and the said to myself '1mm is also 1°'. But when I think about it, it is clear to me, that that is not correct and that I also incorrectly lied down the set square.

As shown in this one excerpt, as a consequence they got aware how their explanations using mathematically inappropriate language led Anna to false actions. These few excerpts show that pupils generated multiple related ideas for this situation. In other words, conceptual change was exhibited allowing children to challenge their written and verbal explanation about the 1° concept. These were exhibited on two meta-levels: language and situative. In the former certain words and formulations were used to describe the distance GV that could be interpreted in different manner. However, the children used the word "distance" in a non-linear context, as they could not find another word for it, such as "angle openness". In the latter, Anna-letter and Anna-video were seen as two independent entities. The ideas from the video were regarded as a new concept that got either accommodated or rejected.

Relationships among identification of 1° angle and its representation – the pupils identified 1° angle on the set square and represented in then on paper. Collectively, the pupils identified 1° angle either along the leg of the set square or on the half-circle scale. More precisely, four pupils identified it as an angle between first two dashes on the leg, whereas only two pupils as an angle between any to dashes lying next to each other on the leg of the set square. Three students referred to the half-circle scale; similar to the above description, two pupils identified it as an angle between any two dashes next to each on the half-circle scale. Surprisingly, one student, Joanna (6^{th} grade), claimed 1° angle not existing on the set square.

Interviewer: And now show me a 1° angle on it [set square].

Joanna: That doesn't work since it [scale] begins with 10°. So, it's a bit difficult to measure.

Interviewer: One cannot see 1° angle on the set square?

Joanna: Nah-ah … This is merely 170°, when one doesn't have an entire half-circle.

Pupils were then asked to draw 1° angle using the set square or the protractor. Collectively almost all pupils identified 1° angle as a part of plane bounden by two half-rays and an arc. That is, as a part of plane bounded by the arc and close to the vertex; what was "outside" of the arc was not identified as 1° angle.

Elaine: That's 1°.

Interviewer: Show it one more time.

Elaine: Here and here, so here behind would be 1°…Well, maybe here would also be 1°, because when one … so I have here… nah, in fact it has to be here in the front.

Figure 2: Elaine's identification of 1° angle through fixation on angle notation.

Hence, 1° angle was identified through notation. Such fixation on notation enabled the conceptual understanding of the angle concept; she thought about extending the rays, but then concluded that 1mm would not come out. At the end 1° angle was again equated to 1mm, where 1mm was the Euclidean distance between the two rays and part of the angle enclosed the by vertex and plane bounded by the 1mm length. Similar behavior was observed with other participants. Thus, interplay between both ideas about 1° angle and its notation allowed for developing deep misconception not only about 1° angle, but angle concept and its main ideas.

CONCLUSIONS

The results presented here show that students have a fragmented understanding of the angle concept as shown in previous studies. However, we have shown that the development from grade 5 to grade 10 continues to cause a rising gap between the angle concept and its main ideas. In particular as a consequence pupils cannot fully grasp what 1° angle is, cannot fully describe it or show it, and reject fallible actions of others. Different misconceptions, such as notation fixation, can severely inhibit identification or construction of adequate images. Secondly, the results have shown that some children do have the mathematical understanding of 1° angle, but missing appropriate mathematical language competencies to communicate their thinking. Thirdly, many of children's misconceptions were directly connected to the set square. Through the routine tasks procedures were learned and practiced, without building a deeper understanding of the tool and its affordances, which then inhibited conceptual understanding of the angle concept and its operations. Moreover, the tool emphasizes the static angle perspective, but is not suitable for developing the dynamic angle perspective.

The pupils cannot assimilate the ideas given by the teacher, nor can these be transferred into the pupil's mind. Children construct their own images, ideas, and models based on the available teaching-learning situations. As a result of this and other research (e.g., Dohrmann & Kuzle, 2013; Kaur, 2013; Mitchelmore & White, 1995, 2000), we advocate for teaching and learning concept oriented towards the student, on the understanding and the application with respect to the angle concept on the basis of a GV-grounded access and by using a didactically more appropriate angle tool. In more details, the process of teaching and learning focused on transposing basic ideas on the one hand, and being sensible for the individual images on the other hand, are didactical means for inviting students developing adequate meaning of mathematical concepts.

We are currently developing materials for teachers and students with these ideas in mind. On the one side, the materials should allow students with the situation that support discovery of fundamental ideas and aspects in order to develop understanding for corresponding operations. Moreover, the students need to development the ability to handle the daily situations for which the angle concept is crucial. Last but not least, we want to support teachers by developing materials that would allow them to analyze children's strategies and mistakes, and hinder misconceptions to enable the further progress. Such appropriate teaching-learning situation would allow for changes, reinterpretations, or modifications to basic ideas contributing to a greater understanding of a multi-faceted angle concept.

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GENERALIZING DOMAIN AND RANGE FROM SINGLE-VARIABLE TO MULTIVARIABLE FUNCTIONS

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The purpose of this paper is to describe (a) multivariable calculus students' meanings for the domain and range of single and multivariable functions and (b) how they generalize their meanings for domain and range from single-variable to multivariable functions. We first describe how students think about domain and range of multivariable functions as inputs and outputs, independent and dependent quantities, and as associated with particular variables. We then use an actor-oriented transfer framework to describe the ways in which students identify similarities between domain and range in single- and multivariable functions, and how they use these similarities to generalize their meanings for domain and range.

INTRODUCTION

While researchers have identified interesting and useful phenomena about how students think about single-variable functions, far fewer studies exist about how these findings might extend to multivariable functions This motivates the first focus of our paper. Multivariable functions form the backbone of multivariable calculus, and are frequently used in physics and other sciences, but research about how students understand multivariable functions and ideas in multivariable calculus is largely preliminary (Kabael, 2011; Martinez-Planell & Trigueros, 2013; Trigueros & Martinez-Planell, 2010; Yerushalmy, 1997). Given the documented difficulties students have with single-variable functions and single-variable calculus, it bears investigating if and how these difficulties appear in multivariable functions and multivariable calculus. We focus on domain and range because researchers have suggested that a robust conception of function begins with students thinking about the correspondence between inputs and outputs; that is, the function's domain and range (Oehrtman, Carlson, & Thompson, 2008).

It is clear to experts that multivariable functions and ideas related to them (e.g., domain, range, rate of change) are extensions of the same ideas in the single-variable function context. However, students do not always make the connections that experts do, and they do not necessarily develop the meanings that instructors intend. This motivates the second focus of our paper. We analyze what students see as similar between the domain and range of single- and multivariable functions. This actor-oriented perspective yields insight into how students generalize ideas; that is, how they develop meanings for ideas in a novel setting by leveraging their meanings from a familiar setting. Though there have been many studies about generalization in algebra (e.g. Amit & Klass-Tsirulnikov, 2005; Carpenter & Franke, 2001; Cooper &

Warren, 2008; Ellis, 2007), there are fewer in undergraduate mathematics topics. At the same time, generalization is a critical component of mathematical thinking (Amit & Klass-Tsirulnikov, 2005; Lannin, 2005; Mason, 1996; Peirce, 1902; Sriraman, 2003; Vygotsky, 1986) and thus it is important to extend knowledge of how students generalize in higher mathematics. If we know the specific ways in which students generalize their ideas about single-variable functions to multivariable functions, instructors can build on connections that appear naturally to students while providing evidence to counter any unproductive generalizations (that is, not congruent with experts' views) students make.

BACKGROUND LITERATURE

While a systematic search of the literature did not reveal studies explicitly focused on domain and range, there are some findings in the function literature that are relevant to the present study. For instance, one way to define domain and range is the set of inputs and outputs of the function, respectively. According to Oehrtman, Carlson, and Thompson (2008), thinking about a function in terms of an input and corresponding output is the beginning of a robust function conception. Monk (1994) found that most calculus students have developed this pointwise view of function but fewer develop an across-time view of function, in which students' conception of function progress to thinking about the function for infinitely many values and understanding how the a change in one variable affects the other(s). That is, a robust function conception involves not only the ability to pair an input with an output, but an understanding of the relationship between quantities. Confrey and Smith (1995) say the beginning of this understanding occurs as students form connections between values in a function's domain and range. However, as function is introduced in algebra and/or precalculus, the functions instructors ask students to reason about are single-variable functions. How students build an understanding of multivariable functions is not known. Investigating students' meanings for domain and range thus extends the literature about students' understanding of single-variable functions, and adds to the body of knowledge that has just begun to develop regarding students' understanding of functions of more than one variable.

When students learn multivariable functions, they must broaden their notion of function beyond the single-variable case; that is, they must generalize their ideas. Note that abstraction is also a key part of this process, but space limits the discussion to generalization. Generalization is a critical component of mathematical thinking (Amit & Klass-Tsirulnikov, 2005; Lannin, 2005; Mason, 1996; Peirce, 1902; Sriraman, 2003; Vygotsky, 1986), and while there have been many studies about generalization in algebra (e.g. Amit & Klass-Tsirulnikov, 2005; Carpenter & Franke, 2001; Cooper & Warren, 2008; Ellis, 2007), but far fewer studies exist about generalization in undergraduate mathematics. Findings from generalization studies typically indicate that generalization is difficult for students; for instance, algebra students' over-generalize of linear relationships interferes with their understanding of quadratic, exponential, and logarithmic functions (Chazan, 2006; Ellis & Grinstead, 2008;

Schwarz & Hershkowitz, 1999; Zaslavsky, 1997). Other difficulties include trouble transitioning from pattern generalization to abstract algebraic thinking (e.g., Moss, Beatty, McNab, & Eisenband, 2006; Mason, 1996; Orton & Orton, 1999; Schliemann, Carraher, & Brizuela, 2001) and shifting from thinking about a pattern recursively to developing a formula for the n^{th} case. If we know the specific ways in which students generalize, instructors can build on connections that appear naturally to students while providing evidence to counter any unproductive generalizations (that is, not congruent with experts' views) students make.

THEORETICAL PERSPECTIVE

We studied generalization from an actor-oriented perspective, which attends to what students see as similar in mathematical situations. This is in contrast to an observer-oriented perspective in which students' ideas are compared to what an expert would see as similar across situations. Such perspectives often find that students cannot or do not generalize ideas from one setting to another, and focus on students' final generalizations rather than generalization as a process. We are interested in *how* students generalize, and the actor-oriented perspective allows us to privilege students' perceptions of similarity, even if those perceptions are not necessarily correct. We follow Ellis (2007) and Lobato (2003) in thinking about generalization as "the influence of a learner's prior activities on his or her activity in novel situations" (Ellis, 2007, p. 225). This was a useful lens for looking at how students viewed domain and range, a topic they had experienced prior with single-variable functions, in the novel situation of multivariable functions. We use Ellis' (2007) generalizations taxonomy as an analytic framework, which is detailed later in the paper.

DATA COLLECTION AND ANALYSIS

We interviewed 20 students enrolled in multivariable calculus at a mid-size university in the northwestern U.S. The students were volunteers from all the multivariable calculus students enrolled during that term, and were compensated for their participation. The course topics included vectors, vector functions, curves in two and three dimensions, surfaces, partial derivatives, gradients, directional derivatives, and multiple integrals in different coordinate systems. Each student participated in a semi-structured interview that lasted about an hour. We recorded audio and written work from each of the interviews using a LiveScribe Echo Pen, which provides a recording consisting of synced audio and written work of the student. These recordings also allowed us to create dynamic playbacks of the interviews during analysis of the data.

The students responded to the following tasks, which were developed to elicit their verbal definitions for the concepts (Q1) and how they operationalized those definition in problem contexts involving single-variable (Q2) and multivariable (Q3, Q4) functions.

Q1. What does domain mean? What does range mean?

Q2. What are the domain and range of $f(x)=4+1/(x-3)$?

Q3. What are the domain and range of $f(x,y)=x^2+y^2$?

Q4. What are the domain and range of $x^2+y^2+z^2=9$?

Each research foci required its own analytic framework. We used a constant comparative analysis Corbin (2008) to investigate students' meanings for domain and range (the first focus), and Ellis' (2007) generalizations taxonomy to investigate how students generalized their meanings for domain and range (the second focus). We discuss the specifics of the constant comparative analysis and its results in the next section, and the specifics of the generalizations analysis and its results in the section following that.

MEANINGS FOR DOMAIN AND RANGE

To perform the constant comparative analysis, Researcher 1 listened to half of the interviews and highlighted phrases students used to talk about domain and range. These phrases included words like input, dependent variable, 'goes with x,' etc. The researcher formed codes from these words (e.g., input/output, independence/dependence, associated with particular variables) and used these to code the second half of the data. The researcher added to and modified the codes based on this data, and then both researchers independently used the codes to code all of the data. They compared results, discussed any differences, and modified the codes a final time. The researchers then used the coded and categorized data to describe students' meanings for domain and range.

These meanings fit into three categories: as attached to specific variables (e.g., Adam), input/output (e.g., Jim), and independence/dependence (e.g., Phillip). We found that students talked about all of these ideas for both single-variable and multivariable functions, as is evident in the selected excerpts below.

Adam: [Q3] It's a helix, or spinny spring looking thing. Domain and range, so the domain of this would be all real numbers for x values, so x can equal any number, and it changes what z equals, but even negative numbers squared equal positive z. And the range is all real numbers because there is no value of y for which the graph is undefined.

Jim: [Q1] Domain is your input values, otherwise known as your x values. It could also represent your independent values. The range is your output, your dependent variables, y values.

[Q3] There would be two different domains. You have your x input and our y input. Your x domain and your y domain give you a range of a different variable. It's the range of z or $f(x,y)$.

Phillip: [Q3] It's a function of two variables. x and y are both independent variables, rather than the dependent variable. You could say the domain is the independent variable and range is the dependent variable.

Students who thought about domain and range as attached to specific variables thought that domain always meant the possible values of x and range meant the possible value of y, regardless of whether the function was $f(x)$ or $f(x,y)$. Other students' meanings for domain and range relied on the notion of function as a 'machine' which generates inputs from outputs; these students' meaning for domain was the possible input values and their meaning for range was the possible output values. Finally, students thought of domain as a set of values for an independent variable, and range as a set of values for a dependent variable. These categories are not mutually excusive, and many students had a meaning for domain and range that incorporated both ideas of input/output and independence/dependence. Having identified students' particular meanings, we then analysed the generalization of those meanings from the single-variable to multivariable context.

GENERALIZING THE MEANING OF DOMAIN AND RANGE

The generalization analysis was based on Ellis' (2007) generalizations taxonomy (see Ellis, 2007, p. 235, 245). The taxonomy distinguishes between *generalizing actions*, which are "learners' mental acts as inferred through the person's activity and talk" (Ellis, 2007, p. 233) and *reflection generalizations*, or students' public statements about a property or pattern common to two situations. The taxonomy includes subcategories that represent specific types of generalizing actions and reflection generalizations. The first researcher coded all of the data according to the descriptions indicated in the tables. The second researcher reviewed the coding and any the researchers discussed and adjudicated any points of disagreement. Not all of the categories in Ellis' (2007) taxonomy appeared in this data. The categories that did fit, their descriptions, and examples from this data are shown in tables 2 and 3.

We found that students primarily used the generalization methods of relating objects (equations and graphs), stating global rules, and using and/or modifying prior ideas and strategies. Students often appealed to the similarities of $f(x)$ and $f(x,y)$ as each being a function "of" something, and stating that in the multivariable case, x,y were inputs or independent variables just as x was an input in the single-variable case. They used this to justify that the domain of $f(x,y)$ was the possible values for x and y, as it was in the single-variable case. Students related graphs by noticing that the range typically had to do with the variable on the vertical axis and domain typically had to do on with the variable on the horizontal axis, and they used this to infer that the range of a multivariable function would be the possible z values and the domain would be for the horizontal plane. Finally, other students stated that domain would mean the input or independent variable no matter how many variables in the function argument, and range would always be the output, the dependent value, or the function value. These similarities allowed them to apply a prior idea or modify their idea as they thought about the domain and range of multivariable functions.

<table>
<tr><td colspan="3">Ellis (2007) framework</td><td>Example in domain/range data</td></tr>
<tr><td rowspan="4">Type I: Relating</td><td rowspan="2">1. Relating situations: The formation of an association between two or more problems or situations.</td><td>Connecting Back: The formation of a connection between a current situation and a previously-encountered situation.</td><td>Domain is your input values. It could also represent your independent values. I am trying to think like in terms of my physics lab where there are independent and dependent variables and you plug in the numbers that you use.</td></tr>
<tr><td>Creating New: The invention of a new situation viewed as similar to an existing situation.</td><td>Say you need to calculate temperature and you have the temperature relative to California and you have some conversion, so the input values are the temperatures in Oregon and the output values are the temperature in California.</td></tr>
<tr><td rowspan="2">2. Relating objects: The formation of an association between two or more present objects.</td><td>Property: The association of objects by focusing on a property similar to both.</td><td>Lets call z the dependent variable here and move the x and y to the other side. Now the domain is x and y.</td></tr>
<tr><td>Form: The association of objects by focusing on their similar form.</td><td>You can’t have negative z but I don’t know if that’s the domain or the range. I’m going to say it’s the range, and treat the z axis like the y axis of the function.</td></tr>
<tr><td rowspan="2">Type III: Extending</td><td colspan="2">1. Expanding the range of Applicability: The application of a phenomenon to a larger range of cases than that from which it originated.</td><td>Domain is your input values, otherwise known as your x values. It could also represent your independent values. The range is your output, your dependent values, your y values.</td></tr>
<tr><td colspan="2">2. Removing Particulars: The removal of some contextual details in order to develop a global case.</td><td>I am a little fuzzy on range in 3D. I think in 2 dimensions, whatever your domain is, you put that in and that’s what your output is. I suppose that’s the same in 3D as well: the array of possible values I can get out of the function.</td></tr>
</table>

Table 1: Generalizing actions for domain and range.

<table>
<tr><th colspan="3">Ellis (2007) framework</th><th>Example in domain/range data</th></tr>
<tr><td rowspan="2">Type IV: Identification or Statement</td><td rowspan="2">3. General Principle: A statement of a general phenomenon.</td><td>Rule: The description of a general formula or fact.</td><td>[Q3] Domain of this would be all real numbers for x values, so x can equal any number, and it changes what z equals, but even negative numbers squared equal positive z. And the range is all real numbers because there is no value of y for which the graph is undefined.</td></tr>
<tr><td>Global Rule: The statement of the meaning of an object or idea.</td><td>[Q3] Z is kind of like the function value. It equals f(x,y) kind of like $y = f$(x). It's the dependent variable, not the independent.</td></tr>
<tr><td rowspan="2">Type VI: Influence</td><td colspan="2">1. Prior Idea or Strategy: The implementation of a previously-developed generalization.</td><td>[Q1] Range is the set of numbers the function can have.
[Q4] I think the range is 9 for this one… because that's the value on the other side of the equal sign. So it can't range to any other values.</td></tr>
<tr><td colspan="2">2. Modified Idea or Strategy: The adaptation of an existing generalization to apply to a new problem or situation.</td><td>[Q3] In this instance the range is z, the output value. So I would say the variables applied to the function doesn't necessarily correspond to domain as x, range as y. So if I looked back to my definitions in question one, I could define domain and range in 3D space with domain as the span of values that can occur on the horizontal plane and I would define range to be the span of values that are dependent on the domain and span the vertical plane.</td></tr>
</table>

Table 2: Reflection generalizations for domain and range.

IMPLICATIONS FOR INSTRUCTION AND SUGGESTIONS FOR FURTHER RESEARCH

The results of actor-oriented generalization research have direct implications for instruction. Knowing what students see as similar allows instructors to build on the productive connections that appear naturally to students. For instance, many of our subjects developed a mathematically correct notion of the domain of $f(x,y)$ by thinking of $f(x,y)$ as having two inputs (and a domain for each) just as $f(x)$ has one input (and a corresponding domain). Others thought about extending the concept of an independent variable (and its domain) to two independent variables (with a domain for each). Therefore, instructors can introduce multivariable functions by referencing students' notions of inputs, outputs, independence, and dependence. They can also point out the generalizations students may make that are not mathematically correct, such as explicitly noting that 'domain is x, range is y' is not necessarily correct for functions of more than one variable.

Our further research plans are to select other topics in multivariable calculus, such as partial derivatives and multiple integrals, and study both students' understanding of these concepts and their generalizations from single- to multivariable calculus.

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EMPIRICAL STUDY OF A COMPETENCE STRUCTURE MODEL REGARDING CONVERSIONS OF REPRESENTATIONS – THE CASE OF FRACTIONS

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Given the key role of conversions of representations for mathematical understanding, it is highly relevant to investigate in detail competencies regarding conversions of representations. In particular, a corresponding competence model should not only be developed theoretically, but also examined empirically. However, such empirical studies are rather scarce, especially concerning content domains other than functions. Consequently, this study focuses on the design and empirical validation of a competence structure model regarding conversions of representations in the domain of fractions using multidimensional item response modelling. The results suggest that the data support the theoretically developed structure of the model and moreover, they indicate a hierarchical relationship which may give rise to a competence level model.

INTRODUCTION

The ability of dealing flexibly with distinct representations of a mathematical concept and changing between them has been shown to be an important factor for successful mathematical thinking and problem solving (e.g. Lesh, Post & Behr, 1987; Deliyianni et al., 2008). Research into students' competencies regarding the idea of using multiple representations is thus highly relevant. Hence, our project "La viDa-M" (c.f. Dreher, Winkel & Kuntze, 2012) aims at investigating aspects of students' competence regarding conversions of representations domain-specifically by focusing on the content domain of fractions. Moreover, La viDa-M examines possible impact factors on such competencies including specific professional knowledge and views of their teachers. Central to the first project phase is the development of a competence model for learners and its empirical evaluation, on which we will report in this paper. Taking into account different research projects and findings concerning students' competencies in dealing with multiple representations, a competence model regarding conversions of representations and a corresponding domain-specific test instrument were designed. In order to validate the developed model empirically the data of 675 students in 29 sixth-grade classes were analyzed using multidimensional item response modelling. The theoretical background, methods and results reported in the following refer to this first phase of our project. In the last section, additionally to the discussion of these results an outlook is given on first findings regarding interrelations between students' specific competencies and teachers' corresponding views.

THEORETICAL BACKGROUND

The significance of using multiple representations for learning mathematics is emphasized in many national standards (e.g. KMK, 2003; NCTM, 2000). This has good reasons: Doing mathematics relies on using representations, since mathematical objects are not accessible without them (Duval 2006). In fact, a single representation standing for a mathematical object is usually not enough, since mostly a representation can merely emphasize some properties of the corresponding object, so multiple representations have to be integrated in order to develop appropriate conceptual understanding (Ainsworth, 2006; Duval, 2006). Consequently, making connections and conversions between different representations is central to the understanding of mathematical concepts (e.g. Lesh, Post & Behr, 1998; Deliyianni et al., 2008, Renkl et al., 2013). For the purposes of this study we chose to focus on conversions of representations in the content-domain of fractions, since it is particularly well-known that different representations of fractions may highlight different core aspects of the concept and that hence changing between them is important (e.g. Ball, 1993).

This key role of conversions of representations for conceptual understanding leads to the research aim of describing learners' competence regarding conversions of representations. Two requirement scenarios can be distinguished: Firstly, a conversion of representations may be given, which has to be *examined*, i.e. one has to check whether two representations match, if they represent the same mathematical object. Secondly, a conversion of representations may have to be *performed*, i.e. one has to construct a matching second representation in a different representation register on the base of a given representation. Similar distinctions have been made by several researchers investigating students' competencies in dealing with multiple representations, who focused however mostly on the content domain of functions (c.f. e.g. Hitt, 1998, Bossé, Adu-Gyamfi & Cheetha, 2011, Nitsch et al., accepted). Bossé et al. (2011) differentiate for instance between "interpretative activity" and "constructive activity" and Nitsch et al. (accepted) use the distinction of "identification" and "construction" referring to them as "elements of cognitive action". However, in the cited studies it becomes not entirely clear whether the notions "interpretative activity" resp. "identification" refer to single representations or to conversions of representations. Yet, it makes a difference whether aspects of one given representation have to be identified/interpreted or if a conversion of representations has to be examined in the sense of identifying/interpreting aspects of both given representations and deciding if they match. Since we focus on learners' competencies regarding conversions of representations, we do not adopt these notions, but use instead the terms *examining a conversion* and *performing a conversion*. As metacognitive activities like justifying, in the sense of reflecting, explaining and giving reasons play an important role for conceptual understanding using multiple representations (c.f. Renkl et al., 2013), learners should also be able to justify why a given or a self-performed conversion of representations is correct or not. Regarding the content domain of functions, Nitsch et al. (accepted) have implemented the actions "description" and "explanation" in their competence structure model, which could however not be separated empirically, but formed a common dimension instead. With respect to the domain of fractions Deliyianni et al. (2008) differentiated between "recognition tasks" and "conversion tasks" within the construct of "flexibility in multiple representations" and they have also taken into account so-called "justification tasks", but those were operationalized as being part of another competence construct, namely "problem solving". However, seeing the ability to justify conversions of representations as an important facet of competence regarding dealing with multiple representations, it appears to be appropriate to include it into the structural modelling of such competence. Hence, our theoretical competence structure model regarding conversions of representations encompasses the following facets: *examining, performing* and *justifying*. In particular, tasks regarding conversions of representations may require *examining* or *performing* these conversions and optionally they may in addition ask for *justifying* the given or self-performed conversions. Since it may be argued that these three abilities differ in their cognitive demands, this suggests a 3-dimensional competence model (3D) regarding conversions of representations which is shown in Figure 1. According to this model *examining*, *performing* and *justifying* of conversions of representations form one dimension each in the sense of being empirically separable (but not necessarily independent) constructs representing different facets of such competence.

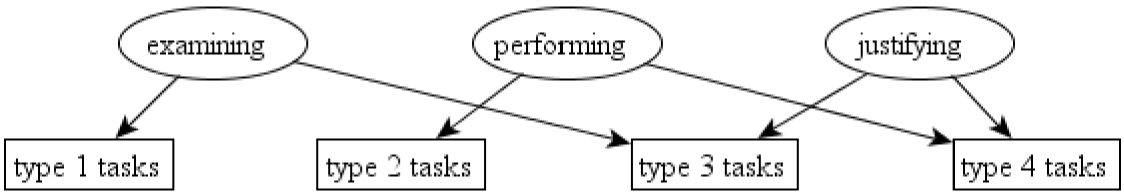


Figure 1: 3D competence structure model regarding conversions of representations

For the purpose of empirical validation of the structure of this model, multidimensional item response theory (MIRT) is used, which is particularly suitable for psychometric modelling of competence taking into account different potentially relevant abilities (Hartig & Höhler, 2008). In this approach possible alternative models are compared to the anticipated model (c.f. Figure 1) with respect to how well the empirical data from our study focusing on the domain of fractions fit them. One of these alternative psychometric models which should be taken into account is the 2-dimensional model (2D), where *examining* and *performing* are not separated, but form a common dimension. This dimension is hence relevant for all tasks regarding conversions of representations and *justifying* represents a separate (optionally relevant) dimension, as it requires metacognition which has to be verbalized. Moreover, a 1-dimensional psychometric model (1D) which assumes that a single dimension represents all three abilities regarding conversions of representations should be tested.

Besides the structure of the competence regarding conversions of representations in the sense of underlying dimensions, the level of difficulty of the abilities encompassed are highly relevant for designing specific learning opportunities and for the diagnosis of learning processes. From a theoretical point of view one may suppose that *performing* a conversion of representations is generally more difficult than *examining* a given conversion of representation, since in the first case a new representation has to be created (c.f. Nitsch et al., accepted). Corresponding assumptions can be found for instance in the context of the theoretical competence level model by Hitt (1998). Since empirical evidence for such a hierarchy is however still lacking, it is a question worth investigating, whether *performing* is generally speaking more difficult than *examining* with respect to conversions of representations.

RESEARCH INTEREST

Examining the model shown in Figure 1 in comparison with other potential models in the content domain of fractions can help to describe the structure of competence regarding conversions of representations. In line with the need for research outlined above, the evaluations presented in this paper are guided by the following research questions:

- Is it possible to validate our theory-based competence structure model regarding conversions of representations in the domain of fractions empirically using multidimensional item response theory?
- Do the empirical data support the theoretical assumption that *performing* conversions of representations constitutes a higher level of difficulty than *examining* conversions of representations?

DESIGN, SAMPLE AND METHODS

For answering these research questions, a test instrument corresponding to our theoretical competence structure model was designed specifically for the domain of fractions. In line with the structure shown in Figure 1, this competence test includes four types of tasks, for each of which Table 1 shows a sample item. The first type is about *examining* given conversions regarding their correctness, i.e. one has to decide if given representations match in the sense of representing the same mathematical object. The second type of tasks demands *performing* conversions of representations. For solving the third resp. fourth type of tasks, it is not enough to examine resp. perform conversions of representations, but they also have to be justified. From each type, three tasks were included in the test instrument, so that it consisted of 12 items in total. Tasks of different types were arranged in alternating order. The paper-pencil test was completed by 675 students in 29 sixth-grade classes at academic track secondary schools in Germany. Within a lesson (45 min.) they were given enough time to solve all the tasks under the supervision of a member of the project team. The answers to each task were scored dichotomously as being correct or incorrect according to criteria established beforehand. Prior to fitting any item response models, one of the type 2 tasks which had been revised after piloting had to be excluded, as a misconception could lead to a correct answer of the item. The modelling of the competence structure was conducted with CONQUEST software (Wu et al, 2007) using multidimensional item response theory (Rasch analysis).

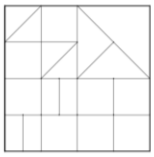
<table>
<tr><th>Examining a conversion (type 1 tasks)</th><th>Examining a conversion and justifying (type 3 tasks)</th></tr>
<tr><td>The second photo was taken shortly after the first one.
1.  2. 
Do the following calculations match what has happened between the two shots?
A. $\frac{6}{8}+\frac{1}{8}$ ☐ yes ☐ no B. $\frac{7}{8}-\frac{1}{8}$ ☐ yes ☐ no</td><td>Class 6c has lost a soccer game 3-4 against class 6d .
Lars considers whether the class 6c has scored $\frac{3}{4}$ or $\frac{3}{7}$ of the total goals. Fatima wants to help Lars: “Only a fraction less than $\frac{1}{2}$ is possible.”
Is Fatima right? ☐ yes ☐ no
Why or why not?</td></tr>
<tr><th>Performing a conversion (type 2 tasks)</th><th>Performing a conversion and justifying (type 4 tasks)</th></tr>
<tr><td>For solving word problems you have to find calculations to given situations. Here you are asked to do it the other way round. Write down a word problem which exactly matches the calculation $\frac{2}{2}+\frac{1}{2}$.</td><td> Take two crayons and color parts of the square so that the calculation $\frac{2}{16}+\frac{2}{16}$ is shown and the entire square is the whole. Explain in detail why the calculation can be seen in your representation.</td></tr>
</table>

Table 1: Sample items for each of the four types of tasks

RESULTS

Focusing on the first research question, we started by fitting the three possible models (1D, 2D and 3D) to the data. Table 2 shows the resulting deviances as a measure of discrepancy and the number of parameters estimated as a measure for the complexity of the model. Since models using more parameters always deviate less (or at least equally) from the real data, both these characteristics of the models have to be taken into account for deciding which one fits best. As the 1D model is a sub-model of the 2D model, which requires two parameters less, the difference between the deviances of the two models follows an approximate chi-square distribution with two degrees of freedom (c.f. Wu et al., 2007). Given the estimated difference of 20.6 in the deviance, we conclude that the extra parameters of the 2D model highly significantly improve the fit ($p<.001$). In the same way we can compare the 2D model with the 3D model, as the 2D model is a sub-model of the 3D model with three fewer parameters estimated. Considering the chi-square distribution with three degrees of freedom shows that the reduction in deviance of 12.36 indicates that the 3D model may fit the data significantly better than the 2D model ($p<.05$).

Model	1D	2D	3D
Deviance	7533.83	7513.23	7500.87
# Parameters	12	14	17

Table 2: Comparison of the fits of the three alternative models

The evaluation of how well the items in the developed competence test fit these models, can be done based on the weighted mean square (MNSQ) fit statistics (c.f. Wu et al., 2007). As this statistic takes the value 1 for a perfectly fitting item, we have checked for each item whether its weighted MNSQ statistic regarding the respective model is significantly different from 1. This analysis shows for the 1D model that not all the MNSQ fit statistics lie inside the ninety-five percent confidence interval for the expected value and thus we have rejected the null hypothesis that the data conforms the model. For the 2D model as well as for the 3D model however, the weighted MNSQ statistic of none of the items is significantly different from 1 ($0.90 \leq$ MSNQ ≤ 1.07, resp. $0.95 \leq$ MSNQ ≤ 1.06), which indicates that the test items fit both of these models very well.

Addressing our second research question, we focus next on comparing the difficulties of the tasks which demand *examining* conversions with those demanding *performing* conversions of representations. The difficulties estimated from the data which are displayed in Figure 2 indicate that in both cases (with or without requirement of justifying) *performing* was more difficult than *examining* with respect to conversions of representations in the domain of fractions. The same pattern could also be found by considering simply the percentage of students who have solved the respective items.

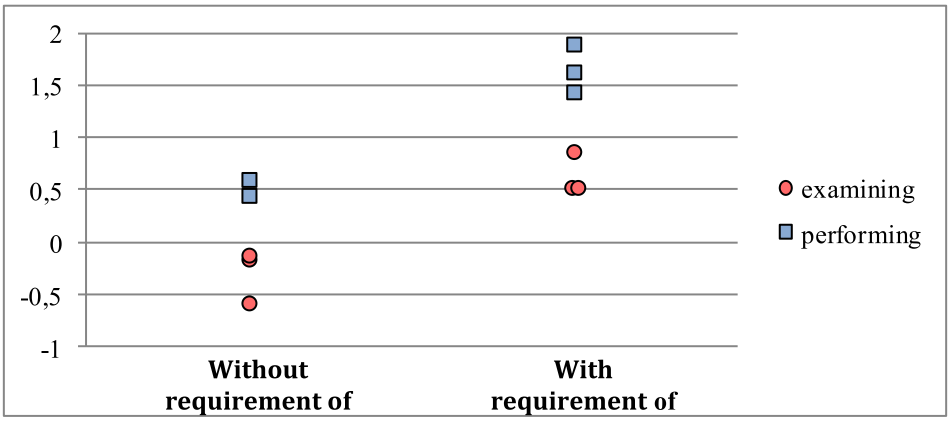


Figure 2: Empirical difficulties of the tasks of the four different types

DISCUSSION AND OUTLOOK

The results of this study may contribute to a better understanding of the construct of competence regarding conversions of representations – with respect to its structure as well as with respect to the differentiation of possible competence levels.

Before these results are discussed in more detail we would however like to recall the limitations of this study which suggest interpreting the evidence with care: Although the sample of this study is reasonably large, it is not representative for German students in sixth grade. Furthermore, even though a spectrum of different items was developed according to the theoretical competence structure model, only a relatively small number of items could be implemented in the test instrument for reasons of feasibility.

Bearing this in mind, the findings however allow answering the research questions and indicate several aspects of theoretical and practical relevance.

Concerning the first research question, the result that the 3D model fits the data better compared to the alternative models backs up the structure of our theoretical competence model regarding conversions of representations. Moreover, seen in connection with similar findings by Nitsch et al. (accepted) with respect to the domain of functions, this indicates that the framework may even be valid across content domains. The finding that the items also fit the 2D model very well suggests that the 2D model, where *examining* and *performing* conversions of representations form a common dimension, may also be used for pragmatic and simplicity reasons. It has the advantage that a joint competence score for both of these abilities may be considered.

Regarding our second research question the results have provided some empirical evidence for a hierarchical relationship of the abilities *examining* and *performing* which was previously merely theoretically postulated. This finding may be an important step towards a model of competence levels regarding conversions of representations and hence it should be replicated by studies using a bigger pool of items and also focusing on additional content domains. From a practical point of view, implications of the findings of this study concern in particular the design of specific learning opportunities, the analysis of the demands of tasks and the diagnosis of learning processes with respect to conversions of representations (in the domain of fractions).

First evaluations focusing both on students' competencies regarding conversions of fractions as well as on their teachers' views on how to use multiple representations for teaching fractions suggest interesting interrelations. For instance, the teachers' view that pictorial representations of fractions should merely be used for the introduction of the concept was significantly negatively related to the mean joint competence score (examining and performing conversions) of his or her students ($r=-.55$, $p<.01$). Despite such significant correlations, multi-level analysis showed that the differences between classes are not significant. This could be due to the fact that individual differences within the classes are much higher than the differences between the classes. However, further analyses have to be conducted in order to explore possible explanations for this interesting phenomenon.

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LINGUISTIC RELATIVITY AND NUMBER

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Linguistic relativity, the idea that language affects the way that people think, and that people who speak different languages think differently, has implications for mathematics education because people use different languages to teach, learn and practice mathematics. This paper reviews research on linguistic relativity and number, looking at languages with very few number words, languages with extensive and regular number systems and the order of composition of numbers. Linguistic relativity appears to involve memory more than perception. Linguistic relativity effects involving number need to be taken into account in designing mathematics education research.

INTRODUCTION

In the science fiction novel *Nineteen Eighty-Four* (Orwell, 1954), the state-imposed language Newspeak is designed to constrain and control the thoughts of the speakers. Another science fiction novel, *Babel-17* (Delany, 1966), focuses on a language which simultaneously enhances speakers' analytic abilities and turns them into political saboteurs. Both these novels explore *linguistic relativity*, the idea that language affects the way that people think, and that people who speak different languages think differently.

The term linguistic relativity was coined by the American linguist Benjamin Whorf (1956) and the idea is also widely known as the Whorfian Hypothesis. The premise is that since different languages have different structures and categorise the world differently, they promote different conceptual developments and practices. Language shapes the way that we see the world.

The linguistic relativity hypothesis exists in two forms. The strong form, that language *determines* and *constrains* the thoughts of speakers, is explored in the above-mentioned science fiction novels. Such "linguistic determinism" has been discredited to the extent that the linguistic relativity hypothesis was out of scientific favour for some time (Brysbaert, Fias & Noël, 1998) and remains contentious today (e.g. Pixner, Moeller, Hermanova, Nuerk & Kaufmann, 2011).

The weak form, as Whorf (1956) himself put it, is that "people act about situations in ways which are like the ways they talk about them" (p. 148). How a language expresses things and what it must express thorough the imperatives of grammar, as opposed to what it may express, has an impact on what the individual is likely to think and to do.

This means that the effects of linguistic relativity apply *to habitual thought* rather than *potential thought* (Lucy 1992). It is not that people cannot understand concepts that are not commonly expressed in their language. Rather, language affects what we pay

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attention to in the world, how we remember it and how we conceive it. Hunt and Agnoli (1991) expressed this in terms of perception and memory:

> although perception may be relatively immune to language, memory is not. Memory can be based on two different records, a direct record of the sensory information at the time that we perceive an event and an indirect, linguistically based record of our description of the event to ourselves. The latter effects, because they are coded by language, are subject to any biases built into the memorizer's language." (p. 381)

Rather than being a true hypothesis, Hill and Mannheim (1992) contend that linguistic relativity is in fact an axiom which "can only be judged on the basis of the extent to which it leads to productive questions about talk and social action, not by canons of falsifiability" (p. 386). Linguistic relativity is significant for mathematics education because it points to possible impacts of the language of students on their mathematical thinking. There is thus a need to look deeply into languages for how they might affect speakers' mathematical thinking.

LINGUISTIC RELATIVITY EFFECTS

Linguistic relativity effects reviewed here consider the impact of speaking languages that have very few number words, of speaking languages with extensive and regular number systems, the order of composition of numbers and grammatical number. In most cases the educational implications of these effects have not yet been described or are somewhat speculative. This review hopes to stimulate such considerations.

Few number words: Australia and Brazil

Some investigations into linguistic relativity effects regarding number have focused on languages which have very few number words. This includes various indigenous Australian languages. Traditionally, Wik Mungan had only a single unique number name: a word for exactly 'one'; the words for 'two', 'three' and 'five' ('hand') had approximate values and fingers and toes could be used to indicate larger number, but without number names (Sayers, 1983). Warlpiri has number names only for very small numbers such as 'one' and 'two' (Hale, 1975). Some other Australian languages traditionally used elements of a base-5 system such as in Yolngu (Cooke, 1990) and Anindilyakwa (Stokes, 1982). However, the larger numbers – numbers above three – were traditionally used in few contexts, such as the division of foods such as turtle eggs (Cooke, 1990; Stokes, 1982). In these cultures, quantification was traditionally not very important outside those restricted contexts (Rudder, 1983).

Experiments in Australia have shown that monolingual Warlpiri- and Anindilyakwa-speaking children were able to match small collections of objects in one-to-one correspondence with an accuracy comparable to urban English-speaking Indigenous Australian children (Butterworth, Reeve, Reynolds & Lloyd, 2008). Butterworth and colleagues claimed that these Indigenous children "with very restricted number vocabularies possess the same numerical concepts" (p. 13179) as the comparison group. However, a similar ability to match small collections of objects in one-to-one correspondence does not necessarily mean that the two groups have the same numerical concepts. Success with small quantities compared to larger ones could be related to having number words for small quantities, or it could because of the use of subitisation, that is, the instant recognition of the size of a small collection without counting. In fact, the Australian language-speaking children used a very different strategy to the English-speaking children. The Warlpiri and Anindilyakwa children were successful with a spatial strategy, reproducing the way the objects were arrayed in the stimulus, rather than using enumeration (Butterworth, Reeve & Reynolds, 2011).

Similar experiments have been conducted in Brazil. The Amazonian Pirahã people speak a language that has number words only for 'one', 'two' and 'many' (Gordon, 2004; Everett, 2005). The Munduruku, also from the Amazon, have number words up to five (Pica, Lemer, Izard & Dehaene, 2004). Studies into their number abilities show that both the Munduruku and Pirahã are able to match small collections of objects in one-to-one correspondence (Gordon, 2004; Pica et al., 2004). The Munduruku are also able to make evaluations of larger collections in an approximate manner, such as telling which collection is larger than another (Pica et al., 2004). Gordon identifies the Pirahã strategy with small quantities as subitisation, which he calls parallel individuation. Although Pirahã speakers performed well on some number matching tasks, language was a factor in reduced performance on numerical tasks involving memory (Frank, Everett, Fedorenko, & Gibson, 2008).

This research demonstrated that people without number words have abilities and strategies for dealing with numerosities. However, different strategies and reduced performance in memory tasks suggest that these people have different numerical concepts from people who count with words.

Regular and extensive number words

There is also the contention that the language features of some counting systems facilitate the performance of certain numerical and arithmetic tasks. Some East Asian languages such as Chinese, Korean and Vietnamese have regular, transparent base-10 counting systems. The spoken number in these languages explicitly corresponds to the base-10 composition of the number, so for example, 14 is said *ten-four*, and 44 as *four-ten(s)-four* (Miura, Kim, Chang & Okamoto, 1988). The regularity and transparency is also reflected in the written symbols used for the numbers. These languages have a minimum of arbitrary number names and complete regularity in the rules generating numbers above ten. This contrasts with languages such as English where the tens numbers in particular show irregularities, and although a number name such as twenty contains roots meaning two-ten(s), the roots are not immediately obvious to most learners. The regularity of the number system in the East Asian languages makes learning to count easier (Miller & Stigler, 1987; Song & Ginsburg, 1988). The short word length of the East Asian number names allows larger numbers to be held in short-term memory, which is another factor that contributes to arithmetic success in speakers of these languages (Geary, Bow-Thomas, Fan & Siegler, 1993; Nguyen & Grégoire, 2011; Wong, Taha & Veloo, 2001). There are many other factors that influence arithmetic success among these East Asian cultures or Confucian cultures including personal, familial and cultural motivation (Leung, 2006; Song & Ginsburg, 1988). It is difficult to separate linguistic effects from effects of these other cultural factors in experiments (Saxton & Towse, 1998). As mentioned above, the linguistic relativity impact of number systems on counting and arithmetic performance is due to differences in memory use in these mathematical activities.

Alternatively, a complex multi-base counting system may facilitate arithmetic computation in quite a different way. The Yoruba counting system of Nigeria uses a primary base of twenty with subsidiary bases of ten and twenty. Yoruba uses subtraction as well as multiplication in numeral composition, thus a number such as 36 is said as *minus-four-plus-(twenty-times-two)* (Verran, 2001). While this system is awkward to write, Verran claims that the multiple bases and multiple ways of composing and decomposing larger numbers assist mental calculation in Yoruba.

Order of composition of numbers

Some studies have attempted to investigate how the order of composition of base-10 numbers may affect cognitive processing, specifically whether the tens proceed or follow the units. Brysbaert, Fias and Noël (1998) found differences in the verbal processing of numbers between Dutch numbers, which are said units first and then tens, and French numbers which are said tens first and then units. This difference disappeared when participants wrote their numbers. The authors fail to give significance to the fact that in writing their numbers, Dutch speakers use the same tens and then units structure as the French. A comparative study of German, Czech and Italian found a small Whorfian effect regarding the compatibility between the written and spoken form, that is, whether the spoken and written forms agreed or not in the order of composition (Pixner et al., 2011). This effect was not taken into account in Brysbaert et al. (1998).

Arabic might be a fruitful language to include in a comparative investigation because its numbers are units-first in both spoken and written form. Alsawaie's (2004) investigation of the linguistic relativity hypothesis and place value with Arabic speaking children did not use natural (in the sense of day-to-day use) Arabic numbers, but instead made the tens more explicit, such that 23, which is usually said *thalathah-wa-ishroon*, (3 and twenty) was said *thalathah-wa-asharatan* (3 and two 10s). The study thus investigated the effect of making explicit the tens in the number rather than the effect of saying the unit first. Interestingly, units first numbers, described by Brysbaert et al. (1998) as "reversed", predated the practice of saying and writing the higher powers first, which began with a reversal of the reading order of numbers adopted from Arabic (Edmonds-Wathen, 2012).

Grammatical number

Grammatical number refers to how and whether a language marks singularity and plurality of objects or actions grammatically. In languages like English, most nouns must be either singular or plural, where plural is any quantity of two or more. In many

Australian languages, there are singular, dual and plural categories. The dual form is used for two objects, and the plural form is used for three or more (Cooke, 1990; Hale, 1975; Sayers, 1983; Stokes, 1982). Hale (1975) speculates that the small number names in Warlpiri are not counting words at all, but are instead grammatical “determiners” or tags, corresponding to the singular, dual and plural the grammatical categories. These Australian languages emphasise the use of small numbers through their dual (and sometimes triple) grammatical categories in addition to the single and plural categories of a language such as English. While English makes a grammatical division between one item (singular) and more than one (plural), these languages must also specify grammatically exactly two and sometimes exactly three items. The cognitive effects of this attention to small quantities have not been investigated.

DISCUSSION AND CONCLUSION

Understandably, the claim of a Whorfian effect seems to generate more controversy when it can be used to suggest a deficiency, as in the case of the Pirahã or Munduruku languages, rather than a superiority, as in the case of Korean or Chinese. The reader may have noted that in most cases English or another European language is used as reference for comparison either directly or indirectly. It is worth considering how linguistic and cognitive norms are constituted within mathematics education as well as in field such as linguistics or psychology. Since when we talk about languages we are also talking about peoples and cultures, we need to be careful that a claim for an increased or decreased ability is not used to reinforce hierarchical ideas about peoples and cultures. The findings of Butterworth et al. (2011) are important because they show different groups of people using different strategies rather than focusing on a lack or deficiency in one group.

The balance of the evidence shows that people who do not have counting words, perhaps because historically they have not felt the need to invent and use them, have different concepts of number than people who have and use counting words. Although speakers of Pirahã, Munduruku, Warlpiri and Anindilyakwa can all subitise small quantities and match concrete collections, their use of memory in tasks involving quantities differs from that of English and French speakers. People with few number words think differently during these tasks than people who have many.

It is difficult to avoid a deficit perspective in a discussion of people not using numbers because Western culture and mathematics education values quantification so highly. Nevertheless, it also does learners a disservice if their prior learning and conceptual development is not taken into account by mathematics educators. This is particularly relevant for remote Indigenous Australian children who enter a compulsory school system that is largely designed and taught by English-speaking non-Indigenous people who learnt their own number words from their parents within their own cultural milieu. Similar contexts exist in many countries and educational systems.

There is extensive scope for further empirical investigations into the effects and implication of linguistic relativity in mathematics education. For example, the studies of the East Asian languages suggest that number naming practices that make the place value structure explicit can be advantageous for learners. The teaching of comparative number systems may also help develop rich and solid conceptual structures of number. Although is it difficult to separate cultural and linguistic factors in learning and practice, investigations that require the use of memory in number processing might better draw out linguistic factors.

At this point it might also be productive to consider the implications for mathematics education and mathematics education research of taking linguistic relativity as an axiom rather than a hypothesis (Hill & Mannheim, 1992) and as a fundamental part of linguistic diversity in mathematics education. There is still the need for carefully designed comparative research. Mathematics education researchers need to avoid making normative and universalist assumptions about language processing in their designs. Linguistic relativity may also offer an explanation of why effects of linguistic diversity cannot be written out of large scale international testing regimes.

The languages that people speak, particularly those they learn as a child, affect their worldview and their thought processes. Mathematics educators and mathematicians need to be thinking about the possibilities created out of these differences between languages. What mathematical practices might be drawn out of the attention to small quantities in Australian languages, from the complexity of multi-base counting systems such as Yoruba or from speaking and writing lower powers before higher powers as in Arabic? People use different languages to teach, learn and practice mathematics, and the differences between these languages matter. Accepting linguistic relativity is part of true acceptance of linguistic diversity.

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STUDYING MATHEMATICAL LITERACY THROUGH THE LENS OF PISA'S ASSESSMENT FRAMEWORK

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Accurate interpretations of large-scale assessment results and sound judgments about students' mathematical literacy depend on these assessments' validity and reliability. One important type of evidence towards this validation is the dimensionality analysis, which explores the conformity between the intended factorial structure (related closely to defining a construct –e.g., mathematical literacy, and its perception) and the statistical structure of the test results. This study investigates the dimensionality of mathematical literacy in PISA. Our results show that the structural relationship between PISA's theoretical (cognitive) and score interpretation frameworks is not at an expected level. These results have important implications for the way mathematical literacy is assessed from mathematics education and psychometric perspectives.

BACKGROUND

This research focuses on the validity of assessment of mathematical literacy at a large-scale through the lens of the Programme for International Student Assessment (PISA) by studying the conformity between the intended structure of the cognitive framework provided for mathematical literacy, and the statistical structure of the results of students' scores in the 2009 implementation cycle. Based on the recommendations from the National Research Council (NRC) (NRC, 2001), the three components of assessment design: cognition, observation, and interpretation, need to be coordinated in a consistent and integrated way, as opposed to having them develop as isolated from each other. Cognition refers to the model of student learning in the domain, or mathematical literacy for our study; observation consists of the evidence provided by the student of the assessed construct; and interpretation entails making sense of this evidence. Our study is centered on the alignment between the theoretical framework for cognition and the score interpretation framework provided in PISA's 2009 assessment of mathematical literacy. There are a limited number of studies investigating the connection between the assessment framework and results. Schwab (2007) found that the multidimensional nature of PISA's science framework was reflected well in the items. Ekmekci and Carmona (2012) studied the students' responses to PISA 2003 mathematics items and detected unidimensionality for the U.S. student population. However, this study extends prior work by conducting a dimensionality analysis using the database for PISA 2009 for all students' mathematics literacy scores from 32 countries in order to better understand the complexities of assessing mathematical literacy at a large scale.

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Mathematical Literacy

The conversations around being mathematically literate began in the early 80's and have continued to gain importance to this day. Furthermore, the standards that have been set for literacy (being able to read and write) have sifted to incorporate mathematics as having equal importance in defining literacy (Jablonka, 2003; Moses & Cobb, 2001). In support of these views, this study is motivated by: (a) the perception of mathematical literacy through assessments; and (b) the reflection of mathematical literacy on assessments, especially in large-scale assessments whose results might have serious impact on education systems globally. In the literature, some math educators (e.g., Kilpatrick, Swafford, & Findell, 2001) focus on proficiencies or competencies when defining mathematical literacy, while others (e.g., Ojose, 2011) describe knowledge and skills. Some others (e.g., Steen, 2001) situate mathematical literacy according to its connection to real life situations (i.e., context). As diverse as multiple approaches taken by different mathematics educators and researchers might be, it seems a consensus that there are multiple dimensions or components constituting mathematical literacy. For this study, mathematical literacy is defined and viewed as "a multidimensional construct composed of distinguishable but related components rather than single, general mathematics ability" (Ekmekci, 2013, p. 1).

Since 2000, the Organisation for Economic Co-operation and Development (OECD) organizes PISA to assess 15-year olds' skills and competencies in reading, science, and mathematics through a worldwide large-scale assessment every three years. In its theoretical (cognitive) framework, PISA presents mathematical literacy as a multidimensional construct. The following is the program's given definition of mathematical literacy.

> An individual's capacity to identify, and understand, the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned, and reflective citizen. (OECD, 2003, p. 24).

PISA's mathematical literacy framework has a multidimensional structure composed of three main attributes: *content*, *processes* and *context*. Content is divided into four sub-dimensions: *quantity*, *space*, *shape*, and *change and relationship*. The process dimension has three sub-dimensions: *reproduction*, *connections*, and *reflection*. Context is composed of four sub-dimensions: *personal*, *educational/occupational*, *public*, and *scientific*. The goal of this study is to show how and to what extent this multidimensional structure is reflected on the actual tests by analyzing dimensionality of the students' responses to PISA 2009 mathematics items for 32 countries participating in the OECD.

Test Dimensionality

One of the most powerful ways to explore the connection and conformity between the framework for mathematical literacy and its assessment is dimensionality analysis. Dimensionality of a test could be informally defined as "the minimum number of examinee abilities measured by the test items" (Tate, 2002, p.182). If items in a test are found to have a unidimensional structure, then this set of items are said to be measuring one dimension of a construct. Similarly, if an assessment is said to be measuring several important attributes of a construct, then it is expected to have a multidimensional structure. Issues in development and use of large-scale assessments such as validity and test fairness are related to test dimensionality. For example, unidimensionality is one of the basic assumptions of some measurement models such as Rasch model (Hattie, 1985). The results of the tests whose items are calibrated according to these measurement models have to produce a unidimensional structure for construct validation of those tests (Rubio, Berg-Weger, & Tebb, 2001). However, it might be the case that a test that is intended to be unidimensional measures more than one latent variable (construct or one dimension of a construct). Conversely, it might be the case that some factors that do not relate to construct being measured, such as item type and format, could introduce multidimensionality to the assessment. Therefore, analyzing the dimensionality of an assessment is important and required for construct validity and to ensure accurate interpretations of test results.

PROBLEM STATEMENT

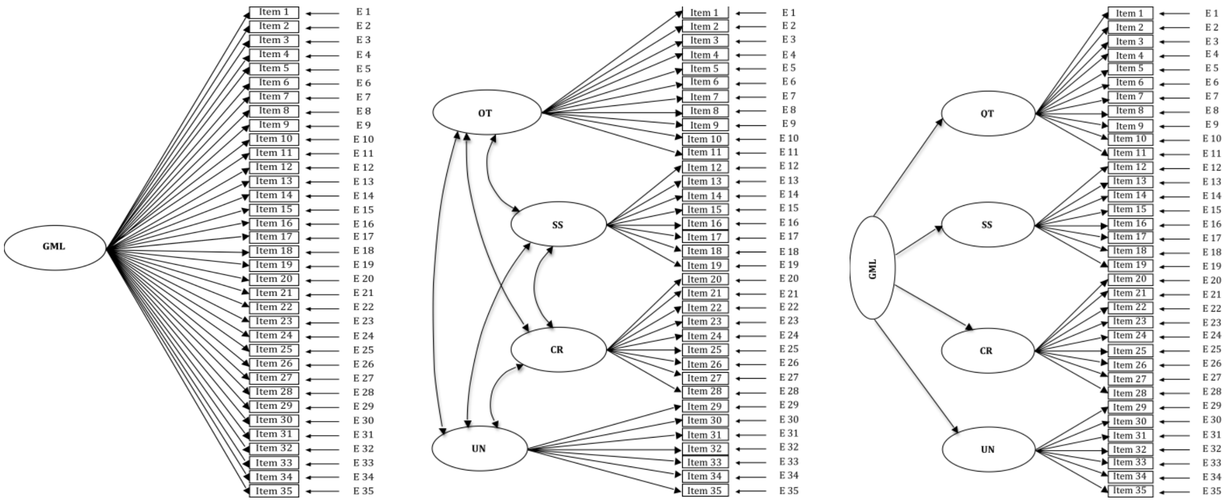
The dimensionality of PISA's mathematical literacy assessment with the inclusion of data from 32 OECD member countries has not been undertaken before. Thus, this investigation is an important contribution to the study of its construct and inferential validity. Moreover, assessing dimensionality of PISA mathematics items is needed to understand the relationship between the important assessment design components of PISA's assessment design for mathematical literacy, as recommended by the NRC (NRC, 2001). The significance of this study comes from the need to provide evidence for validation process of PISA's mathematical literacy assessment. Prior studies (e.g., Ekmekci & Carmona, 2012; Schwab 2007) have set the ground in this direction. However, this study extends prior work by conducting a comprehensive dimensionality analysis incorporating all students' responses to mathematics items from 32 OECD member countries in order to better understand the complexities of assessing mathematical literacy globally and at a large scale. The following are the research questions that guided this study.

1. What is the correspondence between the dimensional structure of PISA's mathematical literacy assessment framework and its score interpretation framework in terms of the content, process, and context dimensions?
2. What is the best representation for the dimensional structure of the PISA mathematics items used to assess students' mathematical literacy?

METHODS

This study entails a secondary-analysis of the dataset from the OECD's PISA database. The data includes student responses to individual mathematics items from 32 OECD member countries in PISA 2009. There is a variety of ways to test dimensionality of tests (see Hattie, 1985, for a comprehensive list). Having a well-developed mathematical literacy framework in PISA means that there is a strong prior expectation about the factorial structure of mathematics items (multidimensionality). In presence of a prior expectation, confirmatory factor analysis (CFA) is considered the best approach to analyze the structure of the assessed construct, i.e., mathematical literacy (Kline, 2010; Tate, 2002).

Seven CFA models were developed based on the mathematical literacy dimensions described in OECD's assessment framework for mathematical literacy. These models include a unidimensional model, three (content, process, and context) correlated factor (1-level) models, and three (content, process, and context) higher order factor (2-level) models. Correlated factors of 1-level models and factors at the first level of level-2 models are the same factors – the sub-dimensions of each main dimension. The latent factors for content dimension are thus *quantity*, *space*, *shape*, and *change and relationship*. The factors for process dimension are *reproduction*, *connections*, and *reflection*. Lastly, the factors for context dimension are *personal*, *educational/occupational*, *public*, and *scientific*. Sample illustrations for different types of models are given in Figure 1 below.



Model 1 (unidimensional   Model 2 (Level-1 Content   Model 5 (Level-2 Content

GML: General Mathematical Literacy, QT: Quantity, SS: Space & Shape,
CR: Change & Relationship, UN: Uncertainty, E: Error Term.

Figure 1: Sample models for the content dimension.

Each CFA model was tested with the student responses to mathematics items. There were 35 mathematics items in PISA 2009. They were dichotomously scored (correct and incorrect). The binary nature of the response data requires using a weighted least squares means and variance adjusted (WLSMV) estimator for CFA (Muthen & Muthen, 2012). The total number of respondents from 32 OECD member countries was 276,142. This large sample size could inflate the power of chi-square tests on which CFA analyses were based (Kline, 2010). Therefore, to avoid Type-I error, a smaller sample was derived randomly using appropriate sampling weights to avoid any bias in the selection. Since the number of mathematics items were large compared to typical CFA analyses, a minimum of 15,000 observations were needed (lower number of observations produced incomplete matrices for CFA calculations). This minimum number also met the criteria for minimum sample size (at least three to five times the number of correlations between items) for CFA with dichotomous items in the literature (Tate, 2002).

The statistical software Mplus 6.12 (Muthen &Muthen, 1998-2011) was used to conduct confirmatory analyses (with WLSMV being the default estimator for categorical data). For each of the three dimensions, the factorial structure of the students' responses and the assessment framework were expected to corroborate each other. This would provide evidence supporting construct validity of the PISA assessment. In other words, multidimensionality was expected in the response data. The first research question addressed how different factorial models (derived from the PISA's mathematical literacy framework) would fit the students' responses to PISA mathematics items. Goodness of fit indices (GFIs) obtained from CFA analyses such as comparative fit index (CFI), the Tucker-Lewis index (TLI), and root mean square error of approximation (RMSEA) were used to evaluate model-fit. Moreover, individual item parameter estimates (factor loadings and R-square values) were evaluated to see how each mathematics item behaved in each model (i.e., the connection between items as observed indicators and their related dimensions as latent factors).

The second research question related to comparing different structural models in order to find the best models that represented the dimensionality of response data. DIFFTEST (alternative version of chi-square difference testing modified for WLSMV estimator) and ΔGFI methods were used to compare models within each three main dimensions (content, process, and context).

Hypotheses

The single-factor model (Model 1) illustrates the hypothesis that PISA mathematics items measure a single construct labelled as general mathematical literacy (GML). The second type of models (Models 2-4) embody the hypothesis that the PISA mathematics items helps explain mathematics knowledge, competencies, and skills in terms of correlated factors of related dimension (content, process, or context) as the latent constructs. The third type of models (Models 5-7) illustrates the hypothesis that the PISA mathematics items measure GML (level-2 factor) by factors (the level-1 latent variables) of related dimension (content, process, or context).

RESULTS

All seven models were found a good fit for PISA 2009 mathematics items. Model fit indices are given in Table 1. All of GFI indices were significant according to the criteria for those indices set by Hu and Bentler (1999). In other words, the responses to the mathematics items do not contradict any of the models proposed for the dimensionality of PISA's mathematics framework. However, high correlations between latent factors in level-1 models (with the lowest correlation coefficient of 0.860) and high latent factor loadings in level-2 models (with loadings of at least 0.841) further supported the unidimensionality. Complete table of these values will be presented in the session. Relating these results to the first research question (response-framework correspondence), overall model-fit results indicate a rather weak reflection of the mathematical literacy framework in the structural representation of the PISA mathematics items. On the other hand, since model-fit indices are relatively strong for all models, multidimensionality also holds. Therefore, results for model fit indices imply that there is evidence supporting both the unidimensionality and multidimensionality of mathematics items in terms of the content, process, and context dimensions.

Secondly, all of the individual parameter estimates were found significant in each model meaning that all models provided a good account for factor loadings and that each mathematics item plays an important role in all models. A complete summary of individual item parameter estimates will be given in the presentation session. This supports that the mathematics framework is reflected in the multi-level models of dimensionality in the PISA mathematics items with respect to the three dimensions.

	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6	Model 7
χ^2 value	743.5	711.2	741.6	729.4	713.7	742.6	731.9
d.f.	560	554	557	554	556	559	556
p-value	0.0000	0.0000	0.0000	0.0000	0.0000	0.000	0.000
CFI/TLI							
CFI	0.980	0.983	0.980	0.981	0.983	0.980	0.981
TLI	0.979	0.982	0.979	0.980	0.982	0.979	0.980
RMSEA							
Estimate	0.005	0.004	0.005	0.005	0.004	0.005	0.005
90% C.I.	[0.004, 0.005]	[0.003, 0.005]	[0.004, 0.006]	[0.004, 0.005]	[0.003, 0.005]	[0.004, 0.005]	[0.004, 0.005]
Prob. ≤ 0.05	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 1: Model fit indices (all statistics are significant)

Lastly, model comparison results revealed that the 2-level model performed better with the PISA 2009 mathematics items in terms of the content and the context dimensions. Therefore, a multidimensional content and context models were more plausible than the unidimensional model. However, this is not the case for the process dimension, where the unidimensional model was preferred to the multidimensional models. Complete results of the model comparisons (including statistical values) will be presented at the conference session. The summary of results is provided in Figure 2.

Content: 2-Level Model > 1-Level Model > Unidimensional Model

Process: Unidimensional Model ≧ 2-Level Model ≧ 1-Level Model

Context: 2-Level Model > 1-Level Model > Unidimensional Model

Figure 2: Model comparisons for PISA 2009.

DISCUSSION AND IMPLICATIONS

In summary, overall results reveal that although the most robust tools identified in the literature were used for analyzing PISA's 2009 mathematics literacy test dimensionality, the results are inconclusive, and in some cases, contradictory. In other words, the connection between the assessment framework and the statistical structure of mathematics items is rather weak in that the intended multidimensional nature of mathematics items is not reflected well enough in the students' responses. PISA is one of the most widely recognized and respected assessments in the world, having a well-articulated and comprehensive mathematical literacy framework and a robust psychometric design. Yet, the major components of its assessment design are not at an expected level of corroboration. This has important implications for mathematics education, measurement, and psychometrics fields.

The authors argue that psychometric methods that are most commonly being used for large-scale assessments (e.g. Rasch models) might be too limiting to provide evidence for the types of constructs the field of mathematics education is interested in and in need of assessing, especially those with multidimensional structure. An important implication for the field of mathematics education is that this area of study is in high need of new assessment designs that can bring to bear other views on mathematics literacy -beyond those addressed in PISA, and that incorporate more current psychometric models that allow for assessment of mathematical literacy in a multidimensional manner. This more consistent alignment between the nature of mathematical literacy construct and psychometric approaches allowing for multidimensionality in assessments can provide a more encompassing perspective and more valid assessments, especially those that are implemented at a large-scale and that have such high stakes decisions in educational systems all over the world.

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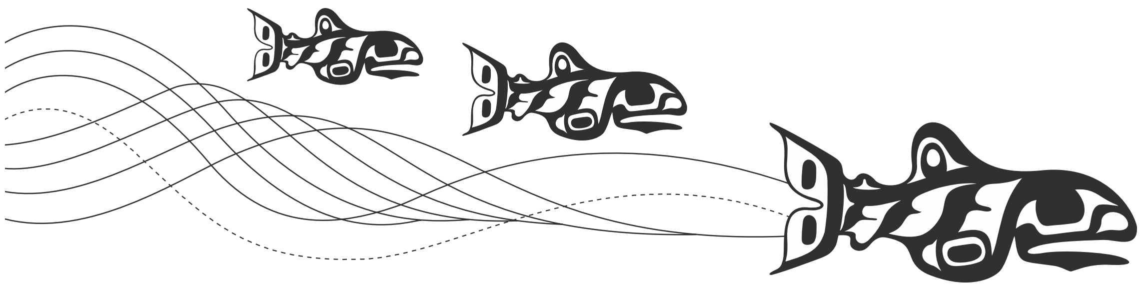

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