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MATHEMATICS
EDUCATION

How to solve it?



PROCEEDINGS OF THE 40TH CONFERENCE OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

EDITORS: CSABA CSÍKOS • ATTILA RAUSCH • JUDIT SZITÁNYI

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International Group for the Psychology of Mathematics Education



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Mathematics Education:
How to solve it?



Proceedings of the
40th Conference of the International
Group for the Psychology of Mathematics Education

Editors

Csaba Csíkos

Attila Rausch

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WELCOME TO PME 40

We are delighted to welcome you to the 40th Annual Conference of the International Group for the Psychology of Mathematics Education, being held in Szeged, Hungary. PME40 is being hosted by the University of Szeged, and the theme of the conference is *Mathematics Education: How to solve it*. This title reminds all participants that 70 years ago the Hungarian Pólya György (George Pólya) published his seminal book entitled “How to solve it?”. This book was used by generations of mathematics teachers as their inspiring source of teaching ideas. Besides commemorating Pólya’s oeuvre, the title evokes the everlasting debate on the role of mathematical problem solving in fostering children’s thinking. We invite all participants to contribute actively to the discourse and analysis of ideas. We also encourage all of you to foster a welcoming and stimulating atmosphere at the conference, that all participants may feel included as members of the PME community. We extend a special welcome to those attending their first PME conference. Our hope is that the conference will provide a chance to attain some pinnacles and to establish some fruitful connections.

It is the second time Hungary is hosting a PME conference. PME 12 was held in Veszprém, and several presenters of that conference are still active members of the PME community. Another special welcome is due to them!

Do extra-terrestrial beings exist? – the Nobel Prize winning Italian physicist, Enrico Fermi, was once asked by his disciples in California. Of course, Fermi answered – they are already here among us, they are called Hungarians... A headline article published in *Nature* in 2000 (“Genius Loci”) claimed that the 20th century was made in Budapest. The article goes on to enumerate all the amazing contributions to progress by Hungarian scientists early in the century. Recognizing and developing mathematical talent has long been and is still a central issue in the Hungarian educational system. However, international assessment projects (TIMSS, PISA) in the last decades warned us that our mathematics education should be reformed according to the principle of evidence-based educational policy. We hope that our educational system will benefit from hosting such a highly prestigious scientific conference.

The Local Organizing Committee has 13 members from different universities, thus making the occasion a national endeavor. The University of Szeged is proud of hosting such a highly prestigious event. Szeged is most famous for its culture, including the University which is among the 500 best universities of the world. The name of the town is also closely intertwined with sport events: the Canoe Sprint World Championships were hosted in Szeged three times. Moreover, the town is a gastronomical and spa event itself worth being discovered.

The Program Committee and the Local Organizing Committee want to express our thanks for the support we have received from members of the PME community, including previous conference organizers and Bettina Rösken-Winter, PME’s administrative manager.

Welcome to PME 40

The Proceedings book has four volumes in accordance with PME tradition. The selection of the contributions involved in these volumes reflects the hard work of our reviewers and the Program Committee, listed in full later.

I am personally grateful to my colleagues whose work went beyond what could reasonably be expected. Attila Rausch was my right-hand secretary available all time during the last ten months; Dóra Prievara managed to answer several hundreds of emails; Katalin Molnár took care of all little details the importance of which cannot be overestimated, and Judit Kléner gave help in handling the financial issues of the conference.

We hope you enjoy your stay in Hungary and find your participation at the conference fruitful and memorable.



Csaba Csíkos

Chair of PME 40

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THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION (PME)

HISTORY AND AIMS OF PME

The International Group for the Psychology of Mathematics Education (PME) is an autonomous body, governed as provided for in the constitution. It is an official subgroup of the International Commission for Mathematical Instruction (ICMI) and came into existence at the Third International Congress on Mathematics Education (ICME 3) held in Karlsruhe, Germany in 1976.

Its former presidents have been:

<i>Efraim Fischbein</i> , Israel	<i>Carolyn Kieran</i> , Canada
<i>Richard R. Skemp</i> , UK	<i>Stephen Lerman</i> , UK
<i>Gerard Vergnaud</i> , France	<i>Gilah Leder</i> , Australia
<i>Kevin F. Collis</i> , Australia	<i>Rina Hershkowitz</i> , Israel
<i>Pearla Nesher</i> , Israel	<i>Chris Breen</i> , South Africa
<i>Nicolas Balacheff</i> , France	<i>Fou-Lai Lin</i> , Taiwan
<i>Kathleen Hart</i> , UK	<i>João Filipe Matos</i> , Portugal

The present president is **Barbara Jaworski**, UK.

THE CONSTITUTION OF PME

The constitution of PME was adopted by the Annual General Meeting on August 17, 1980 and changed by the Annual General Meetings on July 24, 1987, on August 10, 1992, on August 2, 1994, on July 18, 1997, on July 14, 2005 and on July 21, 2012. Here, we have only printed two parts of the constitution. The group has the name “International Group for the Psychology of Mathematics Education”, abbreviated to PME. The major goals of the group are:

- to promote international contact and exchange of scientific information in the field of mathematical education;
- to promote and stimulate interdisciplinary research in the aforesaid area; and
- to further a deeper and more correct understanding of the psychological and other aspects of teaching and learning mathematics and the implications thereof.

The whole constitution can be found at the PME Website: <http://www.igpme.org>

PME MEMBERSHIP AND OTHER INFORMATION

Membership is open to people involved in active research consistent with aims of PME, or professionally interested in the results of such research. Membership is on an annual basis and depends on payment of the membership fees. PME has between 700 and 800 members from about 60 countries all over the world.

The main activity of PME is its yearly conference of about 5 days, during which members have the opportunity to communicate personally with each other about their working groups, poster sessions and many other activities. Every year the conference is held in a different country.

There is limited financial assistance for attending conferences available through the Richard Skemp Memorial Support Fund.

A PME Newsletter is issued three times a year, and can be found on the IGPME website. Occasionally PME issues a scientific publication, for example the result of research done in group activities.

WEBSITE OF PME

All information concerning PME, its constitution and past conferences can be found at the PME Website: <http://www.igpme.org>

HONORARY MEMBERS OF PME

Efraim Fischbein (Deceased)
Hans Freudenthal (Deceased)
Joop Van Dormolen (Retired)

PME ADMINISTRATIVE MANAGER

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INTERNATIONAL COMMITTEE OF PME

Members of the International Committee (IC) are elected for four years. Every year, four members retire and four new members are elected. The IC is responsible for decisions concerning organizational and scientific aspects of PME. Decisions about topics of major importance are made by the Annual General Meeting (AGM) during the conference.

The IC work is led by the PME president which is elected by PME members for three years.

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No.	Year	Location	<i>ERIC number, ISBN/ISSN and/or website address</i>
1	1977	Utrecht, The Netherlands	Not available in ERIC
2	1978	Osnabrück, Germany	ED226945, ISBN 3-922211-00-3
3	1979	Warwick, United Kingdom	ED226956
4	1980	Berkeley, USA	ED250186
5	1981	Grenoble, France	ED225809
6	1982	Antwerp, Belgium	ED226943, ISBN 2-87092-000-8
7	1983	Shoresh, Israel	ED241295, ISBN 965-281-000-2
8	1984	Sydney, Australia	ED306127
9	1985	Noordwijkerhout, The Netherlands	ED411130 (vol. 1) ED411131 (vol. 2)
10	1986	London, United Kingdom	ED287715
11	1987	Montréal, Canada	ED383532, ISSN: 0771-100X
12	1988	Veszprém, Hungary	ED411128 (vol. 1) ED411129 (vol. 2)
13	1989	Paris, France	ED411140 (vol. 1) ED411141 (vol. 2) ED411142 (vol. 3)
14	1990	Oaxtepec, Mexico	ED411137 (vol. 1) ED411138 (vol. 2) ED411139 (vol. 3)
15	1991	Assisi, Italy	ED413162 (vol. 1) ED413163 (vol. 2) ED413164 (vol. 3)
16	1992	Durham, USA	ED383538
17	1993	Tsukuba, Japan	ED383536
18	1994	Lisbon, Portugal	ED383537
19	1995	Recife, Brazil	ED411134 (vol. 1) ED411135 (vol. 2) ED411136 (vol. 3)

20	1996	Valencia, Spain	ED453070 (vol. 1) ED453071 (vol. 2) ED453072 (vol. 3) ED453073 (vol. 4) ED453074 (addendum)
21	1997	Lahti, Finland	ED416082 (vol. 1) ED416083 (vol. 2) ED416084 (vol. 3) ED416085 (vol. 4)
22	1998	Stellenbosch, South Africa	ED427969 (vol. 1) ED427970 (vol. 2) ED427971 (vol. 3) ED427972 (vol. 4) ISSN: 0771-100X
23	1999	Haifa, Israel	ED436403, ISSN: 0771-100X
24	2000	Hiroshima, Japan	ED452301 (vol. 1) ED452302 (vol. 2) ED452303 (vol. 3) ED452304 (vol. 4) ISSN: 0771-100X
25	2001	Utrecht, The Netherlands	ED466950, ISBN 90-74684-16-5
26	2002	Norwich, United Kingdom	ED476065, ISBN 0-9539983-6-3
27	2003	Honolulu, Hawai'i, USA	ED500857 vol.1) ED500859 (vol.2) ED500858 (vol.3) ED500860 (vol.4) ISSN: 0771-100X http://www.hawaii.edu/pme27
28	2004	Bergen, Norway	ED489178 (vol.1) ED489632 (vol.2) ED489538 (vol.3) ED489597 (vol.4) ISSN: 0771-100X www.emis.de/proceedings/PME28
29	2005	Melbourne, Australia	ED496845 (vol. 1) ED496859 (vol. 2) ED496848 (vol. 3) ED496851 (vol. 4) ISSN: 0771-100X

30	2006	Prague, Czech Republic	ED496931 (vol. 1) ED496932 (vol. 2) ED496933 (vol. 3) ED496934 (vol. 4) ED496939 (vol. 5) ISSN: 0771-100X http://class.pedf.cuni.cz/pme30
31	2007	Seoul, Korea	ED499419 (vol. 1) ED499417 (vol. 2) ED499416 (vol. 3) ED499418 (vol. 4) ISSN: 0771-100X
32	2008	Morelia, Mexico	ISBN: 978-968-9020-06-6 ISSN: 0771-100X
33	2009	Thessaloniki, Greece	ISBN: 978-960-243-652-3 ISSN: 0771-100X
34	2010	Belo Horizonte, Brazil	ISSN: 0771-100X http://pme34.lcc.ufmg.br
35	2011	Ankara, Turkey	978-975-429-262-6 ISSN: 0771-100X http://www.arber.com.tr/pme35.org
36	2012	Taipeh, Taiwan	http://tame.tw/pme36 ISSN: 0771-100X
37	2013	Kiel, Germany	ISBN 978-3-89088-287-1 ISSN 0771-100X http://www.pme2013.de/
38	2014	Vancouver, Canada	ISBN 978-0-86491-360-9 ISSN 0771-100X http://www.pme38.com/
39	2015	Hobart, Australia	ISBN 978-1-86295-829-6 ISSN 0771-100X http://www.pme39.com

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Members of PME can reach most of the previous proceedings books at the IGPME website: <http://igpme.org>

Abstracts from some articles can be inspected on the ERIC website (<http://www.eric.ed.gov>) and are listed in the Mathematics Education Database (MathEduc, <http://www.zentralblatt-math.org/matheduc>).

REVIEW PROCESS OF PME 2016

RESEARCH REPORTS (RR)

Research Reports are intended to present empirical or theoretical research results on a topic that relates to the major goals of PME. Reports should state what is new in the research, how the study builds on past research, and/or how it has developed new directions and pathways. Some level of critique must exist in all papers.

The deadline for submission of RR proposals was January 15, 2016. The number of submitted RR proposals was 287, and 153 of them were accepted. Of those rejected as RR proposals, 82 were invited to be resubmitted as OC, and 48 as PP. Re-submitted OCs and PPs underwent the same review process as the OC and PP submissions that were submitted directly.

ORAL COMMUNICATIONS (OC)

Oral Communications are intended to present smaller studies and research that is best communicated by means of a shorter oral presentation instead of a full Research Report. They should present empirical or theoretical research studies on a topic that relates to the major goals of PME. The deadline for submission of OC proposals was March 6, 2016. The number of submitted OC proposals was 132, and 114 of them were accepted. In the end, considering resubmissions of Research Reports as OC presentations 171 OC are presented on the PME 40 conference.

POSTER PRESENTATIONS (PP)

Poster Presentations are intended for information/research that is best communicated in a visual form rather than an oral presentation. The number of submitted PP proposals was 55, and 45 of them were accepted. With the resubmitted Research Reports, 68 posters are presented on the PME 40 conference.

RESEARCH FORUMS (RF)

The goal of a Research Forum is to create dialogue and discussion by offering PME members more elaborate presentations, reactions, and discussions on topics on which substantial research has been undertaken in the last 5-10 years and which continue to hold the active interest of a large subgroup of PME. A Research Forum is not supposed to be a collection of presentations but instead is meant to convey an overview of an area of research and its main current questions, thus highlighting contemporary debates and perspectives in the field.

There are three Research Forum proposals accepted this year:

RF1: *Language mediating learning: The function of language in mediating and shaping the classroom experiences of students, teachers and researchers*

Coordinators: David Clarke, Javier Diez-Palomar and Markku Hannula

RF2: *Mathematics learning and teaching at university level?*

Coordinators: Barbara Jaworski and Despina Potari

RF3: *Understanding obstacles in the development of the rational number concept – searching for common ground*

Coordinator: Wim Van Dooren

DISCUSSION GROUPS (DG)

The objective of a Discussion Group is to provide attendees with the opportunity to discuss a specific research topic of shared interest. The idea for a Discussion Group may be the result of an Ad Hoc Meeting or an intensive discussion of a Research Report during the previous conference. Discussion Groups may begin with short synopses of research work, or a set of pressing questions. A Discussion Group is exploratory in nature, and is especially suitable for topics which are not appropriate for collaborative work in a Working Session because they are not yet elaborate enough or because a coherent research strategy has not been identified. A successful Discussion Group may result in an application for a Working Session one year later.

This year the International Programme Committee approved four discussion groups:

DG1: *Tools, signs, formation of concepts and learning of mathematics*

Coordinators: Yasmine Abtahi, Mellony Graven, Margot Berger, Steve Lerman and Debbie Stott

DG2: *Aesthetics in school mathematics: a hands-on approach*

Coordinators: Esther Levenson and Manya Raman-Sundström

DG3: *Textbook signatures: exploring possibilities*

Coordinators: Ban Heng Choy, Mi Yeon Lee and Angel Mizzi

DG4: *Observations on observing pedagogy: Further discussion of researching the unobservable*

Coordinators: David A. Reid, Richard Barwell and Annie Savard

WORKING SESSIONS (WS)

The aim of Working Sessions is that PME participants collaborate in joint activities on a research topic. For this research topic, there must be a clear research framework or research strategy and precise goals so that a coherent collaborative activity is ensured. Ideas for a Working Session can result from Discussion Group sessions of previous conferences where a topic was elaborated upon and a research framework or strategy was developed. Each Working Session should be complementary to the aims of PME and ensure maximum involvement of each participant.

The accepted Working Sessions for PME 2016 are:

WS1: *The ritual vs exploration conceptual dyad – affordance and open question*

Coordinators: Einat Heyd-Metzuyanim, Mellony Graven, Talli Nachlieli and Nadav Ehrenfeld

WS2: *Special Education and math*

Coordinators: Helen Rachel Thouless, Yan Ping Xin, Ron Tzur, Jessica Hunt and Robyn Ruttenberg

SEMINARS (SE)

The goal of a Seminar is the professional development of PME participants, especially new researchers and/or first comers, in different topics related to scientific PME activities. This encompasses, for example, aspects like research methods, academic writing or reviewing. A Seminar is not intended to be only a presentation but should involve the participants actively. PME can give a certificate of attendance to participants of the Seminar. The proposals of accepted Seminars are included in the Conference Proceedings.

This year the International Programme Committee approved two Seminar proposals:

SE1: *An introduction to electroencephalographic research*

Coordinator: David Maximiliano Gomez

SE2: *Reviewing for the PME: a primer for (new) reviewers*

Coordinator: Anke Lindmeier, Anika Dreher and Michal Tabach

The reviewing process was completed during the 2nd Meeting of the International Program Committee at the end of March 2016. Notifications of decisions of the International Program Committee to accept or reject the proposals were available by April 2016.

LIST OF PME 40 REVIEWERS

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Schubring, Gert
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Simon, Martin
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Stacey, Kaye
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Sullivan, Peter
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Superfine, Alison
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Suurtamm, Chris
(University of Ottawa, Canada)

Swidan, Osama
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Szitányi, Judit
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Sztajn, Paola
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Tabach, Michal
(Tel Aviv University, Israel)

Tatsis, Konstantinos
(University of Ioannina, Greece)

Teppo, Anne
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Thomas, Michael O. J.
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Thornton, Stephen John
(University of Oxford, Australia)

Tillema, Erik
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Tirosh, Dina
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Tomaz, Vanessa Sena
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Trigueros, Maria
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Walshaw, Margaret
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Zazkis, Rina

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Zazkis, Dov

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Zeybek, Zulfiye

(Indiana University, United States)

A circular graphic with a teal background, featuring a close-up of a pen writing on a piece of paper with cursive script.

PLENARY PRESENTATIONS



SOLVING THE PROBLEM OF POWERFUL INSTRUCTION

Alan H. Schoenfeld

University of California, Berkeley

My roots are in Pólya's work, in making sense of what it means to be an effective problem solver. The problem I've worked on for more than 40 years is, how do we build on Pólya's insights? First we had to understand what it means to be a powerful mathematical thinker and problem solver. Then we needed a deeper understanding of teaching. Then the question was, how do we create powerful learning environments, that is, classrooms that produce students who are powerful mathematical thinkers? I discuss what we now know about such learning environments – what they look like from the student's perspective, as well as from the mathematical perspective.

INTRODUCTION AND OVERVIEW

I was a young mathematician when, in 1974, I encountered George Pólya's (1945) book *How to Solve It*. The book was a revelation. Page after page offered descriptions of problem solving strategies that I used as a mathematician. I was amazed, and nonplussed. On the one hand, Pólya's book provided meaningful personal validation. I did many of the things that Pólya said mathematicians do – thus, I must be a real mathematician! On the other hand, I wondered why I had never been taught such strategies. Some preliminary inquiries revealed the following: (a) the mathematicians I spoke with believed that Pólya had indeed identified powerful problem solving strategies; but, (b) reading about the strategies, and trying to teach or learn them, did not seem to improve students' problem solving performance, either during regular instruction or in preparation for problem solving competitions.

Now *that* was a problem! It was a propitious time to work on such a problem, because the cognitive sciences were emerging, with novel ways of examining the strategies used to solve problems (cf. Newell and Simon's classic 1972 volume *Human Problem Solving*.) So, I left the study of pure mathematics and began the study of mathematical thinking and problem solving.

This paper briefly describes the early days, which resulted in the publication of my 1985 book *Mathematical Problem Solving*. It was, in a sense, the first expansion of Pólya's ideas. In hindsight, I can characterize things as follows. Pólya asked the question, "What strategies, in addition to knowledge, enable people to become powerful problem solvers?" The emphasis was on *productive thinking* (which is also the title of a book on problem solving also published the same year (1945) by Max Wertheimer). Part of my work was to elaborate on the *heuristic* strategies Pólya invented and described, but part of it was also focused on unproductive student thinking – the causes of failure as well as success. The question I would up addressing was: "What do people do when they are engaged in problem solving? What contributes

to success, and what contributes to failure; and, in the light of such understandings, how can we support the development of more successful habits?”

The question I could not address at that time was *why* people make the (often inappropriate) decisions they make while they are engaged in problem solving. That called for a theory of decision making, which I decided to study in a very complex but useful domain: teaching. Teaching is a very complex act of problem solving. If you can understand teaching – which, for me, means building models of teachers that accurately characterize the teachers’ real-time decision making – then you can help teachers become more effective. Building a theory of decision making took some 20 years, culminating in the publication of my (2010) book *How We Think*.

But making further progress called for a shift in focus. My previous work had focused on individuals – the individual problem solver, the individual teacher making decisions. But there is a lot more to the classroom than student and teacher: students experience the classroom as a whole. So, for me, Pólya’s original question has evolved into the question, “*What are the properties of powerful learning environments (classrooms that produce students who are effective mathematical thinkers and problem solvers), and how can we help teachers construct such environments?*” The bulk of this paper is devoted to that question.

Before proceeding, I do want to make it clear that I consider the vast majority of my career to have been spent in pursuit of Pólya’s trailblazing ideas. Before I submitted my first major paper on problem solving (which was eventually published as Schoenfeld, 1980) for publication, I sent the very first xeroxed copy of the draft manuscript to Pólya with the following inscription:

“Isaac Newton wrote ‘If I have seen further, it is by standing on the shoulders of giants.’ Now that I work in problem solving, I truly understand the meaning of his words.”

UNDERSTANDING PROBLEM SOLVING

As I mentioned above, I – and virtually every mathematician I know – resonates to Pólya’s descriptions of problem solving strategies, and to the richness of the mathematics that Pólya emphasized in his problem books and expository texts (Pólya, 1954, 1962/65; Pólya and Szegő, 1925). We would read his books, work the problems (when we could solve them!), and say, “yes, that is rich and important mathematics; and yes, that is how I solve problems.” The challenge was that students did not appear to learn to become more effective problem solvers by reading Pólya’s books. In the mid-1970s, when I began research on problem solving, researchers in artificial intelligence (AI) were working on problem solving, in different ways than Pólya, who relied mostly on introspection, had worked. AI researchers such as Newell and Simon (1972) made observations of people solving (very well defined) problems, and abstracted the patterns of productive behavior that resulted in problem solutions. This was very fine-grained work aimed at the computer implementation of certain strategies (back in the days when computers were not very “smart” and one had to give them very

very specific instructions), but it raised the possibility that I might use similar techniques to find out how people could implement Pólya's problem solving heuristics. The approach helped. I discovered that much more detail was needed: what to the mathematician was a "simple" strategy like "try to solve an easier related problem" was in fact a whole family of strategies, because there are dozens of ways to create easier related problems. Once you know them, you can summarize what you've done by saying "I solved an easier related problem" – but to reach that point you have to learn many different strategies. That's what was missing.

But, the proliferation of strategies led to a new problem: How do you keep so many strategies in mind, and how do you decide which one to use at any particular time? In a positive sense, this became the issue of effective control – picking the right strategy. But there was a negative sense too. When I made videotapes of students working on problems, it became clear that they often made bad choices of direction, and then persevered at them. That's a serious problem: if you're pursuing the wrong direction, then you're not working on something productive! Ultimately, this area became known as "monitoring and self-regulation," an aspect of metacognition. The idea is that one has to use one's resources effectively. Knowing something isn't useful if you don't give yourself the opportunity to use that knowledge.

When I asked students to work problems, I chose problems that they *should* have had enough knowledge to solve – there is no point in giving students problems that they have little chance of solving. But, to my surprise, I found them doing strange things, e.g., making conjectures that violated statements that they had proved. Ultimately, this led me to realize that students' *beliefs* about mathematics – e.g., "proof knowledge is irrelevant when one is working on a 'discovery' problem," or "all problems can be solved in just a few minutes, so if I have failed to solve a problem in five or ten minutes, I give up" – played a fundamental role in shaping their problem solving behavior.

By 1985 I had concluded that four categories of knowledge and behavior were necessary and sufficient to explain someone's success or failure at problem solving: (i) the knowledge they had at their disposal, (ii) the problem solving strategies (à la Pólya) they were able to call upon, (iii) "control," or monitoring and self-regulation; and (iv) their beliefs, which shaped which knowledge and strategies they might or might not use. The development of these ideas was intimately tied to the ongoing development of a course in problem solving, which both served as a test bed for my evolving ideas and as a source of new ones.

STUDIES OF TUTORING, TEACHING, AND DECISION MAKING

The main limitation of my problem solving work was that it offered a framework ("look at these things, and you will locate the source of success or failure") rather than a theory ("this is how and why things happen the way they do"). At the time I was unable to say *why* people made the choices they did while problem solving, and I had no idea of how to make progress on that issue. My attempts to explain such decisions, by building a mathematical model of people's decision making while problem solving, were

unsuccessful. That is, I was unable to succeed at Pólya's first step: "First, you have to understand the problem."

In hindsight, what I did next also made use of one of Pólya's heuristic strategies: "If you cannot solve the proposed problem, try to solve first some related problem." A long-term goal was to understand teaching, but in teaching, the teacher is reacting to myriad events and perceptions at the same time. So, my colleagues and I decided to study tutoring. Tutoring is an act of problem solving – the problem being, how does one help a student learn certain mathematical content? In some ways it is more complex than problem solving, given that conditions can change in the moment (e.g., when the tutee has an insight or reveals a misunderstanding); but in other ways it is more straightforward to study, because the give-and-take of the tutoring conversation is overt, providing a great deal of information about the specific decisions made by the tutor. The results of our work were summarized in Schoenfeld, Gamoran, Kessel, Leonard, Orbach, and Arcavi (1992). See Figure 1 below.

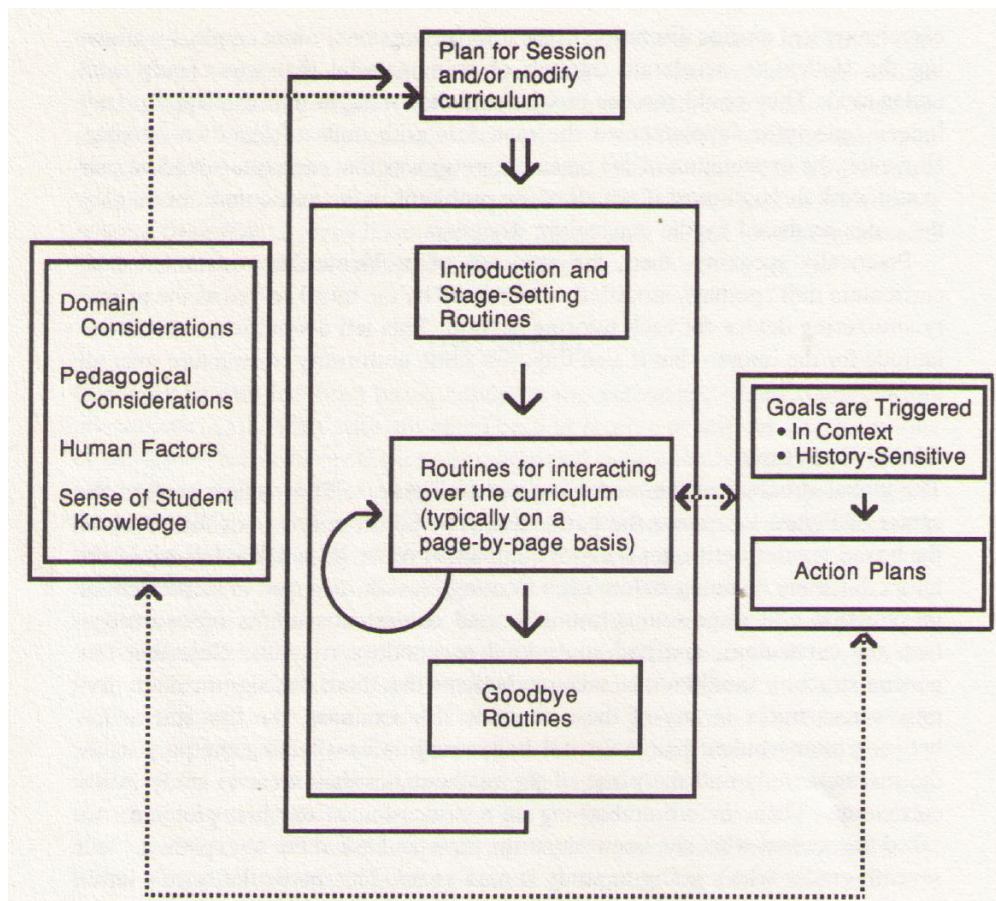


Figure 1. Macrolevel structure of tutor's behavior (unit is one tutoring session)

This model was dynamic, and capable of explaining what happened over time. Yet, the problem solving framework was contained within it. Knowledge and strategies (later to be named "resources") were drawn upon in interaction, as "action plans." Goals took on a more central role than before, in that the tutor had multiple goals, some of which

were mathematical (help the student learn particular mathematics), some interpersonal (e.g., maintain a good relationship with the student). Decision making depended on two main factors: (i) the tutor's sense of what matters, as seen in the box on the left hand side of Figure 1 (this is where beliefs come into play), and (ii) the tutor's sense of what was attainable, which includes metacognitive judgments.

Having built a rough model of the tutoring process, we were ready to work on a model of teaching. It may seem that the leap from tutoring to teaching is huge – after all, the messy complexity of the classroom stands in stark contrast to the comparative serenity of a one-on-one tutoring session. But, in terms of decision making, much is the same. The teacher, like the tutor, has goals, which evolve over the course of an interaction. The teacher, like the tutor, makes choices based on his or her beliefs about what matters. The teacher, like the tutor, is both supported by and constrained by his or her knowledge and the resources he or she can bring to bear during instruction. Finally, the nature of the teacher's decision making, like that of the tutor, is strongly affected by metacognitive actions (monitoring and self-regulation, goal setting and review) during the act of engaging with students. Thus it seemed plausible that we might build a full theory of decision making while teaching.

I should stress that I mean a theory in the scientific sense – a description of objects and relations between them that is specific enough to allow one to build models that serve as a test of the theory's accuracy. One might think of a theory of gravitation, or a theory of heat conduction in a laminar plate, as examples. In the case of heat flow, for example, you specify the initial (boundary) conditions, and the rules of heat flow; the mathematical model then describes the ways that temperature should change (and stabilize) over time. Gravitational theory calls for specifying the mass, direction, and velocity of a number of objects, after which Newton's inverse square law can be used to predict the motions of those objects. So the question for teaching is, could we specify what "counts" – the components of a model – and how those components interact? That would be the general theory. The test would be if we could examine a broad range of specific teachers and model their behavior, in the same ways that the theories of gravitational attraction and heat flow allow one to model particular configurations of objects. The challenge: could we model (say) an hour of instruction, where the input into the model consisted of representations of the relevant aspects of the teacher's goals, resources (mostly knowledge), and orientations (beliefs, values, etc.), with the result being that the model "behaves" like the teacher?

We began the work in the mid-1990s, with the analysis of a short segment of a beginning teacher teaching a traditional lesson. From there we moved to a full hour's instruction, but this time of an experienced teacher teaching an innovative lesson. (Note that this too is in the spirit of "First. You have to understand the problem." Beyond that, if you wish to develop a theory, then it is essential to explore the wide range of circumstances in which the theory is supposed to apply – in this case, beginning and experienced teachers, traditional and experimental lessons, students of different ages,

and lessons with different types of agendas (teacher directed and co-developed with students.)

We spent 3 years analysing the most complex case, a highly interactive lesson taught by Deborah Ball (see Schoenfeld, 2008). But, having modelled that case of teaching, we had evidence not only for a theory of teaching, but for a general theory of decision making in “well practiced” domains. Most human decision making, whether in school, on the job (as a doctor, mechanic, secretary, or teacher) depends largely on skills one has learned on the job and call on with some frequency – hence, “well practiced.” Doctors, for example, come to recognize symptoms, just as mechanics do. All of these fields are knowledge-intensive; in all of them, knowledge matters, as do beliefs. (There is a large literature on medical decision making, for example.) And, the social interactions and contingencies in teaching are at least as complex as in any of these fields. Hence, if one solves the problem of decision making during teaching, one has solved the problem of decision making in general.

That solution, hinted at above, is laid out in detail in Schoenfeld (2010). The basic idea is that teaching is goal-oriented: a teacher enters the classroom with a set of goals in mind, takes the “temperature” of the classroom to see if revisions are necessary, and then, on the basis of (often tacit or unarticulated) beliefs about what is important and possible, selects a routine for implementation. If things are unproblematic, the routine is implemented; when the goal for the routine has been satisfied, then the next highest priority goals come to the fore, and the process is repeated. If at any point in the implementation something problematic does occur (e.g., a student misunderstanding is revealed, or student attention is waning, or...) then new goals (e.g., “deal with this misconception”) may be added to the goal stack, and the prioritization process begins a new.

This may sound simple, but, as always, the devil is in the details. The idea was to build the architecture of the model by defining key constructs (goals, resources, orientations, and a mechanism for decision making) and then to construct models by building representations of all of these constructs for each teacher being modelled. As the case grew in complexity, so did the time for analysis. Yet, the fact that we were able to model lessons of extreme complexity – lessons that some colleagues (Borko and Peressini, 1998; Leinhardt, 1998) had said would be impossible to model – gave credence to the generality of the theory, as did the fact that the work was consistent with work on decision making in other fields (e.g., medicine).

We could claim, then, that we understood decision making during teaching. But then, how to improve it?

SOLVING THE PROBLEM OF WHAT MAKES FOR POWERFUL CLASSROOMS

To this point, my research had focused on the question of how and why people make the decisions they do – students as learners and problem solvers, teachers as decision makers. But, this focus was inadequate for addressing the fundamental problem of

instruction. Learning takes place in classrooms, so the main analytic question is: *How can we characterize what is most important in classrooms with regard to students' learning of mathematics, and then, how can we work to improve instruction?*

The TRU Framework

A foundational insight for thinking about this issue comes from my earlier work on problem solving. In that work, we learned that the outcomes of instruction are much more than students' mathematics learning: Those outcomes include students' beliefs (about self, about mathematics, about learning, about problem solving) and aspects of metacognition. That is, there are many outcomes of the mathematics classroom that go beyond the content of the mathematics being studied. Thus, if one wants to understand learning in classrooms, one has to ask, "What are all the different aspects of classroom environments that might contribute, for better or for worse, to students' development of (a) mathematical content understandings, (b) habits of mind, (c) beliefs, (d) sense of self as a learner/doer of mathematics in particular, and a learner in general? Addressing this question involves a shift in perspective. Typically, when we look at classrooms, we ask: "What do we think of what the teacher is doing?" Here the fundamental question is: "Imagine yourself as a student in the classroom. What does it feel like to experience the instruction?"

I will save the reader the story of the development of the Teaching for Robust Understanding (TRU) framework, which is the focus of the rest of this paper. Suffice it to say that, as in the case of any challenging problem, the path to a solution was anything but straightforward; but each failed attempt yielded insights that ultimately led toward a solution. See Schoenfeld (2013, 2014, 2015) for details. I begin by framing the problem, and explaining why it is framed the way it is. Here is the fundamental question:

If you could focus on up to 5 aspects of instruction in order to improve mathematics teaching (or teaching in general), what would those five things be? And, how would you know they're the right things?

I have asked this question of large audiences many times. In just a few minutes each group has listed dozens of possible sources of improvement. So, why do I limit the list to five? Because you can remember and act on five things (Miller, 1956). A list of twenty or more issues is so large as to be functionally impossible to keep in mind and work with. The next question is, what attributes should these five aspects of instruction have? To begin, they should be necessary and sufficient to characterize powerful mathematics learning environments. By "necessary" in this context I mean that each aspect of instruction must be essential – if things in a particular classroom are not going well along any of the five aspects of instruction, then the students who emerge from that classroom will not be powerful mathematical thinkers and problem solvers. By sufficient, I mean that if things go well along all of the relevant dimensions, then the students who emerge from the classroom will be powerful mathematical thinkers and

problem solvers. In addition, the five dimensions should be framed in such a way that they can be used to support professional growth.

Figure 2 provides a top-level description of the five dimensions of the Teaching for Robust Understanding (TRU) framework. For a literature review substantiating each of the five dimensions, see Schoenfeld, Floden, and the Algebra Teaching Study and Mathematics Assessment Project (2014). In what follows I will briefly elaborate on each of the five dimensions identified in Figure 2. I then describe some tools we have been building, to support teacher growth.

The Five Dimensions of Mathematically Powerful Classrooms				
The Mathematics	Cognitive Demand	Access to Mathematical Content	Agency, Authority, and Identity	Formative Assessment
<i>The extent to which the mathematics discussed is focused and coherent, and to which connections between procedures, concepts and contexts (where appropriate) are addressed and explained. Students should have opportunities to learn important mathematical content and practices, and to develop productive mathematical habits of mind.</i>	<i>The extent to which classroom interactions create and maintain an environment of productive intellectual challenge conducive to students' mathematical development. There is a happy medium between spoon-feeding mathematics in bite-sized pieces and having the challenges so large that students are lost at sea.</i>	<i>The extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core mathematics being addressed by the class. No matter how rich the mathematics being discussed, a classroom in which a small number of students get most of the "air time" is not equitable.</i>	<i>The extent to which students have opportunities to conjecture, explain, make mathematical arguments, and build on one another's ideas, in ways that contribute to their development of agency (the capacity and willingness to engage mathematically) and authority (recognition for being mathematically solid), resulting in positive identities as doers of mathematics.</i>	<i>The extent to which the teacher solicits student thinking and subsequent instruction responds to those ideas, by building on productive beginnings or addressing emerging misunderstandings. Powerful instruction "meets students where they are" and gives them opportunities to move forward.</i>

Figure 2. The five dimensions of mathematically powerful classrooms

Dimension 1 concerns the quality of the mathematics addressed in the classroom. One can use the first part of this article, "Understanding Problem Solving," to think about the kinds of opportunities students should have. Those include the following.

1A. The Knowledge Base. The mathematical content must itself be rich; if it is not, there is no hope for deep mathematical learning. The content should be connected in multiple ways, providing opportunities for making connections between procedures and the underlying concepts, between related mathematical concepts, and in applications. Students should have opportunities to practice mathematical reasoning and to produce proofs, and to communicate with mathematics. (In the United States, various *Standards* documents such as the *Common Core State Standards* (2010) and NCTM's (2000) *Principles and Standards* delineate the kinds of content students should learn.)

1B. Problem Solving Strategies. Instruction should provide opportunities for students to learn and develop their abilities to use the kinds of problem solving strategies (heuristic strategies) identified by Pólya and at the core of my 1985 book *Mathematical Problem Solving*.

1C. “Control,” or monitoring and self-regulation. Instruction should provide opportunities for students to become reflective problem solvers. As documented in *Mathematical Problem Solving*, students can learn to be much more effective in monitoring their work, if attention is given to these issues in instruction.

1D. Beliefs. The literature indicates that students develop their counterproductive beliefs about mathematical and themselves as a result of their experiences in mathematics classes. The challenge, then, is to create curricula and classroom environments that give students the opportunity to experience mathematics as an arena in which they can engage productively, so they come to see themselves as mathematical sense makers and mathematics as a logical and coherent discipline that is accessible to them.

As you read through the brief description of Dimensions 2 through 5, remember that the central question is, *how do the students experience the classroom?*

Dimension 2 has an unusual title taken from the literature, “cognitive demand.” The basic idea is aligned with Vygotsky’s (1978) notion of the Zone of Proximal Development, or ZPD. If students are not challenged, they will not learn; but if the work they are asked to do is not cognitively within reach, they will not learn. The idea is to have students engage in what has been called “productive struggle.”

Dimension 3, “access to mathematical content,” is concerned with the question of which students have the opportunity to experience the mathematics described in Dimension 1. We have all seen classrooms in which the mathematics was beautiful, and in which the teacher advanced the lesson by consistently calling on the three top students, who understood the material. But, what about the rest of the students? From my perspective, a classroom is only successful (or “powerful”) if *every* student has meaningful opportunities to engage with the mathematical riches in Dimension 1. If a few students profit from instruction but many do not, then the classroom is not a powerful learning environment.

Dimension 4, “agency, authority, and identity,” concerns the sense of self (including beliefs) that students construct, as a result of their mathematics instruction. If you are reading this article, you probably think of yourself as a “math person.” But, we are in the minority. Surely you have met someone at a party who asked what you do. When you told them about your involvement with mathematics, the response was, “I could never do mathematics. I’m just not a math person.” Such people learned to consider themselves “not math people” because of the way they experienced mathematics in their classes. So the question is, how can we change that? Where are we providing students opportunities to think mathematically, to do mathematical sense making, to put forth ideas, to explain things, and to have their ideas built upon? The goal is to have

students develop a legitimate sense of mathematical agency (“I can think mathematically”) by virtue of their producing (authoring) mathematics, and thus develop positive mathematical identities (“I am a mathematical sense maker”).

Dimension 5, “Formative Assessment” (see, e.g., Black and Wiliam, 1998) concerns the kinds of feedback students get during instruction. Typically major exams (summative assessments) score students’ work but provide little useful information about how students might improve. The idea behind formative assessment is that, as an ongoing practice, students will reveal what they know and instruction will respond to it in ways that help the students learn – i.e., if a student reveals a misconception during instruction, there is time for the student to be given a mathematical experience that helps to undo the misconception. More generally, ongoing formative assessment reveals what students know, enabling the teacher to establish the right level of cognitive demand for the students.

There is ample evidence for the importance of all five dimensions (see Schoenfeld, 2013, 2014, 2015; Schoenfeld, Floden, and the Algebra Teaching Study and Mathematics Assessment Project, 2014, for evidence.) One can think of them, metaphorically, as basis vectors in a five-dimensional space; the idea is that these five dimensions span the space of important classroom activities. As in the case of vector spaces, there is no claim that the decomposition is unique; merely that the basis vectors span what is important. For example, one colleague asked about the concept of “safety.” In an effective classroom, he argued, students must feel safe in order to venture their ideas for classroom discussion. I agreed, but I noted that safety is an essential precondition for the support of dimension 4, “agency, authority, and identity”: in a classroom where students do not feel safe to try out ideas, they will not be able to develop a sense of agency, become known for their ideas, or feel positive about their contributions. Thus the idea of safety is entailed in the five dimensions, even though it is not foregrounded. I have not encountered an important idea that is not entailed in TRU – indeed, that is most unlikely, since TRU was derived as a set of equivalence classes of attributes of powerful classrooms (See Schoenfeld, 2013). Different choices of basis vectors would highlight different aspects of classroom environments, and in doing so might make it easier or more challenging for people to remember and focus on those things. Our experience has been that TRU is easy to remember in its current form, and that the dimensions have served productively as the basis for professional development. But, it is possible for others to organize things differently.

Tools

Over the past half dozen years, the Mathematics Assessment Project and the Algebra Teaching Study (at <http://map.mathshell.org/> and <http://ats.berkeley.edu/> respectively) have developed a series of tools consistent with the values of TRU, intended to help teachers and administrators improve classroom practice. A range of resources can be downloaded from both web sites.

First among these tools is the set of *Formative Assessment Lessons* (FALs) designed by the Mathematics Assessment Project (MAP). There are 100 such lessons, available for free at the MAP web site, <<http://map.mathshell.org/lessons.php>>.

The design of the lessons is critically important. There is an embedded pedagogy that is entirely consistent with the TRU framework. The lessons focus on important mathematics – on making connections, on reasoning, on explaining (Dimension 1, the mathematics). They are based on extensive field testing, in which typical student thinking (including errors) has been revealed, and used to design the lessons; the activities have been constructed with “low floor, high ceiling” and multiple entry points so that all students can engage in meaningful ways (Dimension 3, access). They provide opportunities for students to work through ideas, to discuss them, and to present them – opportunities for developing productive mathematical identities (Dimension 4). The lesson plans inform teachers about likely incorrect student responses to tasks in the lessons, and make suggestions about ways to respond that enable students to work through their mistakes. In that way they deal productively with Dimensions 5 and 2, formative assessment and cognitive demand.

The Formative Assessment Lessons are intended to provide support for both students and teachers. The students are introduced to rich mathematics in engaging ways. But, the lessons also feature a non-standard pedagogy – teachers learn are helped to do less “telling,” and to give students chances to reveal their understandings and work through ideas. Thus, the FALs function as a pedagogical intervention.

To date, there have been more than five million Formative Assessment Lesson downloads. The lessons are very popular. And, they are effective. An initiative called the Mathematics Design Collaborative, funded by the Gates Foundation, provided professional development support for teachers in selected states and cities, to help them implement the Formative Assessment Lessons. The Foundation commissioned a series of evaluations of the efforts. Here is detail from one of the evaluations:

Participating teachers were expected to implement between four and six Challenges [FALs], meaning that students were engaged only 8-12 days of the school year ... Nonetheless, the studies found statistically significant learning effects... the approximate equivalent of 4.6 months. (Herman et al., 2014, p. 10)

These findings are difficult to believe at face value. How could 8-12 days of lesson implementation result in so dramatic a change in student learning? A suggestion as to the mechanism behind the results is given three paragraphs above. The pedagogy of the FALs is very consistent with TRU. Our hope was that teachers who learned to implement the FALs would learn to teach in a more TRU-like manner, even in their regular instruction. Of course, the actuality is more complex than that simple story: teaching is multidimensional, and teachers (like students!) make choices about what they will work on, and those choices shape what they will learn. So they would learn something, but what? Two recent dissertation studies (Kim, 2015; Seashore, 2015) reveal the complexities of teacher learning catalyzed by the use of FALs. Both studies

reveal that significant changes in teachers' regular practices can result from teachers' use of as few as four or five FALs over the course of a year. However, there can be challenges (including difficulties in classroom management) when teachers adopt new practices, and there are inappropriate modifications to the FALs that render them ineffective. Additional studies are needed to explore teacher learning with potentially transformative curricular materials and other support mechanisms.

That comment leads to our next tool, the TRU conversation guide (Baldinger and Louie, 2014). Before proceeding, I should note that, world-wide, there are very different models of teacher support and teacher professional development. In lesson study, as practiced in Japan, communities of teachers collaborate in the design, teaching, observation, and refinement of lessons – often in collaboration with university faculty. In the United States, many teachers work in isolation, with little opportunity for collegial interaction or support (Lortie, 1975). Our preference is for the kind of rich collegial interactions that support ongoing professional growth – whether in a professional learning community or in teacher-coach interactions. The TRU conversation guide was constructed with such interactions in mind, although it can be used by individuals as well. The issue is: how can one inquire into and reflect on one's teaching, using each dimension of TRU as a stimulus for reflection?

For *Dimension 1, The Mathematics*, the central question is: what are the big ideas in today's lesson, and how do/will those ideas get developed and be connected to what came before and will come after?

For *Dimension 2, Cognitive Demand*, the central question is: What opportunities do/will students have to make their own sense of mathematical ideas?

For *Dimension 3, Access*, the central question is: Who does and does not participate in the mathematical work of the class, and how?

For *Dimension 4, Agency, Authority, and Identity*, the central question is: What opportunities do/will students have to explain their own and respond to each other's mathematical ideas?

For *Dimension 5, Formative Assessment*, the central question is: What do we know about each student's current mathematical thinking, and how can we build on it?

For each dimension, the central question is expanded into a collection of questions that can be used to inquire into one's teaching practice. For example, some of the elaboration questions for Dimension 3, Access, are:

- Who generates the mathematical ideas that get discussed?
- Who gets to explain their ideas?
- Who evaluates and/or responds to others' ideas?
- How does the teacher respond to student ideas (by evaluating, questioning, probing, soliciting responses from other students, etc.)?

- What are the norms related to student participation and what counts as a mathematical contribution, and how are they being developed?

Our hope and expectation is that planning with such questions and reflecting on what happens (with the help of others, if possible) will lead to improvements in instruction.

Along these lines, we have constructed a number of other tools. For research purposes, we have a rubric that scores alignment with the TRU dimensions. The major purpose for this tool is to explore relationships between scores on the TRU dimensions and students' performance on tests of mathematical thinking – the goal being to produce additional evidence regarding the relationships between student learning and classroom scores on TRU. Higher scores on each dimension correspond to richer classroom activities; thus the rubric describes growth trajectories along each of the dimensions. Yet, we want to be careful about the distribution and use of this tool. In the high-stakes testing context in the United States, any tool that can be used to score teachers' classroom performance should be used with caution.

Finally, we are building a series of workshops intended to help teachers and teacher communities use TRU to reflect on their practice. As I have noted, a major aspect of TRU is that it considers the impact of the learning environment as a whole. The idea is that, rather than restricting one's focus to what the teacher does, one should be thinking about how the students experience the instruction. Thus, one of our tools asks teachers to observe a lesson as though they were students.

This shift of perspective (see Figure 3 below) is very useful. An introductory workshop features a range of videos as a basis for learning to “see” the dimensions of TRU. In the next workshop, each of five groups picks a different TRU dimension to focus on. After discussing sample videos each group reports out to the collective, so that everyone hears about all five dimensions. For the next workshop teachers pair up, perhaps with coaches or others, to plan and observe lessons in their own classrooms. Videos of these lessons are then discussed. Our expectation is that as teachers do this over time, TRU will get internalized as a way of thinking.

That, of course, is what we are aiming for – that before and after instruction, it will become natural and automatic for teachers to ask themselves: (1) “What are today's big mathematical ideas? (2) How will my students be engaged in sense making related to those ideas? (3) Have I structured things so that all students have the opportunity to engage in meaningful ways, and that (4) they have opportunities to discuss and explain, to build on each other's thinking? (5) How will I know what my students are thinking? How am I prepared to react to what they say and do?”

Figure 3 provides the student view.

Observe as if you were a student

The Mathematics	<ul style="list-style-type: none"> • What's the big mathematical idea in this lesson? • How does it connect to what I already know?
Cognitive Demand	<ul style="list-style-type: none"> • How long am I given to think, and to make sense of things? • What happens when I get stuck? • Am I invited to explain things, or just give answers?
Access to Mathematical Content	<ul style="list-style-type: none"> • Do I get to participate in meaningful math learning? • Can I hide or be ignored?
Agency, Authority, and Identity	<ul style="list-style-type: none"> • Do I get to explain, to present my ideas? Are they built on? • Am I recognized as being capable and able to contribute in meaningful ways?
Formative Assessment	<ul style="list-style-type: none"> • Do classroom discussions include my thinking? • Does instruction respond to my thinking and help me think more deeply?

Figure 3. Questions from the student perspective

Our goal is for these workshops, and for TRU in general, is to support teachers in enriching their classroom activities. To the degree that we succeed, this will help their students better experience the richness and power of mathematics.

DISCUSSION

Little did I know when I picked up a copy of How to Solve It some 45 years ago that reading the book would be a life-changing event. Pólya's ideas in How to Solve it (1945), Mathematics and Plausible Reasoning (1954) and Mathematical Discovery (1962, 1965) opened up a new universe to me. I have always loved mathematics for its beauty and its power, and I have always been concerned that so few students get to experience mathematics in the ways that I did. In reading Pólya's books (and seeing his movie Let us Teach Guessing, and having heard about his teaching summer workshops for teachers for teachers at Stanford) I was made aware of a career path that could be both intellectually rewarding and which might help more students experience the pleasure and power of mathematics. Pólya was a trailblazer in many ways. It is an honor and a privilege to walk in his footsteps.

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INTERPLAY BETWEEN CREATIVITY AND EXPERTISE IN TEACHING AND LEARNING OF MATHEMATICS

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In this paper I discuss the relationship between mathematical creativity and expertise in teachers and students. I consider developing mathematical expertise and mathematical creativity in each student as two equivalent purposes of school mathematics education. The focus is on relative creativity and relative expertise associated with school mathematics teaching and learning. Several studies are described in the paper in order to demonstrate that (advancement of) students' mathematical expertise (reflected in high mathematical achievements) and their mathematical creativity are mutually dependent. While development of mathematical expertise is not necessarily applies advancement of mathematical creativity, creativity-directed learning advances problem-solving expertise. Moreover, while flexibility and expertise in mathematics and mathematics teaching are strongly linked, originality characterises activities of individuals with special mental characteristics. Finally, development of creativity in students entails the creation of a "creative" learning environment and thus requires expertise and creativity in mathematics teachers.

INTRODUCTION

In his arguments regarding the importance of creativity for child development, Vygotsky (1982/1930, 1984/1931) maintains that creativity (or 'imagination' in Vygotsky's words) is the central mechanism in the development of children's knowledge, since it allows them to construct connections between their existing knowledge and new pieces of information. In contrast, creative processes presume the discovery of new constructs, properties and regularities to expand existing knowledge to the new territory, on the basis of one's existing knowledge. This observation leads directly to a *knowledge-creativity paradox*: creativity is a necessary condition for knowledge construction, whereas knowledge is a necessary condition for creative processing. This intriguing phenomenon leads to a question that is surprisingly overlooked in mathematics education research: Does higher creativity in individuals lead to more advanced mathematical knowledge/expertise, or is mathematical creativity a function of the knowledge/expertise development? Consequently mutual relationships between mathematical creativity and mathematical expertise has been the focus of my research during the past decade.

In his seminal lecture before the French Psychological Society, Poincare (1908) provided analysis of mathematical creation in professional mathematicians. He realized that special intuition and understanding of mathematics rather than memory allow *some individuals* to become creators. These individuals have "the feeling of mathematical beauty, of the harmony of numbers and forms, of geometric elegance.

This is a true esthetic feeling that all real mathematicians know, and surely it belongs to emotional sensibility" (ibid. p. 92). Whereas Poincare connected mathematical expertise with creativity, and the creativity in professional mathematicians was analyzed (Hadamard, 1945; Sriraman, 2009), the relationship between mathematical creativity and mathematical ability has remained unexplored in school children. My research examines this connection.

I believe that mathematics teaching is aimed at the advancement of students' thinking skills, problem-solving expertise and mathematical creativity. I agree with Sawyer's (2004) claim that *teaching is a creative performance*, due to the ambiguity and uncertainty of the teaching process. However, creativity in mathematics teaching as a research topic was also overlooked for many years (Haylock, 1987, Leikin, 2009a). In light of these observations I explore teachers' creativity in mathematics teaching as an indicator of teachers' expertise.

BACKGROUND

Mathematical potential and expertise

Mathematics education is aimed at providing equal learning opportunities to all students (NCTM, 1989) which enable the realization of learners' mathematical potential to the maximal extent. Milgram & Hong (2009) stressed the danger of "talent loss" which is "the failure of individuals to realize the potential for the extraordinary achievement in a specific domain that they demonstrated in their early years" (p. 149). Clearly, students differ in their mathematical potential and, thus, according to the equity principle of mathematics education, "reasonable and appropriate accommodations (should) be made to promote access and attainment for all students" (NCTM, 2000, p. 12). Realization of mathematical potential is a function of mathematical abilities, motivation, personality and the learning opportunities open to the students at different stages of their development (Leikin, 2009a). Dowker (2005) argues that mathematical abilities are represented along a continuum that starts from a low level, proceeds to an average level, and then up to exceptional proficiency.

Ericson (1996) defines "expert performance as consistently superior performance on a specified set of representative tasks for a domain" (p. 277) and stresses that "it is generally assumed that outstanding human achievements (i.e. expertise) reflect some varying balance between training and experience (nurture), on one hand, and innate differences in capacities and talents (nature) on the other" (p. 274).

An expert has been described as having more robust mental imagery, more numerous images, the ability to switch efficiently and effectively between different images, the ability to focus attention on appropriate features of problems, and having more cognizance of their own thought process and of how others may think (Carlson & Bloom, 2005; Lester, 1994). In contrast to experts, a novice's system of representations of a mathematical concept may be deficient in number and in connections to form an adequate network of knowledge (Lester 1994). Experts differ from novices in the problem-solving strategies they employ (Schoenfeld, 1992) and in their ability to

categorize problems according to solution principles and choose the most efficient ways of solutions for a particular type of problem (Sweller, Mawer, and Ward, 1983). Expert knowledge is organized in hierarchical schemas which are mostly lacking in novices (Chi, Feltovich, & Glaser, 1988). Experts have been shown to spend more (than non-experts) time on features designated as critical to the problem (Morrow et al., 2009) and to rapidly encode features of problems based on goal-relevant representations.

Krutetskii's (1968/1976) seminal study titled "The psychology of mathematical abilities in schoolchildren" introduced components of high mathematical abilities in school children. The study employed several series of powerful problems whose solutions required mental processing that characterize the work of professional mathematicians. These problem series were specifically designed for that study to compare problem-solving performance in "capable, average and incapable" school students. Mathematically capable students were shown to be special in their abilities to grasp formal structures, think logically in spatial, numeric, and symbolic relationships, generalize rapidly and broadly and be flexible with mental processes. According to Krutetskii, students with high mathematical abilities appreciate clarity, simplicity, and rationality and can be characterized by the general synthetic component called *mathematical cast of mind*. Since 1976 no systematic large-scale or longitudinal study that analyzed specific characteristics of students with high mathematical abilities / mathematical experts / mathematically gifted students has been published.

Whereas mathematical ability is often seen as being equivalent to mathematical attainment, these terms are not necessarily synonymous. Even though mathematical attainments are correlated with mathematical abilities, the key difference is that mathematical ability relates to the potential to do mathematics, while attainment refers to the ability to achieve success on school mathematics tests (thefreedictionary.com). Usually, however, school mathematics tests examine students' ability to cope with learning-based problems only. These problems require application of the studied mathematical contents. High achievements on tests related to school mathematics at a high level can be considered to be expertise in school mathematics. I suggest that excellence in school mathematics can be considered to be *relative expertise*.

Mathematical expertise is frequently associated with general giftedness (G) while psychometric definition of giftedness is often determined by Intelligence Quotient (IQ) with 2 standard deviations (SDs) above the population mean (Feldman, 2003). The studies that perceive expertise as an equivalent of giftedness focus on cognitive characteristics of G students and demonstrate that their performance is significantly better than that of non-gifted (NG) individuals on tasks requiring analogical thinking, acquisition of new information, and application of different problem-solving strategies (Steiner & Carr, 2003).

To address the relative nature of expertise in school students and to be precise in the research we conduct, we make a distinction between general giftedness (G) and

excellence in school mathematics (EM). Moreover, due to the linkage between mathematical talent and general giftedness, my past and present research that examines relationship between mathematical abilities and creativity makes a clear distinction between G and EM factors.

Creative talent in mathematics

Milgram & Hong (2009) argue that analysis of the activities of a number of eminent mathematicians demonstrated that inventions and accomplishments in mathematics require creative talent rather than traditional academic ability. Milgram and Hong suggested a comprehensive model of talent development that introduced a distinction between two types of talent: *expert talent* and *creative talent*. The level of talent varies from minimal to profound while talent realization depends on personal-psychological attributes (e.g. motivation, meta-cognition, affective and biological) and on the environmental-social factors (e.g. school, family, technology and socio-cultural characteristics).

There is consensus that professional mathematicians are experts in mathematics. Poincare (1908) linked the activity of a mathematician to mathematical creation that requires a feeling of mathematical order and mathematical intuition, which, in his opinion, cannot be possessed by everyone. According to Poincare (1908) mathematicians possess special intuition that allows them not only to understand mathematics but to "become creators". He indicated three levels of mathematical abilities as functions of different combinations of the abilities to apply mathematics and to create mathematically. Consistently the connection between mathematical giftedness and creativity led to an eight-tiered (from 0 to 7) hierarchy of mathematical gift (Usiskin, 2000; Sriraman, 2005). According to this hierarchy the 'honors' and 'terrific' high school students are at levels 2 and 3, while creative mathematicians are at the 6th and 7th levels. Those from the 7th level are Fields Medal winners in mathematics.

Inspired by Poincare's lecture, Hadamard (1945) performed an empirical investigation of creative processes in professional mathematicians. He suggested that mathematical invention, which is an integral part of the activities of research mathematicians, consists of four stages: *initiation*, *incubation*, *illumination*, and *verification* (Wallas, 1926). Special attention is given here to illumination, which involves a large measure of intuitive thinking that leads to mathematical insight.

Researchers make a distinction between algorithmic, strategic and creative problem solving, while creative problem solving is associated with mental flexibility (Silver 1997; Star and Newton 2009) and with mathematical insight (Krutetskii, 1976; Eryvnck 1991; Leikin 2013). Insight-based problems are defined as problems that have a relatively simple solution which is difficult to discover until solution-relevant features are recognized (Weisberg 2015). The moment of insight is often described as the "Aha!" moment (Weisberg 2015). The relationship between insight in problem

solving and creativity has been stressed by several researchers (Csikszentmihalyi, 1988).

In general, (not necessarily in mathematics) when drawing a line between giftedness and creativity, researchers express a diversity of views. Some researchers argue that creativity is a specific type of giftedness (Sternberg, 2005); others feel that creativity is an essential component of giftedness (Rensully, 1978, 1986); while others suggest that the two are related but distinct human characteristics (Milgram & Hong, 2009). Sriraman suggests that "mathematical creativity implies mathematical giftedness, but the reverse is not necessarily true (Sriraman, 2005). These different theoretical viewpoints require empirical clarification, and thus analysis of the relationships between mathematical creativity and giftedness is an important research question, in general, and in the context of mathematics education, in particular.

Mathematical creativity in school mathematics is usually associated with problem solving or problem posing (e.g. Silver 1997). Following Torrance (1974), Silver (1997) suggested developing creativity through problem solving as follows: *Fluency* is developed by generating multiple mathematical ideas, multiple answers to a mathematical problem (when such exist), and exploring mathematical situations. *Flexibility* is advanced by generating new mathematical solutions when at least one has already been produced. *Originality* is advanced by exploring many solutions to a mathematical problem and generating a new one. Ervynck (1991), who considered creativity to be a critical component of problem solving connected to advanced mathematical thinking, pointed to three levels of creativity: Level 1 contains an algorithmic solution to a problem; Level 2 involves modelling a situation; and Level 3 makes use of the problem's internal structure. Ervinck's level 3 of creativity actually refers to the ability of a person to perform original, non-algorithmic and, often, insight-based solutions.

Naturally, creativity in school mathematics differs from that of professional mathematicians. Leikin (2009a) suggested a distinction between *relative* and *absolute creativity* when exploring creativity at school level (cf. objective and subjective creativity (Lytton 1971) and that of Big C and Little C creativity (Csikszentmihalyi 1988)). Absolute creativity is associated with discoveries at a global level ("historical works" as termed by Vygotsky 1930/1982, 1930/1984). The focus of my research is on relative creativity in mathematics students and teachers.

The research literature points out mutual relationships between teachers' conceptions, their practice and their expertise. Teachers' mathematical and pedagogical conceptions determine the quality of their mathematics lessons, while the teaching process shapes those conceptions (Kagan 1992; Thompson 1984).

EXAMPLES OF OUR STUDIES AND RESULTS

In this section I briefly describe several studies that examined mathematical creativity in students and teachers. I start with a description of multiple-solution tasks (MSTs), present a model for the evaluation of mathematical creativity and demonstrate how it

was implemented in research that examines the relationship between mathematical creativity and excellence in school mathematics. I also discuss development of mathematical creativity and expertise in prospective high school mathematics teachers and conclude with a discussion of the components of teachers' expertise directed at the development of students' creativity.

Multiple Solution Tasks

Tasks: Solve the problem in at least 3 different ways	
Learning-based solutions	Insight-based solutions
P1. Calculation: $2\frac{1}{4} \times 1.75$	
1.1 The distributive law: in decimal numbers; in common fraction;	1.4 Reduced multiplication $\times 1.75 = \left(2 + \frac{1}{4}\right) \left(2 - \frac{1}{4}\right)$
1.2 Vertical multiplication;	
1.3 Net multiplication	
P2. Jam problem: Mali produces strawberry jam for several food shops. She uses big jars to deliver the jam to the shops. One time she distributed 80 liters of jam equally among the jars. She decided to save 4 jars and to distribute jam from these jars equally among the other jars. She realized that she had added exactly $\frac{1}{4}$ of the previous amount to each of the jars. How many jars did she prepare at the start?	
2.1 System of equations with two variables;	2.4 Diagram based
2.2 Equation with one variable;	2.5 Elegant equations
2.3 Fractions/ Percentages. See students' collective solution space in Lev & Leikin (2013)	2.6 Logical: <i>4 jars include $\frac{1}{4}$ of all the jam. Thus there were 20 jars.</i>
P3 $\begin{cases} 4x + 5y = 18 \\ 5x + 4y = 18 \end{cases}$	
3.1 Algebraic combination.	3.6 Symmetry based consideration <i>The exchange of variables does not change the system which has only one solution:</i> $x = y = 2$ (Polya, 1973)
3.2 Substitution.	
3.3 Subtraction/ addition of equations	
3.4 Graphing.	
3.5 Solutions with determinants.	
P4 Half-Time -- Half-Way problem: Dan and Tal walk from the train station to the hotel. They start out at the same time. Dan walks half the time at speed v_1 and half the time at speed v_2 . Tal walks half way at speed v_1 and half way at speed v_2 . Who gets to the hotel first: Dan or Tal?	
4.1 Table-based	4.4 Logical <i>If Dan walks half the time at speed v_1 and half the time at speed v_2 and $v_1 > v_2$ then during the first half of the time he walks a longer distance than during the second half of the time. Thus he walks at the faster speed v_1 a longer distance than Moshe. Dan gets to the hotel first.</i>
4.2 Graphical	
4.3 Area-based	
	4.5 Diagram

Figure 1: Four MSTs from the test reported in this paper.

During the past two decades, I have systematically implemented multiple-solution tasks (MSTs) as a didactical and research tool in the majority of the studies that focus on the identification, development, and role of creativity in the teaching and learning of mathematics to students and teachers. During the last decade these studies have mainly analyzed creativity-related differences in learners with varying levels of excellence in school mathematics or in teachers with varying levels of expertise. Insight-based solutions in MSTs are in contrast to learning-based solutions of the problems (Figure 1).

A *multiple-solution task* (MST) is one in which learners are explicitly required to solve a mathematical problem using multiple solution strategies. The distinctions between the solution strategies can be based, for example, on (a) use of different representations of a mathematical concept; (b) use of different properties (definitions or theorems) of a mathematical concept; (c) use of mathematical tools from different branches of mathematics; and (d) use of tools from different fields (see examples in Figure 1; Figure 3). In this context *solution spaces of MSTs* (cf. example spaces in Watson & Mason, 2005) are used for the analysis of problem-solving performance associated with MSTs. Expert solution spaces are the most complete collection of solutions of a problem at a given moment. Individual solution spaces include solutions that a participant may produce on the spot or after several attempts. Usually individual solution spaces are subsets of expert solution spaces; however, sometimes they can broaden expert solution spaces. Collective solution spaces characterize solutions produced by a group of solvers. They are usually broader than personal solution spaces and within a particular group and are one of the main sources for the development of individual spaces. All the solution spaces can include conventional (i.e. learning-based) and unconventional (not learning-based that usually require insight) solutions (see Figure 1).

Model for the evaluation of creativity using MSTs

Runco and Acar (2012) highlighted the significance of developing scoring systems for divergent thinking tasks. We developed a model for the evaluation of creativity using MSTs which integrates the views of Erynck (1991), Krutetskii (1976), Polya (1973) and Silver (1997), who claim that solving mathematical problems in multiple ways is linked to mathematical creativity. The model was first introduced by Leikin (2009b) and then employed and validated in the work of Levav-Waynberg & Leikin (2012). Validation procedure demonstrated very high ($r > .95$) correlation between creativity and originality in all the MSTs in all our studies. This correlation attests to the validity of our model and is consistent with the view of creativity as an invention of new products or procedures in both absolute and relative creativity.

The model includes definitions of components of mathematical creativity as reflected in solving MSTs and the scoring scheme. Following Torrance (1974) we examine fluency, flexibility and originality of the solutions. *Fluency* score (n) is the number of ways in which a problem is solved by a participant. *Flexibility* score is given on the

basis of the differences between the solution strategies employed by an individual. *Originality* score reflects the newness of a solution with respect to the learning history of the individual (of his/her reference group) and the insight embedded in the solutions. Detailed description of the scoring scheme is presented in (Leikin, 2013). Figure 2 summarizes the scoring scheme.

Fluency	Flexibility	Originality	Creativity
Scores per solution	$Flx_i = 10$ for the first solution	$Or_i = 10$ $P < 15\%$ or for insight/ unconventional solution	$Cr_i = Flx_i \times Or_i$
	$Flx_i = 10$ solutions from a different group of strategies	$Or_i = 1$ $15\% \leq P < 40\%$ or for model- based/ partly unconventional solution	
	$Flx_i = 1$ similar strategy but a different representation	$Or_i = 0.1$ $P \geq 40\%$ or for algorithm- based/ conventional solution	
	$Flx_i = 0.1$ the same strategy, the same representation		
Total	n $Flx = \sum_{i=1}^n Flx_i$	$Or = \sum_{i=1}^n Or_i$	$Cr = \sum_{i=1}^n Flx_i \times Or_i$
n is the total number of appropriate solutions			
$P = (m_j / n) \cdot 100\%$ where m_j is the number of students who used strategy j			

Figure 2: Scoring scheme for evaluation of creativity (based on Leikin, 2013)

Creativity in mathematics, mathematical expertise and general giftedness

"Multidimensional investigation of mathematical giftedness" was conducted with the generous support of the Templeton Foundation and the University of Haifa in the framework of the University's RANGE Center. The study was conducted in collaboration with Miri Lev (creativity dimension), Nurit Paz-Baruch (basic cognitive traits dimension), Ilana Waisman (neuro-cognitive dimension) and Mark Leikin (co-supervisor). The goal of the study was to better understand the nature of high mathematical abilities.

A sample of 200 students who varied in the level of excellence in school mathematics (EM factor) and in general giftedness (G factor) were tested in the three dimensions of the study (e.g. Leikin, Waisman, & Leikin, 2016). Four major research groups were designed according to various combinations of G and EM factors (G-EM, NG-EM, G-NEM, NG-NEM), while the fifth group of nine students were considered to be 'super mathematically gifted' (S-MG). To examine differences in creativity of students from different groups, all the participants were asked to solve (P1, P2, P3 and P4, Figure 3) in at least 3 different ways. Evaluation of the creativity components was performed according to the scoring scheme for evaluation of creativity (Figure 2). Additionally we examined correctness of the solutions.

The results of the study (e.g., Lev & Leikin, in press) appeared to be task dependent, as revealed by different significant effects of the G and EM factors on correctness, fluency, flexibility, and originality in different problems. MANOVAs demonstrated effects of EM and G factors on all the examined criteria: No significant effect of G and

EM factors was found in correctness on P1 and P3 and in flexibility on P1. At the same time, EM factor and G factor have significant effects on fluency in problems which are usually solved in one particular way in the classroom (Fluency on P1, P2, P4 – Table 4), whereas when different solutions to a system of equations (P3) is a classroom routine, EM does not affect the production of multiple solutions. These results demonstrate that evaluation of correctness -- that reflects mathematical expertise in school mathematics -- in problem-solving performance on simple problems with several algorithm-based solution does not always provide complete information on the differences in students' problem-solving expertise.

Interestingly, flexibility on P2, P3 and P4 is affected by both EM and G factors. More precise examination of these effects demonstrates (Lev & Leikin, in press) that G factor increases flexibility both in EM and NEM students (not always significantly) whereas EM factor increases flexibility mostly in G students. Moreover a significant interaction between EM and G factors was found for flexibility associated with P3. Our other studies (e.g. Levav-Waynberg & Leikin, 2012) demonstrate that flexibility can be increased both in EM and NEM students when they are provided with creativity-oriented learning opportunities (e.g. project-based learning, systematically coping with MSTs, and solving non-routine tasks). By combining the studies results I suggest that, whereas our findings related to the effects of EM and G factors on flexibility when solving the problems demonstrate that general giftedness provides an effective springboard for the development of mental flexibility through the development of students' knowledge and skills, this finding might be related to the differences in learning environments in G and NG classes.

G factor had a major significant effect on the originality component for problems P1, P2, P3 and both EM and G factors had a main effect on the originality of solutions of the system of equations, while EM factor significantly increased the effect of G factor on students' originality when solving this problem. Levav-Waynberg & Leikin (2012) demonstrate that, at the group level, originality decreases as students' flexibility increases. At the same time, on the individual level, originality increases in very few cases (in less than 1.5% of participants considered by their teachers to be exceptional students). Thus, contrary to my suggestion that flexibility can be increased in all students when appropriate learning opportunities are provided, I argue that originality is an innate characteristic which can be developed in a specific population group.

Note here that only Half-way – Half-time problem (P4, which was the most complex one in the test) revealed significant differences between students from three excelling in mathematics groups: S-MG, G-EM and NG-EM students, including differences between S-MG and G-EM students. S-MG participants were significantly more fluent, flexible and original (i.e. more creative) than both G-EM and NG-EM students. Examination of correctness of the solutions in G-EM and S-MG students did not demonstrate differences between these participants.

Note that our research findings in the neuro-cognitive dimension (Leikin, Waisaman & Leikin, 2016) support findings outlined herein that mathematical insight is a specific characteristic unique to generally gifted students. We hypothesized that when solving insight-based problems, G students would start the solving process immediately upon appearance of the question and arrive at the "Aha!" through attention-related and information-integration processing. Consistent with the findings of Schoenfeld (1992) our study revealed that when solving learning-based problems, experts are able to predict the problem question based on the problem givens. We hypothesized that this prediction could be considered an insight-based process related to learning-based tasks and that the insight-related component is involved in the experience-based problem solving by experts at the stage of understanding the problem.

Developing expertise by means of creative mathematical activities

I cannot think of even one [solution] and you ask [me] to produce two....

A mathematical challenge is an interesting and motivating mathematical difficulty that a person can overcome (Leikin, 2007). Mathematical challenge is a core element of mathematical instruction aimed at fulfilment of the learners' mathematical potential through integration of cognitive demand (Silver & Mesa 2011) and positive affect (joy, interest and motivation) in the learning process. Jaworski (1992) stressed importance of mathematical challenge when defined teaching triad which characterises expert teachers by ability to combine mathematical challenge, sensitivity to students and management of learning in every instructional situation.

Challenging mathematical tasks can require solving insight-based problems, proving, posing new questions and problems, and investigating mathematical objects and situations. Investigation tasks (ITs) are the most inclusive (and thus the most challenging) type of tasks directed at conjecturing, examining the conjectures, proving, and posing new questions (Figure 3). MSTs and ITs are unique since they are challenging for novices and experts alike as *creativity-directed tasks* since their solutions require flexibility when finding additional solutions and raising different conjectures as well as originality when finding new mathematical facts and new mathematical solutions.

The analysis presented herein is based on a case study that focuses on the individual and collective solution spaces and spaces of discovered properties of Prospective Mathematics Teachers (PMTs) who are considered (in the present study) non-experts in geometry problem solving. The PMTs participated in 56-hour courses directed at the development of their knowledge of geometry, problem-solving expertise and the ability to create new geometry problems through investigations in DGE (see also Leikin, 2014). ITs and MSTs, which served as a core element of the courses, were completely new for the PMTs at the beginning of the course. The sessions with PMTs were videotaped and artefacts of their works were collected. Additionally, the PMTs presented their investigations to the whole group of PMTs and these presentations were also video-recorded. We analyzed (Leikin & Elgrably, 2015) the PMTs' solution and

discovery spaces using the solution and discovery spaces of Sharon (pseudonym) who is an expert in solving geometry problems at a very high level.

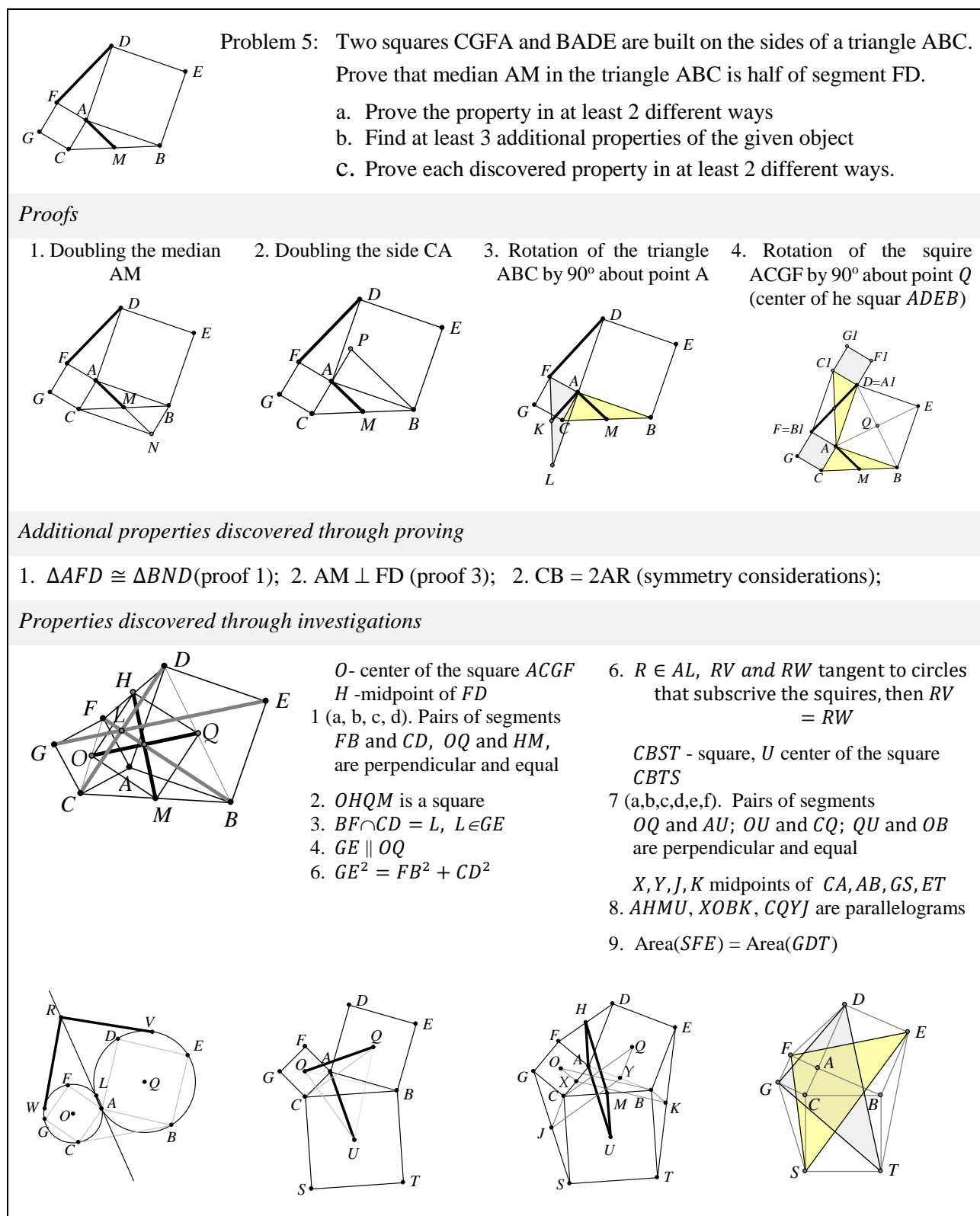


Figure 3: Collective solution spaces of solutions and discoveries of the investigation task produced by non-experts after training

The *discovered properties* were analyzed by all the participants according to: (a) *the newness of the property* discovered in the course of investigation, (b) *the complexity of the auxiliary construction* performed for the investigation and (c) *the complexity of the proof* of the new property. We identified seven *types of discovery strategies* (1) Discovery by chance (through measuring in DGE), (2) Discovery through association with another problem; (3) Discovery in the search for a proof; (4) Discovery based on previous knowledge of related properties; (5) Discovery in the course of proof; (6) Discovery by symmetry considerations; and (7) Intuitive discovery.

The comparison of the expert and non-expert spaces emphasized the *relative nature of the discoveries*. Many of the properties discovered by the PMTs were deemed "trivial" by Sharon (since they "can be proved in two stages") or "familiar" (since "this is a common problem that appears in the textbook"). We also found that the experts' spaces of the discovered properties were usually richer; however, some of the properties were discovered by the PMTs and overlooked by Sharon. The major difference between expert and non-expert investigations was in the investigation-strategies applied. Most of the discoveries by PMTs were found "by chance" whereas Sharon exhibited a variety of conscious strategies.

Based on this teacher education experiment, we argue that problem-solving expertise is a core element in the development of investigation skills in geometry and in advancing PMTs' creativity. Figure 3 demonstrates that, collectively, PMTs produced 4 different solutions to the given problem and formulated more than 10 new problems (requiring proof of discovered properties) most of which were cognitively demanding. I invite readers to prove the discovered properties depicted in Figure 3.

I would like to stress the importance that collective solution and discovery spaces play in the development of individual solution spaces, that is of the PMTs' problem-solving expertise and flexibility. The affective component that accompanied the advancement of PMTs geometry knowledge and problem-solving expertise was evident in the change that occurred in the questions the PMTs asked:

From: Why make life so difficult? I can hardly find one solution while you are asking for two. If I could, I would leave the course. It's unfair to demand these things of us. (SMK, affective)

How can I find a problem that has three solutions? I can see that different people can have different solutions. But how can I do it alone?

To: First of all, it's fun. At some point you feel you enjoy it -- enjoy solving and enjoy knowing. You say, "Wow, I can do it!". ...Why didn't we learn this way in school?

First, you have to force yourself to stop looking for which theorem the problem relates to; otherwise, you are confined to the theorem. Later you realize you are not interested in knowing on what page the task appears in the textbook. You simply solve it.

When solving the tasks in the group, one is always surprised by how differently people think. We always had at least 3 or 4 solutions for a problem. So you start to believe that this really is possible.

The relative nature of the expertise also can be seen clearly in the question raised by one of the best students at the end of the course: *"How do mathematicians discover properties without DGE?"*

Several comments on teachers' expertise through the lens of creativity in mathematics teaching

Analysis of teachers' expertise demonstrates that it is strongly connected to their conceptions of creativity and its role in teaching mathematics. In Lev-Zamir & Leikin (2011) we introduced a model of creativity in mathematics teaching that was shown to be a powerful tool for the analysis of teachers' conceptions and their practice. The model, which implements Torrance's (1967) definition of creativity as being composed of flexibility, originality and elaboration, suggests distinguishing between mathematical versus pedagogical conceptions and teacher-directed versus student-directed conceptions of creativity. Teacher-directed creativity is usually linked to the transformations of mathematical tasks (mathematical conceptions) or changing the design of a lesson setting (pedagogical conceptions) performed by the teacher during lesson planning and lesson management. They are always aimed at making the learning process more effective, yet do not address the promotion of students' creativity as a goal of mathematics teaching. Student-directed creativity is expressed in teachers' actions aimed at fostering students' creativity, including their mathematical flexibility, mathematical originality (imagination), and mathematical elaboration. We argue that teachers' expertise is expressed in the compatibility between their declarative conceptions and their conceptions-in-action, while the student-directed nature of declarative conceptions of creativity is an accurate predictor for teacher's creative expertise (Lev-Zamir & Leikin, 2013).

Note here that there are distinctive characteristics of teachers' expertise emerge from the analysis of characteristics of mathematics teaching that are especially important from the perspective of G-EM students as compared to NG-EM students. Additionally to the ability to explain clearly mathematical content, to raise interest in mathematics and to be sensitive to the students which are especially important to NG-EM students, G-EM individuals point out that mathematics teaching should be challenging, and this requires the teacher to exhibit a genuine interest in mathematics combined with profound mathematical knowledge that enables him to be prepared for any challenge coming from the students. Mathematics teaching should be joyful, practiced by an enthusiastic teacher who has sense of humor, who explicitly expresses a love of math and excitement from engaging in math, a teacher who has broad world knowledge beyond mathematics. Mathematics teaching is inspiring, open, critical and creative when the teacher is flexible and has knowledge and skills that allow for improvisation. In such an environment students develop deep and broad mathematical knowledge as well as flexible, critical and independent thinking.

SUMMARY

Following the literature review and based on our studies (some of which are outlined in this paper) I agree that relative creativity can be fostered in the majority of students in a creativity-directed learning environment.

Mathematical flexibility is a dynamic personal characteristic that can be considered as a function of expertise mathematical, whereas originality (as related to mathematical insight) seems to be a more innate characteristic and can be seen as function of the combination of mathematical expertise and general giftedness.

At the same time learning environment directed at promoting students' creativity is also effective for the development of their problem-solving expertise.

Creation of such a learning environment requires teachers to be creative experts in mathematics teaching.

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DYNAMIC CYCLE DRIVEN BY THE DIALECTIC CYCLE OF TWO COMPLEMENTARY REFLECTIONS IN LESSON STUDY ON SCHOOL MATHEMATICS

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The one of fundamental problems in mathematics education is “what and how can we do for students in enhancing their mathematical ability and performance?” In this paper, we will try to find a better promising solution to the problem. For that purpose, the author focuses on the lesson study and adopts the same way in which George Polya reflected on his experience and described methods of mathematical problem solving. As a mathematics educator the author will share his experience of two different types of lesson study on primary school mathematics, and propose the dynamic cycle driven by the dialectic cycle of two complementary reflections in lesson study for promoting both mathematics teachers’ and mathematics educators’ professional development that may contribute to enhancing the students’ mathematical ability and performance.

INTRODUCTION

The conference theme of PME 40 is ‘Mathematics Education: How to solve it?’ As one of fundamental problems to be addressed and solved by us in mathematics education, the author poses the educational problem: “what and how can we do for students in enhancing their mathematical ability and performance?” In this paper, we will try to find a better promising answer to this problem. For that purpose, the author focuses on the lesson study approach because we may share such philosophy that the quality of education cannot exceed the quality of teachers. In this approach, the author adopts the same way in which a mathematician George Polya reflected on his mathematical experience and described methods of mathematical problem solving (Polya, 1945).

The lesson study on school mathematics has been recognized as an important cultural and collaborative means of the continuous professional development of teachers for a life-long process in Japan (National Association for the Study of Educational Methods, 2011). The Japanese lesson study model is well known internationally (Stigler & Hiebert, 1999; Lewis, 2002; Isoda, et al., 2007; Shimizu, 2010; Takahashi, 2010, 2014). During lesson study, teachers collaborate to: 1) formulate long-term goals for student learning and development; 2) plan and conduct lessons based on research and observation in order to apply these long-term goals to actual classroom practices for particular academic contents; 3) carefully observe the levels of students’ learning, their engagement, and their behaviours during the lesson; and 4) hold post-lesson discussion with their collaborative groups to discuss and revise the lesson accordingly (Lewis, 2002). The success of lesson study can be found in two primary aspects; improvements in teacher practice and the promotion of collaboration among teachers. Lesson study

provides Japanese teachers with opportunities to make sense of educational ideas within their practice, to change their perspectives about teaching and learning, and to learn to see their practice from students' perspectives (Takahashi, 2010).

The lesson study can link together and promote both the pre-service teacher training at universities and the in-service teacher training at schools, local districts, or nationwide levels (Koyama, 2008b; Corey, et al., 2010). The lesson study as an in-service teacher training takes different forms. In the following sections, as a mathematics educator who has been seriously involved in various lesson studies in Japan, the author will share his experience of two different types of lesson study: a school-based and district-wide lesson study (Koyama, 2010b) and a school-based and cross-district lesson study (Koyama, 2015) on the problem solving lesson of primary school mathematics. Then we will reflect on the whole process of those case studies of lesson study by focusing on the relationship between collaboration and reflection. Finally, as a better promising answer to the problem posed in this paper, the author will propose the dynamic cycle driven by the dialectic cycle of two complementary reflections in lesson study for promoting both mathematics teachers' and mathematics educator' professional development that may contribute to enhancing the students' mathematical ability and performance.

EDUCATION SYSTEM AND THE PROBLEM SOLVING LESSON IN JAPAN

Before sharing the experience of lesson study, we note some information about education system and the problem solving lesson model of the teaching and learning of school mathematics (Koyama, 2008a, 2010a) in order to understand and reflect on the lesson study on the problem solving lesson of primary school mathematics in Japan.

Education system in Japan

In Japan the school education system is composed of six-year primary school, three-year lower secondary school, and three-year upper secondary school. The first two levels for nine years, age 6 year-old to 15 year-old, are compulsory education for all children. The national curriculum standard is prescribed in the Course of Study determined and issued by the Ministry of Education (e.g. Ministry of Education, 2008a, 2008b, 2009). The school textbooks must be approved by the Ministry of Education in line with the Course of Study. Public school teachers are local prefectural or municipal officials and appointed by the respective local prefectural or municipal boards of education. Primary school teachers teach almost all of school subjects at their own grade, while secondary school teachers teach their major school subjects.

The problem solving lesson as a typical model of good practice in Japan

A typical model of good practice recognized by many Japanese educators and teachers is the problem solving lesson in the mathematics classroom (Becker & Shimada, 1997; Stigler & Hiebert, 1999; Burghes & Robinson, 2009; Koyama, 2012). The problem solving lesson is structured and progressed through four distinct phases: presentation

of the problem, developing a solution, progress through discussion, and summarizing the lesson (Burghes & Robinson, 2009, p.56). It might be characterized as the parallel and collaborative version of four steps identified by Polya (1945) in solving a mathematical problem: understand the problem, devise a plan, carry out the plan, and look back and extend.

Why do many Japanese educators and teachers recognize the problem solving lesson as a good model of the teaching and learning of mathematics? As Sawada (1997) points out, the advantage of using the problem solving lesson model for teaching and learning mathematics in the classroom are;

Students participate more actively in the lesson and express their different ideas or solutions more frequently. Students have more opportunities to make comprehensive use of their knowledge, skills, and ways of thinking. Even low achieving students can respond to the problem in some significant ways of their own. Students are intrinsically motivated to give their justifications or proofs. Students have rich experience in the pleasure of mathematical activities and receive the approval from peer students in the classroom (Sawada, 1997, p.23).

The essence of doing mathematics is the process of solving a problem mathematically rather than its product. If they are acquired in the process of solving problems mathematically, then we believe that the mathematical knowledge, skills, and ways of mathematical thinking are applicable in a new or unfamiliar situation for learners. Therefore in the mathematics classroom, mathematics lesson as an integration of the teacher's teaching activity and the students' learning activity should be structured and progressed in the whole process of solving problems mathematically. The problem solving lesson is a student-centred model for teaching and learning mathematics that could encourage students to construct mathematics collaboratively in a mathematics classroom by using their naïve conceptions as well as their acquired mathematical knowledge, skills, and ways of mathematical thinking.

In the following three sections, the author will share his experience of two different types of lesson study at primary schools in which he had been involved as an external or internal expert of mathematics education: a school-based and district-wide lesson study (Koyama, 2010b) and a school-based and cross-district lesson study (Koyama, 2015). Then we will compare those two different types of lesson study on school mathematics with an example of research lesson in the problem solving lesson model from each of the two case studies of primary school mathematics.

CASE STUDY 1: SCHOOL-BASED AND DISTRICT-WIDE LESSON STUDY

In this section, as the first case study, we look at the school-based and district-wide lesson study on primary school mathematics at an Onomichi City Public Primary School (OCPPS) where the author as an external expert of mathematics education had been collaboratively working with the school teachers for three years (Koyama, 2010b). The mathematics lessons at the OCPPS were structured by using the problem solving

lesson model for the teaching and learning of primary school mathematics in order to foster students' ability to think and represent mathematically.

Characteristics of the OCPPS

The OCPPS is located in an urban city in Hiroshima Prefecture. The school had 260 students and 14 teaching faculties: ten classroom teachers including two teachers for the students with special needs, one chief of the instruction department, one chief of the research department, one school vice-principal, and one school principal. The school was designated by the Onomichi City Board of Education as a pilot school for primary school mathematics in the city for three academic years from April 2008 to March 2011. The pilot school for mathematics was expected to do a practical research on mathematics education through lesson study in order to improve both the students' mathematical achievement and the teachers' classroom practice, and to share the result and experience with many teachers from other primary schools in the city district.

Whole process of the lesson study at the OCPPS

The whole process of the lesson study at the OCPPS consisted of identifying practical research theme for the lesson study, planning the lesson study for a year, making two groups of teachers for the lesson study, conducting some cycles of the lesson study, and opening classes to teachers from other schools in the city district as follows.

Identifying practical research theme for the lesson study

In April 2008, the committee organized by a principal, a vice-principal and a chief of the research department identified the practical research theme of the lesson study for three years: fostering students' ability to think and represent mathematically. The theme reflected the emphasis by the Course of Study at that time (Ministry of Education, 2008a). It also took the students' issues in learning mathematics into consideration by analysing the result of mathematics achievement tests.

Under the theme for three years, for the first year the committee decided to explore how to improve mathematics lessons for students in enhancing mathematical thinking ability. Especially the committee focused on students' mathematical activity, devising mathematics notebooks, and collaborative learning to accomplish the first-year goals.

Planning the lesson study for a year

The school adopted the lesson study cycle of: 1) investigating a variety of teaching and learning materials; 2) developing a lesson plan; 3) conducting a research lesson; 4) having a post-lesson discussion; and 5) revising the lesson plan for improving classroom practice. By incorporating the lesson study cycles into one academic school year calendar, the school made the schedule of the lesson study for one year. At the beginning of a new academic year, there were several sessions for all teachers to share the research theme and goals for one year in the school, to understand students' actual performance and issues in learning primary school mathematics by analysing the results of national and school tests, and to make an agreement among all the teachers on doing regularly their research lessons for one year at the school.

Making two groups of teachers for the lesson study

The school had twelve teachers with different teaching experience of from just two years to over thirty years. Therefore all the teachers were divided into two groups with due regard to teachers' teaching experience and their students' grade. In the third year of the project, one group of teachers was expected to do lesson studies on some topics in the content areas of "Numbers and Calculations" and "Geometrical Figures" and another group was expected to do lesson studies on some topics in the content areas of "Quantities and Measurements" and "Mathematical Relations" even though in primary school every teacher teaches all the mathematical topics in the four content areas by using primary school mathematics textbooks and teacher's resource books and tools.

Conducting some cycles of the lesson study

Before conducting a research lesson, the lesson planning team developed a lesson plan for a particular topic. In the process of developing the lesson plan, first a teacher who would conduct a research lesson made the draft of lesson plan. Then in a session on developing the lesson plan, all members of the same team examined the draft from their own viewpoints and exchanged their ideas and experience among the team members in order to brush up the draft of lesson plan.

During the research lesson conducted by a classroom teacher of the team, all the participants observed the research lesson and collected data of the lesson in order to evaluate and improve the classroom practice and the lesson plan. The school made a special sheet for participants to check the observed lesson in terms of objectives of the lesson and to take notes on it both the teacher's classroom practice and the levels of students' learning and their behaviours during the lesson. The author as an expert of mathematics education was regularly invited to participate in the lesson studies.

In the post-lesson discussion session, copies of the record of research lesson were distributed to all participants. There was one teacher as a chair of the session and another teacher as a note taker of the session. After a brief explanation of the lesson plan and a conductor's short comment on his/her teaching practice, all the participants were divided into two groups to discuss the lesson in terms of the good practice, the issues to be improved, and the proposal of possible ideas and strategies for improving the classroom practice and the lesson plan by using the data noted on their evaluation sheet and the records of the lesson. After the group discussion, the leader of each group reported to all participants the essence of group discussion with a poster. At the end of the discussion session, the author as an outside expert of mathematics education was given an opportunity to make a summarizing comment on the lesson study.

Opening classes to teachers from other schools in the city district

As a responsibility of the pilot school, the school once a year opened all classes to teachers from other schools in the city district. After several cycles of lesson study, all the teachers developed their lesson plan for the open classes. During summer vacation for students, every teacher made a draft of lesson plan, and then the group of teachers

examined and discussed their drafts each other to improve the drafts of lesson plan. The author was asked to check all the drafts for open classes and visited the school for discussion with the teachers in order to brush up their drafts of lesson plan. In the process of developing lesson plans for the open classes, teachers had opportunities to reflect on their lesson studies done in the previous semester and to making use of their ‘lessons’ and experience learnt through doing lesson study in the school. On the other hand, the teachers from other schools in the city district had opportunities to observe lessons and participate in the post-lesson discussion sessions to share different ideas for improving mathematics instruction in their classroom. At the end of school year the school made a final report by summarizing the result and tasks not only for the reflection on the one-year lesson study but also for the preparation of the next year.

CASE STUDY 2: SCHOOL-BASED AND CROSS-DISTRICT LESSON STUDY

In this section, as the second case study, we look at the school-based and cross-district lesson study on primary school mathematics at the Hiroshima University Attached Primary School (HUAPS) where the author as a colleague has been collaboratively working with all the mathematics teachers of the school for several decades since 1990s (Koyama, 2015). The mathematics lessons at the HUAPS has been structured by using the problem solving lesson model for the teaching and learning of primary school mathematics in emphasizing students’ continuous awareness of learning mathematics.

Characteristics of the HUAPS

The HUAPS is located in Hiroshima City. The school has been attached to Hiroshima University for more than 100 years to accomplish its special mission of taking a part of practice teaching for pre-service teacher training at Hiroshima University, doing new developmental research for the next generation education, and doing lesson study and opening classes to teachers from all districts in Japan. The school is a national and very unique primary school because it has three mathematics teachers who teach only mathematics at some different grades, while primary school teachers usually teach almost all of school subjects at their own grade in local public school. More than 100 years the school has been issuing the own monthly journal *School Education* which includes a pair of articles of the lesson study on primary school subject by both one HUAPS teacher who conducted a research lesson and one Hiroshima University educator who is a specialist in the school subject education. In addition, we have been organizing the collaborative study team for doing the periodic seminar on primary school mathematics with not only mathematics teachers and mathematics educators but also graduate school students of mathematics education PhD and Master courses at Hiroshima University.

Whole process of the lesson study at the HUAPS

We show one example of lesson study conducted by the collaborative study team at the HUAPS from November 2012 to February 2013 in the third semester of academic year 2012. The lesson adopted the problem solving lesson model in the teaching and learning of “Triangle” for the 2nd graders at the school. The whole process of the lesson

study at the HUAPS consisted of seminar before the research lesson, planning the research lesson, conducting the research lesson and post-lesson discussion, and reflection on the lesson study as follows.

Seminar before the research lesson

In November 2012, our study team called *Primary School Mathematics Seminar* began the collaborative study on teaching materials to be used in the teaching and learning of “Triangle” in the teaching unit “Triangle and Quadrangle” for primary school the 2nd graders. We had total four 90-minute sessions from November 2012 to January 2013 before the research lesson. The team was organized by the mathematics teacher who would conduct the research lesson, three graduate school students of mathematics education PhD and Master courses, and two mathematics educators at Hiroshima University. The team had done collaborative studies on the teaching materials in order to help the mathematics teacher design the teaching unit “Triangle and Quadrangle” and develop the research lesson plan of “Triangle” for his 2nd graders.

Planning the research lesson

After the four sessions, Mr. Maeda, the mathematics teacher of the research lesson, designed a series of 15 lessons in the teaching unit “Triangle and Quadrangle” for his 2nd graders of the school. Then he located the research lesson at the 5th of 15 lessons for the teaching unit and developed the lesson plan of the research lesson.

Conducting the research lesson and post-lesson discussion

The research lesson was conducted with his 40 classroom students by Mr. Maeda on 9th February 2013 as one of open classes in the annual Open School Conference in which about 200 teachers from all districts participated in order to observe the lesson. Immediately after the research lesson, the post-lesson discussion with the participants was held in such way that the teacher explained briefly the objectives of research lesson and gave his short reflection on the lesson followed by the 30 minutes’ question and answer in his classroom. The research lesson and the post-lesson discussion were video-recorded and pictured by the graduate school students of collaborative team, and the author also observed the lesson and took a field note during the lesson observation.

Reflecting on the lesson study and writing articles

After the Open School Conference, the detail transcript of the research lesson was made to be used for both Mr. Maeda and the author in writing their own paper for a pair of articles of reflection on the lesson study submitted to the monthly journal *School Education* issued by the HUAPS. About one year later the paired articles were published in the January issue 2014 (Maeda, 2014; Koyama, 2014). The publication of the paired articles of the teacher’s reflection and the educator’s reflection on the lesson study is very important in the sense that it can contribute to promote the lesson study on school mathematics not only as a means of the continuous professional development of teachers but also as an authentic research area in the science of mathematics education.

COMPARISON OF TWO TYPES OF LESSON STUDY

Table 1 shows the comparison of two types of lesson study with an example of research lesson from each of the two case studies on primary school mathematics.

The research lesson in the Case Study 1 was conducted by a female teacher with a few years of teaching experience. Before conducting the lesson, the lesson planning team of school teachers developed the lesson plan for “Multiplication in Vertical Form” in the grade 3. The objectives of the lesson were to develop students’ understanding that multiplication can be calculated by using the multiplication rule: the product of multiplication does not change even if we multiply by decomposing a multiplicand or a multiplier into some numbers. The features of the lesson were to posing the problem of how to calculate multiplication 213×3 by applying a previously-learnt method of multiplication in vertical form by focusing on numerical positions, to using Japanese yen coins in order to calculate multiplication based on the place value system of decimal notation, and to giving two types of exercises at the end of the lesson such as multiplications of a three-digit number by a one-digit number and a missing-value problem to work in pairs to solve the problem and explain their solution strategy. From the observer’s perspective, it can be said that the female teacher had improved her classroom practice through the collaborative lesson studies with her colleagues.

Table 1: Comparison of two types of lesson study with the examples of research lesson

<i>Characteristics</i>	<i>Case Study 1: OCPPS</i>	<i>Case Study 2: HUAPS</i>
Type of Lesson Study	School-based and district-wide	School-based and cross-district
Whole Process of Lesson Study	<ol style="list-style-type: none"> 1. Identifying practical research theme for the lesson study 2. Planning the lesson study for a year 3. Making two groups of teachers for the lesson study 4. Conducting some cycles of the lesson study 5. Opening classes to teachers from other schools 	<ol style="list-style-type: none"> 1. Seminar before the research lesson 2. Planning the research lesson 3. Conducting the research lesson 4. Post-lesson discussion on the research lesson 5. Reflecting on the lesson study and writing articles
Teacher of Research Lesson	A female teacher with a few years of teaching experience	A male teacher with eighteen years of teaching experience
Author’s Position in Lesson Study	An external expert of mathematics education in giving suggestions for a lesson plan, observing a research lesson, and making a comment on the lesson in the post-lesson discussion	An internal expert of mathematics education in making a lesson plan, observing a research lesson, and writing an article of the reflection on the lesson study issued in a journal

<i>Characteristics</i>	<i>Case Study 1: OCPPS</i>	<i>Case Study 2: HUAPS</i>
Teaching Unit and Grade	<ul style="list-style-type: none"> ▪ “Multiplication in Vertical Form” in the grade 3 ▪ 37 students 	<ul style="list-style-type: none"> ▪ “Triangle and Quadrangle” in the grade 2 ▪ 40 students
Objectives of Research Lesson	<ul style="list-style-type: none"> ▪ To develop students’ understanding that multiplication can be calculated by using the multiplication rule: the product of multiplication does not change even if we multiply by decomposing a multiplicand or a multiplier into some numbers 	<ul style="list-style-type: none"> ▪ To fostering students’ ability of logical thinking especially by the activity for stimulating students think about the definition and properties of geometric figures ▪ To deepen students’ recognition and understanding of triangle
Features of Research Lesson	<ul style="list-style-type: none"> ▪ Using Japanese yen coins in order to calculate multiplication based on the place value system of decimal notation ▪ To understand and explain the meaning of multiplication through such activities as the explaining with pictures and the calculating separately for each position of the multiplicand 	<ul style="list-style-type: none"> ▪ Using different figures as teaching materials and showing them one by one in order to stimulate students to remember the definition of triangle and to use the definition in explaining the reason of their judgment ▪ To incorporate the key-questioning “Why don’t you say that the figure connected three points is a triangle?” in order to shake students’ recognition of triangle and deepen their understanding the definition of triangle
Viewpoints for the Analysis of and Reflection on Lesson Study	<ul style="list-style-type: none"> ▪ How to calculate a multiplication such as 213×3 by applying a previously-learned method of multiplication in vertical form by focusing on numerical positions ▪ Two types of exercises at the end of the lesson such as multiplications of three-digit number by one-digit number and a missing-value problem to work in pairs to solve the problem and explain their solution strategy 	<ul style="list-style-type: none"> ▪ Viewpoints of a figure: To classify and identify the given figures from different viewpoints of the figure ▪ Using counter-examples: To understand the definition of triangle through the discussion with counterexamples ▪ Different roles of teacher: To play two different role of ‘supportive’ and ‘active’ interventions to promote social interactions among students

The research lesson in the Case Study 2 was conducted by a male teacher with eighteen years of teaching experience and master's degree of education. After the four sessions of seminar by the collaborative study team, he designed a series of 15 lessons in the teaching unit "Triangle and Quadrangle" for his 2nd graders of the school. Then he located the research lesson at the 5th of 15 lessons, and developed the lesson plan of the research lesson. The objectives of the lesson were to fostering students' ability of logical thinking especially by the activity for stimulating students think about the definition of triangle. The features of the lesson were to using different figures as teaching materials, to showing those figures one by one in order to stimulate students to remember the definition of triangle and to use explicitly the definition in explaining the reason of their judgment, and to incorporating the key-questioning "Why don't you say that the figure connected three points is a triangle?" in order to shake students' recognition of triangle and deepen their understanding the definition of triangle.

ANALYSIS OF AND REFLECTION ON LESSON STUDY

In the following, we look at the lesson study in the Case Study 2 from three viewpoints: the collaborative study on the teaching materials for the research lesson, the lesson plan of the research lesson, and the analysis of and reflection on the lesson study.

Collaborative study on the teaching materials for the research lesson

In the first session of the seminar, the mathematics teacher proposed the students' activity of identifying whether a figure is triangle or not so that his students capture triangle in terms of its components, and shared with the team members his expectations to be realized through this activity. The main issue of the team discussion was what should be the criterion for the students in judging whether a figure is triangle or not. Finally they recognized the necessity to investigate the mathematical background related the definition of triangle and the research findings about the merit and demerit of using counterexamples for the students' concept formation of triangle.

In the second session, as a result of the team discussion, it was confirmed that the activity proposed by the mathematics teacher in the last session could help the students learn a fundamental mathematical activity of going back to the definition if needed. Through the activity the students might be promoted to think about the reason why the triangle is not defined as a figure connected three different points. Then the team members discussed the opinion that alternative teacher's activity of asking the students "Can you see a triangle in the given figure?" would be more appropriate for the students than the asking them "Is the given figure a triangle or not?". At the end, they recognized the necessity to design the development of research lesson in more detail.

In the third session, it was confirmed that the planned research lesson could be theoretically supported on the basis of both the philosophy of mathematics education and the cognition theory of geometric figures in psychological research. The team members agreed that the research lesson could contribute to deepening the students' recognition and understanding of triangle. Then they discussed how the students would interpret the teacher's questioning in "Let's classify the given figures into two groups".

Finally the mathematics teacher proposed eight different figures to be used in the research lesson. The team members discussed the effective way of showing those figures to the students in the research lesson.

In the fourth session, the main issue of the team discussion was how to deepen students' understanding the definition of triangle that the geometric figure surrounded by three straight lines already learnt in the previous lesson in the 2nd grade. In the discussion the team aimed at how to organize both the teacher's activity of shaking students' recognition and understanding of triangle and the students' activity of rethinking the definition of triangle. In order to promote these activities, they discussed the possible way of showing the prepared eight figures, and the expected merit and the demerit of giving counterexamples of triangle to the students in the research lesson.

Lesson plan of the research lesson

After the four sessions, the mathematics teacher of the research lesson developed the lesson plan. We must remember the fact that before this research lesson the 2nd graders already learnt the definition of triangle in the teaching unit "Triangle and Quadrangle" such that the geometric figure surrounded by three straight lines is called triangle. There are three remarkable features in the lesson plan. The first feature is the use of different figures including delicate figures of Fig.(c) and Fig.(d) as teaching materials. The second is the way of showing those figures to the students one by one beginning from Fig.(a), Fig.(b), Fig.(c), and so on (see Figure 1) in order to stimulate the students to remember the definition of triangle and to use the definition in explaining explicitly the reason of their judgment. The third is the incorporated key-questioning "Why don't you say that the figure connected three points is a triangle?" in order to shake the students' recognition of triangle and to deepen their understanding the definition of triangle. It might be said that those three features are crystallized as a result of the collaborative team study on teaching materials before conducting the research lesson.

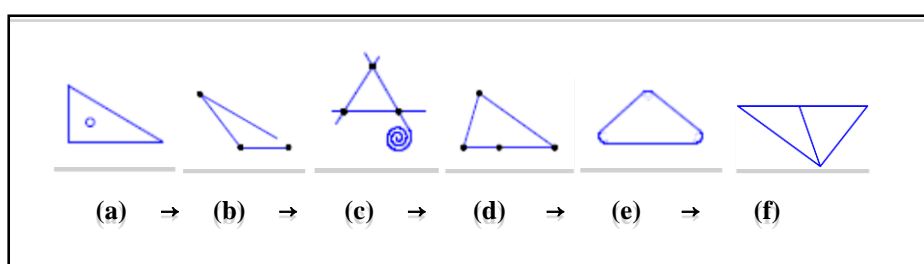


Figure 1: Intended order of showing figures to the students in the research lesson

Analysis of and reflection on the lesson study

The research lesson was analysed by using the detailed transcript and the picture of the blackboard of the lesson. As a result, the author found the importance of emphasizing viewpoints of figure and using counterexamples and the different roles played by the teacher in two notable scenes in the lesson (Koyama, 2014, 2015). In the first notable scene where Fig.(c) was shown by the teacher, the students had opposing judgments about whether the figure is triangle or not. After one-minute pair talk, the students

exchanged actively their viewpoints of Fig.(c) in the whole classroom discussion. Some students presented his/her opinion in their own words. For example, a student said “Those who judged Fig.(c) is not triangle see the figure as a whole including the part of spiral. On the other hand those who said Fig.(c) has a triangle see the inside part of the figure surrounded by three straight lines”. During the classroom discussion, the teacher did not make any comments about the students’ opinions but wrote down only some keywords related to their viewpoints of the figure on the blackboard such as “If we change our viewpoints”, “It depends on the viewpoint of figure”, and “If we see the figure like this” (see Figure2). In this scene, the teacher’s supportive intervention functioned effectively for the students to exchange their judgments by referring to the definition of triangle and to share the different viewpoints of one figure Fig.(c).

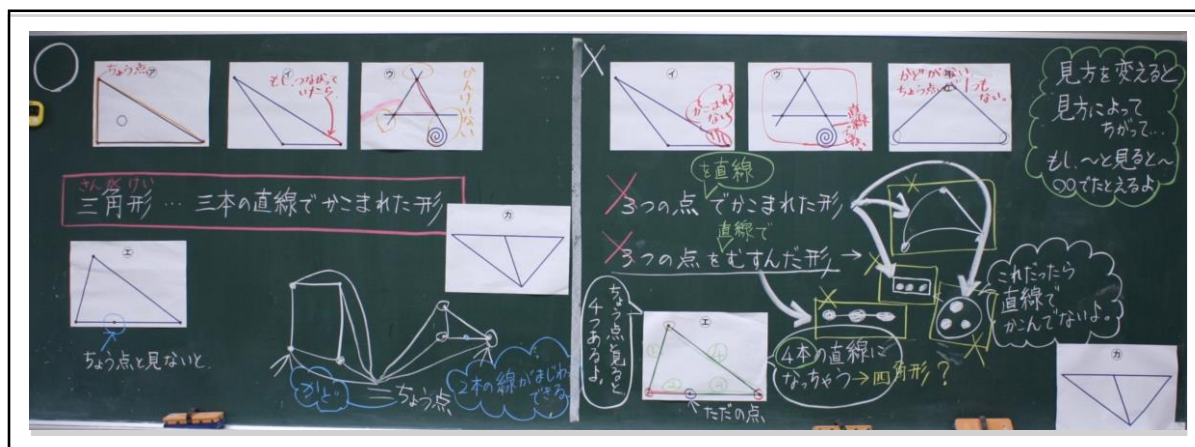


Figure 2: Picture of the blackboard in the research lesson

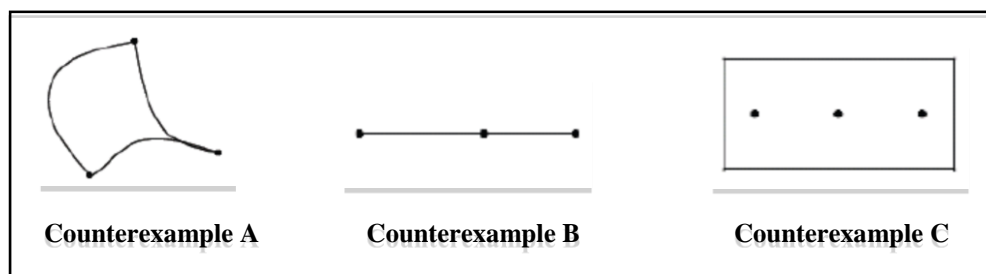


Figure 3: Counterexamples presented in the research lesson

The second notable scene in the lesson is related to the discussion between the teacher and the students. The scene began with the teacher’s questioning to the students “Why don’t you say that the figure connected three points is a triangle?” Immediately many students objected to what the teacher said. A student explained the reason why the figure connected three points is not a triangle by drawing the figure (counterexample A) (see Figure 3) in saying “If the figure is surrounded by three points does not make a triangle. Three points must be connected by straight lines”. Immediately the teacher counterattacked again by saying “I see! Do you agree that the figure connected three points by the straight line is a triangle” to the students. Then another student did not agree with the teacher, and refuted what the teacher said by drawing the figure (counterexample B) with three points on the same straight line in saying “I do not agree.

If three points connected by the straight line was a triangle, then this (counterexample B) should be a triangle. Therefore the triangle must be the figure surrounding three points by straight lines”. The student insisted on the importance of being surrounded by the straight lines. At that moment, many students seemed to be satisfied with their peer’s refutation. However, the teacher with a smile drew the figure (counterexample C) on the blackboard. In this scene, it can be said that the teacher’s active intervention functioned effectively for the students to deepen their understanding the definition of triangle through the whole classroom discussion with counterexamples.

DYNAMIC CYCLE DRIVEN BY TWO COMPLEMENTARY REFLECTIONS

As a result of the reflections on the two case studies of lesson study by focusing on the relationship between collaboration and reflection, the author insists that two kinds of complementary reflection function dialectically in the whole process of lesson study in both cases. The author calls them *collaborative reflection* and *individual reflection*. The *collaborative reflection* functions when a group of people reflect collaboratively on what the group did. On the other hand, the *individual reflection* functions when a person reflects individually on what the person did.

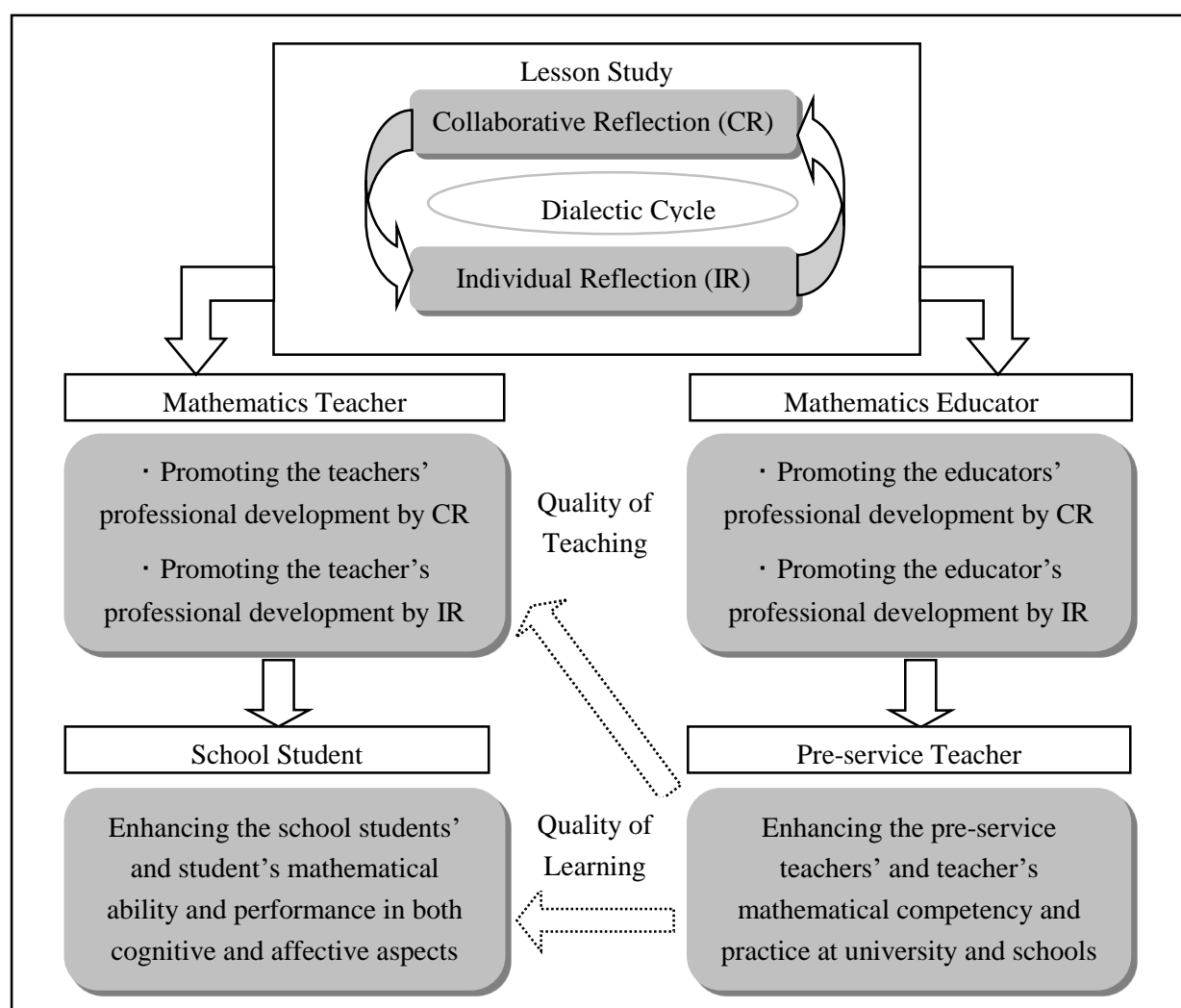


Figure 4: Dynamic cycle driven by the two complementary reflections in lesson study

For example, in the lesson study the *collaborative reflection* functioned when all members of the lesson planning team examined the draft from their own viewpoints and exchanged their ideas and experience among the members in order to brush up the draft of lesson plan. And in the post-lesson discussion, the *collaborative reflection* functioned when the group of teachers discussed the observed lesson in terms of the good practice, the issues to be improved, and the proposal of alternative ideas and strategies for improving their classroom practice and lesson plan. On the other hand, in the lesson study the *individual reflection* functioned when after the team discussion the individual teacher elaborated his/her lesson plan for the research lesson, and when the teacher wrote an article of the lesson study by using the detail transcript and the picture of blackboard of the lesson. The dialectic cycle of complementary *collaborative reflection* and *individual reflection* works as a driving force for promoting the group and the one professional development through the lesson study on school mathematics.

As a better promising answer to the problem posed in this paper, the author proposes the dynamic cycle driven by the dialectic cycle of two complementary reflections in lesson study for promoting the teachers' and teacher's professional development that may contribute to enhancing the students' mathematical ability and performance. This aspect is represented in the left side flow of Figure 4 with the two complementary reflections put at the heart as a driving force in lesson study, while the right side flow implies that the dialectic cycle promotes the mathematics educators' and educator's professional development, and then it produces a powerful effect on enhancing the pre-service teachers' and teacher's mathematical competency and practice teaching.

CONCLUSION

In this paper the author shared his experience of two different types of lesson study on the problem solving lesson of primary school mathematics in Japan. As a result of the reflections on those case studies by focusing on the relationship between collaboration and reflection, the author insisted that two kinds of complementary *collaborative reflection* and *individual reflection* function dialectically in the whole process of lesson study in both cases. The author characterized the dialectic cycle of two complementary reflections is a driving force. As a better promising answer to the problem posed in this paper, the author proposed the dynamic cycle driven by the dialectic cycle of two complementary reflections in lesson study for promoting the teachers' and teacher's professional development that may contribute to enhancing the students' mathematical ability and performance. It was also suggested that the dialectic cycle promotes the mathematics educators' and educator's professional development, and then that the cycle produces a powerful effect on enhancing the pre-service teachers' and teacher's mathematical competency and teaching practice at university and schools.

The author insists that the lesson study is not only an important means for the continuous professional development of teachers but also an authentic research area in the science of mathematics education. Therefore, in our further research, it is a critical issue for all of us in mathematics education to find out the way of certifying that the

dynamic cycle driven by the dialectic cycle of two complementary reflections in lesson study functions effectively and productively as we expected in mathematics education.

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Endnotes

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HOW TO SOLVE IT: WITH A FOCUS ON PROBLEMS IN MATHEMATICS TEACHING

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I address the theme of PME 40 – ‘How to solve it’ – starting from the famous Polya work and that of others following in Polya’s footsteps, addressing ways in which problem-solving heuristics contribute to students’ learning of mathematics. I then take the theme more broadly, asking, with a focus on mathematics teaching, “what is the IT?” -- what are the problems relating to mathematics teaching that are or have needed to be solved over my years in PME conferences? How can a focus on problem solving enable us to address issues in mathematics teaching? I suggest that seeing teaching development to encompass three levels of inquiry involving teachers and others offers a direct analogy with the heuristics of problem solving. The theory of critical alignment, emphasising tensions in development, constitutes an important heuristic.

PME 40, held in Szeged, Hungary, celebrates the work of the famous Hungarian Mathematician George Polya, who is remembered particularly for his book entitled “How to Solve it” (1945). Polya famously introduced a four step model for the problem solving process in mathematics: 1) Understanding the problem; 2) Devising a plan; 3) Carrying out the plan; 4) Looking back. Within each of these steps we see the use of a range of heuristics. I start this paper with considerations of heuristics in mathematical problem solving and move to considerations of problems in mathematics teaching, with an analogous use of heuristics to address teaching development.

HEURISTIC(S) OF/IN MATHEMATICAL PROBLEM SOLVING

I begin with a (familiar?) mathematical problem, expressed by Mason, Burton & Stacy (1982, p.76):

Consecutive Sums

Some numbers can be expressed as the sum of a string of consecutive positive numbers. Exactly which numbers have this property? For example, observe that

$$9 = 2+3+4 \qquad 11 = 5+6 \qquad 18 = 3+4+5+6$$

The authors write, “Try lots of examples; Try changing the question, extending its scope in some way; Be systematic in your specializing and try several different systems; Look for patterns” (p.76). We might see such injunctions as examples of ‘heuristic strategies’ for mathematical problem solving.

Polya (1945, p. 13) defined “heuristic” as follows: “The aim of heuristic is to study the methods and rules of discovery and invention. ... Heuristic as an adjective means “serving to discover””. Polya writes about *heuristic strategies* and *heuristic reasoning*, and Alan Schoenfeld (1985, p. 23) observes “Heuristic strategies are rules of thumb

for successful problem solving” and indeed “heuristics have become nearly synonymous with mathematical problem solving.”

Mason, Burton and Stacey (1982, p. 19) offer a framework or “rubric” for problem solving -- what we might call ‘a problem-solving heuristic’,

Specialise → Generalise → Conjecture → Convince,

in which it becomes clear rapidly that this is not a linear progression of strategies but a complex winding of mathematical processes, a set of cycles within cycles as the solver asks questions, tries out particular cases, notices patterns, checks them against further special cases, makes and tests conjectures, refutes or proves them and so on.

For example, working with the Consecutive Sums problem:

The question “what numbers can be written as consecutive sums (CSs) of 3 numbers” might lead to the following special cases:

$$9 = 2+3+4 \text{ (given) } \dots 1+2+3 = 6 \dots 3+4+5 = 12 \dots$$

This leads to a sequence of 6, 9, 12 ...: these are multiples of 3, so *can the ‘next’ number be 15* (we start to see some generality)?

$$4+5+6 = 15 \text{ indeed!}$$

Further, *if I start with a multiple of three, (such as 27, or 102) can it provide a CS of 3 numbers?*

$$\begin{array}{ll} \text{e.g. } 27? & \text{Well } 27=3 \times 9, \text{ so I can write } 8+9+10 = 27 \\ 102? & 102=3 \times 34 \text{ so I can write } 33+34+35 = 102 \end{array}$$

Aha – I now see a general pattern. Will every sum of three consecutive numbers produce a multiple of 3?

Let the three consecutive numbers be $(n-1)$, n , $(n+1)$ where n is a positive integer: then the CS of 3 numbers is $(n-1)+n+(n+1) = 3n$

Since this sum reduces to $3n$, I am convinced.

Can we do something similar for CSs of 4 or 5 numbers?

Figure 1: Example of exploring consecutive sums of three numbers

We can notice in the exploration exemplified above, that the explorer, or the inquirer, is constantly asking questions:

- *can the ‘next’ number be 15?*
- *if I start with a multiple of three, (such as 27, or 102), can it provide a CS of 3 numbers?*
- *Can we do something similar for CSs of 4 or 5 numbers?*

The heuristic process of constantly asking questions that lead to new mathematical insights seems to be basic to the problem solving process. Of course conjectures are

not usually made and proved so easily; for example Mason et al. make an early conjecture: “Powers of 2 are not the sum of consecutive numbers” (p.77). However, it takes 7 more pages, in their book, of conjectures and refutations, before this is proved to their satisfaction. They write “Articulating a vaguely sensed idea gives the mind something concrete to examine critically. Once articulated, it is important not to believe your conjecture (p. 74); and fancifully but aptly, “Once conjectures begin to flow, they tend to come like a cloud of butterflies, elusively, flying off when they are approached” (p.92). This heuristic cycle is captivating and satisfying; it can be challenging and infuriating; it results in the solver feeling in touch with the mathematics with a deep level of understanding, but an awareness also that there is much more not yet explained. Polya writes, “Heuristic reasoning is reasoning not regarded as final and strict, but as provisional and plausible only, whose purpose is to discover the solution of the present problem”. John Mason invites problem solving participants to engage in a ‘conjecturing atmosphere’, or a “Mathinking Atmosphere” (Mason et al, 1982, p. 176) which is questioning, challenging and reflective and builds mathematical confidence.

The essence of a conjecturing atmosphere is that problem solvers learn to live with uncertainty, and rather than being afraid or deterred by it, come to relish it and to believe they CAN do mathematics. This is akin to the activity of a research mathematician, as Paul Ernest has written:

The processes of problem posing and solving, of representing, conjecturing, testing and of mathematising in general, resemble the processes of knowledge generation applied in the context of research mathematics. Many mathematics educators argue that this is the culture that it would be most productive to emulate in school mathematics. (Ernest, 1990, p. 126)

When I worked at the Open University with John Mason and Nick James (in the 1980s) we produced a video compilation focusing on ‘low attainers’ doing mathematics. It showed low attaining students engaging in heuristic reasoning as exemplified above, and gaining confidence in doing mathematics. We titled it “Turning I can’t into I can and I did”. Yet we see in schools many students who do not believe that they CAN. I think particularly of the students in research by Elena Nardi and Susan Steward who experienced mathematics as T.I.R.E.D. (Tedium, Isolation, Rote Learning, Elitism, Depersonalisation). For these students there seemed to be no pleasure or satisfaction, no challenge or stimulation. Why? Can this be a reflection on the teaching they experienced?

While I could write much more about many years of working with students on mathematical problem solving, this has been only an introduction to the main focus of this paper which is about mathematics teaching, and some of the problems that have arisen in mathematics teaching over the years. I first indicate some of these problems, drawing on previous work in PME, and then use the idea of problem solving heuristics to address “How to Solve IT”. “IT” being one or more of the indicated problems.

So, first to return to Polya's title: "How to solve it", I ask, what is the "it"? In the case of consecutive sums above the 'it' changes with every new conjecture. Perhaps it is the conjecture about $3n$; perhaps it is the conjecture about powers of two. How to solve 'it', in either of these cases, requires some means of convincing oneself or others of the truth of the conjecture. Tim Rowland (2000, p. 108) suggests that

the "beauty of the deictic 'it' lies in its function as a conceptual variable. It (i.e. 'it') conveys the message "I have something in mind. I know what I mean, and I think that you know what I mean." It can be a linguistic pointer to a shared idea, to an understood but unnamed mathematical reference at the deep structure level.

Rowland explains 'deictic' as being concerned with ways in which language draws on and points to context (p. 78). In Polya's case the 'it' refers to many aspects of mathematics problem solving. In what follows I extend the meaning of 'it' and its associated context, and use ideas from mathematical problem solving to address issues in teaching.

PROBLEMS IN MATHEMATICS TEACHING: AS CAPTURED IN RESEARCH IN PME

Here I draw on some of the literature within PME to highlight ways of seeing mathematics teachers and teaching. In each case I point to some problem in the ways teachers and teaching are referenced or conceptualised, which in its turn suggests some 'problem' to be solved, some 'it'. The 'it' stands here for a *context* in which the way of seeing teachers/teaching is rooted. I return to the 'it' a little later.

Table 1 indicates the number and titles of papers, relating to teaching, at PME conferences over 30 years in PME. We see the increasing importance of research into teaching as reflected in PME papers. The development in the titles of papers, reflects a development in the ways in which teachers are characterised and referenced¹.

In her 1992 Plenary, based on papers at PME up to this time, Celia Hoyles suggested that up to 1987 the teacher was considered hardly at all other than as an adjunct of students' learning of mathematics, "a passive conveyer of facts and information". According to Hoyles, early papers on teachers and teaching reflected the teacher's focus as being on curriculum delivery and differentiation of students according to ability. Many papers focused on the beliefs of teachers: "there appears to be an implicit view that teachers have 'beliefs' in some decontextualised sense which need to be accessed and changed" (p. 3-266). The focus on beliefs led to suggestions of "true beliefs which may not be enacted in practice" (p. 3-265). Hoyles had perceived a strong focus on the influence of teacher beliefs on classroom activity, and perceptions

¹ In the following sections I have drawn extensively on an article in the book emerging from the Topic Study Group in ICME 11: In-service Education, Professional Life and Development of Mathematics Teachers, edited by Nadine Bednarz, Dario Fiorentini and Rongjin Huang, in which I trace the development of research into mathematics teaching and the professional development of teachers of mathematics over 2 decades. (Jaworski, 2011)

Table 1: Thirty years of PME conferences 1986–2016

PME #	Year	Papers, their content, and other key events
PME 10	1986	1 paper related to teaching
PME 11	1987	10 papers in themes headed as “instruction” and “teacher training”
PME 14	1990	No thematic categories designated as teaching or teacher education. Papers on teaching, or learning to teach, spread throughout other categories. Three working groups in areas of mathematics teacher education
PME 15-27	12 years 1991- 2002	Development of mathematics teaching and teacher education as major research themes. Two plenaries surveying the subfield: Hoyles, 1992; Ponte, 1994. Three working groups, resulting in books: Zack, Mousley & Breen, 1997; Jaworski, Wood & Dawson, 1999; Ellerton, 1999.
PME 28	2004	50 papers in themes headed “teacher classroom practice”, “teacher education and professional development”, and “teacher knowledge”.
PME 30	2006	27 papers in themes on “in-service teacher development”, “pre-service teacher development (elementary)”, “teacher content knowledge”, “pedagogical knowledge”, and “teacher thinking”. Teaching subsumed in other categories, such as Classroom Culture; Social Activity Theory.
	2006	Handbook: A review of 30 years of PME; 2 chapters on ‘Mathematics Teachers and Teacher Educators as Learners’; and ‘Mathematics Teachers’ Knowledge and Practices’.
PME 31-40	10 years 2007- 2016	530 Papers on keywords such as teacher knowledge, teacher beliefs, teacher education, educator education, professional development and professional growth (Lin & Rowland, 2016)
	2016	Handbook: Another decade of PME research. One chapter on ‘Preservice and Inservice Mathematics Teachers’ Knowledge and Professional Development’

of a gap between espoused and enacted beliefs. She reported on research that saw inconsistencies between teachers’ beliefs and their practice as obstacles to the success of in-service development and therefore as something to be changed. This suggested a need to investigate the interaction of teacher beliefs and curriculum innovation. She marks a shift in the late eighties towards a focus on social interaction and construction in which teachers’ classroom activity was seen as heavily influenced by classroom and school cultures and those of wider society.

Two years later (1994), again in a PME Plenary, Joao Pedro da Ponte examined the place of the teacher in mathematics education research, particularly mentioning negative perspectives – seeing the teacher as an instrument, as a deficient professional, “a person with deep misconceptions, lack of mathematical knowledge, and inappropriate and inconsistent beliefs, contradictory with current reform efforts” (p. 198). He recognised research which focuses on teachers’ knowledge of mathematical concepts and how to teach them, the assumption being that teachers who do not know their subject cannot do a good job in teaching it. He pointed also to more didactically oriented studies focusing on subject matter and pedagogical knowledge, but lacking a clear view of how such knowledge develops and works in practice. Hoyles (1992) had pointed towards teacher research as an emergent focus; here Ponte acknowledges research into reflective practice and teachers as researchers, where teachers are seen as playing an important role in defining the purposes and goals of their work as well as how to attain them. A working group in PME also focused on Mathematics Teacher as Researcher and published an associated book in 1997 (Zack, Mousley & Breen, 1997).

By the time of PME 30, and the associated review of research in PME up to 2006, there has been considerable development but with some disappointing conclusions, as reflected in the review papers from the PME 30 compilation (Gutierrez & Boero, 2006). The book has two relevant chapters: Ponte and Chapman focus on teachers’ knowledge and Llinares and Krainer on teachers (and teacher educators) as learners. From the wide range of papers reviewed, Joao Ponte and Olive Chapman suggest that “the most common conclusion is that teachers need further learning to carry out “better” practices, *more aligned with the researchers’ espoused perspectives*” (p.485, my emphasis). They offer one disappointing conclusion, that still, “the emergent image of the teacher is that of a professional with deficient knowledge”. They recognise also a research emphasis on the complexity of teachers’ knowledge and its “intimate relation” to practice (p. 486). A challenge for research is to find methodologies to explore this relationship without defining the teacher as deficient according to theoretical perspectives espoused by the researcher. Salvador Llinares and Konrad Krainer focus on studies relating to teachers’ learning, mainly “as a consequence of having participated in some kind of programme or course”. Such programmes have “content” goals (e.g., Simon, 2008,) which arise from the ways in which teacher educators perceive the learning needs of teachers. Indeed research into teaching development is largely conducted by the teacher educators who design such programmes. We can see the “content” to include both mathematics per se and ways in which the learning of mathematics can be approached, for example in problem-solving environments; also examples of students’ classroom activity as a basis for learning through analysis of students’ ways of interacting with mathematics. Focuses on reflection and on collaboration and community building can be seen in the ways content is structured and activity designed for teachers’ learning in these programmes. Llinares and Krainer also draw attention to the role of teacher educators in these programmes, to their knowledge and practice in working for teacher development, and indeed to their own professional development.

In the period between 2006 and 2016 (year of writing this paper), papers on teaching, teacher education and teaching development continued to be numerous at PME conferences. In the new PME Handbook in 2016 (Guteirez, Boero and Leder, 2016), Fou Lai Lin and Tim Rowland (2016), in a chapter entitled ‘Preservice and Inservice Mathematics Teachers’ Knowledge and Professional Development’ reported on 530 papers (Research Reports, Plenary presentations, Plenary Panels and Research Fora) from a search on keywords such as teacher knowledge, teacher beliefs, teacher education, educator education, professional development and professional growth (p. 484). Their Final Reflections point to fundamental ‘paradigmatic differences’ between cognitive and social perspectives on both teacher knowledge and teacher professional development.

The above sections are necessarily brief: more detail can be found in Jaworski, 2011. From the above, I extract five cases of problem areas regarding teachers, teaching and teaching development over the years. Although the cases are chronological and so in a way chart the development of perspectives on teaching which emphasise the associated problems, there is considerable overlap in the characteristics painted. For example, we see teachers as researchers emerging alongside studies which still paint teachers as deficient professionals.

1. Teacher hardly considered other than as an adjunct of students’ learning of mathematics, “a passive conveyer of facts and information”. Purely a facilitator—to dispense facts and information (Hoyles, 1992, p. 3: 263).
2. Seeing the teacher as an instrument, as a deficient professional, “a person with deep misconceptions, lack of mathematical knowledge, and inappropriate and inconsistent beliefs, contradictory with current reform efforts” (Ponte, 1994, p. 198).
3. Teacher as a “professional with deficient knowledge”. A research emphasis on the complexity of teachers’ knowledge and its “intimate relation” to practice (Ponte & Chapman, 2006, p. 486).
4. Shift to teacher education: teachers’ learning, mainly “as a consequence of having participated in some kind of programme or course”. Focuses on reflection and on collaboration and community building for teachers’ learning. The centrality of the teacher educator in such programmes and questions about the ways in which teacher educators themselves learn. (Llinares & Krainer, 2006)
5. Shift to Teacher as Researcher: teacher participation in inquiry practices or research programmes, often in partnership with educators/didacticians. (e.g., PME working group and resulting book – Zack Mousley and Breen, 1997; PME Handbook – Lin and Rowland, 2016).

In [1], the focus (the IT) is on early (for PME) research into mathematics learning. The teacher is hardly considered at all. Teaching is considered as passive conveyance of facts and information. In [2] and [3], now focusing on research into teaching, the teacher is seen as someone who is knowledge deficient; perhaps as someone whose

deficiency leads to the inadequate learning of mathematics by students. The context here (the IT) is of the knowledge needed for teaching and its relation to teaching practice, with the suggestion that teachers lack this knowledge so that their practice is deficient. The shift indicated in [4] indicates a focus now on teacher education, particularly through programmes or courses for teachers through which their teaching knowledge can grow and their practice benefit from reflection, collaboration and community building (the IT). Here we see the growing importance of professional development for teachers to enhance their knowledge and practice, and perhaps avoid labels of deficiency. Those leading these programmes are usually teacher educators, and it becomes important to consider teacher educators' knowledge and practice, so we see a further shift to considering teacher educator development (a further IT). Finally, in [5], we see teachers themselves engaging in research, perhaps seeking to promote their own development through exploring aspects of their own practice and their students' learning, perhaps in collaboration or partnership with teacher educators.

The development that we see above has been charted through research presented at PME and by PME working groups. It seems reasonable to suggest that, as knowledge grows through research, awareness and practice of educational practitioners is concomitantly enhanced. Thus, as we start to see research papers at PME addressing teacher, teaching and teacher education, we see very soon an explosion of such papers. Research by its very nature uses systematic inquiry to address issues in teaching practice. Such inquiry, leading to new knowledge can be seen to enter into practice. Thus, rather than teachers being seen as deficient professionals, failing to fulfil the theoretical perspectives espoused by the researchers, teachers are drawn into research inquiry, becoming partners in exploring the nature of teaching, ways of engaging students in/with mathematics, and of fostering students' understanding of mathematics concepts. Here research becomes a tool for teaching development (Jaworski, 2003).

It is with these ideas of inquiry as a means of focusing teaching on "the problems of practice" (e.g., Lampert, 2001) that I return to ideas of problem solving and the 'it' that needs solution. One approach to the problems of practice, seen in terms of the deficiency of teachers, was for teacher educators to present programmes and courses for teachers, suggesting that the teacher needed to be co-dependent on the teacher educator (e.g., Dawson, 1999). "Mathematics teachers need someone to fix them, and mathematics teacher educators need someone to fix" (Dawson, 1999, p. 148). Here perhaps the 'it' is the question of how the educator gets the teacher to act in ways that satisfy the educator's theoretical perspectives. And as long as this 'it' is not addressed, the language is of 'deficient' practitioners. If we change the approach to an inquiry perspective in which teachers and educators engage together in asking questions about effective practices in mathematics teaching and exploring these through systematic inquiry (research), the perspective becomes different. Responsibilities change. Educators start to respect teachers' knowledge. Despite considerable knowledge of both teachers and educators, all recognise there is something unknown, something to be explored. Here the theories of educators become conjectures to be tested and

researched together with the teachers. Thus I move from a perspective on getting teachers to perform in the ways educators' theories suggest, towards a joint focus of teachers and educators exploring together the questions of practice (the new IT).

PROBLEM SOLVING HEURISTICS AND INQUIRY IN MATHEMATICS, LEARNING AND TEACHING

Teachers' engagement of their students in problem solving and inquiry-based tasks has permeated research into teaching development. More recently, the use of inquiry processes has expanded beyond the inquiry-based classroom mathematical tasks to inquiry into the processes of mathematics teaching itself and inquiry in developmental research into teaching (e.g., Jaworski, 2006). In this section I elaborate one approach to teaching development that parallels the heuristics that Polya pioneered.

A thought experiment

Let us imagine a teacher, with a group of 14 year old students of varying attainment levels, using the Consecutive Sums problem outlined above. Students sit together in mixed attainment-level groups and work together on the problem as the teacher circulates. The students engage in dialogue on the problem and the teacher enters into their dialogue, making comments, asking questions, encouraging their engagement and mathematical progress. She can see that different groups take up different questions and work in different directions. She encourages trying special cases and seeking patterns with which they are familiar from previous experience. When a pattern is spotted she encourages a clear articulation of a conjecture and asks what they can do to decide if the conjecture is true. Some groups go further than others in the sophistication of their conjecturing and some use symbolisation to express conjectures.

As well as the teacher, there are two other adults in the room – researchers from a nearby university. One is recording the lesson on video with a hand-held machine, capturing the words, facial expressions and writing of students in the various groups. The other has a camera on a tripod, capturing the classroom activity in a more holistic way. Periodically the teacher stops to talk with one or other of these researchers to ask the researcher to shift the camera, to comment on what she is noticing from the students, or to express uncertainty about her response to some of her students' activity.

After the lesson, the teacher, some of her colleagues and the two researchers sit together and view some extracts from the video. They have done this before and are comfortable with each other's commenting on what is recorded. The teacher talks about her intentions for the lesson, about her students' work on the problem, about ways in which she has been surprised or frustrated by their discussion and reasoning. Her colleagues chip in and ask questions. The researchers also ask questions. They all discuss issues related to learning and teaching. The researchers record this discussion.

For example, one group of students has been intent on finding strings of consecutive numbers and their sums. They have seen quickly that pairs of numbers result in an odd

number; triples prove more challenging, but someone spots that they are multiples of three. They go onto sums of 4, 5 These are harder to pin down. The students make no attempt to symbolise. The teacher wonders whether to suggest symbolisation. What might she say, ask? Should she suggest forms of notation they might use?

Another group has started to list the numbers they can obtain from consecutive sums. They very quickly suggest all the odd numbers and justify this using n and $(n+1)$ and their sum $2n+1$. The teacher helps them to see that $(2n+1)$ represents an odd number. They get multiples of 3 for the triples, which fills in some of the even numbers. They experiment by asking can we get 4, can we get 10, can we get 14? They have all numbers up to 20 except 1, 2, 4, 8, 16. What is special about these numbers? Eventually they settle on powers of two. Why can they not find consecutive sums for these numbers? A process of elimination, trying all combinations of sums up to 20 satisfies them that there is no sum for these numbers. The teacher asks about 2^5 , 32? They are getting tired and can't be bothered checking all sums that might give 32.

Watching the video, the teachers discussed this position. What might they do to encourage students to try a strategy other than elimination? They have read the strategy offered by Mason et al (1982) – of course they could show and explain this strategy. Two of them say that this is what they would do. They argue that this would demonstrate to the students an algebraic way of seeing the solution and would be a valuable part of their experience for use in future problems. They nevertheless struggle with the question of how they might encourage or challenge the students to reach such a solution themselves – perhaps in another lesson when less tired. One of the researchers offers a possibility.

Another issue which they raise is the construction of student groups. The teacher says she always encouraged students to sit in their friendship groups. Most of the groups had worked well together, keeping focused on the problem. A couple of groups had been sidetracked by getting bored and stuck, or by engaging in silly behaviour. One colleague suggests that friendship groups might not always be the best arrangement – perhaps students should learn to work with others, and perhaps be helped or challenged in valuably different ways.

All teachers are concerned about the curriculum and the coming standardised tests for their students. What was this kind of work contributing to their students' ability to do well in these tests? What areas of the curriculum were being addressed through the Consecutive Sums problem? They all agree that such a problem gives valuable experience in the use of symbolisation, contributing to students' experience with algebra. They see a clear contrast between the need for symbolisation in this problem, and the standardised approaches to algebraic expression and the rules of algebra. They decide to find another problem that would afford similar experience with algebra and decide to plan a lesson based on encouraging students to express patterns symbolically. They agree to look at questions involving algebraic expression on the standard tests and think about how to use open-ended problems to enable students to gain the

necessary algebraic skills and understanding. The researchers agree to participate in this planning activity and to record the resulting lesson(s).

After this discussion, the two researchers drive together back to the university. Their dialogue while driving focuses on the lesson and subsequent discussion in which they had participated. They discuss the issue of the powers of two, how they each might have handled that with the students, and the pros and cons of various approaches. They recognise that there was no 'best' way. They say they each learned from the ways the teachers had discussed the issue, gaining insights into the teachers' perspectives and wonder whether they could/should have presented their own (research-informed) perspective to the teachers. They expect to analyse the video recordings from the classroom as part of a wider project looking into the use of inquiry-based tasks in classroom mathematics.

Two weeks later one of the researchers calls one of the teachers to ask how their planning is going for the lesson on symbolising, and whether the researcher can be of help. The teacher is very apologetic: so far nothing has happened; pressures of day to day school life have taken all their available time. However, perhaps the researcher might come next Tuesday and this would spur the teachers to set aside some time for the planning.

Use of heuristics in the example above

The example I have offered above is hypothetical, but typical of many classrooms I, and other colleagues, have observed when working with teachers on inquiry approaches to mathematics, learning and teaching (e.g., Potari & Jaworski, 2002). In the use of the CS problem in the example we see teacher and students engaging in Mason et al.'s rubric of 'specialising, generalising conjecturing and convincing'. Students drew diagrams, composed tables of sums, sought patterns, and reformulated when a conjecture proved incorrect. The lesson raised questions and issues for the teacher concerning the ways she interacted with, helped and challenged her students. Viewing the video gave the teachers a basis for discussing issues in teaching related to mathematics, to their students, and to the demands of curriculum and standard tests. The experience of engaging in the lesson as observers, and of joining the teachers in their viewing of video and their discussion provided researchers with insights into teachers' perspectives and what motivated their teaching activity.

I see here three layers of inquiry (involving use of heuristics): at the centre is the classroom inquiry, the problem solving, the use of heuristics; a second layer is inquiry into the teaching process, addressing issues which arise for the teachers and their students; a third level is research inquiry, inquiring into the two inner layers to offer more general insights to teaching development and the roles of teachers and researchers in promoting development.

We can see heuristics at play in these two further layers. The use of video is one heuristic. It affords the teachers a common experience involving students and mathematics; the problem-solving environment, raising issues for students'

mathematical development; broader issues such as ways of organising the classroom, dealing with the demands of curriculum and tests. The teachers are able to enter into *special cases* of teaching and learning, for example the issue of the powers of two and how to enable students to use symbolisation. They are able to *generalise* from this particular case to think about planning a lesson explicitly to promote students' facility with algebraic formulation. The researchers are also afforded insights through their participation: they see how the teacher deals with the activity of the classroom, gaining insight into the teacher's thinking and the issues behind her actions. These serve as special cases, from which the researchers generalise to teaching issues more broadly and the constraints that teachers have to deal with in their regular practice.

THE PROBLEMS OF TEACHING DEVELOPMENT – WHAT IS THE “IT”?

In 'How to Solve It', the heuristics in Polya's 4 steps deal with not just one problem which needs solution, but the entire process of problem-solving in mathematics, capturing the generalities in such problem solving. From Mason et al.'s rubric of

Specialise → Generalise → Conjecture → Convince

we move from generalising in one problem such as Consecutive Sums, to generalising in problem-solving more widely. Of course every problem is different – how to solve the one is different from how to solve another – however, the heuristics operate at a higher level of generality and can be applied appropriately in a range of problems. The problem solver can generalise from the experience of individual problems to becoming experienced through the application of heuristics. In How to Solve IT, the IT can be any individual problem, solved through experience with the use of heuristics.

When it comes to the problems of teaching, a similar frame of reference can be used. Analogous with the individual problem is the individual lesson. The teacher plans the lesson with objectives for her students and her own strategies for achieving these objectives. They involve design of mathematical tasks, ways of working on the mathematics and encouraging her pupils to do so; pedagogic approaches such as use of questioning, commenting, challenging; decision-making such as when to make an input, to explain some mathematics, to push student to go further than they are inclined to do. She can see (perhaps through watching the video) how things have gone in this lesson to help her plan for other lessons, as she generalises from experience. Of course the ways the teacher works with this lesson has similarities with the ways she works in other lessons. Her own reflections may allow some modification of her approach, some development of her teaching, but perhaps this is constrained by the pressures on the teacher from many directions.

The video conversation with her colleagues and researchers opens up alternative possibilities. Reflection can become more challenging as other perspectives are aired. However, it also opens up other possibilities that might not have been considered, providing new insight and energy for teaching objectives and their fulfilment. If the teachers are able to work together, a new dimension is added to the teaching – approaches become more explicit and the associated issues more cogent as they are articulated and shared. The researchers have an important role – their presence

encourages the giving of time to articulation, reflection and planning, and their alternative insights can inspire and help sustain activity.

THE ROLE OF HEURISTICS IN THE PROBLEMS OF TEACHING

In problem solving, both in mathematics and in teaching, the explorer, or the inquirer, is constantly asking questions. In these questions we see alternative meanings of “IT”. Some of these meanings relate to what the student does to solve a problem, and is the basis for teaching (as the teacher has to consider the heuristics of the problem solving of the task given to the students). Alternative meanings address the solving of problems in teaching (as represented in the example). At a more general level of development, tackling the IT comes closer to the goals of an action research approach to teaching development. Here, perhaps the generality that the teacher seeks comes closer to the one that the researcher wants to achieve.

I have quoted above, “The aim of heuristic is to study the methods and rules of discovery and invention. ... Heuristic as an adjective means “serving to discover””. Polya writes about *heuristic strategies* and *heuristic reasoning*, and Alan Schoenfeld (1985, p. 23) observes “Heuristic strategies are rules of thumb for successful problem solving”. Adapting what I wrote above to the problems of teaching, we might say, ‘This heuristic cycle is captivating and satisfying; it can be challenging and infuriating; it results in the solver feeling in touch with the *teaching* with a deep level of understanding, but an awareness also that there is much more not yet explained. As Polya writes, “Heuristic reasoning is reasoning not regarded as final and strict, but as provisional and plausible only, whose purpose is to discover the solution of the present problem”. This can be as true in addressing the problems of teaching as in addressing the problems of mathematics.

The essence of a conjecturing atmosphere, for developing teaching, is that the inquiring teacher learns to live with uncertainty, and rather than being afraid or deterred by it, comes to relish it and to believe they CAN aspire to teach in ways more effective for their students’ mathematical learning.

In terms of Mason et al’s rubric to teaching development, we can see specialising as enacting a few lessons, or just one, with a particular focus or question about teaching in mind; reflection on these special cases provokes generalising (what did or didn’t ‘work’) and articulation of conjectures (what will or wouldn’t work as a rule). Convincing would typically be by systematic, empirical trials of the practices articulated in the conjectures about teaching. Such an approach is of course firmly embedded in social science research methods, and puts ‘learning from experience’ on a more secure, scientific foundation.

In Summary

- Starting from heuristics of mathematical problem solving as introduced by Polya and developed by other researchers, I have asked: *How to Solve it* – what is the IT?

- The IT stands for one problem, but it also presages a generality – as a result of solving some problems, the heuristics become familiar and can be applied to others
- I shifted from problems in mathematics to problems in the practices of teaching, recognising, down the years (in PME), that teaching has been seen variously as problematic (some problems elaborated with respect to the literature in PME).
- I asked, what are the heuristics in solving the problems of teaching?
- Through the hypothetical case, and specialising within this case, I highlighted issues (problems – the IT) that arise for a teacher in the classroom, including mathematical problems for her students and the problems for her in addressing teaching issues (such as when and how to introduce symbolisation).
- Recognising that the solutions to these problems are complex – the researchers do not have quick and ready answers – I suggest that teachers and researchers are drawn into a problem-solving cycle, an inquiry cycle, in which they ask questions and as they address the problems together.
- The activity and actions that result from teacher/researcher collaboration (itself a heuristic) lead to deeper levels of knowing within their joint community and potentially to developments in teaching.

CREATING INQUIRY COMMUNITIES IN TEACHING DEVELOPMENT

In my earlier work with Simon Goodchild and colleagues in Norway, didacticians (mathematics educators) in the university worked with teachers in a range of schools from lower primary to upper secondary, over 3 years to develop mathematics teaching in schools. Didacticians aimed to create inquiry communities with mathematics teachers in which knowledge of both didacticians and teachers could be enhanced through their collaboration, resulting in teaching development and thus enhancing students' learning of mathematics in classrooms. The voice of teachers became essentially important in directing activity in the project, and tensions arose between the two groups resulting in changes to the activity of both groups. Theoretically, didacticians sought to engage in *critical alignment*, looking critically at the practices in which they engaged while engaging in these practices, and facilitating development and change. Both groups looked critically at their own practices and shared their reflections with the wider community (Many papers speak to these matters – see for example Goodchild, Fuglestad and Jaworski, 2013).

I see in this project a real (not hypothetical) example of the use of heuristics that I have been describing above. My reason for referring to this project is to point out that the processes of inquiry in three levels are not presented as a panacea for removing the problems of teaching. I believe that addressing the tensions in a community of inquiry was one of the most valuable heuristics for increasing knowledge and developing practice – not the most comfortable by any means, but who suggests that development, either in mathematics or in teaching, is comfortable?

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IS PROBLEM SOLVING TEACHABLE?

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The development of the discipline of mathematics through its long history has been characterized by solving problems. Currently, many contemporary curricula place problem solving as a central feature of school mathematics learning, and indicate that the capacity to put mathematics to use in routine and novel situations is a vital outcome of schooling. However, such documents rarely indicate how to teach problem solving, but seem to assume that it is possible to do so. This raises the important question: Is problem solving teachable? In this preface for the plenary panel I recount a personal experience with problem solving in order to provoke some questions about the teaching and learning of problem solving.

BACKGROUND TO THE PLENARY PANEL

Following the success of last year's PME plenary panel debate in Hobart, this year's plenary panel will also take the form of a debate, to address a provocative and perhaps contentious issue in mathematics education. To this end, the members of the panel have been invited to consider the statement that "It is impossible to teach mathematical problem solving." Because the statement is phrased in a way that asserts the impossibility of teaching problem solving, this makes it confusing to talk about our "affirmative" and "negative" teams, the traditional antagonists in a debate. So, to clarify: the affirmative team — whose paper is presented first in the proceedings and who have the first turn at presenting their case in the debate at the conference — are supporting the claim that it is impossible to teach mathematical problem solving; whereas the negative team — who must wait their turn — are arguing against this. As is traditional in debates, it is quite conceivable that the members of a team may not actually believe the proposition for which they are arguing, but argue they must, with whatever reasoning and evidence they have at their disposal.

A SCENARIO WITH SOME PROVOCATIONS

In order to set the scene for the debate I would like to recount a personal experience. For research purposes this makes my account a case study, noting that self-study is a legitimate form of practitioner research (Borko, Liston, & Whitcomb, 2007). I will use this experience to raise some provocative questions that seem pertinent to the debate.

At a recent conference, the following problem came to my attention:

A customer buys four items from a 7-eleven store. When she reaches the checkout the clerk tells her that the cost is \$7.11, but she notices that he multiplied the numbers instead of adding. She gets him to add them up properly. They are both surprised when the answer turns out to be \$7.11 again. What are the prices for the items?

I had not seen this problem before, although it is over 15 years old (my earliest reference is Robertson (1998), although I found suggestions that it may have originated with Doug Brumbaugh). This problem appealed to me when I heard it; it seemed surprising that the product and sum of four numbers could yield the same result, and I wondered how it could be so. Two of my colleagues were at the lunch table with me when the problem was shared, and all of us were interested in what was happening. In the great tradition of problem solving, we set to work making notes on the napkin.

Provocation 1: Why were we interested in such a problem? Was being interested something we had “learned”?

I did not ask my colleagues why they were interested in the problem, but their interest and engagement were evident in the next several minutes as we discussed ideas. However, I have posed this same problem to other people, and they are not so interested, with most not attempting the problem. What causes this lack of interest? Maybe I did not make it sound very interesting. Maybe these other people could not tell that the situation described in the problem was unusual and so, for them, it had no surprise or curiosity factor, since perhaps not many people know it is rare to have the sum and product take the same value. My knowledge of this rarity, due to my extensive experience of mathematics, likely increased my interest in the problem’s solution. Another reason for disinterest may be that the problem was not appealing to them. I know that there are certain problems that simply do not capture my own interest. I doubt, however, that I can articulate (a) exactly which kinds of problems appeal to me, and (b) what makes me decide — often quite quickly — that I am interested enough to engage when I encounter a problem. So, “interest” seems to be a key component of successful problem solving, but one which may be influenced by complex factors.

I have suggested a possible reason for my interest above, but it seems worth further exploring the source of the reasons for my interest. Was my interest something I had “learned”, and, if so, how had I learned it? The English language distinguishes between “learn” and “teach”. “Learn” refers to changes in the learner; “teach” refers to the work of an other, whose actions are intended to cause learning. This distinction provides the humour in the well-known cartoon of a boy claiming to have taught his dog to talk, but then conceding that the dog had not actually learned to do so. I think I can say that there were experiences that I had had that increased my interest in the problem, so, in some sense at least, I had *learned* to be interested in problems like this. But was I actually “taught” to be interested? Who did the teaching? More importantly, how was it done?

As my colleagues and I engaged with the problem I noted that we varied in the kinds of contributions we could make to our search for a solution. Our early discussions led us to a particular area of mathematics that seemed to be useful. This area of mathematics was not particularly advanced, and was familiar to all three of us, but two of us probably had greater fluency in the area than the third. This seemed to have an impact on the later progress and contributions of this third person. Indeed, as the

solution became a little more complicated — although still involving accessible content — the third person had more difficulty keeping up with the ideas and arguments.

Provocation 2: Does problem solving ability depend solely on what mathematics you already know and how fluent you are with it? Is it sufficient (or, at least, more important) to teach mathematics techniques, rather than problem solving per se?

We laboured on the problem over lunch. When the break ended we had a few ideas that seemed relevant but no feeling of certainty that any of our approaches would yield success. We had tried a few paths that seemed to be dead ends, and had clarified some other ideas, but although I felt we had a possible solution strategy to test the second person expressed some doubts and indicated that he would be trying something else.

Provocation 3: What are the strategies of good problem solvers? Can these be taught?

When I was a Grade 7 or 8 student at high school — some 40 years ago — I vividly remember spending a fortnight's worth of mathematics lessons on "problem solving." The rest of our work in mathematics that year — at least, as far as I recall — seemed to revert to archetypal conventional maths lessons, teaching us standard techniques and giving us opportunities to practice on routine exercises. I acknowledge that not all of my high school years were so traditional in skill and drill, but this special two-week problem solving program was a memorable point of difference from much of my school maths learning experience. In that two-week period we tackled a wide range of problems of the kind that often end up in collections of recreational mathematics challenges, and were introduced to Polya's collection of heuristics for problem solving (Polya, 1957). These were presented as a list of seemingly random strategies: make a table, try a simpler case, draw a diagram, consider cases, try to recall a similar problem, and so on. The problems we were given were often aligned with these strategies, and so we had opportunities to put them to use.

During lunch I had considered some of these strategies for the 7.11 problem: I had expressed the relationships algebraically, I had dismissed drawing a diagram as a possible approach (why?), and a simpler case did not seem helpful (and what should I simplify: the numbers? the number of numbers? I didn't know). I was not expecting that I could routinely apply Polya's list and come up with a solution; nor did I expect that knowing how to solve one problem is necessarily specifically helpful for solving another unrelated problem. Nevertheless, I believed that these were useful strategies. What is interesting to me, on reflection so many years after first meeting them, is that this list of strategies still comes to mind quite readily, obviously aided by some repeat experiences in the intervening time but still firmly anchored in that foundational event. Moreover, that two-week problem-solving unit was a profoundly positive experience in my mathematical development, and so it seems that *something* about problem solving can be taught ... and yet, there I was at the end of lunchtime without a solution to the 7.11 task. Furthermore, I could not help wondering what had happened to all my classmates who did that same problem solving unit back in high school: were they now

in their 50s, distracted by an interesting maths problem, happily applying the strategies they had been taught so long ago?

Provocation 4: Can these strategies guarantee the solution of any given problem?

As I worked on the problem — frustrated that, although it seemed to be relying only on elementary mathematics, I was not making headway with my strategies — I did have some moments of insight. Ideas would pop into my head, seemingly out of nowhere, accompanied by a sense that the ideas would be useful. The occurrence of these moments of insight is intriguing from a psychological point of view. Where do they come from? What is necessary for them to occur? Can I make them happen? Is it always possible to solve a problem without them, perhaps just by working through Polya's list and trying out the strategies until a solution "comes out" in some sort of programmatic way, or do some problems actually *require* the "Aha!" moment? If insightful "Aha!" moments are required for some problems, then what, as teachers, can we do to help students have them? Can this really be taught?

Provocation 5: Can students be taught how to stimulate the "Aha!" moment?

Another thought crossed my mind during the lunchtime exploration of the 7.11 problem with my colleagues. I seemed to be having what I felt were useful ideas that I wanted to share. I recalled, however, the experience of another colleague who reported that one of her problem-solving experiences had been ruined by the hints offered by a well-meaning friend. There have been many times when I, as teacher or fellow problem-solver, have known what I could say that might help another person solve a problem. Does this mean I should say it? Does the provision of hints, guidance, and scaffolding undermine the learning of problem solving? It is clear that scaffolding by the teacher or knowledgeable other *alters* the problem solving experience for the learner/problem solver, but is this beneficial and when and how should it be used? (You will have noticed that I have tried to tell you very little about my actual solution to the 7.11 problem, because I do not want to scaffold your own problem-solving experience for this challenge unless you want me to.)

Provocation 6: What is the role of scaffolding? Can students be taught to solve problems without needing scaffolding? How do we teach them to deal with the "zone of confusion"?

[The "zone of confusion" is a useful term coined by Sullivan et al. (2015), to describe the fact that there may be a period of uncertainty in the solving process.] I confess that the conference sessions following lunch did not receive my full attention as I sought the solution. Some dead ends were pursued, some cases were considered, and slowly progress was made. I stopped work to enjoy the conference dinner, but the next day I continued my problem solving efforts, testing possible solutions in my notebook, slightly worried that — in my tired state — I would make an arithmetical error and miss the solution. At last, however, I found it; a success celebrated with weary elation and a quick email to my colleagues. I was also a little disappointed, because it had been

a lengthy solution to achieve, and I had had no moment of insight that might reduce the tedium of the final stages of my approach.

Provocation 7: What is the role of perseverance? Can this be taught? Is teaching perseverance the same as — or only a vital component of — teaching problem solving?

As I contemplated my success, I thought about this plenary panel. What things had been taught to me, over the years, that enabled me to solve the 7.11 problem? Yes, I needed some mathematical knowledge, and I had certainly been taught that. But perhaps my success — involving my strategies and perseverance — was merely due to the fact that I had tackled many problems over the years, and had learned useful skills in the process but had not actually and purposefully been *taught* anything.

Provocation 8: Is problem solving success solely dependent on having had lots of experience solving problems? Is this experience “teaching” problem solving?

One final provocation remains, relating to my first. Yes, I was interested in the specific problem posed, but my engagement with the task arose from more than just interest. Many of my friends think I am a little crazy, because as I engage with the world I *notice* scenarios that are explained and solved by mathematics. I actively look for mathematical situations/problems and wonder about them. I may not *solve* them all, but I do try to think about the nature of the solutions. From whence did I gain the disposition to embrace and actively seek out problems to solve. Can this be taught?

Provocation 9: Can we teach students the disposition to be problem solvers?

CONCLUSION AND INTRODUCTION

The provocations here are based on reflections which are anecdotal, personal, and subjective. Perhaps the account of my experiences resonates with you to some extent, but whatever my experiences may be, they almost certainly differ from yours and those of others, and the outcomes are likely different too. Where, after all, are the problem-solving classmates of my youth? Why, after all, do I still prefer some types of problems over others? However, these reflections provoke in me the very important question that is central to the work of the plenary panel: What, after all, can I do as a teacher to help my students be better problem-solvers?

It is, therefore, appropriate now to introduce Szilárd András and Markku Hannula who will argue the case that it is impossible to teach mathematical problem solving. Based on my experiences, I already think I agree with them. Their case will be followed by the arguments of Berinderjeet Kaur and Miriam Amit who assert the opposite. Curiously, I cannot help but suspect that I will agree with them too.

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IT IS IMPOSSIBLE TO TEACH PROBLEM SOLVING

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In this short note we consider three major challenges to the idea that it is possible to teach problem solving: do the students learn to be better solvers of novel problems, or do they just learn standard solutions for new sets of routine tasks; developing problem solver's affective disposition is not teaching of "problem solving"; and simply solving tasks is not "teaching". Moreover for each challenge we outlined several particular issues that need to be clarified.

RATIONALE

We mathematics educators have identified problem solving as a central part of mathematics curricula. As our jobs require us to plan new curricula and to teach future mathematics teachers, we have a strong wish to find evidence that our task is possible. The affirmative team acknowledges that there is plenty of anecdotal evidence for students learning to become better problem solvers. However, if we want to find robust evidence for certain *teacher actions* promoting better problem solving skills among students, the situation is less clear. In a recent review of the field, Lester and Cai summarize that

although there is a great deal of consensus within the mathematics education community, that the development of students' problem solving abilities should be a primary goal of classroom instruction ..., there is no consensus about what we teachers should do in classrooms to reach this goal (2016, p. 118).

Furthermore (p. 120) research evidence shows that “teaching students to use general problem-solving strategies and heuristics has little effect on students being better problem solvers”. We see it as the task for the other team to show evidence to show that teaching problem solving is possible. In our contribution, we will pose three challenges and we think that the evidence from the second team needs to convince a critical audience. We present three areas of contention for the idea that problem solving is teachable.

The first challenge: Do the students learn to be better solvers of novel problems, or do they just learn standard solutions for new sets of routine tasks

Beyond dispute we would like to emphasize that “problem solving” refers to solving novel, non-routine problems that are unknown to the solver. In order to clarify these aspects we also want to emphasize that the term “unknown to the solver” can refer to a problem for which the conceptual framework needed to solve the problem is known by the solver, but it also refers to problem situations where the conceptual framework is also completely unknown. Teaching mathematics on different levels sometimes reduces to constructing strong conceptual frameworks, sketching universal strategies

with a lot of examples and counterexamples. It is known that mathematics learning is highly contextual and there is little transfer. When there are claims to be teaching "problem solving" the outcome is often limited to students learning new solution methods to certain types of problems. That is, some of the tasks that were previously problems to students are now routine tasks for them. When we teach differential equations we can teach how to solve several types of differential equations, we can teach the main problems related to the equations (modelling with differential equations, existence, uniqueness, qualitative properties, etc.) and our students will become better problem solvers in this specific area, they will understand the conceptual framework, and they will have several universal strategies (procedural knowledge) to solve specific types of differential equations, but most of them will fail in solving completely novel problems.

The situation is the same at lower levels too. As an example we mention Xin (2016) where the author discussed and criticized the use of keywords to choose the operation for solving arithmetic word problems and introduced a more sophisticated model that can be taught to students for solving arithmetic word problems. Yet, it is still essentially about transforming the problem into a routine task. We are convinced also that even if the student is familiar with the necessary conceptual framework, there is a complexity barrier for the problems he is able to solve. As a simple example we can take a look at geometry problems from the International Mathematical Olympiad. These problems are beyond the complexity barrier for a lot of students who possess the necessary knowledge to understand the solutions. These problems require also very high level proficiency on specific areas. We claim that from this viewpoint teaching problem solving should also include pushing the complexity barrier on an individual level. Currently we do not have evidence that this can be done on a large scale.

On the other hand, from the history of mathematics we have a lot of examples where apparently simple problems (problems with simple statements) required hundreds of years to construct a conceptual framework where they can be solved, such examples are Fermat's last theorem (proved in 1995 by Andrew Wiles) or the weak Goldbach conjecture (proved in 2013 by Harald Helfgott). This is a clear sign that on one hand producing cognitive dissonance and frustration is in the intrinsic nature of mathematical problems and on other hand solving mathematical problems sometimes needs a huge effort and commitment. In András and Nagy (2010) we found that average students usually do not have sufficient perseverance. Moreover, in András (2016) we described a series of activities with gifted students, where we experienced the same issue. These aspects raise an important question about raising affective disposition and maintaining a high level of curiosity/interest in case of teaching specific mathematical contents. In Romania our experience is that student population is changing too fast (see Molnár et al., 2013), there are major cognitive differences between students having 1-2 years age difference, so we would like to see some evidence about adapting teaching strategies to the rapidly changing student population in the context of raising affective disposition and maintaining level of curiosity/interest. So far we have evidence that

practitioners (and even researchers and policy makers) do invest a lot of energy in this direction, but we do not see the effects on a large scale.

In the process of constructing a proof or a solution it is a common issue to transform our personal intuitions and ideas to formal proofs or solutions. Our intuitive thoughts belong to our “analogical mind” (Gentner, Holyoak, & Kokinov, 2001), whereas the formal proofs/solutions are strongly related to a fixed conceptual framework, and require our “conceptual mind” (Margolis & Laurence, 2015). The switch between analogical and conceptual mind is an essential process of problem solving. The concept image theory (Tall & Vinner, 1981) is a useful tool to understand some key features of this process. On the other hand, several other phenomena may influence problem solving. A major difficulty is that usually the conceptual and procedural difficulties do overlap. This can be seen quite clear when 9-10 years old students try to solve logical puzzles like the Zebra puzzle. As long as they have only paper and pencil for this, the problem seems almost impossible to solve. However, when they have to solve the same puzzle using a set of specially designed cards and houses (see the IQ-game design problem on www.mascil-project.eu), they can find the solution. It is clear that they did not have the necessary procedural knowledge in order to deal with a huge amount of information, but if the information is converted into concrete puzzle pieces, they are able to manipulate it and to find the correct configuration. The same problems appear in teaching geometrical proofs for 13-14 years old students. They usually do not have sufficient knowledge about formal logic, about what a “proof” means (so they do not have the necessary procedural knowledge), and at the same time they are not yet familiar with the geometrical concepts/properties (so they do not have the necessary conceptual knowledge). Students sometimes may get blocked because these two difficulties are overlapped. During the teaching/learning activities this overlapping can be avoided if we design special activities, where students first have to fit the pieces of some proofs as puzzles and they have to invent the pieces only after they know how to construct a proof. But we are convinced that we do not have the instruments for teaching students how to unsquash such kind of overlapped difficulties in novel situations.

From a broader point of view (see Hepner & Lee, 2002) problem sensitivity is a subskill for general problem solving. In an inquiry based setting the problem sensitivity is crucial for formulating relevant, yet accessible problems. In a series of inquiry-based activities performed within the European Projects Primas and Mascil our experience is that lower and upper secondary level students formulate a lot of “hard” problems (see András, 2011) that cannot be properly solved with their knowledge level. During a concrete inquiry-based learning activity this can be handled, but this is a sign that at this level the problem solving skills and the problem sensitivity skills are not at the same level. This discrepancy can cause problems on the affective side and if, during a longer term teaching process, it is neglected it can cause the regression of the sensitivity skill. This seems to be a major inconsistency because in order to improve our problem

solving skills we need to solve problems that are above our actual knowledge level. We think this is impossible on a large scale.

The second challenge: Developing problem solver's affective disposition is not teaching of "problem solving"

Above, we have already discussed that some aspects of productive problem solving disposition are impossible to train — at least in large scale. Here, we will continue to argue that many of the interventions that claim to improve problem solving skills, may be improving general affective disposition instead.

The difference between successful and unsuccessful problem solvers is largely caused by differences in their overall disposition (for a review, see Hannula, 2015). It has been known for long that mathematics anxiety (Hembree, 1990), negative attitude (McLeod, 1992) and lack of intrinsic motivation (e.g. Middleton & Spanias, 1999) are related to poor performance in mathematics. More specifically, the role of affect has been widely acknowledged by such key scholars as Polya (1957), Mason, Burton, and Stacey (1982), Schoenfeld (1985), McLeod (1988), Goldin (1988, 2000) and Cobb, Yackel, and Wood (1989) in their analyses of mathematical problem solving.

For example, Polya (1957) addresses determination and hope in his short dictionary of heuristics, mentioning also the necessity to become familiar with all emotions related to the problem solving process. As an example of a more elaborated account of the role of emotions in the problem solving process, let us review the role of *affective pathways*, as discussed by Goldin (2000). In an idealized case of struggling with a complex problem, the solver initially senses curiosity, puzzlement and bewilderment. The next stages of the process are crucial in differentiating between successful and unsuccessful problem solvers. Even a successful solver may experience moments of frustration, but they are followed by elation and satisfaction as the problem begins to yield. Unresolved frustration, on the other hand, may lead an unsuccessful solver to experience anxiety and possibly even fear or despair.

Hope, determination, and persistence in the face of struggle are almost synonyms to self-efficacy. Self-efficacy is a belief that one is competent to perform a certain task, and there is empirical evidence that higher self-efficacy is related to better problem solving success (e.g. Marcou & Philippou, 2005; Michael, Panaoura, Gagatsis, & Kalogirou, 2010; Panaoura, Gagatsis, Deliyianni, & Elia, 2009)

Moreover, many of the case studies showing improved performance in mathematical problem solving have a strong element of affect included in the change process (e.g. Hannula, 2002; Liljedahl 2005; Schukajlow & Rakoczy, 2016).

Based on both the importance of affect in major theoretical frameworks on problem solving, as well as on the empirical evidence of the role of affect in improved problem solving performance, we argue that what is claimed to be successful teaching of problem solving is, in fact, successful intervention to improve overall affective disposition and emotion regulation. When the outcome of teaching is that the student

is more confident, relaxed, and persisting, that does not mean that they have learned anything about problem solving. Interventions that improve student problem solving performance through reducing their anxiety and increasing their self-efficacy and motivation should not be considered as evidence for successful teaching or problem solving. Such interventions have improved student affect, not their problem solving skills.

The third challenge: Simply solving tasks is not "teaching"

We have already discussed the kind of teaching that does not teach problem solving skills, but rather teaches routine techniques for certain types of problems or is training the affective disposition. The third type of instruction that is called “teaching” of problem solving often does not include teaching at all. A key element of problem solving instruction is that students solve problems. Lester and Cai (2016), summarize that “The most effective way for students to learn to solve problems is for them to solve a variety of problems” (p. 121). Moreover, the teachers are often explicitly forbidden from helping the student, as it is considered important for the student to reach the Aha!-moment by themselves, rather than the teacher revealing the secret for them. As a consequence, teachers circulate in the class, giving emotional encouragement to the students and asking them to explain their thinking. Or, sometimes, just giving the tasks to the students.

And it is an effective method, indeed. Kapur (2010) describes a quasi-experimental study, where one group of students experienced a *productive failure cycle*, where they worked on complex problems without any teacher support except for the last lesson for the unit. These students performed significantly better in problem solving than students who had received traditional lecture and practice type teaching. In the additional study (Kapur, 2011), he added a third condition, where complex problem solving was accompanied by teacher facilitation (e.g. question prompts that engender student elaboration and explanations). The facilitated problem solving was no better than the traditional teaching style and the productive failure condition produced again the best results.

It seems that teacher facilitation is only making the problem solving less effective. So is there any reason to claim that teaching improves problem solving? We can claim that simply leading students to websites or books with interesting problems produces equally good results as any teaching. Where is the teaching in that?

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IT IS POSSIBLE TO TEACH MATHEMATICAL PROBLEM SOLVING

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This paper sets the context for the debate to oppose the motion that “It is impossible to teach mathematical problem solving”. It first clarifies the definition of a problem and clarifies the process of problem solving. Next it gives an overview of ideas about how to teach mathematical problem solving and draws on several studies to make the case that mathematical problem solving can be taught.

INTRODUCTION

This paper sets the context for the debate to oppose the motion that “It is impossible to teach mathematical problem solving”. In the context of this debate a problem is defined as a task for which:

The person confronting it wants or needs to find a solution.

The person has no readily available procedure for finding the solution.

The person must make an attempt to find a solution. (Charles & Lester, 1982, p. 5)

This definition emphasizes three crucial components of a problem. Firstly, a desire or need on the part of the problem solver to find a solution to the problem, secondly the solution cannot be obtained directly or immediately by mere recall of knowledge, and thirdly the problem solver must make a conscious attempt to arrive at the solution.

In solving a problem one has to engage in a complex process known as problem solving, which requires an individual to coordinate previous experiences, knowledge, understanding and intuition in order to satisfy the demands of a novel situation. In simple terms, it is the mental journey one takes to arrive at a solution, starting with the “givens” of a situation.

According to Charles and Lester (1982), three factors generally influence the problem-solving process of an individual. They are: (i) experience factors, both environmental and personal, such as age, content knowledge, familiarity with solution strategies, familiarity with problem context and content; (ii) affective factors, such as interest, motivation, pressure, anxiety, tolerance for ambiguity, perseverance etc.; and (iii) cognitive factors, such as reading ability, spatial ability, analytical ability, logical ability, computational skill, memory, etc.

HOW DO WE TEACH STUDENTS MATHEMATICAL PROBLEM SOLVING?

There exist some significant ideas related to the instruction of mathematical problem solving (MPS). One of the most well-known models for MPS is Polya's (1945/1973) four phase model for problem solving: Understand the problem; Devise a plan; Carry out the plan; and Look back. It is important to note that these phases are not linear: a problem solver may proceed from the first to the second then return to the first to re-check his understanding, or he may proceed from the first to the second and on to the third before returning to the first again to clarify some doubts that may have surfaced due to the nature of the resulting answer.

While Polya's model is helpful in providing a general guide on how to proceed with solving a problem, it does not always assure the problem solver success. Schoenfeld (1985) noted that success in problem solving was dependent on four aspects of a solver's cognition, namely (i) resources – the knowledge base that one possesses related to the problem at hand, (ii) heuristics – the strategies that one deploys for solving problems, (iii) control – the decision-making process employed when utilizing and choosing the appropriate resources or heuristics, and (iv) beliefs – the views one holds about mathematics and doing mathematics.

EVIDENCE FROM THE MPROSE PROJECT THAT IT IS POSSIBLE TO TEACH PROBLEM SOLVING

A group of mathematicians and mathematics educators in Singapore have used a practical paradigm to teach students mathematical problem solving. Their project, **Mathematical Problem Solving for Everyone (MPROSE)**, adopts a mathematics practical lesson approach to provide students with opportunities to solve problems, make and test conjectures, make new problems, and explore different solutions. The lessons were conceived to function similar to how science practical lessons in laboratories serve to initiate and develop in students the skills and disciplinary appreciation involved in the work of scientists. MPS is the main feature of the mathematics practical lessons and a Practical Worksheet (Toh et al., 2011) that draws on Polya's problem solving model and Schoenfeld's framework provides the student with a systematic model when solving a problem. The practical worksheet is an instructional tool to encourage students to walk through the recommended stages of problem solving. The worksheet contains sections explicitly guiding the students to use Polya's stages and problem solving heuristics to solve a mathematics problem. We draw on three studies in the MPROSE project to make the case that it is possible to teach mathematical problem solving. The studies are as follows:

Teachers solving mathematics problems: Lessons from their learning journeys (Tay, Quek, Dindyal, Leong, & Toh, 2011).

The subjects of this study were 21 teachers enrolled in a Master of Education course entitled Discrete Mathematics and Problem Solving, which was conducted for 13 sessions of 3 hours each over a semester. Part of the content of the course was Polya's

model of problem solving (Polya 1945/1973) and Schoenfeld's problem solving framework (Schoenfeld, 1985). Specifically, the teachers were introduced to the Practical Worksheet as a tool for mathematical problem solving. Teachers also did a final term report entitled "My Personal Journey in Mathematical Problem Solving". The instruction from the report was as follows:

Write a report titled: My personal journey in mathematical problem solving. You should discuss at least one model of mathematical problem solving and several heuristics. For the purpose of this module, ALL examples must be from Discrete Mathematics.

The study drew on the data of teachers' self-reports and found that:

- i) Many of the teachers reported that they had learnt a lot from the course about how Polya's problem solving model works. Of interest was the low level of understanding of Polya's model or other suggested systematic approaches to problem solving with which they had started off.
- ii) The Practical Worksheet was apparently instrumental in bringing about awareness of the problem solving process. It was clear from the accounts that the Worksheet played a critical role in "cutting a groove" at the initial stages of learning problem solving, along which the learner could roll again in later problem solving attempts. The Worksheet highlights the essential stages involved in mathematical problem solving, and may be dispensed with once the desired mental discipline towards problem solving is learned.

Use of practical worksheets in teacher education at the undergraduate and postgraduate levels (Toh, Toh, Ho, & Quek, 2014)

This study reports on the use of the Practical Worksheet with two groups of students – an undergraduate group and two groups of postgraduate students. In this section we report on the undergraduate group only, though the whole study affirms the use of the Practical Worksheet as a viable way to teach mathematical problem solving.

One of the co-authors (the lecturer) taught a 36-hour course introducing number theory to 59 undergraduate students. The course's content was typical of similar courses taught elsewhere and included divisibility, congruence, Diophantine equations, Euler's generalization of Fermat's little theorem etc. Most of the students were in the first year of their B.A. (Ed.) or B.Sc. (Ed.), and had so far been mainly learning content mathematics. About 85% of them had not undergone the "teaching of mathematics" component that would have introduced to them Polya's problem solving framework.

The lecturer, having taught calculus to the same group of students, recognised that many would face difficulties in number theory because it was atypical of the mathematics they were used to in their pre-university education. When faced with a problem like "prove that if m is a composite number, then $2^m - 1$ is also composite," most of them would not go beyond using heuristics like "substitute numbers (for m)" and "search for patterns".

To engage students in a holistic approach to mathematical problem solving, the lecturer tried to teach *through* problem solving (Shroeder & Lester, 1989) using Polya's model. He began with the first three stages of (1) understanding the problem, (2) devising a plan and (3) carrying out the plan, without explicit mention of Polya. During the lectures, the lecturer demonstrated how he understood the problem, what kind of heuristics he would use, as well as possible plans for solving the problem, before finally carrying out the plan. This departs from the usual theorem–proof, theorem–proof type of exposition that is commonly used in teaching advanced mathematics. Gradually, the job of solving the problem was passed on to the students and the lecture notes would only have the names of the Polya stages, followed by spaces for students to work on. About a third of the way through the course, Polya's model, including Stage 4 which we renamed as “Check and Expand”, was formally introduced. Students were also given the practical worksheet to be used for their problem solving assignments.

Data from one of the problem solving assignments is presented here. The problem was:

Find a million consecutive composite numbers.

This was “non-routine” for the students as there were no similar examples in the lecture notes and the prescribed textbook. Among the 57 assignments received, 51 students managed to solve the problem, 50 showed evidence of using the Polya stages in their solution, and slightly less than half (26) used the heuristic of working with a smaller number of consecutive composites first before going on to solve the problem for a million. The lecturer was pleasantly surprised that so many students were able to successfully use Polya's framework. It was also encouraging to see that 37 students went on to Stage 4 and attempted to generalize the problem. Some students also demonstrated higher order thinking skills.

Relooking “Look Back”: A student's attempt at problem solving using Polya's model (Leong, Toh, Tay, Quek, & Dindyal, 2012)

This paper reports on secondary school students' engagement in mathematical problem solving as part of the MPROSE project. This section presents just an excerpt - a case of successful mathematical problem solving instruction. Way Nam is a student in the school who has taken the problem solving module as part of his mathematics lessons. The data presented is part of an interview conducted with Way Nam for the study.

During the interview, Way Nam spoke at different junctures about how he benefitted from the problem solving module. Overall, he found that the Pólya's model provided him with a structure to approach problems, and a way to move forward when he was stuck: "So when I take this module I realised [that] there are actually some defined steps for me to use. ... Because—okay—I know Step One is like that, Step Two is—, Step three, Step four. I'll just do it like I'm very very familiar with it. Because [for] all these questions [I] have to think about using it. [In the past] we just read the question and just rush into it. Even when we don't know, we just think of everything we can think of first. That's not the correct way. Pólya's method make[s] us know what to do when we don't know how to solve the problem" (p.8). Way Nam's favourite part of

the problem solving module was reserved for the last of Pólya's stages: "If can—, can totally concentrate on expansion lah. [It] is really fun. Like you give us the question, let us think more ... [t]hen let us expand. This would be very abstract and very fun. Yah, it's Part Four [of the Practical Worksheet, corresponding to Pólya's Stage Four] that is exciting."

THE KIDUMATICA EXPERIENCE: ARGUMENTS AND EVIDENCE FOR SUCCESS IN TEACHING PROBLEM SOLVING

Studies about teaching problem solving have appeared in the research literature for many years, ranging from Polya (1954) to English and Sriraman (2010). Some of these studies have been theoretical, and others have described empirical research that has succeeded in teaching problem solving. We claim that problem solving is a skill that can be taught, and that doing so successfully relies on three components: the teaching method, the problem type, and supportive environmental conditions (Amit, 2009).

Below we present two examples of successful experiments in teaching problem solving that were conducted within the framework of the "Kidumatica" mathematics club. The first of these (Portnov & Amit, 2015) focused on the explicit teaching of problem solving strategies. The second (Gilat & Amit, 2014) focused on problem solving that requires the modeling of situations from daily life. Both examples are based on the assumption that problem solving is a skill that does not arise automatically from learning the topics on the school mathematics curriculum, and that its development must be based on a foundation of problem solving experience.

Using explicit teaching to provide students with problem solving strategies

Polya (1954) was the first to describe the topic of strategies for solving mathematics problems. He claimed that to solve a problem it was necessary to find the strategy or the synthesis of strategies that would help in its solution. He further claimed that students needed to gain as much experience as they could in solving problems on their own. Finally, he emphasized the importance of the teacher's role, and the impact of how teachers approached the task of teaching students how to solve problems.

The teaching experiment described in this study made use of a structured, methodical and effective form of explicit teaching, in which students were directly and unambiguously provided with basic tools for acquiring new knowledge. Its teaching method made use of a five stage model composed of: (1) Orientation; (2) Presentation; (3) Structured practice; (4) Guided practice and (5) Independent practice. The experiment focused on four strategies ("working backwards," "proof by contradiction," "recursion" and "trial and error") using problems like: *There are 15 students in a class. The teacher asks them to write the day of the week (Sunday-Saturday) of their upcoming birthday. Prove that there is one day that was selected at least 3 times* (Proof by contradiction); or: *David's farm has a total of 50 cows and chickens. The number of feet of the cows and chickens together is 128. How many cows and chickens are at the farm?* (Trial and error).

The study was conducted with 227 sixth grade students in the "Kidumatica" mathematics club. It followed a six-month learning process that was repeated over four years, using a study unit specially developed for teaching learning strategies. The data was analyzed using a mixed methods approach. Findings showed that in three of the four learning strategies, the students who were taught using the explicit method did significantly better than those who were taught using a non-explicit approach. Furthermore, we found that using the explicit approach did not restrict the students' thinking; quite the opposite – 61% of the students used the strategies in a different way than that they were shown. The students became more active problem solvers who understood the purpose of the strategies and their solution stages, developing them as they saw fit, freeing themselves from the restraints of the strategy when necessary and soaring to heights of virtuosity.

Using modeling and creativity to solve problems based on real life situations

The goal of this research project was to explore how introducing students to authentic complex mathematical problems through modeling activities affected their general and mathematical creative abilities. MEAs (model eliciting activities) are heuristic tasks that require students to develop a clear and well-defined mathematical model or a solution to an authentic problem that tolerates an undefined number of possible interpretations, and to explore several different pathways towards solving the problem by utilizing different procedures (Chamberlin & Moon, 2005).

In the spirit of Polya (1957), experience played a crucial part in the development of the modeling abilities of the 220 children who participated in a three stage structured experiment conducted in the informal environment of the Kidumatica mathematics club:

- (a) Stage 1: Warm-up activity - serves to stimulate interest and motivation, providing initial knowledge necessary for solving the modeling task.
- (b) Stage 2: Model Eliciting Activity (for groups of 2-4) - asks students to solve a mathematically complex problem situation for a 'hypothetical client', requiring them to construct the data, recognize the important variables and discover the relations between those variables in order to mathematize the situation.
- (c) Stage 3: Closure activity - poster presentations, critique and discussion.

The study used two orthogonal questionnaires to assess the development of students' ability to solve problems based on real life situations, and the development of their general and mathematical creativity. The MEA teaching unit was taught to hundreds of 11-12-year-old students from the Kidumatica math club over a period of several years. Pre- and post-tests were administered to control and research groups of equal skill levels, and the development in their problem solving was assessed based on the extent to which their solution met the needs of the problem's hypothetical client. The students were also given general and mathematical creativity assessment tests that had been validated by previous studies. Finally, the students' creativity in solving problems

was assessed using a specially designed tool, the **Mathematical Creative Abilities Manual**, which analyzed three components of students' solutions: appropriateness, originality, and mathematical resourcefulness.

The results showed that the students in the experimental group progressed significantly in their ability to solve problems based on real-life situations using modeling. Their models were more mathematically sophisticated and showed greater utility. Moreover, while all of the students showed some improvement in their general and mathematical creativity, those who participated in the mathematical modeling teaching unit, with its open, “real-life”, challenging problem-solving activities, showed significantly higher improvement in their general creativity. Finally, the results indicated that the students had improved over the course of the unit in their appropriateness and mathematical resourcefulness.

CONCLUSION

We have shown that problem solving is not merely a natural, intuitive skill (though intuition does play a central role in learning mathematics). Students need to be provided with a suitable foundation for learning to solve problems. This foundation should consist of three components: the proper problem type, the proper teaching approach, and the proper learning environment. The problems must be challenging, non-routine, and if possible they should have some connection to the students' daily lives. They must, of course, also be mathematically powerful. The teaching approach must be goal-oriented — aimed at the development of students' problem solving abilities. It can take the form of problem-based learning, structured experience, or the explicit teaching of problem solving strategies (as shown above). Finally, the learning environment must encourage problem solving (not just completing exercises) and value experimentation more highly than simply "covering the material." We have backed our case with some successful experiments, and more will be illustrated in the debate.

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MATHEMATICS EDUCATION IN HUNGARY

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MATHEMATICS AND MATHEMATICIANS IN HUNGARY

The history of the Hungarian mathematics dates back to the late Middle Ages, but the first really significant (no exaggeration to say that sensational) results were produced by the Transylvanian Bolyais. The father, Farkas Bolyai (1775–1856) was a friend and a fellow student of Gauss. He was a very versatile mathematician and a school creator. However, sensational results were done by his son, János Bolyai (1803–1860), especially with the creation of the first non-Euclidean geometry (Weszely, 2013; Stäckel, 1913), published in the supplement called “Appendix” produced to his father's book.

The next great era of Hungarian mathematics was launched at the end of the 19th century with such giants as Lipót Fejér, Frigyes Riesz, Alfréd Haár, who taught in Cluj-Napoca, Szeged and in Budapest as well. Their students were György Pólya and Gábor Szegő among others, who spent the second half of their lives in California, at the University of Stanford. Pólya and Szegő published a book together that has become world-famous (Pólya & Szegő, 1976), and Pólya's other works have a significant impact on mathematics education, too (Pólya, 1945, 1954). Their student was János Neumann as well, who started similar revolution as Bolyai in the first half of the 20th century by creating the theory (Vancsó, 2015), and the construction of the first real computer (ENIAC). In addition, the birth of the game theory is primarily linked to him (Neumann, 1928; Neumann & Morgenstern, 1944), as well as establishing the axiom system of set theory or the mathematical foundations of quantum mechanics (Neumann, 1955).

The University of Szeged became a kind of successor of the University of Cluj-Napoca after the Trianon loss and many mathematicians arrived from Cluj-Napoca to Szeged to work at the university. Gyula Szőkefalvi-Nagy and his son Béla worked here. Béla Szőkefalvi-Nagy's fields of expertise were the linear operators of Hilbert spaces, by extension, the entire functional analysis. His book published in 1942 on the mentioned topics was a great success all around the world. His work with Frigyes Riesz titled “Leçons d'analyse fonctionnelle” was published in 1952, and translated into Russian, English, German and Hungarian (as well as e.g. Chinese and Japanese), with its extent of almost 500 pages. The simplified version of the book titled “Real functions and

function series” became a university textbook. He also had outstanding achievements in the theory of Fourier series. We also must write about László Rédei as well, whose most significant results were achieved in the field of algebra and number theory, especially in the theory of finite Abelian groups. His theorem is considered to be one of the most beautiful pearls of Hungarian mathematics, which has also some important implications in number theory, and surprising results were made in coding with its help.

A number of great mathematicians taught at the Péter Pázmány University in Budapest, and also at its successor, at the Eötvös Loránd University. Pál Turán’s field was the analytic number theory, György Hajós taught geometry and algebra while Alfréd Rényi’s expertise was probability theory, but Lipót Fejér and Frigyes Riesz also retired from here.

Among the Hungarian mathematicians of the 20th century, Pál Erdős (a student of Fejér) was an exceptional personality and world traveller even with his fragile physique. Erdős was not only the supporter of the talents in Hungary, but in many parts of the world he also embraced and gave adequate intellectual ammunition for those mathematicians who were seen promising. He lived in several hotels in the world, but mostly with mathematician friends. He had a colleague in America, who set up in his own house a comfortably furnished small apartment, and Erdős was welcomed there at any time.

Wherever Erdős appeared, he invigorated the mathematical life there. Everyone was proud who could write a common publication with him. These fortunate ones were nominated that their “Erdős number” is 1. The Erdős number of those who published with people who had written a common paper with Erdős was 2. Continuing this thought, it is easy to see whose Erdős number could be infinite. This can also be read in his book titled “Primeman”.

One of his last lectures was held in 1996 in Szeged. Some of the authors of the current PME proceedings book participated there, thus carrying on Erdős’s influence on mathematics education.

RENEWAL OF HUNGARIAN MATHEMATICS EDUCATION: THE WORK OF TAMÁS VARGA

The 1879 curriculum is an important milestone in the history of the Hungarian mathematics teaching. Gyula König mathematician, professor of polytechnic, also participated in the development of the mathematical content of the curriculum. His opinion related to the curriculum reform was that high schools could not prepare their students sufficiently for the university studies. His work was later continued by Manó Beke, also a well-known professor of mathematics (Ambrus, Vancsó, & Csapodi, 2016).

After the World War II, Tibor Gallai’s and Rózsa Péter’s works have crucial importance, they prepared course-books for secondary grammar schools (1949–50, see

Vancsó, 2014). The authors declared war on formalism, they attempted to present mathematics in its development, in its conformation and to turn to reality where it was possible. The main point of the textbook series was to show the development of the problem: why it is important to solve, how many different solutions there are, etc. When the problem became known, the series shows how it can be used, what new problems it evokes and goes through the related new problem similarly. In these textbooks we can also realize the influence of György Pólya.

In the renewal of mathematics teaching, Tamás Varga played a crucial role, who explained for the first time in public a professional approach to the teaching of mathematics on the II. Hungarian Mathematical Congress held in 1960 in Budapest. Among the congress speakers Zoltán Dienes was participating as well, whose performance confirmed that Varga was not alone and was not on the wrong tack with his notions. “As a result, he felt that he could start a new educational experiment in the next year..., in order to develop a new curriculum for primary schools.” (Császár, 2005).

In 1962, the International Symposium of Teaching Mathematics was organised in Hungary by the UNESCO. Tamás Varga, Zoltán Dienes, and Richard Skemp were also among the congress speakers. The presentation of Tamás Varga (“On some curriculum problems of school mathematics”) was such a huge success, so he was asked (with Willy Servais) to prepare the documentation of the conference (Servais & Varga, 1971).

In 1963, the Complex experiment of teaching mathematics started and led by Tamás Varga (1966). Since 1968, the complex experiment gained more space, so in 1971, 146 classes studied by using Tamás Varga’s and his research team’s method. In 1973 with the leadership of János Surányi a high school experiment started to pursue primary school complex mathematics in high. The new temporary curriculum was introduced in 1974, and mathematics curriculum and syllabus were based on the complex experiment. Four years later, in 1978, mathematics was compulsorily taught in primary schools under the new curriculum, in upward arrangement, and for 1982 the whole lower grades (1–4) of primary school learnt mathematics based on the new curriculum (Gosztonyi, 2016).

The introduction was not problem-free, in addition to the resistance of the general public and the teacher society, errors in the curriculum and defects of teaching practices also hindered the transition. Meanwhile, further experiments were conducted to transform the upper primary mathematics education, and the development of methodological materials, curriculum and textbooks for secondary schools had already started based on the spirit of complex mathematics. However, they have relatively less effect than the ones for lower grades.

The part, related to mathematics education of lower graders in primary schools, of the National Curriculum introduced in 1995, shows great similarity to the complex experiment of teaching mathematics, and it hasn’t been changed by the amendments

during the last twenty years. So we can say that the methodology and curriculum developed by Varga's group still has a great influence on primary school teaching of mathematics.

Varga's complex mathematics tried to meet the requirements stemming from science, the expectations of the society and economy, and used the findings of modern psychology, the practical realization of the attempted modern learning theories at the same time. Not only he tried to improve and change some subfields of teaching mathematics, but also he considered the subject, the school, the teacher, the student and the family as one unit. His aim was to reform the whole school education of mathematics. This included scientific research, the renewal of the subject in the aspect of structure, content, methodology and education tools, as well as the experiment of the new tools, further training of teachers, meanwhile informing the general public and the scientific society. This is the reason why his work is called as a complex one (Varga, 1966).

In terms of content only such materials could be in the curriculum that are relevant to the whole of mathematics, playfully accessible through problems, they are related or prepare to introduce other topics, and have a role in modern application of mathematics. The complex mathematics paid special attention to differentiation. The tasks themselves are generally differentiators, problems are formulated in such a way that the right answers can be given in many levels, in different ways and by various tools. In this way, at the same time there is the possibility of development of slower and faster students while teaching the same curriculum to each of them.

The effects of Tamás Varga's work and the Complex experiment of teaching mathematics aroused international attention (Andrews, 2003). However, the experiment did not bring about unambiguous success in Hungary. The psychological examination of the experiment was done by Sándor Klein (1980), but the credible, comprehensive examination of its effects did not happen. Nowadays the most important aim of the methodological competition announced by the Hungarian Academy of Sciences is to think over our mathematics education in such a way to save our own values at the same time.

PSYCHOLOGICAL AND SPECIAL EDUCATION RESEARCH ON NUMERICAL COGNITION AND MATHEMATICAL LEARNING

Psychological and special education research in Hungary contributed to some basic understanding of numerical knowledge and offered tools to diagnose and remedy developmental dyscalculia.

Basic research on numerical cognition is run by the research group of Attila Krajcsi. While these studies mainly focus on basic research, many of those results can be applied readily in education. For example, the role of place-value system (e.g. Indo-Arabic notation) and sign-value system (e.g. Roman number notation) was investigated, and it has been shown that contrary to the widely expected view, sign-value number notations are easier to acquire (Krajcsi & Szabó, 2012), and this notation

could also be used to introduce multi-power number notation for children. As another example, it has been shown that preschoolers can use empty sets (i.e. zero) to solve simple numerical tasks, e.g. comparison, addition and subtraction, as soon as they understand symbolic numbers, but they are not sure whether zero is a number (Krajcsi, Lengyel, & Kojouharova, under revision), which findings set a way to develop efficient methods to teach zero.

A very successful and efficient numerical cognition research group was led by Dénes Szűcs in Hungarian Academy of Sciences, which ceased to function after the lab leader moved to the University of Cambridge to continue his fruitful research there. His research is often directly relevant in mathematical education, for example discussing the general viewpoints how developing educational methods might rely on cognitive research, or how bases of number understanding should be measured (Szűcs, Nobes, Devine, Gabriel, & Gebuis, 2013).

Neuropsychological research was widely popularized among professionals by the neurologist Attila Márkus (1998, 2000, 2007). Additionally, the comprehensive diagnosing test for acquired numerical impairments, the Number Processing and Calculation test (Delazer, Girelli, Graná, & Domahs, 2003) was adapted to Hungarian a few years ago (Igács, Janacsek, & Krajcsi, 2008).

Special education research is also active in Hungary. One focus is to offer appropriate tests for diagnosing developmental dyscalculia. In the recent decades official developmental dyscalculia screening was done using a test developed by the special educator Dékány (1999). While this test proved to be useful, and it has been the only available reliable test in Hungarian at that time, it had some shortcomings: e.g. an objective score system was missing, there were no clear standards, it was applicable only for children up to 10 years old.

To overcome these problems the Cognitive Developmental Skills in Arithmetics test (Desoete, 2006) was adapted to Hungarian (Krajcsi & Hallgató, 2012). Additionally, the Hungarian version of the already mentioned Number Processing and Calculation test could partly be used for developmental numerical problem diagnoses (Szilágyi, 2007). Importantly, the original test by Dékány was published with updated and extended tasks and with appropriate standard values (Csonkáné Polgárdi & Dékány, 2013). Beyond offering appropriate tools for diagnosis, developing tools are worked on to improve the performance in children with mathematical difficulties, e.g. using origami (Krisztián, Bernáth, Gombos, & Vereczkei, 2015).

THE EFFECTIVENESS OF HUNGARIAN MATHEMATICS EDUCATION

In the field of mathematics literacy, four major fields of research can be identified among Hungarian studies: (1) measuring mathematical knowledge related to curriculum requirements, (2) measuring mathematical competence, (3) examining mathematical task and problem solving research based on word problems, and (4) examining basic mathematical skills.

TOF-80 was the first step of establishing large scale knowledge assessments, on the basis of the international system-level studies, following their approach and methodology in 1980. Opposed to later measurements, this was a purely subject-related, curriculum requirements-based one. The Monitor study in 1986 can also be considered as a curriculum evaluation study, but as well as the first step of the monitor-system tests conducted regularly until 1999 (Báthory, 2002).

One of the typical strands of county-level knowledge measurement, which was running at the same time as the national measurements, can be followed in Orosz (2001) study series. These studies drew attention to the continuously increasing differences between schools, and to the progressive polarization. The studies also showed great overlaps between students' achievement, who got the same grades.

Switching to online testing can also have a significant change in measuring mathematical knowledge. In the frames of the project Development of Diagnostic Assessments coordinated by the University of Szeged, a mathematical item bank was set out (Csapó & Szendrei, 2011). Detailed examinations were carried out (among others) in the equivalence of online and paper-and-pencil tasks (Hülber, 2013), and in connection with the time-on-task characteristics of solving online tasks (Vidákovich, 2015).

Most of the surveys examining mathematical competences can also be related to the international system-level comparative studies, namely to the Hungarian participation in PISA. The Hungarian adaptation of PISA is the National Assessment of Basic Competences, substituting the earlier Monitor examinations, in which reading comprehension and mathematical literacy are assessed. The first competence assessment was conducted in 2001, among 5th and 9th graders. Since 2004, it measures students in 6th, 8th and 10th grades (Balázsi, Rábainé Szabó, Szabó, & Szepesi, 2005).

The sample selection in the case of National Assessment of Basic Competences is usually fully comprehensive (the task sheets are filled out by all of the students of a particular grade), and the analysis and the documentation of the results follow the methods of PISA. The models of probability test theory are used in the measurements of competences, in accordance with the widespread practice in the international system-level assessments. Schools get detailed feedback about the results, and they have the opportunity to analyze the results at local level, as well (Balázsi, 2007).

Researches on mathematical word problems date back several decades (Szendrei, 2007). In the beginning of the surveys word problem tasks only appeared as instruments for the assessment of mathematical knowledge. The word problem item bank for lower graders, created by Nagy and Csáki (1976), was built with the intention of more objective evaluation of task solving processes. This was the first Hungarian item bank, which covered the entire operation area of a skill system, and also used modern methods for test development. Re-done the studies in 1997, and comparing the results of the two assessments, very important conclusions were made (Vidákovich & Csapó, 1998).

The scope of research on word problem tasks continues to grow. The survey on the Hungarian use of an internationally widely analyzed task series by Csíkos (2003) showed important results. The survey was repeated later on with the closed versions of some of the tasks, thereby providing important information on the role of the task types influencing effectiveness (Csíkos, Kelemen, & Verschaffel, 2011). The newest studies on word problems also discovered the effect of some other factors on effectiveness, like drawings (Csíkos, Szitányi, & Kelemen, 2012).

The first Hungarian empirical analysis of mathematical skills and school development was carried out by researchers of the University of Szeged. Among József Nagy's studies in the development of basic skills, the assessment of elementary and basic operational numeracy skills is of paramount importance.

He also created the test system Preventive Assessment of School Readiness and out of its tests counting and quantity subtests can also be linked to basic mathematical skills. The revised version of the assessment system is the Diagnostic Assessment of School Readiness (Nagy, Józsa, Vidákovich, & Fazekasné Fenyvesi, 2004). This test battery has been a suggested and widely used instrument for examining the school readiness of children for several years. Lately, an online version of the test system was also developed and piloted (Csapó, Molnár, & Nagy, 2014).

Among skills and ability tests, some specific studies can be found examining special fields of mathematical skills and abilities. The latest studies examined the connection between the strategies and the performance in combinatorial reasoning (Szitányi & Csíkos, 2015) and in mental addition (Csíkos, 2016).

TEACHER TRAINING IN MATHEMATICS

In Hungary there is a tradition for training mathematics teachers, which forms changed in the last decades, but the quality is still high. Out of the several universities only six can carry out mathematics teacher training (Eötvös Loránd University – Budapest, University of Debrecen, Eszterházy Károly University – Eger, College of Nyíregyháza, University of Pécs and last but not least University of Szeged), each in different parts of the country. Since 2005, when the Bologna-system was introduced, students have to complete a mathematician bachelor degree before entering the teacher training master program².

The precise structure and content of the BSc program vary from university to university. At the University of Szeged, students have to learn the same materials and complete the same main courses regardless their specializations. Out of 180 credit points only 60 points are responsible for the specialization, so mathematics teachers are also required to know the most important fields of mathematics, as well as mathematicians. In the master program teachers only have to complete 40 credits out

² <https://www.felvi.hu>

of 120 in the field of mathematics, which are mostly methodology courses, but three courses must be related to higher mathematics³.

It is very important to remark here, that in Hungary every teacher candidate has to choose at least two fields, which are either related (e.g. maths and physics) or not so much (e.g. maths and English or even history). In 2013 the Bologna-system has been replaced by the “undivided” teacher training program, which means that the candidates have to decide which two public education subjects they choose before applying for the program, although in the Bologna-system it was enough to decide on the second subject at the end of the first academic year⁴.

The goal of the master program is to train mathematics teachers who are highly professional and have a methodological competence in the field, who are capable of teaching mathematics in a high quality at all levels of public education, and to develop an appropriate attitude towards mathematics considering students special mathematical abilities, who recognize talented students and encourage them to solve problems and to have original ideas, and who are able to critically assess the current mathematics education and scientific research works. The undivided form of teacher training differs from these aims, because candidates have to choose between being a primary school or a secondary school teacher, and after three years of common study they have different courses in the second half of the training. At the end of the master program candidates have a whole semester long teaching practice in school environment, which is preceded by a shorter practice. In the undivided program it is modified, candidates have a two semester long individual practice, which helps them to get ready for the real life and teaching as a profession⁵.

After graduation it is relatively easy for a mathematics teacher to find a job in a school. Based on the statistics of the last years (2012–2015) only 45–50 new mathematics teachers graduated in each year. If we take a closer look, we could see that more than 290 mathematics teacher places are unfilled in Hungary in this summer⁶.

Newly qualified teachers get a mentor for two years when starting working, who help them to use the best methods, be aware of their own mistakes, and constantly develop their competences. Self-evaluation is very important for teachers. It is also really hard to be prepared for the reaction and the construction of the class, a teacher has to get to know their students not only as individuals, but also as a group. We have to find out how the group is dynamics working in a particular class, and it can occur that one method we could perfectly use in one group, will not work in another one. A solution

³ <http://cab.math.u-szeged.hu/index.php/en/>

⁴ <http://www.u-szeged.hu/tanarkepzes-index>

⁵ <https://www.math.u-szeged.hu>

⁶ <https://kozigallas.gov.hu>

or help could be the two semester long practice of the undivided teacher training program⁷.

The other significant difficulty for a mathematics teacher in teaching is to make mathematics interesting. Some of the Hungarian studies also proved that mathematics is one of the least preferred subjects (Csapó, 2002; Sebestyén, 2009). The Hungarian attitude for mathematics is usually about that it is really hard, and we teach so much, which cannot be used after the school leaving exam, so they are not really interested. We should make the learning process enjoyable, change their attitudes towards the subject. And this is a very big challenge, because from this perspective it is not enough for a teacher to be able to explain properly the particular material, they also have to be competent enough to motivate them, help them in the learning process, make the material or the way of elaboration student friendly considering their interests and skills, and last but not least manage to give them different ways of thinking modes.

Fulfilling these requirements is helped by the teacher career model. At first, newly graduated teachers have to spend two years as trainees, and after that they must take a certification exam in order to step forward to the next level (teacher I). The exam consists of a portfolio, self-, colleagues' and students' evaluation and classroom observations made by a committee of experts. After six years the certification exam must be retaken in order to step to the next level (teacher II). After 14 years of active teaching a teacher can re-take the exam if she or he meets the requirement for one of the two following categories: research professor (a PhD and publication activity is needed) or master teacher (professional examination is needed with special requirements, specified every year).

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⁷ http://net.jogtar.hu/jr/gen/hjegy_doc.cgi?docid=A1300326.KOR

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A CASE STUDY ON UNDERGRADUATE STUDENTS' FOCUS WHEN SOLVING INTEGRATION TASKS: THE ROLE OF RELATING TASK PROPERTIES AND POSSIBLE APPROACHES

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Mathematics undergraduates have difficulties solving non-routine calculus tasks even when they have learnt the relevant approaches (Selden et al., 2000). Even when tasks can be solved through standard manipulations and algorithms, students may not think of a fruitful approach. This study aims to gain insight into patterns in students' thinking when solving this kind of calculus tasks, by investigating what students focus on when trying to find an appropriate approach. An approach is defined to be a mathematical procedure that is aimed at solving the task. For integration tasks, approaches are e.g. direct integration or using an appropriate substitution.

Research questions are: (1) Which patterns can be discerned with respect to students' focus on mathematical properties of the task, on possible approaches, and/or on relating them? (2) Which relationship can be found between solution patterns and success? During task-based think-aloud interviews, twelve first-year mathematics students of KU Leuven (Belgium) and the University of Groningen (the Netherlands) have attempted to solve tasks, among which either $\int (1+x^{1/2})/(1+x^{1/3}) dx$ or $\int 1/(1+e^x) dx$. These tasks can to some extent be considered problems in the sense of problem solving, since most students had to search for an appropriate approach to solve these integrals. Only few of the participating students were successful in solving these tasks. For this report, three very different solution attempts of the two tasks were selected.

These solution processes were split into episodes and coded based upon perceived focus on task properties, on possible approaches and on relating them. This revealed three distinctive patterns: (1) an unsuccessful solution process, where the student goes back and forth between task and approaches but has difficulty recognizing which task properties are relevant for solving the task or which approaches can be applied to the task, i.e. unable to relate task properties and approaches; (2) a successful one, where the student recognizes the problem type and reconstructs the solution process; (3) a successful one, where the student goes back and forth between task properties and approaches, discerns some relevant aspects of the task, relates them to appropriate approaches, moving forward step by step. These results suggest to improve students' integration abilities by training them in relating task properties and approaches.

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LINKS BETWEEN MATHEMATICAL KNOWLEDGE FOR TEACHING AND QUALITY OF IMPLEMENTATION OF TASKS: A STUDY OF PRE-SERVICE MATH TEACHERS' EXPERIENCES

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In Turkey, as a result of recent changes in mathematics curricula there is a sustained emphasis on mathematics tasks focusing on conceptual understanding, reasoning and problem solving. When investigating mathematics tasks, researchers address that teachers' implementation of mathematics tasks determines to a large extent students' learning outcomes. Despite curricular efforts to promote cognitively demanding tasks, teachers often choose tasks that are dependent on procedural skills or memorized knowledge and even when teachers have tasks that have the potential for developing students' reasoning and problem solving skills and conceptual understanding, teachers can turn such tasks into routine mathematical exercises, which mainly focus on procedural skills. There are ongoing efforts to study factors influencing in service teachers' use of tasks, often focusing on teacher knowledge.

The aim of this study is to investigate pre-service mathematics teachers' quality of implementation of tasks and how it is related to teachers' levels of mathematical knowledge for teaching. The sample of the study consists of 41 (38 female, 3 male) pre-service middle school math teachers studying at a university having the highest student acceptance scores in this field in Turkey. Participants are enrolled in the final year practice teaching course in Spring 2016 semester. As a measure of their mathematical knowledge for teaching (MKT), they took the Turkish translation of the primary TEDS-M released items test at the beginning of the semester. Quality of implementation of mathematics tasks is conceptualised in accordance with Stein and Kaufman's (2010) triadic framework where high quality implementation is accepted as maintaining the cognitive demand throughout the implementation of tasks, attending to students' thinking and establishing mathematical reasoning as the intellectual authority. Data regarding teacher candidates' implementation of tasks are collected by using the Classroom Observation Coding Instrument (Stein & Kaufman, 2010), from their microteaching experiences at the university and practice teaching in their practicum schools. Quantitative analysis will be conducted, and a positive relationship between teacher knowledge and their implementation of tasks is expected. The findings are expected to contribute to teacher educators' efforts to make sense of the complex relationships within teachers' practice.

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INVESTIGATION OF HUMANITIES STUDENTS' LOW MATHEMATICS SELF-EFFICACY ROOTS IN IRAN

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According to Schoenfeld (1985), one of the main factors which effects on students' mathematics learning is their belief system. There is a great body of research on students' belief which can be categorized to three main groups: beliefs about mathematics education, about the self, and about the class context (De Cote & Op't Eynde, 2002). This study focuses on students' beliefs about self mainly "mathematics self-efficacy" which is defined by Hackett and Betz (1989) as an individual's belief about one's mathematical skills and abilities. As we have noticed in Iran, many humanities students feel low mathematics self-efficacy and so we would like to investigate the roots of this phenomenon.

This study is among qualitative ones. Data of this study were gathered through interviews with students while teaching in the authors' classes and informal discussions with students outside of the classes. Data analysis revealed that there are three main roots for low mathematics self-efficacy among humanities students: society's beliefs about humanities students' mathematical abilities which send negative signals to the learners and persuade them that they are not good in mathematics; the humanities students' disciplinary beliefs, originated from high school, which says mathematics is not of priority for them; and the mathematics university teachers' beliefs about humanities students' mathematics abilities. Learning the roots of the problem, now a further research is required to find the ways of improving humanities students' mathematics self-efficacy.

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DEFINITIONS ABOUT TWO-DIMENSION GEOMETRICAL CONCEPTS BY PRE-SERVICE CLASSROOM TEACHERS

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It is inevitable that concepts of pre-service teachers about geometric shape would effect concepts of their future students. Therefore, in this study it was aimed to put forward how classroom teaching 4th class pre-service teachers define concepts about geometric shapes and analyze how correct these definitions are. Eighty three pre-service teachers who attend 4th class at Education Faculty Classroom Teaching department of a state university in Turkey. Pre-service teachers were applied a form in which they are required to make definition of 11 geometric concepts. The form was developed by Cilavdaroğlu (2012). In the analysis process of data, a scale of 4 categories was used including correct, partially correct, incorrect and blank which were prepared as a result of analysis of definitions in the related literature.

The findings of the definitions developed by the participants about the concept of angle 7% were correct, while 71% partially correct. Concerning their definition about polygon 21% were correct whereas 73% partially correct. Participants' definitions about the concept of triangle 48% were correct, while 41% partially correct. Concerning their definition about quadrangle 39% were correct whereas 53% partially correct. Participants' definitions about the concept of trapezoid 38% were correct, while 34% partially correct. Concerning their definition about parallelogram 46% were correct whereas 45% partially correct. Participants' definitions about the concept of rhombus 20% were correct, while 59% partially correct. Concerning their definition about rectangle 39% were correct whereas 46% partially correct. Participants' definitions about the concept of square 35% were correct, while 55% partially correct. Concerning their definition about deltoid 30% were correct whereas 34% partially correct. Participants' definitions about the concept of circle 37% were correct, while 31% partially correct.

Consequently, the participants generally provided partially correct definitions of geometrical shapes. It draws attraction that especially in mathematic teaching programs, pre-service teachers gave correct answer with quite low percentage in the definition of term angle which is commonly seen. The percentage of correct answers given for all the geometric shapes can be an indicator of the fact that pre-service teachers have lack in learning conceptual learning.

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TEACHERS' ATTITUDE TOWARDS TEACHING COMBINATORICS

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The teachers' attitude has a fundamental influence on the way of teaching and the efficiency of learning. This topic was also brought up in the didactic research in recent years, in connection with the mathematics teachers as well, but a study that investigates this specific area (combinatorics) was not included. Our aim was to examine some aspects of teachers' attitudes towards Combinatorics with the help of a questionnaire.

T. Varga's "Complex Mathematics Education Experiment" has an important effect on Hungarian mathematics teaching primarily in the first eight grades. Our research group's⁸ goal was to rethink traditions based Varga's teaching methods, and to implement his ideas in secondary schools to a greater extent, now on the example of the Combinatorics, that on one hand had also a priority in Varga's experiment. On the other hand this area of mathematic is particularly suitable for comparing the problem solving ability of different age groups (Hoffmann, 2005).

A part of our research dealt with teacher's attitude toward teaching Combinatorics. We developed a test (questionnaire) for teachers' attitude about teaching Combinatorics and how much they like teaching combinatorics and how successful they do this. The questionnaire were filled out by 136 primary or secondary school teachers (31 male; 102 female, and 3 not declared). This relatively small number set some limitations in our results. The test is available and can be filled out in Hungarian <https://goo.gl/RZo526h>. According to our results, among others teachers think combinatorics is useful and not only the standard problems and their substantial majority thinks that combinatorics is not only for individuals who excel in mathematics. We examined how much teachers like to teach combinatorics, compared to 16 other areas of mathematics, too.

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IDENTIFYING MATHEMATICS TEACHER ROUTINES THAT MAY FACILITATE STUDENT LEARNING: A CASE STUDY

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This study has two main objectives. The first is to identify learning processes that take place during a lesson; the second is to identify actions (routines) that mathematics teachers undertake to facilitate learning.

To achieve the study goals, we used a methodological tool (Nachlieli & Tabach, 2011) that combines SFL (Halliday, 1978) and the commognitive approach (Sfard, 2008). The basic premise of SFL approach is that any use of language represents the user's choice, with respect to three meta-functions: ideational; interpersonal; and textual. According to the commognitive approach the development of a discourse can be explored by tracking the changes of four characteristics: words and their use, visual mediators, routines and endorsed narratives. A 90-minute lesson on the concept of a circle was taught and recorded in a sixth grade class comprising 20 students. During this lesson The teacher introduced the basic terms related to a circle: definition of a circle, radius, diameter, chord and center and the relation between chord and diameter. Data analysis took place in five stages: Identifying the teacher's pedagogical routines, categorizing these routines, identifying key moments that attest to learning, identifying the mathematical narratives of the students and the teacher (to trace learning process) and identifying the teacher's routines that seem to have facilitated the learning process. The findings show that learning occurred in two parts of the lesson, when students' narratives changed into acceptable ones. Moreover, the teacher used three types of pedagogical actions (routines) to facilitate the learning: mathematical (e.g. repeating previously learned concepts, asking questions, using visual mediators), interpersonal (e.g. addressing a student, revoicing) and textual (e.g. filling student's attendance). Lipowsky et al. (2009) showed similar findings.

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GROUP PROJECTS AND STATISTICAL THINKING OF UNDERGRADUATES

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RESEARCH GOALS

This study aimed at extending, and at the same time also investigating, the development of statistical reasoning in a group of undergraduate students taking introductory statistics. We examined the ways in which students used the basic techniques of mainly descriptive (but also informal inferential) statistics, as they formulated ideas about real-world applications of statistics. We also investigated the impact of the use of technological tools, and specifically of the dynamic statistics software TinkerPlots®, on the development of students' statistical reasoning.

METHODOLOGY

The site for the study was an introductory statistics course at a university in Cyprus. There were 51 students in the class majoring in primary education. The course design was guided by the conceptual "Framework for Teaching Statistics within the K-12 Mathematics Curriculum" (Franklin et al., 2007), which focuses on building learners' conceptual understanding of statistics by engaging them in authentic educational activities that give them the opportunity to experience the whole statistical problem solving process. The study participants were asked to work in groups of 3-4 in order to conduct a survey on a subject of personal interest. They were responsible for formulating the project questions, gathering their data, analyzing them using suitable software, and communicating their findings in a report.

RESULTS

Analysis of the data collected during the study provided rich insights into how the study participants thought and learned about statistics and how technology impacted their statistical reasoning. In sum, findings indicate weak knowledge of statistical concepts and relatively deterministic epistemological sets among the vast majority of study participants. Nonetheless, engagement in collaborative, technology-supported, project-based learning activities brought about important changes in learners' ways of approaching statistical problems. There was some evidence for higher cognitive involvement, for improved overall comprehension of statistical concepts.

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A COMPARISON OF THE PYTHAGOREAN THEOREM IN BRAZILIAN AND TAIWANESE TEXTBOOKS BY ANALYSING THE NUMBER OF EXERCISES AND COGNITIVE DEMAND

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Textbooks have a strong influence on how much students learn and what they learn. Some international assessments, such as PISA, have shown that students from different countries have different proficiency in mathematics, such as the case of Taiwan (very good) and Brazil (very poor). This research makes an analysis of this two countries textbooks in order to answer if the way mathematics is addressed in those countries' textbooks have influenced how much students from both countries have learnt reflecting on their performance.

Since most of the textbooks chapters are composed by exercises, exercises are an important variable to define whether students learn or not a certain topic when using a specific textbook, been able to apply the knowledge learned later on. In order to understand the level of demand required from each student to solve certain exercises in both textbooks leading to the understanding of how much students are able to comprehend and learn according to the cognitive demand level of the question, it was used the framework developed by Stein and Smith (2011). Hence, the quantity and the cognitive levels of each exercise were analyzed in the Brazilian and Taiwanese textbooks.

The results showed that Taiwanese textbook presented two times more exercises than the Brazilian textbook, most of the exercises presented by the Brazilian textbook have higher-level demand for its solution, while most of the Taiwanese textbook exercises are lower-level demand. This study concludes that Taiwanese textbook asks more from students than Brazilian textbook, since the Taiwanese textbook requires many practices in order to assist the students to build a strong concept image and obtain a better understanding about the validity of the theorem, letting students been able to better apply the knowledge about Pythagorean theorem on assessments such as PISA, obtaining better results. More details about this research are going to be introduced in the conference.

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MANIC AND PHOBIC DEFENCES IN THE BIG BANG THEORY

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A growing body of research has sought to understand how popular culture influences students' mathematical identities in the classroom (Moreau, Mendick & Epstein, 2010). For example, mathematicians are often portrayed in film and television as gifted white men. Psychoanalytic techniques have been used to further understand how students' identities emerge from unconscious responses towards education including conflicting feelings of anxiety, fear, and adoration (Bibby, 2011). My research explores how stereotypes negate the psychic complexities of being mathematically competent by analysing the psychic defences of the white, male protagonist, Sheldon in the television series *The Big Bang Theory*. This study complicates the view that white males exist comfortably within the trope of mathematical competence only if they exhibit undesirable characteristics such as neurotic behaviour and social isolation.

Using Nimier's (1993) typology of six psychoanalytic defences in mathematics, three scenes were read to understand how: 1) defences against mathematics become a way of confronting the anxiety surrounding the need for total competence in the field, and/or 2) mathematics becomes an instrument for defending against other anxieties by providing a place of comfort and reliability. Scene analysis revealed a psychic splitting of anxieties. Sheldon's insistence that there are right answers to subjective opinions (that there is "best number" that is better than all the others) plays out the rationalization that defends against the possibility of the unknown in mathematics. However, when teaching, Sheldon's reversion to dynamics of power and authority over his helpless students becomes a manic defence behind which the authority of mathematics justifies Sheldon's poor pedagogy (his answers are always right even though his teaching is aggressive and alienating). I conclude that the characterization of mathematics in television is a false binary of competence/social inadequacy that limits our understanding of mathematically gifted students' unconscious struggles for social and intellectual acceptance in which manic and phobic defences might coexist.

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AN IN-SERVICE MATHEMATICS TEACHER'S TPACK DEVELOPMENT IN ASSESSMENT THROUGH MATHEMATICS COACHING

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Teachers' knowledge of technology integration is essential for effective teaching; hence teachers should be equipped with knowledge of not only pedagogy and content but also technology. Niess, Sadri, and Lee (2007) proposed a model to describe a development of TPACK for teaching mathematics. This developmental model of TPACK is based on a sequential process which moves through the stages of recognizing, accepting, adapting, exploring, and advancing. The aim of this study is to analyze the development of elementary a mathematics teacher's TPACK according to the levels defined by Niess, Sadri, and Lee (2007) in the TPACK Development Model for the assessment descriptor.

Researchers collected qualitative data from elementary mathematics teachers to determine the development of TPACK in geometry during mathematics coaching as a professional development. Mathematics coaching included three phases: pre-conferencing, observation, and post-conferencing. In this study, the teacher designed his mathematics lessons with the guidance of researchers during the pre-conference. In the observation session, researchers observed the implementation of the lesson to determine how the teacher integrated technology in the assessment of students' knowledge. In the post conference, the teacher reflected on the strengths and weaknesses of his instruction. During the four weeks of coaching, the teacher's TPACK levels were assessed. The findings indicated that the teacher demonstrated growth within the TPACK development model in terms of the assessment descriptor. More specifically, at the beginning of the implementation, the teacher did not use Geogebra to assess students' understanding of polygons although he used Geogebra during the teaching and learning process. He resisted the use of technology in the assessment process. During mathematics coaching the teacher moved to a higher level of the TPACK development model and integrated Geogebra into the assessment process. At the end of the coaching process, he designed assessment tools to reveal students' understanding of geometrical ideas using appropriate technology that extended beyond the paper and pencil questions.

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AN INVESTIGATION OF 7TH AND 8TH GRADE STUDENTS' REASONING AND MISCONCEPTIONS IN ORDERING DECIMALS

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Decimal is a key concept for later algebra learning. However, many students and even teachers encounter some difficulties with this concept (Steinle & Stacey, 2004). The purpose of this study is to examine 7th and 8th grade students' achievement, solution strategies and misconceptions in a task involving 2 questions related to ordering of decimals. The participants were 24 7th graders and 32 8th graders enrolled in a public middle school in the capital city of Turkey. These students were given a paper pencil task which included two questions related to ordering of decimals which were taken from the previous studies. The first question required students to decide on the bigger decimal between the given four decimal pairs. The second question included four decimals to be ordered from the smallest to the largest. Results regarding the achievement of the 7th and 8th graders indicated that participants' achievement varied between 50.0% and 70.0%. The findings also showed that students used three main strategies for ordering the decimals. The first strategy was converting the decimal to fraction and using previous knowledge in fraction concept while ordering. The second strategy was based on the knowledge of place value in such a way that the numbers at the same digits are compared beginning from the tenths and moving to the hundredths and so forth. The last strategy was to equate the number of digits of the decimals by adding zeros to the end and comparing the decimal part of numbers as if they were whole numbers. Findings regarding students' misconceptions centered on selecting the decimal with longer digit as bigger than the decimal with shorter digits. Another misconception was ignoring the zeros at the beginning of decimal portion without a consideration of the function of zero as a place holder in decimals. We could infer based on these misconceptions that students treat decimals as whole numbers and overgeneralize the properties of whole numbers to the decimals (Durkin, Rittle-Johnson, 2015). In order to empower students' decimal thinking and overcome their misconceptions, teachers might use multiple representations and focus on the concept of place value.

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TEACHERS' ORCHESTRATION OF MATHEMATICS

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BACKGROUND

In England, students are usually allocated to mathematics classes (sets) on the basis of prior attainment, and lessons for low attainers tend to have distinctive features that may inhibit progress. Through the development of an observation framework that foregrounds the mathematics made available to students and the pedagogical moves made by the teacher to bring this about, this study aims to develop an understanding of how teachers orchestrate mathematics for different groups of students.

ORCHESTRATION OF MATHEMATICS FRAMEWORK (OMF)

Variation theory has been used to draw together mathematically significant features, such as the use of multiple representations, multiple solution strategies and the sequencing of examples (Marton & Pang, 2006). This structures the analysis of tasks to identify the mathematics made visible to students. Cobb, Gresalfi, and Hodge's (2009) interpretative framework, involving classroom norms and normative identity, has been drawn on to interpret classroom interactions. Crucially for this study, these constructs can take account of individual student contributions whilst maintaining the focus on the teacher. Building on the notion of hypothetical learning trajectories, these dimensions have been synthesised into the OMF, where teachers' planning, activities and assessment of student reasoning form an interrelated network.

RESEARCH DESIGN

A pilot study has been conducted, based on three pre-existing lesson videos, to test the efficacy of the OMF. Distinctive features emerged; for example, the lack of systematic variation in a teacher's examples was reflected in the restrictive way the students tackled an open task. The mathematics made visible was further limited when classroom norms indicated there was no expectation that students should engage with peers' solutions. The main study will be a multiple case study, with sets taught by each teacher forming nested cases. It is anticipated that the OMF will offer a powerful window on the mathematics made visible and the associated pedagogical practice of teachers, so that discernible shifts in practice between classes of different attainment could be observed.

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INVESTIGATING PROSPECTIVE MATHEMATICS TEACHERS' LESSON PLANNING SKILLS IN A CASE OF DATA ANALYSIS

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Knowledge of designing “effective” mathematics lessons is an important aspect of mathematics teachers’ pedagogical content knowledge (PCK), particularly knowledge of content and teaching (KCT) (e.g., Ball, Thames, & Phelps, 2008). Mathematics method courses should be considered as a starting point for learning how to plan effective mathematics lessons. The aim of this study was to investigate prospective mathematics teachers’ (PTs) lesson planning skills during a mathematics method course, which was offered to PTs who would become middle grade teachers (in grade 5-8). 47 PTs in their third year enrolled in this course. The course was taught by one of the researchers during fourteen weeks in the fall semester of 2016. During the semester, PTs were engaged in learning about how to plan effective mathematics lessons, middle school mathematics curriculum and various effective teaching strategies specifically regarding to teaching algebra, geometry and data analysis. After learning how to plan effective mathematics lessons, PTs were divided into three groups (16 PTs in each group) based on three strand areas (geometry, algebra, data analysis) and assigned to submit a lesson plan in their assigned strand area in groups of four. After their first submissions, PTs were provided feedback based on a rubric developed by the researchers according to the literature (e.g., Ball et al. 2008; Smith & Stein, 1998) and asked to revise their submissions accordingly. The rubric consisted of five dimensions with each dimension was divided into four hierarchical levels. These dimensions were: (1) cognitive levels of tasks, (2) knowledge of student thinking, (3) teacher and student role, (4) assessment, and (5) connections among parts of lesson plan. The same revision process was followed for the other two submissions as well. PTs were also interviewed as groups after each submission. The data for this study came from lesson plans and focus group interviews with 16 PTs who worked in data analysis strand area. The preliminary analysis of the study revealed that PTs' lesson planning skills changed progressively throughout the semester, especially in the dimensions of “cognitive levels of tasks” and “teacher and student role”. In the third lesson plan, PTs started to use the real life tasks and student-centered activities, which were for data summarizing, representing, and making interpretation, by using appropriate graphs and charts. The results suggest that the method course should provide PTs with multiple opportunities for exploration and discussion, systematic feedback, and contexts for collaboration.

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PROVING ABILITIES OF PRE-SERVICE MIDDLE SCHOOL MATHEMATICS TEACHERS: THE FUNCTION CONCEPT CASE

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Proving has been studied within various frameworks from elementary to university grades through recent years. Proving ability is closely related with usage of mathematical concepts, logic, mathematical language, and also problem solving ability, hence analysing of proving ability is quite complex. The proof construction is also viewed as a problem solving task (Weber, 2001). When considering the investigation of pre-service teachers' proving abilities' importance and scarcity of studies on investigation of proving abilities via problem solving frameworks, the purpose of this study which is to investigate proving abilities of pre-service middle school mathematics teachers in the context of one variable function concept within Carlson and Bloom's (2005) problem solving framework gains importance. This is a qualitatively designed study in which fourteen participants were selected from thirty pre-service middle school mathematics teachers attending Analysis II course to conduct clinical interviews. In selection of the participants, a prepared test which consisted of open-ended questions on one variable functions was carried out to evaluate their conceptions of function. Obtained data from test and clinical interviews was analysed by using Miles and Huberman's (1994) three-phase qualitative data analysis method. According to results, it was seen that the participants could understand the proving tasks about whether a given relation was a function or not but were not be able to prove them because of the lack of strategic knowledge (Weber, 2001) and inadequate use of mathematical language (Moore, 1994). Moreover, it was seen that the participants considered various representations of the relation (e.g. graph), when they had difficulties in proving process. In addition, the participants were able to assess that their constructions were not a mathematical proof.

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THE VIEWS OF PRE-SERVICE TEACHERS ABOUT THEIR PROFESSIONAL SELF-EFFICACY IN A DYNAMIC LEARNING ENVIRONMENT

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The purpose of this study is to investigate the effect of GeoGebra software on the views of the pre-service mathematics teachers regarding their professional self-efficacy. 23 pre-service teachers in the Department of Mathematics Education of a state university in Turkey constituted the participants of this study which was conducted with single-case (holistic) design (Yin, 2003). Before the data collection tool was implemented, the mathematics concepts were taught in a dynamic learning environment by using GeoGebra software for 14 weeks that included a two-hour lecture per week. While studying the software and the mathematics concepts, the pre-service teachers were encouraged to prepare dynamic materials which they could use in learning and teaching. An open-ended questionnaire which was developed by the researchers with reference to the literature (Bars, 2016; Tschannen-Moran, Hoy, & Hoy, 1998) was used as the data collection tool. The data were analysed by using the content analysis technique. As a result of the study, the pre-service teachers indicated that GeoGebra software positively affected the attention of the students at the beginning of the lecture increased students' motivations, active learning and their effective use of time in the classroom. The fact that the software has an important effect on teachers' self-efficacy with regard to the managing of the classroom, encouraging student participation and the effective use of education strategies was evident in the views of the pre-service teachers.

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AN EDUCATIONAL EXPERIENCE OF CULTURAL TRANSPOSITION IN PRIMARY SCHOOL: PROBLEMS WITH VARIATION

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Since several years, politicians, researchers and educators from different countries are debating on possible reasons for the success of Confucian area students in interactional assessment projects (such as PISA). Starting from the study of teaching practices developed in this East area, in Italy we developed an educational research based on the process of *cultural transposition* (Mellone & Ramploud, 2015). With this term we mean the process of change that develops considering two or more cultural-educational backgrounds with the aim to maintain their differences without “translating” them from one culture to another, but rather highlighting these in order to review their meaning processes and daily use in classrooms. In this article we present two educational experiences developed in a second and fifth grades Italian classroom on the cultural transposition of the *pictorial equations approach* of the Chinese curriculum inserted in the typical structure of Chinese word problems, called *problems with variation* (Bartolini Bussi et al., 2013). According to Cai and Knuth (2011) in China this symbolic representation is used since the early school years as a possible support for “constructing” a bridge to the “informal algebra”. In the Russian curriculum it is possible to find an analogous approach called *intermediate strategies of graphic representation* (Davydov, 1982) referred to similar pictorial equations. Our experience of cultural transposition in the Italian context have shown the potentialities of the use of problems with variation as well as the crucial role of the figural equation.

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LANGUAGE AS A RESOURCE IN SECOND LANGUAGE MATHEMATICS CLASSROOMS: A COMPARATIVE ANALYSIS

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There is now an established body of work looking at the learning and teaching of mathematics in contexts of language diversity (e.g. Barwell et al., 2016), although there is little research that examines the relationship between patterns of language use and specific forms of language diversity. In much of this research, the use of more than one language as seen as productive, rather than as a problem, characterised by the conceptualisation of language as a resource (e.g. Planas & Setati-Phakeng, 2014). In recent work, I have drawn on a Bakhtinian perspective to argue that language use in mathematics classrooms includes three related forms of resource (Busch, 2014): multiple discourses, multiple voices and multiple languages. In this communication, I present preliminary results of a comparative analysis of language as a resource in second language mathematics classroom using this framework, with the goal of understanding how patterns of language use relate to the particular form of second language setting.

In 2008-2012, I conducted an ethnographic study of mathematics learning in four different second language settings in Canada. Settings focused on second language learners of English or French in a mainstream mathematics classroom, in a class of indigenous students, in a class of immigrant students and in a French immersion class. Data included fieldnotes, audio recordings of classroom interaction, interviews with students and teachers and copies of students' work. These data were analysed to compare the use of multiple discourses, multiple voices and multiple languages in the different settings. The results show the significance of multiple voices in learning mathematics in second language settings, while institutional constraints mean that multiple languages, while present, are not always the most significant form of resource. These findings highlight the significance of contextual and institutional factors in understanding language diversity in mathematics classrooms.

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RELATIONSHIP BETWEEN TURKISH MATHEMATICS TEACHERS' VIEWS ABOUT COLLABORATION AND THEIR PERCEIVED CONFIDENCE IN TEACHING

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Research has provided persuasive evidence that effective teaching is a critical school related factor to determine student learning outcomes. Teachers' beliefs about and attitudes towards teaching and learning of mathematics have the capacity to influence their classroom practices (Pajares, 1992). Confidence, as a psychological construct, is expressed as a dimension of attitude. Researchers found that teachers' confidence in teaching mathematics is associated with their practices in the classroom (Beswick et al., 2006), and high levels of pre-service elementary teachers' confidence in teaching mathematics is associated with decreased mathematics anxiety levels (Brady & Bowd, 2005). However, many teachers felt unconfident about their mathematics teaching skills concerning several issues involved in the school mathematics curriculum (Beswick et al., 2006). It is, therefore, important to sustain the development of teachers' confidence to teach mathematics as part of their continuous professional development. Lam et al., (2002) contended that collaboration among teachers is one of the most effective means of developing effective teaching. To this end, in this study we aimed to investigate the relationship between mathematics teachers' views about their level of collaboration and their perceived confidence in teaching mathematics in Turkish school context. Several preliminary (ANOVA and t-test) and inferential statistics (multiple regression) were employed, using Trends in International Mathematics and Science Study (TIMSS) data collected in 2011. The results indicated that mathematics teachers' collaboration to improve their teaching was a significant predictor of their confidence in teaching mathematics. Mathematics teachers who collaborated more frequently were more likely to improve their confidence to teach.

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RELATIONS BETWEEN LINGUISTIC FEATURES AND DIFFICULTY OF PISA TASKS IN DIFFERENT LANGUAGES

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The results of international comparative studies are discussed broadly in media and are often used to influence public opinion. To solve mathematical PISA tasks, students have to read and understand the task text. Since different natural languages have different inherent properties, a task translated between languages might vary in difficulty. Readability formulas often connect task difficulty to word length and sentence length (Lenzner, 2014). In addition, both word length and sentence length often differ between different natural languages. For example, “the bus station” can be written as “busstationen” in Swedish, making a longer word but a shorter sentence by word count. This type of differences also exists in PISA tasks (e.g., Puchhammer, 2007). This study is part of an overarching project examining the relation between the language used in mathematical tasks and both the tasks’ difficulty and demand of reading ability. In this study the research question is: How do word length and sentence length for mathematics PISA tasks correlate with task difficulty in different languages?

English (USA), German, and Swedish language versions of 83 mathematical tasks of the PISA 2012 assessment were analyzed. Five variables were determined: average word length measured both as letters per word and syllables per word, number of words longer than 6 letters, number of words longer than 1 syllable, and average sentence length measured as words per sentence. Using student results, the solution frequency was used to determine task difficulty. Correlations between each length variable and task difficulty in each language version were calculated.

The analysis showed no significant correlations between any length variable and difficulty in any language. One reason that differences between the languages do not result in differences regarding the correlations, could be that the students at this age are used to the specific features of their own language. Also, assuming that the length variables are indicators of other sources of difficulty (e.g., complex sentence structure and unfamiliar words), for the length variables to correlate significantly with difficulty, it might be necessary to use different cut-off values in different languages. For example, words with more than 6 letters in one language, but more than 8 in another. Therefore, further studies regarding differences between languages are needed.

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WHY IS LEARNING VIA CREATIVE REASONING EFFECTIVE?

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When students get to practice mathematics tasks in a creative way (meaning that they are not given any solution procedures or algorithms) they significantly outperform students that practiced in a more traditional way (meaning that they are given algorithms or formulas for the solutions) (Jonsson et al., 2014). The students of first type of are called CMR-students (CMR stands for Creative Mathematical Reasoning), and the second group AR-students (AR stands for Algorithmic Reasoning). In the Jonsson et al. (2014) study, both groups got the same practice tasks with the difference that no solution procedures were presented to the CMR-students. The test tasks were identical between the two groups and the difference in performance was statistically significant in favor of the CMR-students. In the LICR research program (Learning by Imitative and Creative Reasoning) we are starting to get insights in several aspects of how students learn by creative reasoning, but there still are many aspects remaining to explore. One aspect is that we need more evidence concerning *why* the CMR-students are more successful. It could be that they actually learn the content better, that they have to work harder to solve on the practice tasks (a higher cognitive load), or that they during the practice session become more familiar with the task format (they know directly that they have to think).

The purpose of this study is to get more information about what the students are doing during the test. What are their strategies when engaging in the test tasks? Are the AR-students trying to remember the formula from the practice session? Are the CMR-students directly aware of a general strategy? What similarities and differences can be found when comparing AR- and CMR-students? The setup of the study is identical to the one used by Jonsson et al. (2014), with the exception that the test session was now carried out under a video camera in a think-aloud situation. A researcher was present to assure that the students described their solution attempts including their strategies for solving the tasks. The analysis builds on the theory of hypothetical learning trajectories (Simon & Tzur, 2004).

The study will be carried out in March and April 2016 and the results of the study will be a central part of the presentation. We have very tentative indications from a pilot study where it seems that familiarity with the task format might be central to the differences between the two student groups.

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POTENTIAL OF DIFFERENT MATHEMATICAL PROOF TYPES

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Proving is an important but challenging activity of compulsory mathematics education (cf. CCSI, 2012). It is often done with less students' participation although the students' involvement into the dialogue and their active participation is important too from a perspective of learning and enculturation. There are different types of proof with a different potential for the involvement and active participation of the students. Harel and Sowder (1998) distinguish between an empirical and an analytical scheme of proof beside an external conviction. This distinction is realized and expanded by Wittmann (2009), as well. He describes an empirical, a formal-deductive and an operative proof. These types of proof are related to different levels of abstraction and forms or modes of argument representation (Stylianides, 2007). Formal-deductive proofs call for cogent reasoning and inferences. The relation between the premises and the conclusion or a generalization of particular instances is fixed in a symbolic language. Operative proofs, by contrast, make use of a constructive approach that works with enactive or iconic means, and can hence be assigned to a medium level of abstraction. This makes it easier and more straightforward to gain an insight into the mathematical rationale behind a certain inference or formula. Experimental arguments rely on particular examples without striving for abstraction or generalization. Each of the three types of proof has a different potential for student participation and/or letting them work independently. From the experimental proof to the formal-deductive one we can describe an increase of abstraction on the one hand and on the other a decrease of participation. This is the reason we should focus on dialogue in relation to the specific content. Because of their high degree of formalization, formal-deductive proofs require a relatively large degree of teacher support and, as a consequence, do not provide much scope for independent work, whereas operational and, in particular, experimental approaches facilitate student activity considerably.

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ENGAGING MATHEMATICS THROUGH APPS: HOW MIGHT THEY INFLUENCE THE LEARNING?

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This oral communication reports on an aspect of a larger research project examining the ways iPad apps were used in three mathematics classrooms with 7-11 year olds. It focuses on the interplay between the affordances of mobile technologies and asks how the assemblage of social and digital elements (e.g., Meyer, 2015) might influence mathematical learning in particular ways. The glass interface of an iPad presents a haptic affordance that enables direct interaction with the phenomena (Sinclair & Heyd-Metzuyanim, 2014). *Math Shake*, an example of an app that uses screen casting, the digital recording of the computer screen, introduces a further affordance. It allows students to record mathematical processes, and so create a multi-representational recording of their strategies and solutions.

The project used an interpretive methodology involving the collaborative analysis and critical reflection of classroom practice. In this report we present findings from interviews with groups of students that show their perceptions of using *Math Shake*.

Students made references to the features and affordances of the app: *You can record your learning and you can see what stage you are working.* There was also a reflective aspect to this approach: *It helped to solve my problem...you can record and pause and think about what you're saying.* Some students noted specific instances of learning: *I learnt how to use the reversing strategy on the number line.* The opportunity to record their voices whilst writing and drawing seemed important: *It is hard to explain without writing down. You can write it down as well as explaining it while you're recording.* As one student said: *It's just like making a movie for maths.*

The screen-casting feature of the *Math Shake* app was seen to enable the students to create a dynamic aural-visual representation. The recordings might be considered as socio-technical assemblages that in turn influence the perspectives of those that view them, as well as those that compile them. Further study of these assemblages, and how they might influence the learning experience, is needed to understand their potential in reshaping the learning of mathematics.

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TURKISH STUDENTS' MATHEMATICS ACHIEVEMENT: THE PREDICTIVE POWER OF MOTIVATIONAL BELIEFS

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The purpose of the present study is to investigate the trend of the predictive power of Turkish eight grade students' motivational beliefs for determining mathematics achievement based on Trends in Mathematics and Science Studies (TIMSSs) in 2007 and 2011. Students' motivational beliefs were measured through the student background questionnaires in TIMSSs. However, as it is usual, the content of the questionnaire in 2007 was revised in 2011. Only eight motivational items were determined as common. The psychometric properties (factor analysis, reliability of factors) of these common motivational items were investigated. Same items were loaded into two factors in both of the data sets. These two motivational factors - students' confidence in mathematics (SCM) and students valuing mathematics (SVM) – were associated to the expectancy-value theory as a theoretical background of the study (Wigfield & Eccles, 2000; Wigfield, Tonks & Klauda, 2009). SCM reflects students' perceptions about their ability in mathematics and perceptions of task difficulty and SVM refers to whether students perceive mathematics achievement as advantageous to their future education (Martin, Mullis & Foy, 2008). These two factors represent the expectancy aspect and utility value aspect respectively in expectancy-value theory (Wigfield & Eccles, 2000). The data were obtained from the 4498 and 6928 Turkish eight grade students in 2007 and 2011 respectively. Descriptive analyses of the factors showed that the majority of the students (85.9% in 2007; 75.4% in 2011) placed high value on mathematics. However, there is no common pattern for confidence in mathematics. In other words, while most of the Turkish students perceive mathematics as important for their lives, they are not so confident in their mathematical ability. The multiple regression analyses were then used to examine the relative contribution of SCM and SVM toward the explanation of mathematics achievement. Accordingly, 27% of variances in mathematics achievement were explained by students motivational beliefs in both databases ($F_{2007}(2, 4172)=816.15$, $F_{2011}(2,6617)=1228.54$, $p<.001$). Individual contributions of the factors showed that SCM has a stronger predictive power than SVM for students' mathematics achievement scores in TIMSS 2007 and TIMSS 2011. The possible explanations related with these results will be discussed during the presentation based on the expectancy-value theory (Wigfield, et al., 2009)

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DOES READING PURPOSE OF GEOMETRY TEXTS INFLUENCE 7TH GRADERS LEARNING GEOMETRY BY READING?

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Studies (e.g. Bråten & Samuelstuen, 2004; Narvaez, van den Broek, & Ruiz, 1999) show that reading purpose provides students a clue to pay attention to the relevant segments, retrieve the suitable schemata, and strategically organize the new and old knowledge. Since geometry content involves not only visualization but also informal and formal reasoning, it makes geometry learning more complicated.

In order to investigate students' approach to learning geometry, we conducted one reading experiment to examine how middle school students learn geometry by reading texts with different given reading purposes. Participants were 252 7th graders (avg. 13.1 years), randomly assigned to the 4 groups, one control and three experimental groups with separate reading purposes: defining, proving, and problem solving. In each group, students have to read the uniform texts of one geometry topic introducing the properties of intersectional angles of parallel lines, which they have not learned yet, and then to complete the test of this denoted topic. The texts and test are composed of three sequential contents: (1) concepts: definition of angles, (2) justification: reasoning and validation of various properties, and (3) problem solving.

The results showed that there is no significant effect of reading purposes on the *whole* test, $F(3, 248) = 1.27$, $p = .290$, $\eta^2 = .015$, and *problem solving* test, $F(3, 248) = 0.24$, $p = .886$, $\eta^2 = .000$, but marginal effect on *concept* test, $F(3, 248) = 2.11$, $p = .099$, $\eta^2 = .025$, and *justification* test, $F(3, 248) = 2.30$, $p = .078$, $\eta^2 = .027$. The further post hoc comparison on concept test reveals that those receiving reading purpose of problem solving ($M = .69$) performed significantly greater than those of proving ($M = .60$), $p = .036$, and marginally greater than those of defining ($M = .61$), $p = .064$. Regarding to justification test, it reveals that those receiving reading purpose of problem solving ($M = .46$) performed significantly greater than those of proving ($M = .34$), $p = .024$; those receiving reading purpose of defining ($M = .44$) performed marginally greater than those of proving ($M = .34$), $p = .058$. The results expose that emphasizing problem solving seems not the most efficient approach to learning geometry by reading. In future study, it is worth investigating the effective instruction of proving and defining.

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AN EXPLORING OF FIRST GRADE STUDENTS' ATTITUDE TOWARD MATHEMATICS IN CLASSROOM USING OPEN APPROACH

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This study aimed to explore students' attitude toward mathematics in classroom using Open Approach. Target group was 41 first grade students in 2014 academic year. This classroom taught by student teacher who studied at Mathematics Education Program and had teaching practice at school for a year. He taught mathematics by using Open Approach based on Inprasitha's framework; posing open –ended question, students learning by themselves, whole class discussion and comparison and summarized by connecting students' ideas. Data collected by using questionnaire and interviewing students. Data analysis based on attitude framework of Hannula (2002); the emotions that the students experiences during mathematics related activities; the emotion that the student automatically associates with the concept mathematics; evaluation of situations that the student expects to follow as a consequence of doing mathematics; and the value of mathematics-related goals in the student's global goal structure.

The result revealed that students strongly agreed with 12 items such as I feel good when I try to solve a problem and I could complete it; mathematical activities make me exited to do mathematics; I enjoy to participate in mathematical activity; I like to solve mathematics problem with various material. They agreed with the items such as I feel proud to have the opportunity to explain my idea within a group. They moderately agreed with the items such as I feel that the group would not accept the idea that I presented.

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VALIDATING DESIGN PRINCIPLES OF DIAGNOSTIC CONJECTURING TEACHING

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Mathematical conjecturing and diagnostic teaching can be integrated by the proceduralized refutation model as an intermediate framework (Chen, Lin, Hsu, & Cheng, 2014). This kind of teaching approach called diagnostic conjecturing teaching (DCT). The DCT facilitates students to simultaneously develop mathematical conjecturing competence and mathematical concept understanding.

In this study, we further constructed design principles of DCT, which contained: (1) entry principle highlights the opportunity for students to engage in mathematical situation contained a false proposition or misconception with some supportive examples in order to elicit their relevant views; (2) test principle points the opportunity for students to generate examples purposefully in order to support or oppose the false proposition or misconception; (3) discrimination principle concerns the opportunity for students to discern both supportive and opposite examples systematically in order to find a common property among supportive or opposite examples; (4) revision principle emphasis the opportunity for students to modify the false proposition or misconception in order to correct and express it; (5) verification principle stresses the opportunity for students to provide argumentation in order to verify revised proposition (or concept). These design principles of DCT had be valided theoretically and empirically. In design-based professional development, how to facilitate teachers to design specific teaching for promoting their students' performance is crucial. We argued that DCT not only facilitates students in developing the competence of conjecturing and argumentation, but also helps students to enhance mathematical concept understanding by actively coordinating examples, concepts and results. Similarly, the design principles of DCT not only scaffold teachers easily to develop the competence of designing, testing and revising of teaching, but also promotes teachers actively to learn professional knowledge by enactment and reflection, such as pedagogical content knowledge about conjecturing and diagnostic teaching. Moreover, the design principles of DCT helps teachers to changes their views of teaching, such as students' misconceptions can be resources of teaching, students' active participation and active thinking can be goals of teaching. But, the DCT also had some limitations and weaknesses.

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AN EXPLORATIVE STUDY ON TEACHERS' CONCEPTION AND INTENTION OF MATHEMATICS READING

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“Enhancing mathematics reading comprehension ability of young students” is now a new movement of mathematics curriculum for the next decade in Taiwan. In such a steam, teachers’ *conception* and *intention* of math-reading then play a crucial role in school math teaching. Here, conception means the teachers’ understanding of the function of math-reading as a learning strategy, and intention means the teachers’ pedagogical aim when they use reading strategy in math teaching.

This study interview sixty-six elementary and junior high school teachers. All of them engage in the master degree program of mathematics education. The interview includes two tasks. The first task asks the teachers to read an article of unfamiliar proof of Pythagorean Theorem and then asks them to answer the question “what questions do you think the professor will ask you to answer in this article?” The second question asks them to read the proof of the property of angles of a transversal cuts parallel lines in textbook and then asks them to answer the question “what questions will you ask your students to answer when they read this article?” The teachers will show their understanding of the function of math-reading as a learning strategy in the first task, and their pedagogical aim when they use reading strategy in math teaching in the second task. The teachers’ response are coded and analysed in two dimensions. One is the *width* of mathematics content, which categorised into concepts, properties and reasoning. The other is *depth* of learning function, which categorised according to the clusters of PISA into reproduction, connection and reflection (Stacey, 2012).

The results show that more than 90% of teachers perform well in the first task both width and depth. That is, the teachers realize the learning function of math-reading in math learning. However, although the width in the second task is well performed, the depth is averagely one level lower than the first task. Almost all teachers’ intention only focuses on reproduction and connection level. The explanation of teachers shows that most teachers, according to their successful experience of learning mathematics, regard math-reading as a kind of self-effort work. They think the young students have not yet enough competencies to do it well, because their knowledge of mathematical language and math reasoning strategy is poor.

This study shows Taiwanese teaches understand the function of math-reading well. But they choose the low level challenge when implement math-reading in school teaching because they concern students’ learning loading more than possible learning function.

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REPRESENTATIONAL FLEXIBILITY AND ADAPTIVE EXPERTISE IN THE SOLUTION OF ARITHMETIC PROBLEMS

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The study of *adaptive* vs *routine* expertise (Hatano, 2003) in the context of whole number concepts and operations continues to be a dominant theme in research (Verschaffel, Greer & De Corte, 2007). While the notion of adaptive expertise has been examined from several perspectives, Greer (2003: 697) suggested that ‘there is a clear connection between degree of flexibility and Hatano’s distinction between routine expertise and adaptive expertise’. We conceptualise *flexibility* in terms of *representation*, and argue that strategy choice for an operation is a function of the nature of problem representation. A representational notion of flexibility also affords a stronger operational base to visualise the linking of procedural and conceptual knowledge – an important aspect of development of adaptive expertise (Baroody, 2013). Following this line of reasoning, in this preliminary study, we aim to contribute to the field of adaptive expertise by analysing evidence of representational flexibility among a cohort of 26 Malaysian children as they attempted to solve one-step 2-digit addition and subtraction word problems. Results of this design study showed that the children tended to launch into an algorithmic solution to both problem types with limited effort invested in representing the problems. An algorithmic approach might be the quickest strategy to solve the problem (regardless of correctness of outcome) but provided limited insight into connectedness of their knowledge (Lawson & Chinnappan, 2015) that girds adaptivity. The results of the study has implications for examining the development of adaptive expertise in the context a learning culture that is promoted in Malaysian mathematics classrooms.

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THE PHYSICAL CONCEPT AND THE INDETERMINATE TOOL

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The role of tools and their relation to mathematical learning remains a significant field of study in mathematics education. To better understand the entanglement of tools, humans and concepts, I draw on inclusive materialism (de Freitas & Sinclair, 2014) to gather insights and challenge representationalist ways of knowing. An inclusive materialism posits that the tool comprises an integral part of the “assemblage” from which mathematics “emerges”. A concept, therefore, is not purely abstract but partakes of the physical world. Contrary to assumptions about mathematical concepts as ideal, inert, Platonic forms, the inclusive materialist framework sees the mathematical concept originating from the animation of a life force in which temporality and mobility are key aspects.

In this study, high school students were using the open-ended exploratory environment of Geometer’s Sketchpad (Jackiw, 2001) to construct various mathematical objects. The question of this research explored how a mathematical concept was entangled with the students, particularly when the tool was unavailable. One student, who had constructed a “well-constructed” square (when it is dragged by its vertex it will retain its shape), was asked to elaborate what it means that the square moves, something she had written in her notes. At first she was quiet, then mumbled inaudibly and finally put her arms out in front of her as if she were grabbing and turning a steering wheel.

The student’s gesturing into the space before her is not a representation of her idea of square, but a narrative of movement that involves improvisation. In this way, the square participates in the same material plane as the student and the tool, emerging as it does out of a fundamental mobility that is actualised in the wheel-turning gesture. If one takes away the material tool, the concept is affected (a square constructed from compass and straight-edge has no business turning). This gesture is a re-making that changes her and the square yet again—this is the sense in which the square is always becoming and always inscribed in particular material, mobile conditions. This raises the question of whether every new explanation, movement, gesture gives rise to a new concept. If each new description, movement, gesture re-makes the concept, then the concept is never finished, never exhausted.

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DEVELOPING THEORETICAL CONSTRUCTS FOR STUDYING MATHEMATICS TEACHER REFLECTION IN TAIWAN

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The idea that teacher reflection is a component of teaching expertise is well documented. The importance of teacher reflection, as theoretical construct for understanding teaching, extends well beyond the knowledge base for teaching. The theoretical constructs for studying the reflection of pre-service teachers should not be identical to those for in-service teachers. This study aims to develop a theoretical construct for studying the reflection of pre-service mathematics teachers in Taiwan.

The initial constructs involved six components: (1) factors causing reflection (why), including noticing (Sherin, 2011), mathematics teaching-related knowledge, and disposition (Schön, 1983); (2) occasions for reflection (when), including preparing lesson plan, conducting lesson, and observing others' teaching; (3) content of reflection (what), including teaching skills, teacher-student interaction, classroom management, teaching materials, and evaluation; (4) people causing reflection (who), including mentors, school administrators, students, other teachers, and peers; (5) sites reflection occurred (where), including backing in universities, in classroom teaching, and workshop sittings; (6) effects of reflection, including productive or non-productive changes of behaviours, attitude, and competences, as well as no changes. Some of the items mentioned had various parts, for examples, noticing involved attending to particular events and making sense of events; and disposition whole-heartedness, student-centred (directness), open-mindedness, and responsibility (Rodgers, 2002).

This study is an ongoing case study employing the idea of natural inquiry with intense fieldwork and interviews. The sample includes three pre-service high school mathematics teachers currently (will have more). Initial findings have suggested the major parts of the constructs, such as noticing, knowledge, and teacher-student interaction. Other finding includes (1) knowledge base is critical to productive reflection; (2) "making sense of events" plays a more prominent role than "attending to events" in changing of behaviours; (3) student-centred disposition frequently prompts productive reflection.

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THE EFFECTIVENESS OF TEACHING NUMBER THEORY IN PRIMARY AND SECONDARY SCHOOLS IN HUNGARY

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Our research is a part of a larger survey on the status of Number Theory in Hungarian elementary- and highschool education. The initial motivation of our research is an observation that the first-year math students at Eötvös Loránd University claim not remember basic notions of number theory.

We overviewed the Hungarian National Core Curriculum and interviewed many teachers and students about what is taught in Hungarian primary and secondary schools. Number theory appears every year in primary school, for example in grade 2 with the division algorithm, in grade 6 for the addition of fractions they need to find the least common denominator. After grade 9 curriculum does not contain any number theory. According to the interviews, everything is taught from number theory included in the curriculum. Therefore, it is even more surprising that math students do not remember that.

Hence, we started to investigate what is behind the (non) knowledge of first-year math students. Other research projects examine the knowledge of university students (e.g. Zazkis & Campbell, 2006) and the importance of number theory (Ball, 1990), as well.

Our first goal was to make a survey about what primary and secondary school students remember about what they were taught from number theory. We did not care about their actual knowledge, only what they remembered to study from this topic. We put together a questionnaire asking if they have heard about basic notions of the area. The 16 questions covered a wide range of lexical knowledge. We have the tests filled in all over the country by 1231 students from ordinary and vocational highschools and universities. Our hypotheses were the following: grade 10 students' knowledge is still fresh and remember what they learnt. Grade 12 students have already forgot what they learnt at primary school and in grade 9.

Then using hierarchical cluster analysis, k-means method and Tukey's method, we compared the answers to the questionnaires by age and by type of study/school. The following were proved: the class of students who is the most diverse from the others are the grade 12 students who did not study extensive mathematics. They remembered the less. The grade 10 pupils and grade 12 extensive mathematics pupils' results were very good and similar to each other.

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COPING WITH THE ERRONEOUS USE OF THE INTUITIVE RULE “MORE A – MORE B” USING MATHEMATICAL CASES

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The present study investigates fifth-grade students' use of the intuitive rule *more A - more B* (Tirosh & Stavy, 1999) in tasks related to the sum of interior angles of polygons. Furthermore, to cope with the erroneous use of the rule, we used mathematical cases. Doing so, we examined whether the use of these cases influenced students' tendency to answer according to the intuitive rule *more A - more B*.

Thirty five fifth-grade students, whose previous learning included the sum of the interior angles of a triangle and the sum of the interior angles of a quadrilateral, participated in the research. We created two questionnaires: a comparison questionnaire and a case questionnaire, both about the sum of the interior angles of a polygon. The questions in the two questionnaires present systems of polygons in which one salient property (area) is greater in one polygon than in another. Students were asked to compare the polygons with respect to the sum of their interior angles. The questionnaires were distributed to the participants, and a week later they answered the case questionnaire. After yet another week, the students were asked to answer the comparison questionnaire again.

We used frequencies and percentages to analyze the research data quantitatively. The research findings indicate that when solving comparison tasks related to the sum of the interior angles in polygons, students tend to depend on the intuitive rule *more A - more B*. When doing so, students considered more than one salient feature of the polygon, such as its area and the number of its obtuse interior angles. Presentation of the mathematical cases to the students influenced them only temporarily, indicating that the mathematical cases did not lead to a cognitive conflict (Mason, 2001), or that the cognitive conflict did not lead to a conceptual change (Dekkers & Thijs, 1998).

It is concluded that if we were to introduce the students to a sequence of mathematical cases instead of just one, it might influence them in a more lasting manner. Furthermore, students' use of intuitive rules in a given task may be determined by the interaction of various factors, including their previous formal knowledge (Tirosh & Stavy, 1999).

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LOW ACHIEVING STUDENTS' ROUTINES IN LEARNING THE EQUIVALENCE CONCEPTS

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The mathematics education of low-achieving students has attracted educators' attention for decades. To support the mathematics learning of these students, one of the recommendations is to conduct a classroom environment conducive to learning (Leone, Wilson, & Mulcahy, 2010). This can be done, among other things, by giving students authentic activities and dynamic tools, and, at the same time, by maintaining effective teaching, for example through questions. In the present study, we followed these principles by giving low-achieving students authentic activities related to equivalence relations. Specifically, the students worked with an applet for equivalence concepts and relations; issues that have been indicated as critical to algebra (e.g., Stacey & Chick, 2004).

Using Sfard's commognitive Framework (2008) with its four components of mathematical discourse (words, visual mediators, narratives and routines), we analysed the evolution of the mathematical discourse of two pairs of 16-year-old low achieving students as they worked in a conducive environment characterized by its authentic activities, its dynamic tools and its effective teaching. The main research question is: what are the characteristics of low achieving students' routines in the course of learning the equivalence concepts in a conducive environment?

We observed that the pair of low achieving students used a sequence of routines: teacher's request, students' actions with the applet, students' construction of a narrative, teacher's questioning and students' substantiation of the narrative. It was also observed that a learning environment that combines teacher's initiation and questioning, technology and authentic activities support low achieving students' routines for arriving at mathematical narratives. It seems that the teacher's initiation of students' construction and of substantiation routines was a prompt for them to follow routines that supported their successful construction of equivalence narratives. The applet allowed them to perform actions that supported their construction of the equivalence narratives.

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WHAT DOES SUCCESS MEAN?: APPROACHING FROM THE POINT OF PROOF AND PROBLEM SOLVING

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Proof is asserted to be the process, including two sub-processes of ascertaining and persuading, employed by the individuals to remove or create doubts about the truth of an assertion (Harel & Sowder, 1998). Being the promotion of mathematical understanding, proof is seen as an important part of doing mathematics among mathematics educators (Hanna, 2000). However, can academic achievement and the examinations assessing this achievement be regarded as the indicators of students' proof skills? To put it another way, are the academically successful students also successful in proving? These questions led us the way for planning this study. As problem solving is also one of the fundamental aspects of mathematical thinking, this study adopts two main aims: (i) to examine the problem solving and proving processes of successful students, and (ii) to reveal the relationship between problem solving and proving skills of the students. The participants of this qualitative study are 153 eighth and 173 ninth grade students having 15 or more right answers out of 20 questions in the national exam conducted in Turkey. The data collection tool includes three problem-based questions asking students to prove and justify three different problems. The answers of the students were analyzed in four different views of point for each of the three problems: (i) the truth of the answer, (ii) problem solving strategies, (iii) representations used, and (iv) proof processes of the students. The findings of the analyzed data revealed that most of the students have difficulties in proving and problem solving. Although the questions additionally asked for an additional solution, the students answering the questions correctly did not provide an alternative solution to the problems. The students use a limited number of strategies, most of which include trial-and-error approach. The students mostly see the proof as "validation". They hardly justify their claims and mostly present externally formed proofs, in other words they have authoritarian proof schemes as they base their justifications upon the books, teachers or the rules. Their use of representations is limited to the verbal or numerical ones rather than multiple representations. The students mostly do not have difficulties in understanding the problem but most of them did not come up with a right solution for the questions asked.

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NETS OF THE PYRAMIDS: FROM THE VIEW OF THEORY OF FIGURAL CONCEPTS

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When it comes to geometry, concepts and images are two undetachable elements of the instruction. Fischbein (1993) asserts that geometrical figures have both conceptual and spatial representations. Geometrical figures' conceptual side focus on the generality, essentiality, abstractness and ideality, while the spatial facet focus on the shape, distance and position. This is what and why Fischbein (1993) refers to as “figural concepts”. This bridge between the concept and image may cause some difficulties for students to have a geometric understanding of solid shapes as much as it may cause for the 2D shapes (Mesquita, 1998). In addition to conceptualizing the plane representations of 3D shapes, students have also difficulties in recognizing and constructing the nets of those shapes. In this context, adapting the figural concept term to the 3D shapes, this study aims to investigate the thinking processes of ninth grade students about the nets of the pyramids. The participants of this study are 15 students formed by including five students from each of high, medium and low achieving group. The data collected through clinical interviews using a questionnaire with four open-ended questions, each of which asks for the net of specific pyramids, i.e. an equilateral triangle-based pyramid, a triangular pyramid except an equilateral base, a square pyramid and a square pyramid having non-congruent faces. All the interviews except one were videotaped. According to the findings, it has been found that the students mostly have a prototype view of the pyramid, which include a regular pyramid with a square or triangle base. Most of the students drew the nets of the first three pyramids, but they had difficulties imagining the square pyramid with non-congruent faces. However, what's interesting is that the students attempted to draw the net of this pyramid, eventhough they thought such pyramid as “impossible”. When asked if this net constitutes the requested pyramid, the students mostly expressed that “the top of the pyramid would be open” trying to close the faces at the centre of the square base. Only four of the students figured out that this last pyramid could be oblique. Although they had a little bit difficulty thinking of the second pyramid, since it was a triangular one they were able to imagine it easier than the last one. One of the striking findings of the study is that, although nearly the half of the students noticed that the corresponding sides of the triangles which form the faces should be equal to have a pyramid so that they can concurrently meet at the top corner, they neglected other conditions required to have such a pyramid such as the length of sides or the interior angles of the faces.

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TRUMATH IN ESPANOL: EVALUATING LESSON QUALITY AFTER PD IN CHILEAN MATHEMATICS CLASSROOMS

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Chile compares poorly to other countries in the TEDS-M international comparative study on teacher education (Kaiser & Blömeke, 2013). It is likely the limitations of this initial education contribute to extremely traditional teaching in the primary mathematics classroom; in which problem solving methodologies are rare. An eight month professional development program was developed to promote learning mathematics through problem solving, with students learning in peer groups and the teacher in the role of facilitator. The teachers who completed the 2015 program reported very high levels of satisfaction with the course and espoused new goals for their teaching. However it was not clear from these evaluations whether there were changes or improvements evident in their regular mathematics lessons. The goal of the research reported here was to measure whether there was an observable increase in quality of mathematics teaching subsequent to the professional development.

A sample of twelve teachers was randomly selected to provide a range of data, including video recorded mathematics classes at the start and end of the program. The lessons were observed by the authors and coded using a Spanish language adaptation of the TRU math rubric (Schoenfeld, 2013).

At the time of writing data from only four of the twelve teachers have been completely analysed. However initial results suggest that adoption of a new pedagogy within regular mathematics lessons is not automatic. More often the scores of lesson quality were similar pre and post PD. However the TRU math rubric provided an informative analysis of typical practices in Chilean classrooms. For example the consistencies in scores regarding “access” speak to the importance of maintaining classroom control for these teachers, sometimes at the expense of quality teaching. Low scores in “uses of evaluation” were a particular concern. The initial results suggest the need to make more explicit during PD which aspects of a problem solving pedagogy may be applicable generally, in order to promote transferral of these ideals to other lessons.

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EVEN TIE PROBLEMS ARE NOT ALL RETRIEVED

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Students' ability to efficiently perform basic arithmetic operations has been reintroduced as a key goal of early mathematics instruction, encouraging to go beyond "routine expertise" (National Council of Teachers of Mathematics, 2000). Moreover, recent studies question the supposed retrieval level in the arithmetic acquisition, even in adults and for very small problems (Fayol & Thevenot, 2012). The only problems that are consensually considered as being solved by retrieval are ties, leading to the so-called "tie-effect" (solution times to ties are faster than expected given their size), reason why numerous studies exclude them from their tasks or analysis. But some data on adults also suggest that small (operand from 2 to 5) and large (operand from 6 to 9) ties are not equally processed (Campbell & Gunter, 2002). Our study, conducted on a group of 58 typically developing children exposed to single digit additions, aims at testing whether all single ties are mostly solved by retrieval among children at two times of the 2nd grade.

Our repeated measures ANOVAs show that small ties are solved more quickly than small non-ties, but that it is not the case for larger problems. Our analyses also show that more errors occur on large problems when they are ties, whereas they give rise to fewer counting strategies than large non-ties, which altogether suggest that large ties are "guessed" rather than retrieved. A qualitative analysis of these errors shows that from $8+8$, errors mainly result in one or more steps above the target. Therefore, our study together with previous findings suggests that at this time of the acquisition, very few problems are really retrieved. This supports the recommendations to explicitly teach and train conceptual understanding of operations as well as fluent execution of procedures instead of pursuing much direct memorization.

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PROBLEM SOLVING AND ARGUMENTATIVE SKILLS IN MULTICULTURAL ITALIAN CLASSROOM

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In the last decades the population in Italian classrooms has become increasingly more multicultural. Even if the law guarantees to all students the access to school, the gap between Italian students and migrant ones is becoming more and more evident in the results of national surveys in Italian Language and Mathematics and in the high dropout rates. In wide spectrum of Italian research studies, to date, a great deal of attention has been focussed on integration processes and on the subject of interculturalism. Relatively little, however, has been dedicated to specifically didactic and disciplinary matters.

A preliminary investigation was carried out to highlight the main difficulties faced in mathematics by these students, particularly related to problem solving and argumentation. The research aimed therefore to the development and testing of an educational intervention that, taking into account cultural and linguistic differences, aims at strengthening the aforementioned skills. It was tested in an experimental research design with a control group. The sample comprises 453 fourth and fifth graders. The intervention took place once a week, for twenty sessions of two hours. All activities comprising the intervention aim to: reinforce problem solving by developing cognitive processes (memory, comprehension, reasoning, creativity and critical thinking) (Coggi, 2015); encourage comprehension, representation, categorization, planning, monitoring, identification of multiple solution strategies; develop argumentative skills, both as a strategy for communicating knowledge and also for the purposes of self-clarification (Boero et al. 2008). For this purpose, both quantitative and qualitative analysis was conducted (pre-post-test specifically constructed, video recordings, interviews with teachers). Results showed that argumentation in multicultural classes can become the moment in which it's possible to share different reasoning and solving strategies, in order to constitute an enrichment for all, Italians and immigrants.

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HIERARCHICAL AND NON-HIERARCHICAL CLUSTERING METHODS TO ANALYSE AN OPEN-ENDED QUESTIONNAIRE ON ALGEBRAIC THINKING

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In recent years, some papers have tried to develop detailed models of the reasoning competences of the student populations tested, or to subdivide a sample of students into intellectually similar subgroups, by using quantitative or qualitative analysis methods. It is worth noting that research papers using quantitative analysis methods to study student responses to open-ended questionnaire can be found in Science and Physics education (Springuel et al., 2007), but the same cannot be said for research in Mathematics education. In this paper we focus on the application of hierarchical and non-hierarchical clustering methods referred to *dendrograms* and *k-means* approaches (Everitt, et al., 2011), trying to make sense to answers given by 118 Tenth Grade Italian students to six open-ended questions on algebraic thinking. In particular we discuss the results on the study of typical students behaviour in tackling the algebraic resolution of word problems and, at the same time, at understanding how the student semantically and syntactically control questions containing symbolic algebraic expressions (Radford & Puig, 2007).

The two methods (K-means and dendrograms) both allowed us to partition and characterize our student sample, without making any a priori assumptions and giving as output student's behaviour interesting for the researcher in Education. The first method identified 3 groups of students, the second one 5. The results we found are largely coherent with the ones already reported in the literature obtained by means of qualitative methods. For this reason, we can consider the use of both hierarchical and non-hierarchical clustering a valid tool to complement the use of qualitative analysis to study a large number of students with respect to the way they give answer to the questionnaire.

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METAPHORING AND ENACTING IN MATH EDUCATION

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We argue that an approach to mathematics teaching based on *metaphoring* and *enacting* (Soto-Andrade, 2014; Proulx, 2009) may significantly help in alleviating the cognitive abuse millions of children worldwide suffer when exposed to mathematics. Indeed we cognize through metaphoring and enacting, processes shaped by a long evolutionary history that are usually thwarted by traditional teaching. In mathematics education metaphors appear as powerful cognitive tools, that help us in grasping or building new concepts, as well as in solving problems in efficient and friendly ways. Enaction (introduced as “learning by doing” by Bruner), connotes bringing forth a world by concrete handling. Here mathematical strategies emerge continually in the interaction of solver and problematic situation.

We exemplify our approach for mathematical problem solving. Our methodology is qualitative, using didactical engineering in case studies involving primary and secondary students, in service primary and secondary school teachers experimenting in their classrooms, as well as prospective secondary school teachers and first year social sciences and humanities university students, in 2014-2015. Learners, working in groups of 2 to 4, were observed by the facilitator and an assistant. Snapshots of their written output, videos of their enacting and transcripts of turning points in work sessions were made. Data processing involves contrasting a priori and a posteriori analyses. We prompted learners to metaphorize and enact first when tackling problems like: 1. What about the sum of the exterior angles of a polygon? 2. Partition a square in four equal shares in four different ways! 3. Figuring out probabilities for repeated coloured ball drawings without replacement from a composite urn.

In general, we observed that group enacting facilitated insight. In 1, enacting various metaphors for a polygon enabled students and teachers to *see* and *feel* that the requested sum is a whole turn. In 2, metaphoring (even unconsciously) and enacting different ways of carrying out the partition process facilitated the emergence of a wide spectrum of solutions. Our results highlight the big difference in understanding that a metaphoric and enactive approach can make, compared with “blind” calculations. Enacting entails making explicit choices (as in 3. above, when storing drawn balls) that allow students to notice hitherto unperceived aspects of the situation. Moreover the way they enact determines the ideas that may emerge.

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ASSESSING KINDERGARTEN CHILDREN'S KNOWLEDGE OF REPEATING PATTERNS: TEACHERS' CHOICES

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For the past several years, we have been providing professional development for preschool teachers aimed at promoting their knowledge and self-efficacy for teaching mathematics in preschool (e.g., Tirosh, Tsamir, Levenson, Tabach, & Barkai, 2015). This presentation reports on a group of 43 preschool teachers who participated in a section of the program devoted to patterning concepts. The aims of this study are (1) to investigate the types of tasks teachers choose to implement when assessing kindergarten children's knowledge of repeating patterns, and (2) to investigate the reasons behind teachers' task choices (e.g., Do teachers focus on the structure of the pattern or the task activity?)

Teachers were presented with the following six repeating pattern tasks and asked to choose three tasks they would use to assess kindergarten's children knowledge of repeating patterns, and explain their choices.

1. Complete the pattern: □○△ □○△ □○△ _____
2. Complete the pattern: □□□○○○□□□○○○□□□○○○□ _____
3. Find the mistake in the following repeating pattern.

4. Find the mistake in the following repeating pattern.

5. Is the following drawing an example of a repeating pattern? ☺△☺♥☺△☺♥☺△☺♥☺△☺♥
6. Is the following drawing an example of a repeating pattern? ☺△☺♥☺♥☺♥☺♥☺♥☺♥☺♥☺♥

Results are presented in Table 1. The most frequent reason for choosing a task was the type of elements used in the pattern. Teachers did not mention the structure of the pattern.

Task activity	Extend the pattern		Find the mistake		Is this a pattern?	
Pattern structure	Task 1	Task 2	Task 3	Task 4	Task 5	Task 6
	ABC	AAABBB	ABC	AAABBB	ABCD	ABAB
Freq. (%)	33 (77)	31 (72)	13 (30)	18 (42)	16 (37)	9 (21)

Table 1: Frequency of teachers’ choices (%) for each task (N=43)

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PRESERVICE TEACHERS USE OF LESSON STUDY IN THE PLANNING OF A PROBLEM SOLVING LESSON

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Lesson study (LS) is an acknowledged form of professional development for teachers, with historical roots in Japan. The idea of lesson study is simple: collaborating with fellow teachers to plan, observe, and reflect on lessons (Takahashi & Yoshida, 2004). Regarding research lessons, Lewis (2000) has highlighted five characteristics: (1) research lessons are observed by others; (2) research lessons are planned for a long time, usually collaboratively; (3) research lessons are designed to bring life in a lesson, a particular goal or vision of education; (4) research lessons are recorded; and (5) research lessons are discussed. Generally, lesson study research has looked at work with in-service teachers and not so much with pre-service teachers.

In this case study, three pre-service teachers (PSTs) with no prior experience with lesson study, worked together to plan a problem solving lesson for a grade three class. This paper documents how the teachers selected the problem, planned the lesson, delivered the lesson, and reflected on the videotaped lessons, focusing more on the implemented lesson. The focus will be on the following questions: (1) how were the PSTs initiated to LS? ; (2) what were some of the issues that the PSTs faced and how did they cope with them? ; and (3) how did the PSTs approach the planning of the research lesson and the delivery and post-lesson reflection?

PSTs are learning how to teach and do not necessarily demonstrate the same level of expertise as established in-service teachers in LS. The PSTs in this case study demonstrated a good understanding of the basics of the LS approach to plan, implement and reflect on the problem solving lesson. The presentation will also highlight some issues about the viability of LS in an already packed in-service teacher education programme.

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COLLECTING, ANALYSING, AND REPORTING, TEACHER'S PRACTICE SELF-REPORT DATA

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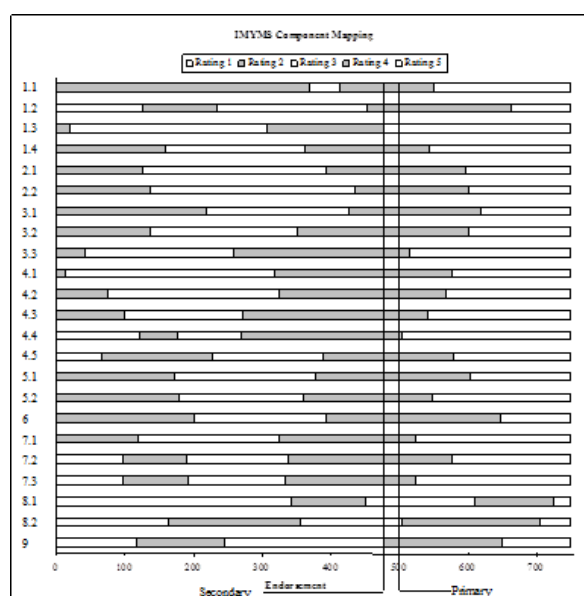
INTRODUCTION

The aim of the *Improving Middle Years Mathematics and Science* (IMYMS) project was to investigate the rôle of mathematics and science subject cultures in mediating change processes in the middle years of schooling.

IMYMS used Component Mapping (CM) as the key tool for capturing teacher practice. The IMYMS Component Map centred around an interview between the school IMYMS co-ordinator and individual teachers. Each sub-Component is discussed, and teacher and interviewer agree on a score out of 5, that represents the degree of exemplification of that sub-Component in the teacher's practice. In all, there were nine major components, with all but two having sub-components: altogether 23 components and sub-components. In total, 83 Primary teachers and 80 Secondary mathematics teachers were interviewed and their Component Maps completed.

A Partial Credit analysis was performed which provided information on the quality of the CM statements as well as on the overall position of respondents with respect to the focus of the CM interview.

An important aspect of a research project is to communicate its findings, and in this case, the results of the Component Mapping analysis were transformed into a visual mapping following the procedures developed by Doig and Groves (2006). This transformation preserves the information but provides an easy to understand format for the lay reader.



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WHICH TEXTUAL FEATURES ARE DIFFICULT WHEN READING AND SOLVING MATHEMATICAL TASKS?

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In this study a combination of statistical and qualitative methods are used to explore the potential role the presence of, and interaction between, different semiotic resources have for how difficult mathematics tasks are to read and solve. The semiotic resources of interest are natural language, mathematical notation, and different types of images. Two different dependent variables are used: one that explains a general task difficulty and one that explains the tasks demand on a general reading ability. T-tests have revealed that tasks with four particular combinations of semiotic resources are solved to a significantly lower frequency than other tasks. Moreover, chi-square tests reveal that the same tasks are overrepresented in the group of tasks for which a general reading ability is not beneficial to use in the solving process. The results of those statistical tests do however only contribute an understanding of what presence of and co-occurrences of particular semiotic resources mean for the reading and solving of mathematics tasks.

The second step in this study is therefore a qualitative analysis of a few tasks from two particular groups of tasks identified in the statistical analyses, namely i) tasks that are more difficult to solve *and* for which a general reading ability is not beneficial to use, and ii) tasks for which a general reading ability is highly beneficial to use in the solving process. The purpose of the qualitative analysis is to explore relations within and between the semiotic resources in tasks that are identified as extremes in the statistical tests (the ones mentioned above). Therefore the method for the qualitative analyses is based on theory about texture (Liu & O'Halloran, 2009) and Kress' concepts translation and transduction (Kress, 2010). The results will contribute to an understanding of the role that multisemiotic features in task text have for aspects of task difficulty.

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COMPARING MATHEMATICS TASKS IN DIFFERENT LANGUAGES

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We report on progress towards the construction of a framework of linguistic properties that can be used to compare versions of mathematics test tasks in different natural languages. Where differences in item functioning (DIF) have been found in test tasks, our goal is to identify if and how specific language properties are causing the DIF. The study is based on the ideas that mathematics and mathematical communication cannot be separated; that language is functional, having evolved along with our human needs, including the specific need of communicating mathematically (Halliday, 2004); and that different languages have different inherent properties for expressing mathematics.

Our first step has been to compile a list of language properties which might affect the reading and/or mathematical difficulty of the task, based on our own previous research and preliminary literature review. The initial list of properties is being refined with respect to the results of a structured literature review investigating connections between the text properties and task difficulty. For each property to be included in the framework, the review needs to show empirical evidence of difficulty which may affect task performance. We are also making textual comparisons of PISA mathematics tasks in several languages (English, Swedish, German and Spanish) to see which properties vary between language versions. Information about each property in the framework will include methods used to measure the property, how the property is connected to aspects of difficulty, and relevance for mathematical tasks.

The framework will be structured to enable analytic comparison of mathematics task versions in different languages. The structure will align with the metafunctions of Systemic Functional Linguistics in terms of what is special to mathematics texts (ideational), what is special to test tasks (interpersonal), and what the generic language features are (textual) (Matthiessen & Halliday, 2009). We will present examples of how a language function may be expressed through different language properties in different languages. A language property may also perform different functions in different languages and in different contexts. Besides being of use to researchers to explain DIF, the framework may also be of use to writers of tests by providing advice on how to formulate mathematical tasks that take account of what we know about how language properties are connected to difficulty, do not have unnecessary demand of reading ability, and can be well-translated.

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MATHEMATICS TEACHERS AND TEACHING ASSISTANTS: DEVELOPING EFFECTIVE CLASSROOM ENVIRONMENTS

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INTRODUCTION

The professional development of practising mathematics teachers is widely researched internationally (for example, Ponte and Chapman, 2008). In England, there is another set of classroom professionals who support both teachers and students in the secondary school classroom. The professionals in these support roles are named in various ways, depending on their specific roles – teaching assistants, learning support assistants or special needs assistants. While there exists some extensive research on the role, deployment, and management of teaching assistants and their impact on student learning (for example, Blatchford et al 2009, Muijs and Reynolds 2003), there is very little research on how teachers and teaching assistants proactively work together, and with pupils, in mathematics classrooms in secondary schools to co-construct an effective learning environment.

THE STUDY

Within the framework of a community of practice, this study explored the relationships between mathematics teachers and their teaching assistants through three in-depth embedded case studies. Using interviews with each of the case participants separately and then jointly, and detailed recordings of minute-by minute observations of relative positioning in the classroom of the mathematics teacher and the teaching assistant, we built up a set of factors which contribute to a positive partnership between the mathematics teacher and the teaching assistant. We present these factors as an interrelated network for one of the in-depth case studies as a model of how we approached all three case studies to ultimately develop a professional development tool for increasing effectiveness of impact on students' mathematical learning.

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THE DEVELOPMENT OF CREATIVITY THROUGH HEURISTIC STRATEGIES

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We have dealt with the development of creativity within a teaching experiment conducted with 62 Czech pupils aged 12–18 years, lasting for a period of 16 months. The aim of the experiment was to find out if pupils, within the time period given, are to learn to use selected heuristic strategies in problem solving. We have also followed to analyse the development of some individual's characteristics determining their ability to solve problems. The most valuable results have been achieved with creativity.

The key role of creativity in problem solving has been discussed by many authors. Nadjafikhah, Yaftian, and Bakhshalizadeh (2012) stress the role of creativity while successful problem solving. By contrast, Silver (1997) shows that inquiry-oriented mathematics instructions which include problem solving and problem posing may develop pupils' creativity.

The research question was: Will active mastery of heuristic strategies cause a significant increase in pupils' creativity?

The level of creativity was measured using Guilford test. After the experiment had finished, pupils showed a significant increase in the creativity index. On average, the index increases up to 2.4 times of the original. The psychologists conducting this survey claim that the above mentioned increases are considerably higher than what the natural increase related to the increase of the age of the pupils would indicate. A more detailed analysis indicates that the most significant growth can be observed within our research sample in the area of fluency and flexibility.

We assume one of the reasons for the increase in creativity may be attributed to the manner in which pupils worked when solving problems using heuristic strategies. The pupils were not only asked to find the result, they had to not only choose a suitable strategy, but also select an effective mode of its usage – arithmetical, algebraic or graphical mode.

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CONCEPT IMAGES' RELATED TO STUDENTS' DIFFICULTIES WITH REGARD TO DENSITY OF IRRATIONAL NUMBERS

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A concept image is the whole of the cognitive structures that are formed in an individual's mind in relation to a mathematical concept (Tall & Vinner, 1981). The irrational number that can be explained by different expressions can also be envisaged in the students' mind. When teachers talk about 'irrational number', some expressions come to the students' mind (such as π , square root, non-periodic decimal, not a fraction and so on). By considering that irrational number representations in the students' concept images can cause difficulties, in this research, it was tried to be determined that the disabilities of 8th grade students related to density of irrational numbers and the effect of irrational number concept images on these disabilities.

For this purpose, "Irrational Number Concept Test" which was composed of open-ended questions was applied to 58 eighth-grade students and semi-structured interviews were made with 4 of them. The students were divided into 3 groups according to their images: *CI1. Decimal Representation*, *CI2. Non-rational*, *CI3. Square Root*. It was observed that more than half of the students thought that rational and irrational number sets were equal or rational numbers were much more. In addition, most of the students stated that there might not be a number between any two irrational numbers. It was also seen that the students' different irrational number concept images had also an effect on these mistakes. The students who have the concept image of "Decimal Representation" could create a new number by changing the digits in the decimal part. This provided them to act comfortably while finding numbers between any two irrational numbers. The students who have the concept image of "Non-rational" thought that irrational numbers cannot be written in fractions but rational numbers can be written in any way. So, according to them, rational numbers were much more than irrational ones. The students, who have root numbers in their concept images, stated that irrational numbers were much more because of the fewness of perfect square numbers. However, it was observed that they thought through the narrow concept of square root numbers while finding numbers between any two irrational numbers.

Consequently, it was seen that the students have some difficulties about the density of the irrational numbers and the students' different irrational number concept images had also an effect on these difficulties.

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IMPORTANCE OF SPRALITY IN TEACHING NUMBER THEORY

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ELTE

In the frame of a larger research we investigated the knowledge of number theory of Hungarian high school pupils concerning the most important basic notions, like notion of primes, divisibility rules, divisibility properties, etc.

By our previous result it was shown that Hungarian students finishing high school do not remember the notions of number theory. We have shown that pupils of grade 10 and math students at the university have similar knowledge and are much better than students of grade 12. We composed a quiz about what pupils remember from number theory. To confirm, we created a test (with practical exercises) covering 7 key areas of number theory. We have evaluated the deviations between solution of quiz and exercises. In this paper we would like to mention the three most important questions of the framework of number theory: fundamental theorem of arithmetic, greatest common deviser, the least common multiple.

Studies have shown that these notions are essential for example at adding fractions (Ball, 1990). Our questions were the following:

Questions	Tests	Exercises
Divide primes multiplied by the following number: 420!	78%	72%
What is the greatest common divisor in 1320 and 504?	91%	46%
How much is the least common multiple of 8 and 12?	97%	54%

In the table the first column shows how many students claimed to remember the corresponding knowledge, in the second one we showed how many of them answered the question properly.

The Hungarian curriculum is very large, there is not enough time in high school to repeat materials. Therefore, the fundamental tool, spirality is broken. Number theory is not repeated in higher classes. So our main goal was to find problems, examples that facilitate refreshing the application of number theory in different areas like in geometry, analytic geometry, analysis, etc. For example, the question $2 \cdot \lg 3 + \lg 11 = 2$ can be checked by a pocket calculator or applying the logarithmic identities. But how about the following, longer one:

$$3 \cdot \lg 3 + \lg 7 + \lg 11 + \lg 13 + \lg 37 + \lg 101 + \lg 9901 = 12?$$

Entering these expressions into a pocket calculator or the Windows calculator the two numbers look equal. Using the definition of logarithm and the fundamental theorem of arithmetic it can be decided as a snap.

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LEARNING TO NOTICE STUDENTS' MATHEMATICAL THINKING: SOME CHARACTERISTICS

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Jacobs, Lamb, and Philipp (2010) conceptualize professional noticing of children's mathematical thinking as three interrelated skills, *attending* to student's strategies, *interpreting* student's mathematical thinking, and *deciding how to respond* on the basis of student's understandings. From this perspective, research has focused on how prospective teachers learn to notice students' mathematical thinking in different mathematical domains. This study focuses on how prospective teachers learn to notice students' mathematical thinking in relation to the limit concept in a learning environment designed ad hoc that progressively nests the three interrelated skills of professional noticing. The participants were 25 prospective secondary school teachers. The learning environment consisted in 5 sessions of two hours, where prospective teachers had to: (i) solve three problems related to the limit concept in order to unpack the important mathematical elements; (ii) anticipate students' answers to these problems reflecting different characteristics of conceptual development; (iii) analyze a set of high school students' answers to the limit problems; and (iv) propose new activities to help students to progress in their understanding.

Results show that after participation in the learning environment, prospective teachers gained expertise in the three component skills since 20 prospective teachers were able to interpret the students' mathematical thinking linking students' answers with the important mathematical elements of the dynamic conception of limit and provide specific activities to help students progress in their conceptual understanding. The different ways in which prospective teachers engage in unpacking the students' mathematical thinking for deciding the new appropriate activities show that prospective teachers need to develop efficient ways of attending the relation between the mathematical elements and the characteristics of student's understanding of the limit concept.

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ANALYSING EMERGING AND DEVELOPING STRUCTURE IN NONSTANDARD ARITHMETICAL REPRESENTATIONS

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Visuospatial representations (VRs) of quantities and their relations are widely used in school mathematics, and learners struggling with arithmetical thinking often employ drawn imagery, concrete manipulatives, and gestural/embodied representations in nonstandard ways, particularly when working on unfamiliar scenario-based tasks. Much of past research on VRs has used broad categorisations. However, Karsenty et al (2007), amongst others, have found a more descriptive approach to analysis of pupil-created graphics and models to be particularly effective for identifying mathematical misconceptions and partial understandings. The aim of this research was to develop a framework for rigorous qualitative microanalysis of pupil-created VRs, allowing for comparison across a number of different aspects, and in particular, the identification of emerging and developing arithmetical structure in learners' thinking.

Data derived from 13 linked microgenetic case studies of low-attaining pupils aged 11-15, who each participated in a series of five individual interactive 'Piagetian' problem-solving interviews. Each of these included multiple arithmetical problems designed to be nonroutine and challenging for these participants. Coloured pens, plain paper, and multilink cubes were provided. A semi-grounded multi-stage approach was taken to analysis, with a bidirectional ongoing relationship between theory and data: analytical aspects were allowed to emerge from patterns detected in the multimodal data examined, and the developing framework repeatedly tested against further data.

Educationally significant structural changes were detected in the representational and metarepresentational strategies, and thus arithmetical thinking, of all participants over the course of the interviews. It was found that the VRs could be compared and contrasted in the necessary level of detail via a set of interlinked aspects: *visual (media, mode, resemblance)*, *spatial (motion, unitariness, spatial structuring)*, *numerical (enumeration, completeness, success)* and *interactive (consistency, errors, verbal/visuospatial prompts)*, which might have differing rates of change or orders of causality in different pupil/task instances. The complete framework is explained fully in the presentation, illustrated with examples from the data to demonstrate its utility for research and pedagogical practice, and which highlight key moments of individual mathematical microprogression.

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MATHEMATICAL LITERACY (NUMERACY) AND TEACHERS

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Numeracy is the capacity to make effective use of mathematics in contexts related to personal life, the workplace, and in exercising civil responsibilities. (Geiger, Forgasz, & Goos, 2015, p. 611)

There is increasing recognition internationally of the need to be numerate (mathematically literate). Australia is among countries with a curricular expectation that all teachers (not only mathematics teachers) will develop students' numeracy skills across curriculum areas. It is already expected that teacher education programs prepare all graduates to "[K]now and understand literacy and numeracy teaching strategies and their application in teaching areas" (AITSL, n.d.). This numeracy expectation will soon be tested (and must be passed) prior to graduation. Little is known, however, about the numeracy skill levels of already practicing teachers. Exploring this was the focus of the study reported here.

The model for numeracy in the twenty-first century (see Geiger et al., 2015) served as the theoretical framework for items included in an online survey. Teachers (not just teachers of mathematics) at all grade levels from around the world were recruited via Facebook (see Forgasz, Leder, & Tan, 2015 for details on how this is done) to complete the online survey.

In all, 1198 teachers from 16 countries responded; the majority came from Australia, Canada, India, UK, and USA. Overall, 81% were female, 77% were aged over 40, 62% were elementary teachers (26% secondary, 11% 'other'), 57% had over 15 years teaching experience, 18% did not teach mathematics, and 48% had studied tertiary level mathematics. Participants were asked if they believed there were differences between mathematics and numeracy and explain their response. They were also asked to complete items drawn from the publicly released PISA items and the 2010 grade 9 (Australian) National Assessment Program: Literacy and Numeracy [NAPLAN] test.

In this session, we present findings on the primary and secondary teachers' views on perceived differences between mathematics and numeracy, and their responses to one of the PISA items included in the survey – an item based on combinatorics.

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YOUNG CHILDREN'S COMPREHENSION OF THE INVERSE RELATION IN DIVISION

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Children's initial understanding of division is based on the schema of sharing, but understanding division implies understanding of the inverse relation between quotient and divisor (Correa, Nunes & Bryant, 1998). Understanding inverse relations is the "first step towards an understanding of division" (Bryant, 1997, p.65). The question arises whether hearing and deaf students' comprehension of this inverse relation would improve by an intervention (Here we present data on hearing children only). We provided feed-back in two conditions: making the child to explain her answer to a problem, or explain the interviewer's answer, which would provide cognitive conflict. We hypothesized children would improve comprehension through interventions, and those in this last condition would make more progress. Participants were 19 children in Year 1, 4 female, 18 male. Mean age=7:06 (Range 7:11-7:00), and 24 Children in Year 2, 10 female, 14 male. Mean age=8:05 (Range 8:10-8:00) who had not received formal school instruction on division. All children were submitted to a Pre-test, but only the children who did 25% or more errors participated in the Intervention -13 children (Year 1), and 9 children (Year 2)-. The task used was adapted from Correa, Nunes & Bryant (1998). Example: Eight problems about sharing carrots between rabbits invited to two parties, with the same number of carrots (dividend) allocated to each party. The number of rabbits (divisor) may be the same or different. The children had to answer if one rabbit in one party would eat more, the same, or less than one rabbit in the other. For the Intervention we used the same task but providing feedback in the two conditions explained above. The Pre-test and the Intervention took place every ten days, with each child individually. The results show that both Year 1 and 2 children seem to improve their performance through interventions and mainly between first and second session. Percentages of errors in Year 1 children were: Pretest: 35.20; 1st session: 28.36, and 2nd Session: 17.79. In Year 2 children, percentages of errors were: 14.32; 10.42 and 5.55 respectively. Regarding feed-back, explanation of experimenter's answer seems to improve understanding of the inverse relation only in Year 2, while in Year 1, explanation of own answer proved better.

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TEACHER NOTICING FOR PRODUCTIVE GEOMETRICAL REASONING

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It was Fischbein (1993, p. 144) who coined the term “productive reasoning process” whereby, in geometry education “images and concepts interact intimately”. In our ICME-10 paper (Jones, Fujita, & Kunimune, 2012) we showed how teachers use various instructional techniques and strategies to promote productive geometrical reasoning. Within a theory of teacher expertise, what a teacher ‘notices’ during classroom teaching is emerging as an important component (e.g. Sherin et al., 2011). Yet, in terms of how context-specific is noticing expertise, as Schoenfeld (2011, p. 237) queried, there are questions about whether the specificity is “the mathematics being studied”, the students’ perceived capability, the school and its “socioeconomic characteristics”, or “some combination of these, and perhaps other things”.

In our study we focused on what aspects of ‘noticing expertise’ can be identified in geometry lessons taught by an expert teacher of school mathematics at grades 7 and 8. In order to address this research question, we analysed transcribed lesson records from a sample of ten successful lessons taught by Mrs M (a teacher of 20 years’ experience). We particularly examined how she: a) attended to students’ strategies; b) interpreted students’ understanding; and c) responded on the basis of students’ understanding.

We found the following components of ‘noticing expertise’ relating to productive geometrical reasoning: a) selecting different types of solutions, including incorrect constructions; b) interacting with students who were reasoning using visual images or measurements rather than deduction; and c) and interacting with students to encourage what properties can be used to deduce a solution. Such noticing was identified in the all ten lessons. These findings illustrate the form of ‘noticing expertise’ in the context-specificity of the mathematics teacher promoting productive geometrical reasoning.

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THE FLOW OF A PROOF – RHETORICAL ASPECTS

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The ‘flow of a proof’ (Gabel and Dreyfus, 2013) encapsulates aspects of proof presentation, and is determined by a lecturer’s choices how to present the proof. We designed a method to analyse global and local aspects of the flow of a proof, adopting Perelman’s New Rhetoric (PNR, Perelman and Olbrechts-Tyteca, 1969) as theoretical framework. In this paper, we present part of a case study from a Number Theory class, focusing on a lesson where a theorem about linear Diophantine equations was formulated and proved. We analyse two aspects: the scope and organization of the argumentation, and the presence (in the PNR sense) with which the lecturer endows mathematical and didactical elements in the proof.

We present findings from classroom observations and lecturer interviews. The scope and organization analysis is based on the order, duration, transitions and interrelations of the proof modules. The presence analysis was performed on 24 elements that emerged from the lecturer post lesson interview, some mathematical and some didactical. In this paper we focus on two elements identified by the lecturer as central in a post-lesson interview: linking the linear Diophantine equation $ax+by=d$ with the $\gcd(a,b)$ and enriching the concept of $\gcd(a,b)$. We demonstrate how the lecturer endows these elements with presence by using multiple warrants and backups (as analysed using a Toulmin scheme) and other rhetorical techniques (metaphors, examples, narration style and repeating ideas in different words).

A year later, the same lecturer taught the same theorem after an intervention consisting of a researcher-led discussion. The scope and organization of the lesson changed: The lecturer added new, less formal, modules, characterized by a shared classroom discussion, in an effort to smooth transitions, clarify the proof structure and motivate the use of previously learned theorems. In addition, the lecturer endowed several ‘core’ elements with increased presence, both locally (module level) and globally (lesson level). The analysis of these changes clarifies global-local flow interactions.

In conclusion, PNR appears to be an efficient framework to discuss the rhetoric of proof presentation. Future research should examine this issue as well as use PNR to address other aspects of flow.

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SORTING ALGORITHMS FOR 5TH AND 6TH GRADE STUDENTS: GREEDY OR COOPERATIVE?

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Our main research goal lies in a proposal to discuss the lack of, and improve, activities in the Italian school curriculum about discrete mathematics, computer algorithms and cryptography, especially for 3rd to 8th grade students. Activities of this kind are missing almost entirely, both in the school programs and in textbooks, despite many agree that they can be really useful to improve *both general skills, such as reasoning and modeling, and skills particular to discrete mathematics, such as algorithmic and recursive thinking*. A survey among various grades teachers confirmed this.

The activity we are going to describe fits into a wider research project. Design research, chosen as the methodology to use, feels quite appropriate, as we are facing a brand new experience in an environment that we need to analyze carefully, i.e. on a local scale, considering all the different elements in the learning environment.

The activity exposed in our paper is part of some lessons about sorting algorithms for 5th and 6th graders. From simple ordering to more efficient algorithms (quicksort as an example), we get to sorting networks and other variants.

We focus on one single activity which is presented to students as a group task, giving them the rules, but with the goal that they find themselves the algorithm for the solution. The methodology used follows the principles of the Guided Reinvention of Mathematics and of RME (Freudenthal, 1973) and allowed us to highlight one particular and very important aspect discussed with the student: the relation between algorithms called greedy and others which are more group-oriented.

A qualitative analysis of the results, through some videos recorded in the classroom, shows, according to Vygotsky's perspective on the zone of proximal development (Vygotsky, 1981), how children, playing together, realize that some greedy algorithms might never work, if we want to achieve the group success. Some dynamics in which the game cannot finish if they seek to optimize their own result over the group result are shown.

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MATHEMATICAL AND COMPUTATIONAL MODELING OF EYE-TRACKING DATA TO PREDICT SUCCESS IN A PROBLEM SOLVING TASK

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We present a mathematical and computational analysis of the optical tracks obtained during a graph interpretation task which allows us to predict, with a high degree of certainty, whether a child succeeds in correctly solving the problem.

THE PROBLEM SOLVING TASK

113 students, approximately 8.67 years of age, from Brisbane, Australia, were shown a bar chart depicting the number of hours worked each week, by Sarah. This included a labelled coordinate system, where the x -axis has the week number labels, and the y -axis indicated the number of hours worked. Beneath the graph, was information about Sarah's hourly wage and the question, asking students to determine how much Sarah earned in Week 3. A Tobii TX300 eye tracker recorded the locus of focus of their eyes throughout the activity. The threshold for fixations was set at 100 milliseconds.

MATHEMATICAL AND COMPUTATIONAL MODEL

The graph task was partitioned into areas of interest. The areas of interest were categorised based on the relative importance in completing the problem solving task. For each optical track a sequence was obtained consisting of the areas of interest and durations (in pairs) of the fixations.

We first used the *Classify* function in *Mathematica* (a supervised machine learning function) which uses a Markov model to classify the sequences on a training set consisting of 90% of the entire data sample. This model correctly predicted the solution given by each student, with an accuracy of over 99%.

As a second approach, we have devised 6 separate models based on the area of interest sequences that students followed. These separate models predicted the results with an accuracy of 60%. When these separate models are used in combination, the accuracy jumps to 99%. Further research is required in order to extend this model so it can be used with other data from similar problem solving scenarios.

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FACTORS INFLUENCING THE DEVELOPING EFFECTIVENESS OF EARLY CAREER PRIMARY TEACHERS' MATHEMATICS TEACHING – INITIAL FINDINGS

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Students start a UK primary teacher education course with a range of academic qualifications in mathematics, a range of experiences in school mathematics and a range of attitudes and beliefs about the subject. During the course they develop their subject and pedagogical knowledge and gain experience in mathematics teaching. Students then enter a complex and changeable situation in schools in terms of provision for their ongoing development (ACME, 2015). Literature suggests that key influences on the effectiveness of mathematics teachers are their beliefs and attitudes (Ernest, 1989) and the nature of their subject knowledge (Ball et al, 2008). These in turn are impacted by the teacher's ongoing professional development.

The aim of this study is to gain a deeper understanding of how the effectiveness of early career primary school teachers' mathematics teaching develops and what impacts on this development, with a particular focus on teachers own perspective. A multiple case study approach is being employed to follow the trajectories of a sample of student teachers as they progress into their first two years of teaching. An initial interview at the end of their course focuses on their relationship with, and attitude to, mathematics and their progress in teaching the subject as a student teacher. Twice yearly interviews and discussion of evidence of their progress as early career teachers provide evidence of their ongoing development.

In the presentation I intend to describe the early findings of a comparative analysis of the trajectory to date of two of the teachers. These findings suggest that there are two related and interwoven, but distinct, categories of factors that might influence an early career teacher's trajectory in relation to the effectiveness of their mathematics teaching - those that are related to the teacher themselves, in terms of beliefs, attitudes and subject knowledge and those that are related to their teaching context.

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EXPLORING PRESERVICE TEACHERS' VISION OF PRACTICUM EXPERIENCE IN SCHOOLS

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Field experiences and teaching practice (Zeichner, 2010) have been powerful in helping preservice teachers make sense of what they are doing and seeing in the mathematics classroom. Preservice teachers (PT) report the time spent in classrooms during internships to be the most influential and useful part of their preparation programs (Guyton & McIntyre, 2010; cited in Schwartz, 2015). However, fewer studies thoroughly examine elementary mathematics PTs' beliefs and experiences of this practicum in Turkey (Eraslan, 2009). This article presents findings from research on a practice teaching course designed to help elementary preservice teachers to learn from the practices of the school teacher and the supervisor. For this reason, the aim of this study is to search the contribution of practicum course on teacher candidates' sense of teaching process.

The participants in this study were 61 preservice teachers, enrolled in a required six-hour teaching practicum course taught in the last year of an elementary teacher education program at a large state university located in the north coast of Turkey. They were sent to the placement for the purposes of observing the classroom teacher and teaching the actual class by themselves with the guidance of the school teacher for at least six hours. To answer the research question, information was primarily obtained from reflection reports prepared by the supervisor. Content analysis was made to reveal the emerging themes and codes. The results indicate that practicum have made several contribution to novice teachers view of teaching. These are preparing profession (*professional development, gain experience, motivation and desire, self confidence, teacher responsibilities, comparing teachers, outside activities*); application of theory-practice (*observing classroom, making plan, teaching, classroom and time management, using activity and material, method and techniques, evaluation*) ; knowing student (*communication, ingratiate math, student approach, individual differences, reach the level*), being aware of the difficulties (*drawing attention, making mistakes, seeing deficiency, coping with the unexpected situations*).

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COGNITIVE CONFIGURATION SOLVING ANTIDERIVATIVES

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The objective of this study is to identify and characterize the meanings that civil engineering students from two countries (Mexico and Colombia) mobilize in their mathematical practices on antiderivatives. For the purpose of evaluating the understanding of the antiderivative, a questionnaire was designed, which gathers three elements: 1) the use of diverse meanings of the antiderivative, 2) diversity of representations and 3) the mathematical relationship between the antiderivative and other mathematical objects. The Onto-semiotic Approach (OSA) to mathematical cognition and instruction (Godino, Batanero & Font, 2007) was the theoretical model set to analyze the students' answers. This model provided us with 'theoretical and methodological tools' that allowed to describe in a systematic way, the students' mathematical practices as well as the elements, and their meanings, regarding the *cognitive configuration* associated with such practices (i.e., linguistic elements, concept/definitions, propositions/properties, procedures and arguments).

As a result of this study, some relevant aspects of the community of engineering students' mathematical knowledge of the antiderivative were highlighted, for example, that the partial meaning of the antiderivative fluxions-fluents (Gordillo & Pino-Fan, *in press*) was one of the most activated meanings in their mathematical practices. On the other hand, the engineering students' deficiencies of mathematical knowledge were also evident through the questionnaire and justify the pertinence of designing specific formative actions to facilitate the understanding of the antiderivative for engineering students. Such formative actions should consider the complexity of the holistic meaning as well as the diversity of representations for this mathematical object.

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ARTEFACTS IN THE ZPD: MULTI-DIRECTIONAL ANALYSIS

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This communication focuses on a multidirectional analysis of the role of a remote control (artefact) in the emergence and sustaining of a 5 year old child (L)'s ZPD as she talks to her mother (M) about her discovery of counting in threes from noticing the layout of numbers on the remote; by saying "I am going to count in threes look (*holding up the remote control*)". ZPD is described by Vygotsky as the distance between independent problem solving and problem solving under guidance (1976). The stimulus comes from earlier writing where we shared the transcript of the child and her engagement with the remote control and her mother as she makes and extends her discovery. In that article (Graven & Lerman 2014) we talked about the bi-directionality of interactions and the emergence of the ZPD between mother and child. Abtahi (2014) responded challenging us to ask 'who/what is the more knowledgeable other in such a learning event?', and 'what was the role of the artefact throughout the interactions in the learning event?'

Taking up this challenge we provide a multidirectional analysis, with particular focus on the role of the artefact in the emergence and sustainment of the ZPD. We argue that L, at times, used the guidance provided by physical properties of the remote control and sometimes used the suggestions provided by her mother to think about counting in threes. For example, when asked by M "how did you work that out? Show me". L said: "Cause everyone is 3. Three, six, nine, twelve (As she *pointed to the unnumbered button under the 7*) and that's 11 (*points to the unnumbered button under the 8*) and that is 12 (*points to the unnumbered button under the 9*) to show her how she has worked it out. A multidirectional ZPD thus emerged as the role of the more knowledgeable other alternated between the mother and the remote control. Such a multi-directional analysis contributes to challenging bi-directional notions of the ZPD when artefacts play a key role in learning and illuminates ways in which an artefact can be usefully considered as the more knowledgeable other in such a learning event.

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PRESERVICE TEACHERS ANALYZE CHILDREN'S ANSWERS TO GEOMETRIC PATTERN PROBLEMS

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Mathematics preservice teachers' (ps-teachers hereafter) training has to include the evaluation of students' outcomes, grounded on specialized mathematical content knowledge and noticing of students' thinking.

Geometric pattern problems (gp-problems hereafter) offer students contexts where they may start handling literal symbols in meaningful ways. Gp-problems have proved to be a successful teaching methodology, implemented in ordinary schools, even from early primary grades (Rivera, 2013).

We present a teaching experiment where primary and secondary ps-teachers were asked to evaluate children's answers to gp-problems. Our research objective was to classify the justifications provided by the ps-teachers, based on certain given criteria, to support their evaluations of children's outcomes. We obtained an emergent categorization which differs from other categorizations found in the literature.

This study was based on the answers of 33 primary ps-teachers and 23 secondary ps-teachers, who had not received any previous training related to gp-problems.

The experiment had three parts: i) We informed the ps-teachers about the gp-problems, their learning aims, and the aspects of students' solutions that they were to analyze. ii) The ps-teachers solved some gp-problems, that were posed to children in grades 6 (primary) and 8 (secondary). iii) We selected some primary/secondary children's answers and asked the primary/secondary ps-teachers to analyze them. The gp-problems asked for immediate and near terms, verbal and algebraic generalizations, and a reversing process.

To analyze the ps-teachers' answers, we followed a cyclic process of identification of types of justifications, clustering of similar justifications, and refinement of the categorization. The main result was a list of types of ps-teachers' justifications.

A surprising result was the big number of wrong analysis by both primary and secondary ps-teachers, which is in the line of results by other researchers. It suggests that ps-teachers are not used to analyze children's answers to problems.

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A REVIEW OF THE COMPUTATIONAL ESTIMATION SKILLS OF PRESERVICE PRIMARY SCHOOL TEACHERS

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Estimation is a tool employed frequently in cases where the actual result eludes assessment (Rubenstein, 1986) or where an approximate result would do. In real life, individuals often need to use estimation skills to overcome a problem. Furthermore, estimations are among the means frequently utilized by specialists (i.e. scientists, engineers, mathematicians) before actual use of measurement tools (Jones & Taylor, 2010; Jones, Gardner, Taylor, Forrester & Andre, 2012). That is why it is necessary to instill skills of estimation and interpretation of a result (in terms of validity and coherence), among primary school students. Instilling estimation skills at an early age, however, requires primary school teachers to assume substantial responsibility.

This study aimed to review the computational estimation skills of preservice primary school teachers, and to see if they are able to use the correct strategy when making estimates. For this purpose, the "Estimation Skills Test" comprised of 41 questions allowing estimates using 14 distinct estimation strategies was used. 209 preservice primary school teachers took part in the study. The responses provided by the preservice teachers were scored on the basis of the accuracy of the estimation. Each response was scored in a scale of 0 to 3, and then total scores were calculated for each preservice teacher. The responses to the questions in the test were scored with reference to the applicability of the estimation method, on a binary scale of 0 and 1.

The results of the study reveals that just 2(1%) preservice teachers had "outstanding" computational estimation skills, while 81(38.8%) had "good" skills, 102 (48.8%) had "mediocre" skills, and 24(11.5%) had "poor" skills. 178(85.2%) of the preservice teachers failed to make computational estimates using the optimal strategy; just 31(14.8%) responded using applicable strategies. The study revealed that the preservice primary school teachers had mediocre computational estimation skills, and that majority of them failed to use the optimal strategy. In this context, the offering of undergraduate courses to preservice primary school teachers to improve their estimation skills may be recommended.

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THE NEED FOR CONJECTURING AND IT'S INFLUENCE ON PROCESSES OF CONSTRUCTING JUSTIFICATIONS

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The research presented here is part of a wider study in which we explore the influence of students' need for justification on processes of constructing justifications. We refer to processes of construction in the sense of Abstraction in Context (Hershkowitz et al., 2001). Boero et al. (2007) stressed that while conjecturing, students reconstruct their knowledge and improve their ability to justify and prove. We observed several aspects of students' needs for justification. For this purpose, we designed nine activities. Each activity has been carried out by at least three pairs of students as task-based interviews. For each activity, we conducted an a priori analysis, in which we attempted to determine the elements of knowledge that are expected to be necessary or useful to complete the activity, as well as the connections between these elements of knowledge. We analysed each interview by the model of the Abstraction in Context framework for exploring constructing processes (Hershkowitz et al., 2001).

Here, we exhibit the students' need for conjecturing. We present a case study of one pair of students, Ram and Etay, grade 12 in the advanced mathematics stream in a high school in Israel. The task was designed to encourage them to find the extremum of the product of two linear non-constant functions and explain and justify their way of solving the problem. The findings show that the students' need for conjecturing leads them to construct two new constructs: An algorithm for evaluating positive and negative domains of the product function, and an algorithm for examining whether the product function has extremum at its root. Using the new constructs the students formulate conjectures, evoked incorrect ones, and reached the conjecture they justified. There was no connection between constructs from the conjecturing process and constructs from the justification process. However, this doesn't detract from the importance of the conjecturing process. While conjecturing the students improved their ability to justify. Furthermore, the new constructs of knowledge helped them to maintain the process of conjecturing. Since conjecturing and justifying are inter-related, by investigating the need for conjecturing, we took another step toward exploring the influence of students' needs on processes of constructing justifications.

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CAN STRATEGY KEYS HELP 3RD AND 4TH GRADERS ENGAGE IN MATHEMATICAL PROBLEM SOLVING?

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BACKGROUND AND METHOD

Explicit instruction in heuristics is time-consuming and difficult to transfer to other tasks (Schoenfeld, 1992). Strategy keys were developed to provide an alternative way of enabling students to develop the capacity to use heuristics. This study aims at identifying affordances initiated by strategy keys and perceived by 3rd and 4th graders.

12 children (7 to 10 yrs.) have been videotaped while solving the ‘farm’ task: ‘On a farm is an open-air enclosure for chickens. In this enclosure also live rabbits. Jens stands by the fence and counts 20 animals with 70 legs in total. How many chickens are there?’. The students were provided 8 different strategy keys and asked to think aloud while working. We analysed the data using Gibson’s Theory of Affordances.

Gibson’s psychological framework enables us to systematically analyse how the keys are used by students. Gibson’s (1979) notion of affordances has been used to analyse the way in which the keys affect students’ problem solving. Gibson argued that, when a person perceives an entity in their environment – such as a chair – they perceive how the chair can be used. Environmental entities ‘afford’ opportunities for action in that, when a person perceives a chair, they also actively perceive that the chair affords an opportunity to sit or – depending on the current situation – to defend themselves. If the keys are considered as part of students’ physical environments then, by examining students’ interaction with the keys when solving problems, the affordances of the keys can be identified.

RESULTS

Across the 12 interviews, 15 incidents of interaction with the keys were identified. In all of these incidents, students did not select the first key they interacted with. So their selection of a key was not random. From data analyses we could identify five different affordances: the opportunity (a) to act as suggested on the key, (b) to consider a different mathematical perspective, (c) to name solution strategies, (d) to motivate oneself and (e) to gain thinking time. In 4 incidents of interaction no affordance was perceived.

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BELIEFS AND PRACTICES REGARDING THE ASSESSMENT OF STUDENT ACHIEVEMENTS

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THEORETICAL BACKGROUND

Academic interest in teachers' beliefs is based on the assumption that these beliefs influence their actions in class. But the nature of the connection between beliefs and practice is still a subtle and complicated issue that remains hotly contested by researchers (Beswick, 2007).

METHODOLOGY AND SELECTED FINDINGS

This study examines teachers' beliefs in the area of novice elementary (**EMT**) and secondary (**SMT**) mathematics teachers' assessment of student achievements, and the extent to which their actions correspond to their beliefs.

Two questionnaires were composed, one to check beliefs regarding assessment (beliefs questionnaire), and one to check the way teachers assess student achievements (practice questionnaire). Each questionnaire contained 77 statements designed to check six aspects of assessment, based on the assessment standards recommended by NCTM (2000): (a) purposes of assessment, (b) norm or criterion referenced test, (c) cognitive and affective aspects in the assessment process, (d) position of assessment in the teaching process, (e) attitude toward external assessment and its impact on internal assessment, (f) means of assessment. Both questionnaires had the same structure and contained nearly identical items, so as to enable a comparison between beliefs and implementation.

The two questionnaires were given to 32 novice EMT and 34 novice SMT. Each was asked to mark the measure of his/her consent to each of the statements on a scale of 4 = "absolutely agree" to 1 = "definitely do not agree" for the beliefs questionnaire, and 4 = "to a great extent/regularly" to 1 = "very little or not at all", for the practice questionnaire.

The results show that for each group a significant difference was found between their beliefs and their implementation of those beliefs. The gap is huge for SMT but smaller for EMT. These results might reflect differences in aims between elementary and secondary school teachers and the possible destructive influence of the annual external assessments held only in secondary schools.

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DISTANCE TEACHING AND LEARNING OF MATHEMATICS: EPISTEMOLOGICAL AND DIDACTICAL VALIDITY ISSUES

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Here it is setting a discussion about the main characteristics of new online educational applications for the learning and teaching of mathematics, namely MOOCs (Massive Open Online Courses), based in revisiting and applying the theoretical notions of epistemological and didactical validity of the learning computational environments to the case of distance teaching and learning of mathematics. That is supported by the analysis of new data on the resolution of mathematical complex tasks accomplished by in-service secondary teachers enrolled in a distance program for professional development on mathematics and technology.

The application of the afore mentioned theoretical concepts to the use of computational devices for the teaching and learning of mathematics has already been accomplished (see Balacheff 1994, 1999, 2004; Balacheff and Sutherland, 1994; and Sutherland and Balacheff, 1999) to illustrate different contributions that certain software has in different virtual learning environments. Now the emergence of new MOOCs (see, for example, <http://www.coursera.org>) provokes consideration of their characteristics under these concepts.

While the use of computational environments for learning mathematics raises questions of complex character from an educational point of view, particularly when the focus is centred on the teacher as an administrator or manager of these media in learning situations (Sutherland & Balacheff, 1999); new educational trends, now materialized with the massive open online courses (MOOC) call for free online education without any tutorial intervention, occurring outside of school and that pedagogical means are managed by the digital ones. For example, through these courses, users are expected to assume the responsibility of their own appropriation of knowledge, and their learning and possible development is promoted primarily through a series of opinion exchanges among peers. In this work it is discussed how it is not enough to have access to mathematics technology and/or Internet free resources to achieve expertise on the mathematical content addressed.

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WHAT BEHAVIORS OF ASSESSMENT SHOULD A GOOD MATHEMATICS TEACHER HAVE? PERSPECTIVES OF JUNIOR HIGH SCHOOL STUDENTS IN TAIWAN

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This study regarded mathematics assessment behaviors, which somewhat indicate the implicit knowledge of assessment teachers possess, as one of the appropriate aspects to describe a good mathematics teacher. In addition, student-centered learning has always been an important aspect in describing a good mathematics teacher. The aims of this study is to provide national representative lists of, latent factors of, and types of ideal teaching behaviors of mathematics assessment from junior high school students' perspectives.

This study was conducted in two stages. First, a qualitative study employing open-ended questions was conducted on 238 high school students. A content analysis of the students' responses was performed to obtain the teaching behavior items in this stage. In the second stage, a questionnaire with dichotomous items was developed and administered. The sample consisted of 1037 students which were randomly selected nationwide in Taiwan. The EFA (in M-plus) was used and two factors contributed to ideal mathematics-assessment behaviors were identified, which were named as *mastery-demanding* factor (MD) and *diagnosis-understanding* factor (DU).

In addition, K-means cluster analysis was used to group participating students into categories according to their preferences of teachers' assessment behaviors. Four types of students were identified and the distinction between the types is related to the strengths of the two factors.

Type 1: Comprehensive Mastery and Diagnosis: a great teacher should employ assessment behaviors embedding both MD and DU factors.

Type 2: Aggressive Diagnosis for Understanding: a great teacher should orient assessment on learning with understanding.

Type 3: Mild Diagnosis for Understanding: the performance of this type are similar to type 2, but they did not consider a great teacher should perform it ($\mu=0$).

Type 4: Overall Exclusion of Assessment: this type did not approve most of the assessment behaviors in either MD or DU factors.

This study further compared students' perspectives of teachers' teaching process in different types. The results indicate that, though mathematics education in Chinese culture is demanding, most students favored assessment behaviors embedding diagnostic and formative orientations for promoting their understanding rather than demanding and mastery oriented approaches. In addition, students who preferred mild assessments also preferred slow-paced teaching.

FOURTH GRADERS' UNDERSTANDING OF INCLUSION RELATIONS IN QUADRILATERALS

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Inclusion relations are the core of structuring mathematics in a logical-deductive manner. Previous studies have explored students' conception of inclusion relations in general without the consideration of characteristics specific to different geometric shapes. The first aim of the study was to explore fourth graders' understanding of each individual quadrilateral definition and inclusion relations established based on the definitions. We particularly focuses on students' conception of inclusion relation between square and rectangle and the one between square and rhombus; the first is established based on the attribute of interior angles, while the second is generated by the lengths of edges. The second aim was to examine to which extent dynamic geometric software (DGS) scaffolds students in realizing the inclusion relations.

A class of 10 disadvantaged fourth grade students located in a remote district participated in this study. 80% of class students are aborigine and 70% are from single-parent or grandparents rearing families. The participating students were first asked to recall the definitions of the three quadrilateral shapes and describe corresponding inclusion relations. Then, they were required to explore the inclusion relations in DGS environment. If the students could not successfully recall definitions of quadrilaterals, another DGS activity designed for exploring quadrilateral definitions were also implemented. After completing the activities, each individual student had to describe inclusion relations and the reasons for determining the relations again.

The analysis show the cognitive differences in recalling quadrilateral definitions. Students performed better on recalling properties related to lengths of edges rather than angles. Regarding the mediation tools, DGS allowed students to observe the variants and invariants which are the core to establish inclusion relations. However, the observation does not guarantee the understanding of inclusion relations. 8 out of the 10 students could not use the variants and invariants as the warrants to evaluate the inclusion relations. Another important result is students' belief in the relation between square and rectangle. All the students refused to accept the inclusion relation even though they could identify the variant and invariants across square and rectangle examples in DGS. One of the main reasons is that the students firmly believed square and rectangle are disjoint subsets of quadrilaterals. Students' beliefs can consequently influence their learning in argumentation and proving (de Villiers, 1994).

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MENTORING FOR CULTURALLY RESPONSIVE AND AMBITIOUS MATHEMATICS PEDAGOGY

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Mathematics teaching and learning in New Zealand has undergone many changes over recent decades and each change has brought shifts in achievement for some groups of students. However, a concerning number of Māori and Pāsifika students remain at levels well below that of their Asian and European fellow students. Increasingly, research has illustrated that when teachers use ‘ambitious’ mathematics pedagogy (Lampert, Beasley, Ghouseini, Kazemi, & Franke, 2010) and culturally responsive teaching (Gay, 2010) such diverse learners are able to achieve equitable outcomes. But, we know that for many teachers enacting ambitious and culturally responsive teaching is challenging and the complexities of what is required can cause teachers to resist change. We know also that mentors can support teachers to critically reflect on their practices, experience dissonance and as a result transform their current pedagogical practices. To this direction the current study draws on the work of Lampert et al., (2013) and builds on their framework of actions. In particular, the paper focuses on the actions mentors focussed teacher attention on as they co-constructed lessons aimed to elicit and respond to student reasoning within a culturally responsive framework. Particularly, the research project aims to explore the effect on teacher practices when they are mentored to use Pāsifika focused culturally responsive pedagogy. The sample was comprised of 10 teachers, 3 mentors and 250 students aged between 8 and 11 years. Predominantly the students were of Māori and Pāsifika ethnicity. Data included video recorded observations of mentored sessions and teacher interviews. For one year the teachers and mentors had been involved in developing mathematical inquiry learning communities embedded within culturally responsive practices. The findings illustrated that when mentors and teachers worked together to co-construct ambitious mathematics pedagogy the teacher’s pedagogical skills to draw on culturally appropriate actions increased as did their skill at noticing and responding to student reasoning.

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THE SPECIALIZED CONTENT KNOWLEDGE OF A PRE-SERVICE TEACHER ABOUT THE DOMAIN OF LEARNING ALGEBRA: A CASE STUDY ON SOLUTION OF FIRST DEGREE EQUATIONS OF ONE UNKNOWN

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The studies about defining the teacher knowledge conducted by Shulman et. al., in 1986 and 1987 created a widespread interest in the education community. The reason of this intensive interest is that it forms a bridge between content knowledge and teaching practice (Ball, Thames & Phelps, 2008). In this context, Ball and his team established the theoretical framework of Mathematical Knowledge for Teaching (MKT) in order to research the specific knowledge needed in teaching mathematics. Ball and his team stated that the focal point of their study was the specialized content knowledge (SCK) component (Ball, Thames & Phelps, 2008; p. 400). SCK is important from the point of view that in the studies performed with teachers and pre-service teachers, it reveals what the special knowledge needed by them is in teaching mathematics and therefore it is worth analysing. In this direction, the research problem of the study is as in the following: In terms of the Specialized Content Knowledge (SCK) component of the MKT, how is the algebra teaching knowledge of the pre-service teacher?

The research is conducted with a pre-service teacher studying in a state university in Turkey. For the selection of the teacher, the grade point average and volunteering were taken as the criteria. The teaching processes of the pre-service teacher were observed in a state school where she is conducting her intern-ship. The research had been conducted using the case study, which is one of the quantitative research methods. The data of the study were obtained by the interviews carried out with the teacher about the specialized content knowledge component of the MKT model and video taping of the four-hour teaching process of the teacher. The data obtained from all of the data collection means were collected in a pool and analysed with respect to the specialized content knowledge component of the MKT theoretical framework. In terms of the indicators of specialized content knowledge, it can be said that the pre-service teacher has some deficiencies. These deficiencies are seen especially in the indicators of using the mathematical content language correctly and ability to make changes in the teaching process. Subject-specific micro-teaching studies can be conducted in order for the pre-service teachers to have better levels of specialized content knowledge

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IMPORTANCE OF SITUATED LEARNING THEORY IN REALIZING CULTURAL AIMS OF MATHEMATICS EDUCATION

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Cultural aims, which aim for students to enjoy, inherit and develop mathematics as culture, are considered part of the aims of mathematics education in Japan. However, they have not been realized yet. One of the reasons would be the lack of Japanese teachers' and researchers' recognition of the goals of mathematics education at the stage of designing lessons. Situated learning theory (Lave & Wenger, 1991), a viewpoint on learning which emphasizes paying attention to 'activity', is considered to have potential to solve this. Focusing on activity requires teachers to have a clear image of ideal students in terms of mathematical activity while designing mathematics lessons. The purpose of this paper is to illustrate the importance of the theory in realizing cultural aims of mathematics education in Japan. General principles of lesson design based on the theory (Imai, 2010) to realize the above-mentioned aims can be interpreted as follows: (1) Choose a mathematical activity from a real situation in the history of mathematics in order to stage it in the classroom. (2) Make students involved in the actual mathematical activity staged by the teacher. Based on these principles, case-study lessons on the trisection of an angle, one of the three famous problems of the ancient Greeks, for three classes of seventh graders (first graders in junior high school in Japan) were conducted by the author of this paper in 2016. Looking back on the history of the problem, the problem attracted significant interest among amateur researchers, who one after another claimed to have found how to do the impossible - construct trisectors, and troubled mathematician even after the proof of impossibility. In the lessons, students worked on the trisection of 90° , 45° and 60° , and many of them could construct the trisector of the former two angles in collaboration with each other. The students' feedback regarding the lessons obtained with the use of a form at the end of the lessons revealed that about 50% of them clearly showed the motivation to explore the problem in the future and about 40% of them described they enjoyed solving the problem. It implies that cultural aims of mathematics education can be expected to be attained by recreating the real situation in the mathematics lessons based on the principles of situated learning.

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UNDERGRADUATE STUDENTS' PERCEPTIONS REGARDING THE FINAL EXAMINATION IN ABSTRACT ALGEBRA

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BACKGROUND

Research in the learning of Abstract Algebra proves significant, since novice students consider this module as one of the most demanding subjects in their syllabus. (Dubinsky et al., 1994). Gueudet (2008) suggests that many pedagogical issues emerging in undergraduate Mathematics Education are based on the transition from secondary to tertiary Mathematics. In fact, student difficulties in Abstract Algebra may be an indication of problematic transition, mainly due to the particular nature of this module.

ANALYSIS

The revision for the final examination is a 'renewed contract' for mathematical learning, and undergraduate mathematics students need to develop and/or apply certain techniques. They are invited to revisit what they have been taught, localise the conceptual gaps and overcome the remaining misconceptions. The majority of students adopt a similar approach towards revision for the final examination. Usually the revision process initiates by revisiting the lecture notes. The predominant aim is to engage again with the definitions, theorems, lemmas and proofs, both to improve their understanding and memorise the ones that are most likely to appear in the exams. The next step of revision is to study the coursework questions in parallel to the given model solution and then attempt to solve past papers, or vice versa. Revisiting the coursework, for many students, is an important step in their learning process. Students have the opportunity to compare their own solutions with the model solutions and precisely localise any inaccuracies. Regarding the solution of past papers, many students at this stage try to specifically identify the definitions, theorems and proofs that are likely to be included in the examination paper, to pinpoint possible mathematical tasks that they may be asked to prove or solve, and to enrich their learning experience.

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MODELS OF UNDERSTANDING: A STUDY OF SIXTH GRADERS' UNDERSTANDING OF FRACTION INTERPRETATIONS

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Mathematical understanding is always the goal of mathematics teaching and learning. Previous studies state that mathematical understanding is characterized as hierarchy (e.g. Pirie & Kieren, 1989). It was defined as different categories, stages or levels. However, some of these categories ignore that mathematical understanding involves cognitive constructions and translations among multiple representations; some are difficult to be applied in teaching practices. Fraction is a rather complicated mathematics concept for elementary students predominantly due to its multiple interpretations: part-whole, measure, ratio, quotient, and operator (Pantziara & Philippou, 2012). The research questions are: Is there a practical understanding model which reveals the characteristics of mathematical understanding? To what extent are most students' understanding achieved in the part-whole and measure subconstructs?

The study categorizes the mathematical understanding into five levels: intuitive, image, abstract, formal, and creative understanding. In order to examine this model, students' understanding of part-whole and measure subconstructs of fractions were investigated. A 36-item test was designed to reflect the characteristics of the first four understanding levels. 333 sixth graders from five different primary schools in Beijing participated in the test. Rasch model was applied to analyse the data. The reliability of the person ability and item is 0.84 and 0.98 respectively. Students' abilities distribute normally.

The item-person map shows three significant gaps between item allocations on the logit scale, which means that students' understanding of interpretations of fractions is hierarchical in line with the model. Analysis of variance also indicates that the difference between different levels is significant ($p=0.000$). Therefore, the understanding model above is acceptable and practical for teaching and learning of fractions, including intuitive, image, abstract, formal, and creative understanding. Furthermore, for part-whole subconstruct, about 90% of sixth graders are located in the formal understanding level, while only about 3.87% achieve in this understanding level of the measure subconstruct. These suggest that students perform better in the part-whole subconstruct than in measure subconstruct. More details will be discussed during the presentation.

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CORRELATIONS BETWEEN CREATIVITY IN GEOMETRY AND ALGEBRA

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BACKGROUND

A growing area of interest for mathematics education is that of creativity (e.g., Leikin & Lev, 2013). Several test instruments have been developed in order to both identify and quantify this construct (Leikin & Lev, 2013). Previous studies have shown that creativity is not general but domain-specific, which indicates that there is a special kind of mathematical creativity (Kattou et al., 2015). However, there is little discussion on subdomain-specificity, addressing possible differences between creativity in geometry or algebra. In this paper, we discuss and give first indications for such a subdomain-specificity of mathematical creativity. Further, we take a look at the internal consistency (reliability) of common tests for mathematical creativity. Our *research question* is: To what extent do students' achievements in creativity differ, depending on the subdomains geometry and algebra?

THE STUDY

Following Leikin and Lev (2013), three multiple solution tasks (MSTs) – two geometric and one algebraic – were used in a group of 15 mathematically interested upper secondary school students, aiming at measuring creativity in the above-mentioned two subdomains of mathematics (see Joklitschke et al., 2016 for details). The non-parametric analysis indicates that the two geometric problems are stronger correlated to each other than to the algebraic problem. This supports our hypothesis regarding the subdomain-specificity of mathematical creativity. Because of the small database of the study, we will address this phenomenon in further investigations.

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THE NATURE OF EMBODIED MATHEMATICAL OBJECTS

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There are a lot of expressive, multimodal materials in a mathematics classroom. Students learn to do mathematically to form mathematical objects through bodily interactions with these ones. This article tries to explain the nature of the objects from the embodiment theory perspective (Radford et al., 2009) on the assumption of different fields for teachers' and students' doings. In particular, we focus on multimodality which refers to "the range of cognitive, physical, and perceptual resources that people utilize when working with mathematical ideas" (Radford et al., 2009, p.91) because it is needed to examine what are possible bodily doings if embodied mathematics is both enacted from and restricted by doings. To accomplish the goal, we proceed (1) to categorize possible doings –perceptual or not, for example– by referring to the expressive products and its affordances (Edwards et al., 2014, pp.7-19), (2) assume some different fields for these categorized doings, and finally, (3) identify the nature of embodied mathematics.

As a result, we suggest that there are at least three fields to make doings possible:

- *The experience-based field* (E.F.) is a physical, perceptible, and heterogeneous one such as 3D space and 2D paper or screen in actual life. Students and teachers see, touch and manipulate concrete objects to perceive them in this field.
- *The possible field* (P.F.) is an ideal, non-perceptible, and homogeneous one as a reflection of *the experience-based field*. Students and teachers image and operate objects which are originally invisible or untouchable, and experiment with them both repeatedly and freely in this field.
- *The theoretical field* (T.F.) is a fictional, creative, and rule-driven one constructed in an arbitrary way. Students and teachers use language and transform mathematical notations to construct and deal with mathematical objects.

These fields have own characteristics but aren't necessarily separated from each other. For example, a concept of a geometrical transformation –translation, rotation, and reflection– is connected with actual moving (in E.F.), abstracted, extended and integrated (in P.F.), and formulated by using notations and reduced to symbolic operations (in T.F.). Mathematical objects are generated by recognizing the isomorphism of doings throughout these fields. Therefore, a doing in a field is founded and effected by other ones in other fields. Embodied mathematical objects are enacted from the intertwinement of different doings in each field.

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HOW ELEMENTARY STUDENTS INTERPRET EQUAL SIGN WITH SYMBOLIC EQUALITY SENTENCES

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Equal sign symbol forms the basis of algebraic thinking. Understanding equality concept and equal sign is inseparable from understanding algebra (Knuth, Stephens, McNeil & Alibali, 2006). In this research, students' interpretation of equal sign from second grade to sixth grade has been investigated.

Participants of the study were 236 elementary school students (48 second grade, 53 third grade, 43 fourth grade, 42 fifth grade, and 50 sixth grade) from four public schools in Konya, Turkey. Data were collected through Mathematical Equality Questionnaire (MEQ) which includes two items measuring students' interpretation of equal-sign in symbolic equality sentences as shown in Fig. 1. After implementing questionnaire, semi-structured interviews were conducted with two students selected from each grade.

Fill in the blank with the value that makes the given statement true.

a. $14 = \square +$

Figure 1. Symbolic mathematical equality items

Analysis of data revealed that students' achievement level for item (a) was higher than item (b) with a gradual increase from the second grade to the sixth grade. In addition, almost three quarter of students from all grade levels could interpret equal sign relationally in item (a) and wrote 9 into the box. However, percentage of students who could interpret relationally item (b) ranged from 8,3 to 30. It showed consistencies with results of previous studies (Stephens et al, 2013). Moreover, almost 75% of students either added all numbers together in the statement and wrote 18 for item (b); or, viewed equal sign as do-the-operation signal and wrote 13 by ignoring 5 on the other side of equal sign. In other words, students, no matter of grade level, mostly interpreted equal sign operationally. It showed the resistance of operational interpretation of equal sign as grade level increased.

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CHALLENGES IN INTRODUCING MATHEMATICS PROBABILITY IN ELEMENTARY SCHOOL MATHEMATICS CURRICULUM IN IRAN

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One of the challenging studies about the probability was published by Piaget and Inhelder (1975). They believed that learning mathematics probability is beyond the capabilities of younger children. Some studies supported this belief but there were growing numbers of studies which contrasted that. For example Shaughnessy (1992) said: "Probability concepts can and should be introduced into the school at a fairly early age" (p. 481). Moreover, Principle and Standards of NCTM (2000), as one of the important documents in mathematics education, considers probability as one of its content standards.

In Iran, math probability is a new topic in elementary school mathematics curriculum. The aim of this study is to investigate the challenges in introducing this topic in Iranian elementary schools. To this aim, we planned a study in three phases which two of them will be reported here.

In the first phase we gathered data from mathematics elementary textbook. We found out that probability starts from grade one implicitly and it continues explicitly in grade two where it comes right after the topic of fractions. From grade three to five probability and statistics come together.

In the second phase we interviewed elementary school teachers. Teachers at grade five believed that introducing probability in earlier grades is helpful to teach this subject at higher levels as they have not noticed any serious problem with students' understanding. On the other hand, teachers at grade two reported lots of problems in teaching probability, so they believed this subject should not be taught at such an early stage. In the last phase, we are going to observe teaching process and study the students' understanding directly. Hopefully we are finished the last phase by the time of the conference, PME 40.

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SUPPORTING MATHEMATICAL LITERACY BY MEANS OF REASONING BY USING TEACHING WITH VARIATIONS

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Mathematical literacy is a crucial competency which can enable an individual to use mathematics in real-world situations. As an individual's mathematical literacy level increases, the ability making benefit of the fundamental mathematical capabilities also increases (Turner & Adams, 2012). On the other hand, there is an outstanding mathematical literacy performance of East Asian countries in PISA (Programme of International Student Assessment) reports and one of the common properties of these countries is that using teaching with variations in their mathematics education programmes (Park, 2012). The purpose of this study is to investigate eight graders' mathematical literacy progress by means of reasoning and argument competency as a fundamental mathematical capability by using teaching with variations. With this purpose, an open-ending test -including PISA mathematical and problem solving literacy items- was administered to 237 eighth graders. After the analysis of data quantitatively, seven students who were thought to have different reasoning and argument competency levels according to competency scheme (Turner & Adams, 2012) were chosen for the second stage of the study. In this stage, a teaching experiment in which the data was collected through two open-ending tests, clinical interviews focusing on solving them and four teaching episodes conducted with seven eighth graders. Data was analysed qualitatively by using content analysis technique. The findings indicate that the learning environment which is designed based on the teaching with variations supports mathematical literacy by means of reasoning and argument from the aspects of making ideas public, evaluating the accuracy of mathematical claims, making and testing assumptions, trying to determine the accuracy and sufficiency of the solutions and attempting to convince yourself. On the other hand, some aspects of this competency such as trusting your own reasoning, justifying the accuracy of solutions against an authority and making complex reasoning chains were not supported sufficiently.

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THE TEACHING AND LEARNING OF STATISTICAL PROBLEM SOLVING THROUGH THE USE OF STATISTICS POSTERS

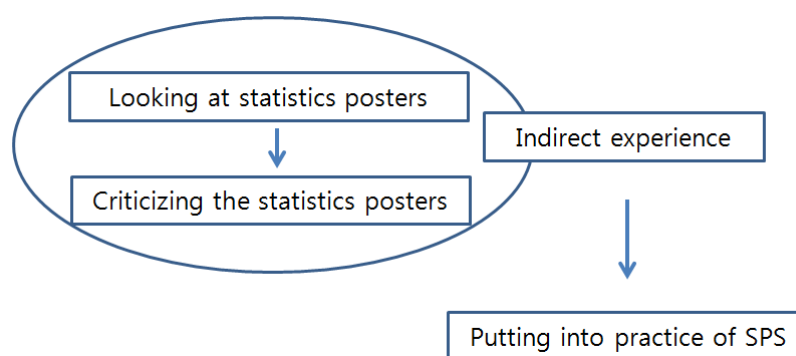
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This study proposes the alternative model for the teaching and learning of statistical problem solving (SPS). SPS has steps such as problem, plan, data, analysis and conclusion and they include main issues (Wild & Pfannkuch, 1999). For example, problem has defining problem as a main issue and data includes data collection and data management as main issues. These issues can be learned through the experience on SPS.

This study proposes indirect experience through statistics posters as an alternative approach for teaching and learning of SPS. Interaction with context is important for understanding of the issues. Statistics posters provide useful and faithful information about the context in which statistical enquiry is carried out. The indirect experience consists of two activities, looking at statistics posters and criticizing the statistics posters. First, students look at the given statistics posters containing some errors or something to be desired which can be included in the posters themselves or made by teachers. The students, then, criticize the posters against main issues related to the steps of SPS. After the indirect experience, the students put into practice of their own SPS. The indirect experience provides a foundation for their own SPS.

At the stage of the indirect experience, it is important that students face various errors or something to be desired and discuss them with their peers or teacher. It helps the students make their understanding of the issues deep.



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DESIGNING E-BOOKS TO FOSTER CREATIVE MATHEMATICAL THINKING: THE CASE OF CURVATURE

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In this study we look at a digital medium, called 'c-book' (c for creativity), affording the meshing of narrative with dynamic constructionist media and examine the ways in which a community of professionals jointly designed a c-book unit on Curvature with the aim to foster creative thinking in their prospective students. We focus on the emergence of Social Creativity (Fischer, 2005) which builds on the wealth of diverse perspectives in addressing a complex design problem of common concern and focuses on the interactions occurring in socio-technical environments (i.e., among the individual members of a community and between them and particular technologies). Our designer communities were engineered so as to include diverse educational professionals in line with Fischer's 'communities of interest' (CoIs). To understand the context particularly of teachers as resource designers we put in use the *documentational approach of didactics* (Gueudet & Trouche, 2009). Moreover, we traced *boundary crossing processes* (Akkerman & Bakker, 2011) employed by CoI members to (re)establish continuity in interaction across diverse practices. The C-book environment provides a tool for online discussions (the CoIcode) and a platform designed to incorporate pages with dynamic widget instances and corresponding narratives. Our data were the 124 contributions uploaded in the CoIcode during the design process of the c-book unit on curvature. The analysis focused on the boundary crossing interactions and the role of the narrative as a key resource for the development of social creativity. Two boundary crossing processes, coordination and reflection, enhanced social creativity establishing communication between different communities of practice and enabled the fertile synthesis of diverse views. Moreover, the narrative versioning process allowed for intense debate and idea exchange to occur: it created common ground for CoI members to unfold their expertise, as well as the meshing of narrative with constructionist artefacts-widgets on curvature.

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DEFINING QUADRILATERALS: RELATED LANGUAGE USED BY MATHEMATICS TEACHERS

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It is seen that studies on mathematical language are conducted in the fields of algebra and analysis and mostly at secondary school education and high education levels. For instance, in algebra, letters are generally used to represent unknown quantities, and interpretation of mathematical language could lead to different meanings of letters such as variable or parameter. However, in geometry, when objects are represented by figures, it is not possible for such a case to occur (Mesquita, 1998). Moreover, geometry is not made up of just objects/figures. Definitions, theorems and axioms constitute the milestones of the deductive structure of geometry. In geometry, information can be provided via visual representations and certain semantic networks (Duval, 1998). It could be stated that especially the semantic structure of mathematical language, or semantic relationships, should be identified to understand the inclusive and exclusive definitions of quadrilaterals. Duval (1998) defines three cognitive processes: visualization, construction and reasoning. The researcher points out that the mathematical language used in the processes of “purely configurational process”, “natural discursive process” and “theoretical discursive process”, which are all related to the reasoning process has influence on understanding and learning. The present study structured based on cognitive processes aimed at determining the mathematical language used by secondary school mathematics teachers not only for the definition of quadrilaterals but also in the process of solving the related problems. In line with this purpose, first, three secondary school mathematics teachers were asked to respond to an open-ended test regarding the definition of quadrilaterals. Following this, clinical interviews were held with the teachers, and in-class observations were done. The findings obtained via the open-ended test revealed that two of the teachers described the quadrilaterals rather than defining them; that they failed to establish the hierarchical relationship between quadrilaterals; and that they thus failed to make verbal definitions to demonstrate the inclusive relationships between the quadrilaterals.

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WRITING COLLABORATIVELY THE NARRATIVE OF A C-BOOK UNIT AIMING AT FOSTERING CREATIVE MATHEMATICAL THINKING

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Cultural, linguistic, social and technological changes result in new conceptualisations and approaches to the way language in general and narrative in particular can be capitalised upon in mathematical teaching and learning (Kynigos, 2015). Only now are we beginning to understand the ways in which narrative practices are affected by the use of multiple modes and media, while it is completely vague how ‘contemporary’ narrative can catalyse mathematical knowledge construction and mathematical creativity. This paper investigates the core issues and practices that emerged as one group of professionals, with diverse experience and background, collaborated in order to write the narrative of an interactive e-book (using the c-book technology) aiming at fostering creative mathematical thinking. The c-book technology is an integrated and beyond the state-of-the-art digital system which aims at facilitating creative design processes and practices, developed in the framework of MC2 project (<http://mc2-project.eu>).

The qualitative data collected were subjected to thematic analysis (Braun & Clarke, 2006). The results show that the affordance of interweaving small software pieces with narrative into a concise whole is a new and definitely demanding practice. Designing digital mathematical resources and writing stories come in the foreground as interlinked processes in the framework of digital literacies. The interplay between linearity, hypertextual structure and multimodality is crucial and results in a shift of attention from print to visual layout. The c-book technology supported crossing of different expertise and knowledge fields facilitating collaboration and argumentation while mediating semiotically social creativity. It seems that the new digital literacies afforded by the technology used rather change the way readers/students of mathematics and writers/designers of mathematical resources are conceived and related, which is an issue that should be further investigated.

The research leading to these results was co-funded by the European Union, under FP7 (2007-2013), GA 610467 project “M C Squared”. This publication reflects only the authors' views and the Union is not liable for any use of the information contained therein.

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AN ITEM ANALYSIS ON MATHEMATICAL COMPETENCY IN TEST FOR KEY COMPETENCIES FOR THE WORKFORCE

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The information age necessitates workers to equip themselves with professional and complex skills for work performance. Among the key competencies for work performance, mathematical competency is being emphasized for effective task performance (Bakker et al, 2006). Although considered pivotal in competency tests, the current state of the evaluation and effective evaluation methods of mathematical competency are not well known. Accordingly, this study aimed at analyzing the questions testing the mathematical competency in competency tests by focusing on the mathematical competency which is actually used by workers in the workplace and by suggesting future directions for research.

This study analyzed the questions on mathematical competency of six tests in two different perspectives. First, a framework to analyze the content was devised based on the frameworks used by the National Competency Standard created by the Human Resources Development Service of Korea and in Australia, Canada and the United Kingdom. To analyze the job context of the questions, this study used the job context axis in the framework for basic competency test from specialized high schools. Interviews were also conducted with 20 workers currently working in five different fields of work relevant to the competency test among the 10 occupational clusters classified by Statistics Korea. Through the job analysis, this study identified the mathematical competency used by workers and based on a detailed analysis.

The results of this study showed that there are difference between mathematical competence in the test and the mathematical competence used by workers. For example, Employees at the workplace were using proportion and ratio of basic math calculation skills to compare the numbers and quantity in a given situation by collaborating with co-workers. But, there were only few questions assessing the knowledge on proportion and ratio. Also, worker made a decision by mathematical competency at the problem situation of the workplace. Employees at the workplace were using basic math calculation skills for budget planning and measurement for producing goods, data analysis for managing sales, using formula for calculating fraction defective. But the test just assessed a result, not the process. This study presented a large picture depicting the actual condition of the assessment and also identified problems and presented methods for improvement.

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EXPLORING HOW A KINDERGARTENER COMPARED

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Baroody (1998) indicated that numbers and numerals are essential tools for mathematics. A thorough understanding of number concepts plays an important role in children's mathematics learning. Studies have showed that children can develop number concepts without adult assistance (Smith, 2006). The purpose of this study was to explore how a kindergartener compared the number of objects in two groups.

The subject for this study was a five-year-old kindergartener. In this study, the kindergartener received an individual clinical interview while working on a series of five mathematics tasks as follows. The interview was audiotaped and transcribed. Transcripts of the interview and the interviewer's observation notes were analyzed to investigate how the kindergartener compared the number of objects in two groups.

In Task 1, there are two groups, A and B, and each group has ten candies. The interviewer asked the kindergartener: Which group has more candies?

In Task 2, the interviewer took one candy away from A and then put the candy in B. The interviewer then asked the kindergartener: Which group has more candies?

In Task 3, the interviewer took one more candy away from A and then put the candy in B. The interviewer then asked the kindergartener: Which group has more candies?

In Task 4, the interviewer took four candies away from B and then put them in A. The interviewer then asked the kindergartener: Which group has more candies?

In Task 5, the kindergartener was asked to make groups A and B have equal amounts of candies.

The following are the findings of the study. In Task 1, the kindergartener compared the number of candies in A and B directly by counting. In Tasks 2, 3 and 4, she counted the number of candies in A and B each time after the interviewer moved a candy or candies between A and B; she, however, didn't realize the relationship of more and less caused by the interviewer's moving candies between A and B. Consequently she got stuck in Task 5. The results of this study suggested that the kindergartener tended to compare the number of objects in two groups by counting without realizing the changing more-and-less relationship between A and B. It would be beneficial to conduct studies on the causes of the kindergartener's failure on Task 5.

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PRESERVICE TEACHERS' QUESTIONING: SUPPORTING AND EXTENDING STUDENTS' MATHEMATICAL THINKING IN PROBLEM SOLVING

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Teachers' using of supporting and extending (S/E) questions plays a critical role in problem solving activities. However, Jacobs, Ambrose, Philipp, and Martin (2011) found that preservice teachers (PSTs) "generally imposed their own thinking when supporting children in the interviews and interacted in a limited way when extending [children's mathematical thinking]" (p. 40). Therefore, the current study aims to investigate PSTs' questioning over a course of an 8-week series of one-on-one interviews with children. The Teacher Moves of Supporting and Extending Mathematical Thinking (TMSEMT), proposed by Jacobs and Ambrose (2008), serve as a framework for categorizing teacher questioning. The research questions are:

- How do PSTs quantitatively employ questions during the interviews?
- What factors influence PSTs questioning in the interviews?

This study draws data from four interviews conducted by two PSTs in their first mathematics methods course. The data collected for this study include the PSTs' video recordings and transcripts of the videos. The questions PSTs asked in the interviews were analysed using the TMSEMT. The results show that both PSTs employed a higher percentage of extending questions in the final week compared to their usage as observed in the beginning weeks. Both participants demonstrated that their use of multiple questioning types in the interviews was strongly influenced by students' positive and negative reactions (e.g., first graders might become frustrated in trying to verbalize thoughts in detail or to articulate the procedure of solving a problem). In addition, extra effort to pose unprepared, strategic S/E questions is indispensable for ensuring students' understanding of the mathematical relationships among the elements (e.g., the quotient and remainder) involved in the tasks.

Based on the results, the author concludes that experience in one-on-one interviews significantly reinforced PSTs' ability to dynamically employ and balance questioning types with improvised modifications to facilitate student thinking in problem solving.

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ADDITIONAL INFORMATION AS A DIFFICULTY GENERATING FACTOR: AN EYE MOVEMENT ANALYSIS

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There are several approaches to explain the difficulty of a mathematical task. Current research about the so-called *difficulty generating factors* provides schemes to predict the complexity of an item and as a consequence its difficulty. These schemes are either depending on the specific content of an item or are independent of the content. These general content models include aspects like complexity of language or of calculations as well as the cognitive complexity of tasks. (e. g. Sijts, 2005).

However, items may encompass additional pieces of information, which are not necessary for the solution, but may be used to find it. These pieces of information require mental resources. According to cognitive load theory (e. g. Plass, Moreno, & Brünken, 2010), this can have an adverse effect on the solution process and thus may be a difficulty generating factor. Our research aims at the question whether these additional pieces of information can be considered as difficulty generating factors.

We used 24 items, 12 of them provided an additional piece of information, and 12 did not. The participants were instructed to decide whether a function given as a polynomial of degree 3 is the derivation of another function given in a graph. The additional piece of information was the polynomial term of the plotted function, which was not necessary to solve the task, but enabled people to solve the task, however, in a more difficult way. We analysed the eye movements of 17 undergraduates solving these 24 items.

We found out that the time students needed to solve the items did not differ significantly between the two types of items ($p = .71$). The average total number of fixations on the items did not differ significantly either ($p = .78$). However, in items that provided the additional piece of information, the number of fixations on the most central piece of the item (the graph) was significantly lower ($p = .02$).

If eye movements were related to the attention foci (which is the basic assumption of eye tracking) there seemed to be a shift of attention: The main part of the item was less in focus if there was additional information. In line with cognitive load theory, this suggests to consider additional information as a difficulty generating factor.

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THE CHANGES OF TEACHER VERBAL FEEDBACK IN CLASSROOM INTERACTION: A CASE STUDY

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Classroom interaction is usually created by teachers and students and these alternations are characterized by interactional sequences of three interconnected parts: initiation, reply/response and feedback or follow-up (Sinclair & Coulthard, 1975). Teaching questioning, as one of the initiation strategies, has been focused so much while less on teacher verbal feedback which also have considerable effect on students learning (Zahorik, 1968).

This study categorized the types of teacher verbal feedback and examined the changes through documenting frequency and duration in one teacher's professional development. Two video lessons of one female primary school teacher were analyzed. One lesson was taped in 1998 when she was a novice teacher and the other lesson was obtained when she has already been an experienced teacher in 2011.

Based on one framework developed for knowing mathematics teachers' role in classroom communication (Li, 2009), a revised framework was established to outline the types which including 4 categories: a) Not accept; b) Accept with a neutral attitude; c) Accept with an encourage attitude ;d) Accept, encourage, explore and apply students' ideas in classroom teaching. This study found that the most frequency was the second type in the two lessons while the least was the fourth. Regarding to the duration, the most was the fourth type whereas the least was the first one. Although the most significant changes occurred among the two lessons were the second and the fourth types, these changes were different. The frequency and duration of the second type decreased greatly. Instead, the frequency of the fourth type decreased but its duration increased apparently.

The framework may be used in analyzing primary mathematics teachers' verbal feedback for other research. The statistical results indicated that the teacher intended to leave much more time for students to articulate their ideas in classroom teaching with the year of teaching experience growing. However, other factors, such as the curriculum policy, may also play a role in the changes.

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GERMAN AND TAIWANESE STUDENTS' EYE MOVEMENTS WHEN SOLVING STATISTICS ITEMS

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Statistical text, used in daily life and in international assessments, is edited with descriptions of variables, formulae, and graphs that consist of axes, scales, names of variables, and central graphs (Carpenter & Shah, 1998; OECD, 2013). Reading comprehension of statistical text can be regarded as a cornerstone for life as a reflective citizen, for effective learning, and for the international comparison of learning effectiveness. In Taiwan, the statistics curriculum addresses collecting and categorizing statistical data, and reading the data involved in graphs as well as generating graphs. In Germany, the statistics curriculum focuses on understanding statistical graphs, particularly reading the data in the graphs (Kultusministerkonferenz, 2004). Therefore, different perspectives have led to differences in the design of the statistics curriculum between Taiwan and Germany. Hence, this study aims to compare reading patterns of 9th grade students between Taiwan and Germany while solving problems accompanied with statistical text and diagrams.

The study comprised an eye movement experiment, including a pilot study, in addition to a formal experiment. Participants of the main experiment were 9th grade students in Taiwan (N=21) and Germany (N=25). Tobii and EyeFollower both with 120Hz sampling rate were used for collecting eye movement data. Experiment materials included three PISA items with eight sub-items, including structured and semi-structured ones. There was no significant difference between the test performance of in the German ($M=6.60$, $SD=0.81$) and the Taiwanese sample ($M=6.13$, $SD=0.90$); $F(1, 45) = 2.46$, $p = .08$. Nevertheless, eye movement revealed significant differences between two countries. German participants showed longer reading time, and total fixation duration than Taiwan participants. Moreover, they spent a higher ratio of their reading time on the figure and exhibited fewer counts between text and figure than Taiwan participants. Data analysis on reading patterns and their relation to performance is still ongoing. We expect that the results can inform reading strategy trainings for statistical texts with diagrams in both countries.

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DEVELOPING A MENTORING MODEL TO FOSTER A NOVICE TEACHER'S KNOWLEDGE OF INQUIRY TEACHING

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This article presents how a senior exemplary teacher plays a mentor of a novice teacher for learning mathematics inquiry teaching and investigates the growth of the mentee's mathematical knowledge of teaching. This is a narrative analysis-oriented case study. Firstly, an inquiry-oriented mentoring model is constructed based on grounded theory by means of collecting and analysing data from classroom teaching observation, discuss, and reflection. Secondly, the novice teacher's knowledge of mathematics inquiry teaching is analysed based on the framework of Mathematical Knowledge for Teaching (Ball et al., 2008), for examining how the inquiry-oriented mentoring model influences the novice teacher's knowledge of mathematical inquiry teaching.

The centre of the inquiry-oriented mentoring model is that the mentor constructs an inquiry-oriented learning environment, so that the mentee is able to grasp the spirit of inquiry teaching when learning the inquiry teaching strategy. There are two phases of the inquiry-oriented mentoring model: Classroom Observation and Co-teaching. In the phase of classroom observation, the mentee poses her questions and ideas about the mentor's inquiry teaching during the cycle of the classroom observation, after class discussion and reflection; the mentor guides the mentee to construct her understanding of inquiry teaching by continuous questioning. In the phase of co-teaching, the focus is on guiding the mentee to transfer her inquiry teaching knowledge to practice, while the other design is parallel to the phase of classroom observation. (see Figure 1)

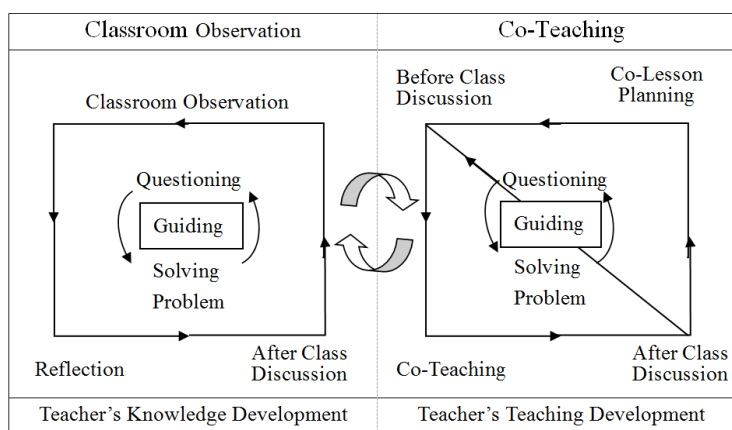


Figure 1: the Inquiry-oriented Mentoring Model

The data analysis reveals that KCS (Knowledge of Content and Students), KCT (Knowledge of Content and Teaching), and KCC (Knowledge of Content and Curriculum) (Ball et al., 2008) of the novice teacher develops significantly during the academic year the research is conducted. It is suggested that the inquiry-oriented mentoring model could be applied to the teacher practicum mentoring.

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COLLECTIVELY BUILDING KNOWLEDGE ON FRACTIONS THROUGH DISOURSE

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Classroom discourse is a central instructional method in mathematics teaching (Chapin et al., 2003). Discourse occurs when both the teacher and students actively contribute ideas to collectively build new knowledge (ibid). Studies on classroom discourse show positive effects on students' conceptual learning, and they also provide examples of effective ways in which mathematical knowledge can be collectively built. However, little is known on the different ways (patterns) in which teachers and students can collectively build mathematical knowledge. In this study, we examine how knowledge on fractions was collectively built using selected videos from two provinces collected in a nationwide study on mathematics pedagogies across Canada (see Reid et al., (2015)). Our goal is to observe patterns in which teachers and students contribute to the knowledge building at each part of the discussion actions.

Leikin and Rota (2006) view discourse as a sequence of discussion actions cycles containing three parts: the stimulating initiation (start of the action), the stimulating reply (continuing the action), and the summarizing reply (closing the action). Both the teacher and the students can contribute to each part of the discussion action.

We observed four videos of teachers teaching fractions and broke down the discourse into discussion actions. We observed how teachers and students contributed to the knowledge for each part of the discussion actions. After we analysed the different patterns that occurred in the collective knowledge building in all the discourses.

Results revealed very few instances of discourse. Most of time time, the teachers would ask “evaluative” questions to students instead of collectively building the knowledge. During discourse, the teachers initiated the discourse actions, except for one case in which it came from students. Teachers and students shared ideas in the replies, but this was mostly done between the teacher and one student. The conclusions were built by both, but the teachers usually made the final remark.

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IMAGINE: SOLVING THE “ATTITUDE EQUATION” TO BE A “POSITIVE” TEACHER?

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To achieve success in statistical education the need to know teachers attitudes towards Statistics is recognized. This knowledge could improve the chances of teachers having a predisposition, willingness, and commitment to teaching and learning processes in statistics and also their training. Consequently they will influence students' academic success in statistics (Gal, Ginsburg, 1994).

This study focuses on the measurement and characterization of attitudes towards Statistics by teachers of the 1st and 2nd cycle of Portuguese basic education (ages 6 till 12) using a tested scale – EAEE (Estrada, 2009). It also aims to determine if there are differences between teachers' attitudes of these two cycles and the effect of some others variables. The Portuguese version of the EAEE scale, validated by a board of experts, was applied to a sample of 1098 teachers. The attitudes at the level of the overall score were found to be positive, especially in cognitive and social components. The reliability can be considered high (Cronbach alfa=0,869). There were found significant differences in relation to the global score in the variables: teaching cycle, time of service, initial area of training, level of Statistics studies and if already taught Statistics.

As a conclusion we can emphasize that the studied teachers don't do a conscientious use of Statistics in theirs daily life; express a disbelief feeling about the use of Statistics, especially in the media and in politics; don't share Statistics difficulties and knowledge with colleagues; and also assume that have some lack of Statistics knowledge. This information about variables influence in the teachers' attitudes towards Statistics and the aspects linked to their less positive attitudes may allow us to find answers to how to solve the “attitude equation” to be a “positive” teacher.

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INFORMAL PROBABILISTIC REASONING OF HIGH SCHOOL STUDENTS CONCERNING BINOMIAL DISTRIBUTION

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In this work, we are interested in observing how high school students reason with or ignore variability when faced with a situation of prediction/uncertainty in which the underlying distribution is $b[x, 2, \frac{1}{2}]$.

The Conceptual Framework for this work is formed by concepts related to probabilistic reasoning and informal reasoning. The first one is the reasoning in which one of the 'big ideas' of probability occurs. These big ideas are *randomness*, *variation*, *independence* and the pair of complementary ideas, *predictability* and *uncertainty*, Gal (2005). The informal reasoning is a process in which the student builds a model of the situation, articulating several of its elements and obtaining consequences with the help of common sense and previous knowledge. Informal probabilistic reasoning is informal reasoning that involves some of the big ideas of probability.

The method consisted of four-step study with 37 high school students (15-16 year olds) who had not previously taken a course in statistics and probability. The task of predicting the results of drawing 1000 times the random variable $b[x, 2, \frac{1}{2}]$ and giving its distribution, was administered in four steps: 1) Students were asked to respond according their actual knowledge. 2) Students simulated the situation with manipulatives. 3) Students simulated the situation using the software Fathom. 4) Students were asked to respond again the same questions.

As a result, we found that there are two main patterns of responses. 1) Responses that consider the theoretical distribution but without integrating variability of the results, for example: "probability distribution: $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$. Prediction: 250, 500, 250 [expected frequencies]". 2) Responses that consider empirical distribution where variability is reflected but without consider the theoretical distribution, for example: "probability distribution: 0.23, 0.52, 0.25; prediction 234, 521, 245". Only in one case, in the final step, a student identified the distribution $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$, and predicted frequencies different of expected frequencies. In closing, some students choose the frequency approach to probability but without considering the underlying distribution. Other students calculate classic probabilities and propose determined expected frequencies without considering variability. Articulating classic and frequency approaches, including the role of variability, requires larger processes of conceptualization.

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CALCULUS STUDENTS' INTERPRETATIONS OF RATE OF CHANGE AND ACCUMULATION FROM GRAPHICAL DATA

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THEORETICAL BASIS AND RESEARCH QUESTIONS

In this paper, I examine students' interpretations of rate of change and accumulation of functions presented graphically. Prior studies have shown that students often expect a function and its derivative to have similar graphical forms (Nemirovsky & Rubin, 1992) and even students who are successful in calculus courses and correctly answer typical calculus problems often hold weak understandings of the concepts that underlie the algorithms they use (Selden et al., 2000). The research question I address here is: How do calculus students make sense of rate of change and accumulation when graphical data is provided and no explicit equations are available?

METHOD

Participants are undergraduate students at a medium-sized research university in the United States enrolled in introductory-level calculus courses. The data sources are students' written work on a problem that asks students to reason about the position, velocity, and acceleration of two cars given a graph of the cars' velocities, as well as semi-structured clinical interviews centred on a similar problem.

RESULTS AND DISCUSSION

Preliminary analysis reveals that many students in early calculus courses have difficulty interpreting rate of change and accumulation from graphical data. In the absence of explicit equations, students appear to rely on interpretations that often appear inconsistent. For example, given a graph of speed over time where Car A travels consistently faster than Car B until time t at which point they have the same speed, students will routinely claim that at time t , both cars are traveling at the same speed and also have travelled the same distance. We present case studies of student thinking, including instances when students interpret the slope of the secant lines to the speed graph to indicate average speed over time because "slope shows speed," and interpret these responses in the context of students' sense-making endeavours.

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A CASE STUDY OF PILOTING FLEXIBLE GROUPING IN MATHEMATICS IN FINNISH SECONDARY SCHOOL

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Since the level courses were removed in 1985 in Finland most schools have had fixed, heterogeneous groups in mathematics. Some schools have however started to use flexible grouping. Flexible groups can be formed by ability, activity level, motivation, future plans, learning style or group dynamics. The composition of groups can be quite established or vary even during individual lessons. The variation between schools in Finland is small but there is diversity between the students in schools. According to the research of Finnish National Board of Education 16 percent of schools used different types of flexible grouping. It seems, according to the research, that learning outcomes were slightly higher when using flexibility in forming teaching groups compared to fixed, heterogeneous groups (Metsämuuronen, 2013). It is known there is variety when flexible grouping is put into practice but the research field is almost unexplored.

This case study is from teachers' view about piloting flexible grouping at one school in Helsinki. Here we report how the participating teachers plan to form the groups, in what ways they are flexible and what difficulties and successes they expect to occur. The first interview was when they were planning the pilot for two general education classes with 47 students, age 13-14. They asked both students' and guardians' preferences in which of the three groups formed by ability and motivation student should participate. The second interview was when teachers were dividing students into the flexible groups after they had taught one course using heterogeneous groups. Teachers were debating between academic and social aspects. Should they separate friends if their academic achievements were very different? Teachers had prepared themselves the grouping would raise some emotions among students. They hoped negative feelings would be decreased because everybody had an opportunity to influence grouping, groups could be changed and all groups were using same materials and tests.

I will interview teachers again after they have taught two courses with flexible grouping and results will be discussed in the presentation more detailed.

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EFFECTS OF MATH SELF-EFFICACY ON BEHAVIORS OF ENGINEERING STUDENTS WITH POOR MATH PREPARATION

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Math preparation and math self-efficacy (math SE) have been shown to be relevant factors in students' decision to pursue an engineering major. According to Bandura's (1986) Social Cognitive Theory, self-efficacy refers to "people's judgments of their capabilities to organize and execute courses of action required to attain designated types of performances" (p. 391). For engineering students, having adequate math preparation is a key factor to complete their major. However, there is much variability in students' level of math knowledge. Analyzing the relationship of these two factors and how they influence engineering students' math performance and behavior in class could help math educators to address problems related to students' poor math preparation like college attrition rates. This study is led by this research question: How does math SE and knowledge of engineering students with poor math preparation influence students' performance and behavior in their college math courses?

This study is the first phase of a mixed methods research, and it follows a grounded theory approach to develop a theory that describes math SE perceptions of engineering students with poor math preparation, and how these beliefs could be related to students' behavior and attitudes in math classes. Six participants were randomly selected from the lowest level math courses offered at a U.S. university. Participants were interviewed about their past math experiences, reasons for choosing engineering, persistence, and their math SE using a scale from 1 to 10 to rate their confidence in performing different math-related activities. Students' high school and college test grades were collected as indicators of students' math knowledge to examine how well it supports their math SE.

Students' math SE was relatively high for most of the students, ranging from 6.5 to 8. But these students described different behaviors in their college math courses when their math knowledge did not seem to support their math SE. Findings suggest that engineering students that showed a high math SE level with lower math knowledge were more likely to be overconfident, procrastinate and struggle in their math courses due to lack of effort. On the other hand, students that showed a math SE more aligned with their math knowledge were more likely to work harder and look for extra class resources such as tutoring and office hours to resolve doubts about their math work when they struggled to understand their course material.

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PROBLEM-BASED LEARNING ASSOCIATED BY ACTION- PROCESS-OBJECT SCHEMA THEORY IN MATHEMATICS INSTRUCTION

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Problem-Based Learning process is organize in following steps: connecting with the problem, setting up the structure, visiting the problem, revisiting the problem, producing a product or performance, and evaluating performance and the problem (Delisle, 1997: 26). All of the steps in mathematics instruction are setting up by the teacher to make students constructing new mathematical knowledge. Theories about constructing new mathematical knowledge that is part of the constructivist view begins with what Piaget stated (Dubinsky, 2002) about the process of reflective abstraction. Reflective abstraction as a method of knowledge construction is the core of APOS (Action - Process - Object - Schema) theory from Dubinsky.

The purpose of this study was to obtain answers about: whether the university students as a prospective teacher can apply the Problem-Based Learning theory associated by APOS theory, whether senior high school students can be actively involved in learning, and whether the students can demonstrate the ability to acquire new mathematical knowledge. In order to get these answers, we developed a teaching materials of mathematics for high school students course and construct the instruments to get the information about prospective teacher students' ability in teaching mathematics and senior high school students to involved actively in learning mathematics and the ability to acquire new mathematical knowledge

This study uses mixed methods that consists of a research and development in its development of teaching materials and experimental methods in the implementation of lecture for students of Mathematics Education Program of Faculty of Teacher Training and Education and their teaching practice in high school.

Results of this research can be used as a supporter of the "mathematics for high school" course because it is considered able to overcome the problem of adversity to teach mathematics for high school students.

The purpose of the research can be achieved given the excellent properties of the PBL approach with the support of a complete constructing knowledge theory like APOS.

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ANALYSING CHALLENGES OF ENGAGING STUDENTS IN GENERATIVE INTERACTION WITH DIAGRAMS DURING GEOMETRIC PROVING

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Malawi National Examinations Board (2013) reports mistakes which show that students fail to construct geometric proofs because they fail to interact with diagrams. Herbst & Arbor (2004) identified four modes of how students interact with diagrams; empirical, representational, descriptive, and generative interaction. Herbst & Arbor (2004) encourage teachers to use generative interaction because it makes students responsible for their arguments (Herbst & Arbor (2004)). Generative interaction involves giving students a diagram containing little information for exploring a proof. Although Herbst & Arbor (2004) acknowledge that involving students in generative activity is difficult, they have not explored its challenges. The question that guided the study is; what are the challenges of engaging Malawian tenth-grade students in generative interaction with diagrams during geometric proving? The study used a single qualitative case study design and it is part of PhD project which aims at exploring knowledge for teaching geometric proofs. A 120 minutes lesson was observed and video recorded in grade ten in a Malawian secondary school. The teacher drew on chalkboard a circle diagram with two angles at the centre and one angle at the circumference. He asked students to work in groups to prove that angle at the centre is twice angle at the circumference. Analysis of the transcribed lesson was guided by Stylianides and Ball (2008) three components of a proof; true statements, valid modes of argumentation, and appropriate modes of representation. The findings showed that students were unable to construct proofs which fulfilled the three components because they did not know the correct angle to focus on at the centre. In response, the teacher asked the students to suspend the proving activity and do some measurements to decide on angle at the centre. This resulted into postponement of other planned activities.

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REVISITING DEMOCRATIC COMPETENCE THROUGH MATHEMATICS EDUCATION: FROM THE VIEWPOINT OF “FOR WHOM AND WHOSE BENEFITS?”

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In 1997, Keitel delivered a lecture in Japan entitled “Perspective of Mathematics Education for 21st Century Mathematics Curricula: For Whom and Whose Benefits?”. Her lecture was focused on issues such as the political dimensions of mathematics; gender; ethnomathematics, etc. In current unequal society, more focus is placed on the subtitle, because it will be the starting point for social investigation. The development of these issues is regarded to have contributed to the development of critical thinking. I confirmed that democratic competence cannot be only defined simply by explicit mathematics, it must also involve the reflective knowledge necessary to reconstruct implicit mathematics, while explaining the interests and intentions which brought it into existence (Keitel, 1997). The purpose of this study is to clarify the democratic competence by project-based teaching.

Keitel asserts, it is important to ask where students have an opportunity to reflect upon and make judgements about implicit mathematics, and to explore issues related to the mathematization and mechanization of our society. Usually, project-based teaching is used in such cases. This is where personal research is undertaken by the learner, using reference material, and making presentation in report form. In my practice, high school graduation thesis report has become the typical example of development of critical thinking. Here I will discuss one such report entitled "Degree of Cutting Corners" of Six Televisions Stations Rebroadcast Programs. The student was motivated by a newspaper article about broadcasting programs on TV. The article stated that NHK educational television is cutting corners. “Cutting corners” means rebroadcasting of programmes, and is deemed not necessary. The student wanted to know how many hours a day NHK educational TV rebroadcast programmes. For each TV station, he made a table that reveals the number of rebroadcasting time from 6:00 A.M. to 12:00 P.M. each day. Also, he found the percentage of rebroadcast number for each day of the week. This case study can be summarized as follows: while the analysis was insufficient, the student was able to achieve reconstruction of implicit mathematics. He failed to reach the conclusion that the viewers were disadvantaged. But, his critical motivation, aim, and effort contributed to his development of democratic competence.

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CHILDREN'S SOCIAL VALUES EXPRESSED IN A NUMBER ACTIVITY IN A JAPANESE KINDERGARTEN

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The interest toward mathematics education for young children has been growing for the last two decades (Meaney, et al., 2016). The article describes the implementation of a kindergarten mathematics project in Japan and reports the results, aiming at examining the effectiveness through the perspective of children's social values (Shimada & Baba, 2015). The main purpose of the activity is to find a pair of numbers that combine to make five or ten, using sweet potatoes. The objective of our kindergarten mathematics project (Mathematics for Preschool Teachers and Parents: MPTP) is to propose mathematics curriculum in the stream of the current *Course of Study for Kindergarten* in Japan. So far, we have developed several programs presenting children's activities in different areas of mathematics.

The activity was conducted in a public kindergarten in Tokyo in October 2015. Data were collected by observing and recording it. The activity consisted of three-time trials. The data were transcribed and analyzed qualitatively. The number of participating children was sixteen, aged five to six.

There are three findings in the analysis. First, Japanese children were able to compose/decompose the number five using concrete objects and that the majority of them can also count numbers up to ten (and some up to twenty). Thus, the planned activity worked effectively to foster students' cognitive development in mathematics. Second, the focus on children's social values in the activity helped demonstrate that children met the developmental requirements of the Course of Study, because they could present their ideas and listen to one another. Third, the planned activity worked effectively in terms of the continuity of the children's lives: It was not a mathematical activity but in line with their daily life. Learning context is crucially important, as much or almost as much as learning content.

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SHORTENING THE SHADOW BETWEEN GENDER AND MATHEMATICS: PERCEPTIONS OF GIRLS IN UGANDAN SCHOOLS

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Uganda's formal education terrain is fraught with major gender disparities in enrolment, dropout and general attainment, especially in basic sciences. These arise from historical and cultural factors that regard the male sex as paramount (Amodio & Devine, 2006). As Tripp and Kwesiga (2002) argue, obstacles to gender parity are embedded in the cultural norms and practices valued by the patriarchal arrangements of Uganda's society through which the policy and implementations have been modelled. While statistics on gender differences in mathematics are available, little understanding exists of the underlying factors from the students' own perspectives.

A large-scale survey of over 3000 female students in 13 selected secondary schools across Uganda was carried out in order to identify the factors affecting differential achievement in mathematics examination by girls and boys. Qualitative and quantitative data were collected and analyzed. The overall focus was to determine from the students why the gender gap exists and what could be done about it. The study ascertained the girls' beliefs about the roles of their own efforts, their relatives, the girl-boy and, girl-teacher relationships in mathematics classes. The results showed that gender disparities in mathematics performance were not perceived to stem from innate differences in aptitude, but rather arose from the students' attitudes towards the subject and their confidence in their own capability. The findings will inform a recommendation to Uganda's Ministry of Education that mathematics performance could be boosted through measures aimed at improving attitudes among girls towards studying mathematics, including introducing female role models and parents encouraging girls to consider careers involving mathematics. Although it might be hard to eradicate a counterproductive cultural condition in schools, significant steps could be taken towards minimizing it and it is hoped that this will boost girls' confidence in mathematics. The study has implications for mathematics education in other countries where a gender disparity persists.

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USING PROOF IMAGE AND EPISTEMIC ACTIONS TO TRACE THE PROOF PROCESS OF A PROSPECTIVE MATHEMATICS TEACHER

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As the importance of proving activities has been emphasized in mathematics education researches, it can be said that studies which enable the proving process to be analyzed in terms of different dimensions are needed. In this context, this study aimed to examine a prospective mathematics teacher's proof process in terms of both cognitive and affective dimensions by using the notion of proof image (Kidron & Dreyfus, 2014), a particular kind of affect – nonemotional cognitive feelings- (Clore, 1992) and the RBC (Recognising–Building with–Constructing) framework (Hershkowitz Schwarz & Dreyfus, 2001).

Data were gathered through multi-stage sampling method. The scale consisting of two open-ended questions was applied to 120 third-year pre-service mathematics teachers. Task based interviews were conducted with three students who answered those questions correctly. Because of the space restriction, only one student's proving process was presented in this paper.

The findings indicated that the student had a proof image. In the analysis of her proving process, not only the emergence of the proof image but also its transition to the formal proof was carried out. Furthermore, the epistemic actions were observed during the process and she was able to construct the knowledge that was necessary for formal proof. Besides, it was also observed that non-emotional cognitive feelings in this process played an important role in constructing the proof.

Based on this analysis we concluded that to analyze the proving process in the light of different characteristics can provide rich information how different components give directions to proving process by interacting with each other, and with the studies in this direction, the necessary steps can be identified to overcome difficulties that students have in proving process.

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HELLO FROM PROBABILITIES

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The usefulness of probability for daily life, the way in which probability reasoning support decision making and the instrumental role of probability in various curricular areas and professional work has been stated by Batanero, Contreras, Díaz, and Sánchez (2015) but other authors have written also about its teaching and learning.

In this work we present the semiotic conflicts of an explanatory study with 66 Portuguese secondary school students (ages from 15 to 17) in the northern Portugal, in the academic year 2013/2014, in solving two problems that involve the use of conditional probability, events independence, the theorem of total probability and the Bayes formula. The Onto-Semiotic Approach (Godino, Batanero & Font, 2007) was adopted in order to identify and describe the semiotic conflicts – the disparity between the student's interpretation of a mathematical expression and the meaning of the same expression in a mathematics or school institution (Batanero, Contreras, Díaz, & Sánchez, 2015) – that arose in the answers of the students in a survey. The survey was presented in a 50 minute class and had 11 multiple choice questions and students were asked to write the reasons of their choice. In this paper we analyse the answers to questions 4 (based on absolute frequencies) and 9 (based on relative frequencies), and both of them were designed to detect the semiotic conflicts of the students' answers that are referred in the literature.

As a result the semiotic conflicts referred in the literature were identified and analysed in order to support a plan proposition for classroom work and its further research.

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EXPLANATIONS DO NOT IMPROVE ALGORITHMIC REASONING TASKS

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Previous research has shown that creative mathematically founded reasoning (CMR) renders higher test results than algorithmic reasoning (AR) in a laboratory setting (Jonsson, Norqvist, Liljekvist, & Lithner, 2014). Still, most teaching and textbooks do not only provide algorithms but also explanations of the principle behind the algorithm. This is not the case in the AR condition. The tasks were designed using the mathematical reasoning framework by Lithner (2008) utilizing an a-didactical practice situation (Brousseau, 1997). However, Lithner's framework does not include 'AR with explanation' (XAR) as a category. The aims of this study are therefore to 1) explore if the efficiency of AR tasks will increase when explanations are provided and 2) evaluate a new category (XAR) for the reasoning framework.

The sample comprised 104 natural science students in Swedish upper secondary school (16-17 year olds). The participants were matched into three groups (32 AR, 38 XAR, and 34 CMR) based on mathematics grade, gender and a cognitive composite score (CPI). The students did a computer based training session with 14 task-sets that took about 40 minutes to complete. The AR and XAR groups got five tasks per task-set with a presented algorithm and with an additional explanation for the XAR group. The CMR group got three tasks per task-set without algorithms. After a retention interval of 6-8 days all students did a test with 14 task sets, where the first two tasks for each set were limited in time to test memorized information and the last was given extra time to allow for eventual (re)construction.

Analysis of students' solving frequencies show that the added explanation in XAR has no positive effect on test result compared to the AR group (as predicted by Brousseau (1997)). There are also indications that the result presented in Jonsson et al. (2014) still holds, i.e., that CMR is more efficient than AR and in this case also better than XAR.

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PRINCIPALS' TRUST AND INVOLVEMENT IN PRIMARY SCHOOL TEACHERS' ASSESSMENT PRACTICES

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This paper presents a small-scale study of principals' (N = 22) involvement in primary school teachers' work with a national mapping test in numeracy. According to national guidelines, test outcomes should be used for assessment for learning (AfL) and to plan teaching interventions with low-achieving students. AfL has been adopted as a national educational policy in Norway and is included in the Education Act. The research for this study question regards how principals involve in the teacher's work with the test.

Many teachers struggle to identify what their students can do and they struggle with as well as what the teacher can do next (Wiliam, 2007). TIMSS 2011 revealed that Norwegian principals spend little time on developing school academic and curriculum goals in mathematics and monitoring how teachers and students perform. Principal involvement, such as cooperation with the teachers, might help teachers use assessment in ways that support student learning (Goddard, Goddard, Kim & Miller, 2015) and might also lead to shared accountability and decision making, and thus, positively influence student learning (Hallinger & Heck, 2010). Analyses of data from interviews (N = 4) and surveys (N = 19) focused on how principals delegated responsibility, to what extent they trusted the teachers and how they became involved in the teachers' work with the mapping test. Nearly all principals reported that they trusted teachers to administer and score the mapping tests according to national guidelines. Furthermore, many principals reported that they collaborated with teachers to analyse assessment outcomes and plan mathematics interventions for low-achieving students. This practice has the potential for capacity building (Hallinger & Heck, 2010), which could lead to improved student achievement. Although principals recognised that teachers might lack knowledge or competence to plan good teaching interventions, many viewed the mapping test as a possible resource for professional development and capacity building within the school that could support teachers' AfL practices.

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ASSIGNMENTS OF MATHEMATICAL PROBLEMS AS TEXTS

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Reading comprehension and the ability to translate assignments of word problems into language of mathematics is seen as one of four components of Culture of Problem Solving (Intelligence, Creativity, Reading texts with comprehension and Ability to use the existing knowledge) as presented in (Eisenmann, Novotná, Příbyl, 2014). Pirie (1998) distinguishes six means of mathematical communication classified as follows: “ordinary” language (the language current in the everyday vocabulary of any particular child), *mathematical verbal language* (verbal here means “using words”, either spoken or written), *symbolic language*, *visual representation*, *unspoken but shared assumptions* and *quasi-mathematical language* (usually, but not exclusively, that of the pupils; it has, for them, a mathematical significance not always evident to an outsider, even the teacher). This variety of means of communication holds implications for pupils’ understanding of problem assignments. There are also other variables at stake when analysing assignments as a text. We focus mainly on text comprehension, on the impact of collision of ordinary and mathematical languages and on the possible impact of presence of mathematical terminology in a text. One of the important variables to be taken into account is how the assignment is presented to pupils.

Written text vs. oral text: In case of a written assignment, pupils must have some knowledge of vocabulary and grammar. The focus is on the meaning more than on the form. Input is unlikely to be modified. The activity is subject-centred and requires reading with comprehension. An oral assignment requires listening comprehension skills. The focus is mixed: both on the form and on the meaning. The input is very likely to be modified. The activity is learner-centred.

Closed vs. open problems: In case of closed (multiple-choice) problems, pupils’ understanding is checked by their choice of answer. Monitoring the pupils’ thought processes is usually impossible. Open problems require the teacher to deal both with the product and the process, taking into consideration mathematical correctness and language appropriateness. Moreover, they can also be used for diagnostic purposes.

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UNIVERSITY TEACHING ASSISTANTS' SELF-EFFICACY

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The aim of this study is to start a longitudinal research project on the teaching assistants (TAs) working at the Department of Mathematics and Statistics in the University of Helsinki. In the first phase of the project, we investigated the TAs' teaching self-efficacy. The results enable us to plan and develop the research project. The overall goal is to improve the TAs' training and practices.

In many universities, TAs have an important role in mathematics education. At the Department of Mathematics and Statistics in the University of Helsinki, the TAs are usually undergraduate, master's degree or doctoral students. The duties of the TAs vary. Lecture courses have TAs who meet with a group of students in a tutorial. There are also TAs who help students in drop-in sessions. Since 2011, a new student-centered teaching method, Extreme Apprenticeship (XA), has been used on many undergraduate courses (Rämö, Oinonen, & Vikberg, 2015). In XA, the TAs' role is different from the more traditional TAs. They coach and scaffold the students' work and model the thinking processes and practices of mathematicians. In most of the tutorial and drop-in session TAs have a brief voluntary training in the beginning of semester. The XA TAs have a pedagogical training that last throughout the semester.

We investigate the teaching self-efficacy of TAs, which is a valuable predictor for student achievement, teacher retention, and persistence in the face of teaching difficulties. The first phase of the study was conducted in the autumn 2015, and the participants were TAs at the Department of Mathematics and Statistics in the University of Helsinki (n=35). The instrument was adapted from DeChenne (2010).

The preliminary results suggest that although the XA TAs have less teaching experience than other TAs, their teaching self-efficacy is not lower. In addition, the XA TAs experience the departmental teaching climate more positively, a factor that is known to be linked to self-efficacy. The results might be caused by the more intensive training the XA TAs receive.

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SOLVING THE PROBLEM OF PROFESSIONAL LEARNING IN A DEVELOPING COUNTRY

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This paper summarises the reasons for the effectiveness of PNG lecturers teaching school teachers ways of implementing changes in mathematics education.

In Papua New Guinea (PNG), five good teachers (researchers) were training teachers who generally ‘taught’ by talking with a few demonstration objects and children copying down mathematics from the board with little understanding. To challenge this, teachers need to participate in alternatives, try them, and know where to start in planning with support from valued community leaders. In PNG, 4 to 10 day workshops with manuals or 3 days with take-home computers were possible as a catalyst for change. During these workshops, teachers were able to conceptualise the changes and differences to their current practices and implement new or modified approaches in practice. The workshops focussed on culture, mathematics (problem solving), and early mathematics education bringing them together through an inquiry approach.

After assisting in workshops, each PNG researcher/teacher/valued community leader undertook workshops on their own and provided reports and video on how they carried out the workshop and how teachers tried ideas out afterwards. A significant ‘other’, Australian team leader, analysed data and noted key issues and took the ‘story’ back for a response and confirmation to the PNGians to build the story of this paper.

The computers were a key reason for both the involvement of the PNG researchers and the teachers. These trainers were leaders maintaining their status in community by assisting their community. PNGians are proud of their cultures despite changes to it, so emphasis on culture engaged them as they saw new insights into their cultural mathematics. For the teachers, they wanted professional training on good ways of teaching (most had only 6 weeks training). They particularly appreciated use of open-ended questions and small group game activities which were implemented into classroom practice after the workshops. They also valued grappling with the problem of how to incorporate culture and language into mathematics.

As a result all PNG trainers are continuing with further study/research. All developed aspects of the workshops: organization; the resource; the video training for use of computer, solar power and battery; and a simplified model of the key values and their integration. Despite the same materials, they emphasised and implemented differently.

Having PNG researchers taking the workshops was a really important part of the workshop success: speech and metaphor, good questioning and modelling, and respect.

PRE-SCHOOL TEACHERS' CONCEPT IMAGES OF QUADRILATERALS

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Before young children begin primary school, they are at the first stage of the van Hiele levels and begin to recognize figures by appearance alone but may not realize which attributes are critical. As time goes on some difficulties and failures occurs in students' geometric achievement. In this case, we need to first focus on pre-school education.

In pre-school education period, the early-years teacher has a very important role in children's mathematical learning experiences. If the teacher does not use correct mathematical knowledge and mathematical language even at daily routines, it can lead to misconceptions later on. Teachers knowledge and language may shaped by their concept images.

The aim of this study was to find out early years teachers' concept images of quadrilaterals. In order to find out their concept images, how teachers' define and determine inclusion relations of quadrilaterals were sought. This research was a qualitative study. It was conducted with 22 early-years teachers working at public and private schools in Turkey. The data were collected by semi-structured interview. Teachers were requested to define each figure orally and then to draw one example and other two examples different than first one. Finally they were requested to determine the inclusion relations about the four quadrilaterals. The obtained data were analyzed by inductive content analysis. Results indicated that in general, teachers were able to identify only square correctly but had difficulties defining other shapes especially parallelogram and rhombus. In addition to this, all the teachers were misidentified rectangle although being very sure of themselves. Some teachers drew parallelogram as only two parallel line segments. Furthermore, the participants did not prefer inclusion relations for the classifying quadrilateral. In contrast, they often see them as independently of one another. There are studies found similar results (Turnuklu, Alaylı & Akkas, 2013; Tsamir, et al., 2015).

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5TH GRADE GIFTED STUDENTS' VIEWS ON DIFFERENTIATED MATHEMATICAL TASKS

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Societies aim to provide with opportunities to discover and reveal their gifted students' full potentials (Fıçıcı & Siegle, 2008; Maryland, 1972). Although regular classroom environments have some limitations for gifted students (Dimitriadis, 2011), their educational needs of them are crucial. Therefore, the aim of this study is to explore the 5th grade gifted students' views about the effects of usage of differentiated tasks on meeting their mathematical needs in classrooms. Data collection process of this study consists of two stages. In the first stage, Test of Mathematical Abilities of Gifted Students (Ryser & Johnsen, 1998) was administered to three classrooms in a private school in Marmaris, Turkey. Fifteen grade students were assessed based on the criteria defined in the original test. Among 56 students, 12 of them were identified as high or very high possibility of mathematically gifted. After this process, differentiated mathematical tasks which were developed with the help of design based research process were applied in classrooms. After completion of the tasks in classrooms, which lasted through 6 weeks, students' views on the effects of material usage on meeting their educational needs were explored by using semi structured interviews. Constant comparative analysis of transcribed data led to meaningful and structured categories; students' views on the content, need for learning, need for challenge and motivation. Students stated that the content of the task were different and more comprehensive than regular mathematical tasks that provides opportunities to meet their need for learning and discussing. In addition, due to including more challenge and need more thought provoking process, students were motivated towards not only doing the activities in classroom and in break but also at home as homework. This study could be beneficial to address a way for meeting special needs of gifted students by filling the gap both in practice and theory.

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HOW TO DESIGN AN APPLICATION INCLUDING PROBLEM POSING TASKS BASED ON MATHEMATICAL MODELLING

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The purpose of this study is to design an application including problem posing tasks based on mathematical modelling and to explain how we interfere with it and what the interventions are. We adopted an educational modelling (Kaiser & Sriraman, 2006) which elicits learning processes and fosters understanding of concepts and used mathematical modelling as a purpose of mathematics education. In accordance with our purpose, we used the design based research method (Van der Akker, Gravemeijer, McKenney & Nieveen, 2006). The first step of design based research is to introduce students with mathematical modelling and to give them experience with mathematical modelling tasks and their solutions. Then, we initially designed a prototype application including problem posing tasks based on mathematical modelling and then applied it to ninth graders. In the prototype application there are three tasks and it lasts three weeks. We gave photographs as premises, since in mathematical modelling tasks, photographs can be reminder, visualiser, helper, and trigger for constructing mathematical model. We gave a photograph with a 'book fair' theme to all groups in the first task. In the second task different photographs were given to each group. In the last task we set them free for choosing theme/subject they wanted. For each task we asked students to pose at least three modelling problems, to choose the most suitable problem for mathematical modelling and to solve it. During the application we did some interventions. Most of them are about out of classroom activities. We carried out some out of classroom activities with different groups for each activity. The application designed took seven weeks during which the first and second tasks were divided into two parts, the problem posing and solving. Whereas the last task was divided into three parts, the problem posing, solving and presenting of the solution. With this application designed, mathematics teachers/educators can apply problem posing tasks based on mathematical modelling. These interventions and tasks can be easily adapted/changed/improved for different perspectives and different visual tools are used instead of photographs.

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FEATURES AFFECTING SUCCESS IN FLEXIBLE EQUATION SOLVING – A CASE STUDY

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Comparing different solutions is a good way to gain flexibility (e.g. Rittle-Johnson & Star, 2007) and student collaboration is widely recognized as a core instrument in mathematics. In the Flexible equation solving (FES) project, peer collaboration alternates with teacher-directed sessions. The FES material includes comparison tasks and emphasises conceptual understanding. In this case study, Finnish seventh graders Matti and Anni study linear equation solving. They were chosen based on the great difference in their behaviour before and after the FES experiment. The data consists of selective transcriptions of 11 lessons, pre- and post-test results, interviews and observation notes. Seven goals of the FES guided the classification of the data while 10 new categories arose from the analysis. The purpose of this study was to identify the features affecting two students' ability to adapt to the FES material and method.

Before the FES experiment, Matti's mathematics grade was 7 (on a scale of 4 to 10) and Anni's 9. During the FES, Matti's low calculations transformed into an ability to discuss mathematical problems and to advise others. Anni, instead, had enjoyed quiet individual work during normal lessons, and her difficulties arose in small group work. Content analysis of the data identified several typical patterns related to behaviour and learning. For Anni, they were: difficulties with concepts; lack of verbalising; answer-orientation and reluctance to compare different solutions. For Matti, they were: emphasis on concepts; verbalizing even when nobody actively listened; advising others; process-orientation and engagement in comparison tasks. According to the post-test, Matti gained flexibility of 16/20 and Anni 7/20 (class average 8.3; SD 1.4).

Matti and Anni, two very different learners, succeeded differently in FES compared with their normal lessons. Star et al. (2015) have studied statistically different student characteristics related to flexibility. In their study, good prior knowledge, low flexibility in pre-test and the female gender connected with high flexibility in post-test. This case study has identified several other potentially important features behind the flexibility in equation solving. These features should be the object of further study.

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ONLINE DIAGNOSTIC ASSESSMENT OF CLASSIFICATION IN THE BEGINNING OF SCHOOLING

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The ability of classification has long been recognized as one of the fundamental building blocks for mathematics and science learning and for formal reasoning (Piaget and Inhelder, 1958). Due to the face-to-face nature of assessment methods in early childhood carrying out regular diagnostic assessments is often hard to realize. Technology-based assessment may provide solutions for developing efficient, reliable and easy-to-use instruments even in early ages (Csapó et al, 2014). The purpose of this study is to develop an online assessment tool for classification and to examine the psychometric properties and the usability of the test.

Participants were students beginning school in September 2015 who were selected for a new sample of the Hungarian Educational Longitudinal Program (N=6013, age Mean=7.10, SD=.49). The 8 computerized tasks to assess classification were part of an inductive reasoning test. In the first four classification tasks students had to classify five figural elements into two sets. After this their task was to classify eight figures into two, and again two (applying a different rule), then three and four sets. Students could listen to the instructions via headphones and they could move the objects on the screen by drag-and-drop function. Instant feedback was given after test completion. Before the assessment an ICT familiarity test was also administered to provide opportunity for practicing basic mouse use skills. The data collection was carried out in schools' computer rooms by the eDia system.

The reliability of the test was acceptable Cronbach alfa=.75. Our test proved to be moderately difficult (M=45%) and there were large differences between the children at school entry (SD=29%). The most difficult task was when students had to classify the 8 figures in a different way into two sets (M=21%). More detailed picture can be gained about students thinking by the analyses of the content of the responses. IRT analyses revealed the need for further item development for low and high achievers as well.

Our study demonstrated that the innovative features of technology-based assessment such as pre-recorded instructions, manipulative items and automatic scoring made our tool an easy-to-use diagnostic instrument in everyday classroom context.

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ADULT PRE-SERVICE PRIMARY TEACHERS' MATHEMATICAL IDENTITY WORK

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Research on mathematical identity work of adult pre-service teachers has not been explored so much. This research project focuses on prospective primary pre-service teachers' mathematical identity work for adults with negative feelings about mathematics. Here mathematical identity is understood through those lived experiences by which pre-service primary teachers explain their relationship to mathematics and their mathematical lives (cf. Kaasila, 2007).

The data is collected between years 2011–2015 on 190 adult pre-service primary teachers (average age 33–39 years) during mathematics education course. The data consists of 'Me and Mathematics' –essays written in the beginning of mathematics education course, reflective learning diaries, and results of arithmetical surveying. For this paper I have purposively chosen 4 pre-service primary teachers with negative feelings about mathematics as research participants. The selected cases are particularly information rich and the expressions in learning diaries are vivid and rich. The aim of this paper is to understand these purposively chosen 4 pre-service primary teachers' identity work during mathematics education course in a hermeneutic phenomenological framework.

From the data analysis I have found *empowering*, *irresolute*, and *future connected factors*. *The empowering issues* were mainly social learning atmosphere, active working methods, and the support families and friends. *The irresolute factors* were mostly pre-service primary teachers' abilities to learn mathematics, and mathematics teaching as a teacher. *For the future* these pre-service primary teachers felt that mathematics course had given them confidence to mathematics learning and teaching. From the development point of preservice teachers' mathematical identity work I agree with Di Martino, Coppola, Mollo, Pacelli, and Sabena (2013) that teacher educators' role is a fundamental in supporting pre-service primary teachers' efforts to re-build their relationship towards mathematics, mathematics learning and teaching.

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REALIZING MACRO-SCAFFOLDING FOR PERCENTAGES IN MATHEMATICS CLASSROOMS – A FIELD EXPERIMENT

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The influence of academic language proficiency on achievement in mathematics has repeatedly been shown. Due to that, several approaches for *fostering students with low language proficiency* in math classrooms were developed. Some of them follow the *design principle of macro-scaffolding* with the aim to coordinate a conceptual learning trajectory with well-structured language learning opportunities (Gibbons 2002).

In several iterative design research cycles, a *macro-scaffolding intervention on percentages* was designed which intertwines a conceptual (adaption from van den Heuvel-Panhuizen 2003) and a lexical learning trajectory (Pöhler & Prediger 2015). They start from students' everyday resources and aim at flexible use of learned concepts.

The research was conducted as a quasi-experimental field experiment in *pre-post-test-design* with *whole classes* of grade 7 and investigates the *functioning of the intervention* by comparing the effects on students' performance in a standardized percent test in an intervention and a control group (n = 29 students each). The effects are investigated separately for different problem types (Find base, Find amount, Find base after reduction) and formats (pure, visual, text). Both groups are comparable with respect to fluid intelligence, language proficiency and mathematical pre-knowledge.

The empirical results show that the intervention group tends to outperform the students who attended the traditional course, especially with respect to more complex problem types and the pure format, although the intervention rarely focuses on items without any context. The good results of the control group in the visual item with the (possibly unknown) percent bar, can be interpreted as an indicator for a good accessibility of this visual model.

The study is limited by a small sample size, which leads to not yet significant group differences. However our results encourage us to continue the study with bigger samples. The insights on different problem types and formats will be taken into account for optimizing the instructional design and supplemented by qualitative analyses.

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UNDERSTANDING THE EMERGING COMPLEXITY OF PROFESSIONAL DEVELOPMENT

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In this paper, accounting for the complexity inherent in mathematics teaching practices within technology, we focus our attention, from the research point of view, on educational programmes for supporting teachers to integrate the use of digital technologies in their practices, aiming to help with their professional development. We examine the challenges of differentiating between micro and macro levels of complexity when dealing with unpacking teachers' professional development from a dynamic point of view through specific examples, contextualised in teachers' education at secondary school level, with the use of digital technologies. We use a theoretical framework comprising the Meta-Didactical Transposition model developed by Arzarello et al. (2014) that considers the evolution of the teachers' and researchers' praxeologies over time, and the notion of emergence that provides us with better insights into how the praxeologies' components have been constituted at the micro level. We applied the co-working of these two models in a cross-national analysis of a teaching experiment, centred on the introduction of GeoGebra in Australian and Italian secondary teachers' educational programmes. Through the lens of the co-working of these two models, we observed that for making the component of praxeology internal to teachers' communities, researchers fostered the implementation of activities involving GeoGebra and the mathematics laboratory practice as agents. Independent agents that were active during teachers' practices interacted with other agents at the micro-level to cause dynamic changes in the teachers' praxeologies when using GeoGebra for a particular activity, shaping their components, and leading to the development of new and/or shared praxeologies, as emergent phenomena arising from the process of teachers' development. However, not all the teachers reacted and changed in the same ways their praxeologies: for some of them change could not happen, for others it was immediate as an unexpected new awareness of teaching practices with GeoGebra, for others it took more time. But even if the final product could be different, the occasion of meetings between the community of researchers and that of teachers is nonetheless fruitful for the interaction of agents.

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GEOMATECH PROJECT: THE PROFESIONAL DEVELOPMENT OF MATHEMATICS TEACHERS

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In recent studies the professional development of teachers is observed through the joint work of researchers and teachers. In the particular context of the Geomatech, an EU-Funded project focuses on integrating technology into the entire Hungarian education system. The collaboration between researchers, trainers and teachers helped both to build innovative situations and to develop technology resources for the K1-12 mathematics and science curriculum in Hungary together with a range of pedagogical approaches.

The Geomatech project involved multiple collaborations among researchers, teacher trainers, teachers and programmers. The project involved almost 250 staff including 75 trainers, who gave 60-hour training for 2500 teachers in 950 Hungarian schools in close collaboration with the other teams. The training was developed and trialled during an 8-month period, when the research team closely worked with 25 mathematics and 20 science teachers on developing technology resources and pedagogies for their use. During this period, participating teachers and their students were interviewed periodically, and at least 5 classes were videotaped for further analysis. As a result Geomatech developed close to 2000 high-quality technology resources and related pedagogies based on the analysis of the trial period data and good practices from Hungarian and international initiatives. Furthermore, a questionnaire was sent to all participating teachers to gauge the development of their technology use and connected conceptions. The data was analysed using the theoretical framework of the theory of didactic incidents and documentational genesis (Gueudet and Trouche, 2012) that allows to analyse and better understand the dynamics of the joint action of reserachers, trainers and teachers to enhance teachers' professional development.

In our talk we will outline key findings of the analysis of multiple collaborations based on the theoretical frameworks and highlight some examples of Geomatech. In addition, we will show how Geomatech is being developed in other countries and how international comparative research emerging from the project.

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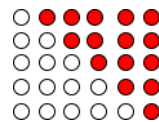
NOT JUST AHA!, BUT WOW!: CHILDREN'S AESTHETIC JUDGEMENTS OF MATHEMATICAL EXPLANATIONS

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In discussions of mathematical aesthetics, much focus has been on the dispositions of mathematicians (e.g. Inglis and Aberdeen 2015; Burton 2004). This leaves open a more foundational question of how aesthetical dispositions develop in the first place. Are young children capable of making similar kinds of judgments as mathematicians about the mathematics they do? At what age can a student reliably experience the beauty of a piece of mathematics? This is an exploratory study, focusing on fifth grade students (age 10), which examines how children judge the aesthetics of different kinds of mathematical explanations. Students were interviewed about the aesthetic merits of different explanations, as part of a larger project that maps aesthetic judgments and experiences from school age children to mature mathematicians.

As a work in progress, our results are tentative. One example comes from children working on a task about triangular numbers. Four students were given a task involving triangular numbers, represented with dots (1, 3, 6, 10). They were asked how many dots will be in the tenth triangular number. The hundredth? The students then were shown several ways to derive the formula, from simple counting to more sophisticated reasoning using a diagram like the following.



Students were asked to compare the different explanations, and they preferred this one with the rectangular diagram. Reasons they gave included seemingly superficial reflections such as “because it is more fun”, and “children like colors”, as well as structural/aesthetic properties, such as “simpler and easier”, and “draws attention more”. These results are similar to those we found for mathematicians comparing different proofs of the Pythagorean theorem. In that case the favourable proofs were “simpler,” “easier”, and “generalizable.” While students do not perhaps have the maturity to make all the same kinds of judgments of mathematicians (e.g. about generalizability) it would be wrong to say they made no aesthetic judgments at all.

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EFFECTS OF LOGICAL REASONING AND SELF-EXPLANATION TRAINING ON PROOF COMPREHENSION

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Proof plays an important role in mathematics, and it is vital that students learn not only to construct proofs, but also to read them. It has been noted that students sometimes read proofs ineffectively and are not able to evaluate deductive arguments (Weber & Mejía-Ramos, 2014). To improve students' reading, Hodds, Alcock and Inglis (2014) designed a self-explanation training booklet and showed that self-explanation training can significantly improve proof comprehension.

In this research, we investigate whether self-explanation training has different effects for students with different logical reasoning abilities. The data were collected in two phases. In September 2016, first year mathematics students at University of Helsinki were given a logical reasoning test (adapted from Evans, Clibbens & Rood, 1995). In November 2016 the same students were given a proof comprehension test. Before completing the proof comprehension test, participants were randomly assigned to receive self-explanation training or a control activity.

At the conference we will report our analysis of these data. Our hypothesis (following Hodds, Alcock & Inglis, 2014) is that students with different initial capabilities in logical reasoning might benefit differently from self-explanation training. One possibility is that the students with good logical reasoning abilities already employ self-explanation skills, and training does not change their behaviour much. In this case we might expect the students with poorer logical reasoning skills to benefit more. Another possibility is that the students with good logical reasoning skills do not employ them as well as they might, so that training will help them to put their knowledge to more effective use. In this case, we might expect students with better reasoning skills to show greater gains. Understanding these effects would be useful both theoretically and in designing undergraduate programs.

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ONLINE ASSESSMENT OF EARLY NUMERACY AT SCHOOL ENTRY

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Early numeracy is an important basis of the basic arithmetic skills which develop during the first years of primary school (Aunola et al, 2004), and it predicts students' later mathematics performance (Aunio & Niemivirta, 2010). Therefore it is crucial to identify problems in time by assessing early numeracy, and to promote successful school entry. In the 21st century technology-based assessment could be the key to carry out large scale school readiness assessments without excessive effort (Csapó, Molnár, & Nagy, 2014).

Our aim is to develop an online early numeracy test which can be a reliable tool for teachers to assess their students' early numerical skills. We have created a new, colorful, and carefully designed testing environment to assess early numeracy in age 5-7 which grab and retain young children's attention.

The online early numeracy test included 6 subtests (Magnitudes and number words, Number word sequence, Relations, Basic counting, Numeral recognition, Magnitudes and numerals), overall 40 items. Our sample consisted of first grade students representatively selected from primary schools of Hungary (N=5,149), the mean year of age was 7.09 (SD=.48). Data collection was administered through Internet in the school's computer laboratories during the first three months of school. Class teachers supervised the testing process.

The early numeracy test proved to be reliable (Cronbach- α =.89). And the 6-dimensional CFA model based on the subtests showed an acceptable model fit as well (CFI=.933; TLI=.928; RMSEA=.033). The average achievement was 80.32%p (SD=15.92), due to some subtest (e.g Magnitudes and Numbers) happened to be easy for the first graders. Based on these results we are planning to revise some subscales and develop new items to improve the test.

With further improvements the online early numeracy test can be a useful educational tool to assess early numeracy and provide valuable information for teachers to design their teaching process.

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DISCUSSING A PRIMARY PROSPECTIVE TEACHER PRACTICE AND ANALYSIS ON A MEASUREMENT EPISODE: THE ROLE OF VIDEO ANALYSIS

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Research reveals the need for designing resources for teacher education. One of such resources aiming at developing, from one side, teachers' awareness and knowledge for teaching and, from other side, the educators perception of such revealed knowledge and awareness concerns the video analysis of one's practice. Teachers' reflection and discussions when interpreting, analyzing, and reflecting upon the recorded interactions (e.g., Sherin & van Es, 2002), is perceived as a pathway for developing their knowledge and professional noticing of children's mathematical thinking. In the case studies we have been working on, the MKT conceptualization (Ball, Thames & Phelps, 2008) is perceived as an analytical tool with two aims. To analyze, from one side, Ana's (a primary prospective teacher) revealed knowledge when in practice and, from other side, the role of video analysis when she analyzes her own practice – in an episode aimed at exploring the use of non-standard length measurement units. We discuss the results concerning Ana's practice and the video analysis process. In practice, Ana reveals knowledge on anticipating students' difficulties, but also difficulties in interpreting a student use of non-standard measurement units in a non-standard way. When focusing on Ana's analysis of own practice the results enhance the need for improving the use of video-based tasks in teachers' education the and impact of prospective teachers' analysis in and for educators' professional development.

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MATHEMATICS TEACHERS' SPECIALIZED KNOWLEDGE ON FEEDBACK: A THEORETICAL APPROACH

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Teachers knowledge has been one of the areas in which research has increased substantially in the last decades, including the development of several different teachers' knowledge conceptualizations. Such increase is also reflected on the amount of papers presented at the last PME's. On other hand, feedback is perceived as a crucial aspect for students' learning (Nicol & Macfarlane-Dick, 2006). A review of the works presented at PME revealed a scarce amount of research focusing on feedback, focusing the large majority essentially the effects of feedback on students learning (e.g., Santos & Pinto, 2009). In that sense, the content of teachers' knowledge in and for providing fruitful feedback seems to be left aside. Aiming at fulfilling some of these aspects, a research project has been designed, aimed at, amongst others, developing a broader and deeper understanding of the content of teachers' knowledge (assuming the Mathematics Teachers Specialized Knowledge conceptualization—Carrillo et al., 2013), specifically linked with feedback, as well as its influence in practice.

We have to note that it is not an evaluative approach to teachers' knowledge (revealed and expressed in practice), but radar to understand and contribute for/with ways for its improvement. We will present and discuss some aspects of the intertwined nature both of teachers' knowledge subdomains and feedback. For doing so, a theoretical approach (model) is being developed, which will be the focus of discussion, complemented with some examples from practice – involving both primary prospective and practice teachers.

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TEACHER PRACTICES THAT FOSTER ELLs' CONCEPTUAL UNDERSTANDING OF INITIAL FRACTION CONCEPTS

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Although educational reform aims to develop conceptual understandings, students identified as English Language Learners (ELLs) face the difficulty of developing language and mathematical understandings simultaneously. This struggle is reflected in the 2015 NAEP results, where only 14% of fourth graders identified as ELLs performed at or above proficiency level, in contrast with 40% non-ELLs doing so. Teacher practices (TPs) have been shown to influence the conceptual development of learners with a wide range of language and learning needs, as they profoundly influence the ways in which students learn, and the type of knowledge they construct (Confrey, 2013). But how TPs provide access to and support for ELLs' conceptual understandings, such as fractions, remains unexplored. Two groups of researchers have contributed to our understanding of this issue. Language and mathematics education scholars have mostly used a socio-cultural perspective on learning, identifying TPs that support ELLs to increase their participation in mathematical practices. Although those TPs engage ELLs on problem solving practices, how TPs also support conceptual understandings of a particular topic remains unexplored. A second group has used a socio-constructivist perspective on learning, identifying TPs that support conceptual understandings; however, most practices place high language demands as they are rooted in discursive action (i.e., asking students to justify and explain their thinking). Understanding how TPs might support both participation and concept development will provide insights into how instruction might meet support conceptual understandings with ELLs. This study plans to observe and document TPs in two fourth grade classrooms over a twelve-week period, as they teach initial fraction concepts to a class with a high number of ELLs. Teachers will be selected following an extensive set of criteria; student's achievement, teacher preparation and class observations. Class and group discussion transcripts will be analysed using the methodological framework proposed by Steffe, Thompson & Von Glasersfeld (2000). First, I will conduct a line-by-line analysis in which inferences will be made about the ways teachers provide access to and support ELLs' conceptual understandings. Based on the literature review and pilot observations, I hypothesize there are arrays of practices that support, to varying degrees, ELLs' conceptual understanding of initial fraction concepts. To determine the utility of TPs, I will relate them with evidence of conceptual advances on students' actions and interactions around mathematical activity.

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THE MATHEMATICS DISCOURSE IN INSTRUCTION FRAMEWORK: FROM ANALYTIC TO PEDAGOGIC TOOL

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The Mathematics Discourse in Instruction (MDI) framework (Adler & Ronda, 2015) has developed out of the question: what are key tools and processes of mathematics instructional discourse? The MDI framework is based on socio-cultural theories of learning and empirically informed by mathematics teaching data in the project schools of the Wits Maths Connect Project in South Africa. The framework characterises the teaching of mathematics as involving the mediation of an *object of learning* via *exemplification* and the accompanying *explanatory talk* together with the opportunities provided for *learners' participation* in mathematics discourse. One of the research aims of the project was to describe mathematical discourse in instruction with respect to the degree to which it makes possible the development of scientific concepts (Vygotsky, 1978). Thus, MDI as an *analytic tool* was developed to enable descriptions of (a) whether and how the *examples* in a lesson, and the *tasks* in which they are embedded accumulate towards generality; (b) whether and how the mathematical content is *named* and how criteria are transmitted to *legitimate* what counts as mathematics; and (c) the nature of *learners' participation* in the discourse. We have shown the potential of the MDI Framework for systematic analysis of what is made available to learn in teachers' lessons and in textbook lessons. The present study describes the use the MDI Framework as a *pedagogic tool* in the development of a sequence of lessons on function in the Philippines. As a pedagogic tool it asks different questions but the constructs in the framework nevertheless remain the same. The results show the potential of the tool as discursive resource or artefact in teachers' professional development especially in foregrounding key elements of the mathematics discourse into which the learners need to be initiated.

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A PROJECT BASED LEARNING APPROACH: DEVELOPING MATHEMATICAL COMPETENCES IN ENGINEERING STUDENTS

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DESCRIPTION OF THEORY AND THE PROJECT

Mathematics applications are present in an engineering environment; in the particular case of mathematical models, there are several applications where an industrial process can be idealized and taken to a mathematical model, this in order to control or predict its behavior. A pedagogic theoretical framework that supports various techniques for teaching is constructionism, which is a powerful design tool for transforming passive activities into highly engaging, thought-provoking, educationally rich experiences (Ackermann, 2010), one technique we can use is called Project Based Learning (PBL) which provides several advantages: Students become co-responsible for their learning, it focuses on the concepts and principles of a discipline, involves students in investigations troubleshooting and other meaningful tasks, allowing them to work independently to build their own knowledge and culminates in realistic products.

PBL aims to introduce students in mechatronics engineering to develop their math skills and integrate knowledge engineering in general. Due to the innovative and unconventional nature, a strategy was planned to create a prototype that allows students to explore, simulate and verify the behavior of differential equations. In this case, the problem of coupled as tanks course project was selected. Learning activities and evaluation were guided in order to review the system theoretical and practical content, allowing students to engage in research and understanding of mathematical and simulation elements. Students thought the project was a challenge, and allows them to develop their technical skills and creativity. The PBL has added a several new dimensions to the educational experience of engineering undergraduates at Polytechnic University of Chiapas. These include vertically-integrated, large scale, multidisciplinary teams. Multi-year student participation that enables them to experience all phases of the design, programming and construction process, and large-scale design projects in a learning context. Data from student evaluation demonstrate that PBL is a model where professional skills, including teamwork and communication, can be learned while participating in realistic projects.

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TEACHERS' AND RESEARCHERS' VOICES CONCERNING THE MATHEMATICAL MEANING CONSTRUCTED IN THE CLASSROOM

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Mathematics classroom has been the focus of the research attention in the recent years due to the recognition that it constitutes a crucial learning site for both students and teachers. Considerable part of the relevant research aimed to identify teaching practices which shape pupils' learning while many studies inquired into 'teaching as learning in practice' within communities involving teachers and educators (e.g., Jaworski 2015) and/ or the quality of teaching mathematics (e.g., Ball et al 2008).

The study reported here is part of series of studies looking at students' mathematical meaning construction as a result of teaching practices (e.g., of dealing with pupils' misunderstandings) in varied mathematical contents (e.g., algebra versus geometry) and in critical educational contexts (e.g., in implementing a new curriculum) (e.g. Kaldrimidou et al 2008). The results of the above studies highlight the close relation between students' mathematical meaning-making and teachers' management of the subject matter and its pedagogy in action.

The study concerned focuses on three primary teachers' reflections with respect to the mathematical meaning shaped in the contexts of their reformed classroom teaching as well as on the respective analysis carried out by educators working with the teachers for the reform of their practice in an attempt to identify critical features of the alignment process. The results indicate significant tensions between the design of the reformed teaching practices and the outcomes of the practice due to contextual factors as well as to weaknesses related to teachers' knowledge for teaching mathematics.

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SELECTION AND ORCHESTRATION OF DIGITAL RESOURCES FOR GEOMETRY BY PRIMARY-SCHOOL TEACHERS

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We present two case studies from a research that focuses on the selection and orchestration of digital resources by primary-school teachers experienced in the use of technological tools. The case studies (of teachers from a public school in Colombia of 1st and 5th grades, respectively) describe the paths that each of these teachers followed when working in a lesson related to geometry topics, and illustrate possible paths of orchestration and selection of resources.

Our research is based on the documentational approach (Pepin, Gueudet & Trouche, 2013). Thus, we understand by resources, everything that supports and gives meaning to the teacher's professional activity (e.g. textbooks, different mathematical representations, digital media, classroom discussions and reflections, etc.). We are interested in how these are orchestrated (i.e. how, when and where they are used; and articulated within the classroom dynamics). Our two teachers from the case studies have been through several teacher-training programmes for the use of digital technologies, and usually use digital resources in their classrooms. We observed each of these teachers in one of their geometry classes, and conducted pre- and post-interviews with each one, in order to analyse their documentational processes and resource orchestration. First, we analysed how each of the teachers selected their resources, in terms of which criteria they used: technical, curricular, mathematical or didactical. We then looked at the orchestration of the resources during the class. The 1st grade teacher had a "local" aim for his lesson, asking students to identify different types of lines using as resource a virtual game, which he selected mainly through technical and didactical considerations, rather than mathematical; his orchestration centred on the interaction between each individual student and the resource (the game). The 5th grade teacher had a "global" aim, where students would design a house over several sessions, and required measuring lengths and areas, using as resource an interior design application, which was selected using mostly technical and mathematical criteria; his orchestration emphasized collaboration and the interaction between pairs of students with the resource. We observed that the paths of selection and orchestration of the resources depend on the teacher's didactical aims and requirements, and involve decisions related to their professional expertise.

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TROUBLING ASSERTIONS OF PEDAGOGICAL CONTENT KNOWLEDGE

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Current debates on the nature of pedagogical content knowledge (PCK) have centered on whether, and how, the boundary between PCK and other forms of knowledge should be drawn. What has been missing from the discussion, however, is a careful examination of how what lies within the boundary of PCK itself has been determined. In this paper, a critical reflection of the fundamental assumptions underlying PCK is offered. Through an analysis of key writings on PCK (see Shulman, 1986, 1987) and accompanying research programs (see Ball & Bass, 2000) that support them, three central orientations about knowledge, learning, and teaching are explored. It is argued that, in conceptualizing PCK as the “capacity of a teacher to transform the content knowledge [...] into forms that are pedagogically powerful” (Shulman, 1987, p. 15), Shulman seemed to hold an objectivist view of subject matter knowledge, one in which such knowledge is given and grounded in objective reality. Further, the philosophy of ‘transforming the subject matter in ways accessible to students’ has fostered the association of subject matter as an object of teaching (rather than an object of learning). From this perspective, subject matter is considered as a sort of package, where the quality of ‘making the content accessible’ depends on the quality of “deconstruct[ing] [...] [the subject matter] into a less polished and final form” (Ball & Bass, 2000, p. 98). This language masks some of the most significant assumptions underlying PCK: an implicitly linear, remarkably narrow, and technical transmission view of teaching that was challenged more than a decade ago since the first appearance of PCK and that is decidedly out of keeping with contemporary understandings of learning. Questions are raised about what these orientations mean for teaching as a profession, for conceptualizing teacher professional knowledge, and for scholars’ understanding of the teaching-learning complexity. It is concluded by sketching a reorientation of the basic assumptions underlying PCK that embraces the complexity of the teaching-learning process and acknowledges teachers as agents in promoting students’ learning progression.

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STRATEGIES FOR COPING WITH MATH INQUIRY ASSIGNMENT

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There exists a call for a vision of math education, one that will prepare young people for the new realities of the 21st century. This had lead to new math curriculums, including the new standards by the NCTM (2000) and the Israeli math curriculums (2006 - in Hebrew). Problem solving otherwise inquiry assignment, is one of the key components of such high-quality school mathematics program. Problem solving, according to Schoenfeld (1992), refers to a question for which students have no immediate apparent resolution, nor an algorithm that can be directly apply to get an answer. This type of assignments forces the students to collaborate, in order to create new knowledge and makes them learn how to think critically, creatively, and to discover new techniques.

Our research deals with one type of inquiry assignment – emanating from the Square Exercise first appeared at Patkin (2011). We added to the basic Square Exercise more and more stages, checking each new stage with students, until we built a new inquiry assignment that 6th and 8th grades students are able to tackle. The inquiry assignment was presented to 20 Students from the 6th grade and 20 students from the 8th grade. Each time we asked the class' math teacher to provide us with 4 to 5 students of different math proficiency levels to cope with the assignment, and taped the groups' conversations while they were solving the problem. In addition, after each assignment, we interviewed 2-3 students from each group, for better understand their answers and their reasoning.

Our preliminary research goals and hypotheses were as follows: (1) To identify and rank students' strategies for their work upon inquiry assignment. No hypothesis. (2) To contrast and compare strategies applied by the 6th grade students with those applied by the 8th grade students. Our hypothesis was: High ranked strategies applied by the 8th grade students. We identified 3 levels of strategies: (1) A "trial and error strategy" with 2 separate sub-strategies. (2) A "generalization strategy" with 3 sub-strategies. (3) A "generalization and proof strategy" with 4 sub-strategies. We detected no difference between the strategies that the 6th grade students used and those the 8th graders used.

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THEORETICAL PERCEPTION OF THE CARTESIAN PLANE: A DUAL EYE-TRACKING STUDY THROUGH DOUBLE THEORETICAL LENSES

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Our study concerns the advantages of dual theoretical analysis of teaching-learning processes, from the Husserlian phenomenological approach and Vygotsky's cultural-historical approach. The paper takes further Radford's 'poetic moment' (Radford, 2010) in merging opposed theoretical perspectives in the study of the emergence of mathematical perception. In particular, the focus of this research is *the role of cultural means*, namely verbal expressions and gestures, and of the *intentionality* of a child in the formation of joint attention and of theoretical perception of the Cartesian plane, by first grade children. For the closest possible understanding of the interaction between a child and a parent with gestures and verbal expressions we combine eye-tracking data with synchronised verbal protocols and video material. A unique feature of our research is that we have put *two eye-trackers side by side* and managed to calibrate two participants at the same monitor, so that they would both look at the same diagram; thus they had actual *shared* space, as is the case in common teaching processes. They were also able to manipulate a pointer, which allowed natural gesticulation. 5 pairs of a parent and a Grade 1 child participated in the research. The Cartesian coordinates were chosen as material within the zone of proximal development: the first grade children had never met the Cartesian plane before but they were able to solve the tasks with the parent's help. The 'moment' of shared understanding was achieved in each analysed case: we observed situations of *joint attention* as a crossing of a child's and an adult's gazes. The children were not simply 'following' the already enculturated adult but they were actively accepting and taking hold of the 'invitation' of culture themselves. Hence we tracked close encounters of the phenomenon of joint attention, as linked to the intersubjective, original *intentional synthesis* of the *learner's intentionality* (Husserl, 1970) and the *adult's cultural practice*. (Vygotsky, 2004). Thus, the analysis from the Husserlian phenomenological approach and Vygotsky's cultural historical approach demonstrated the dialectical synthesis of *personal intentionality* and *cultural practice*, and it accentuated their interplay as a principal source of learning.

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PRESERVICE TEACHERS' CONCEPTION AND METAPHOR OF PROBLEM-POSING AND THEIR PROBLEM POSING PERFORMANCE

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The purpose of this study is to examine preservice elementary teachers (PSTs)' conception and metaphor of problem-posing. We also investigate how their conception of problem posing is reflected in mathematical problem posing performance. Data from this study came from 96 PSTs from two different university sites. A written task was used for the study, which consists of two parts (see Fig. 1). In the first part, they were asked to answer three open-ended questions regarding their conception and metaphor of problem-posing. The second part is to create a word problem when given a fraction multiplication expression (improper fraction \times whole number) and then to solve their problem. We found that out of 96 PSTs, 70 PSTs defined problem-posing as a feature of mathematical teaching. However, unlike Silver's first perspective, the PSTs in this study did not specifically mention the type of instruction (e.g., inquiry-oriented instruction) in defining problem-posing. In addition, 30 out of 96 PSTs defined problem-posing in light of its connection to problem solving. Among them, half of PSTs simply focused on the aspect of finding a solution as a final goal of problem-solving rather than problem-solving which consists of successive re-formulations of an initial problem through the use of self-questioning technique and metacognitive skills. One interesting thing from this result is that none of the PSTs viewed problem-posing as a window into students' mathematical understanding. This finding makes sense when considering that PSTs did not have enough teaching experience to work with students. Furthermore, we found that there appears to be a positive association between PSTs' notion of problem posing and their ability to pose and solve a fraction multiplication problem. This study contributes to the current literature on problem posing and the knowledge base of teacher education. This study suggest elementary mathematics teacher education programs needs to include more problem-posing activity so that PSTs can experience the benefit of problem-posing from a variety of perspectives (Cai, & Hwang, 2002; Son, 2016). This study also suggests the importance for future research to continue to investigate PSTs' conceptions of problem-posing.

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MENTAL REPRESENTATION AND MATHEMATICAL ABILITY IN SOLVING ILL-STRUCTURED PROBLEMS

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The mathematical ability and the mental representation are psychological constructs that integrate mathematical thinking. Knowing them, can support the teaching practice by proposing maths problems to students for them to solve. The mental representation, in Johnson's Laird (1983) view, adapted to the case of mathematics, it can be: propositional (P), analytical (A) or propositional-analytical (P-A), as they are based on language strict rules (propositional) or without rules/informal rules (analytical). The mathematical ability, according Krutetskii (1976), is a complex mental structure which assesses following the acquisition, processing and retention of information. Students may be, temporarily, classified as having more ability (MH) or less ability (mh), depending on the speed of the solution, independent thinking, creativity and establishment of efficient relations between the elements of the problem, according to the perspective mentioned above.

Correlational studies on these variables can increase understanding and teaches acting on them, thus, leading to the following research question: 'Is there a relationship between mental representation and math ability demonstrated by students engaged in ill-structured math problems?'

The qualitative study was carried out on twelve undergraduate Computer Science students, aged between 17 and 22. The aim of the study was to get the students to individually solve (out loud), five Krutetskii's ill-structured problems. Ill-structured problems for which there are no known algorithms to solve them with. The mathematical ability and the mental representation were analysed from the record audio and video-tape of their oral and written protocols with use of speaking aloud technique and under theoretical premisses of Krutetskii (1976) and Johnson-Laird (1983).

The analysis revealed: two students (MH) with representation (P-A); six (MH) with (P); one (MH) with (A); three (mh) with (P). Highlight from the analysis, the lack of representation flexibility offered by (mh). The conclusion is that there was diversity of mental representation for (MH) and uniqueness of representation (P) for (mh), which leads to other questions and recommendation to continue the study.

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TEACHERS ENCOUNTERING CHALLENGING WORD PROBLEMS: HOW DO THEY SOLVE THEM?

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‘Problem solving’ is an important aspect of mathematics learning and teaching. In fact, Cai et al. (2014) stress that problem solving competence ‘reflects students’ degrees of performance proficiency in mathematics’ (p.233). Nevertheless, school students fail to connect problem solving with the real world (De Corte et al., 2000) and thus fail to successfully solve them. This tendency has been observed in primary school student-teachers as well (Verschaffel et al., 1997). The purpose of this pilot research project is to fill a gap in the relevant literature by extending the results mentioned above to in-service primary school teachers.

The study’s sample consisted of 32 primary school teachers (some of them having more than 15 years of experience). A paper-and-pencil test containing ‘problematic’ (from a realistic point of view) and ‘non-problematic’ items was delivered to them.

The results indicate that most of the teachers of the sample (even the experienced ones) solve word problems without taking into account the realistic requirements of their context. Amongst the outcomes of the statistical analysis, two tables are offered. The first categorises the teachers according to their ‘problem solving approach’. The other classifies word problems in order to create a ‘problematic context’ scale. Jointly, these tables are offered as an informative and powerful tool for both research and practice. Their role in the fields of teacher education and teacher professional development is particularly stressed and exemplified.

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NUMBER SENSE IN CHILDREN FROM DIFFERENT LIVING CONTEXTS

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Children acquire their mathematical knowledge from the activities they perform at home, at school and on the streets. The present study has investigated number sense in 100 low-income Brazilian elementary-school children. Number sense refers to the general understanding of numbers, their properties, uses and meanings in a variety of situations (Greeno, 1991). Half of the participants lived with their families, and half lived in an orphanage. Following from previous research (Spinillo & Batista, 2009), the children were asked to make judgments about situations involving both numbers and measurement. Also, some of them were observed in their living environment so as that we could identify the types of mathematical activities they were engaged in when at home. The hypothesis was that there would be differences between the two living contexts and that the two groups of children would differ in their mathematical knowledge. The children who lived with their families performed better than those who lived in the orphanages. The children in the orphanage performed poorly both in the tasks involving numbers and in those involving measurement. The children living with their families performed better in the tasks where they were required to deal with measurement than in those where they had to deal with numbers. The research has also revealed differences in the frequency and types of mathematical activities performed by the children in the two living contexts. It can therefore be concluded that the mathematical activities children perform in their home environment, just like those they perform at school and on the streets, play an important role in the acquisition of mathematical knowledge. Further research is needed in order to better understand the impact of institutionalization on children's mathematical knowledge.

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COMMON ALGEBRA ERRORS IN UNIVERSITY COURSES: EXISTENCE, PERSISTENCE AND SIGNIFICANCE

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The purpose of this project was to identify common algebraic errors students make in university level mathematics courses that challenge their ability to succeed in higher level mathematics courses. For many years, university instructors have viewed the final problem solving steps in their respective disciplines as “just Algebra”, but a weak foundation in algebra may be the cause of failure for many students. Although, research on students’ difficulties with algebra at school has been well documented, methodical studies on the presence of these difficulties and their impact at university level are scarce.

This project employed Tall’s (2013) framework of embodied, symbolic and formal mathematical thinking in an effort to construct a model of mathematical thinking for investigating students’ conceptual and procedural understanding of algebra. In an effort to identify common algebra errors at the university level and gain insight about how best to develop appropriate interventions, this project is focused on the following research questions: 1) In what areas of algebra do the major errors occur? 2) What are the most persistence errors and their significance in calculus courses? 3) How would algebra interventions affect students’ understanding of calculus and help them with more symbolic ways of interpreting mathematics? 4) As students advance into various STEM disciplines and encounter more formal mathematics, how does the lack of understanding algebra affect them and what interventions would help them? 5) What are some of the pedagogical challenges in the symbolic world? For the purposes of this paper we will present research findings related to the first and second questions.

This study was conducted in a mathematics department at a large research university in Southwest of the United States. Data were gathered from approximately 3076 students’ examinations (final and midterm exams) from a variety of first year mathematics courses in Fall 2014. The data revealed numerous errors with *fractions* type error, in finding the limits, derivatives and domains of functions as well as calculating for max and min points, which suggested this error is still dominant and persistent at this level. In many cases, students are working to learn concepts that are new to them in calculus courses and the results on assessments, formative or summative, are often more reflective of student difficulties with algebra than the newer concepts such as limits and differentiation. The frequency of such errors creates frustration for both students and their instructors and may create barriers to student advancement in mathematics.

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FAMILY MATHS EVENTS RUN BY AFTER-SCHOOL PROGRAMMES: EMERGENT RESULTS FROM A 3-YEAR PROJECT

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As early as 1974 Bronfenbrenner, wrote about the importance of family involvement in child development, particularly with regard to the success of intervention programmes. Since 2013, the South African Numeracy (SANC) project at Rhodes University has supported a number of schools in setting up and running Family Maths Events which aim to get families talking and enjoying maths together as well as encouraging a ‘maths is fun’ ethos. Following positive experiences, in 2015 as part of a larger 3-year partnership project with 5 local after-care centres, I extended this work by coaching and supporting maths club facilitators at to run such events at their centres. Both the research and development work are premised on a Vygotskian (1934) perspective of learning which emphasises the critical importance of both language and social interaction (including peer and adult mediation) in learning. Methodologically, this research takes an interpretive, qualitative approach and explores possible benefits of running such Family Maths Events. Written questionnaires were administered at end of the 1st year of the three-year project to 12 after school maths club facilitators who ran the events from the 5 after-care centres.

Initial results suggest the facilitators perceive the following benefits: 1) The events encourage families learning together both at the centres and at home; establishing closer links between the centres and the families as well as promoting dialogue between children and their parents about maths. 2) The children gain a sense of achievement in participating in the events and give them the opportunity to show their parents what they know and can do (mathematically). 3) The events allow the centre staff to show parents how simple resources such as dice and cards can be used at home to support their children in their mathematics learning. 4) Facilitators report that both children and parents would like these types of event to happen more often. These findings cohere with anecdotal evidence from running Family Maths Events in the SANC project. It is hoped that the findings will be stronger as the three-year project continues.

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DIFFERENCES IN ADULTS' ATTENTION ON NUMBERS DURING THE READING OF MATHEMATICS ITEMS

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Students in their first years of school tend to show a certain focus of attention to numbers when solving word problems. This accumulates e.g. in the phenomenon of solving unsolvable 'age-of-the-captain'-problems by just inserting the numbers presented in the problem into an operation that seems to produce a sensible result (Verschaffel, Greer, & De Corte, 2000). A way to reveal the focus of attention in reading word problems is the method of eye tracking (De Corte, Verschaffel & Pauwels, 1990). We investigated if specific attention to numbers persists in adults when solving complex word problems and if it depends on academic achievement, mathematical self-concept, gender, or age.

17 adults (11 female, 6 male, aged $M = 31.3$, $SD = 4.36$) solved six PISA items presented on a computer screen while their eye movements were recorded. Areas within 1° of the visual angle of numerical information were defined as 'number areas'. The percentage of characters in these areas in relation to the complete task was 13.6%. In regular reading, the number and duration of fixations on these areas should somewhat match this ratio (Rayner, Pollatsek, Ashby, & Clifton, 2012). Fixations longer than 300 ms were excluded as they were not part of a regular reading process. A questionnaire about final grades, age, and gender was answered.

On average, participants fixated the task 880.8 times ($SD = 245.0$). 267.4 of these fixations occurred within the number areas ($SD = 88.4$), which is 30.4%. Fixation times show a similar pattern: While the complete task was on average fixated for 145.5 s ($SD = 44.1$ s), the number areas account for 44.8 s ($SD = 14.7$ s), which is 30.8%. Both ratios differ significantly from the baseline percentage ($p < .001$). No significant correlations were found between these ratios and neither the number of correct answers, final grades, age, gender, nor mathematical self-concept. We conclude that a strong focus on numerical information seems to persist in adults irrespective of these individual characteristics.

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SYLVESTER'S PARTITION THEOREM AS A TEACHING UNIT OF MATHEMATICAL PROOF

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Every student in junior high school in Japan must learn how to prove a geometrical proposition. However, only 20 percent of these students understand how to construct such a proof fully, and almost all advance to senior high school. Therefore, there should be another curriculum for teaching mathematical proofs throughout secondary education. We focus on a theorem proposed by James Joseph Sylvester: The number of representations of a natural number n as a sum of consecutive natural numbers is equal to the number of odd divisors of n (Wittmann, 2001). The theorem is considered instructional material for pre-service elementary-school teacher education.

As the first teaching experiment, the authors conducted a survey related to constructing representations of the sum of consecutive natural numbers, targeting students of a junior high school. The survey revealed that there was a gap between the students' awareness of the existence of the theorem and the students' awareness of the uniqueness of the theorem. It is thus advisable for senior-high-school students to pay attention to uniqueness. In the second teaching experiment, we carried out two types of lessons in two senior-high-school classrooms. In the first class, students learned to find the sum of consecutive natural numbers from odd divisors. In the second class, students learned to deduce odd divisors from the sum of consecutive natural numbers. We regard two types of learning content as mappings of a relation to two finite sets. This offers the key to an understanding of the relation between the numbers of representations of a sum of consecutive natural numbers and the numbers of odd divisors. If the number of sums of consecutive natural numbers is not equal to the number of odd divisors, we would be concerned about injection or surjection. In the two cases of injection and surjection, the contents of anxiety are different. Analysis of the two lessons reveals that there was little quantitative difference in the awareness of uniqueness between students of the two classes. However, we found that some students, as evidenced in their comments on the lesson, made interesting comments related to anxiety.

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ON TEACHING BASIC KNOT THEORY TO SEVENTH GRADERS

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In view of the increasing applications of knot theory in certain area of sciences, a project was launched that explored if basic knot theory could be taught to seventh graders in Taiwan by way of organizing an extracurricular course on tying traditional Chinese knots. During planning, it was hypothesized that students' prior experiences with physical knots might impede with their learning of basic concepts in knot theory. This paper concerned itself with the topic on understanding the equivalence among mathematical knots (Adams, 2004). We examined what prior concepts were held by students regarding the discernment of the equivalence between two knots, and whether such concepts would change after instruction.

In accordance to the nature of extracurricular course, eight classes with hands-on experience in tying knots were organized with twelve students attending. The lessons focused on the definition of knot, drawing knot diagrams and an informal explanation of the equivalence of knots. Prior to instruction, interviews were conducted in which each student followed the same procedure in tying a string and a wire, respectively, into a knot. They were then asked to explain if the two knots were the same. Afterwards, the two end points of each knot were sealed and the students were again asked to explain if the two objects were knots and whether they were the same knots. The mathematical meaning of a knot and the equivalence between two knots were introduced subsequently in classes with instructional sheets and hand-on activities related to knots. As a posttest, students were each given a left-handed and a right – handed trefoil knot diagrams and asked to explain if they were equivalent.

Due to limitation of space, we briefly reported overall findings here. It was found that both prior to and after instruction, most students relied on whether the tying processes were the same to discern the equivalence among knots. If the tying process was unknown, most students relied on the similarity of the shapes of the knot diagrams for discernment. In the posttest, six students actually made the two trefoil knots but discerned their differences based on physical features rather than by deformation. Two students thought the two trefoil knots were not equivalent based on opposite crossings of their knot diagrams. The implication from this study is that though basic knot theory can be taught in junior high school, support must be provided to help students make concept change from the concept images of physical knots to the concept definition of mathematical knot and the equivalence of knots (Vosniadou, 2005).

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PROCEDURE AND UNDERSTANDING IN NUMBER THEORY

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ELTE

Our research is a part of a larger survey on the status of Number Theory in Hungarian elementary- and highschool education. The research originated from an observation that Hungarian university students do not remember basic notions and theorems of number theory from the high school curriculum. Earlier, surveys on the importance and use of Number Theory were proceeded in (Zaskis, 2006), or (Ball, 1990) where it was shown that problems with the addition of fractions arise from the deficiencies of the basic concepts of the number theory.

First we assessed what was taught to the students from number theory and what they remember from these. In this part of our research we compiled two tests. Then we wanted to check if the pupils know what they earlier stated to remember. The tests involved elementary questions from number theory. We formulated the questions in a way that they should be answered by pupils of grade 6. The questions of the test were related to the fundamental theorem of arithmetic, greatest common divisor, least common multiple, divisibility rules, divisors, division algorithm and relative primness. We asked about the same concepts in different ways, for example:

- In a bath there are two pools. One of them is cleaned every 24 days, the other one every 28. If both of them are cleaned the same day, the bath is closed the whole day. When will be the bath closed next time, if it is closed today?
- Find the LCM of 12 and 8.

The tests were done by more than 1200 pupils of grade 8-12 from different types of schools throughout the country. The tests were graded by the quality of the solution, e.g: started but did not finish, made a numerical mistake, etc. Students made many typical errors and we collected them. In our talk we analyse the many typical errors and find what is behind them. For example there were many problems by finding the greatest common divisor of two numbers. Pupils typically factored both numbers, but obviously, they did not know why.

Students forget the most fundamental issues of number theory because they learnt only the procedure behind and did not understand the concept. Our next move would be a series of case-studies on what pupils have in their mind about Number Theory.

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HOW DO FINNISH AND HUNGARIAN STUDENTS SOLVE COMBINATORIAL PROBLEMS?

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A comprehensive study was started in Hungary and Finland to assess the basic elements of different types of mathematical thinking. In Hungary the first step was a research of the combinatorial problem solving strategies of students in elementary and secondary education. The following study is a part of this wider research, in which we compare the combinatorial thinking of Finnish and Hungarian students. The 179 Finnish and 429 Hungarian students, both in elementary and secondary education, did the same combinatorial test (16 problems) of enumerative problems.

The curricula of both countries emphasize the importance of problem solving. Sriraman & English (2004) highlight the idea that through combinatorial problems students can learn mathematical processes of representation, reasoning, abstraction, generalization and forming connections which are important features of mathematical thinking and problem solving. However, a substantial difference can be found between the curricula of the two countries in the topic of combinatorics.

Regarding these differences in the curricula, it is not surprising that the results of the tests were substantially different than we had estimated. Consequently our analysis focused on the two following questions. Q1: What kinds of problem solving strategies are used by students in the elementary schools without applying formulas? Q2: To what extent the use of formulas is emphasized among secondary school students?

From the first data analysis we have found that the elementary students in both countries are more creative and have more visual solutions than the other students of the target group. They can develop their own strategies for solving problems without using formulas or knowing the common structure of problems in different contexts (Sztányi & Csíkos, 2015). We have also found differences in Finnish and Hungarian students' problem solving skills, e.g. in the problem solving persistence.

The first results of this project give an insight into the problem solving strategies and its teaching in both countries.

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MULTIMODALITY IN MATHEMATICS CLASSROOM DISCOURSE: AN ETHNOGRAPHIC STUDY

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Mathematics classrooms are becoming linguistically diverse – for example, in Canada, 17.5% of the total population reported speaking at least two languages at home. This context prompts us to seek for understanding the characteristics of language used in mathematics classrooms. Conceptualizing language, I bring the perspective of multimodality in mathematics classroom discourse. Mathematics in the classroom consists of multimodal resources including linguistic, symbolic and visual forms of representation (O'Halloran, 2015). In this paper, I will present findings from an ethnographic study in an urban school in Canada. I focused on two Grade 4 mathematics classes and data collection lasted over an academic year. The data collected include video recordings of classroom interactions, interviews with the teacher, and ethnographic fieldnotes taken during the school visits. Based on the discourse analysis of video-recorded interactions between the teacher and students, I will demonstrate how different modalities of classroom mathematics discourse communicated different concepts. The representative excerpts demonstrated the complexity of classroom mathematics discourse, even when the teacher minimized the use of written texts. My analysis of interactions showed how spoken language and visual forms of representation highlighted different aspects of key mathematical concepts. For example, when the teacher and students were communicating the concept of equivalent fractions, visual representations highlighted the relationship among fractions whereas spoken language emphasized the algorithm of dividing a numerator with a denominator to understand the equivalence. Different visual representations of fractions, such as a rectangle and a circle, afforded different meanings of fractions. Despite the teacher's efforts to reduce written texts from mathematics lessons, the classroom discussions still presented linguistic complexities. Academic literacy in mathematics classroom goes beyond merely words but involves mathematical proficiency, practices, and discourse (Moschkovich, 2015). The representative interactions highlighted in this paper show the necessity to make explicit connections among different modalities of classroom mathematics discourse.

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INDIGENOUS INTERCULTURAL EDUCATION: UNVEILED CONTRADICTIONS IN SCHOOL MATHEMATICS ACTIVITY

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This article examines the dialectical contradiction involved in indigenous school mathematics activity developed by Brazilian indigenous undergraduate students, from different languages and logic of life, who participate in the Indigenous Teacher Intercultural Education Course with emphasis on Mathematics Education. The contradiction was identified when students were asked to forecast a personal budget for a scholarship, filling a spreadsheet.

I used the Cultural-Historical Activity Theory (Engeström & Sannino, 2010) as a theoretical and methodological approach to highlight the tensions and contradictions triggered in the activity when each student had to predict his/her future expenses and record them in a spreadsheet, using the academic mathematics language, which was aligned with the capitalistic logic. The data were collected during the formative intervention (Engeström & Sannino, 2010), including ethnographic records.

The analysis focused on a specific activity system; *Planning and management of scholarship*. When I identified its *object* “the filling of the spreadsheet by students to plan the expenses on course”, *tensions* were perceived between this and other components of the activity system. These tensions accumulated and resulted in a *contradiction*: a personal budget is required for each subject, whose life primarily focuses on the present and is grounded in collective logic. The contradiction was overcome only after a complex negotiation in which the students created *new rules* to the activity. Teachers and students sharing the authority moved toward greater autonomy for the students to fill in the spreadsheet, since they could incorporate some aspects of their culture. Thus, the budget was considered as a wide phenomenon of culture, taking into account its use, collective necessity, and rituals.

The study showed that the spreadsheet was a powerful *artifact* to regulate students' expenses, but changes occurred in the activity system because the *object* of an intercultural activity is unstable and resists attempts at control and standardization.

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A CASE STUDY ON TAIWANESE HIGH SCHOOL GIFTED STUDENTS' PROGRESS OF MATHEMATICAL THINKING

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The main purpose of this study is to investigate the progress of six Taiwanese high school gifted students' mathematical thinking by means of Tall's framework of "three worlds of mathematics" which includes embodied, symbolic, and formal worlds (Tall, 2013). Tall thinks that the development of individuals' mathematical thinking is based on their set-befores and met-befores, while the knowledge structures can be encapsulated through mental compression, refinement, to thinkable concepts, making links and constructions of knowledge structures within these thinkable concepts to form crystalline concepts by means of expansion, deconstruction and reconstruction, specialisation and generalisation (Tall, 2006, 2009, 2013). For the learning of mathematics, mathematical thinking is the core of problem solving. Hence, studying students' mathematical thinking in their problem solving process should be beneficial for teachers to understand their students' mathematical learning and make necessary adjustment of their teaching.

Case-study approach is adopted in the study. The six research subjects are all high school students with at least PR97 of math scores of the Basic Competence Examination in Taiwan (BCE is the national exam taken by all 9th graders before entering high schools). The study lasts four months. Qualitative data, such as classroom observations, students' exam and worksheets, interviews, are collected and analysed while they participate in an extra advanced mathematics course.

The research results show that students' problem solving strategies and cognitive levels could be clearly interpreted and predicted by means of the framework of the three worlds of mathematics. Besides, how their mathematical knowledge and thinking progresses from the concrete to the abstract is also revealed. It is suggested that this framework could provide a reference model for interpreting the progress of the development of individuals' mathematical cognitive structures not only on the theory-based but also the practice-based. The study might imply that teacher's purpose of teaching, especially to the gifted students, should be focused on inspiring students to discover mathematics and enjoy the pleasure of success. It requires the teacher to have deeper understanding of students' cognitive development, and how they apply their mathematical thinking while learning, as well as the structures of mathematical crystalline concepts.

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MULTIPLICATION PROBLEMS POSED AND MODELLED BY CLASSROOM TEACHERS ABOUT FRACTION

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One of the most difficult mathematical topics for both students and teachers is multiplication in fractions. One of the reasons for this difficulty for students is that fractions have limited use in daily life. For teachers the reason may be related to their limited information about fractions. If teachers enrich the teaching of fractions through problem posing activities and modelling, teaching may result in conceptual learning instead of surface level learning of the topic. Therefore, teachers should have skills of problem posing and modelling. In this context the aim of the study is to examine the problems about multiplication in fractions posed by classroom teachers and the models they use in the problem solving process.

The study was designed as a case study in which a descriptive approach was employed. The participants of the study were ten classroom teachers. The data of the study were collected through three operations which involved multiplication in fractions. These operations were “ $\frac{1}{2} \times 4$, $\frac{1}{4} \times \frac{3}{5}$ and $1\frac{1}{3} \times 2\frac{3}{5}$ ”. The participants were asked to write down a problem for each operations and to solve each problem using modelling. The data obtained were examined using descriptive analysis method which is part of qualitative research techniques. In order to identify which type of problems were posed by the participants the problem types developed by Işık (2011) were employed. In revealing models used by them the models in fractions were taken into consideration.

In regard to the first operation the participants mostly posed problems in which repeated additions were used. For the second operation they calculated the $\frac{1}{4}$ of $\frac{3}{5}$ of a plurality or a whole. In regard to the last operation the problems posed in the type of simple exercises were used, but there were participants who did not answer it or who incorrectly produced a solution. Concerning modelling of the first problem they mostly employ a whole (bread, bagel, pizza etc.). However, for the second and third problems they tried to employ a region or area. In fact they could not model the third problem. The findings of the study indicated that classroom teachers should be informed about modelling through in-service training activities.

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GENERALIZING IN EARLY CHILDHOOD

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Abstraction and generalization lies at the heart of mathematical activity and attracts the interest of research in mathematics education. Abstraction is considered as a process during which students directly reorganize former structured mathematics in a new mathematical structure (Hershkowitz et al., 2001), while generalization could be identified as the level where students, starting from specific situations, proceed to more general conclusions and further identification of patterns, structures, relationships, rules etc. (Sriraman, 2004). Most of current research focuses on generalizing processes in algebra and addresses older students (Zazkis, et al., 2008; Lannin, 2005), while relevant studies in younger children examine partially generalizing abilities mainly related to patterns and structures (Mulligan, 2013). Thus, there is less evidence concerning generalizing in early ages.

In our study, we present a part of a wide research aiming at examining young children's generalizing abilities. As earlier approaches questioned this possibility, we wanted to examine if an appropriate teaching methodology could achieve this. For this reason, 23 preschoolers participated in a four month teaching intervention with appropriate tasks in varied topics (figures, patterns, measures and numbers) and were pre and post examined in tests designed to examine their abilities. During the intervention, the children were involved in "generalizing experiences" during which they were systematically encouraged to identify common characteristics, relationships and patterns in different situations, express more general ideas and formulate conclusions or rather overall rules.

The results coming from the comparison of the tests in all aforementioned topics indicate that these young children improved significantly their abilities to reflect on their own activity, to express their ideas and reach to conclusions, initially 'locally' related to their own personal experiences and later, after a series of activities and relevant discussion, at a more general level.

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MINUS SIGN IN ALGEBRAIC EQUATIONS: STUDENTS' STRATEGIES AND DIFFICULTIES

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Algebra plays a crucial role in middle school mathematics curriculum but, as mentioned in the literature, many students have difficulties when negative integers are included in algebra (e.g., Şandır, Ubuz, & Argün, 2007). Students' difficulties in negative integers are not only related to their conceptions of negative numbers but also related to their poor conceptions of minus sign regarding its functions: unary (e.g., $x+6=-7$), binary (e.g., $3x-2x=4$), and symmetric (e.g., $-(x+1)=4$) (Vlassis, 2008). With an aim to examine how middle school students use and interpret minus sign in algebraic equations, we explored 46 7th grade (13-14 years old) Turkish students' strategies and difficulties in solving algebraic equations. Data collected through students' written responses to a test, including three additive (e.g., $6-x=8$) and three multiplicative (e.g., $-4x=24$) structured equations, revealed that few students could provide the correct response, particularly for question 2 ($6-x=8$) (9%) and 5 ($-4x=24$) (24%) where the solutions required were negative in the additive and multiplicative structured form, respectively. However, more students provided the correct response for question 1 ($3-x=1$) (74%) where the solution required was positive in the additive structured form. Students' written responses to the test and semi-structured interviews with five students indicated two main difficulties: omitting the minus sign in solving equations and solving multiplicative structured equations as if it was additive structured. In solving equations, the minus sign was omitted because of students' limited understanding on the changing function of minus sign when crossing the number to the other side of the equation or focusing on numbers rather than the function of minus sign. Multiplicative structured equations were solved as if they were additive structured equation because of the ignorance of the multiplication operation or changing the sign of the multiplier number when crossing the number to the other side of the equation as if in the additive structured equation. Besides that no big difference was observed on successful and unsuccessful students' preferred solution strategies, arithmetical or algebraic. Generally speaking, students experience difficulties while differentiating unary and symmetric function of minus sign. In sum, the focus in teaching practices related to the equations should be on understanding the meaning of the procedures rather than procedures itself.

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HOW MENTAL ROTATION TRAINING INFLUENCES CHILDREN'S ARITHMETICAL SKILLS

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Several studies conducted in cognitive science have shown concurrent relationships between math skills and other skills in elementary students. Its results are important in determining general teaching guidelines and educational policies. However, to be coordinated with educational practices in the classroom, those results should be complemented with research that accounts for both the genesis of those skills at the individual level and the nature of their relationships. In that vein, we propose to study how spatial training influences children's arithmetical skills taking into account two opposite and complementary approaches: abilities as traits that can be statistically measured versus skills as systems of operative and discursive practices that a person uses to solve certain type of problematic situations (Gonzato, 2013). We assume two presumptions based on related research (Dowker, 2005): First, there is not a unique arithmetical skill, but several diverse skills that interact with each other. Second, understanding of arithmetic concepts implies the capacity of applying them in word problems and re-interpreting them flexibly in different representation systems. A pre-test post-test methodology with experimental and control group will be used with around 50 third graders. Arithmetic tests containing addition problems in both numerical format and word problems will be applied to both groups. Children in the experimental group will be trained between tests on mental rotation exercises. Meanwhile, children in the control group will solve crosswords. Based on the analysis of test results, problematic situations with selected students will be proposed in order to better understand how mental rotation training influences arithmetical skills.

Two pilot studies have been realized. Some results show that children who make lot of mistakes (measured as wrong responses) in certain numeric calculations, improve notably after the mental rotation training session; nevertheless, they rarely make mistakes when the same kind of arithmetic problems are posed as word problems, even before the training session. These results support the idea that mental rotation exercises can influence procedural arithmetical skills whose conceptual counterpart is not problematic. We hope to draw attention on the importance of investigating the nature of the relations between skills and not simply stating such relations.

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UNIQUE ATTRIBUTES OF YOUNG TALENTED STUDENTS IN PROBLEM SOLVING-VIRTUALIZATION & IMPRONOVATION

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"No mathematician should ever allow himself to forget that mathematics, more than any other art or science, is a young man's game" (Hardy 1992, p. 6). According to Hardy, anyone who has mathematical aptitude must work to develop their talent while still young, before the decline in creativity begins to take hold with the progression of age. 'Kidumatica' was established in pursuit of this goal– as a framework for the detection, development and research of young talented students (Amit, 2012). The purpose of this study is to examine the cognitive characteristics of the youngest students in 'Kidumatica'. It employed a mixed method, combining qualitative and quantitative data collection and analysis. Data collection was based on 100 hours of observations focusing on the Problem Solving (PS) activity of 19 gifted students, ages 9-10, who participated in 'Kidumatica'- the math club for excellence and creativity. The analysis included three stages: A. Identifying cognitive characteristics that arose during PS activity. B. Defining a 'Critical Event' (CE) when students demonstrated a solving method that reveals one of the characteristics. C. Quantifying the characteristics according to the frequency of the CEs.

The study found two unique characteristics that do not appear in the research literature. The first is "Virtualization" (15 CEs): the ability to look at a problem as if it was a virtual reality while reaching a solution. The second is "Impronovation" (Improvisation & Innovation) (15 CEs): the ability to improvise and find an innovative way of solving a problem while lacking any prescribed mathematical tools. We also found five characteristics that were discussed in previous studies: Creativity (45 CEs), Reflection (24 CEs), Generalization (20 CEs), Argumentation (18 CEs) and connectivity (15 CEs). The study shows that the young gifted students do not just share the characteristics of talented adults, but that they have added value that arises from the fact of their being young. Instead of resorting to familiar algorithms taught at later stages of mathematical education, these students rely on imagination and improvisation. This study provides additional validation for the creation of programs such as 'Kidumatica', which are aimed particularly at younger gifted students.

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OUT-OF-FIELD TEACHING AS BOUNDARY CROSSING

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Out-of-field teaching of mathematics is a reality in many secondary schools in Australia, especially in ‘hard to staff’ schools that is, schools in rural and remote locations or that service low socio-economic provincial or metropolitan communities. Out-of-field teaching of mathematics occurs around the world and is of increasing international concern and interest due to its association with reduced student outcomes, and increased teacher attrition. Perceptions of out-of-field teachers and teaching practices is often enshrined in deficit thinking (author, 2010), but an alternative perspective using theories of boundary crossing (Akkerman & Baker, 2011) provides a means of understanding the processes and factors enabling teachers qualified in another discipline specialisation to learn to teach mathematics. The *Out-of-Field Teaching: Sustaining Quality Practices Across Subjects* project (funded by the Australian Research Council) is a longitudinal study, which investigates the changing landscape of out-of-field teachers’ perceptions and practices over three years using case study methodology.

Akkerman and Baker (2011) describe domains as socio-cultural discourses with characteristics of expertise and practice such that boundaries bring these cultural and knowledge differences to attention. Teachers in our study are not only crossing boundaries when teaching out-of-field but may also be crossing boundaries of cultural difference when teaching in a rural school. Context, support mechanisms and personal resources are factors which contribute to teachers’ identity as out-of-field teachers (Hobbs, 2013) and therefore impact on teachers’ capacity to cross boundaries and transform their practice and identity.

The cases presented in this short oral communication arise from the first year of the three-year study and provide evidence of discontinuity and continuity of discourse and practice across discipline specialisations including mathematics and the dialogic processes contributing to learning to teach mathematics through boundary crossing.

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THE DOUBLE ROLE PLAYED BY EXAMPLES IN A THIRD-GRADE CLASS: THE CASE OF THE GOLDBACH'S CONJECTURE

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Examples play an important role in the activity of *reasoning-and-proving* (Stylianides, 2008); however it has not received enough attention by researchers (cf., Lockwood, Ellis, Dogan, Williams & Knuth, 2012). Moreover, one of the most common limitations reported in the literature is the production and acceptance of empirical evidence as sufficient evidence to validate universal true statements.

In this short oral communication we present the analysis of two classroom episodes to show that: (1) exploring examples at elementary school level might contribute to the reasoning-and-proving activity, and (2) third graders can be aware of the insufficiency of empirical evidence to provide valid support to mathematical claims.

The two episodes are part of one 90-minute session in a third-grade class in which the first author was a guest teacher who has worked with a group of 24 third graders as part of a research project focussing on aspects of proof. The episodes are from one of the later sessions, after around 28 hours of work with these students.

We shed light on the relationship between empirical evidence and proving, and young students' perceptions. We show that grade three students are able to understand the role of counterexamples in refuting general statements, at least in the given context, and that the students recognise that examples are insufficient to justify a general statement that applies to an infinite set.

While there is a large body of evidence in the literature that other students in other contexts do not understand the relationship between empirical evidence and proving, our results suggest that this lack of understanding is not universal, and more research is needed into the factors that support or interfere with this activity.

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HEURISTICS EMPLOYED BY PROBLEM SOLVERS ENGAGED IN A ROBOTICS-BASED TASK

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Despite efforts to improve students' mathematical problem solving, the majority of students still face great difficulty with problems that require analysis or creative mathematical thinking; that is, nonroutine problem solving. In *How to Solve It*, Polya provides a list of heuristics that he advances as problem-solving strategies that often prove fruitful in solving nonroutine problems (Polya, 1945). Recently, robotics-based task have received attention for their potential to facilitate the learning of heuristics. In this study, we sought to investigate the purported potential of robotics-based task to stimulate problem solvers to use Polya's heuristics by exploring the following research question:

- What heuristics are employed and how by problem solvers during their attempt to solve a robotic arm task designed to stimulate the applications of trigonometry in its solution?

The sample consisted of four participants who enjoyed solving problems: a high school student, an undergraduate student, a graduate student and a university professor. Structured, task-based interviews, audio and video recorded, were coded using a modified version of Kilpatrick's problem-solving coding system. Data was collected during an initial and final interview to gather participants' mathematics background and affect towards robotics, and to capture the participants' awareness of the heuristic behaviors they implemented. A checklist matrix was used to analyse data from the initial and final interviews.

Each of the 13 heuristics in this study was used by at least one participant with nine heuristics used by all participants. Despite the overlap in usage of heuristics among the participants, the heuristics were implemented in a different manner across participants. These differences highlight the complexity of Polya's heuristics, and the journey of proficient problem solvers to polish heuristics over time. The used of the heuristic "Varying the Problem" and the "Looking Back" problem-solving behavior during the problem-solving episodes were encouraging results because these behaviors, although crucial in the problem-solving process, occur rarely. Thus, we believe robotics-based tasks do have the potential to stimulate problem-solvers to use Polya's heuristics. A larger study will be worthwhile.

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INTERDISCIPLINARY COMMUNICATION BETWEEN MUSICIANS AND MATHEMATICIANS: AN EXPERIENCE WITH STOCHASTIC MUSIC

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We present partial results of a research focused on the analysis of communicative and cognitive processes in instances of interdisciplinary collaboration between musicians and mathematicians. Our interest lies in two global issues: (1) recognize the inherent complexity of human knowledge, whose development is based on the specialization of communities in different areas of knowledge as well as the integration and interaction between the communities; (2) analyse the music-mathematical relationship from individuals who act and interact, not just from the perspective of disciplinary knowledge. That is why we have focused on communication and cognitive processes that occur between individuals specialized in knowledge and language from different communities, who meet to achieve a common goal. Interdisciplinarity will be understood as a local phenomenon and described from the interaction between participants' actions.

So far, we have conducted two experiments where students from both areas were invited to face a challenge of musical composition inspired on the Stochastic Music, developed by the architect and composer Iannis Xenakis (1992). From the two experiences we highlight some aspects that could contribute to the characterization of an interdisciplinary communication between both disciplines. During the experience, music and math students took different roles worrying about those objects that are part of their communities' discourses, however they achieved a major integration when recognize common elements and when they took off their formal and specialised discourses in favour of the common goal. They were able to create a data communication system: for one group, the graphics line allowed visualization of melodies from a set of random data and for the other, a system they called "baggies", based on the metaphor of the math concept of set. To undertake this research we sustain on the notion of *Commognition* (Sfard, 2008), a contraction between cognition and communication. For Sfard, thinking is an individualized version of interpersonal communication, in other words, any act that could be characterized as thinking is an act of self-communication. We believe that the "baggies" was a resource for interpersonal communication but also a cognitive resource, helping the participants to understand what they were looking for. That mean, it was a *commognitive resource*.

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DEVELOPMENT OF 11-12 YEARS-OLD CHILDREN'S MULTIPLICATION STRATEGIES

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In recent decades there has been a steady growth in mathematics education research investigating how students solve multiplication word problems (Mulligan & Mitchelmore, 1997). The models developed by Siegler (2000) can be used for investigating the development of pupils' mathematical thinking (Jacob & Mulligan, 2014).

In this study we monitor how successful Hungarian students are in solving one-digit and two-digit multiplication word problems and whether their effectiveness actually depends on adaptive strategy use.

The month long developmental program included a group of six grade students (n=270). We used for the pre- and posttest two mathematical achievement tests, background factors questionnaires and the questionnaire Mathematical Beliefs (Kelecsényi & Csíkos, 2013), interview. There are significant differences in the strategy use of children. The collected data we analysed by help SPSS 17.

Teaching strategies were successfully integrated into a teaching experiment conducted with 30 children in the developmental research. The usefulness of improving pupils' adaptive calculation and multiplicative skills for everyday life and real-life situations will be discussed. For more, adaptiveness as an approach comprising also metacognitive components of mathematical thinking may have relevant transfer effects from one domain to others.

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ESTONIAN AND FINNISH BEGINNING UNIVERSITY STUDENTS' VIEWS ABOUT PROOF

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It is broadly considered that proof and proving belong essentially to mathematics. Quite often the amount and importance of proving increases considerably when a student starts studies in mathematics in tertiary education. In literature, several functions for proof have been presented. Traditionally, it has been considered that the most central function of proof in mathematics is to *verify* the truth of the statements. However, Hanna (2000) emphasizes that *explaining* –enhancing mathematical understanding– is the most important goal for the use of proofs and proving, whereas Hemmi (2006) suggests a *transfer* of techniques, methods, or ideas as an important function of proof. In addition, *systematization*, *discovering* and *communication* as well as *aesthetic experiences* and *intellectual challenges* are considered as functions of proof (de Villiers, 1990; Hemmi, 2006).

In this study, university students' views about the importance of proof and its different functions were explored. Altogether 97 students in Finland and 215 students in Estonia participated, all at the beginning of their university mathematics studies. The data was collected by applying a questionnaire. The results show that the students in both the countries quite highly appreciate the importance of proof both in school mathematics and in mathematics in general. However, the Estonian students seemed to be more critical toward proof and proving and the usefulness of the functions of proof than the Finnish students. Support for understanding and development of logical thinking skills were reasons that the students considered the most important for studying proof and proving. In general, the students in both the countries seemed to appreciate most highly reasons that referred either to the explanation or the transfer function.

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USING MATHEMATICAL MODELLING ACTIVITIES TO MOTIVATE BIOLOGY STUDENTS TO LEARN MATHEMATICS

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We report on the first phase of a developmental research project (Goodchild, Fuglestad & Jaworski, 2013) aimed at increasing biology students' motivation for studying mathematics through the use of mathematical modelling. So far the activities in the project consist of three hour sessions with small groups of students working on mathematical modelling tasks framed in a biological context. We have met with two groups of students – one pilot group consisting of 10 students that had finished their mandatory mathematics course, whom we met on one occasion, and one group of 11 students whom we met on four occasions in parallel with their mathematics course. At the beginning and end of the first sessions we distributed questionnaires consisting of Likert scale and open items, aimed at exposing the students' attitudes towards mathematics and its role in biology, as well as their response to the activities in the session, and how these might affect their motivation to pursue studies in mathematics. We emphasize that the groups are small, and we do not see these observations as more than indications of directions for future research. The students in the two groups mostly responded in a similar way to the questions. They found mathematics moderately interesting and enjoyable, and rated their own mathematical competence as relatively low. The activity was seen as interesting, enjoyable and challenging, and had contributed to their understanding both of mathematics and its role in biology, which was encouraging given the overall aims of the project. Students in both groups strongly indicated the relevance of modelling tasks of this type in their regular mathematics course, thereby supporting a continuation of the project on a larger scale. Only the item concerning the relevance of their mathematics course to biology stood out as getting very different responses from the two groups, with the group having already finished the course being much less convinced than those currently taking it. This begs the question of whether the general content of the mathematics course, and the fact that it needs to cater to the interests of a very diverse body of students, might negatively influence students' opinions of its relevance, and this is something that we want to investigate further. In the next stage of the project, we hope to be able to continue the modelling activities on a larger scale, involving all the first-year biology students as well as their regular mathematics and biology teachers. We believe that the work done so far shows the feasibility of such a continued and enlarged project.

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ENGAGING STUDENTS AT THE BEGINNING OF LESSON INSTRUCTION: FOUR CHINESE ELEMENTARY MATH TEACHERS' PRACTICES

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There is a common saying in China: "Well begun is half done." Chinese teachers often spent much time to think about and design different ways to start a lesson. However, there is very limited research available about Chinese math teachers' strategies on students' engagement, especially at the beginning of a lesson. As a part of the project to explore how Chinese elementary math teachers engage students, this study is designed to examine instructional activities at the beginning section of four teachers' lessons. In particular, the following research questions are addressed: (1) What are general characteristics of four Chinese elementary math teachers' practices in the beginning section of their lesson instruction? (2) What are the strategies of these four Chinese elementary math teachers used in the beginning section?

Although the content topics of these four lesson video clips vary across grade levels, beginning section is identified as including two parts: introduction of a topic and task presentation. It is commonly occupies 3-6 minutes of a lesson. To answer these two questions in this study, we analyse instructional video clips with a focus on teacher's practices in terms of three aspects: teacher-student(s) interactions, teacher's questions, and tasks used. The interaction-coding system aims to examine how teachers interact with individuals and whole class; teachers' questions are analysed in terms of their nature of challenges with a revised Blooms' Taxonomy. The tasks that teachers used are examined in terms of their cognitive demands.

The results suggest that a lesson's beginning was carefully designed to effectively engage students into learning as soon as possible. Some general characteristics in terms of these three aspects include: (a) these teachers mainly used oral language, but not much math instruction or math language; (b) teachers' communication pattern varies likely due to their self teaching styles and habits; (c) teachers mainly asked the whole class with low-level questions that are readily accessible, and asked individuals a few high-level questions to 'push' the lesson development; (d) all these teachers used tasks with high cognitive demands. Moreover, by putting these three aspects together holistically, these teachers' practices show three strategic moves that help engage their students: (i) *varying for engagement*: teachers used various strategies for engagement based on student situation and learning content through reviewing, questioning, using game situation and manipulative, etc.; (ii) *connecting for coherence*: teachers used the initial section as paving the way for the follow-up new content study to build lesson coherence; (iii) *questioning for scaffolding*: teachers used special questions to lead student to the next step learning, esp. "fill-in" questions and "confirming" questions as revealed in this study.

THE ANSWER PATTERN OF JAPANESE STUDENTS IN PISA2003 AND PISA2012 MATHEMATICAL LITERACY SURVEY

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Suzukawa et al. (2008) found that Japan has unusual answer pattern among 13 countries and area (namely, Australia, Canada, Finland, France, Germany, Hong Kong, Ireland, Italy, Japan, Korea, Netherland, New Zealand and the United States) by making secondary analysis of PISA2003 data. Following this research, Watanabe (2012) focused on the area of probability and statistics, and then mentioned that the items of outlier and sample collection have different patterns compared to other items thorough analysing PISA2003 data of the same 13 countries and area.

When PISA survey was conducted in 2003, the Japanese course of study did not cover the area of probability and statistics in junior high school mathematics education. The Japanese course of study, however, was revised with the area of probability and statistics in 2008. PISA2012 had area of probability and statistics as well as PISA2003. Then, what kind of changes of answer pattern do the Japanese students have between PISA2003 and PISA2012 mathematical literacy servery, especially in the area related to probability and statistics?

The aim of this study is to reveal the secular changes of answer pattern that Japanese students have between PISA2003 and PISA2012 by making cross-national comparison with above 13 countries and area focusing on the area of probability and statistics. In order to consider the answer pattern of Japanese students, the method of multiple group item response theory is used to detect uniform differential item functioning.

Analysis revealed that the answer pattern of Japan does not have observable changes between PISA2003 and PISA2012 in whole. This characteristic is applicable to other 12 countries and area. Looking at common items between PISA2003 and PISA2012 to examine the answer pattern in more detail, one of the items of probability and statistics has remarkable change in only the result of Japan. The results suggest that revised course of study has a gradual effect on students' answer pattern.

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WHAT POSSIBILITIES DOES SOCIAL REALISM OFFER FOR UNDERSTANDING TEACHERS' MATHEMATICS PEDAGOGICAL PRACTICES?

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This short oral communication emerges out of my PhD research which seeks to explore the structural (material), cultural (ideational) and agential mechanisms that give rise to Foundation Phase (grades R – 3) teachers' mathematics pedagogical practices. Central to this is the social realist notion of emergence, that two or more objects give rise to a phenomenon that is not irreducible to the parts of the original objects (Archer, 1995; Sayer, 2000).

Six months were spent in four grade 3 teachers' classrooms, observing and interviewing teachers about their mathematics pedagogical practices. In two of the classrooms, the teachers taught a particular method for solving addition and subtraction equations. The method is presented as follows $215 - 95 = (200 + 10 + 5) - (90 + 5) = (200) + (10 - 90) + (5 - 5) = (200^{100}) + (1^{10} - 90) + 0 = 110$

Data from interviews and an analysis of the Curriculum and Assessment Policy Statement (CAPS) for Foundation Phase mathematics suggest that this method emerged from the interaction of two cultural mechanisms: (1) the inclusion of a method in the Foundation Phase CAPS that decomposes numbers; and (2) the teachers' competence with the traditional algorithm. The research suggests that in an attempt to use the methods in CAPS, but with the teachers' experience and expertise dominated by the traditional algorithm, a 'new' method emerged.

It is through the process of identifying the structural and cultural emergent properties that condition teachers' mathematics pedagogical practices, and the personal emergent properties of teachers to 'act back', that researchers are able to (1) understand why teachers' mathematics pedagogical practices are as they are; and (2) consider how to motivate for changed practices. As such this oral communication elaborates the potential of a social realist framework, in particular Archer's (1995) morphogenetic approach, to provide fresh insights in the field of mathematics education.

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SAME PROBLEMS, DIFFERENT CONCEPTS

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THE CONCEPT BUILDING

The different parts of the same problems could cause severe problems in different student groups. If the interpretation of one part of the group is on a language baseline and it is significantly different from the interpretation of the others, who prefer the mathematical language could cause an inevitable problem. What would be the right way to teach mathematics? How can we lead the students to it? I strongly believe that the perfect attitude of a mathematics teachers is to have multiple personalities, but that sounds a little schizophrenic though. The ideal would be to handle the text both as a mathematics problem as well as a literacy problem. My dream would be if the teachers could represent both viewpoints in their mind and would have the ability to handle these problems and explain the difference.

Problems for 10-year-old pupils

I examined two text-based problems from the Hungarian school entrance tests for ten-year-old students. Everybody awaited proper instructions in the test (Li, Kaiser, 2012). The official solution said that the answers are clear, none of the test writers thought that there could be multiple ways of representing the problems. I have collected more than 200 solutions from 10 year old children, 50 solutions from university students and 20 solutions from mathematicians. It is easy to see and to prove with statistical methods that there are significant differences between the groups. The different groups have different representations for the problems.

Problems for university students

I have prepared the same probability problems for three different groups to examine the different understandings about basic concepts. The problem originated from inheritance, so we had to repeat a lot of basic knowledge from that area. The three groups were: a regular student group from the compulsory probability lecture, the students from a voluntary special course and a professional teaching group with practicing teachers. The main problem was to differentiate the realized events, the sure events and the impossible events in the experience. As I awaited the way of thinking of the special course university students was the most flexible, but the practicing teachers were the most educated in mathematical questions.

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DOES EXPLORATION HELP WRITING FORMAL PROOFS?

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Pedemonte's (2007) stated that induction is constructed by a process pattern generalisation which was made on the steps sequence. Hsieh, Horng, and Shy (2012) claimed that carefully developed hands-on exploration could help students in constructing not only conjectures but also generalised proof schemes, thus benefit the production of proofs. This study aims to investigate the relationships between using explorative activities with paper-folding and the writing of inferring steps in proofs.

A teaching experiment with paper-folding exploration was conducted using worksheets and hands-on sheets to collect students' responses and constructions. Interviews were implemented. Inductive and content analyses were used to analyse data. The sample included 52 ninth-graders who were not being advanced students and had not learned the property that the measure of a tangent chord angle is a half of its intercepted arc, which was the topic taught in the experiment.

A total of 73% students may come up with the results that the tangent chord angle was a half of the intercepted arc, and 47% of them provided formal proofs, despite having not been previously taught the concept of proof formally. The relationships in concern were shown in figure 1. Results included: (1) Steps of creating creases in exploration were in time sequence, which formed sequences of mentally inferring steps during the exploration; (2) Mentally inferring steps were sieved at exploration stage according to students' judgements of usefulness; (3) Mentally inferring steps were either packed as larger and fewer steps or unpacked; (4) Writing proofs were to persuade the processes and results of exploration; (5) Proving statements contained context- or content-bound statements in accordance with explorative steps.

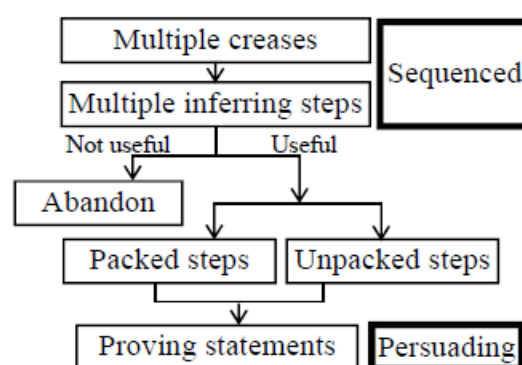


Figure 1.

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TEACHERS' MATHEMATICS CURRICULUM DEVELOPMENT DURING CLASSROOM TEACHING IN CHINA: A CASE STUDY IN BEIJING

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Teachers' mathematics curriculum development transforms the formal mathematics curriculum into an operational curriculum, which is closer to students' experiential curriculum. It converts mathematics curriculum from a static state to a dynamic state during the classroom teaching. As the forefront of education reform and cultural center of China, Beijing teachers' mathematics teaching has aroused much attention and research interesting. However, there is few qualitative research in this field.

The framework of Brown (2002), the Design Capacity for Enactment (DCE), will be applied as the initial theoretical framework to explore the research questions: a) How do experienced teacher and novice teacher develop mathematics curriculum during classroom teaching in Beijing? b) What factors influence mathematics teachers' curriculum development?

Considering Berliner (1988)'s five-stage model of teacher's pedagogical expertise development and teacher's ranking system in Mainland China, the study has selected one senior teacher with 18 years' teaching experience and one intermediate level 2 teacher with 3 years' teaching experience in grade 8 from one public secondary school of Beijing. Extreme case sampling method and reputational case sampling method have been mixed applied. Classroom teaching about the *The Pythagoras's Theorem* will be observed. Classroom observation, semi-structure interview and document analysis will be applied in different research stages for data collection.

Uncovering the features of different teachers' mathematics curriculum development of different contents during classroom teaching in Beijing may shed some light on Chinese mathematics education. Factors influencing teachers' curriculum development will be explored from perspectives of education system, teacher traits and curriculum resource. And the findings may provide some clues for teacher education and textbook publishers. Since it's an on-going research, further results will be discussed in details in the presentation.

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THE EFFECT OF TEXT ORGANIZATION ON LEARNING FROM READING A GEOMETRIC CONSTRUCTION TEXT

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Reading serves as an approach to learning mathematics actively and meaningfully, when we intend to help our students become lifelong learners of mathematics. However, we know few about how text organization can influence students' learning by reading. In this study, three factors: *text structure*, *the position of figures*, as well as *the construction goal of each step*, are taken into account to design eight versions of geometric construction texts, including construction steps and reasoning (proof).

For *text structure*, we know that it is helpful for readers to construct a well mental model from applying their knowledge of text structures to organizing and integrating information (e.g. Kintsch & Yarbrough, 1982). In mathematics textbooks, it is common to present one proof of a property prior to its application (proof-first). An alternative to this proof-first structure is the application-first structure, in which the application task is presented prior to its proof. We wonder whether the alternative structure is better for understanding. Geometric construction is selected as text content because it essentially involves both application and reasoning (proof).

In geometric text, it is important to integrate verbal and non-verbal information. We wonder whether figures prior to verbal information (figure-first) is better to facilitate students' concept images and then understand geometric construction steps than verbal information prior to figures. Thus, *the position of figures* is the second factor. By referring to the isolated interacting element effect in cognitive load theory (Pollock, Chandler, & Sweller, 2002), we justify whether construction steps with the supplementation of *the construction goal of each step* is better for understanding than construction steps without it.

Eighth graders who have not been taught the geometric construction task in each text are randomly assigned to read one of the eight text versions. We mainly found that the text in which the application-first structure is better for understanding the reasoning embedded in the read text and alternative geometric construction steps in a new text. Students have poorest performance when reading the figure-first text with the supplementation of the construction goals. The results suggest that text design using the application-first structure is more effective for students to read to learn geometry.

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A FRAMEWORK FOR INVESTIGATING PRINCIPLES OF DESIGNING GEOMETRIC TEXTS

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Although the significance of reading to learn mathematics is not a new research issue, little attention has been paid to principles of designing mathematical texts for improving mathematics learning. Herein, we propose a framework for investigating principles of designing geometric texts in which the coordination of verbal and non-verbal information are particularly necessary.

In order to make meaning of rich geometric texts, our framework involves two dimensions and three categories of factors considered to design texts. The two dimensions involve types of understanding mathematics by reading as well as types of geometric content. Referring mathematical proficiency (Kilpatrick, Swafford, & Findell, 2001), the types of understanding include conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. The types of geometric content include concepts, problem solving, constructions and proofs in geometry.

Most studies about reading mathematics have focused on the content of worked examples and proofs. In addition, we focus on concepts (definitions and properties) and constructions (steps and reasons) because concepts are important to learn geometry, and constructions are a special genre of text and complicated. The findings in reading problem solving and proofs cannot be necessarily generalized to reading concepts and constructions. As for understanding, not only problem-solving strategies but also reading strategies are included in strategic competence.

By referring to Peirce's triadic model of sign: the object, the representamen and the interpretant (Radford, Schubring, & Seeger, 2008), we identify three categories to classify factors considered to design texts: structures or histories of mathematics (e.g. triangles or quadrangles as generic examples), organizations of texts (e.g. arrangements of verbal and nonverbal information), and readers (e.g. low and high prior knowledge). Based on the framework, a series of research questions have been proposed to investigate effects of *text design* on various *types of understanding* as to each *type of geometric content* by an integrated project. Although this framework is developed conceptually, principles will be derived from empirical studies.

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STRATEGIC FLEXIBILITY OF PROSPECTIVE TEACHERS

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The flexible use of problem solving strategies is an important aspect of mathematical competence that mathematics classrooms should address (Heinze, Star, & Verschaffel, 2009). A remarkable emphasis on strategic flexibility in last decade makes this skill an important ability for teachers. Therefore, this study aims to investigate whether prospective elementary school teachers can exhibit strategy flexibility in solving non-routine problems. Four pairs of prospective teacher separately worked on six non-routine problems and all of their procedure was recorded by keeping their scripts and videotaping them. All pairs had taken an instruction about non-routine problems for nine lessons in the scope of a mathematics teaching course in previous semester. No intervention was made while they were solving problems unless they had difficulty in understanding problem. Although there was not any time limit, all pairs spent about one hour for solving all problems. To evaluate the data, four criteria were established to determine strategic flexibility (Arslan & Yazgan, 2015): C1. Selection and use of the most appropriate strategy, C2. Changing strategies when it does not work for the solution of a problem (intra task strategy flexibility) C3. Using multiple strategies for the solution of a problem (intra task strategy flexibility) C4. Changing strategies between problems (inter task strategy flexibility). With regard to first criterion, all pairs were able to choose and use most appropriate strategies for problems. However, students did not need to change their strategy since they were able to complete the solution with the first strategy they tried. Therefore, second criterion could not be observed in detail. Two of the problems used in this study were aimed at using multiple strategies, and pairs managed to use different strategies together for one problem. As to last criterion, we found that all pairs had intra task flexibility, namely, they could change strategies between problems based on the contexts of problem. Generally speaking, prospective teachers exhibited strategic flexibility as a consequence of the instruction they received. Still, future studies with more participants are needed.

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ANALYSIS OF SOCIAL CONSTRUCTS IN WORD PROBLEMS IN MATHEMATICS TEXTBOOKS OF EARLY 20TH AND 21TH CENTURY OF TURKISH EDUCATION

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A content analysis was carried out to illustrate the internal aspects reflected in the contexts of mathematical word problems in both the late Ottoman period (20th century) and contemporary Turkish education. Internal aspects will be based on the social, economical and political culture of the empire and the republic, the alterations of the expressions in the word problems will be presented with a brief background of republican reforms (the measurement units, dates, language reforms). Dogan (1994) conducted a comparative research titled “textbooks and socialization” between Abdulhamid II and the second constitutional era yet math textbooks were excluded from the subjects investigated. This paper aims to show that the contexts of math textbooks are directly related to social and political constructs of the era in question. Mathematics is perceived as a relevant subject area because “Mathematics as human and social activity requires mathematical literacy to be functional and to prepare to live, understand, [...], mathematized society” (Yore, Primm & Tuan, 2007, p. 574).

The books; Talim-i Hesap, Ilmin Hesabi and Basic Mathematics were selected from the reign of Abdulhamid II (Islamism), the second constitutional era (Liberalism and Ottomanism), early republican period (nationalism) and contemporary era (depoliticization) respectively. These books are selected in conformity with the historical timeline. Our initial content analysis was focused on 1500 real life word problems asked. The preliminary findings indicated a context shift in the problems that ranged from the necessities of militarist (i.e. logistic, geography, historical events), and traditional (i.e. gender issues, means of transaction) society to constructed modern society of the early republic and 21st century's. Socially, the language of the problems signifies the social status of the individuals mentioned in the problems; and in a similar way the problems mentioned historically, the reign of the Sultans; militaristically, the logistics of a battalion etc.

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ANALYSIS ON THE MENTAL STRUCTURE OF STUDENTS LEARNING GEOMETRY: BASED ON APOS THEORY

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Various research has been conducted in order to improve instructions on geometry. Yet it is hard to find one that looks at how mental structures of students are formed when they are learning geometry.

Meanwhile, as a theory that enables one to analyze the mental structures of students, APOS Theory explains the mental structure of students in a sequence of ‘Action-Process-Object-Schema’ (Arnon et al. 2014). Despite many researchers have been presenting the experimental results from applying APOS Theory, most have focused on topics regarding arithmetic, algebra and functions, while only a few number of research is on geometry. Since APOS Theory starts from the Action of a learner and appropriately explains the process that happens through reflective abstraction, it does not blend in well with the fields of geometry, which focuses on the characteristics of itself(Tall, 1999).

This study aims to show that APOS theory is applicable in a geometry lesson. For this, the data collection was carried out through class discussion, written assessments and interview.

As a result, in the ‘Action’ stage, the incenter and circumcenter of a specific triangle and its properties could be found through folding papers and utilizing Geogebra. In the ‘Process’ stage, students could intuitively know the existence of the incenter and circumcenter of an arbitrary triangle with properties previously found. In the ‘Object’ stage, students were able to find the center of a given circle applying the incenter and circumcenter of triangles.

Hence, one should concern on the twofold meaning of the Object when considering concepts in geometry; Object as the Object that a learner has constructed as well as the Object that means itself. It was possible to see that APOS Theory is applicable when taking the geometric object as a constructive object through action.

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THE EFFECT OF GEOGEBRA SOFTWARE ON PRE-SERVICE TEACHERS' VIEWS ABOUT MATHEMATICS HOPELESSNESS

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The purpose of this study is to evaluate the effect of dynamic mathematics software GeoGebra (Hohenwarter, Hohenwarter, & Lavicza) on the pre-service teachers' views about mathematics hopelessness. Within this scope, GeoGebra software was presented to pre-service teachers for 12 weeks that included a two-hour lecture per week and mathematical constructs were developed for them to use the software in the process of learning and teaching mathematics. In addition, dynamic materials regarding basic concepts like limit, derivative and integral were developed by the pre-service teachers. Concepts were addressed by using these dynamic materials. This study was conducted through a single-case (holistic) design (Yin, 2003). The participants of this study consisted of 24 pre-service teachers in the Department of Mathematics Education of a state university in Turkey. Data were collected with an open-ended questionnaire that was developed by the researchers. While developing the data collection tool, the researchers referred to the studies of Beck, Weissman, Lester and Trexler (1971) and Çetin, Bars and Bars (2015). The data were analysed by using the content analysis technique. As a result of the analysis of the collected data, pre-service teachers' views regarding the use of the GeoGebra software in teaching mathematics were considered. These views indicated that as a result of using the software, concepts were better and more easily understood. In addition, the pre-service teachers stated that the software decreased the anxiety that they experienced regarding the difficult subjects in mathematics and it enabled learning to be far from involving memorization. Based on these views, researchers concluded that the software decreased the pessimism of pre-service teachers regarding mathematics and it enabled them to have positive future expectations.

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FORMATIVE ASSESSMENT IN INQUIRY BASED PRIMARY EDUCATION

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Inquiry-based education (IBE) means that the teacher in the classroom creates the conditions for pupils to inquire independently a part of knowledge which they are supposed to learn. The process has to cope with problems (a) with the formulation of learning objectives relevant to the valid curriculum, (b) with the management of pupils' experiments designed to accomplish these objectives, (c) with respecting the individual differences of pupils. All these obstacles play together with the assessment traditions and competence orientation in different educational settings (Harlen, 2013).

The study is a part of a European project ASSIST-ME (*Assess Inquiry in Science, Technology and Mathematics Education*) investigating the implementation processes of various formative assessment methods related to IBE. In the Czech Republic six teachers of primary mathematics worked with researchers on inquiry tasks and methods of peer assessment and implement them in their classes during 8-12 hours long teaching units lasting for two months. The teaching units were videotaped and transcribed, as well as the interviews with teachers and pupils before/after them. Data analysis was based on grounded theory approach.

Data analysis showed that the main concern in implementation of formative assessment in IBE was the teachers' feeling of loss of control on the instruction. The teachers enter an unknown terrain and have to react to unexpected or unclear pupils' contributions in classroom discourse. In case that the formulation of objectives of the lesson and criteria of achievement are too general or unclear, the teacher may fail to register the evidence of a pupil's understanding and may fail to provide feedback that would move the learners forward. In the discussion we will mainly focus: (a) on the interplay of teachers' intention, subject matter and learners in inquiry; (b) the teachers' role in support of pupils' learning and on how they use assessment; (c) the pupils' role in their own learning and learning of peers.

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ON THE MENTAL PROCESSING AND REPRESENTATION OF FRACTIONS: IS THERE A SNARC EFFECT?

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The SNARC effect (Dehaene et al., 1993) consists in a non-intentional speedup of judgments about natural numbers, when correct responses to relatively small numbers are associated with the left hand and those to relatively large numbers are associated with the right hand. Similar effects have been found for some types of magnitudes (e.g. Prpic et al., 2013), but no studies so far have explored in depth whether a SNARC effect holds for fractions. This issue is relevant because it is intimately related to how fractions are mentally represented and processed. A study by Bonato et al. (2007) provided a first approach to the question of whether fraction magnitudes can be processed automatically (i.e. non-intentionally). Other groups have followed their lead, in spite of the great difficulty that, to evaluate a true SNARC effect, the task used must be strictly unrelated to magnitude (Prpic et al., 2013).

We present data from two experiments. The first one asked university students of Mathematics and Physics (N=14) to classify fractions as reducible or irreducible (a task where magnitude is irrelevant). An analysis of response times showed no effects related to fraction magnitude, and therefore no SNARC effect. The second experiment asked participants (N=15) to explicitly assess if the presented fractions were larger or smaller than $1/2$ (a task where magnitude is relevant). Clear effects of numerical magnitude were observed related to a distance effect, but again no SNARC effect was detected. Our data is consistent with previous findings showing that fraction magnitude is processed only when it is relevant to the task at hand (e.g. Gabriel et al., 2013), even in the case of persons with strong mathematical expertise.

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THE EXPERIENCE OF BRAZIL IN INTEGRATING TECHNOLOGY IN TEXTBOOKS: A MULTIMEDIA LEARNING ANALYSIS

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This paper is part of a broader research project that aimed to obtain a deeper understanding on how Geometry is presented in the textbooks and what kind (nature and goals) of proposals are made to integrate the technology. Among ten sets of textbooks written to Middle School that have been evaluated and approved by the Brazilian Government, to be distributed in schools (freely), only three of them have "digital learning objects" (DLO), in the first initiative of integration of technology in this traditional resource. For High School, in total of ten collection, only one set has the DLO. Mayer (2009) presents a 'cognitive theory of multimedia learning' considering how people learn from words and pictures. The technology evolution prompted new efforts to understand the potential of multimedia as a means of promoting human understanding – a potential that Mayer (2009) called the promise of multimedia learning. The need for intertwine research on textbooks and the DLO included on them is also justified by the fact that such DLO expand Valverde et al.'s (2002) comments that textbooks are the printed resources most consistently used by teachers and students in the course of their joint work. The project considers a qualitative research with an interpretative approach and it discusses some aspects of the DLO present in all this 4 sets of textbook to Middle and High School. The results showed that the required level of interaction with pupils is very low and it can be considered only has a detractor for students, and a 'domestication of media' (Borba & Gadanidis, 2008). The conference will be an opportunity to share a deeper analysis centered in understand what is really the role of DLO in textbooks and discussing some possibilities to contribute for improving its conceptualization in and for facilitate the multimedia learning.

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GIFTED STUDENTS VERBALLY COMMUNICATING VISUAL INFORMATION IN A VIRTUAL ENVIRONMENT

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To promote learning of mathematically gifted (mg) children, teachers should pose them challenging tasks (Diezmann & Watters, 2001). Grouping mg students together to solve problems in collaborative teams is not always possible in schools, but virtual environments allow such interactions. Researchers have reported benefits for mg students of collaborative problem solving in online environments using e-mail, chat..., but there is little information on virtual real-time interaction (Alagic & Alagic, 2013).

We report here a case study of two pairs of mg students (12-15 years old), living in different cities. We analyse the interactions of each pair while collaboratively solving a task by communicating in real-time via Skype. The use of visualization is important for mg students (Ramírez, 2012), so the task concerned the development of this ability: a set of buildings had to be built on a squared grid, to fit with the four orthogonal projections provided. Each student had only two projections, which could not be showed to the other student. They were provided with grid paper and Multilink cubes. The researchers were next to the students, but they only participated when the students were blocked. Data were screen captures and video recording of students.

The virtual environment proved to be useful for our mg students, and provided us with information on characteristics of mathematical talent related to verbal communication of information, and exceptional verbal and reasoning abilities while finding, sharing and justifying the placement of buildings. The oldest pair had no problems of communication, and showed a higher level of use of cognitive strategies, including analytical reasoning to discard impossible configurations of buildings. The youngest pair had difficulties due to a misunderstanding when fixing a coordinate system, and they preferred the visualization of multilink cubes for solving the task. Both pairs showed a high commitment with the task, even in the most critical moments, and they were motivated by the collaborative character of the task.

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AN EXPLORING TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING ALGEBRA

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Teachers' mathematical knowledge for teaching is important to improve of teaching and learning (Ball et al., 2008). However, teachers lack essential knowledge for teaching mathematics (Hill et al., 2005). The teaching and learning of algebra continues to be a major policy concerned around the world (e.g., National Mathematics Advisory Panel, 2008). Hill et al. (2005) found that teachers' mathematical knowledge for teaching affects student achievement. Mason and Sutherland (2002) suggests that concerns about school algebra have changed little over time. Hodgen, Küchemann, Brown, and Coe (2009) compared students' understandings of algebra currently with that of the 1970s and found that the understandings of 14-year-old students are no better than 30 years ago. The purposes of this research was to explore teacher's mathematical knowledge for teaching algebra. This research is part of *U.S.-Thailand Research Roundtable: Advancing Research Networks in Algebraic Reasoning* Project. The first year of the project were conducted in 12 elementary schools that had 41 mathematics teachers were participants. The MKT test was primarily designed to describe teachers' mathematical knowledge for teaching algebra, and consists of 37 tasks which taken from the database by the Learning Mathematics for Teaching Project, LMT (2009). The research results found that teachers pass less than 50 percent and average score is 12.91, the teachers have varying score from 5.53 until 19 (37-full score), this indicates that teachers' mathematical knowledge for teaching algebra were unsatisfied.

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ESTABLISHING CRITERIA FOR TEACHERS' REFLECTION ON THEIR OWN PRACTICES

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The objective of this poster is to present a case study that deals with the analysis of the reflection process of a teacher, who is addressed as Mr. Lopes, for the improvement of the implementation of new contents related to the Riemman integral in the Elementary school level, which he proposes as his end of thesis project (ETP) in the Professional Master's Program in Mathematics in the National Network, Brazil (PROFMAT)

We used the *indicators of didactical suitability* proposed by the Onto-Semiotic Approach to mathematical knowledge and instruction (OSA) (Godino, Batanero & Font, 2007) as theoretical model to analyse the teachers' reflections on how to improve their own teaching practices, connected to the implementation of the didactical activities proposed as part of their ETPs: epistemic suitability, cognitive suitability, interactional suitability, mediational suitability, emotional suitability and ecological suitability.

As in (Breda & Lima, 2016), the teacher, implicitly or explicitly, uses all of the indicators of didactical suitability proposed by OSA (Godino, Batanero & Font, 2007). A problem that is important to address here is that the teacher shows difficulties in finding a balance among the suitability indicators; on the one hand, the author plans an innovation project with high epistemic suitability and, in his reflection, it is clear that he is also concerned about achieving high cognitive demand. However, in order to achieve this, he leaves aside some contents that had previously been planned and, in particular, he is not able to solve the initial problem that he had suggested. In this sense, the learning was not complete and the teacher argues that it was due to lack of time, so in other words, he could not attain good mediational suitability.

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WHAT GHANAIAN STUDENTS VALUE IN SCHOOL MATHEMATICS LEARNING: PRELIMINARY FINDINGS

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INTRODUCTION AND METHODOLOGY

In the latest Trends in International Mathematics and Science Study, Ghanaian eighth grade students ranked last in mathematics achievement. Investigating the practices/norms of education systems that face challenges is as important as research studies of exemplary mathematics education systems. This paper reports on a study of what Ghanaian students valued and found important.

Values in mathematics education are the convictions associated with mathematics learning and teaching. This social variable is believed to interact with cognitive and affective variables to regulate how mathematics is learnt / taught (Seah & Andersson, 2015). Other conceptualisations (e.g. Bishop, 1996) also regard values similarly.

The convictions which Ghanaian students held with regards to mathematics learning were assessed using the validated 'What I Find Important' [WIFI] questionnaire. 1,256 returns were collected from public schools across Ghana, amongst which 69% of the respondents were female, reflecting the local enrolment student gender distribution. A third of the respondents were studying at each of the primary, junior high school, and senior high school levels. Data from the 64 Likert-scale items was analysed with SPSS. A Principal Component Analysis (PCA) was performed with the significance level set at .05, and a cut-off criterion for component loadings of .45.

RESULTS AND DISCUSSION

The PCA identified 13 components which accounted for 52.47% of the total variance. In the light of the 38 questionnaire items which constituted them, the values were labeled *achievement, relevance, application, accuracy, ICT, versatility, learning environment, strategies, feedback, communication, knowledge, luck, and instruction*. This knowledge can further develop teachers' capability to plan engaging lessons and to negotiate cognitive conflicts in class, and facilitated a more complete understanding by researchers of the state of mathematics education in Ghana.

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A STUDY OF THE KNOWLEDGE OF PRE-SERVICE ELEMENTARY SCHOOL TEACHERS REGARDING THE DIVISION OF FRACTIONS

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Fraction is a crucial mathematical concept for students in primary education and the teaching of division of fractions is an important topic for teacher education. According to Van Steenbrugge, Remillard, Verschaffel, Valcke, & Desoete (2015) math activities involving fractions should 1. understand the meaning of dividing fractions and when to apply it or give examples, 2. provide clear representation, and 3. require students to use strategies to solve and check answers. In this study, we followed these three issues and considered 3 research questions: 1. Do pre-service teachers know the meaning of dividing fractions and can they give examples? 2. What kinds of representations do pre-service teachers use to present division of fractions? 3. What kinds of strategies do pre-service use to solve division of fractions?

The researcher collected data from 132 pre-service teachers. The tasks are 1. Pose a word problem for $3\frac{3}{4} \div \frac{3}{8}$. 2. Use the representations you have learned to draw a picture illustrating the meaning of $3\frac{3}{4} \div \frac{3}{8}$, and include a text description. 3. Use adequately strategies to solve $3\frac{3}{4} \div \frac{3}{8}$. Write also the calculation process.

Quantitative and qualitative methods were used to analyze the knowledge of the data. We find the proportions of correct are 41% pose problem, use presentation 42% and solve problem 73%. The qualitative analyze results are three: 1. Pre-service teachers were able to pose problems using divisor as a definite unit and combine the relationship between dividend and divisor to give examples. 2. When asked to use representations, they used counters, straight lines and area presentation models to present division of fractions. 3. When solving problems regarding division of fractions, problem strategies exhibited were: invert-and-multiply algorithm, repeated subtraction using unit number, and, adopt divisor equal to "1".

In order to upgrade teachers' knowledge of division of fractions, the investigators suggest to emphasis the concept of unitilize and to enhance the use of multiple representations.

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SUBJECTIVE NORMS IN MATHEMATICS: A COMPARISON OF MACAO'S ACADEMIC RESILIENT AND NON-RESILIENT STUDENTS IN PISA 2012

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According to the findings of OECD's Programme for International Student Assessment (PISA) 2012 mathematics study, Macao-China has a high share of academic resilient students (17%) in its 15-year-old population amongst the 65 participating economies. In this study, academic resilient students are hypothesized to be influenced more than their ESCS-disadvantaged counterparts by their friends' or parents' positive view of mathematics learning. In the socio-psychological literature, students possessing high level of subjective norms are prone to be mentally stimulated and spiritually supported, especially by parents and the very close friends, to advance in academic performance in spite of the unfavorable ESCS-disadvantaged home conditions. In PISA 2012, examples of subjective norms in mathematics (SUBNORM) measured on a 4-point Likert response scale are: (1) *My friends enjoy taking mathematics tests*; (2) *My parents believe it's important for me to study mathematics* (OECD, 2013). For the purpose of valid comparative education, the PISA index SUBNORM has been further anchored by vignettes in the student questionnaire in order to adjust for possible response style effects of the respondents. Drawing data from the PISA 2012, this study seeks to examine the similarities and differences between Macao's ESCS-disadvantaged resilient and non-resilient students with regard to their perception of the subjective norms in mathematics. The following hypothesis is postulated for statistical significance testing: *Macao's academic resilient students have a higher level of subjective norms in mathematics than their ESCS-disadvantaged non-resilient counterparts.*

The independent *t*-test establishes that there is statistically significant difference in the means of the anchored PISA index of SUBNORM between Macao's ESCS-disadvantaged resilient and non-resilient students ($t = 2.240, p < .05$). Specifically, compared with ESCS-disadvantaged non-resilient students, Macao's resilient students are found to be more prone to agree with that *parents believe it's important for me to study mathematics* ($p < .001$), but *not* with that *My friends enjoy taking mathematics tests* ($p < .05$). Implications of study are proposed to render help to the non-resilient students for the betterment of their motivation in their learning of mathematics in Macao's schooling contexts.

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THE PERCEPTION OF TURNED CARTESIAN COORDINATE SYSTEM: AN EYE TRACKING STUDY

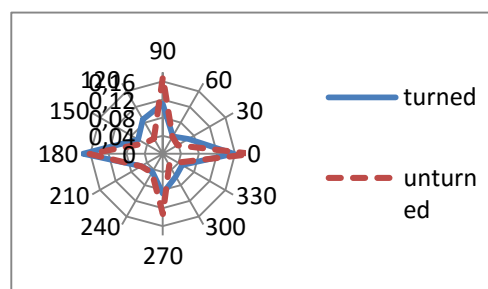
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Radford in his 2010 paper raised the question of theoretical perception: how mathematic education transforms the perception? In 2014 we conducted research on perception of Cartesian coordinates by novices and experts (Chumachenko et al., 2014). The task was to locate a target point in the Cartesian coordinate system. We registered the dominance of vertical and horizontal saccades in all groups of participants. There are at least two possible explanations of our results: (1) eye-movements are conditioned by the task and the specific actions, which lie under the cultural perception of the coordinate system; (2) the predominance of right angles in our life (buildings, trees, skyline, screens and so on) leads to the preference for vertical and horizontal saccades. In order to distinguish these factors we turned all our tasks on 30 degrees and conducted new research with these stimuli.

Since there were no difference between distributions of saccade directions of experts, intermediates and novices (related to the whole number of saccades, of course, absolute numbers are different) we compared mean distribution from the first experiment with distribution received on turned tasks (see Fig. 1).

The relative numbers of saccades directions were compared by repeated measures ANOVA with turning tasks or not as a between group factor and saccade direction sector as within subject factor. It showed the significant interaction between these factors ($F=17.130$, $p<0.001$). So the structure of saccade directions changed, when we had turned the coordinate system, but in non-obvious way: we still registered the dominance of vertical and horizontal saccades, but we also observed the big amount of axial (along the axes) saccades.



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FACILITATION OF AHA! MOMENTS IN MATHEMATICS CLASSROOMS

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Proposed poster informs about the introduction of bisociation, the theory of Aha! Moment of Arthur Koestler (1964) from his Act of Creation into Mathematics Education. It reports and develops further a long term research conducted in the Bronx Community colleges of CUNY focused on formation of the Creative Learning Environment among the “underserved” student population of the Bronx described in Czarnocha et al, (2016). Strong arguments are presented suggesting particular usefulness of facilitated Eureka experience for the promotion of mathematics creativity among “underserved” students leading to the development of positive attitude towards mathematics and its understanding and learning.

Closely related research questions analysed in the poster are: (1) what are the processes of facilitating Aha!Moments? and (2) how to assess the increase of the mathematical understanding occasioned by the Aha!Moment?

Bisociation is the spontaneous flash of insight which connects the previously unconnected frames of reference and makes us experience reality at several plane at once. (Koestler, p.45) This definition suggests that a theory of schema formation, Triad of Piaget and Garcia (1989) will be a useful theoretical framework to answer research question 2. The poster will contain the short description of the PG Triad as well as the analysis of chosen fragments with its help. As there aren't many descriptions of Aha!Moments in math ed professional literature, we collected known cases in the Library of Aha!Moments at www.hostos.cuny.edu/mtrj and will use two examples from that collection. The central conclusion from the research of literature as well as from several teaching experiments (Czarnocha et al,2016) is that facilitation of Aha!Moments in regular classrooms depends on the concept of the “bisociative framework” that is the presence of two or more unconnected frames of reference with enhanced possibility of “*unearthing hidden analogies*” within bisociative Aha!Moments. Consequently we will explicate such frameworks in these examples while answering research questions 1 and use them as a criteria to analyse methods used by different teachers.

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CREATIVITY, AHA!MOMENTS AND TEACHING-RESEARCH – OUTCOMES OF THE DISCUSSION GROUP OF ICME 2016

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The Discussion Group *Creativity, Aha!Moments and Teaching-Research* at the ICME 2016 in Hamburg was proposed to investigate the nature of Aha!Moments in mathematics through the coordination between the bisociation theory proposed by Koestler (1964) in the Act of Creation and recently published accounts of Aha!Moments in literature of mathematics education (e.g. Palatnik and Koichu, 2014).

Posing bisociativity at the centre of the creative act opens doors to new theoretical and didactic possibilities of understanding creativity in mathematics and mathematics education. This led Prabhu and Czarnocha (2014) to propose bisociation as the new definition of creativity in mathematics education in response to often met assertion.

Koestler definition of creativity uncovers creativity's cognitive aspect in the creation of new schema of thinking and connects the cognitive and affective aspects through the principle of cognitive/affective duality of the Aha!Moment (Czarnocha, 2014). Presence of an affective dimension of the Aha!Moment has been empirically observed and discussed by Liljedahl (2013). The poster will present the content of the DG discussion and questions raised, while providing the general, unified to the degree possible, view on the subject Aha!Moment creativity.

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IMPROVING ELEMENTARY STUDENTS' MENTAL THREE-DIGIT ADDITION: RESULTS OF A CLASSROOM INTERVENTION PROGRAM

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The focus of this study is on improving 4th grade elementary school students' mental three-digit addition in a two-week brief intervention program. In general, a promising tool for fostering students' thinking is to make their mental strategies cognizant for them, and making them aware of the selection and usage of different strategies (Heinze, Marschick, & Lipowsky, 2009). According to previous results (Csíkos, 2016) students find it difficult to adjust their strategy use to the current task, and they tend to insist on using one strategy throughout a series of mental three-digit addition tasks.

The experimental group comprised 78 students; the control group comprised 67 students. The 4th grade students involved were 10-11 years old. The written arithmetic test used both in the experimental and control groups and both as pre- and post-test proved to be reliable (Cronbach- α = .90 as pre-test, and .83 as post-test, 32 items, N = 134 and 133, respectively.) The two-week long brief intervention program consisted of a series of practicing mental calculation tasks solved at the beginning of each math lesson. Teachers explicitly named the possibly used mental calculation strategies (e.g., “walking”, “completing”), and they tried to keep children’s strategy use flexible, i.e., no winning strategy for a given task was declared.

The main results of our intervention program are shown here.

	Pre-test (available: 32)	Post-test (available: 32)
Experimental group	19.80 (7.12)	20.90 (6.59)
Control group	22.95 (5.67)	23.52 (5.68)

Cohen’s unbiased d estimate proved to be .14.

The main novelties of the current research include (1) feasibility of developing an intervention program to foster students’ arithmetic skills, (2) revealing a small effect size for their arithmetic skills after a two-week brief intervention.

This study was supported by the Hungarian Scientific Research Fund (OTKA 81538).

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CAN FINANCIAL MATHEMATICS PROMOTE FINANCIAL LITERACY?

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Financial illiteracy is widespread and has important economic consequences (Lusardi & Mitchell, 2014), so the need to increase youth financial literacy is well-recognized. In this respect, the Organisation for Economic Co-operation and Development recommends that financial education starts at school to ensure exposure at an early age (OECD, 2005). There is however no consensus on how to implement this policy recommendation. Indeed, the optimal manner in which financial literacy goals can be integrated in curricula and the best way to expose children to financial literacy at school are still open for debate. We add to this discussion by setting up an exploratory study aimed at empirically examining the possible contribution of financial mathematics in the curriculum of secondary schools.

Eighty-four 16-18-year-old secondary school students was subjected twice to a multiple-choice test on financial proficiency: a pretest – taken just before the start of the financial math course – and a posttest just after they had taken the financial math course. Post- and pretest were equivalent, as certified by experts. For further analyses, questions were categorized into the following three categories: ‘Methods’, ‘Concepts’, and ‘Spillovers’. The first two categories refer to different aspects of financial literacy related to the content of the financial math curriculum. ‘Spillovers’ consisted of questions that are not part of the learning outcomes of financial mathematics, but are part of financial literacy. Categories were not disclosed to participants and the questions were randomized over categories and tests.

With the exception of the category ‘Spillovers’, the average value in the posttest was significantly higher than in the pretest (p -value of a paired sample T -test was lower than 0.001 in each instance). This result suggests that financial mathematics can help promoting financial literacy, both in terms of skills and knowledge. Results of this exploratory study are promising. In particular, the holistic approach adopted by financial math teachers ensures that the positive effect is not limited to a better understanding of math-technical aspects but also encompasses broader applications of the concepts to the financial world.

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SOME COMPONENTS OF GEOMETRIC KNOWLEDGE OF PROSPECTIVE ELEMENTARY TEACHERS

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Geometric experience, spatial representation, spatial visualization, understanding the world around us, and developing spatial reasoning ability are fundamental aims in the teaching of mathematics. (Freudenthal, 1972) Learning is a process which involves advancing from level to level. In primary school the focus is on the first two levels of Van Hiele model (1. recognizing figures; 2. analyzing figures), on laying the foundation of basic terms in geometry. The curriculum should include developing spatial visualization, spatial reasoning, and the ability to recognize figures in different settings. Pupils should be able to accurately describe figures, shapes, and the properties of these, using appropriate geometric terms.

The aim of the research is to investigate the knowledge of teacher training students in the area of associating and building relations between geometric content and real life objects (e.g. buildings, sculptures, fountains); to provide a means for revealing and studying some components of the students' geometric knowledge and point out any lacuna, as well as facilitate the filling in of existing lacunas.

The participants involved in this research (N=115) had to identify two-dimensional and three-dimensional shapes on the basis of 12 photos of real three-dimensional objects (selection criteria: a multitude of shapes and bodies). I have focused on the accuracy, frequency and variation of terminology in students' answers, as well as on analyzing mistakes. (Pjanic & Nesimovic, 2015)

The students are not familiar with the correct geometric terms for figures and shapes, they cannot identify these and are not familiar with their properties. They are able to identify more two-dimensional shapes (60%), rather than three-dimensional ones (40%). The shapes students find the easiest to identify are the *circle*, the *square*, the *rectangle*, and the *triangle*, as well as the *sphere*, the *cube* and the *cuboid*. Identifying truncated shapes posed the most problems for students. They found it difficult to identify shapes and figures placed in unusual positions. The lacuna in students' knowledge of identifying shapes and figures needs to be filled in within the course that focuses on methods for teaching geometry, with the help of as many concrete tasks as possible.

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THE DEVELOPMENT OF CHILDREN'S ADDITIVE AND MULTIPLICATIVE REASONING IN OPEN PROBLEMS

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Additive reasoning was traditionally assumed to be a precursor of multiplicative reasoning. This was postulated after numerous studies indicating that young children erroneously reason additively in multiplicative word problems (for an overview, see e.g. Van Dooren, De Bock, & Verschaffel, 2010). However, this assumption has been recently questioned by the finding that young children already show some multiplicative reasoning abilities (Nunes & Bryant, 2010). Moreover, older children erroneously reason multiplicatively in additive word problems, despite their additive reasoning abilities (e.g. Van Dooren et al., 2010). Children's incorrect reasoning in word problems seems not merely dependent on their *(in)ability* to reason additively or multiplicatively, but also on their *preference* for additive or multiplicative reasoning.

We studied the development of third to sixth graders' preference for additive or multiplicative reasoning by means of schematic problems that were *open* to both additive and multiplicative reasoning, i.e. arrow schemes wherein three numbers were given and a fourth one was missing. While children in Study 1 were asked to fill out the missing number in an open answer format, in Study 2 another group of children was asked to indicate all possible answers amongst a set of given alternatives.

In both studies, most answers were additive, but a substantial number of multiplicative answers was given too. This indicates the existence of a multiplicative preference besides an additive preference. Second, additive answers decreased, while multiplicative answers increased across grades. Third, problems with integer number ratios evoked fewer additive and more multiplicative answers than non-integer problems, especially in fifth grade. Study 2 moreover revealed that children rarely considered both the additive and the multiplicative answer. This occurred more often by older children in upper primary education and in integer problems. In sum, our results resemble previous findings of word problem research (e.g., Van Dooren et al., 2010), suggesting that getting a view on preference next to ability is indispensable in order to fully understand the development of additive and multiplicative reasoning.

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RESOLUTION AND FORMULATION OF PROBLEMS SINCE INTERDISCIPLINARY PROJECTS

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This study is the result of a research conducted by an educational intervention developed with students of the 3rd year of a polytechnic high school of the 3rd Regional Coordination of Education of Vale do Taquari – RS/Brazil. Throughout the meetings, we aimed to understand “What is the influence of interdisciplinary projects in the resolution and formulation of mathematical problems?”. The main aim of this research was to analyze and explore the formulation and solving mathematical problems from interdisciplinary projects. The theoretical reference follows assumptions which are based on Dante’s (2010), Polya’s (1978) and Smolle and Diniz’ (2001) ideas, who talk about problem solving. The methodology used to achieve the aims of this study was qualitative. The data survey was obtained from a field diary, from the students’ search projects, an initial and a final questionnaire and from shoots of the thirteen meetings with the subjects of the intervention. In those meetings, the formulation and resolution of problems, the mathematic issue present in each student’s search, the resolution of problems formulated by them and the activities socialization were explored. Based on answers and reports of the students involved in the research, it was possible to verify that the problem solving can be explored through projects and interdisciplinary topics. Students approached mathematics present in their daily lives through moments of reflection and dialogue.

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ARGUMENTATION BY PRESERVICE MATHEMATICS TEACHERS DURING THE PREPARATION FOR THEIR PRACTICUM

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Due to the complexity of teachers' argumentations in the classroom, it is required diverse skills, specifically, to argue during teaching. Several studies have shown that not only students have difficulties arguing about their mathematic ideas and comprehension but also prospective and in-service teachers (Barkai, Tsamir, Tirosh & Dreyfus, 2002) in regard to explaining. The main objective of the research is to study preservice teachers' argumentations when explaining geometry tasks. The argumentations are analysed in regard to four features: a) Structural qualities (logic-substantive, rhetoric and dialectic); b) The warrants (Toulmin, 1958) (a priori, empiric, institutional and evaluative); c) Modal qualificators and d) rhetoric resources (illustrations, examples and models). This reports draws on the data of four preservice teachers that attended the 'Practicum' course, offered to prospective mathematics teachers in the School of Education, Antioquia University, Medellín, Colombia. During the first year the preservice teachers design and choose geometry tasks, solve and present them to their fellow colleagues, who commented and suggested improvements; in the remaining term, the teachers acted as teachers in the classroom. The data is taken out from classes planning, videos, audios and the notes taken during the sessions. The paper informs about the first year. It was found that the teachers prefer to argue using logic-substantive- that account for a logic stance. The warrants used are mainly a priori and empiric-that refer to previous knowledge and examples taken from daily life. The modal qualificators are used according to the type of questions. If the questions posed by prospective teachers ask for previous knowledge, the modals refer to formal logic, but if the questions are guided by a teaching intention, the modal qualificators point out to doubts. In regard to the rhetoric resources, teachers use only illustrations-graphics- and examples.

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CREATING REPRESENTATIONS OF PRACTICE IN TEXT, COMIC AND VIDEO FORMAT FOR PROFESSIONAL LEARNING AND TEACHER ASSESSMENT

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In order to foster mathematics teachers' noticing and analysis (Sherin, Jacobs, & Philipp, 2011; Kuntze, Dreher, & Friesen, in press) in pre-service and in-service professional development, representations of practice can be rich learning opportunities. Correspondingly, the way how teachers analyse classroom situations can for example provide insight into their professional knowledge. Professional requirements related to representations of practice can thus be opportunities for both professional learning and teacher assessment. However, video recordings of authentic classrooms, for instance, frequently do not exactly reflect the specific goals of a professional learning unit and can bring a large amount of unnecessary context information. For this reason, designing "artificial" classroom situations may be a good alternative. Texts, comics and videos are possible representations of classroom situations, each connected to specific strengths and limitations. Thus, different formats should be given careful consideration before creating representations of practice.

As there are still hardly any established guidelines available for creating representations of practice in different formats, this poster aims to structure key questions related to this topic. With a focus on assessing resp. fostering teachers' analysis of how representations are dealt with in the mathematics classroom (cf. Dreher & Kuntze, 2015), we present how an exemplary classroom situation was created in a research project as transcript-like text, how a comic strip was subsequently developed and how a video was produced on the base of the comic strip. On the "way back", comic and text formats of the classroom situation were adapted to avoid inconsistencies between the different formats. Still, the different formats differ in the information contained in them and with respect to several general properties connected to the specific formats – and of course, none of them is the classroom situation itself. The poster discusses these questions and concludes with a collection of criteria for the design of representations of practice in text, comic and video format.

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CHILDREN'S REASONING DURING FRACTION COMPARISON IN A NUMBER LINE CONTEXT

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Educators and researchers strive to understand why learning fractions is so difficult for children. From a cognitive perspective, a possible source of many misconceptions and errors is known as the Natural Number Bias (NNB), for instance the use of strategies that are appropriate for natural numbers in the context of fractions. In this work, we ask how a task that emphasizes the fractions' numerical magnitudes modulates the effects of the NNB. A number line task, where participants mark the location of a given number in a number line (e.g. Siegler & Opfer, 2003), is a good candidate to fit this requirement. We explored fraction comparison by means of a number line task by providing a reference fraction already positioned in the line. Hence, locating the target fraction to the right of the reference is equivalent to stating that the target fraction is larger than the reference, and *vice versa*.

We presented sixth-grade Belgian children ($N = 74$) with a fraction comparison task and an adapted number line task, to compare the strength of NNB effects between both tasks. Overall performance in the number line task was 87%, significantly higher than the 78% obtained in standard fraction comparison ($p < .0001$). A clustering analysis revealed the presence of four clusters of children, with patterns of responses similar to those observed in previous quantitative (Gómez & Dartnell, in press) and qualitative (e.g. Stafylidou & Vosniadou, 2004) data. Importantly, the number line context induced changes in the strategies used by children to compare fractions, for instance some of them switched from a "smaller components, larger fraction" strategy to the opposite ("larger components, larger fraction") in the number line context. Our results show that the use of a number line context for a fraction comparison task can substantially improve the results and provoke a change in children's reasoning.

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ARITHMETIC EXPRESSIONS WITH MULTIPLE OPERATIONS – HOW TO SOLVE IT?

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Imagine that you are not aware of the mathematical laws, rules and conventions. How would you then solve the problem of calculating an arithmetic expression? And, if there were conventions that you felt somewhat acquainted with, how would you know when to apply them? In this poster, it is discussed what students do, their actions, when students solve arithmetic expressions with multiple operations.

The aim of the study presented in this poster is to describe how students solve arithmetic expressions, and possibly to find out whether there could be other actions, or possibly other errors, not captured by previous studies, in students' solutions. Several previous studies have shown that students misinterpret or misuse arithmetic conventions (Blando, Kelly, Schneider & Sleeman 1989; Glidden, 2008; Headlam, 2013; Pappanastos, Hall & Honan, 2002). Particularly, Blando et al (1989) and Headlam (2013) characterized the types of errors students made. However, I argue that none of the studies above challenged the order of operations convention alongside with the left-to-right convention. Instead, in this poster the following task is discussed:

“Calculate the following: $5 \times 3 + 2 \times 4 \times 5 + 3 \times 2$ “

The task was one of 10 tasks included in a pencil-and-paper test given to 84 students (Yr 8) in eight classes in four different schools in Sweden. Data from the tests show that the students use a set of different actions, sometimes related to arithmetic conventions and sometimes seemingly at random. Particularly, we have found that the central double multiplication ($2 \times 4 \times 5$) causes students to solve the task in several unconventional ways and to reveal previously unreported actions.

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TRADITIONAL ARCHITECTURE IN IRAN: AN ETHNOMATHEMATICS STUDY

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This paper presents some of the underneath mathematical ideas that are embedded in the everyday activities of a group of traditional architects in Iran. The first author is doing her MSc in Mathematics Education and this research is part of her master thesis. Considering an ethnomathematics approach, she wanted to learn more about the kind of very sophisticated mathematics that her father -the head architect- is using, mathematics that no one taught him and he learned it by heart. However, his co-workers believe that mathematics solely belong to intelligent and educated people and has no place in their profession. Ethnomathematics approach, was first introduced to mathematics education literature by D'Ambrosio (Millroy, 1992), an approach that displays the relations between mathematics and culture. So researchers seek to response following questions: How traditional architects with minimal formal schooling, use mathematics in their everyday activities? How can we find meaningful ways to bring ethnomathematics into school mathematics curriculum?

The methodological framework of the present study was ethnography. Participants consist of a traditional architect and his 16 co-workers with almost no formal training in mathematics. For the data collection, the first author acts as participant observer to be able to interact with them. In addition, she has conducted many interviews with these architects under the supervision of her thesis supervisor. For 4 months, the first author worked with this group as an apprentice and recorded their activities and took field notes at the end of every working day.

Analysing the data collected through observations and informal conversations, revealed that many mathematical concepts such as symmetry, geometrical shapes such as circle, rectangle, triangle and octagon and concepts of ratio and proportion, are used by architects. They are doing a kind of mathematics that is quite efficient, but is different from formal content of mathematics textbooks. For them, touching and seeing are two strong senses of measurement, estimation, approximation and many other mathematical skills. What we tried to depict from this real story is that it is useful to uncover various aspects of mathematics that is embedded in our culture and integrate them in formal mathematics curriculum, using ethno- mathematics frameworks. In this presentation, further results with pictures and some ethnomathematical tasks which are designed for school mathematics classroom will be discussed in some details.

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“WE MUST SHOW THAT THE MAPPING IS WELL-DEFINED”: GESTURE AND DIAGRAM IN ABSTRACT ALGEBRA

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TOPIC

Checking that a mapping is well-defined is a critical component of the proofs of fundamental results in group theory. Failure to be well-defined is the obstruction to coset multiplication that the definition of normal subgroup was created to overcome.

RESEARCH QUESTION

How does the concept of well-definedness manifest itself in the gestures and diagrams of a professor of undergraduate group theory as he models for them his own understanding of this important concept?

THEORETICAL BACKGROUND AND METHOD

A transcript was made of 35 videotaped lectures of an undergraduate course on group theory. Using elementary corpus linguistic tools, it was determined that there were a few episodes containing clusters of the use of the term ‘well-defined’. I adopt from Châtelet the perspective that gestures and diagrams can be mathematically highly significant: they can capture a move or action that is not as easily captured in words; actually performing a gesture or drawing a diagram oneself can help one develop one’s mathematical understanding of some concept. I take as an assumption that gestures and diagrams are highly imitable.

RESULTS AND CONCLUSIONS

The lecturer uses dramatic, large gestures in order to explain the relationship between a mapping being well-defined and a mapping being one to one. He draws a diagram with a question mark on it (the only such diagram in a course of roughly 125 diagrams) in order to capture this subtle notion of imagining that we might have 2 outputs to our function, when in reality there may be only 1. The professor’s gesture of well-definedness both resembles the gesture he makes for a mapping (because after well-definedness has been proven, we do in fact have a mapping) and avoids being identical to the mapping gesture (because when we are checking well-definedness, we don’t know yet that it is a mapping). This epistemologically tense state is what called for the Châtelet perspective in the first place. The lecturer’s discussion of well-definedness differs from his discussion of every other concept: he resists giving an explicit prescription of exactly when to check that a mapping is well-defined.

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USING ORIGAMI FOR TEACHING GEOMETRY IN A RURAL AREA OF MASHHAD

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As a mathematics teacher, and before I started a few Origami classes, I barely took into consideration the connection between Origami and mathematics. These Origami classes are a part of a larger program, launched and fully funded by Mashhad Municipality for deprived students aged between 10 to 14 years old who live in rural areas, to help them spend their summer in a more fruitful way. These classes are an extra-curricular activity and are all offered free of charge. The experience of these Origami classes was so unique for me that it encouraged me to arrange a participant observation about the aforementioned connection between Origami and Mathematics as well as the impact of Origami on the motivation of students towards mathematics. By consulting our advisor, we decided to design our research methodology as participant observation and firstly pilot it with one of these classes. The chosen class had 22 female students aged between 10 to 12 years old living in a rural area in Mashhad, Iran. In these classes we taught some easy, preliminary step-by-step samples. One of these samples is shown in figure number 2. Although this research was conducted unsystematically, it led us to the development of two research questions for further studies, whose findings will hopefully contribute systematically to the community.

Our theoretical framework during this study was Duval's (1998) cognitive process as;

- visualization processes, for example the visual representation of a geometrical statement, or the heuristic exploration of a complex geometrical situation.
- construction processes (using tools)
- reasoning processes - particularly discursive processes for the extension of knowledge, for explanation, for proof

To better picture our class, it is helpful to indicate that these students were so interested in doing hands-on, engaging tasks with paper via paper folding. They knew how to calculate the square's area and perimeter but it was surprising to us that they could not make a square with a certain side by paper folding. They knew what the bisector is but they could not create the bisectors of a given square by folding it. In summary, they had theoretical knowledge about geometric shapes but were challenged when applying it practically.

We noticed that by proceeding with the classes, and specially after 3 or 4 sessions, the students become more acquainted with the geometric shapes and their visualization ability grows so that when folding some more complicated shapes many of them can guess the next step without any given instruction or before it would be taught by the teacher. The other aspect that really surprised us was their interest towards learning geometrical conceptions related to the shapes they had worked on. The other aspect which was prominent to us was motivation of creative students to share their ideas with the other students, which led to the creation of few working groups.

With these observations, we formulated 2 research questions, as How students conceive geometric conceptions using Origami? and How these informal types of education can contribute to the formal education of deprived students? We would like to continue working with this same class as our case study.

Based on our preliminary findings of these classes we are willing to explore these issues more systematically with deprived students, since a considerable number of them had bad experience with mathematics and many of them have to work due to their parents' financial

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HOW DO PUPILS INTERPRET THE RESULTS OF DIVISIONS WITH REMAINDER IN MATHEMATICAL MODELING?

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Mathematical modeling is a means of interpreting real life problems in mathematical terms. It involves the following activities: creating a model which involves mapping from the real world to the mathematics, manipulating the model, obtaining relevant results from manipulation, and checking the usefulness of the results in the context of the real world (Lesh & Zawojewski, 2007). Previous studies have focused mainly on educational stages above middle school (e.g., Blum & Borromeo Ferri, 2009; Niss, 2010). However, pupils in elementary school may also be able to interpret a situation mathematically to some extent if they are not required to use advanced mathematics. The purpose of this study is to explore how pupils interpret the results in a modeling activity. To achieve this purpose, the current study developed a set of problems, including one involving division with remainders, analyses pupils' responses to them.

The theoretical framework of this study is based on the 'Models-and-Modeling Perspective on problem solving' which focuses on the process of interpreting a situation mathematically in problem solving (Lesh & Zawojewski, 2007). This study identified phases in the process of interpreting a situation. This paper presents a problem of division with remainder and explores pupils' interpretations. In total 266 pupils of grade 3, 4, 5 and 6 participated in the study. They worked on the problem that asks to find the number of benches if there are 69 pupils and a lot of benches for four persons. Pupils' solutions were analysed in each phase of problem solving.

The results showed that pupils are able to interpret the results not mechanically but freely, for example, based on own feeling, experience or mathematical means. The findings suggest that pupils interpret the results primitively from various perspectives, but a majority of their interpretations can be related to ideal mathematical modeling.

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RESEARCH ON INSTRUCTION IN EARLY STAGES OF PROBABILITY

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Among the countries of East Asia, the students learn early probability at the elementary school. The purpose of this research is to clarify the conditions with which the problem and question should be equipped. At the 1st step, we arranged about the worth of teaching materials. Sriraman & English (2004) mentioned that there was hardly any topic which could improve the skills of pupils so extensively as combinatorics. As a result, we pointed out worth of permutation and combination. At the 2nd step, we surveyed about the method of the present instruction. As a result, we paid our attention to the relation between asking for the number of permutation and combination, and calculation. At the 3rd step, we considered the difficulty of the instruction. Szitanyi & Csikos (2015) concluded that college students often chose one of the algebraic expressions from their high-school repertory, and use it without sense-making of the problem. As a result, we pointed out that calculation was interfering with achievement of the purpose of the lesson. At the 4th step, we examined the method for the instruction to improve. As a result, we proposed the instruction which does not ask the number of permutation and combination. At the 5th step, we carried out the lesson by the improved teaching method. As a result, we recognized the importance of the problem. At the 6th step, we analyzed the lesson qualitatively. As a result, we specified the effective problem and question.

By all the steps, we clarified the following two points. The 1st point is not asking the number of permutation or combination. That is, it is important to ask how to put permutation and combination in order. The 2nd point is that the student searches for neither permutation nor combination by calculation. That is, it is important that the student sets up the problem which is not called for by calculation. When carrying out instruction in early stages of probability, we need to fill these two points.

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FACTORS INFLUENCING STUDENT SELECTION OF SENIOR SECONDARY SCHOOL MATHEMATICS SUBJECTS

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INTRODUCTION

Declining numbers of Advanced Mathematics (AM) students at secondary school are seen as a major issue for the future of STEM in Australia and internationally (Noyes, Wake, & Drake, 2011; Office of the Chief Scientist, 2014). Few large-scale research studies have investigated why students choose particular mathematics subjects.

THE STUDY

The aim of this empirical study is to identify the reasons why students choose or do not choose AM in the last two years of secondary school. Quantitative data were collected via surveys from secondary school mathematics students and teachers, and university mathematics lecturers. The surveys contained 20 statements on reasons for choosing/not choosing AM, covering intrinsic and extrinsic motivational factors.

RESULTS

Surveys were returned from 60 teachers, 20 lecturers, and 1000 students, comprising 350 AM students and 650 students who did not choose AM. Those who chose AM typically stated intrinsic reasons whereas those who did not typically stated more extrinsic ones. For example, AM students said they enjoy mathematics and AM would be good for their lives, whereas non-AM students said they would not get good marks in AM, they do not need AM for university, and there were other subjects they wanted to study. There are also differences between what students *say* are the reasons and what teachers and lecturers *think* are the reasons. For example, the latter thought friends would influence student decisions not to choose AM; however, the students said this was not the case.

These results provide rich information to address the decline in AM numbers. Knowing why students do not choose AM allows teachers and lecturers to accurately target future students by promoting a range of intrinsic and extrinsic motivational reasons to convince students to choose AM for the last two years of secondary school.

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INVESTIGATING PRE-SERVICE MIDDLE SCHOOL MATHEMATICS TEACHERS' QUANTITATIVE REASONING AND THEIR SUPPORT FOR STUDENTS' QUANTITATIVE REASONING IN THE PROBLEM SOLVING PROCESS

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Quantitative reasoning has a remarkable value in middle school mathematics. Recent studies stress that middle school mathematics teachers have a significant role in developing the quantitative reasoning of middle school students. Promoting the middle school students' quantitative reasoning in the learning environment depends highly on middle school mathematics teachers who are able to use well-structured questioning, such as posing appropriate questions, leading students to consider carefully about quantitative relationships in the problem and think quantitatively (Ellis, 2007). The purpose of the study is to examine pre-service middle school mathematics teachers' quantitative reasoning and their support for students' quantitative reasoning in the problem solving process. The teaching experiment was used as the research method in this study. Data for this study were collected through a series of exploratory teaching interviews with focus group participants and each participant conducted a clinical interview with one middle school student. The clinical interviews performed by the researchers and the participants were analyzed with qualitative content analysis. According to results, there is strong relationship between pre-service middle school mathematics teachers' quantitative reasoning and their support for students' quantitative reasoning in the problem solving process. Pre-service middle school mathematics teachers with poor quantitative reasoning were not able to support their middle school students' quantitative reasoning in the problem solving process. Even though these pre-service teachers took course in relation with the quantitative reasoning and its pedagogy, they were not able to put what they learned into practice. On the other hand, pre-service middle school mathematics teachers with strong quantitative reasoning were able to support their middle school students' quantitative reasoning in the problem solving process. Since these pre-service teachers with strong quantitative reasoning took course in relation with the quantitative reasoning and its pedagogy, this might have also helped them to provide awareness in quantitative reasoning and its teaching that allows them to put their knowledge into practice. The results suggest that pre-service middle school mathematics teachers' quantitative reasoning is associated with their support for students' quantitative reasoning in the problem solving process.

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A CHALLENGE OF VISUALIZATION IN DIFFERENTIAL EQUATION EDUCATION

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In this study, the first author started with “particular belief” that Differential Equation Visualization (DEV) was almost the “easy factor.” A number of the Schoenfeld’s R/O/Gs framework in teachers’ considerations given below, indicated he emphasised on the visualisation and especially on the need for it to be easy factor in DE education; R1: “the visualisation is one of these things that I see happened a lot in school mathematics but you don’t see it so much at the university level”, O1: “visual approach is far easier than analytical approach” and G1: “I hope to give a flavour for just how easy the DEV education is.”

However, by scrutinising the details of teaching and learning process, and collaboration with the second author his previously held views challenged as given below; R2: “my students had a strong preference for the analytic mode of thinking, but did not feel a need for a visual mode of thinking, though they preferred to expose their well-developed analytic abilities”, O2: “students did not write the prediction of long-term behaviour as they did not consider the visual coordination of their sketch and the $f(y)$ graph, which I thought it was easy to see” and G2: “yeah, I guess one of my guiding principles in teaching was that it should be easy by visualisation, now, if I think it’s going to help.”

We suggested that what follows was representative of a “particular belief” which should be taken into account, as the “easy factor” in DEV education was a dominant consideration that may it overshadowed other considerations of teachers’ considerations. Although our goal was to maximize information, not to facilitate generalization (Lincoln & Guba, 1985), however this “particular belief” may was contradiction to the most of DEV teachers as discussed by (Whitehead, 2000). This give rise to a conjectural characterization of DEV teachers’ considerations, with the suggestion that the R/O/G can serve as the foundation for the pedagogical development.

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RECONTEXTUALIZING MATHEMATICAL PROCESSES THROUGH A CURRICULUM REFORM

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The adoption of the sociological notion of ‘recontextualizing knowledge’ can provide an operational framework for understanding and interpreting the implementation of a new curriculum. In relation to this, Bernstein (2000) distinguished the Official Recontextualizing Field, established and dominated by the State, and the (official) Pedagogic Recontextualizing Field, in the formulation and management of which agents, such as teachers’ trainers, more or less independent from the State, are involved. The interaction between the fields of recontextualization forms the resources used by teachers to legalize their practices in classroom (Morgan, 2010). The Greek recontextualization field can be considered to consist of three official sub-fields: (a) the Official Recontextualization Field, (b) the Official Pedagogical Recontextualization Field, and (c) the Local Pedagogical Recontextualization Field, where schools and teachers interpret the new curriculum through additional resources produced locally.

Utilizing Bernstein’s theoretical framework, the aim of the present research is to study the ways in which primary teachers, who participated in the one-year pilot implementation of the new curriculum developed in the Greek compulsory education, recontextualized mathematical processes such as mathematical reasoning and argumentation, creating bonds between concepts, communication through the use of different tools and metacognitive awareness. The sample consisted of 13 primary teachers who worked in three schools in the north-east part of Greece and they had considerable teaching experience (10-25 years). A semi-structured interview took place with each of the teachers aiming to study the recontextualized process that possibly took place in relation to the mathematical processes promoted by the new curriculum. For the data analysis, techniques from content analysis and grounded theory were employed. Teachers drew mainly on the resources provided by the Official Recontextualizing Field when arguing about the four mathematical processes promoted by the new curriculum. At the same time, the Local Pedagogical Recontextualization Field fuel teachers’ resources for understanding the four mathematical processes, having an impact on the options available to teachers and adopted by them in the classroom practice.

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INTEGRALS: NON-ROUTINE QUESTIONS

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The integral is an essential part of high school calculus. The prevalent approach to teaching the integral in Israeli high schools is an initial presentation of the indefinite integral as antiderivative; then, the definite integral is presented as a tool for calculating area; the connection between the two kinds of integral is rarely presented. The literature (e.g., Sealey, 2006) leads to the conjecture that many high school students acquire formal techniques using integrals for the solution of some classes of problems but fail in tasks that require comprehension of the concept of integral. We empirically study this issue. The research questions are how advanced level high school students

- respond to non-routine comprehension questions about integrals?
- understand the concept of the integral?

We developed a questionnaire and interviews. We administered the questionnaire to 269 Israeli advanced level high school students having completed their study of integrals. The questionnaire contains 8 non-routine comprehension questions that do not address what students are doing but how and why they are doing it. The questions required (a) providing an answer and (b) explaining or arguing the answer. Hence, part (b) required the students to think in depth about what they had learned and gave them opportunities to reflect on their knowledge. Following the questionnaire, we conducted 14 semi-structured interviews in order to clarify students' responses.

Fewer than 40% of the students answered more than 8 of the 16 sub-questions correctly. Only 4 sub-questions were answered correctly by more than 50% of the students, and only one of these was a (b) sub-question.

The analysis of the interviews confirmed that the students remembered formulas and procedures, and demonstrated routine calculation skills. However, the majority consistently failed in conceptual assignments. Their knowledge regarding the concept of the integral was messy, incoherent, and confused. It seems that high school students who learn about the integral according to the prevalent approach, even those who learn mathematics at the advanced level, acquire formal-procedural knowledge that includes some techniques and formal notions but is not coherent. Therefore, the development of an alternative approach that could help the students form an essential understanding of the concept of integral is indicated.

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INVESTIGATING ELEMENTARY MATHEMATICS TEACHERS' GEOMETRY COURSES TOWARDS GEOMETRIC HABITS OF MIND*

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Geometric thinking is one of the most important skills which has a significant place in the development of mathematical thinking. Besides, it is recommended for teachers to foster geometric thinking in their mathematics classes by the curriculum developers. Therefore, teachers have an important role in fostering geometric thinking. Teachers are expected to teach geometry to support geometric habits of mind and also to choose activities and problems that they use appropriately in the classroom. For this reason, examining elementary mathematics teachers' geometry courses is significant.

In this study, it is aimed to investigate how teachers teach geometry in their routine classroom environment after attending a professional development project about fostering geometric thinking. Participants of the study are five mathematics teachers working at four different elementary school. During the project the participants attended the seminars based on the Geometric Habits of Mind (GHOM) framework (Driscoll, DiMatteo, Nikula, Egan, Mark & Kelemanik, 2008) for a month. Following these seminars, lesson study model was applied for geometry courses among teachers for three months.

About two months after the completion of the implementations of lesson study, the activities and problems teachers used for fostering geometric thinking in his/her own geometry class were observed during two weeks. Data about instructional explanations, problems and activities used by teachers was obtained from these observations and was analysed by qualitative data analysis methods. At the end of the study, it was found that teachers organize their geometry class by considering geometric habits of mind. In addition, it was also seen that the mathematical language and representations they use and their questioning skills were gradually getting better.

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CRITERION AWARENESS AND PROFESSIONAL KNOWLEDGE AS PREREQUISITES FOR TEACHER NOTICING AND ANALYSIS

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In a growing number of recent studies, mathematics teacher expertise is described with a focus on how teachers make use of their professional knowledge (e.g. Kuntze, 2012) in situation contexts (e.g. Kersting et al., 2012; Sherin et al., 2011). In these approaches, constructs such as “Usable Knowledge” (Kersting et al., 2012) or “Noticing” in the sense of “selective attention and knowledge-based reasoning” (Sherin et al., 2011) play key roles. A common characteristic of these studies is their phenomenological perspective which concentrates on describing what teachers notice or what knowledge they use. This raises the question how processes of noticing or analysing are started, what guides and triggers their development and why specific elements of professional knowledge come into play. There is hence a need for models which can explain the use of professional knowledge in these processes.

Responding to this need, this poster presents and explains a process model which has been used in the project ANAKONDA-M (e.g. Kuntze, Dreher & Friesen, in press) and which describes the teachers’ analysis of classroom situations as a knowledge-based and awareness-driven circular process. Criterion awareness is thus key for what we understand by knowledge-based analysis – a notion which we describe as “an awareness-driven, knowledge-based process which connects the subject of analysis with relevant criterion knowledge and is marked by criteria-based explanation and argumentation” (Kuntze et al., in press). Criterion awareness influences the readiness and ability of teachers to use specific elements of professional knowledge in instruction-related contexts and hence makes professional knowledge accessible, so that it can be used in the process of analysing classroom situations, for instance. Consequently, the process model presented and discussed in this poster can help to interpret and to better understand results from phenomenological studies.

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SYSTEMATICAL AND MATERIAL BASED DEVELOPMENT OF PROBLEM SOLVING COMEPTENCE OF MIDDLE SCHOOL STUDENTS

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The (inter-)national educational standards have strongly endorsed the inclusion of problem solving in school mathematics. The empirical studies, the PISA and TIMS studies, and quality analyses, however, portray a different picture: Students are often unable to solve problem tasks and teachers lack practical teaching materials with didactical comments to foster students' problem solving abilities, and at the same time to consolidate their own competence in the area. Efforts to design, use, and do research on materials in a real setting through a collaborative work between researchers and practitioners can promote adoption of innovations.

In this poster presentation I demonstrate the possibilities for developing materials using design-based research ([DBR]) focusing on a systematical development of mathematical problem solving competence. More concretely, I focus on the question: How can material supporting pedagogies and theories on the development of problem solving competence using DBR be developed in praxis? In accordance with DBR, the design process was informed by theoretical and empirical results with respect to different areas in problem solving: teaching approaches and concepts on problem solving (Bruder & Collet, 2011); theories of self-regulated learning (Zimmerman, 2002); theories on self-regulation in problem solving (Pólya, 1957; Schoenfeld, 1987) and activity theory (Giest & Lompscher, 2004). In addition, I show the results based on six DBR-cycles with respect to the designed materials, how the theoretical ideas got implemented, and what design and theoretical principles are important for the future development of materials relating to teaching and learning of problem solving.

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LEARNING HOW TO MEASURE ANGLES WITH TEACHER DESIGNED PROTRACTORS AND VARIATION IN TOOLS

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We report a research lesson in grade 7 where the teacher designed a set of simple protractors and tasks to introduce the concept and technique of angle measurement. This is a case study among others in a research project focusing on the design and use of tools and tasks in primary and secondary mathematics teaching. The ongoing project consists of researchers and teachers collaborating in design of lessons that can help each other to rethink the affordances of tools and associated pedagogical possibilities.

We utilise and refine a framework of tool-based task design, which emphasizes the semiotic potential of artefacts (Bartolini Bussi & Mariotti, 2008) and “nested epistemic modes of practices, discernment and discourse” (Leung, 2015). Study of this lesson aims at evaluating students’ response in initial stages of tool use as well as the teacher’s development of task design.

The lesson design began with research meetings for identifying students’ potential difficulties in skills and concepts of angle measurement using ordinary protractors and textbook exercises. From past experience, students tend to mechanically read from the scale without paying attention to the size of an angle as a geometric attribute. A new form of simple protractors was made by printing on transparent sheets some circles with indication of equal partitions in various ways but without any numerical scales. Tasks were designed to allow groups of students to choose from several versions of such protractors and compare their effectiveness in measuring specially prepared angles in a task sheet. Various forms of signs and representations were produced by the students from their measurement. Students shared and explained their views and productions of signs after using those protractors. Based on lesson observation and students’ work on the measuring tasks, we found that the students can relate the tools to geometric features relevant to angle measurement. They were also found to acquire the skill of using ordinary protractors easily and confidently in the subsequent lesson.

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EXPLORING TAIWANESE STUDENT TEACHERS' PERSPECTIVES ON MATHEMATICS AND CREATIVITY

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Developing students' creativity is crucial in today's world (Sriraman, 2009). Nadjafikhah, Yafian, and Bakhshalizadeh (2012) indicated that "nature of mathematics provides a suitable platform for developing creativity" (p. 285). It is important for student teachers to hold such perspective because their perspective on mathematics helps determine the learning environment they will create (Riedesel, Schwartz, & Clements, 1996). Little, however, is known about Taiwanese elementary student teachers' perspectives on mathematics and whether they believe mathematics is creativity. The purpose of this study was to explore Taiwanese elementary student teachers' perspectives on mathematics and whether they believe mathematics is creativity.

The participants of this study comprised thirty elementary student teachers in Taiwan. They were asked to circle the words that they believe best describe mathematics in a questionnaire based on Riedesel, Schwartz, & Clements (1996). The words included certainty, right answer, creativity, logic, estimating, fast and precise, listening, entertaining, rules, projects, exciting, abstract, remembering, exploring, making, intuition, inventing, useful, telling, meaningful, pure, practicing, basics, difficult, symbols, important, individual, and patterning, and there was no limit on the number of circled words for each participant. The findings of the study indicated that all of the thirty elementary student teachers circled logic as one of their views on mathematics and only six of them circled creativity as one of their views on mathematics. More research needs to be undertaken that investigates how elementary student teachers' perspective that mathematics is creativity could be promoted in Taiwan. Such information is important because the better student teachers think of mathematics as creativity, the better they will be able to help their students develop creativity through mathematics learning.

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IMPROVING STUDENTS' LEARNING EFFICIENCY AT THE SIXTH GRADERS BY MAKING MATHEMATICAL SENSE

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The objective of current research is to improve sixth graders' achievement on mathematics learning and performance on attitudes. The theoretical basis is the Making Mathematical Sense and the teaching method is the optimization of multiple teaching strategies. Lee & Lin (1998) defined mathematical sense as "a higher-order thinking that one can extract the intuitive meaning from mathematical materials", and defined optimization of multiple teaching strategies as "a teacher should choose appropriate teaching method based on the difficulty level of the targeted content and students' level of understanding".

The research method is experimental teaching. There are 35 sixth graders (two classes of two elementary schools) as the experimental group, 129 sixth graders (other classes of the two schools) as the control group. The duration of experimental teaching is one year. For experimental group, at first, there have two-day workshop to make teachers understand how to make students' mathematical sense and the concept of optimization of multiple teaching strategies. Monthly workshops were held to solve the difficulties teachers encountered in making students' mathematical sense. The data collection includes 4 mathematics exams (T1~T4) for experimental and control group, and 3 attitude questionnaires (A1~A3) for experimental group. The exam scores were applied with ANCOVA and the questionnaire on attitude was applied with paired-samples t-test. The value of Cronbach's Alpha is between 0.93~0.95.

The result shows the mean(SD) of T1~T4 is 86.06(12.78), 83.94(11.58), 71.14(18.82), 89.95(11.61) for experimental group and 85.04(14.03), 78.07(20.08), 65.31(25.49), 76.29(23.44) for control group. The experimental group reduce the gaps among students' scores. The Johnson-Neyman method used to determine the significance region, because F values go against the test of the homogeneity. The results showed that the current research indicated significant instructional effectiveness for middle and low-level students. Mean (SD) of A1~A3 on attitude is 2.68(0.75), 2.73(0.53), 2.73(0.51), there is no significant difference. This indicates that there is no quick change for students' attitude. And there is also some room for improvement on the current research. We listen for recommendations for school teachers, principals, educational inspectors and mathematical educators and make them as the important reference for research modification.

Note: This paper is a partial result of the Project MOST 103-2511-S-845 -007 -MY3.

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INTEGRATING HISTORY OF MATHEMATICS IN CLASSROOM TEACHING: THE CASE OF USING ZU GENG'S (CAVALIER'S) PRINCIPLE

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Studies have found that using history in mathematics education can provide benefits to both students' learning and teachers' teaching (Jankvist, 2009; Wilson and Chauvot (2000)). Thus, it is considered as a pedagogical tool for learning (Jankvist, 2009).

This study provided a case which was a second year of mathematics beginning teacher in a key high school in Shanghai. She obtained a bachelor's degree in Mathematics and Applied Mathematics and master's degree in mathematics education, especially in HPM (history and pedagogy of mathematics). Through one lesson with the integration of Zu Geng's principle (Cavalieri's principle), the two questions were addressed: a) How history of mathematics was integrated in the case teaching? b) What was the teacher knowledge reflected in using of history in teaching?

According to Jankvist (2009) and Tzanakis and Arcavi's (2000) work, multiuse history of mathematics was shown in the case teaching. The connection between their works can be seen in the figure 1 through analysing this case.

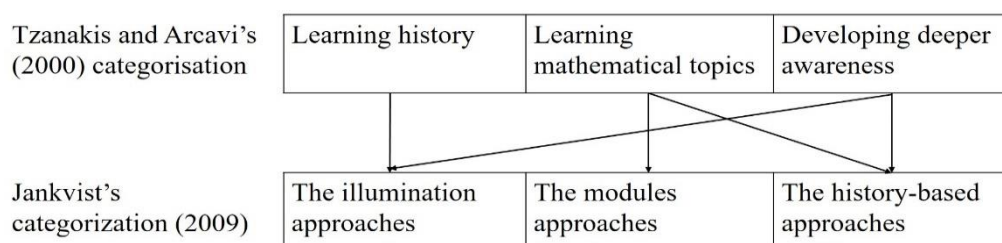


Figure 1: Categories of approaches that used to integrate history

In this case, she had developed the ideas that her teaching integrated history would provide additional benefits. Even though, she still insisted on the teacher-centred practice, which was regarded as a dominant teaching pattern in China because of the centralised curriculum system and extra pressures from the national or large scale public examinations e.g., Gaokao.

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COGNITIVE ACTIVATION IN THE MATHEMATICS LESSONS: A COMPARISON OF THE FOUR CHINESE-SPEAKING ECONOMIES IN PISA 2012

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According to the findings of PISA 2012 study, the four Chinese-speaking economies (i.e. Chinese Taipei, Hong Kong, Macao, and Shanghai) top the league table of mathematics performance amongst the 65 participating economies (OECD, 2014). Attested to the spirits of the new mathematics curriculum standards put forward for the 21st century, the mathematics curricula implemented in these four places called forth building up of students' higher level of cognitive functioning. PISA 2012 provides a golden opportunity for the educational researchers to examine whether higher degree of cognitive activation in the mathematics lessons (as measured by the COGACT index) is related to higher student mathematics performance (as measured by the student mathematics score MATHPVs). In PISA 2012, there are altogether nine cases of cognitive activation to be examined for their frequency of occurrence in everyday mathematics lessons. Because of the classroom environments in the four places are culturally similar but socially different it is of considerable interest in this study to examine whether there are similarities and differences in the classroom practices of cognitive activation in the mathematics lessons. Drawing data from the PISA 2012, the following two research questions are put forward for examination in this study:

1. Is cognitive activation in mathematics lessons positively related to the student mathematics performance in each of the four Chinese-speaking economies in PISA 2012?
2. Amongst the nine cognitive activations in mathematics lessons examined in PISA 2012, what are the similarities and differences amongst the four high-performing Chinese-speaking economies?

Through regression analyses of COGACT on MATHPVs, the finding is that there are statistically significant independent effects ($p < .05$) of cognitive activations on student mathematics performance for all the four Chinese-speaking economies. Through analyses of variance (ANOVA) of the mean of each of the nine cases of cognitive activations in mathematics lessons across the four Chinese-speaking communities, the finding is that there are a number of cognitive activations the means of which are statistically significantly different amongst the four Chinese-speaking communities. In the light of the findings, recommendations for the betterment of mathematics education in each of the four Chinese-speaking communities are given.

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ENHANCING MATHEMATICAL MODELLING COMPETENCIES THROUGH TEACHING INTERVENTION

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In the current era, known as the technology age, mathematics has an important and extensive role in a way that society's movement alongside this fast-progressing scientific development requires a deep and fundamental understanding of mathematics (Clayton, 1999, cited in English, 2004). This paper aims to describe the effect of mathematical modelling teaching intervention on 6th grade students' mathematical modelling competency. In sum, 300 students (12 years old) participated in eight weeks course. These students allocated in experimental group and control group randomly. After a short explanation about modelling process for all students, a pre-test containing a modelling activity with context of travelling was applied. Intervention program (contain five modelling activities) was organized seven sessions only for experimental group. The post-test, which had same context as the pre-test, was administered in a separate session for all students. Preliminary results show that the experimental group had better performance in solving modelling activities than the control group. Obviously, performance of students in experimental group show dramatically enhancement in upper levels of modelling competencies. Also there are different aspects and especially in this study that they play remarkable role in mathematics education. Firstly, elementary school students the same as high school students can participate to solve modelling problems. Secondly, the Who want to solve this type of problems solve, they encounter with variety of factors in each problem and they try to use them in problem solving. The Third, they should share freely their ideas and they have to communicate in Solutions, and this leads a deep understanding from the mathematical concepts. The Fourth, Solving in groups give them opportunity to acceptance each student role in during solving the problem as a result they help each other and offer a better way of solving. Finally, one of the best ways is that students have to listen to the criticisms of others for their solution that results in they can in assessment their solutions. There are many positive points in education problems modelling. On the one hand, there are many positive points in education problems modelling. In the other hand, the modelling problems are time-consuming. In addition, some students do not like those and they do not participate in the group and because of that get introverted.

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A MATHEMATICS FOR TEACHING OF PROPORTIONALITY

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We present mathematics for teaching as a model for the teaching of a certain concept. The model allows us to gather a variability of ways to communicate a concept and, thus, (re)introduce it through a theoretical structure that organizes the ways of concept occurrence, in this case, proportionality. Being inspired by Sfard (2008), we define concept as realizations associated with the name that refers to it or is able to designate it. In turn, realizations refer to what has been communicated by speech and writing with regard to the concept of proportionality. From this premise, the general aim of this study was to build a mathematical model for the teaching of the proportionality concept. We have identified three different ways to communicate this concept through the use of three sources: scientific papers, a group of teachers and mathematics textbooks. Emphases from the Study of Concept (Davis & Renert, 2014) were mobilised as an inquiry tool. The appropriation that we made from this tool along with concepts of Sfard's (2008) theory were used for modelling theoretically the proportionality concept. The results have showed diversity of realizations for the concept of proportionality, which were distributed in three different landscapes. In the first landscape, the concept of proportionality was related to ratio and it was held as rate, scale, division, probability, trigonometric ratio, percentage, proportional division and partition, vector and music intervals. In the second one, it was described as the equality between ratios through the use of the rule of three, the proportional division of segments and percentage. In the last landscape, this concept was presented as a variation rate of a function and it could be identified as a proportionality constant, a scale factor, an angular coefficient or a declivity. We hope this model will foster discussions in the scientific field, collaborating with other concepts as a model for teaching as well as in the training of teachers to present an overview of the concept of proportionality.

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I CAN DO THAT TOO! TEACHERS DISCUSSING HOW THEY REPRESENT FRACTIONS

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Teachers are expected to use high-quality teaching practices in mathematics classrooms to support students' conceptual learning. One of those practices consists of using multiple ways to represent mathematical ideas (Stylianou, 2010). In this study, we used video clubs, a type of professional development in which teachers videorecord themselves teaching and discuss their practices with colleagues and experts (Borko, Jacobs, Eiteljorg, & Pittman, 2008), as a medium to discuss representing fractions. Our goal was to investigate: 1) the types of representations teachers used in their lessons to highlight ideas related to fractions; and 2) what teachers noticed about the practice of representing fractions.

We used data from a nationwide study on mathematics pedagogy in regions of Canada (see Reid et al., 2015). Three groups of four teachers from two Canadian provinces participated in separate focus groups lead by two of the authors. Each teacher videorecorded a lesson on fractions. During the focus groups, the participants watched the videos and discussed their teaching practices and representations of fractions. The focus groups were videorecorded and transcribed. We used the videos of the lessons to identify the representations used by the teachers and the transcripts of the focus groups to analyze the teachers' reactions to these representations.

Results showed that teachers drew circular area models most often to represent fractions. They also drew rectangles, measuring cups, number lines, and showed YouTube videos. Teachers agreed that the number line should be used more often. They also noticed the complementary use of circular and rectangular representations: "I see you [the other teachers] often using rectangles to represent fractions. I don't do that. Maybe it's something that's lacking in my own practice."

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USING VYGOTSKY TO UNDERSTAND THE WAYS GTAS APPROPRIATE PROFESSIONAL DEVELOPMENT

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In the U.S., students' experiences in Calculus I are a main contributing factor to students' decisions to leave the STEM (Science, Technology, Engineering, and Mathematics) disciplines (Seymour & Hewitt, 1997). One way departments have been working to address student attrition is to offer professional development (PD) to the Graduate Teaching Assistants (GTAs) and to reorganize the structure of the GTA program. Little research has been done on how the GTAs are appropriating and transforming the professional development activities.

In our study, the GTA program around Calculus I and II has a professional development component. In addition, the GTAs are observed by a lead TA and they meet weekly with the coordinator of the course. We are interested in how exactly GTAs are appropriating the professional development in which they are engaged.

Data consists of recordings of the professional development sessions, weekly coordinator meetings, selected classroom observations, and the discussions the lead TA has with the observed GTAs. We utilize a framework known as the Vygotsky Space. The Vygotsky Space framework (Harré, 1983) is used to describe the ways the individual appropriates ideas from the social plane into their individual plane, transforms these ideas, and then makes the ideas public again to the social plane. There are four main quadrants within the Vygotsky Space framework created by the public-private axis and the individual-social axis through which individuals move continuously as they engage with others. While it was originally created for the use of understanding the development of personality, it has been used successfully in other areas of social science, such as coaching at the K-12 level (Gallucci, DeVogt Van Lare, Yoon, & Boatright, 2010). We investigate how the GTAs appropriate and transform PD and what role the lead TA plays in publicizing and the conventionalization of the professional development activities. In our poster, we will discuss the initial findings from our study using the Vygotsky Space framework.

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UNDERSTANDING EQUIVALENCE THROUGH VISUAL REPRESENTATIONS ON AN INTERACTIVE WHITEBOARD

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Equivalence is a difficult concept to grasp for many students when starting to learn the formal rules of algebraic syntax. A relational understanding of the equal sign is essential for perceiving algebraic transformations as meaningful activities, and for succeeding with algebra and mathematics in general. Fostering algebraic reasoning early, prior to formal algebra instruction, is important for facilitating in-depth algebra learning (e.g. Carraher & Schliemann, 2007). Digital technologies can offer unique visual mediators for describing mathematical concepts, ideas and relationships (Sinclair & Baccaglini-Frank, 2016), and an interactive whiteboard has important potentialities for supporting the development of mathematical concepts and improve understanding (De Vita, Verschaffel & Elen, 2014). This study aims at studying how whole-class interactions based on visual representations on an interactive whiteboard can contribute to primary school students' relational understanding of the equal sign.

As part of a larger Norwegian project, *ARK&APP*¹, all the algebra lessons of 23 5th-grade students and their teacher were video-recorded for three consecutive weeks in December 2014. Digital resources for interactive whiteboards gave basis for exploratory whole-class interactions. The poster will present examples of interactive pictures, objects and animations used by the teacher in order to illustrate the equivalence relation represented by the equal sign. Further, the poster will highlight how the teacher facilitated for conceptual learning through building a bridge between visual representations related to contexts that are relevant to students' everyday life, the classroom dialogue, and equivalence.

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¹ <http://www.uv.uio.no/iped/english/research/projects/ark-app/>

PEER ROLE MODELS IMPROVE SELF-PERCEPTION

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Persistence in the STEM disciplines continues to be a problem in the U.S., especially for women and under-represented minorities. Nationally, female Calculus I students are over one and one-half times as likely as male students to opt out of taking Calculus II (Rasmussen & Ellis, 2013). One factor is the absence of female peer role models in the classroom, leading to feelings of not belonging (Lockwood, 2006). For under-represented students, including women, belonging is crucial (Walton & Cohen, 2007). We present preliminary results of an ongoing study involving the use of peer role models to increase women's, particularly Latinas', persistence in the calculus sequence. Peer role models are inspiring in-group members who defy stereotypes (i.e., mathematically-capable female students).

In a large university, half of the Calc I break-out sections were visited twice for 15 minutes by female peer role models (treatment) presenting two applications of Calculus, whereas the other half were not (control). At the start and end of the semester, we administered a Likert survey on feelings of belongingness, personal mathematical ability, attitude toward mathematics and intention to continue in the calculus sequence. The results were statistically analyzed as a function of section type (peer role model/control) and student gender (male/female). As a measure of persistence, we examined how many students enrolled in Calc II/repeated Calc I.

In the control condition, female students' beliefs about their math ability were significantly lower than male students ($p < .051$, $n=252$). In the treatment condition, female students' beliefs about their math ability were as high as that of male students. Regardless of student gender, students who saw a role model were more likely to be enrolled in Calc I/Calc II than those not exposed to a role model. This shows evidence that exposure to a female peer role model may improve women's mathematics experiences. We will also report on the nature of the intervention, as well as the replicated scaled-up intervention the following semester.

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MATHEMATICAL MODELLING ACTIVITY FOR IN AND OUT OF CLASSROOM: THE WALKING-TRACK

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Mathematical modelling can be described as a process which starts and finishes in the real world and during which mathematics are used in every step. We adopted an educational modelling (Kaiser & Sriraman, 2006) which elicits learning processes and fosters understanding of concepts. Also we used mathematical modelling as a purpose of mathematics education. We conducted a design based research (Van der Akker, Gravemeijer, McKenney & Nieveen, 2006) including problem posing tasks based on mathematical modelling. As interventions we carried out some out of classroom activities with different groups. The purpose of the study is to introduce one of them named “The Walking-Track” and compare in and out of classroom activities. In the problem posing task, we gave students ‘walking-track’ themed photographs to pose modelling problems. They posed at least three modelling problems and they chose the most appropriate for mathematical modelling. In the following lesson they solved it in the classroom and asked us to carry it out outside the classroom. Then we decided to carry out this modeling task as an out of classroom activity. During the activity we recorded their voice with a tape recorder and students took photographs that could be used as a reminder, visualiser, helper and trigger for constructing mathematical model. After the out of classroom activity we compared in and out of classroom activity solutions. We realized that out of classroom activity provides students more freedom and mobility. Further, they can make more realistic and logical assumptions. Because of the unlimited time-frame, they enjoyed the activity more than the classroom activity. We recommend teachers to do some out of classroom activities that can include students’ real life context.

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MATHEMATICS READING COMPREHENSION OF FIFTH GRADE STUDENTS IN SOUTHERN TAIWAN

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Many researchers reported that students with reading and math difficulties are characterized by weaknesses in problem solving (e.g., Jordan, Hanich, & Kaplan, 2003). However the detail of word problem reading comprehension of southern Taiwan's students was little to be discussed. Thus, the purposes of the study was to explore the relationship among mathematics reading comprehension (MRC), Chinese reading comprehension (CRC) and mathematics performance (MP), to examine the predictors of MP and to investigate the performance of MRC of 5th grade students.

The number of participants in this study was 868 fifth grade students (8 schools from 4 counties/cities). Students were required to have Chinese reading comprehension standard test and the mathematical word problem comprehension and performance test which was developed by the author. Each word problem in this test consisted of five questions which sequentially examined student's abilities of the interpretation and integration of sentences (IIS), the understanding of the problems' goal (UPG), reasoning (R), solution planning and execution (SPE) and MP. The sum of IIS, UPG and R was MRC. The Pearson correlation, regression analysis and ANOVA were used in data analysis process.

The findings indicated that the correlations between MRC and CRC, MRC and MP, CRC and MP were all statistically significant ($p < .01$), a largest correlation was shown between MRC and MP. The correlations between IIS and UPG, IIS and R, UPG and R were also statistically significant ($p < .01$). The results of the regression predicting MP indicated that IIS ($\beta = .495$), UPG ($\beta = .143$) and R ($\beta = .157$) were significant ($p < .01$). IIS, UPG, and R significantly accounted together for 53.7% ($R = .734$) of the MP variance respectively and IIS was the most important predictor of MP. The 5th grade students' abilities on IIS, UPG and R were significantly different ($IIS > UPG > R$).

In summary, the MRC skills (IIS, R and UPG) are better predictors of math performance than the general CRC skills. This has important implications for instruction of math word problem reading comprehension. The instruction of IIS, R and UPG skills should be conducted in the word problem reading comprehension of 5th grade students, especially interpreting and integrating sentences.

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HOW DO THE FIRST GRADERS LEARN ADDITION AND SUBTRACTION?

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Addition and subtraction are the basic operations that young students learn. Whereas students often know how to compute such operations, they do not necessarily understand their meanings and properties. Against this trend, it has been emphasized that young students should experience various meanings of addition and subtraction with mathematical models, explore the properties of operations (e.g., commutativity and associativity), and employ such properties to multiple problem contexts (Caldwell, Karp, & Bay-Williams, 2011). Given this background, we have developed an electronic book in which students are urged to understand the meanings of addition and subtraction, represent the given contexts into addition or subtraction, explore the basics of two operations, and add or subtract whole numbers less than 20 in an effective way. These topics cover at least 15 lessons in the first grade. The electronic book includes interesting stories, interactive components, and various tasks tailored to students' different abilities.

A questionnaire with 18 items was developed to assess students' understanding of addition and subtraction. Two groups of the first graders from 8 classrooms were participated in this study. The first group who had learned the topics with mathematics textbooks completed the questionnaire at the end of their first grade. The second group completed the equivalent questionnaire before and after learning the topics with the electronic book. A preliminary analysis showed that the first group of students were successful in solving typical textbook-type problems but had difficulties in other types such as interpreting the given contexts and representing them into equations. In comparison, the second group of students were more successful in solving the difficult problems than the first group. Our proposed poster displays the main characteristics of the electronic book, sample assessment items, students' overall performance, and typical responses per different types of items. As such, this poster is expected to illustrate materials for meaningful learning of addition and subtraction as well as various understandings related to the basic operations by young students.

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WHAT IF WE LIMIT THE OBVIOUS AND CHALLENGE THE MIND?

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The central idea developed here is how visualization (Gutiérrez, 1996), geometric reasoning (Battista, 2007), argumentation, the use of DGS, and task designing in the teaching of geometry, interplay with each other promoting students' geometric and deductive reasoning. Visualization and argumentation based on the properties of geometrical objects are integral parts of geometric reasoning. But, there needs to be a continuous interplay between visualization and argumentation, for students' geometric reasoning to evolve. The correlation of properties between geometrical objects of different dimensions is the thought that generated the idea of what I call *D-transitional tasks*; tasks involving transitions from 2D to 3D (or vice versa) geometrical objects. "DG environments may provide opportunities for the creation of uncertainties, leading students to seek for explanations" (Hadas et al., 2000, p.128). A way to create this desirable uncertainty are the *Black Box* activities (Laborde, 1998).

I designed an activity with *D-transitional tasks* in GeoGebra 5 (*3D Graphics*) dealing with incisions performed by a plane on solids. The documents given to the students to work with, have everything "hidden" except for the axes of the 3D space, the plane and the incision. The question set to the students is: "What solid do you think this could be, judging from its incisions?". This is a Black Box activity, since the whole construction is hidden and the students need to turn to the properties of both the solid and the 2D incisions in order to discover the solution. Creating a feeling of uncertainty during the exploration, this activity aims to "cause" the students the feeling of the need for theoretical mathematical justification trying to promote students' property-based geometric reasoning, visualization and deductive reasoning.

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RECIPROCAL LEARNING IN MATH EDUCATION: A COMPARATIVE STUDY ON THE TEACHING OF PROBLEM SOLVING AT TWO CANADIAN AND CHINESE ELEMENTARY SCHOOLS

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Contemporary research literature depicts a high number of comparative studies in mathematics education, which dwell on general teaching/learning theories, assessment/evaluation practices, and test achievements/results in the East-West educational paradigms and arenas (Kaiser & Blömeke, 2013). In this study, we go beyond the back-and-forth debates that often arise from these comparative studies, and take a reciprocal learning approach to explore in-depth the commonalities and differences in mathematics education between two Canadian and Chinese elementary schools as part of a larger seven-year Reciprocal Learning Partnership Project between China and Canada. In the study, we pay particular attention to the teaching of problem solving, not only because problem solving has long been a crucial component of school mathematics, but also because the development of students' abilities to solve mathematical problems has remained one of the fundamental goals of mathematics education in the two research schools.

Research data were collected through direct and indirect interactions between the pair of research schools. These interactions included Skype meetings; formal and informal conversations with teachers and administrators in both schools; and the sharing/exchange of documents, texts, teaching materials, and resources. Data coding techniques based on grounded theory were utilised to explore and analyse the differences and commonalities of the teaching of problem solving in the two schools.

Our results show that there is a degree of commonality across the two schools in the teaching of problem solving with a common emphasis laid on having connections with real life situations, and encouraging students to use multiple strategies to solve problems. On the other hand, we also observed some differences between the two schools with regard to teachers' teaching strategies, students' learning tendencies, and sources of mathematical problems. Based on our findings, we proffer suggestions on what the two Canadian and Chinese schools could learn from one another.

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STRATEGIES FOR PLACE-VALUE MATCHING IN MULTI-DIGIT NUMBER ADDITION

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There are multiple cognitive processes such as visual perception, arithmetic fact retrieval, or the handling of carry over (Trbovich & Le Fevre, 2003), and also a variety of elements in the presentation format (e.g. visual orientation and spacing), that may impact how additions are cognitively performed. Conceptually, we can decompose multi-digit addition into at least two broad processes. The first identifies the position of digits within the number strings and defines which digits should be added to one another, a *place-value matching*, while the second computes the actual additions. In the present work we investigate the processes underlying the place-value matching stage.

Participants (N=40 Psychology students) were asked to calculate the result of horizontally-presented additions. Items consisted of a 3-digit number (“large” number) and a “small” number. Items varied in terms of: the location of the large number (left: $610+3$ or right: $3+610$); number of digits of the small number (one: $610+3$ or two: $601+30$); and the digit within the large number to which the significant digit of the small number had to be added (zero: $610+3/601+30$ or two: $612+3/621+30$). To control for working memory demands, we only included additions with no carry over.

An analysis of response times showed, among others, significant effects for the location of the large number ($p = .002$): when the large number was at the left, answers were quicker. Comparing our results with Trbovich and Le Fevre’s, we conclude that this effect of the location of the large number does not depend on: the modality of the response (oral or manual), nor to the presence or absence of carry over in the additions. This implies that we can discard accounts specific to the oral modality, and that this effect is not due to phonological working memory load produced by carry over. These data suggest that additions where the larger addend is presented first are cognitively easier for adults, similar to previous findings on children (De Corte & Verschaffel, 1987). Ongoing work is exploring the involvement of both phonological and visuo-spatial working memory.

This research was funded by the Chilean programs CONICYT Basal (grant FB0003) and CONICYT PAI/Academia (grant 79130029).

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CHANGES IN PERCEPTION OF GRAPHS OF FUNCTIONS

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INTRODUCTION AND THEORETICAL BACKGROUND

Our poster shows results of small scale qualitative research focused on student's interpretations of graphic representation of functions and their assessment from the learning perspective. We concentrate on the evolution of the perception of graphs of functions using a framework of hypothetical and individual learning trajectories (Simon, 1995) through observing student's perception of examples of graphs of functions (Watson, Mason 2005). The study was supported by Charles University in Prague, project GA UK No 227-364.

RESEARCH GOALS

The goal of this study is to identify properties of graphical representations of functions which students use when solving tasks of a non-mathematical, epistemological character (like "Separate good examples from bad ones") and find out how the usage of these properties evolves in the course of education from late primary to early tertiary level.

METHODOLOGY

Semi-structured clinical interviews were conducted with three groups of respondents (twenty five in total) from late primary to early university level on three tasks focused on evaluating and ordering of fourteen graphical representations of functions from mathematical textbook. Interviews were analyzed using properties of graphs of functions we found important for students in previous experiments.

RESULTS

As preliminary results, we observed (1) changes in perception of relative importance of different constructs identifiable in picture (e.g., auxiliary constructions vs. graph itself), (2) change from procedural to conceptual perception of pictures, (3) changes in perceived relation between abstract function and its graphical representation, especially regarding continuity.

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PERCEIVED RESPONSIBILITY FOR FAILURE IN MATHEMATICS: A COMPARISON OF MACAO'S GRADE REPEATERS AND NON-REPEATERS IN PISA 2012

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According to the findings of OECD's Programme for International Student Assessment (PISA) 2012 mathematics study, Macao-China has the highest percentage (~42%) of grade repeaters in its 15-year-old student population amongst the 65 participating economies. Three mediation mechanisms, taken together explaining 0.7 grade level of progression in schooling, have been identified to explain the effects of grade repetition on student mathematics performance (Sit, et al., 2015). One mechanism identified is: *Inadequate self-regulation of students*. Grade repeaters are likely suffering from inadequate self-regulation of mental processes, and face problems with their perceived responsibility for failure in mathematics. In PISA 2012, examples of perceived responsibilities measured on a 4-point Likert response scale are: (1) *Not very good at solving mathematics problems*; (2) *Teacher does not explain the mathematics concepts well*; (3) *Make bad guesses on the quiz*. PISA 2012 developed an index (i.e. FAILMAT) and higher values (positive) of the index correspond to the higher level of external attributions of failure such as bad guesses or the teacher; whereas lower values (negative) correspond to the less external attributions to failure such as mediocre abilities or the difficult course contents. Drawing data from the PISA 2012, this study seeks to examine the similarities and differences between Macao's grade repeaters and non-repeaters concerning their locus of control for their academic failure in mathematics. The following hypothesis is postulated for statistical significance testing: *Grade repeaters have a higher level of external attributions of failure in mathematics, such as bad guesses or the teacher, whereas the non-repeaters have a higher level of the less external attributions of failure in mathematics, such as mediocre abilities or the difficult course contents*.

Statistically significant differences ($p < .01$) are established for two of the six perceived responsibility for failure in mathematics. Compared with the non-repeaters, grade repeaters are found more prone to attribute failures to "*make bad guesses*" ($p < .001$) and "*not very good at solving problems*" ($p < .01$). Implications of the study are proposed to render help to Macao's grade repeaters for the betterment of self-regulated learning in mathematics.

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ADAPTIVE SUPPORT OF COLLABORATIVE LEARNING IN SMALL GROUP USING TECHNOLOGICAL TOOLS

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The promotion of collaborative learning has become a central educational goal in the last decade. It belongs to the 21st century skills and is now incorporated in tests such as the PISA tests. However, its ethical aspect – the fact that people learn to coordinate actions and eventually to help each other, bestow to collaborative learning a genuine educational vision. Yet, the implementation of collaborative practices in schools is problematic. As noted by Webb (1991, 1995), arranging students in small groups rarely leads them to collaborate, even when students are given scripts or instructions in advance for collaborating. Even when appropriate conditions are created to trigger collaboration, students have difficulties in maintaining it for the long run.

In this research, we describe the first steps of a research design program aimed at helping teachers to orchestrate small group collaborative learning. Our research goal was to identify critical moments that might foster the learning processes of geometrical concepts. We hypothesized that identifying these critical moments would be a crucial step for helping teachers facilitating multiple group working in parallel.

The identification of critical moments is difficult for human analysers and a fortiori for helping on-line teachers, especially when multi students working in parallel in small groups. Consequently, we elaborated a system based on artificial intelligence that automatically recognizes critical moments. The purpose of the system is to detect these key interaction patterns during the group interaction and to present them to teachers in real-time, allowing them to intervene when necessary and supporting them in their goal to facilitate collaborative learning in parallel small groups.

This research is a mutual effort between mathematics educations and learning analytics researchers to overcome the challenges of orchestrating the learning processes of small group in mathematics. The learning analytics researchers used the interactions' analysis done by the mathematics education researchers, as an input data for developing the computer algorithm aims to automatically recognizing the critical moments. In this poster, we present the technological system, which may enable the orchestration of multiple small groups of students working in parallel.

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RECOGNIZING THE RELATIONSHIP BETWEEN FIELD WORK AND UNIVERSITY COURSES

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Preparing teachers to teach upper secondary school mathematics at a high mathematics level is challenging. As experienced providers of teacher education, we decided to adapt a more clinically oriented approach (CA) to teacher preparation, emphasizing the integration of theory and practice (Zeichner, 2010). The clinical approach (CA) was enacted with a sub-group (about one-third) of the prospective teachers enrolled in the university teacher preparation program. The aim of this mid-program study is to investigate if participants recognize a relationship between their university studies and field work and if so, how may this recognition be characterized.

At the university, both groups of prospective teachers participated in the same mathematics and methods courses. Both groups were required to spend 130 hours in the field supervised by a teacher mentor. However, the non-CA group (30 participants) were spread over 11 schools and were divided between 12 teacher mentors with little shared field experience. The CA group (17 participants) were divided into three groups of 5-8 participants, each group assigned to one teacher mentor, observing at least one lesson per week together as a group, and each group spending 90 minutes discussing this lesson together with the mentor. In addition, field experiences reported by the CA participants were discussed and analysed during the university courses, helping to bridge the gap between university courses and field work. In one course, focusing on examples and non-examples in mathematics lessons, participants analysed an observed geometry lesson and were then asked to reflect on the contribution of the analysis to their understanding of the use of examples. Here are some remarks written by CA participants: "I realize that I need to build a wide base of examples for each topic;" "I understand that to minimize mistakes, an example must be brought from beginning to end and only later bring additional examples;" "From the discussion (of the observed lesson) I realized that the lesson didn't include non-examples and this was missing;" "I saw that different examples allowed the students to see different possibilities." These remarks exemplify participants' integration of the field experience into their course work and the impact this integration can have on their university learning.

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TREATMENT OF FAKE MATH-DISLIKES AMONG JAPANESE JUNIOR HIGH SCHOOL STUDENTS

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PURPOSE

We defined “fake math-dislikes” as those who claimed they disliked mathematics explicitly while they accepted it implicitly. We aimed to examine the following two hypotheses. 1) There would be fake math-dislike students in Japanese junior high schools. 2) Informing them of their positive attitude would prevent them from becoming real dislikes.

METHOD

We used a two-way between-subject design (fake vs. real math-dislike \times with vs. without intervention). To detect fake math-dislikes, the questionnaire for like/dislike of math and the paper-based IAT (Mori, Uchida, and Imada, 2008) with “math” as the target were administered to 204 junior high school students. As for the preventive intervention for the fake math-dislikes, we informed them that their implicit attitude toward math was positive. We examined the effects of the intervention by assessing their math achievement scores at pre- and post-intervention with a one-year interval.

RESULTS AND DISCUSSION

We found 38 fake and 24 real math-dislike students and randomly assigned them to Experimental and Control conditions. Then, we informed only the former of their positive implicit attitude toward math. One year after the intervention, we assessed their math achievement scores and found that 15 of the 16 Experimental students improved while only eight of the 17 Control students did. As for the real math-dislike groups, six and four of the 12 students in each condition showed improvement. The frequencies of improved students in four groups were statistically different ($X_{(3)} = 12.53$, $p < .01$, $\phi_c = 0.47$). The combination of questionnaires and paper-based IAT successfully detected fake math-dislike students. As expected, informing their implicit attitude worked as a preventive intervention for the fake math-dislikes to become real math-dislikes.

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STUDENTS RESOLVING CONJECTURES ABOUT FRACTIONS

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As part of a long-term research project into students' development of mathematical ideas, fourth-grade students were introduced to the use of Cuisenaire rods for representing fractions (Yankelewitz, 2010). Students used the models to resolve conjectures they made about comparing fractions. For this problem, the context was sharing chocolate bars. Meredith's group of nine students shared one bar, so each student in her group got $\frac{1}{9}$ of a chocolate bar; another group of eight students shared two bars, so each student in that group got $\frac{1}{4}$ of a chocolate bar. All students agreed that $\frac{1}{4}$ was larger than $\frac{1}{9}$. In answering the question of how much larger, Meredith conjectured that the difference would be $\frac{1}{5}$, because "nine minus four equals five, so they got one fifth bigger." Many students agreed. By building models, some students disproved this conjecture. We discuss how one student, James, created a model for "one" that consisted of four blue Cuisenaire rods (equivalent to nine purple rods or 36 white rods). This showed that the difference between $\frac{1}{4}$ (one blue rod) and $\frac{1}{9}$ (one purple rod) is five white rods or $\frac{5}{36}$, as shown in Figure 1.



Figure 1. James' model for comparing one fourth (blue) to one ninth (purple)

This example illustrates the challenges that students face when exploring fraction ideas and the opportunities for developing fraction concepts when they use models. Students often naturally try to apply whole number operations to fractions; but when they build the model, they have visual evidence that operations on fractions require different strategies and that addition and subtraction require a common denominator. The discussions that went along with making conjectures and testing them with models helped students develop quantitative reasoning skills.

The activities described here can be viewed at videomosaic.org (the RU Analytic), where viewers can also participate in discussions about these and other videos of students learning mathematics (Agnew, Mills, & Maher, 2010).

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STOCHASTIC MUSIC AS INTERDISCIPLINARY MUSICIAN AND MATHEMATICAL SCENARIO

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In this poster we will present some interdisciplinary experiences between music and mathematics, inspired by the methods of stochastic and electroacoustic music composition. These were developed in collaboration with a musical student, who together with the author, maintain a blog of musical-mathematical creation. The aim of these explorations was to create scenarios in which an interdisciplinary collaboration between the two areas of knowledge be promoted. We'll show explorations in music composition where mathematical objects were based inspiration for the creation of sound-musical pieces, in a search that led to deepen knowledge both mathematical and musical and to seek new relationships as a challenge to our curiosity and the creation ability.

In this blog were recorded explorations in musical composition, where the use of technology was of fundamental piece, we use computer programs from both disciplines, such as Excel, Scilab, Finale and Csound. We're going to present three of these explorations: (1) use of graphical representations of random data to generate melodies, something that we can describe as a translation of random geometric shapes into sound forms; (2) generation stereo panning effects inspired by the parameterization of curves in the plane; and (3) creation of a random melody, where much of the musical decisions depended on mathematical models and notions. The procedure was inspired by the methods of Stochastic Music composition developed by Iannis Xenakis (1992) and it was brought to the field of Electroacoustic Music. To do this, probability models were defined from a set of musical options associated to four aspects of the sound: pitch, time, rhythm, and intensity.

In general, it has been an experience that has paid attention to the multimodal character of the concepts, understanding that thinking is not based solely on the language and symbols, but also in the senses (Radford & André, 2009). In addition, these experiences have incorporated the auditory sense, little explored in contexts of teaching and learning of mathematics.

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SPATIAL ABILITY EXPLAINED THE GENDER GAP IN MATHEMATICAL SELF-CONCEPT

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Previous studies have shown huge gap between male and female in mathematical self-concept. The girls tend to assess their mathematical self-concept lower than boys with similar mathematical grades and test scores (e.g., Skaalvik & Skaalvik, 2004). Gender differences in mathematical self-concept are most frequently explained in terms of gender stereotypes and differential gender role socialization patterns (Cvencek, Meltzoff, & Greenwald, 2011). The current study tried to investigate the cognitive, social and emotional sources for the gender gap.

The participants were 307 pupils aged 9-10 years old (166 boys, 141 girls) from 11 primary schools in the greater Beijing area. Approximately 30 children were randomly selected from each school. Tests were administered to the students in a classroom collectively, which consist of five cognitive tasks (simple subtraction, complex subtraction, mental rotation, number series completion and word sentence completion) and three emotion or attitude tasks (mathematical anxiety, mathematical gender stereotype and mathematical self-concept). A series of hierarchical regression analyses were conducted to examine the independent contribution of gender to mathematical self-concept, before and after controlling for performance on each other 7 tasks respectively.

Results showed that boys outperformed girls in mathematical self-concept as well as in mental rotation and number series completion while girls outperformed boys in word sentence completion. There were no significant gender differences in arithmetic (i.e., simple subtraction, complex subtraction), mathematical anxiety and mathematical gender stereotype. The amount of variance accounted for by gender in explaining the mathematical self-concept can be partially explained by mental rotation. Our research provides support for the idea that general spatial ability could greatly contribute to the gender differences in mathematical self-concepts. Further results will be discussed in the presentation.

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VERBALIZED MATH IS FUNDAMENTAL TO MATH ACHIEVEMENTS

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Studies have shown the close relation of reading ability and math performance. (e.g., Siegler & Booth, 2004; Laski & Yu, 2014). Math knowledge in symbolic math is typically expressed with math symbols, numbers and letters in textbooks and academic literatures, but it can also be represented by language (e.g., “Exchanging the positions of addends in an addition expression does not change the result”). Little research has been conducted to evaluate the role of the verbalized math on math performance. The verbalized math is assumed to have close relation with the math performance in the current investigation.

Participants for this study were totally 132 undergraduates (64 males, 68 females, mean age = 21.3, SD = 2.6) recruited from colleges and universities in Beijing of China. They were asked to perform tests on processing speed, visuospatial processing, reasoning, basic numerical processing, computation, verbalized math term, verbalized math rule and math achievements (self-adapted math achievement and math problem solving). These tests were computerized with the “Online Psychological Experiment System (OPES)” (www.dweipsy.com/lattice, Wei et al., 2012; Zhou et al., 2015).

Results showed that verbalized math term or verbalized math rule could independently contribute to math achievements after all the cognitive processing and symbolic math were controlled for. The result suggests that the mathematical thinking could be supported by the language that is highly embedded in math, and math education might benefit from the extensive practice of verbalized math.

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A PROPOSED MODEL FOR DEVELOPING TEACHERS' KNOWLEDGE OF ANALYZING STUDENTS' MATHEMATICAL ERRORS

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Students' mathematical errors are a world-wide phenomenon during their mathematics learning. Students of any age, any country, any era, irrespective of their performance in mathematics, have experienced getting mathematics wrong. Due to the complexity and variety of students' mathematical errors, both teaching practices and research results show that teachers' analysis of students' mathematical errors is a critical factor influencing the effectiveness of mathematics teaching.

Given the importance of analyzing students' mathematical errors, teacher knowledge of analyzing students' mathematical errors has been attracted increasing attentions in recent years. Researches focus on what knowledge mathematics teachers had or need to have for identifying, interpreting and remediating students' mathematical errors in different learning topics (Peng, & Luo, 2009). Among those researches, some researches show that mathematics teachers lack of enough knowledge to analyze students' mathematical error appropriately, the results of which address an important issue that more professional intervention are needed in order to prepare mathematics teachers to analyze students' mathematical errors better. In this presentation, we present a model for developing teacher knowledge of analyzing students' mathematical errors, which is mainly developed from two lines of researches, professional development models and mathematics teacher knowledge.

The proposed model includes four elements which is centered by students' mathematical error, namely, selection and presentation of students' mathematical error, individual reflection and sharing of students' mathematical error, collaborative analysis and discussion of students' mathematical error, collaborative teaching design centered by students' mathematical error.

The proposed model has been used in several situations for in-service mathematics teachers training, and the feedbacks from the trainees show that the model enables mathematics teachers to develop their knowledge of analyzing students' mathematical errors. In the conference, we will present one of the professional development situations where the model was applied.

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AN INTERNATIONAL COMPARISON OF THE DEGREE OF LINKING BETWEEN MATHEMATICS AND SCIENCE KNOWLEDGE: FOCUSING ON PISA2003, 2006 AND 2012

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Separating mathematics and science knowledge, it is difficult for students to solve the environment and social issues through the use of mathematics and science (e.g., Frykholm & Glasson, 2005). Considering this notion, the viewpoint of linking between mathematics and science knowledge deserve to be focused on. On the other hand, it is generally said that developing countries have lower mathematics and science achievement according to the international educational surveys such as PISA and TIMSS. We question that the degree of linking between mathematics and science knowledge in the lower achievement countries, namely developing countries, may be weaker than high achievement countries.

The aim of this study is to examine the question mentioned above by making cross-national comparison thorough analysing PISA2003, 2006 and 2012 dataset. In the PISA survey, the mathematical and science literacy are measured every three years from the year 2000. It can provide the rationale that the degree of linking between mathematics and science knowledge can be analysed including its secular changes. In this study, in order to clarify the degree of linking mathematics and science knowledge within participating countries of PISA survey took place, hierarchical linear model (HLM) is used especially to focus on random effects to reflect the domestic characteristics of each country.

Analysis reveals that the higher the countries perform mathematics and science achievement based on the average score, the stronger the countries have the degree of linking between mathematics and science knowledge. Concretely speaking, the correlation coefficient between the average score of mathematical literacy and the degree of linking is 0.761, 0.800 and 0.814 in respective PISA2003, 2006 and 2012. The results show robustly that the low achievement countries, more specifically developing countries, have weaker degree of linking between mathematics and science knowledge than high achievement countries. This study gives affirmative answer to the question mentioned above.

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IMPROVING STUDENT ANALYTICAL JUSTIFICATION THROUGH PD

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Multiple policy documents have stressed the importance of increasing student sense making and reasoning (e.g., NCTM, 2014; NRC, 2001). One way of engaging students in expressing their mathematical reasoning is to encourage them to justify their mathematical work. We adopt Staples, Bartlo, and Thanheiser's (2012) definition of justifying as making "an argument that demonstrates (or refutes) the truth of a claim that uses accepted statements and mathematical forms of reasoning" (p. 448). This study explores changes in the number of student justifications produced during mathematics lessons before, during, and after the teachers are engaged in professional development (PD) that focused on increasing student engagement in justifying their mathematical thinking. Our project engaged 74 teachers working in grades 4-12 in PD that focused on students' use of high-level analytical justifications to solve mathematics problems. Throughout the four years of PD, classroom observations captured five levels of justification: 0) No justification; 1) Justification by showing work; 2) Empirical justification; 3) Justification based on admissible mathematical actions; and 4) Analytical justification. Over the course of four years, we collected 572 observations of mathematics lessons, during which we recorded the highest level of justification made by a student. Lessons were categorized as containing low-level justification (categories 0-2) or high-level justification (categories 3-4). As seen in Figure 1, the percentage of classroom observations where students were seen making high level justifications increased over the four years of the PD project. We suspect this is due to the intense PD focused on building teachers' mathematical content knowledge and use of justification in their mathematics lessons. This poster presents details regarding the PD in which teachers were involved that resulted in an increase in high-level justifications observed in math lessons.

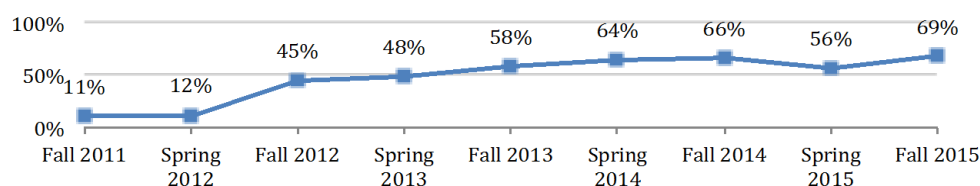


Figure 1: Instances of high-level justification during classroom observations

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INQUIRIES IN PROBLEM SOLVING WITH CONTRIBUTIONS FROM LESSON STUDY

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Polya (1981) claims that Mathematics classes based on problem solving should be driven by inquiries that stimulate and direct students' thinking. On the other hand, the Lesson Study (LS) method (Fernandez & Yoshida, 2004) seeks to improve the teacher's practice by a collaborative and reflective way. Thus, we were interested in investigating which inquiries would generate eases/difficulties to stimulate the deductive reasoning of students, according to the premises of Polya and in an environment where we use aspects of the LS method.

Four Mathematics Teachers had participated in the investigation, one of them proposing an unknown ill-structured problem (no previous algorithms for solution) to the other three, who sought to solve it based on inquiries formulated by the first one. All teachers are knowledgeable of the LS and were committed to identifying and providing feedback on the favor or not of the inquiries for the reasoning development after the end of the class.

Problem: in a soccer championship in which each team plays the same amount of games, every winning is worth three points, a tie just one point, and defeats are worth nothing. In case of a tie between teams, the organizers would consider winning to those that had more defeats instead of the old standards of more wins. Are these criteria equivalent?

The analyzes revealed that general inquiries as "What does the tie tell us about the teams?"; "Can we use this data to infer something about the defeats?" And "How could you test the new criteria?" Did it generate blockages in the reasoning of the teachers and were they, later, replaced by others that facilitated the reasoning flow: "How to write with these variables that the teams tied in the league?"; "How could you assume in the equation the amount of wins of a team is greater than the other?"; "Have we used the fact that the teams played the same amount of games?".

Wide inquiries like the ones that we presented before, which are commonly used in the classroom by Mathematics teachers, did not fertilize the creation of efficient strategies to solve the problem by them. Replacements showed to be effective in reapplications of the problem, highlighting the methodological potential of LS.

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THE COMMUNICABILITY OF FIGURE SIGNALING FOR GEOMETRIC PROBLEM SOLVING

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The figure accompanying a geometric problem displays considerable information. Which signals communicated via labels on the figure accompanying such a problem provide insight into the solution? This study investigates the signals mathematics teachers try to convey when labeling figures, and whether readers find those signals communicable.

The participants in our study were 54 mathematics teachers and 18 college students in Taiwan. The mathematics teachers had an average of 14.5 teaching years (ranging from 2 to 34 years) in junior high school. The college students were not majors in mathematics but were participating in an elementary educational program. There were 12 geometric problems (e.g., Fig. 1), which used to be teachers' materials, and which were separated into two versions to reduce participants' load. The teachers were asked to label the geometric figures using signals to facilitate students' problem solving. Each signaled figure (e.g., Fig. 2) was rated by six students as portraying *high communicability*, *low communicability*, or *undecided*. If a signaled figure was awarded a high rating by more than five students, it was categorized as *well signaled*.

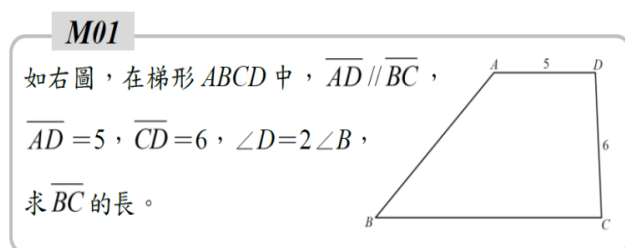


Fig. 1 One of the geometric problems

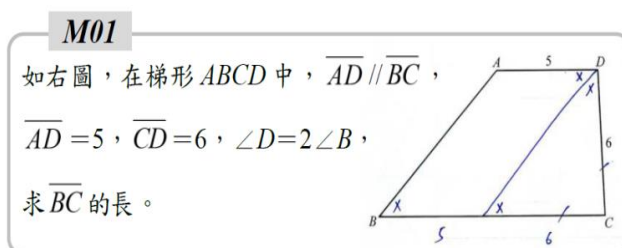


Fig. 2 Figure with labeled signals

The results showed that 58% (186/324) of labeled figures were well signaled. Those figures labeled by the most signals illustrated the corresponding solution steps. For example, the auxiliary line created a parallelogram, shown in Fig. 2. The “x” signals indicate that the opposite angles and the corresponding angles in the parallelogram are equal, and that the base angles of the isosceles triangle are congruent. In addition, the “/” and “6” symbols show that the legs of the isosceles triangle are equal, and the value “5” displayed on the opposite sides of the parallelogram indicate that they are congruent. Most teachers could label communicable signals on the figure of most problems except for three problems. For instance, a number of teachers neither illustrated the given conditions about “ $DE : EC = 1 : 4$ ” nor did they provide that “the angle exterior to the vertex angle of an isosceles triangle is twice as large as the base angle of this isosceles triangle”. One explanation for this is that there were not common signals for proportions. Thus, communication in the mathematics classroom requires teachers and their students to agree on singles for geometric figures.

ONLINE INDIVIDUALIZED FEEDBACK FOR IMPROVING MATHEMATICS LEARNING: A COGNITIVE DIAGNOSTIC MODEL APPROACH

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Meaningful feedbacks that can facilitate learning effectiveness are timely feedback and instructional feedback. The challenge for teachers on providing feedbacks is how to efficiently define the weakness of students during the instruction process. Cognitive diagnostic model aims to provide examinees with information concerning whether or not they have mastered each of a group of specific, discretely defined attributes. The attribute is the categorical latent traits that tests are developed to measure. Hence, cognitive diagnostic model is an efficient approach which teachers can quickly access detailed information of students' learning strengths and weaknesses so as to improve students learning.

Taking fraction as an example, based on the cognition diagnostic model, DINA model (deterministic input, noisy "and" gate model) (de la Torre, 2009), an online system with immediately feedback and personalized intervention program is developed. A pre-test/post-test non-equivalent group design is conducted to explore the effectiveness of the individualized intervention program based on the feedbacks of cognitive diagnostic model. The eight attributes such as f1: identify proper, improper, and mixed fractions; f2: conversion between whole numbers, mixed fractions and improper fractions, and 30 items, which are the foundation of online cognitive diagnostic assessment of fraction is developed by experts. A sample of 593 4th grade students is collected to evaluate the quality of fraction test. Eighty three 4th grade students from four classes participated in the field experiment (data is under collection). In the Experimental group ($n=49$), based on the personal diagnostic report given by the online system of the pre-test (fraction test), students take on individualized digital remedial courses on the attributes they did not master. For example, (1,1,1,0,0,1,0,0) represent the student have mastered f1,f2,f3 ,and f6, and need to learn f4,f5, f7, and f8 .In the control group($n=34$), the group remedial instruction based on the classroom report of the pre-test is performed by the teachers.

The reliability of fraction test is 0.88, indicating a significant internal consistency. The content validity for the test was reviewed by seven domain experts. The average guessing index and the average slipping index of the items estimated by DINA model are 0.14 and 0.22, respectively, showing high quality of fraction test. The result of ANCOVA reveals significant differences in post-test scores between the experiment (mean: 84) and control groups (mean: 80) ($F=5.05$, $P<.05$). The online system with immediately feedback and personalized intervention program can help students learning mathematics effectively.

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A QUANTITATIVE FRAMEWORK FOR THE COMPARATIVE STUDY: EXERCISES OF MATHEMATICAL TEXTBOOKS OF SENIOR SECONDARY SCHOOL FROM TEN COUNTRIES

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Textbook as the main text resource for instructional design and guiding the teachers' classroom practice (Beaton et al., 1996). Although the structure of mathematics textbooks varies in different countries, exercises are still as one of the main components in textbooks. Hence, the focus of this study was to develop a framework to compare mathematical textbooks of different social and cultural backgrounds from the perspective of exercises. Textbooks of grades 7 to 9 were selected from the following ten countries: China, Australia, the United States, Britain, France, Germany, South Korea, Singapore, Japan and Russia. The quantity, type, openness and difficult levels were considered in comparing these exercises which were classified into Number and Operations, Equations, Triangles, Special Shapes and Statistics.

The quantity and types comparison were shown in the following figures (Figure 1& Figure 2) which presented the total number of exercises and the distribution of the types of exercises.

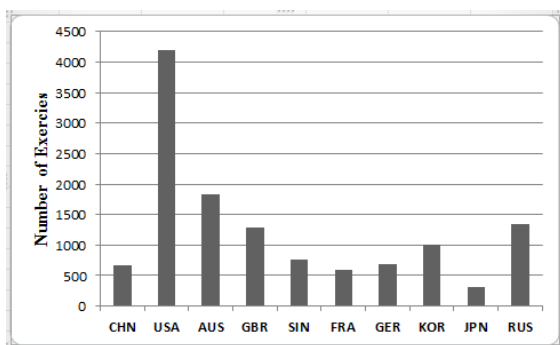


Figure 1 The number of exercises (Total)

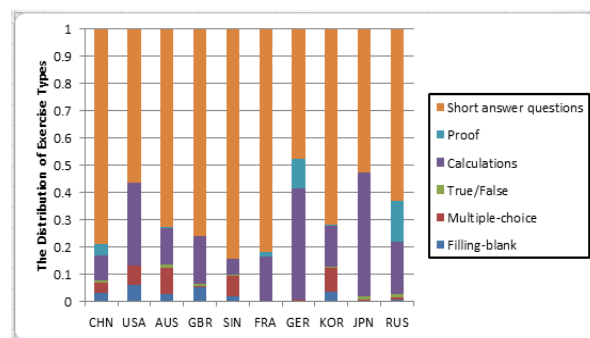


Figure 2 The distribution of the exercise types

This quantitative framework has been developed and used in this study. By comparison, we know more clearly about the features of our own textbooks. Moreover, the foreign counterparts broaden our vision and will serve as good reference for our textbook designing and mathematical class teaching in junior secondary school.

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NUMBER SENSE METHODS USED BY ELEMENTARY SCHOOL TEACHERS IN TAIWAN

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The major purpose of the study was to examine the number sense methods used by elementary school teachers. Number sense has been viewed as a major topic in primary mathematics education internationally (Anghileri, 2006; Berch, 2005; Dunphy, 2007; Verschaffel, Greer, & De Corte, 2007; Authors, 2013). Teachers' mathematical knowledge influences their teaching and their students' learning (Ball, Hill, & Bass, 2005; Ng, 2011). If we intend to increase children's number sense, we must first investigate their teachers' number sense. Six primary school teachers were interviewed by semi-structured interview method. A total of 15 interview questions were constructed, with each of the five number sense components addressed by three questions. Results showed that two elementary teachers majored in math education responded questions by number sense-based method over 70%. However, the range of using number sense-based method to respond questions by the other four elementary teachers with background in Education, Chinese Education, and Sociology were from 27% to 47%. In addition, about 35% of the responses used rule-based methods and about 10% of responses were deemed incorrect. The primary teachers with background in math education outperformed the teachers with background in Education, Chinese Education, and Sociology. Therefore, a question of utmost importance for future teacher professional development is how to design different courses to help elementary school teachers with different background to develop a profound understanding of number sense.

Keywords: Elementary school teachers, Number sense, Rule-based method

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A STUDY TO MATHEMATICS TEACHERS' ABILITY OF ANALYSIS OF STUDENT MISTAKES

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RESAEARCH GOALS

Teachers' ability of analysis of student mistakes is an indispensable and fundamental teaching activities which can distinguish experienced teacher from novice. Previous studies concentrate on finding the reasons of student mistakes, few attempts to evaluate how teachers influence student achievement that based on the analysis of the errors.

METHOD

In this study, we examine the relationship between mathematics teachers' these competences and student achievement at grades 4 and 8. The population is a representative sample of 69916 students and 8006 mathematics teachers. Teachers measure by questionnaire concern with rectangular perimeter at grade 4 designed to four options, eighth grade involve in function. Mathematics results for grades 4 and 8 apply the Item Response Theory ($M \pm SD = 500 \pm 100$). The range of the ability of analysis of student mistakes is represented by three levels: top competence, moderate competences and low competence.

RESULT

At grade 4, 41.6 percent of the teachers are top competences correspond to student average performance (494 points) while 6.2 percent are low competences correspond to student average achievement (472 points). Subsequently, at grade 8, about one in five teachers (19.2%) reach level 4 or 5, teachers at level 1 with 12.5 percent, correspond to the difference of 21 points of student results. There was a disparity between rural and urban areas, at grade 4, top competences of urban areas in mistake-analysis reach 46.9%, while 38.6 percent in rural areas. The average scores in urban and rural areas have large difference (75 points). Regionally, teachers in eastern area show the greatest difference compared with eastern area.

DISCUSSION

Mathematics teachers' ability of analysis of student errors restricts the effect of teaching activities which reflects form the achievement. However, what influences teachers' mistake-analysis, then how to promote immediacy through training, especially in rural or western area.

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LANGUAGE MEDIATING LEARNING: THE FUNCTION OF LANGUAGE IN MEDIATING AND SHAPING THE CLASSROOM EXPERIENCES OF STUDENTS, TEACHERS AND RESEARCHERS

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This Research Forum addresses theoretical, methodological and practical issues associated with language use and the mathematics classroom. All forum components have the function of language as their central and cohering theme, but address different forms of language use within that overarching theme. Language as communicative exchange provides the vehicle for the social construction of knowledge in mathematics classrooms. Language as discourse prescribes the limits of acceptable speech (Butler, 1997), both within the mathematics classroom and among the community of mathematics teachers when discussing the mathematics classroom. Within classroom discourse, different types of talk can be identified, characterised and their function in the learning process investigated. Language also functions as the medium by which the academic community analyse and theorise the phenomena for which the mathematics classroom is the setting. While these functions of language may appear quite distinct, it is our suggestion that they are profoundly intertwined: the language by which teachers shape the practices they orchestrate reflects cultural-historical origins that also sets bounds on researchers' capacity to articulate theory concerning those practices. The Research Forum purposefully combines a variety of cultural and theoretical perspectives in order to interrogate the role of language.

INTRODUCTION

The aim of the Research Forum is to use the findings from recent research to explore issues associated with the function of language in mediating the ways in which three groups interface with the mathematics classroom: students, teachers and researchers. In particular, we examine the role of language in shaping and determining the classroom-based learning of each group. The entry point for discussion is the proposition that the capacity to learn is mediated by the language through which one participates in and reflects upon lived experience. It is our assertion that employing this as an analytical perspective can provide significant insight into the learning of students,

teachers and researchers. Our initial interest is lexical and concerns the actual terms by which students, teachers and researchers name the objects in their respective worlds (Sapir, 1949). We also want to focus on the analysis of speech acts (Searle, 1969) as the minimum unit of analysis to understand dialogue and communication. Austin's (1962) categorisation of locutionary, illocutionary and perlocutionary speech acts makes the useful distinction between what was said, what was meant and the consequences of the speech act. However, mathematics is multimodal by its nature and some of the analyses reported in this forum attend to representational and communicative forms other than speech, including the use of gesture. We acknowledge the important entwining of different modalities in the mathematics classroom, but have chosen to focus primarily on linguistic aspects for this Research Forum due to the specific issues related to verbal language in cross-cultural research. The Research Forum employs both cross-cultural and in-depth research examples to raise questions about the function of language in determining learning for different communities in different cultures.

Learning can be conceptualised in terms of progressively enhanced participation in forms of institutionalised social practice, where discourses form key components of that practice. Students are initiated into the discourse of the mathematics classroom: a discourse with its own technical vocabulary and discursive and social conventions. Mathematics teachers similarly participate in a discourse community in which the mathematics classroom and its objects, agents and events provide the subjects of professional discourse and for which language mediates the experience of the classroom and the professional learning that experience engenders. Classroom researchers' experience of the classroom is similarly mediated by the language available to describe those objects and events occurring in classroom settings. Thinking of the research process as analogous to learning, in that the goal is the construction of new knowledge, we find in the mathematics classroom the nexus of three learning communities: students, teachers and researchers. The learning opportunities available to each group are afforded (and constrained) to a significant extent by the language employed to participate in and reflect on the mathematics classroom.

Goals framing the Research Forum:

- (i) To identify various aspects of the function of language in relation to the mathematics classroom and illustrate with examples drawn from several contemporary research studies;
- (ii) To identify issues associated with the different forms of language use and the contemporary theories relevant to their consideration;
- (iii) To identify ways in which considerations of language in one context (for example, classroom discourse) might reflect or shape other uses of language (for example, theorising about classroom learning);
- (iv) To bring together researchers from a variety of countries, who share an interest in the function of language in all aspects of our research into mathematics classrooms;

- (v) To draw to the attention of PME members some of the issues associated with the various forms of language use with which we as a research community are obliged to interact and the consequences for our research practice and theorising.

The Research Forum addresses the following two issues:

- (i) The nature and function of discourse in mathematics classrooms and among those responsible for what happens in those classrooms;
- (ii) The role of language in shaping teachers' and researchers' capacity to observe, reflect upon and theorise about the mathematics classroom.

The Research Forum is structured so that one session is devoted to each issue. Each issue, in turn, is addressed from two perspectives and this provides the structure for the Research Forum, as follows:

Session A. Researching the discourse of students and teachers: Speaking in and about the mathematics classroom.

Focus A1. Discourse in the mathematics classroom

Focus A2. Discourse about the mathematics classroom

Session B. Language mediating the learning of mathematics teachers and researchers: Naming, Noticing and Selective Attention

Focus B1. Language determining teacher selective attention and learning

Focus B2. Language determining researcher attention and theory construction

Four focus questions are intended to catalyse discussion:

Focus Question 1. *What are the characteristics of classroom discourse in mathematics classrooms (in the same or different cultural settings) and what distinctions within student classroom talk offer insight into the nature of student learning?*

Focus Question 2. *What are the implications of culturally-based differences in classroom-related language use (teacher professional discourse) for our advocacy of instructional practice and our construction of related theory?*

Focus Question 3. *In what ways does language constrain our capacity to recognise, describe, analyse, theorise, optimise and share the practices of the mathematics classroom?*

Focus Question 4. *What is the role of language in shaping teacher and researcher selective attention and consequent learning in and about mathematics classrooms?*

Structure of the Forum

The Forum is structured around three research projects that provide the major sources of illustrative data and findings: (i) The Social Unit of Learning Project (involving Australia, Finland and Spain); (ii) The Lexicon Project (involving Australia, Chile, China, Czech Republic, Finland, France, Germany, Japan and the USA); and (iii) The Learning from Lessons Project (involving Australia, Portugal and the USA). These three projects provide many of the examples cited in the discussion below. Other research is also cited where appropriate.

In each research project, the nature of the data collected and the form of analysis is significantly shaped by the mediating function of language. Whether the focus is on the student or the teacher (or, for the purposes of this forum, the researcher), language can be seen to mediate, shape and even constitute the classroom-based learning of each group.

The intention in this Research Forum is not to report the findings of the cited projects in conventional fashion, but to draw upon the findings and methods of all three projects to demonstrate the need for certain questions to be addressed by the mathematics education community.

This Research Forum aims to implement inclusivity as a general principle with respect to methodology, theory and culture. It is an essential premise of this Forum that international comparative research offers unique opportunities to interrogate established practice, existing theories and entrenched assumptions. We argue that such interrogation is difficult to undertake adequately from within a single methodology, theory or cultural setting.

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DISCOURSE IN THE MATHEMATICS CLASSROOM

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Current research in education suggests that discourse and certain social interactions have a major role in shaping students' mathematics learning, as well as their motivation and engagement. According to Vygotsky (1978), there is a clear connection between speech acts and cognition. Subsequent research has supported his proposition (Díez-Palomar & Cabré, 2015; Mercer, 2010; Webb et al., 2014). After Vygotsky, Bruner and others (Wood, Bruner, & Ross, 1976) developed the idea of scaffolding to explain how teachers (or more capable peers) support students in their learning. Other scholars replicated their findings (Nathan & Knuth, 2003; Rogoff, Turkkanis, & Brackett, 2002; Williams & Baxter, 1996). One perspective on scaffolding is that in the hands of a skilful teacher it represents the nurturing and management of discourse. In this session we draw on The Social Unit of Learning Project to provide three different approaches to studying discourse and learning in the context of collaborative problem solving by middle school mathematics students.

The Social Unit of Learning Project

Previous research has highlighted the need for more sophisticated consideration of the social interactions by which teacher and student realise the instructional and learning affordances of a given task. This project is an experimental study of student social interaction during the completion of open-ended tasks undertaken in individual, pair, small group and whole class social units. The project is conducted in a laboratory classroom at the University of Melbourne, where the social interactions of intact classes of students and their teacher are recorded on 10 high definition video cameras and up to 15 high quality audio inputs. The resultant data are extensive and detailed. Data analysis draws on the expertise of an international team with experience in the analysis of student learning products, classroom discourse, particularly student discourse while working in collaborative groups, and the affective (motivational and emotive) response of students working in the various social configurations orchestrated for the research.

Learning in social settings involves complex processes that require research designs and analytical techniques that are sensitive to its multifaceted nature and open to multiple construal (Clarke, 2001). According to previous research, classroom discourse and forms of students learning are closely interwoven (Clarke 2013; Mercer & Howe, 2012). Yackel and Cobb (1996) interpreted argumentation and autonomy within the mathematics classroom using the idea of sociomathematical norms, which refers to

“normative aspects of mathematical discussions that are specific to students’ mathematical activity.” (Yackel & Cobb, 1996, p. 458)

In this view, sociomathematical norms are interactively constituted, and students and teachers use them to regulate mathematical argumentation. Student and teacher-negotiated common understandings of these socio-mathematical norms exist as the implicit terms of the didactical contract (*le contrat didactique*), conceptualised by Brousseau (1986) as embodying precisely these negotiated socio-mathematical norms in which teacher and students are complicit.

The first analytical approach in the Social Unit of Learning Project investigated the foci of student negotiative events during collaborative problem solving. Chan and Clarke (2016) suggest that meaning negotiation in mathematics classrooms can be usefully distinguished as social, socio-mathematical, and mathematical. Negotiation with respect to each of these employs its own lexicon and can be considered as a distinct mode of interaction. However, to interpret the three modes as a stratification of social process suggests a separation and a hierarchy that was not evident in practice. All three modes co-exist in an entangled form in the negotiative interactions documented in the mathematics classroom. All three are proposed as both constitutive of learning and as providing distinct entry points for teacher instructional intervention (or scaffolding). Each must be accommodated in a social theory of learning and each represents one avenue to improved learning outcomes in our mathematics classrooms. All three must be studied in situ and in relation to each other as they occur in authentic classroom activity. Such postulated interconnectedness poses challenges for theory and for research.

The second analytical approach used the notion of dialogic and non-dialogic talk as a methodological instrument to clarify how interaction works when students solve mathematical tasks. According to Flecha (2000), learning arises from a dialogic process in which “dialogic” means everybody has the same opportunity to participate in a dialogue, presenting and sharing arguments to justify their statements. Flecha (2000) explores the boundaries of human action and defines those “communicative acts” that constitute one of the roots of the egalitarian dialogue established between people involved in a process of dialogic learning.

Discourse analysis suggests that there is a variety of talk that students may use within their interactions. For example, participants within the classroom discourse might use either “validity” or “power” claims to justify their statements, where the use of one or the other type of claim depends on the intentionality of the speaker. In this discussion, we focus on the effects of these different kinds of talk in terms of effective mathematical learning.

Dialogic talk, as employed in this analysis, only occurs when people use arguments based on validity claims, not on the position of power of the person who is talking. Díez-Palomar (2016) employed the categories of dialogic and non-dialogic talk within

the Social Unit of Learning Project to analyse social interaction among students and their teacher within the mathematical classroom.

In one episode analysed by Díez-Palomar, four students engage in both egalitarian and non-egalitarian interaction, employing both validity claims and power claims. In this analysis, the social nature of the learning process is very visible. The distinction is made between the negotiative consequences of validity claims, which consistently advance the group's mathematical understanding, and power claims, which do not. In research currently underway, the teacher's role is to limit the occurrence of student disputational talk based on power claims. Previous research suggests that this should be conducive to more effective learning.

The third analytical approach addresses the affective dimension (Hannula, 2012). Affective aspects may constrain students' efforts to learn and understand mathematics. In the particular analyses reported in this forum, interactions are analysed and compared between a pair of girls, a pair of boys, and then their subsequent interactions when working as a group of four. The context for each set of interactions is collaborative problem solving in 7th grade mathematics. The negotiative exchange related to affect has a different character and calls upon different communicative tools than does more content-oriented negotiation. Voice intonation and gesture, in particular, are important in the interpretation of student affect. The inferences made about student affect in this study depend significantly on the video record. Addressing affect requires attention to more than just speech and suggests the need for a more inclusive interpretation of what constitutes language in classroom settings.

In the episodes analysed, the girls pair established a functional affective micro-culture with each other, while this did not happen as effectively for the boys' pair. The analysis suggests that a functional affective micro-culture is prerequisite to an effective collaboration. On the basis of this analysis, the construct "adaptive affect" can be introduced. The results of the analysis indicate that to make collaboration effective, the participants need to be on the same affective ground: their inter-individual affective levels (Tuohilampi, 2016) should become constructive and adaptive to each other. Being on the same academic level might not help if the students do not feel their group members to be affectively adaptive. However, if the students are affectively adaptive, they could establish a shared solidarity they feel they can rely on, in which case the differences in academic level might become a potent source for the development of a variety of thinking skills.

Our goal in this session is to examine the potential for connection between the three analyses of student interaction. Multi-theoretic research designs have been variously discussed (Bikner-Ahsbahr & Prediger, 2006; Cobb, 2007) and implemented (Clarke, et al, 2012; Tsapralis, 2001). In this particular instance, the Research Forum provides an opportunity to examine how the complementary analyses of the same collaborative problem solving activity might be mutually informing and, in their combination, provide greater understanding and explanatory power regarding the function of

collaborative problem solving and learning in mathematics and how this might be promoted in school classrooms. Our specific focus is the critical role of language in each of the analytical approaches pursued.

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DISCOURSE ABOUT THE MATHEMATICS CLASSROOM

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It has been observed elsewhere (Clarke et al., 2012) that our research interactions with classroom settings are significantly mediated by our choice of theory. But theories are articulated through language and the meanings invoked by the use of one language may go unrecognised in another. In particular, the expectation that publication for an international audience must be in English places a serious impediment between any non-English speaking theorist and the international community of researchers. Different communities, speaking different languages, employ different naming systems to describe the events, actions and interactions of the mathematics classroom.

Naming Our World

The Sapir-Whorf hypothesis suggests that our lived experience is mediated significantly by our capacity to name and categorise our world. “We see and hear ... very largely as we do because the language habits of our community predispose certain choices of interpretation” (Sapir, 1949, p. 162). Marton and Tsui (2004) suggest that categories “not only express the social structure but also create the need for people to conform to the behaviour associated with these categories” (p. 28). Our interactions with classroom settings, whether as learners, teachers, or researchers, are mediated by our capacity to name what we see and experience. Speakers of one language have access to terms, and therefore to perceptive possibilities, that may not be available to speakers of another language. This has implications for international comparative research (Clarke, 2013).

No comparison is legitimate if the parties compared cannot each represent his own version of what the comparison is about; and each must be able to resist the imposition of irrelevant criteria. In other words, comparison must not be unilateral and, especially, must not be conducted in the language of just one of the parties. (Stengers, 2011, p. 56)

Despite this admonition from Stengers, we find ourselves members of an international mathematics education community for which English is the lingua franca and this places significant limitations on what can be said. A pilot project in 2009 (Clarke & Mesiti, 2010) demonstrated that these limitations extend beyond the imprecisions inevitable with translation. In the pilot project and in the larger study currently underway, many instances have been identified of terms existing in one language for which another language has no meaningful equivalent term. This has significant implications for our instructional advocacy, international curricular reform, and for our capacity to theorise about the practices of the mathematics classroom and their connection with student learning.

The Lexicon Project

The Lexicon Project, undertaken in Australia, Chile, China, the Czech Republic, Finland, France, Germany, Japan and the USA, seeks to document and compare these naming systems. The principle focus of the Lexicon Project is the identification of the lexicon employed by middle school mathematics teachers in each country to describe the objects and events in their classrooms. It was hypothesised that comparison of one lexicon with another would provide insight into the pedagogical traditions of each country, the prioritisation in each language of particular objects or events, and the relative absence or invisibility in one community of classroom activities prominent in another.

The composition of the research teams in each country prioritised the voice and perspective of practicing teachers. A team for one country was required to consist of a senior researcher, a postdoctoral or doctoral researcher and, at least, two experienced teachers. In practice, teams were larger than this, with the underlying principle that prominent representation be given to experienced teachers.

It is important to note that in the Lexicon Project the video material did not constitute data in the conventional sense. The intended role of the video material was catalytic and, subsequently, illustrative. The data in this project consisted of the identified terms and the subsequent definition generated for each term.

Following identification of a local lexicon by each research team, steps were taken to obtain national validation of the identified lexicon. The procedure for national validation varied from country to country. Strategies included the use of professional associations and existing networks of mathematics educators. In the discussion below, examples are employed from the lexicon of each of four countries to illustrate findings and issues related to language use by mathematics educators, particularly middle school mathematics teachers, in each country.

Country Lexicons: Australia, China, the USA and the Czech Republic

Australia

A local team of seven Australian researchers and experienced teachers classified video records of mathematics classrooms in order to identify those terms that in combination constitute the national pedagogical lexicon. The essential point was to record single words or short phrases that are consistently and widely used within Australia by teachers with a consistent and agreed meaning.

There are 69 terms in the Australian National Lexicon, these terms are considered familiar and in widespread use (e.g. *Assigning Homework*, *Rephrasing*, *Worked Example*). However, the Australian team found it useful to identify two additional categories:

Phrases (17) that are recognisable and readily understood, describing familiar classroom phenomena for which there was not an institutionalised name (e.g. *Setting a Time Limit*); and

Familiar Activities (25) which are seldom described or referred to, but are sufficiently frequent in occurrence as to possibly be named by other communities (e.g. *Arranging the Seating*).

An interesting feature of the Australian National Lexicon is that not one of the 69 terms identifies a practice that can only be found in the mathematics classroom. The terms all refer to general pedagogical practices. Worthy of note is the prevalence of ‘gerunds’ (noun/verbs) in the Australian National Lexicon. Such terms include: *Rephrasing*, *Reasoning*, *Justifying*, *Scaffolding*, *Questioning*, and *Discipline*. This duality can be seen as usefully inclusive, invoking both process and product, but it can also be used to obscure or simply avoid the necessity of deciding between a noun or verb version of the term (simultaneously invoking both object and activity). This linguistic option is not available in many other languages and highlights both the affordances of particular languages and the difficulties of translation.

China

The research team in China combined experience with expertise and involved teachers from universities, middle schools and a coach/mentor/advisor from the department of education. There are 125 terms in the Chinese lexicon. Some terms can be approximated in English (e.g. Teacher's Feedback) and there are those that have no simple equivalent English term or phrase but can only be represented in pinyin (see Table 1 below) with a lengthy description in English of the named object or activity.

Lexical term	Description
'Ceremony of class beginning' 上课仪式 SHANG KE YI SHI (shàng kè – class beginning; yí shì – ceremony)	This is the beginning of the class and is indicated by a regular pattern of behaviour. The teacher says, "The class begins." the student monitor says, "Stand up." then the teacher states, "Good morning everyone." the students respond, "Good morning teacher," in unison. The act of teachers and students bowing to each other accompany these interactions, and it signals the formal beginning of class.
'Stating teaching objectives' 提出教学目标 TI CHU JIAO XUE MU BIAO (tí chū – to state; jiào xué – pedagogical; mù biāo – objectives)	The teacher outlines the learning content of the lesson and sets detailed objectives for the class, which includes detailed information about the following three dimensions: knowledge and content; process and methods; and emotional attitudes and values. These dimensions are identified in the syllabus.
'Teaching with variation' 变式教学 BIAN SHI JIAO XUE (biàn shì – variation; jiào xué – teaching)	A teaching method whereby the teacher transforms the conditions of the original example or varies the questions asked of the students. This is undertaken with the purpose of strengthening and deepening students' understanding of theory and the execution of examples.

Table 1: Three examples of lexical items from the Chinese lexicon

The Chinese process for constructing a structured lexicon was distinctive: the first stage involved grouping of terms; terms were then distinguished by whether the agent was the teacher (T), the student (S) or the interaction between the teacher and the student (ST); terms were then connected within the T, S, and ST categories through a ‘tree’ structure; and, finally links were identified between terms.

The nature of any connection between the lexical terms is currently under investigation. Such links might be differentiated as follows: causal (inevitable or intended), coincident, associative, sequential, hierarchical, or, canonical (where the existence of one term requires the existence of the other). It is hypothesised that the structure of the lexicon and the nature of the identified connections represent important pedagogical principles, encrypted in the lexicon, reflecting the evolving values and educational beliefs of the community and the culture.

Czech Republic

The Czech team consisted of three university academics, whose interests included didactics of mathematics and pedagogy, and two experienced teachers. There are 89 terms in the Czech lexicon: 61 items describe teacher-student interactions, 17 items describe student activities, and, 11 items describe the teacher’s activities when pupils work on their own. All these items are independent of the mathematical content as they may be used for describing lessons of other subjects without requiring modification. Indeed, Stuchlíková, Janík, and Beneš (2015) state that subject didactics have no specific terminology in Czech and this observation is consistent with the absence of any focus on specific mathematical content in the Czech lexicon.

Particular terms in the Czech Lexicon illustrated the evocative richness of some lexical elements. For example, *učitelská ozvěna* - the “teacher’s echo” refers to occasions when the teacher reformulates a student’s answer. The development of the Czech Lexicon also revealed the need to distinguish between a term in widespread academic use and a related term employed by teachers. For example, the term: *heuristický rozhovor* is an academic term meaning “Heuristic dialogue” or an exploratory dialogue intended to solve a problem. However, it was felt by members of the Czech team that *heuristický rozhovor* was not a term in widespread use by teachers and that most Czech teachers would call it *řízená diskuse* (Guided discussion).

There are 12 categories (and numerous sub-categories) that help organise the Czech lexicon: Classroom management; Introductory communication; Explanation of a new topic; Revision of a previously taught topic; Solving of a problem; Checking individual work; Institutionalisation; Summary; Non-mathematical social interaction; Assessment; Concluding the lesson; and Individual consultation with the pupil. There exists, however, a silence within the Czech lexicon with respect to lexical items with affective associations and many familiar classroom events, easy to describe, are not named.

The Czech team has emphasised the point that use of pedagogical terms varies according to groups of users (authors in different fields of pedagogy, teachers, etc.) and

the lexicon could be used as a tool for teacher training and as part of a unifying process triggering and framing discussion.

USA

The USA Lexicon could be divided into two sub-categories:

- Math-specific (Organisational Structures, Discourse, and Pedagogical Structures); and
- Non-math specific (General pedagogical strategies; Classroom Management; Activities).

And there are nested sub-categories within these categories, so that sub-categories within Organisational Structures are Activity Structures and Participation Structures. And Activity Structures contains “do now,” “review homework,” “guided practice,” and “number talk.” Each research team confronted a similar challenge to represent the structure of the local lexicon in a valid way.

Other features of the USA Lexicon included a distinction between teacher focus, student focus and interactive focus, similar to the Czech structure. It was noted that the lexical terms referred to events that could span very different lengths of time. There were also some terms that referred to very specific points in a lesson.

Pedagogical Traditions

Each particular country’s lexicon reflects a specific pedagogical tradition, culturally and historically situated. The extent to which a lexicon appeared to have an intrinsic connective structure that reflects a locally-situated history of pedagogical practice varied from country to country. Certainly, the variation evident between the different lexicons makes it clear that the teaching communities in the different countries interface with the mathematics classroom in very different ways, mediated by entirely different naming schemes for the things we might find there.

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LANGUAGE DETERMINING TEACHER SELECTIVE ATTENTION AND LEARNING

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Classrooms can be seen as a site for learning not only for students but also for teachers (Leikin & Zazkis, 2010; Margolinas, Coulange, & Bessot, 2005). Rather than using students as the focal point, this section of the Forum focuses on the learning of teachers in their development of teaching expertise. Specifically, this session examines the role of language in shaping the selective attention (and therefore the learning) of teachers. Issues associated with the predeterminant role of language in directing the attention of teachers and researchers are discussed, drawing on examples from three studies that investigated teacher learning.

Learning From Lessons – Chan and Clarke

The Learning From Lessons project aimed to investigate the research question: In what form and by what process do teachers learn from the experience of teaching mathematics lessons? The project design was based on the assumption that teachers' in-class learning is significantly shaped by their selective attention. Through examining their planning of and reflection on their lesson delivery, this project identified the teachers' objects of attention, the meaning associated with those objects, and the consequences of the objects of the teachers' attention as determinant of teacher learning, as evidenced through epistemic claims and their adaptive practice.

A key element in the research design of the project was the provision of purposefully-designed experimental mathematics lessons, which provided the initial teacher stimulus for this study of teacher selective attention, reflection and learning. The data generated from the project included the teacher's adaptation of a pre-designed lesson, the teacher's actions during that lesson, the teacher's reflective thoughts about the lesson, and, most importantly, the consequences for the planning and teaching of a

second lesson. More details of the project can be found in Clarke, Clarke, Roche, and Chan (2015).

This report focuses on three middle school mathematics teachers (pseudonyms: Ashley, Tracey, and Jennifer) from Melbourne, Australia, who participated in the project. All of the teachers had at least five years of classroom teaching experience. Analysis of the transcripts of the interviews with the teachers revealed that the teachers attended to events connected with current educational priorities such as “engagement”, “learning intentions”, and “student thinking/misconceptions”. However, such terms were differently realised in the specific objects of the teachers’ attention and in the way they adapted their practice based on what they had learned from teaching the initial lesson.

Analysis of teacher interviews indicated that when the teachers addressed lesson elements such as “student engagement” and “summary phase,” they actually attended to different observable phenomena and accordingly modified their practice in different ways, based on the different meanings that they attached to the particular term or phrase that signified the object of attention. The project highlights the need to identify not only the focus of teacher selective attention but also the meaning attributed to the objects attended to and the implications of that attention for teacher professional learning. We need a better understanding of what teachers attend to, what meaning they derive from their acts of attention, and what actions are likely to follow. Once this is known, it should be possible to develop programs to recommend what teachers should attend to and to scaffold the teachers’ meaning-making reflective process in ways that are likely to advance the teacher’s learning.

Whereas the Learning From Lessons project focused on examining the selective attention and consequent learning of experienced teachers (five years and more), the study conducted by Jazby used post-lesson video-stimulated interviews with stimulus video footage recorded through head-mounted video equipment to compare selective attention between teachers with different levels of teaching experience in terms of how they described classroom events during the post-lesson interview.

Teacher attention using head-mounted video - Jazby

In order to investigate what primary mathematics teachers actively attend to as they teach, head-mounted, video-cued recall interviews (HMV interviews) were trialled with three primary teachers of mathematics. Each teacher had a different level of experience in teaching primary mathematics: less than 5 years’ experience, 5-10 years’ experience, and over 10 years’ experience. Teachers wore a head-mounted camera as they taught a mathematics lesson that they had planned themselves. All three teachers were considered by their peers to be competent teachers of mathematics. Approximately 10 minutes after the lesson, they reviewed the footage captured on the head-mounted camera in order to re-experience teaching the lesson (Omodei, McLennan, & Wearing, 2005). Teachers were asked in a recall interview to provide commentary regarding what they had been thinking, feeling and attending to during

the lesson. In order to identify whether teachers were employing specific heuristics, video and interview data were analysed side-by-side in order to ascertain what teachers were looking at and what teachers were looking for.

The least experienced teacher (Sam), unlike her two more experienced colleagues, spent more time looking at student body language. The more experienced teachers, in contrast, spent more time looking at mathematical representations that had been created by students using manipulatives or drawings. Miranda, the most experienced of the teachers, looked at students' arrangements of counters to discern if students were 'on task'. HMV interview data suggests that Miranda was actively looking for 'interesting' entities as she engaged in between-desk instruction, and that these 'interesting' entities were 'unusual/uncommon' student responses. Todd and Gigerenzer's (2012) notion of perceptual heuristics enables the development of a language that describes the properties of what Miranda sees as 'interesting'. It is hoped that this approach can answer Grossman et al.'s (2009) call for the development of a language and structure for describing teachers' selective attention that renders it teachable.

The Structured Stimulation of Teacher Reflection - Hollingsworth

Also utilising video in her study, Hollingsworth focused on providing teachers with multiple opportunities to describe and reflect on their own teaching practice, and therefore build their capacity for productive learning. While video is a potentially powerful tool for focusing teacher self-reflection (Hollingsworth & Clarke, in press), as Gaudin and Chaliès (2015) report, "simply viewing video does not ensure teacher learning. An important issue concerns how to facilitate substantive analysis of teaching practice with video so that it becomes a productive learning tool for teachers" (p. 59).

This study investigated the structured stimulation of teacher reflection and the learning consequences of this process. Central to the study were the significant agency that resided with the participating teachers, and the building and couching of theory in the language of teacher learning and everyday classroom practice (Shulman, 1992; Shulman & Shulman, 2004).

Two teachers, one secondary and one primary, selected foci for their own professional learning from a theoretically and empirically grounded observation framework and used these foci to observe, analyse and reflect on video recordings of their mathematics teaching practice. The researcher also observed and analysed the video recorded lessons using analytical software, and then the teachers and the researcher engaged in a video-stimulated feedback conversation to discuss their observations and analyses. During the conversations teachers took the lead role in directing discussion about their practice, with the researcher prompting the teachers for elaboration and asking them to articulate implications for future teaching practice.

As Hollingsworth and Clarke (in press) note, "Our increasing use of video material to facilitate teacher reflection on classroom practice may (i) render visible, for the first time, some of the unnoticed practices of teachers; and (ii) facilitate the development among the teaching community of a new vocabulary by which we might describe

teaching practice. Both of these developments are important”. Study participants reported that the process they engaged in provided them with opportunities to consider their practice in ways not previously possible.

The structured scaffolding of teachers’ use of language to describe, analyse, reflect on, and consider ways to transform their practice is an important element of the design. The research design provides a promising approach for using video to facilitate substantive analysis of teaching practice and also a positive and constructive learning experience for the participating teachers.

Conclusion

The three studies illustrate different ways in which teachers engage in observational, interpretive, and reflective practices significantly mediated by language. The first study (Clarke and Chan) showed the different meanings that teachers could attach to the “same” designated object of attention (e.g. “student engagement” and “summary phase”) with different learning consequences in terms of their adaptations of the lesson plans and reflection on the lessons. The second study (Jazby) found that experienced teachers may use very vague or general language (e.g. “interesting” or “good examples”) to describe the classroom events, which creates a barrier for others to understand their focus of attention and access their expertise. Language may limit both thought processes and their articulation in speech. The third study (Hollingsworth) demonstrated the value of providing participating teachers with a structured vocabulary to review and reflect on their own practice through a video recording of that practice. The three studies highlighted the importance of understanding how language mediates teacher selective attention and learning, as well as suggesting ways in which language can be utilised to facilitate productive teacher learning.

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LANGUAGE DETERMINING RESEARCHER ATTENTION AND THEORY CONSTRUCTION

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The critical role that language plays in shaping our research has been variously acknowledged. Clarke (2001) drew attention to a very specific aspect of language use by researchers, particularly with regard to the construction of theory.

I want to examine the manner in which any understanding of Vygotsky's theorising must be grounded in an understanding of the language available to him, and the limitations (constraints and affordances) that language places on our theorising about classrooms. In particular, it appears that readers of Vygotsky in English have been denied a richness of meaning present in the original Russian text. (Clarke, 2001, p. 312)

Clarke then goes on to contrast two translations of Vygotsky in which the same (Russian) word has been differently translated into English.

From this point of view, instruction cannot be identified as development, but properly organized instruction will result in the child's intellectual development, will bring into

being an entire series of such developmental processes, which were not at all possible without instruction. (Vygotsky, 1982, p. 121, as quoted in Hedegaard, 1990, p. 350)

The translation used by Hedegaard is referenced: “Vygotsky, L. S. (1982) Om barnets psykiske udvikling [On the child’s psychic development]. Copenhagen: Nyt Nordisk.” However, a different translation of the same passage, is provided in the widely-cited 1978 translation published as *Mind and Society*.

From this point of view, learning is not development; however, properly organized learning results in mental development and sets in motion a variety of developmental processes that would be impossible apart from learning. (Vygotsky, 1978, p. 90)

The ‘conflicting’ translations arise because of a duality of meaning in the original term employed by Vygotsky. This duality has been noted previously, but its significance seems to have been given little consideration in the interpretation and application of Vygotsky’s work.

The theoretical framework of Vygotsky entails specific understanding of learning, development, and the goal(s) of development. In Vygotsky’s usage, the term *obuchenie*, frequently translated as learning, more accurately indicates the interaction of teacher and student. (Wertsch & Sohmer, 1995, p. 332)

As we have seen, the same term (“*obuchenie*”) is also translated as “instruction” and shares with corresponding terms in other languages (“*leren*” in Dutch and “*gakushushido*” in Japanese) the capacity to invoke both teaching and learning, as these are named in English. Once this duality of meaning is recognised our reading of Vygotsky and our theorising about the teaching/learning process are greatly enriched.

We need to attend carefully to the constraints that the language of our theories places on the ways in which we interface with the settings of our research.

It becomes essential to consider how the individuals in the classroom are positioned by the discourses in which they participate. The discourse of educational research also acts to position participants in ways that afford and restrict certain interpretations. (Clarke, 2001, p. 296)

As a particular example of cultural or linguistic specificity: The word “effective” (or rather, the translations of that word into different national languages) seems to have value-laden and context-specific connotations that influence how the word is interpreted and how people relate to the word. In the NorBa-project on mathematics teacher’s beliefs in the Baltic and Nordic countries (see, e.g. Lepik, Pipere, & Hannula, 2013), the process of designing the survey instrument for the teachers, involved lengthy discussions whether to label one section of the questionnaire as “Your view about good teaching” or “Your view about effective teaching.” For some members of the project team, the word “effective” had a connotation with economic efficiency and squeezing the most out of the people who work. For some others, there was no such connotation. The agreed solution was to use both words in parallel (“good/effective”) and leave the final decision of the word choice for each national team – with an awareness of the different connotations in different countries.

In Finland the word “effective” is seldom used in educational contexts, due to its engineering and economical connotation . . . This could highlight a specific Nordic emphasis on education as a humanistic endeavour. (Hannula, 2012, p. 89)

Hannula and his colleagues have investigated the metaphors used to describe the practice of teaching (Löfström, Anspal, Hannula, & Poom-Valickis, 2010; Löfström, Hannula, & Poom-Valickis, 2010; Oksanen, Portaankorva-Koivisto, & Hannula, 2014). Metaphors are determined significantly by language and culture and the utility of a metaphor outside its culture of origin may be limited, especially when it has been translated into another language. Yet metaphors are among our most important linguistic tools for the development of theory, supporting us as we employ the familiar in our attempts to capture the less familiar, the emergent and the novel.

Novotná et al. (2016) report significant differences between the terminology used by teachers and researchers in the Czech Republic, particularly in relation to fields that have undergone recent development, such as formative and summative assessment. They suggest that the language used in the subject didactics should form the basis for descriptions of lessons from both a researcher’s and a teacher’s perspective. It is interesting to reflect upon the relationship between the language used by researchers and teachers to describe the practices of the classroom. Should the lexicon for these two communities be identical? Certainly it seems reasonable that there should be significant overlap if the two communities are to communicate effectively with each other. Currently subject didactics in the Czech Republic are in the process of reconstitution. This highlights the evolving nature of the language available to teachers and researchers.

As an example of the evolutionary nature of language use in education, recent developments in the Chinese mathematics curriculum are characterised by the emergent use of key words such as 活动经验 (pinyin version: huó dòng jīng yàn, and English translation: “activity experience”). The significance accorded to these words reflects a research-driven reconceptualisation of valued pedagogy and educational theory. Such a reconceptualisation poses significant challenges for Chinese researchers in mathematics education, who must operationalise newly prioritised constructs if the efficacy of recent curriculum initiatives is to be evaluated.

In the Chinese curriculum, “activity experience” is the fourth basic emphasis after basic knowledge, basic skill, and basic methodology. The basic mathematical activity experience is conceptualised as a mode of the thinking formed over time on the basis of three main procedures: Gaining intuition from observation, drawing conclusions from guessing, and verifying the conclusions through proof. This term “activity experience” is employed to raise teacher's awareness of learning mathematics by performing mathematical activities instead of memorising and repeated exercises. Interestingly, “experience” could be a noun or a verb in the curriculum. When used as a verb, it means the action/process of acquiring the mathematical experience.

The discussion in this section has highlighted examples of culturally-specific terms that, after translation into English, appear to have well-established and widely-shared meanings: teaching, learning, effective and experience. Considered together with the acknowledged cultural and linguistic specificity of metaphor, the possible variation in use and interpretation of key educational terms make clear the ambiguities and uncertainties present in the language in which we frame our theories and our curriculum documents. The underlying emphasis in the discussion is the function of language in framing, shaping and constituting our development of educational theory and the associated conduct of our research in mathematics education.

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MOVING FORWARD

The Research Forum has been structured to provide a narrative pathway from research into classroom discourse, in which language considerations are obviously central, through the language employed by teachers to discuss and reflect upon the classroom to the language available to researchers to classify, connect, model, predict, explain and theorise about those phenomena found in mathematics classrooms internationally.

It is important to acknowledge that use of the term “language” conflates a wide range of communicative mechanisms and purposes. Contrast, for example, the use of language in the sense of “foreign language” and the conversational language employed between students when engaged in mathematical problem solving. Invoking language as a distinct communicative system employed by a community, nation or diaspora, for whom their use of this language is a distinctive attribute (French-speaking, English-speaking, Chinese-speaking, Catalan-speaking, and so on), simultaneously invokes the nation or community (however concentrated or widely dispersed) for whom that language is a defining characteristic. There are politico-historical entailments to any reference to language in this sense. There are also cultural entailments and it is these that have featured prominently in parts of this Research Forum.

By comparison, reference to language as the communicative medium of social interaction calls upon different associations, even though a particular research project might examine cultural differences in social communication in classrooms in different countries. Within a single culture or community, the investigation of the role of language in facilitating, inhibiting or even constituting the learning process could focus on language as an innate human ability or as a formal symbolic system and both analyses might usefully contribute to a study of the nature and function of language in educational settings.

Language as communicative exchange provides the vehicle for the social construction of knowledge in mathematics classrooms. Language as discourse prescribes the limits of acceptable speech, both within the mathematics classroom and among the community of mathematics teachers when discussing the mathematics classroom. Within classroom discourse, different types of talk can be identified, characterised and their function in the learning process investigated. Language also functions as the medium by which the academic community analyse and theorise the phenomena for which the mathematics classroom is the setting.

While all these functions of language may appear quite distinct, it is our suggestion that they are profoundly intertwined: the language by which teachers shape the practices they orchestrate reflects cultural-historical origins that also set bounds on researchers’ capacity to articulate theory concerning those practices. This Research Forum has presented examples of all of these ways in which language as a construct might be enacted and studied in educational settings, particularly in mathematics classrooms.

It is useful to revisit some of the different points made throughout this forum and a suitable mechanism for doing this is to address the focus questions with which the Research Forum was introduced.

Focus Question 1. What are the characteristics of classroom discourse in mathematics classrooms (in the same or different cultural settings) and what distinctions within student classroom talk offer insight into the nature of student learning?

Mathematics classrooms provide a site for the negotiative use of language as students interact with their classmates and their teacher actively or passively, collaboratively or competitively. This negotiation may be concerned with the substantive mathematical content that is the pretext for the social gathering called “a mathematics class” or it may be concerned with establishing a set of social obligations and responsibilities, without which neither a class nor a collaborative group will run smoothly. Student and teacher-negotiated understandings of socio-mathematical norms exist as the implicit terms of a didactical contract in which teacher and students are jointly complicit. We have found it useful to distinguish meaning negotiation in mathematics classrooms as social, socio-mathematical, and mathematical. Negotiation with respect to each of these employs its own lexicon and can be considered as a distinct mode of interaction, offering interpretive possibilities for the researcher and foci for instructional intervention by the teacher.

The distinction made between dialogic and non-dialogic talk within the mathematical classroom appears to offer a productive lens on specific patterns of language use in student collaborative solving of mathematics problems. It appears that the negotiative exchange related to affect has a different character and calls upon different communicative tools than does more content-oriented negotiation. Addressing affect requires a more inclusive interpretation of what constitutes language in classroom settings. By identifying the focus of negotiation, the discursive mechanics by which negotiation is conducted and the nature of the operative affective microculture in which negotiation is undertaken, the Social Unit of Learning Project provides a multi-dimensional portrayal of collaborative problem solving and associated student learning in mathematics.

Focus Question 2. What are the implications of culturally-based differences in classroom-related language use (teacher professional discourse) for our advocacy of instructional practice and our construction of related theory?

It is hypothesised that the structure of the lexicon of a country’s mathematics teachers and the nature of the connections identified within that lexicon represent important pedagogical principles, encrypted in the lexicon, reflecting the evolving values and educational beliefs of the community and the culture. The Czech team, for example, emphasised the point that use of pedagogical terms varies according to groups of users (authors in different fields of pedagogy, teachers, etc.) and suggested that a national mathematics teachers’ lexicon could be used as a tool for teacher training, both triggering and framing discussions, and serve as part of a unifying process among the

mathematics education community to standardise both professional language and practice.

Each country's lexicon reflects a specific pedagogical tradition, culturally and historically situated. The extent to which a lexicon appeared to have an intrinsic connective structure that reflects a locally-situated history of pedagogical practice varied from country to country. Certainly, the variation evident between the different lexicons makes it clear that the teaching communities in the different countries, interface with the mathematics classroom in very different ways, mediated by entirely different naming schemes for the things we might find there.

Among the many language-related issues documented in this Research Forum, the colonisation of teacher talk by the language of the researcher is one phenomenon of interest. For some members of the education community, this reflects the progressive increase in the professionalism of mathematics teachers as they develop an increasingly sophisticated professional vocabulary to describe and engage in their practice.

Focus Question 3. In what ways does language constrain our capacity to recognise, describe, analyse, theorise, optimise and share the practices of the mathematics classroom?

The Learning from Lessons project highlights the need to identify not only the focus of teacher selective attention but also the meaning attributed to the objects attended to and the implications of that attention for teacher professional learning. We require the development of a language and a structure for describing teachers' selective attention that renders it teachable. The structured scaffolding of teachers' use of language to describe, analyse, reflect on, and consider ways to transform their practice is an ultimate goal for which the project represents an initial step.

Imprecision and ambiguity in teacher professional discourse were identified in both the Learning from Lessons project and in the study by Jazby. The study by Hollingsworth constitutes a demonstration of the likely efficacy of the deliberate scaffolding of that discourse through negotiated reflection stimulated by the provision of a framework to structure the language available to teachers for professional reflection. Language may limit or afford both thought processes and their articulation in speech.

Focus Question 4. What is the role of language in shaping teacher and researcher selective attention and consequent learning in and about mathematics classrooms?

Theories are articulated through language and the meanings invoked by the use of one language may go unrecognised in another. This statement may be as relevant if the comparison is between the languages of countries or the idiosyncratic lexicons aligned with particular theories. Language plays a critical role in the construction of theory. The generality of terms central to contemporary theory and curriculum, such as teaching, learning, effective and experience, is problematised once their cultural specificity is recognised. As a consequence, the utility of any construct outside its culture of origin may be limited, especially when it has been translated into another

language. This has significant implications for the portability of theory between cultures.

To conclude: It is hoped that the research reported in this forum has demonstrated the diversity of ways in which language shapes all aspects of activity in and about mathematics classrooms. The mathematics education community is engaged in productive research activity with regard to all of the issues raised in this forum. Differences in language use between classrooms and communities must be seen as opportunities for insight. It is through such comparisons of the various forms of situated language use that we propel the discourse of each community from the insularity of convention to the recognition of difference that leads to insight and innovation.

MATHEMATICS LEARNING AND TEACHING AT UNIVERSITY LEVEL

Coordinators:	<i>Barbara Jaworski</i> Loughborough University, UK	<i>Despina Potari</i> University of Athens, Greece
Presenters:	<i>Chris Rasmussen</i> , San Diego State University, USA <i>Greg Oates</i> , University of Auckland, NZ University of Tasmania, Australia	<i>Oh Nam Kwon</i> Seoul National University
With:	<i>Jessica Ellis</i> Colorado State University, USA <i>Angeliki Mali</i> , Loughborough University, UK <i>Megan Wawro</i> , Virginia Tech, USA	<i>Wes Maciejewski</i> San Jose State University, USA <i>Georgia Petropoulou</i> , University of Athens, Greece <i>Theodossios Zachariades</i> University of Athens, Greece

In this RF we focus on a range of issues relating to Mathematics Learning and Teaching at University level. These include relationships of a number of kinds relevant to enhancing and improving the experiences of students' learning at this level, and to our wider knowledge and understanding. We see relationships between learning and teaching in Calculus; between didactics, pedagogy and mathematics in teaching for student understanding; between mathematicians and mathematics educators who seek to develop their teaching. We address characteristics of inquiry-based teaching, teaching in lectures and in small groups; and conclude with a theoretical vision asking how the theories we use influence what we learn in our research. We invite audience consideration of issues arising in the above.

THE TOPIC OF THIS RESEARCH FORUM

This topic, *Mathematics Learning and Teaching at University Level*, is of growing interest within PME as can be seen by an increasing number of research reports year by year. Events around the world (such as the KHDM “Didactics of Mathematics in Higher Education as a Scientific Discipline” in Germany (2015) and the ERME “INDRUM” conference in France (2016) indicate a strong interest in this topic.

While research into mathematics learning at university level has a very considerable literature, research into mathematics teaching is still at a relatively early stage of development. We hope, in this forum to look at learning and teaching and raise issues about their mutual concerns. Thus we offer a range of perspectives from researchers established in the field and some who are ‘early researchers’.

Research background

Research into so-called “Advanced Mathematical Thinking” dates back to the 1980s, culminating in the now seminal book edited by David Tall (1991) of the same title. In this book, authors dwelt on advanced topics in mathematics that were taught and learned at university level. As well, it addressed theories in mathematics learning and teaching, many rooted in constructivist theory (such as APOS theory, concept image and definition, cognitive obstacles) and questioned the difficulties that students were observed to have in learning advanced level topics.

As a staging post, possibly the next event on the horizon was an ICMI study focusing on “The Teaching and Learning of Mathematics at University Level” and its associated Study Volume edited by Derek Holton (2001). As we might expect of an ICMI study volume, the book is wide ranging addressing Policy, Practice, Research, Mathematics and other disciplines, Technology, Assessment, and Teacher Education. It opens with an essay from Claudi Alsina (Spain) addressing “Why the Professor must be a Stimulating Teacher”, in which the author looked critically at a range of factors, “the myths” associated with university practices in the (then) current day teaching of mathematics. He suggested a “New Paradigm of Teaching Mathematics at University Level” involving innovative approaches, use of technological tools, new pedagogical strategies and modes of assessment. The aim he expressed was to avoid forms of teaching in which “teaching is reduced to lecturing, and learning to an individual after-class activity of assimilating results and practicing techniques” (p. 5). This introductory chapter by Alsina is followed by a chapter on “Changing Contexts” in which Robin Zevenbergen addressed “Implications for Diversity and Equity”. She recognized what we all increasingly know from experience in countries across the world, that students at the university are no longer the mathematical elite, but encompass a wide range of interests and purposes in studying mathematics. Thus learning mathematics takes on new meanings and teaching must adapt to enable these students to become successful learners.

The focus of this research forum

Against this backdrop, we present our research forum which picks up on the issues adumbrated above. The structure of the forum is constructed, around the didactic triangle of mathematics/teaching/learning (Goodchild & Sriraman, 2012; Straesser, 2009). We consider how the three elements Student–Teacher–Mathematics cohere and offer challenges to each other, and expand the didactic triangle to consider the broader institutional contexts in which teaching takes place. In addition we address theories in which research at this level is framed as well as forms of teacher or teaching knowledge that underpin pedagogical development. It is worth mentioning that we had hoped to address the didactic tetrahedron (Straesser, 2009) which recognises that technologies of various kinds play an important role in mathematics teaching and learning at this level. However, we were overambitious in what we had time to include. Perhaps some future RF can take up this challenge.

In terms of the institutional setting, we recognise that mathematics teaching at the university still takes place mainly in the form of lectures and tutorials. There are extremely differing views on the ways in which lectures afford or constrain learning potential (e.g. Pritchard, 2010). Pritchard argues that lectures can be effective in communicating information, modelling reasoning and motivating students. He claims that they can be effective when supported by other activities such as for example more independent work of the students in tutorials and homework. Research into learning in small-group tutorials shows a variety of approaches to support students in making mathematical meaning. The nature of the small group means that teachers or tutors can provide more readily for individual student needs (e.g., Jaworski, 2002; Nardi et al., 2005). However, it is important to see teaching in these two different settings in relation to students' actual or potential mathematical engagement. Recent work by Lew, Fukawa-Connelly, Meija-Ramos, and Weber (2016) shows that even for excellent lecturers, what they see as the main points from their lectures are not what students see as the main point. Speer et al, 2010 point out that there is too little research into what teachers do and think in their day-to-day work in either lectures or tutorials. This is therefore a theme that we address. We also highlight the emerging body of work that is examining inquiry-oriented teaching, and its use alongside lectures and tutorials.

Of course, centrally related to the decisions that teachers make about their teaching is what we know about learning. Research into the learning and teaching of different topics at university level (particularly in the areas of calculus and linear algebra) has focused on the difficulties that students experience (e.g., Nardi, 2008; Nardi, Jaworski & Hegedus, 2005) and the design of courses to address such difficulties. More recently, research in the US has focused on learning and teaching in post-calculus university courses as shown in a recent survey of developments in post-calculus mathematics education research (Rasmussen & Wawro, in press). In the past 10 years the field has witnessed considerable growth, with more and more research moving away from identifying student difficulties to studies of learning and teaching processes and productive ways that instruction moves forward student thinking. We argue that much of this work can be seen as falling in what Stokes (1997) refers to as “Pasteur’s quadrant,” which refers to basic research that seeks to extend the frontiers of understanding but is also inspired by considerations of use.

Yet another theme considers current/emerging practice looking at tertiary teacher knowledge and professional development at the undergraduate level (e.g., Barton, Oates, Paterson, & Thomas, 2014; Rowland, 2009). This theme merges into that of development of learning and teaching through developmental research – that is research which not only studies aspects of learning and teaching at this level but also contributes to knowledge in and development of practices (e.g., Petropoulou, Potari & Zachariades, 2011; Jaworski, 2009).

Theory in developmental research leads to further perspectives on theory underpinning research into learning and teaching at this level. While literature focusing overtly on

theory at this level is in its infancy, theoretical perspectives are starting to emerge. For example concepts and constructs include the emergent perspective (e.g. Rasmussen & Kwon, 2007); sociocultural perspectives including social practice theory and activity theory (e.g., Biza, Jaworski & Hemmi, 2014;); commognition (e.g., Nardi 2014); threshold concepts (e.g., Land, Meyer, & Flanagan, 2016).

While the didactic triangle deals with micro aspects of teaching and learning mathematics – those that are largely researched through considerations of teaching-learning interactions and student cognition, we are interested also in the wider social context that frames mathematics learning and teaching at the university. Teaching at the university level is framed by different conditions that may be addressed through a sociocultural frame (e.g., Hernandez & Harth, 2015; Jaworski, Robinson, Matthews & Croft, 2012). These conditions pose new issues to be considered when we study teaching at the university level from the perspective of the teachers or students. For example, conflicts between student culture and academic culture can affect how students respond to expectation in higher education; whether the university teacher is an active research mathematician or not is related to the type of questions he or she asks of the students; or the ways that the textbooks and lesson handouts transpose theory and practice. Winslow et al. (2014) address these conditions under the theoretical perspective of ATD (Anthropological Theory of Didactics). They claim that “universities do not operate independently from the rest of society, and their mathematical and didactic praxeologies are certainly subject to conditions and constraints coming from outside the university” (p. 98).

The presentations in this RF address some of these issues from a number of perspectives. In our first session, we see a theme of thinking about teaching and preparing to teach from a number of cultural positions. Within the frame of post-calculus teaching and learning in the USA, Ellis and Rasmussen look at the ways in which courses in calculus are coordinated within a range of institutions, and the training of graduate teaching assistants for teaching these courses. Also relating to the teaching of calculus, Potari and Zachariades discuss a study in Greece of the teaching of a mathematician who is also a researcher in mathematics education, focusing on relations between mathematics and pedagogy in encouraging students’ learning. The ways in which mathematicians and mathematics educators teach and think about teaching is the focus of Oates and Maciejewski from New Zealand, developing a theme of how co-researching mathematics teaching can be illuminating for both groups.

In our second session the theme is of teaching related to learning: its innovation through inquiry processes, its ways of fostering student meanings in lectures and small group tutorials, and its analysis through theoretical constructs. A collaboration between mathematicians and mathematics educators is the focus of Kwon and Rasmussen (Korea and USA) who discuss the inquiry-oriented teaching of differential equations. Mali, Petropoulou and Jaworski (Greece and UK) present two characterisations of teaching, one in lectures and one in small group tutorials with examples from the teaching of calculus. Finally, frameworks for analysis of learning and teaching are the

focus of Rasmussen and Wawro (USA), who offer an interpretive framework, entailing four different analytic constructs: disciplinary practices, classroom mathematical practices, individual participation in mathematical activity, and mathematical meanings. In our presentations we will show how these various focuses and themes are linked in and around the didactical triangle.

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Questions that the Research Forum will address;

1. In what ways does research into university mathematics learning and teaching take into account broader departmental and institutional practices and cultures?
2. How are teaching practices involved at the university level framed by the advanced character of mathematical concepts and processes?
3. What pedagogical processes and forms of students’ mathematical engagement does research reveal?
4. In what ways does university mathematics teaching take into account the diversity of students’ needs and backgrounds in different instructional settings?
5. In what ways does/can university teaching develop and what is the role of research in such development?

ORGANIZE THE TROOPS: THE ROLE OF COORDINATION AND GTA TRAINING IN FIRST YEAR UNDERGRADUATE MATHEMATICS

Jessica Ellis and Chris Rasmussen

Calculus is typically the first mathematics course for science, technology, engineering, and mathematics (STEM) majors in the United States (US). In the US, a university-level Calculus I course typically covers limits, rules and applications of the derivative, the definite integral, and the fundamental theorem of calculus. Typically, over half of Calculus I students also took a calculus course in secondary school, which usually focuses on techniques of differentiation and integration. In comparison, university-level Calculus I is usually more rigorous in its treatment of concepts (including limits, graphical interpretations, definitions, etc.) and applications. In the US and elsewhere, first year university mathematics courses such as calculus are consistently credited with preventing large numbers of students from pursuing a career in a STEM area (Seymour & Hewitt, 1997; Wake 2011).

In this presentation we report findings from a five-year national study of Calculus I programs in the US. The five-year project was conducted in two phases. In Phase 1 surveys were sent to a stratified random sample of students and their instructors at the beginning and the end of Calculus I. The surveys were restricted to the calculus course designed to prepare students for the study of engineering or the mathematical or physical sciences. Surveys were designed to gain an overview of the various calculus programs nationwide, and to determine which institutions had more successful calculus programs. Success was defined by a combination of student variables: persistence in calculus as marked by stated intention to take Calculus II; affective changes, including enjoyment of mathematics, confidence in mathematical ability, interest to continue studying math; and passing rates. In Phase 2 of the project, we conducted explanatory case studies at 18 different post-secondary institutions, where the type of institution was determined by the highest degree offered in mathematics (i.e., 2 year degree, Bachelors, Masters, or Doctoral). Here we present findings about the roles of coordination and graduate teaching assistant (GTA) training from case studies at five more successful calculus programs at institutions that offer a doctoral degree in mathematics (Rasmussen & Ellis, 2015; Ellis, 2015).

Coordination and GTA training are two of seven common features of these five successful calculus programs (Bressoud & Rasmussen, 2014). The surface features of coordination include a uniform textbook, syllabus, some common homework assignments (often online), and common exams. It was also typical that instructors have pedagogical autonomy. That is, no one prescribed how faculty would organize and carry out his or her teaching. This is an essential component of coordination. More important than having a uniform syllabus and common exams, however, was the role of the course coordinator in supporting faculty collaboration. An essential action that

the coordinator took was to convene regular meetings of calculus instructors. At some of the case study institutions these meetings were held biweekly and at other institutions they were held less frequently (two or three times per term as well as over email). The meetings, which included from 3 to over a dozen instructors¹, responded to the needs and priorities of those teaching calculus, focusing on practical issues of delivering course content and assessing student learning.

Benefits of such meetings include developing a shared sense of what is valued for students to learn, contributing to a department climate that values teaching as well as research, and fostering trust and open lines of communication among colleagues. Meeting regularly with colleagues about issues of teaching, learning, and assessment is somewhat uncommon in mathematics departments, especially at research universities. It is therefore striking that all five of the selected mathematics departments had faculty who were willing to participate in these meetings. One reason faculty were willing to participate in these meetings was because being part of the coordination actually made their life easier. For example, the coordinator typically took over a number of duties such as exam logistics, administration of online homework, and concerns with graduate teaching assistants.

While handling various logistical and administrative duties is an obvious way in which the coordinator supports his or her colleagues, it misses an aspect of the role of coordinator that is much more nuanced and perhaps even more important. In particular, we see the coordinator as a *choice architect*. A choice architect, a term coined by behavioral economists Thaler and Sunstein (2008), is someone who is responsible for organizing the context in which people make decisions. The power of a choice architect lies in focusing people's attention in a certain direction (e.g., calculus innovations across the country), and nudging them to make particular choices, rather than making the choice for them (e.g., offering only green beans at lunch or requiring the use of clickers). The role of the coordinator shares many of the characteristics of a choice architect, including setting default options, providing feedback, helping faculty understand how certain options or structures map onto different outcomes, and providing information about what others (locally and nationally) are doing (Rasmussen & Ellis, 2015).

We now turn to GTA professional development at the PhD granting programs identified as having a more successful calculus program. GTAs contribute to calculus instruction in two ways: as the primary teacher and as recitation leaders². As teachers, GTAs are completely in charge of the course just as a lecturer or tenured track/ tenured faculty would be, although they lack the experience, education, or time commitment

¹ In the US it is common to use the term "instructor" to refer to anyone who is the teacher of record. This could be a Professor, a post doc, an adjunct faculty, or a graduate student

² A typical format is to have a large lecture three times per week (with 150 or more students) and one or two "recitation sections" (or "discussion sections") per week. Recitation sections are much smaller (30-35 students) and are typically led by graduate students.

of their faculty counterparts. GTAs can also be viewed as the next generation of mathematics instructors. Thus, in addition to their immediate contribution to the landscape of Calculus instruction, GTAs will contribute significantly to the long-term state of calculus in their future occupations. Their professional development is therefore of immediate and long term concern, something all the more successful calculus programs took seriously.

The GTA professional development programs at the more successful calculus programs had well-developed and prolonged training programs, but with different characteristics. In particular, Ellis (2015) distilled the following three different types of GTA training programs: the Apprenticeship Model, the Coordinated Innovation model, and the Peer Mentor model. The Apprenticeship Model is guided by the desire to transition graduate students into the role of instructor, both as part of their immediate role as GTAs and as their (potential) future role as undergraduate mathematics instructors. Embedded within this philosophy is the belief that people learning a new profession must participate in the practices of the profession (Grossman et al., 2009) with growing responsibility.

The Coordinated Innovation model is guided by a view that Calculus I should be taught in an innovative and student-centered way, in small, coordinated classes. This innovation addresses the approach to the content, which is conceptually oriented and application driven, as well as the pedagogical approach, which includes group work and whole class discussions surrounding students' mathematical work. The coordination of these classes ensures that students have similar experiences. Further, this coordination helps to support the secondary guiding philosophy: that graduate students can be prepared and supported to successfully implement innovative instruction, reflecting long term goals that go beyond local needs. This model is also motivated by a third, underlying philosophy: that graduate students can be, as Seymour termed it (2005), "partners in innovation" and that graduate students who are effectively prepared to implement innovative instruction will likely carry these innovative practices into their future roles as undergraduate mathematics instructors.

The Peer-Mentor model is guided by a view that a more experienced GTA is not only capable of preparing and facilitating seminars for GTPD, but that this experience additionally supports interested graduate students in taking a leadership role among the GTAs. Further, that by involving interested and experienced graduate students in the training of novice GTAs, this model fosters a community of graduate students around teaching undergraduate mathematics. The Peer-Mentor GTPD model prepares and supports novice GTAs to lead recitation sections. It also prepares more experienced GTAs to be local teaching experts in their current role as Senior TAs and in their (potential) future roles as mathematics faculty. The recitation sessions are designed to go beyond a question and answer session, instead providing opportunities for students to work together on more conceptually oriented problems related to the lecture.

In addition to distilling these three models of GTA training, Ellis (2016) developed a theoretical framework that draws on K-12 literature and captures the particulars of the three models distilled from the successful calculus programs. The framework, which will be detailed in the presentation, highlights the institutional and departmental context and culture, the structure of the training, and the different ways that the professional development emphasizes various practices and different forms of knowledge.

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BLENDING PEDAGOGICAL AND MATHEMATICAL PRACTICES IN UNIVERSITY MATHEMATICS TEACHING

Despina Potari and Theodossios Zachariades

Studying university mathematics teaching with regard to the practice of the mathematician researchers and their pedagogy has currently received attention because of its potential to contribute to the improving of teaching (Schoenfeld, Thomas & Barton, 2016). Jaworski, Treffert-Thomas & Bartsch (2009) studied university mathematics teaching in a linear algebra course identifying two modes of a lecturer's talk: the expository and the didactic. The first concerns the lecturer's thinking about mathematics and the second about teaching. The above distinction is related to the different sources of experiences that a lecturer utilizes in his planning, teaching and reflecting on the lesson. Similarly, the study of Speer and Wagner (2009) indicated the different aspects of knowledge needed by a university lecturer in order to provide analytic scaffolding during classroom discussions in an inquiry oriented curriculum on differential equations. In the above case studies teaching at the university level was framed by the lecturers' experiences as participants in both mathematics research and university teaching. Skott (2010) theorised patterns of a teacher's participation in simultaneous practices related to the teaching and learning of mathematics when the teacher designs, acts and reflects on her teaching in an attempt to understand how this participation in different practices influence the learning opportunities for students. This presentation is a part of a wider study of university mathematics teaching and concerns the study of first year calculus teaching of one university lecturer in a mathematics department. This lecturer is a research mathematician, a researcher in mathematics education and has long experience in teaching this course. Through this case we will address in what ways the different practices in which the lecturer has participated (university teaching, mathematical research and mathematics education research) frame his teaching decisions and actions. A part of this study has been published in Petropoulou, Potari and Zachariades (2011). In this presentation we focus on the teaching of the derivative of a function at a point of its domain, and the generalization of the concept of the tangent.

METHODOLOGY

The study was based on a semester calculus course taught to freshman students in the mathematics department of a Greek University. The course is compulsory for the students and it is taught in a lecture format in parallel in three classes of approximately 100 students. The lecturer, the second author of this paper, is a mathematician and a mathematics education researcher with more than twenty-five years teaching experience at the university. The other two researchers were a mathematics education researcher (the first author) and a high school mathematics teacher who was also a doctoral student in mathematics education. The content of the course included limits of sequences and functions, continuous functions and the related theorems, derivative

and its applications (e.g., local maximum and minimum, the Rolle's and Mean Value Theorems). The emphasis was on the relevant concepts and on the proof of the theorems while less attention was given to the computations. The first two researchers, observed the 26 two- hour lectures of the course during a semester. The lectures were audio-recorded and transcribed while field notes were also kept. A meeting of the three researchers followed each lecture where the group identified central issues related to the lecturer's teaching actions and discussed about his decisions and the rationale behind them. The discussions in the meetings were also audio-recorded and transcribed. The analysis of the data was ongoing and retrospective. In the ongoing analysis a number of issues emerged concerning mathematics teaching. These issues were traced through our data from the teaching and patterns of teaching strategies were identified. Then, from the lecturer's reflections during the meetings the sources on which the lecturer based his specific teaching decisions were also identified.

RESULTS

In this paper we focus on a teaching episode about the introduction of the derivative and the tangent line of the graph of a function at a point of its domain.

Teaching episode

The lecturer started the lesson by referring to the tangent line of a circle, which the students knew from Euclidean geometry. Using the equation of the circle, known to the students from high school, he considered the tangent line as the tangent of the function $f(x)=\sqrt{1-x^2}$, $-1 \leq x \leq 1$ at a point and asked the following question: "How can we define the tangent of the graph of a function at a point of its domain?"

He asked students to recall the various critical properties of the tangent line of the circle as well as whether some of these properties could be of help to define the tangent of a function. They came to the conclusion that these geometrical characterizations could not be applied for extending the concept of the tangent in the general case. The lecturer pointed out the need for finding another characterization of the tangent of the circle at a point that could apply to extending the concept in the general case. He drew on the board different secants of the circle passing through the given point $(x_0, f(x_0))$ and determined informally the tangent of a point of the cycle as the limit of the secants. Then he determined the slope of the tangent as the limit of the slopes of the secants, $\lim_{x \rightarrow x_0} \frac{f(x)-f(x_0)}{x-x_0}$, and wrote on the board the equation of the tangent line using this limit.

He gave students as homework to verify the above result finding the equation of this tangent line by using analytic geometry methods. Finally, he pointed out that if for any function this limit at a point exists then the tangent of the graph of this function at this point can be defined.

The lesson continued by the lecturer giving the example of calculation of the instantaneous speed of a car and compared this with the slope of the tangent. On the basis of these two examples the definition of the derivative of a function at a point x_0 was given and applied in finding the derivatives of some functions. The lecturer drew

on the board the graph of the function $f(x) = \sqrt{|x|}$ and posed the question to the class: “what is the limit of the secants passing through the point (0,0)?”. A student responded that this limit is the y-axis and then the lecturer defined the vertical tangent line, emphasizing that the image of the tangent in the general case is different from the image of it in the case of circle. He supported this claim by speaking about the tangents at zero of the functions $f(x)=x^3$, $g(x)=x^2$ for $x \leq 0$ and $g(x) = 0$ for $x>0$, and $h(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $h(0)=0$, as well as the tangent of a linear function at any point. He found the equations of the above tangents, he drew the graphs of the functions and their tangents on the board and he pointed out to the students the different relations between the function and the tangent line in these cases and in the case of circle. He emphasized that in general the relation between the tangent line and the function is local but in the case of the circle this relation is global.

The lecturer’s reflection

In the discussion after the lesson, the lecturer explained his rationale about his teaching actions. He commented that he started with a known problem, the tangent of a circle as “in mathematics, every new result has as a starting point a problem. So, when I introduce a new concept or a theorem I try to follow this way”. He also pointed out that in mathematics research the generalization of a concept to stand for a broader class of mathematical objects is a common practice. He added that “one way to address such a problem is to find a characterization of the concept in known cases that it also works in the general case”. From mathematics education, he became familiar with some didactical aspects of those generalizations. He mentioned that the case of tangent line could be seen as a reconstructive generalization in the sense of Harel and Tall (1991) that requires the reconstruction of an existing schema to widen the range of its applicability. He also referred to the construct of concept image from the work of Tall and Vinner (1981) and the importance of helping students to build rich images that are also essential in mathematics research. Moreover, he considered that his personal involvement in research about university students’ and upper secondary school teachers’ understanding of the concept of tangent line (Biza & Zachariades, 2010; Potari et al., 2007) and his teaching experience helped him to know students’ and teachers’ misconceptions and incorrect images that are derived from the tangent of circle. Drawing on the above practice, he designs examples that aim to address these weaknesses and makes explicit references in the lecture on the tangent line at a point of a curve as a “local” concept versus the tangent line at a point of a circle as a “global” one.

In his interaction with students he tried to build on their prior knowledge, to engage them in the process of creating mathematics and to encourage them to participate in the discussions that he initiates. However, considering the institutional context, he claimed that “it is difficult in the context of lectures to engage students actively. Students hesitate to express publicly their views in such a large audience”.

EMERGING ISSUES

In this paper we studied the university teaching of a lecturer who has experience from research in mathematics, mathematics education and university teaching. The data analysis of the teaching observations and the discussions after the lecture showed that the lecturer brought experiences from the aforementioned practices. Concerning the mathematics taught he emphasized the mathematical thinking and its development as well as the important mathematical ideas and processes. Concerning the pedagogical practices, he tried to consider, in his planning and teaching, students' prior conceptualizations and to support their development by linking intuitive and formal ideas and methods. The blending of pedagogy and mathematics appears in different teaching actions such as in the introduction of a new concept, where his aim is the students to understand not only the concept but also the way of thinking that leads to this concept or the use of examples to address specific students' difficulties and to create an appropriate image of the concept. The analysis of the group discussions revealed certain interactions between mathematics and pedagogy in the lecturer's teaching and the development of teaching awareness that grounded on his practice as a researcher in mathematics education and on his practice as a research mathematician. Beyond this particular episode, we try to see different ways that the lecturer draws on these practices in different moments of his teaching and also relate them with other practices (e.g. institutional) that, although they do not relate to communities that are physically present in the classroom, are central in university mathematics teaching.

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RESEARCH MATHEMATICIANS & MATHEMATICS EDUCATORS: COLLABORATIONS FOR CHANGE

Greg Oates and Wes Maciejewski

MOTIVATION

As a developing university lecturer, Martin-Molina (2016) observes there are many challenges facing young researchers when they finish their Ph.D. and want to embark on a career as teachers at the university level. They often receive little or no training on how to teach, they may face a widely diverse array of students in contrast to their own experience as a mathematics major, they may shift universities and hence student and teaching cultures several times in a few years, and are subject to student evaluations the results of which are critical to their future careers. This is the environment within which many of our mathematics colleagues have developed as instructors. However, the landscape is changing with increasing institutional and student pressures for quality teaching, and a growing number of programmes providing either mentoring or explicit training for new lecturers. Given this context we ask, how can mathematics educators and mathematicians collaborate to develop the instructional practices of current and future teachers of post-secondary mathematics?

The University of Auckland has benefited for some twenty five years of having a mathematics education unit within the mathematics department. This presentation argues that this arrangement has provided unique opportunities for mathematics education and mathematics researchers to collaborate in an examination and development of their teaching practice from a mathematically-focused perspective. The two authors of this paper bring both perspectives to bear, with the second author in particular offering his insights in moving from a Ph.D. in mathematics to a developing career as a mathematics education researcher, alongside his continuing role as a teacher of university mathematics.

OPPORTUNITIES FOR COLLABORATION

The Department of Mathematics at the University of Auckland has four major research groups: Applied Mathematics, Algebra and Combinatorics, Analysis and Topology, and Mathematics Education. One advantage of this grouping is that it enables mathematics education researchers and research mathematicians to work closely together and collaborate on research and development of teaching practice at the tertiary level. It should be noted here that all members of the mathematics education unit maintain roles as university-level mathematics instructors, as well as their research and teaching interests in mathematics education. This presentation will consider several aspects of this blooming community of practice, with examples and data emerging from two nationally-funded research projects that have involved such collaborations.

DATUM Project

The *DATUM* project (Development and Analysis of Teaching in Undergraduate Mathematics) began as a longitudinal project to develop a model for professional development, theoretically grounded in Schoenfeld's (2010) resources, orientations, and goals (ROG) model of teacher action (Barton, Paterson, Oates & Thomas, 2014). A DATUM group includes both mathematicians and mathematics education researchers, to stimulate discussion of both mathematical and pedagogical knowledge. Each member of the group has one of their lectures recorded and from this they select a short (3- to 4-minute) segment for discussion, along with a brief written reflection of their ROG. The emphasis is on inclusiveness, collegiality, and shared learning and we believe one of the key dynamics of the DATUM groups is that the flow of pedagogical knowledge is not uni-directional from education researcher to mathematician. Participants are encouraged to reflect on and discuss their teaching episodes and thereby develop their practice organically.

The DATUM study has had an enduring impact on teaching practice in our department, with two independent groups of 6-8 colleagues continuing to meet five to six times per year since the initial research project ended in 2012. Positive outcomes of the study have been widely reported (Barton, Paterson, Oates & Thomas, 2014) and DATUM has shed light on undergraduate teaching practices. For example, work emerging from DATUM has revealed lecturers' internal dialogues as both mathematician and teacher when confronted by unplanned problems in class, weighing up pedagogical issues against mathematical values as they make instant decisions as to whether to deviate from their lecture plan (Paterson, Thomas, & Taylor, 2011). Hannah, Stewart and Thomas (2013) consider the role of language and visualisation in teaching linear algebra, while Barton (2011) describes how DATUM discussions led to a consideration of the value of mathematical content from a *pragmatic, epistemic & heuristic* perspective, focusing on the interplay between aspects of the "mathematical essence" of the lecture and the "learning culture" in which it is embedded

LUMOS Project

The *Learning in Undergraduate Mathematics Outcomes Spectrum* (LUMOS) initiative, started in 2014, aims to increase our understanding of learning outcomes for undergraduate mathematics. Of course, we expect our students to learn "maths", but what else? What mathematical skills, dispositions, affective outcomes, processes, and knowledge do mathematicians hope their students might develop? A large part of the project has been devoted to developing instruments by which we might observe how these outcomes might be measured. Progress to date includes identifying some potential new orchestrations which helped mediate students' movements towards instrumental genesis when engaged in active-technology tasks (Drijvers et al, 2010 and Artigue, 2001, cited in McMullen, Oates & Thomas, 2015) and an evolving instrument for assessing mathematical communication, trialled with students engaged in Team-Based Learning (TBL) activities (Paterson, Sheryn & Sneddon, 2013). One particular

undergraduate learning outcome we wish to highlight here has emerged from a careful analysis of the way mathematicians select and work on mathematical problems

An Example from LUMOS: Mathematical Foresight

Interviews with mathematics colleagues has led to the realisation that mathematicians often anticipate the value/utility/beauty of a problem and chart a likely course to a resolution in advance of embarking on actual rigorous work. This ability is distinct from, but not unrelated to, intuition, strategic thinking, and aesthetic sense and has been termed *mathematical foresight* to highlight its similarities to future-thinking behaviour in other domains (Maciejewski & Barton, 2015). Mathematicians have identified this ability as central to their mathematical work and we ask, should instructors strive to develop this in their undergraduate students? How might we judge the success of such an effort?

Since the development of the initial mathematical foresight model, two further studies of students' mathematical foresight have been conducted. The first (Maciejewski & Barton, 2016), characterises students' problem-planning behaviour through a mathematical foresight lens. The second, presented at this conference (Maciejewski, Roberts, & Addis, 2016), draws analogies between foresight in mathematics and in general daily experience. While acknowledging that an undergraduate education in mathematics is not always or necessarily about producing mathematicians, we believe that an examination of mathematicians' practices can nevertheless lend insight into the implicit/hidden mathematics curriculum.

DISCUSSION

We suggest that mathematicians hail from a strong teaching culture with a long history, emerging from mathematicians' evolving perceptions of the nature of mathematics. This culture is pervasive; many mathematicians have a clear idea of what constitutes a good education in mathematics. These ideas, however, may or may not align with those of a mathematics educator. This leads us to ask, how might we invite mathematicians into a conversation about education, especially in a way that is respectful of their teaching culture and is informed by contemporary mathematics education literature? We propose that there must be a willingness of both parties: both must exhibit a "willing suspension of disbelief" by engaging with practices and literature with different standards and forms of conviction to their respective fields. We view the onus as being on the mathematics educator here: they are the facilitators and simultaneously must not be critical of the mathematicians' approaches and be the champions of change. We have incorporated these perspectives into both our DATUM and LUMOS work.

In DATUM, the conversation is started explicitly – both mathematics educator and mathematician discuss the participant's teaching practice as it unfolds. LUMOS takes a different approach: we look for issues that resonate with mathematicians, on topics they can identify in their own practice, as an invitation into the world of mathematics educators. In both projects, authentic undergraduate educational situations are brought to the fore and made accessible to mathematicians and mathematics educators alike.

Both projects rest on a strong theoretical basis, with an emphasis on authenticity and practical relevance which appeals to the practice of the mathematicians and feeds back to the theoretical work of the mathematics education researcher. This is, in our view, a productive and effective collaboration for change.

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INQUIRING INQUIRY-ORIENTED INSTRUCTION AT THE UNIVERSITY LEVEL

Oh Nam Kwon and Chris Rasmussen

As early as 1972, Donald Bligh first published ‘What’s the Use of Lectures’. Bligh outlined the problems of using lectures in teaching as follows:

- Lectures are as effective as other methods for *teaching information* but are no more effective; forty studies reviewed suggested that *unsupervised reading is better than lecturing*
- Lectures are *ineffective* in stimulating higher order thinking
- Lectures cannot be relied upon to *inspire* or change students’ *attitudes* favorably.

In mathematics classrooms that use a traditional, textbook-dominated approach, effective participation means that students listen to and watch the teacher demonstrate procedures, and then practice what was demonstrated by completing textbook exercises. This prevalent phenomenon of traditional approach to instruction is linked with some general existing ‘myths’ and practices in the teaching of the mathematics at the undergraduate level such as ‘the researchers-always-make-good-teachers’ and ‘the perfect-theory presentation’ (Alsina, 2002). In the last two decades, there has been a growth in educational research on undergraduate mathematics education. Research began by investigating students’ learning and understanding of specific mathematical concepts. The results obtained offered convincing evidence of the limitations of traditional teaching practices. The evidence of the gap between what a teacher tells students and what is learned was noted by several researchers (e.g., Gutierrez, & Boero, 2006). There have been still ongoing calls for innovation in tertiary education, where for decades knowledge has been transmitted mainly through traditional lectures, despite continuous quality concerns (e.g., Kwon, 2015). How can one help tertiary mathematics students come to mathematical thinking? Which instructional efforts might be more productive of mathematical thinking? The aim of this section for the forum is to provide an example of inquiry-oriented mathematics instruction for mathematical thinking and to discuss their implications for the higher-education communities.

While there are clear calls for inquiry in both science and mathematics classrooms, what exactly characterizes an inquiry-oriented classroom is less clear. To clarify the nature of inquiry-oriented classrooms and to provide a more comprehensive perspective on the complexity of teaching and learning, Rasmussen and Kwon (2007) characterize inquiry in terms of both student activity and teacher activity for their Inquiry Oriented Differential Equations (IO-DE) project. The IO-DE project is an example of a collaborative effort between mathematics educators and mathematicians that seeks to explore the prospects and possibilities for improving undergraduate

mathematics education, using differential equations as a case example. In particular, students learn new mathematics by inquiry, which involves solving novel problems, debating mathematical solutions, posing and following up on conjectures, and explaining and justifying one's thinking. The first function that student inquiry serves is to learn new mathematics by engaging in genuine argumentation. The second function that student inquiry serves is to empower learners to see themselves as capable of re-inventing mathematics and to see mathematics itself as a human activity. On the other hand, teachers also engage in inquiry. Teacher inquiry centers on inquiring into their students' mathematical thinking and reasoning. Teacher inquiry into student thinking serves three functions. First, it enables teachers to interpret how their students build mathematical ideas. Second, it provides an opportunity for teachers to learn something new about particular mathematical ideas in light of student thinking. Third, it better positions teachers to follow up on students' thinking by posing new questions and tasks.

It has been shown that the IODE approach positively contributes to students' conceptual understanding, problem solving, retention, justification, and attitudes toward mathematics (Ju, & Kwon, 2004; Kwon, Rasmussen, & Allen, 2005). However, we still have to resolve the notorious dilemma of an inquiry-oriented mathematics classroom for teachers, that is, "how to teach without teaching?" Kwon et al. (2008) investigated what it is that teachers actually do in an inquiry-oriented classroom. We identified four broad categories of teacher discursive moves: (1) revoicing; (2) questioning/requesting; (3) telling; and (4) managing. Revoicing is broadly defined as reuttering – or saying again (could be verbal, symbolic, or gestural) – of someone else's utterances (symbolizing or gesturing). Questioning is a discursive move in which a teacher checks for understanding, requests to explain thinking, requests to justify thinking and so on. Telling is defined narrowly as stating information or demonstrating procedures in the more traditional sense in order to clearly distinguish this form of discursive move from others. The four subcategories of telling are: (1) initiating, (2) facilitating (3) responding, and (4) summarizing. Managing is consisted of arranging, directing, motivating, and checking. The way or nature of teacher's listening is not directly captured in any of these codes. Rather, the nature or way of teacher's listening can be inferred through looking at patterns in teacher – student discursive moves. For example, the IRE (Initiation, Response, Evaluation) pattern suggests that the teacher is listening evaluatively (Mehan, 1979). The Q-A-R (Question-Answer-Response) pattern, in contrast, suggests that a teacher is listening interpretively or even hermeneutically. In addition to developing a fairly elaborate coding scheme to characterize all the various types of the teacher's discursive moves, we also coordinated this coding scheme with our definition of inquiry. The framework that captures this coordination consists of four different discursive moves and the relationships between these discursive moves and the various functions that inquiry serves (three teacher functions and two student functions as previously defined). As such, we present part of the Inquiry-Oriented Discursive Move (IO-DM) framework (see Table 1) as a coordination of two dimensions (discursive move on one axis and

function of the discursive move as it relates to teacher and student inquiry on the other axis). The black cells in Table 1 are teacher discursive moves that we see as most strongly connected to specific functions of teacher and student inquiry. The grey cells are teacher discursive moves that have a secondary connection.

Teacher Inquiry		Student Inquiry	
A – model student thinking		A – engage in student argumentation	
B – learn new mathematics		B – affect, beliefs	
C – next tasks, questions			

Teacher Discursive Move	Teacher Inquiry			Student Inquiry	
	A	B	C	A	B
Revoicing					
Repeating					
Rephrasing					
Expanding					
Reporting					

Table 1. The Inquiry Oriented Discursive Move (IO-DM) framework

Further, Kwon et al. (2008) focused on the teacher's revoicing in an inquiry-oriented classroom, because it is one of the discursive strategies that often occurs in the teaching of mathematics, but which has received limited attention in mathematics education research at the undergraduate level. Our analysis shows that a teacher's revoicing can constitute a major repertoire of his or her discursive moves and carries out critical functions in the context of mathematics practice in class. From that perspective, revoicing serves at least three functions in the classroom. First, revoicing functions to highlight specific mathematical ideas and/or provide mathematical content to move forward the mathematical agenda. Second, revoicing functions to honor and empower student thinking. That is, revoicing facilitates the development of students' mathematical identities. Third, revoicing functions to help students understand what constitutes a sufficient explanation or justification. That is, revoicing can serve to promote certain social and socio-mathematical norms .

Inquiry-oriented instruction is too complex (respecting cognitive, but also social, affective, and cultural) to serve as a likely arbiter in the short term. This paper shows that inquiry-oriented instruction at the university level is becoming the object of promising research and development studies in mathematics education, aiming at a deeper understanding and theoretical basis for educational innovations in the future.

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ANALYTICAL FRAMEWORKS OF UNIVERSITY MATHEMATICS TEACHING

Angeliki Mali, Georgia Petropoulou and Barbara Jaworski

Despite an often subtle recognition and analysis of students' learning needs, mathematics education research on the teaching and learning of undergraduate mathematics only rarely probes teachers' perceptions of these needs (Nardi, Jaworski & Hegedus, 2005). We focus on the teaching of university mathematics, particularly on "what teachers do and think daily, in class and out, as they perform their teaching work" (Speer, Smith & Horvath, 2010, p. 99). In doing so we seek to relate the design of teaching to teachers' perceptions of their students' learning needs. How do teachers promote the mathematical meaning making of their students?

Since we seek deep insights into teachers' thinking, practices and development of teaching, our methodology is largely qualitative, with analyses which explore meanings within the sociocultural frame. We provide examples from two studies, exploring (A) teaching in university lectures and (B) teaching in university small group tutorials. In both cases researchers collect data through observations of teaching and interviews with teachers, and analyse the data taking a grounded approach. Analytical frameworks are often a synthesis of grounded analyses and external theories.

STUDY A: LECTURES AND LECTURING

The issue of how teaching takes students into account is of great concern for mathematics departments in universities in Greece. In mathematics, *lectures*, to large cohorts of students, are the usual instructional activity at the university level. A study of lecturing in two Mathematics Departments explores how teachers conceptualize students' learning needs, and how they enact their conceptualizations in their teaching.

The data of the study derive from observation of first year lectures on Calculus, reflective discussions with the six lecturers (research mathematicians), and group meetings between lecturers and researchers, discussing issues from teaching. Three of these teaching cases have been analysed in depth to characterize the observed teaching. Three layers of data grounded analysis (Charmaz, 2006) were conducted: *First*, analysis of the discussions led to identification of how these university teachers conceptualize their students' learning needs, and of their teaching goals. *Second*, analysis of the observations led to identification of the lecturers' typical teaching actions. By using Activity Theory perspectives and especially the work of Leont'ev (1978) goals and actions were linked in an attempt to characterize the teaching activity. *Third*, a cross-case analysis of the teaching of the three lecturers with the Teaching Triad (Jaworski, 1994) allowed insights into the nature of university mathematics lecturing and the way that it takes students into account.

The results show that the three lecturers conceptualized in different ways the students' learning needs that motivated their teaching. L1 conceived as a main need for the students to be affectively (gain confidence) and cognitively included into the university environment. The main intention of this lecturer was to initiate students into how mathematicians think; how they explore relations, discover connections and produce new knowledge. In his practice, mainly, he interacted with the students and encouraged them to think and contribute their ideas on which he then built the new knowledge. For example he prompted students to think and discuss, based on the concept of series, why the number $0.333\ldots$ is a finite number even if it has infinite decimal digits; something that is taken for granted at high school. Then he used the general form of this example (the geometric series) to facilitate conjectures about when a series converges. L2 had a more social perspective; his perception was that students mainly need "not to delay in getting their degree" and to go on with their lives. His intention was to help students to start their university studies smoothly and to be prepared for the final examinations of the course. Thus he rarely interacted with them but instead he managed time to make

the content relevant to the students by using familiar natural language and by carefully clarifying subtle aspects of the material many times. For example, in order to use the partial sum S_{n-1} of a series in a proof (seen many times in proofs about series), he pointed out that there was a gap in the textbooks regarding this sum which was not explained: “the books always write just ‘consider the partial sum S_{n-1} ’ but what is this sum for $n=1$? Is it S_0 ? It is not defined!” To resolve this gap he explained formally and informally how a new sequence, defined for every n , could be generated from S_{n-1} . L3 mainly considered that students should have a sound foundation, as prospective mathematicians, on what it means to have rigorous mathematical proofs although she expressed her awareness of students’ need to get comprehensive explanations to their questions as well. Her intention was to introduce students into the formal mathematical language and structure; she adapted a traditional style of teaching being concomitantly accessible to students’ questions to which she responded usually with an example or a counterexample.

Briefly, different conceptualizations of students’ learning needs motivated the formation of different goals and the adaptation of different teaching actions for addressing these needs into these lecturers’ teaching. By using the Teaching Triad, it was seen that meeting students’ needs in this context is related to lecturers’ sensitivity to a large group of students from a cognitive, affective and social perspective.

STUDY B: TEACHING IN SMALL GROUP TUTORIALS

An aim of Study B, at an English university, is to characterise mathematics teaching practice in small tutorial groups of 2-8 first year undergraduate students and a tutor. The tutor is a research mathematician or mathematics educator who also lectures in mathematics modules. Twenty six tutors were surveyed and three chosen for in-depth study. Data consisted of audio-recorded interviews with three tutors and observations of their teaching for more than one semester. A grounded approach to the data (Charmaz, 2006) resulted in a set of concepts, characterised as ‘tools’ and ‘strategies’, for each of the three tutors’ teaching. Interview data indicates that the main goal for each tutor’s teaching is students’ mathematical meaning making.

We take the case of Zenobia’s teaching as an example of analysis with ‘strategies’ and ‘tools’ for teaching. Zenobia is a research analyst with a 20-year experience in teaching. Her tutorial group includes five students. The concepts generated as ‘strategies’ to characterise her teaching, are: selecting tasks, creating students’ positive feelings, selecting examples, evaluating the students’ sense making of mathematics, decoding, encoding and explaining the mathematics.

We quote an excerpt from the transcript of her second tutorial, which is coded ‘explaining the mathematics’. In that tutorial, the mathematical task is whether the sequence $n^2(-1)^n$ is bounded. Zenobia draws the graph of the sequence as a function from the natural numbers to the real numbers (Figure 1).

Zenobia:

You do know what the graph of x^2 looks like, and what the graph of $-x^2$ looks like. And this [Figure 1] is really a restriction of points from one or the other of these two curves. Do you see that? So, on the bottom, you've got that downward parabola. And over here, you've got this upward parabola. Now obviously, they should be symmetric. And you're bouncing back and forth here, and you can start to see that this is not going to be a bounded sequence.

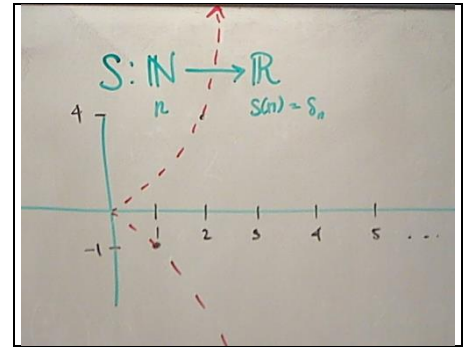


Figure 1: Graphical representation of $n^2(-1)^n$.

The findings of this study indicate that the role of the three tutors' mathematical representations is to explain the mathematics. In other words, the representations are Zenobia's tools for the strategy 'explaining'. Recognition of tools is related to tools in Vygotskian theory (Vygotsky, 1978), where the psychological nature of tools is to influence human behaviour: e.g., analysis indicates that Zenobia acts with the graphical representation of $n^2(-1)^n$ to influence students to "start to see" that the sequence is not bounded.

The strategy 'explaining' is more than a mere exposition of the mathematics. Zenobia uses formal mathematical language (e.g. graph of x^2 , restriction of points, curves, downward parabola, symmetric) to connect the mathematical areas of sequences and functions and to make the conjecture that $n^2(-1)^n$ is not bounded. In this way, she draws students into mathematical language and concepts: representation of a sequence graphically as a set of points, the relation with functions from the natural numbers to the real numbers; and the conjecture of the property to be unbounded.

This study offers a set of 'strategies' and 'tools' for each of the three cases of teaching. 'Explaining the mathematics with mathematical representations' is a common strategy among the three cases of teaching, where the main teaching goal is students' meaning making. The three tutors act differently with 'representations in explaining', attributed to their different views in mathematics and teaching/learning. For Zenobia, the graphical representations are connected with heuristics such as 'sketch a graph', declared by her to be used in her own mathematical research and her teaching for students' mathematical meaning making. In the case of Phanes', a research geometer, explaining with graphical representations is connected with making fundamentally mathematical ways of thinking in geometry transparent to students. Finally, for Alex, a mathematics educator, explaining with graphical representations allows connection of different mathematical representations for students' meaning making. Pedagogy for each tutor can be mapped onto the *Spectrum of Pedagogical Awareness*.

CHARACTERISING TEACHING WITH ANALYTICAL FRAMEWORKS

The two studies develop their own frameworks and use external theory to complete their analysis. They provide evidence that lecturers' conceptualizations of students' learning needs have an impact on what they teach and why they teach it in certain ways.

The teaching approaches studied reflect different forms of pedagogical awareness (Nardi et al., 2005) and corresponding degrees of sensitivity to students (Jaworski, 1994), which include sensitivity in the social domain of mathematics, research in mathematics or mathematics education, and institutional factors.

The theory and terminology here just start to offer a characterisation of teaching. The characterisation links the expressed thinking of the teacher with actions in the observed practice of teaching. Thus, terminology like ‘sensitivity to students in the social domain’ (a generalised finding of Study A) can be seen in terms of the detailed descriptions of episodes from practice. Similarly in Study B, characterisation through designation of ‘tools’ and ‘strategies’ allows teaching acts to be classified under such generalizable terminology which again can be linked to episodes of practice. Both studies offer emergent theory in terms of frameworks for characterisation. These frameworks are the beginning of theory in the domain of teaching, to address "how and why teaching happens in certain ways" (Speer et al., 2010, p. 107) and the extent to which teachers’ actions foster students’ meaning making in mathematics. In addition, awareness of teaching approaches and their relation to students’ learning needs can act as a self-awareness springboard towards teachers’ improving teaching.

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COORDINATING ANALYSES OF INDIVIDUAL AND COLLECTIVE MATHEMATICAL PROGRESS

Chris Rasmussen and Megan Wawro

Recent work in mathematics education research has sought to integrate different theoretical perspectives to develop a more comprehensive account of teaching and learning (Bikner-Ahsbahr & Prediger, 2014; Cobb, 2007; Hershkowitz, Tabach, Rasmussen, & Dreyfus, 2014; Saxe, 2002). An early effort at integrating different

theoretical perspectives is Cobb and Yackel's (1996) emergent perspective and accompanying interpretive framework. In this presentation we expand the interpretive framework for coordinating analyses of collective and individual mathematical progress.

The emergent perspective is a version of social constructivism that coordinates the individual cognitive perspective of constructivism (von Glasersfeld, 1995) and the sociocultural perspective based on symbolic interactionism (Blumer, 1969). A primary assumption is that mathematical development is both a process of active individual construction and a process of mathematical enculturation (Cobb & Yackel, 1996). The emergent perspective's interpretive framework makes use to two analytic constructs to account for students' mathematical progress: classroom mathematical practices (for collective progress) and individual students' mathematical conceptions and activities (for individual progress). Classroom mathematical practices are ways of reasoning that function as if shared (Rasmussen & Stephan, 2008) and represent the collective, discursive development of the class. Analyses of an individual student's conceptions and activities have often been framed in terms of that student's participation in these practices (e.g., Stephan, Cobb, & Gravemeijer, 2003).

In previous work (Rasmussen, Wawro, & Zandieh, 2015) we illustrate a way to expand these two analytic constructs to allow for more robust and comprehensive analyses of classroom learning and teaching. More specifically, we expand the construct for documenting individual progress to include analysis of individual student's participation in the collective development as well as identification of particular mathematical meanings that individual students bring to bear in their mathematical work. This latter approach enables us to draw on, when available, the rich literature on individual student thinking. Our prior work at the undergraduate level has also highlighted the fact that, in comparison to K-12 students, university mathematics students are more intensely and explicitly participating in the discipline of mathematics. However, the notion of a classroom mathematical practice was never intended to capture the ways in which the emergent, normative ways of reasoning developing within a classroom community relate to various disciplinary practices of the broader mathematical community. In order to more fully account for what often occurs at the undergraduate level, we therefore expand the interpretive framework to explicate how the classroom collective activity reflects and constitutes more general disciplinary practices.

To summarize, Figure 2 shows our expansion of the bottom row of the interpretive framework, which now entails four different analytic constructs: disciplinary practices, classroom mathematical practices, individual participation in mathematical activity, and mathematical meanings.

Social Perspective		Individual Perspective	
Classroom social norms		Beliefs about own role, others' roles, and the general nature of mathematical activity	
Sociomathematical norms		Mathematical beliefs and values	
Disciplinary practices	Classroom mathematical practices	Participation in mathematical activity	Mathematical conceptions

Figure 2. Expanded interpretive framework

The left hand side of the bottom row comprises two different constructs for examining the mathematical progress of the classroom community, while the right hand side comprises two different constructs for examining the mathematical progress of individual students. Coordination across these four analyses provides researchers with a more comprehensive account of mathematical progress.

Classroom mathematical practices. Classroom mathematical practices refer to the normative ways of reasoning that emerge as learners solve problems, explain their thinking, represent their ideas, etc. By normative we mean that there is empirical evidence that an idea or way of reasoning functions as if it is a mathematical truth in the classroom. Thus, normative ways of reasoning function in classroom discourse *as if* everyone has similar understandings, even though individual differences in understanding may exist. The empirical evidence needed to document normative ways of reasoning may be garnered using the approach developed by Rasmussen and Stephan (2008). This approach applies Toulmin's (1958) argumentation scheme to document the mathematical progress using two well-developed criteria.

Disciplinary practices. Disciplinary practices refer to the ways in which mathematicians typically go about their profession. The following disciplinary practices are among those core to the activity of professional mathematicians: defining, algorithmatizing, symbolizing, and theoremizing (Rasmussen, Zandieh, King, & Teppo, 2005). Not all classroom mathematical practices are easily or sensibly characterized in terms of a disciplinary practice. This is because classroom mathematical practices capture the emergent and potentially idiosyncratic collective mathematical progress, whereas a disciplinary practice analysis seeks to analyze collective progress as reflecting and embodying core disciplinary practices.

Mathematical meaning. As students solve problems, explain their thinking, represent their ideas, and make sense of others' ideas, they necessarily bring forth various meanings of the ideas being discussed and potentially modify these meanings. For example, we identified the following different meaning for rate of change that a group of four differential equations students leveraged as they reinvented Euler's method: steepness, growth rate length, ratio, fraction, tool, and function.

Participation in mathematical activity. This analysis draws on recent work by Krummheuer (2011), who characterizes individual learning as participation within a

mathematics classroom using the constructs of production design and recipient design. In production design, individual speakers take on various roles, which are dependent on the originality of the content and form of the utterance. These roles include that of author, relayer, ghostee, and spokesman. The recipient design also includes four possible roles: conversation partner, co-hearer, over-hearer, and eavesdropper. Our recent analysis of a differential equations class adds to this theoretical framing by detailing what we refer to as facilitator design roles.

In addition to using various combinations of the four constructs to more fully interpret students' mathematical progress, there exist multiple ways in which coordination across the four constructs is possible. For instance, one could choose an individual student within the classroom community and trace his/her utterances for the ways in which he/she contributed to the emergence of various normative ways of reasoning and/or disciplinary practices. Alternatively, when considering a normative way of reasoning, a researcher could investigate who the various individual students were that offered the claims, data, warrants, and backing in the Toulmin analysis used to document normative ways of reasoning. How do those contributions coordinate with those students' production design roles within the individual participation construct? For instance, does a student ever utilize an utterance that a different student authored as data for a new claim that he/she is authoring, and in what ways may that capture or be distinct from other students' individual mathematical conceptions? We also imagine ways to coordinate across the two individual constructs as well as across the two collective constructs. For example, how do patterns over time in student participation in class sessions relate to growth in their mathematical meanings? Are different participation patterns correlated with different mathematical growth trajectories? In what ways are particular classroom mathematical practices consistent (or even inconsistent) with various disciplinary practices? Finally, future research could take up more directly the role of the teacher in relation to the four constructs. Since we view the teacher as one of the participants in inquiry-oriented classrooms (but of course with unique and important distinctions from the students) we can begin such analyses with the same framing. We employ ways of reasoning that function as if 'shared' is a joint endeavour, which involves all participants, teacher included.

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UNDERSTANDING OBSTACLES IN THE DEVELOPMENT OF THE RATIONAL NUMBER CONCEPT– SEARCHING FOR COMMON GROUND

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FOCUS ON THE DOMAIN OF RATIONAL NUMBERS

A thorough understanding of rational numbers is very important for later mathematics learning and achievement. For example, Thompson & Saldanha (2003) found that learners' fraction knowledge was a better predictor for their algebra readiness than their whole number magnitude knowledge. Still, a large body of research indicates that elementary and secondary school students, educated adults, and even prospective and in-service teachers face serious challenges with learning (or teaching) various aspects of rational numbers.

Research has pointed to a variety of reasons for learners' difficulties with rational numbers. Moss (2005) has summarized the following: a) there are several conceptually distinct meanings attached to rational numbers that need to be understood and coordinated (e.g., fractions as parts of a whole, fractions as ratios, fractions as numbers, fractions as operators); b) new symbols and representations are introduced that, again, need to be understood and coordinated (e.g., decimal representations, fractional representations, percentages); c) learners need to construct a complex knowledge network for number based on multiplicative rather than on additive relations; d) the notions of the unit and the arithmetical operations need to be reconceptualised.

A common source of these difficulties seems to be that learners' prior knowledge and experience in the domain of natural numbers is not always compatible with rational numbers. Researchers continually report on the phenomenon of interference of natural number knowledge in learners' attempts to make sense of rational numbers and deal

with rational number tasks (e.g., Ni & Zhou, 2005; Van Dooren, Lehtinen, & Verschaffel, 2015). It is then argued that learners transfer their natural number knowledge in the domain of rational numbers in a naïve, ‘wholesale’ manner, using it to interpret information coming from instruction, and to solve rational number tasks.

A PHENOMENON WITH MANY FACES

Inappropriate application of natural number properties to rational numbers, also termed “natural number bias” (e.g., Ni & Zhou, 2005) can have a variety of appearances. The current literature often distinguishes four major areas, although more can certainly be identified: Size, operations, density, and representation.

Errors in comparing the *sizes* of two decimal numbers (or in ordering a set of decimal numbers) are often attributed to learners’ idea that, just as it is the case with natural numbers, the size of decimal numbers can be determined by considering differences in their number of digits (“longer decimals are larger,” whereas “shorter decimals are smaller”) (e.g., Resnick et al., 1989). Similarly, errors in comparing or ordering common fractions ($1/n$, or m/n) are often attributed to learners’ tendency to consider the numerical value of each of the components of the fractions separately (e.g., Stafylidou & Vosniadou, 2004).

Regarding *operations*, learners in their first years of elementary education often use arithmetic operations with natural numbers only. For natural numbers, these operations have outcomes that are predictable to some extent: Multiplication (with numbers larger than 1) and addition will always result in outcomes equal or larger than the operands while division (by numbers larger than 1) and subtraction will always result in equal or smaller outcomes. In the domain of rational numbers, however, these rules no longer necessarily apply. Yet, research suggests that some learners still rely on them, leading to mistakes such as thinking that 0.99×5 leads to an outcome larger than 5 (e.g., Fischbein, Deri, Nello, & Marino, 1985).

Regarding *density*, a fundamental difference between natural and rational numbers is found in that natural numbers are discrete, and thus one can always point to the succeeding, larger number after a given number. Rational numbers are dense: they do not have a successor, and between any two rational numbers there are infinitely many others. Studies show that learners struggle with the density property and think, for instance, that there are no (or finitely many) numbers between two pseudo-successive numbers (for example between 1.2 and 1.3) (e.g., Merenluoto & Lehtinen, 2002).

Concerning *representation*, rational numbers can be represented by fractions and decimals (and additionally by percentages, which are not considered here). Even within each of these two representational types, they have an infinite number of possible representations. For example, “one half” can be represented as 0.5, but also as 0.50, 0.500, $\frac{1}{2}$, $\frac{8}{16}$, 50%, etc., and many learners struggle with this representational multiplicity. Learners further tend to think of a fraction as two (natural) numbers

separated by a ‘bar’ rather than as a single quantity and a number of its own right (see for example Stafylidou & Vosniadou, 2004).

Some appearances relate to a combination of the above areas. For instance, in relation to the density idea, learners are also reported to have intermediate conceptions that depend on the representation; for instance, some believe that there are infinitely many numbers between two decimals but not between two fractions, or that there are no decimals between fractions and no fractions between decimals (Vamvakoussi & Vosniadou 2010).

RENEWED INTEREST

Research on students’ difficulties with rational numbers has been conducted for over 3 decades, and frequently presented at previous PME conferences. In recent years, there were several attempts to bring researchers in this area together. A recent special issue of *Learning and Instruction*, entitled *Mind the gap! studies on the development of the rational number concept* (Van Dooren, Lehtinen, & Verschaffel, 2015) was devoted to this topic, and several papers of a special issue of ZDM Mathematics education, entitled *Inhibitory control in mathematical thinking, problem solving, and Learning* (Van Dooren & Inglis, 2015) addressed the interference of natural number knowledge in rational number tasks. Focusing the Research Forum on aspects of the difficult-to-grasp (and teach) domain of rational numbers can thus contribute to the ongoing work while further anchoring it empirically and theoretically.

AT THE CROSSING BETWEEN PSYCHOLOGY, MATHEMATICS, AND EDUCATION

While a great variety of topics could be addressed in relation to the obstacles in understanding rational numbers, this Research Forum specifically focuses on the nature of the influence of individuals’ natural number knowledge on their rational number understanding at various ages. This phenomenon may seem too specific; however, we believe that the work presented here has a broader relevance. It addresses a central mathematical area with attention to a broad range of theoretical frameworks and methodologies. Moreover, some themes will be addressed that in the recent past have drawn the attention from a broader range of PME members.

In 2006, Vosniadou organized a Research Forum titled, *The conceptual change approach in the learning and teaching of mathematics: An introduction*. Many researchers in the field use the conceptual change theoretical framework to try to describe the interference of natural number prior knowledge and the reorganization process that takes place. However, the framework is also helpful to understand the learning of various other mathematical topics, and to design learning environments (see, e.g., Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2004).

In the same year, Ejersbo, Inglis, & Leron (2006) organized a Working Session titled, *Intuitive vs. analytical thinking: A view from cognitive psychology*. This WS addressed the so-called rationality debate in Cognitive Psychology, and its relevance for

Mathematics Education. Building on Fischbein's seminal work on intuitive thinking, the distinction between intuitive/heuristic reasoning and analytic reasoning was elaborated and operationalized by cognitive psychologists in the dual process theory. This theory helps to explain how it is possible that learners who have acquired all relevant knowledge and in principle are able to give correct answers to the rational number problems that they are solving still commit errors on some occasions. It also explains research that indicates that experts – while responding correctly – are still affected by natural number knowledge in their reasoning due to the dual processing involved. Both the intuitive/analytic distinction and the dual processing theory are nowadays being used more and more frequently in various mathematical domains (e.g., Gillard, Van Dooren, Schaeken, & Verschaffel, 2009; Leron & Hazzan, 2009), while also criticized (Tzur, 2011).

Finally, some of the work to be presented in this Research Forum uses cognitive-psychological methods that are attracting growing attention from Mathematics Education researchers. Examples are eye-tracking methods (see the Working Session *The use of eye tracking technology in mathematics education research*, Barnby, Andra, Gómez, Obersteiner, & Shvarts, 2014), and Cognitive Neuroscience techniques (e.g., the Research Forum *Interweaving mathematics education and cognitive neuroscience*, Tzur & Leikin RF, 2015; and the Seminar *An introduction to electroencephalographic research*, Gómez, 2016).

We thus consider this Research Forum to go beyond the specific topic of rational number learning, as it addresses theoretical and methodological issues that are at the intersection of Psychology, Mathematics, and Education.

KEY QUESTIONS

The contributors to the Research Forum will represent a variety of perspectives on the phenomenon under investigation. They will not present specific empirical studies, but rather provide a broad overview of their work in the field, thereby taking an explicit stance on various questions and raising issues for discussion. The following questions will be central in the discussion, while participants may raise others:

- What alternative perspectives on the relation between natural and rational numbers are found in the field? Specifically, how do the “bias” (or interference) and reorganization hypotheses differ—theoretically and pedagogically?
- Are terms used in the literature, such as bias, interference, intuition, etc., appropriate for studying the phenomenon at hand? If so, why? How can they be defined and how can they be operationalized for empirical observation?
- How can we characterize the implications of the bias? Are they similar across learners, or can we find qualitatively different profiles of understanding rational numbers that cannot be simply explained by having a stronger or weaker bias?
- What is the influence of instruction? Is there evidence for its existence in learners who no longer commit errors on rational number tasks? How can it be

observed? What are the implications of such a bias? Can mathematical experts still be biased?

Each contribution is followed by a discussant's reaction. The audience will be engaged in this discussion by addressing these questions.

THE NATURAL NUMBER BIAS: AN ATTEMPT TO MEASURE AND MAP ITS DEVELOPMENT ALONG PRIMARY AND SECONDARY EDUCATION

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The natural number bias is known to explain many difficulties with understanding rational numbers. The research field distinguishes at least three aspects where natural number properties are inappropriately applied in rational number tasks: density, size, and operations. A comprehensive test was constructed to characterize the development of 4th to 12th graders' natural number bias. This test was administered to 1343 students. Results showed an overall natural number bias that was weakest in size tasks, somewhat stronger in operations tasks, and by far strongest in density tasks. An overall decrease— but only a disappearance for size tasks – of the natural number bias with grade was found.

INTRODUCTION

Recently, a lot of authors ascribe many of the difficulties with understanding rational numbers to the tendency to (inappropriately) apply natural number features in rational number tasks (e.g. Van Hoof, Janssen, Verschaffel, & Van Dooren, 2015). While others refer to this phenomenon as the whole number bias (see for example Ni & Zhou who introduced this term in 2005), we favor the term *natural number bias*. The reason is that unlike rational numbers, natural numbers are always positive and also this specific natural number characteristic may be inappropriately assumed when dealing with rational number tasks. Learners are found to make systematic mistakes specifically in rational numbers tasks where reasoning purely in terms of natural numbers results in an incorrect solution (incongruent items), whereas much higher accuracy levels are found in rational number tasks where reasoning merely in terms of natural numbers results in a correct solution (congruent items). Three main aspects of the natural number bias can be distinguished: density, size, and operations. The first aspect concerns the dense structure of rational numbers. Natural numbers are characterized by discreteness: You can always point out the successor number of any given number (for example: after 4 comes 5). Rational numbers, on the contrary, are characterized by a dense structure: There is no successor number of a given rational

number, as there are always infinitely many numbers between any two rational numbers. The second aspect is related to the numerical size of rational numbers. Learners have the wrong assumption that, as is the case with natural numbers, “longer decimals are larger”, “shorter decimals are smaller” and “a fraction’s numerical value increases when its denominator, numerator, or both increase” (e.g., Van Hoof et al., 2015). The third aspect of the natural number bias concerns the effect of arithmetic operations. While addition and multiplication with natural numbers will always result a larger number and division and subtraction will always result in a smaller number, these rules do not longer necessarily apply in the case of rational numbers, but learners still wrongly assume them to be true (e.g., Vamvakoussi, Van Dooren, & Verschaffel, 2012).

THE PRESENT STUDY

The overall goal of this study was to characterize the development of the natural number bias in all three aspects (density, size, and operations) across the wide span between 4th and 12th grade. By doing so, we addressed various issues that – despite the extensive attention that the topic of rational number understanding has recently received – have not been covered by research so far. First, the natural number bias has amply been studied in separate age groups, be it elementary school students (e.g., Gómez, Jiménez, Bobadilla, Reyes, & Dartnell, 2014), secondary school students (e.g., Stafylidou & Vosniadou, 2004), or adults (e.g., Vamvakoussi et al., 2012). However, the evolution over a wider age range spanning primary and secondary education has not been addressed in research so far. Second, while the natural number bias has been thoroughly investigated with items involving rational numbers in both their decimal form and in their fractional form, to the best of our knowledge, no study has yet directly and systematically compared the strength of the natural number bias in both representation types. Third, previous research has typically focused on the three aforementioned aspects separately, which makes it difficult to compare the strength of the various aspects of the natural number bias at a given age. For the above three reasons, the major goal of the current study was to characterize the development of the natural number bias in all three aspects across the span between 4th and 12th grade. Since no comprehensive test instrument was available to measure the natural number bias, a secondary goal of our study was to create such a comprehensive paper-and-pencil test. We administered this test with the aim to investigate: (1) the overall occurrence of a natural number bias, (2) the relative strength of this bias in the decimal vs. fraction format, (3) the relative strength of the natural number bias across the density, size, and operations aspects, and (4) the evolution with age of the natural number bias as a whole and specifically within each of the three aspects.

METHOD

Participants

Data were collected in 21 schools (9 primary schools and 12 secondary schools) from different parts of Flanders, Belgium. This resulted in a representative sample of 1343

learners distributed over 4th grade ($n = 213$), 6th grade ($n = 230$), 8th grade ($n = 293$), 10th grade ($n = 302$), and 12th grade ($n = 305$).

Design

Based on broad literature review and an exploration of the Flemish mathematics curriculum, a comprehensive paper-and-pencil test (the Rational Number Sense Test) was constructed. The test contained 63 items addressing the three aforementioned aspects of the natural number bias, with items presented in fraction or decimal form or using a combination of both. The reliability of the test instrument was high (Cronbach's $\alpha = .87$). Examples of congruent and incongruent items are given in Figure 1.

	Congruent item	Incongruent item
Density	Write a number between $1/4$ and $3/4$	Write a number between 3.49 and 3.50
Size	Choose the largest number: $14/18$ or $29/31$	Choose the largest number: $3/9$ or $2/5$
Operations	Is $50 \times 3/2$ bigger or smaller than 50?	Is 72×0.99 bigger or smaller than 72?

Figure 1: Examples of congruent and incongruent items.

RESULTS

A significant main effect of congruency was found $X^2(1, N = 1343) = 1456.13$, $p < .001$. The accuracy level for congruent items (87.9%) was significantly higher than for incongruent items (66.8%) for the whole group of participants. This result clearly confirms an overall natural number bias. Further, no significant interaction effect between representation and congruency was found, $X^2(1, N = 1343) = 0.03$, $p = .87$, indicating that the natural number bias was equally strong in rational number tasks with decimal numbers and with fractions. Next, a significant interaction effect between congruency and aspect was found $X^2(3, N = 1343) = 439.86$, $p < .001$. The odds ratios and their 95% confidence intervals showed that the natural number bias was weakest in size tasks (OR = 1.48, 95% CI [1.40, 1.57]), somewhat larger in operations tasks (OR = 1.66, 95% CI [1.58, 1.74]), and clearly largest in density tasks (OR = 11.48, 95% CI [10.01, 13.17]). We also found a significant interaction effect between congruency and grade $X^2(8, N = 1343) = 998.95$, $p < .001$. The odds ratios and their 95% confidence intervals of each grade level are shown in the upper left panel of Figure 2. Figure 2 moreover provides an overview of the overall evolution of the strength of the natural number bias for each aspect separately.

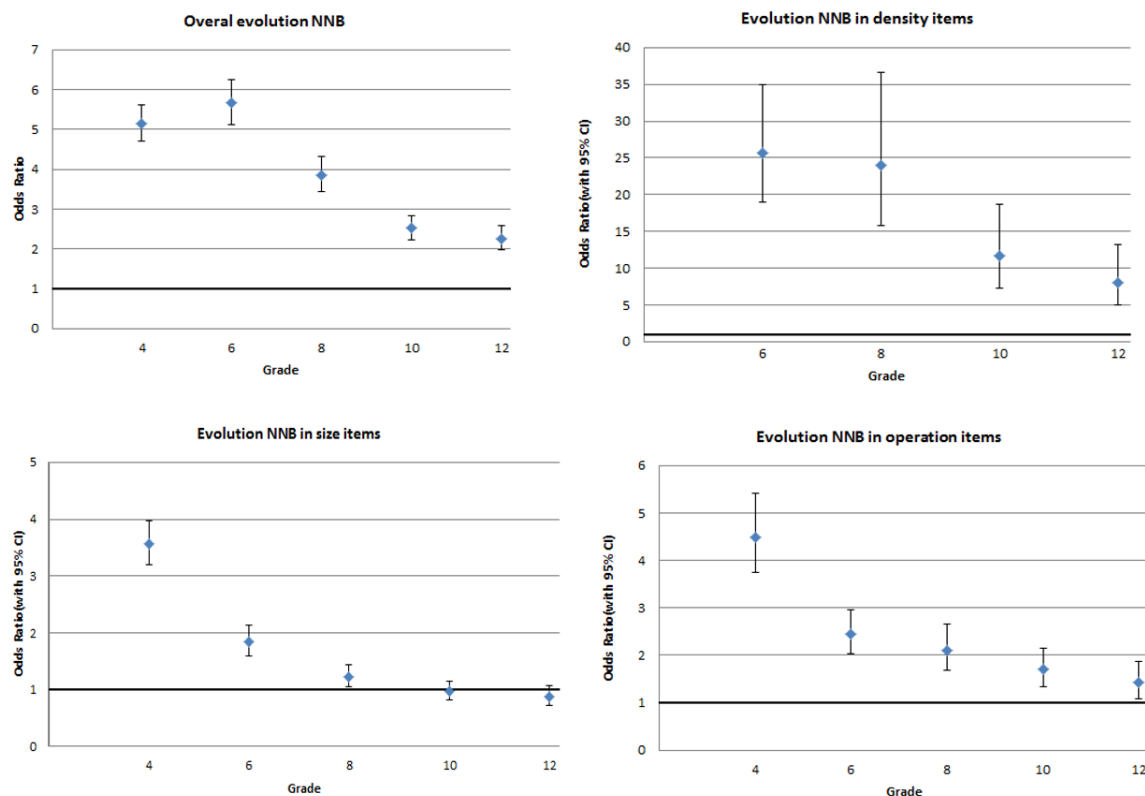


Figure 2: Overall evolution and evolution per aspect of the strength of the natural number bias as represented by the odds ratio (and 95% confidence interval) of accuracy for congruent and incongruent items (an odds ratio of 1 indicates an absent bias)

CONCLUSION AND DISCUSSION

By administering a new comprehensive test instrument to a large group of 4th to 12th graders, we first found that there was a clear natural number bias, as shown in the significantly higher accuracy to congruent than to incongruent items. Second, this bias was equally strong in tasks with decimal numbers as with fractions. This is an interesting finding, particularly because the available theoretical and empirical literature contains evidence that different kinds of natural number-based errors may occur in items involving these two representations. Third, learners' rational number understanding develops over a period of at least four years, but not continuously. The development is characterized by periods of rapid development (for example between 6th and 8th grade) and periods where it tends to stagnate (for example between 10th and 12th grade). This finding cannot be entirely explained by the relative focus on rational numbers in the curriculum. If this was the case, then one would expect a clear evolution between 4th and 6th grade, which was not found. Fourth, the natural number bias was weakest in size tasks, somewhat stronger in operations tasks, and by far the strongest in density tasks. As in our previous study (Van Hoof et al., 2015) this result suggests that understanding the size of rational numbers forms a first step in learners' rational number understanding, which is consistent with the integrated theory of numerical development (Siegler, Thompson, & Schneider, 2011).

Of course, next to cross-sectional data, longitudinal data is also needed to capture how learners' individual rational number understanding evolves over time. Recently, we collected longitudinal data of a large group of 4th and 5th graders. While we are still analysing the data, the results will be presented at the conference.

It is quite worrying that the majority of learners have troubles understanding the various aspects of rational numbers. Consequently, the acquisition of rational number understanding – and particularly of the understanding of the differences with natural numbers – deserves more attention in the mathematics class. In this respect, we note that errors committed by learners may be partly caused by formal instruction. Debou and Verschetze (2012) showed that the most often used mathematics textbooks in Flanders pay almost no explicit attention to the (conceptual) differences between natural and rational numbers, but rather tend to only point to similarities between both. We believe that if textbook designers and teachers have a thorough understanding of the natural number bias, they will be better able to address the natural number bias, for instance by pointing the learners systematically to differences between natural numbers and rational numbers (Depaepe et al., 2015). Moreover, based on the results of this study and previous studies in which was found that the size of rational numbers forms a prerequisite for gaining understanding in the aspect of operations and density (Van Hoof et al., 2015), we suggest that curriculum developers and teachers should focus first on a deep exploration and understanding of the magnitude of rational numbers, before focusing on the other aspects of rational number understanding.

ON THE NUANCES OF THE NATURAL NUMBER BIAS

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The Natural Number Bias has been proposed as a cognitive mechanism underlying many of the errors that learners make when reasoning about rational numbers, and it has been studied from several perspectives and methodological approaches in Psychology and Mathematics Education. In this contribution, we concentrate on the “congruency effect” interpretation of the Bias, distinguish some term meanings that are sometimes confounded by researchers, and discuss the interpretation of the Natural Number Bias and the congruency effects in terms of strategy diversity and usage.

INTRODUCTION

The natural number bias (hereafter, NNB), also known as whole number bias, has been the focus of attention of a wealth of research studies in Cognitive Psychology and Mathematics Education. References to a possible interfering relation between

knowledge of natural numbers and of rational numbers have been present in the literature for at least twenty years, such as Streefland's (1991) reference to the "temptation to deal with fractions in the same manner as with natural numbers" (p. 70). A review by Ni and Zhou (2005) focused on the cognitive origin of the NNB, introducing it as "the tendency in children to use the single-unit counting scheme applied to whole numbers to interpret instructional data on fractions" (p. 27). These authors observed that researchers in Education and Psychology converge in ascribing many errors committed by students to this tendency to reason about rational numbers as if they were natural numbers. Many reports of children's errors demonstrate this point, such as the statements that $18/27 < 18/30$ because $27 < 30$ (Pearn & Stephens, 2004), that $7/8 + 12/13$ is approximately 19 or 21 rather than 2 (Carpenter, 1981), or that $5/6 = 7/8$ "because each has one left" (Clarke & Roche, 2009).

Errors in reasoning about rational numbers extend beyond those related to fraction ordering, comparison, and operations. Rational and natural numbers differ also in other aspects such as their *density*. Rationals are dense in the number line, in the sense that between any two rational numbers there is always another rational number. Density has proven a very difficult property to learn, probably because it goes against the well-known intuition obtained from natural numbers that every number has a unique successor so that there are no other numbers in between (e.g. Vamvakoussi & Vosniadou, 2004).

CONGRUENCY EFFECTS AND THE NATURAL NUMBER BIAS

Several studies about the NNB have focused on a specific type of bias, often named *congruency* or *consistency* (e.g. Gómez, Jiménez, Bobadilla, Reyes, & Dartnell, 2014; DeWolf & Vosniadou, 2015; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; Vamvakoussi, Van Dooren, & Verschaffel, 2012; Van Eeckhoudt, 2013; Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013). To the best of our knowledge, the idea of congruency first appeared in Ischebeck, Schocke, and Delazer's (2009) neuroimaging study of fraction comparison. In this context, congruency was taken to represent a specific type of processing, namely when a person reasons that the magnitude of a fraction is positively correlated to the magnitudes of its numerator and denominator, "*larger components, larger fraction*" (e.g. Gómez et al., 2014). More specifically, a pair of fractions to be compared is called *congruent* if the larger fraction has the larger numerator and denominator, *incongruent* if the larger fraction has the smaller numerator and denominator, or *neutral* otherwise.

The congruency of a pair of fractions has proved to be a significant predictor of children's average accuracy in comparing these fractions (Gómez et al., 2014; Van Hoof et al., 2013). Even response times from adults' and expert mathematicians' judgments seem to align with congruency (Obersteiner et al., 2013; Vamvakoussi et al., 2012), although some studies suggest a more complex scenario. Gómez and Dartnell's (2015) meta-analysis of a series of fraction comparison tasks reported that predictions based on congruency were well supported by the data when the fractions

to be compared shared a common numerator or denominator, but were not when these had no common component. Several studies actually show a reversed congruency effect in this case (DeWolf & Vosniadou, 2015; Gómez & Dartnell, 2015), raising important questions about the nature of the NNB.

CONGRUENCY AS A FORM OF COMPONENTIAL REASONING

Ischebeck et al. (2009) introduced the notion of congruency as a manner of classifying fraction comparison items depending on the applicability of the basic componential strategy described above. Componential processing (also known as segmental processing, e.g. see Barraza, Gómez, Oyarzún, & Dartnell, 2014; Gabriel, Szűcs, & Content, 2013; Meert, Grégoire, & Noël, 2009) refer to strategies that make use of the values of the fractions' components with no regard of the fractions' magnitudes. It is worth noticing that segmental or componential processing of fractions is far from inappropriate, as many sound reasoning strategies rely purely on the components of the fractions to be compared, such as cross multiplication (e.g. reducing the comparison of $\frac{3}{7}$ and $\frac{4}{9}$ to that of 3×9 and 4×7) or transition through a fraction with common components (e.g. comparing $\frac{2}{7}$ and $\frac{3}{5}$ by noticing that $\frac{2}{7} < \frac{3}{7}$ and $\frac{3}{7} < \frac{3}{5}$). From this perspective, reasoning by congruency may be understood as a naïve, extreme form of componential reasoning.

MANY PATTERNS OF REASONING MAY HIDE BEHIND A CONGRUENCY EFFECT

Gómez and Dartnell (2015) distinguished two—sometimes confounded—meanings of NNB and congruency effect. The first meaning refers to the data pattern where congruent items are easier than incongruent items, whereas the second meaning points to the putative cognitive mechanism underlying such data patterns.

Recent work by our group (for a summary, see Gómez & Dartnell, in press) explored the patterns of reasoning of a sample of about 500 middle school children by means of a clustering analysis, where children's scores in a fraction comparison task were sorted into different groups based on a measure of similarity. The results showed the presence of at least five groups, whose patterns of answers were not always predictable by congruency. For instance, 9% of the children answered in a manner consistent with a “*smaller denominator, larger fraction*” strategy, and 8% of them answered in a manner consistent with a “*smaller components, larger fraction*” strategy (also reported in qualitative research, see Stafylidou & Vosniadou, 2004). Notice that the answers of the latter group are the exact opposite of those predicted by the congruency account.

Findings from meta-analysis (Gómez & Dartnell, 2015), clustering (Gómez & Dartnell, in press), and from specific populations (DeWolf & Vosniadou, 2015) lead to questions about the nature of the congruency effect. If congruency represents a deeply ingrained, psychological bias, how can some participants exhibit a reversed pattern? The congruency effect for fraction pairs without common components may well be explained by the use of particular strategies such as those described above, rather than a specific bias towards natural numbers when operating with fractions.

ON THE PRESENCE OR ABSENCE OF COMMON COMPONENTS

Research has shown that the presence or absence of common components influences learners' choice of cognitive strategies to compare fractions (e.g. Meert et al., 2009) as well as the associated neural activity (Barraza et al., 2014; Ischebeck et al., 2009). The presence or absence of common components seems to also play an important role in the expression of the NNB while comparing fractions. Pairs with a common numerator or denominator show robust congruency effects (Gómez & Dartnell, 2015), whereas the evidence for fraction pairs without common components is far from conclusive and it seems to point towards a diversity of strategies rather than to congruency (e.g. DeWolf & Vosniadou, 2015; Gómez & Dartnell, in press).

Even in the case of fraction pairs with common components, the question arises as to whether the congruency effects are due to a specific interference from natural numbers onto rational numbers (a proper NNB), or if cognitively simpler hypotheses may account for the congruency effects. Comparing fractions with common components may elicit a Stroop-like effect, in which pairs with the same denominator may be easier than pairs with the same numerator because in the former case both the larger components and the larger fraction coincide, whereas in the latter case the larger components are associated with the smaller fraction. Although this interference can be interpreted as a NNB, it probably stems from much lower-level mechanisms of cognitive control, for instance those involved in answering the Flanker task (Eriksen & Eriksen, 1974). A usual variant of this task asks participants to decide if the central arrow in stimuli like $\rightarrow\rightarrow\rightarrow$ or $\leftarrow\rightarrow\leftarrow$ points to the left or to the right. This lower-level account suggests that the symbols used to represent fractions may have a relevant role in eliciting a NNB-like effect, in a mixture of perceptual and cognitive—rather than purely cognitive, as suggested by the NNB account—mechanisms. This hypothesis is partly supported by Kallai and Tzelgov's (2012) training study, where the use of arbitrary visual symbols for unit fractions (instead of the usual natural-number-based symbols) improved participants' representation of fraction magnitude.

CONCLUSIONS

In this contribution, we have made the case that the cognitive substrates of the congruency effect in fraction comparison are far from fully understood. More research is needed to understand the scope of congruency effects and their cognitive origin, whether a primitive Stroop-like bias towards low-level information, cognitive interference from the domain of natural numbers, or an epiphenomenon due to common but diverse strategies used by learners. In this endeavor, it is important to keep in mind not only the similarities but also the peculiarities of the many documented manifestations of congruency effects and the NNB. In order to move forward, a detailed understanding of these nuances is crucial for distinguishing the different underlying cognitive mechanisms and their educational relevance.

Acknowledgements

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THE PERSISTENCE OF THE NATURAL NUMBER BIAS BEYOND LOWER SECONDARY SCHOOL

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This section reviews empirical evidence for the persistence of the natural number bias beyond primary and lower secondary school. Recent studies involved older students, educated adults and academic mathematicians. They mainly focused on two types of rational number problems, namely fraction comparison and judging the effect of arithmetic operations. In both problem types, the natural number bias appears to persist until adulthood. Under certain conditions, high mathematical expertise enables people to be completely unaffected by the bias. In conclusion, it seems that the occurrence and the strength of the bias depend on task-related factors, individual factors, and the interaction between them. Theoretical implications of these findings and perspectives for future research are discussed.

THEORETICAL BACKGROUND

Research has documented that many students show a natural number bias when working with rational numbers (e.g., Ni & Zhou, 2005). Students persistently rely on their knowledge about natural numbers even if it is not applicable. The phenomenon that a previously acquired concept can hamper learning of a new concept can be described by the conceptual change approach (Vamvakoussi & Vosniadou, 2004). As learning of rational numbers requires reconceptualising what numbers are and how they “behave”, it is not surprising that students make mistakes during the early phase of learning of rational numbers. It is, however, an interesting question how persistently people rely on their natural number knowledge when working with rational numbers long after they have learned the concept of rational numbers. To address this issue, researchers have investigated whether older students and adults with good mathematical knowledge, who are likely to have no misconceptions about rational numbers, show a natural number bias. Showing a bias means to struggle more with incongruent than with congruent problems. Incongruent problems are problems in which relying on knowledge of only natural numbers leads to an incorrect response (i.e., $1/3 > 1/4$ although $3 < 4$), whereas congruent problems are problems in which relying knowledge of only natural numbers leads to the correct response (e.g., $4/5 > 2/5$ and $4 > 2$).

The question of whether even people with good mathematical knowledge show a natural number bias can contribute to our understanding of the cognitive mechanisms underlying a bias: If people with high mathematical knowledge do not show a bias, then insufficient conceptual knowledge might be the source of the bias, and learning the concept might prevent people from being biased. If, on the other hand, even people with good mathematical knowledge show a bias, then the bias must be deeply rooted in the cognitive system and cannot be overcome just by understanding the concept. In this case, intuitive mechanisms that interfere with analytic mechanisms might be the source of the bias. In fact, many researchers have shown that intuition can guide mathematical problem solving (Fischbein, 1987), and dual-process accounts provide a theoretical framework for the interplay between intuitive and analytic reasoning processes (e.g., Guillard, Van Dooren, Schaeken, & Verschaffel, 2009).

Investigating the natural number bias in educated adults and experts has implications for education. If even educated people show a bias, then the bias must be deeply rooted in the cognitive system, and instruction should—in addition to teaching conceptual understanding—make students aware that even educated people show a bias, and that they should be careful with making quick decisions on specific problems. Teaching conceptual knowledge alone might not prevent students from making typical mistakes.

EMPIRICAL EVIDENCE FOR THE PERSISTENCE OF THE NATURAL NUMBER BIAS

There is empirical evidence that not only students who are just learning about rational numbers but also secondary school students and even educated adults show a natural number bias. While initial studies have investigated adults' problem solving behaviour in a variety of problem types (Vamavakoussi, Van Dooren, & Verschaffel, 2012), further studies have put particular focus on two problem types, namely fraction comparison and judging the effects of arithmetic operations. In the following, the focus will be on these two problem types.

The natural number bias in fraction comparison

In fraction comparison, people are asked to pick the larger out of two fractions. Van Hoof, Lijnen, Verschaffel, and Van Dooren (2013) found that students in their first and fifth year of secondary school showed a natural number bias. That is, these students tended to consider the fraction with the larger components as the larger fraction, although this is not always true. Meert, Grégoire and Noël (2010) and DeWolf and Vosniadou (2011) found the natural number bias in university students, among whom were students at a highly selective university. Obersteiner, Van Hoof and Verschaffel (2013) studied academic mathematicians, using a more strictly controlled set of fraction comparison problems. In spite of extremely high accuracy, these mathematicians showed a natural number bias in terms of response times when the fractions had common components (i.e., common numerators or common denominators), but not when the fractions had no common components. This result suggests that the occurrence of the bias depended on problem type. Further analysis

suggested that people use different strategies for different problem types: While for fraction comparisons without common components the participants predominantly used holistic strategies that take into account the fraction magnitudes, they predominantly used component strategies that rely on the fraction components for comparison problems with common components (Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013). Similarly, DeWolf and Vosniadou (2015) found the natural number bias in a sample of university students, but a bias in the opposite direction in another sample of university students, suggesting that individual factors might influence the occurrence of the bias. Moreover, in their study, the influence of the fractions' natural number components on the participants' comparison strategies was stronger for fraction pairs with small numerical distances compared to fraction pairs with larger numerical distances. As it is easier to rely on holistic comparison strategies when the distance between fractions is large than when it is small, this result is in line with the assumption that the occurrence of the bias depended on whether the participants used holistic comparison strategies (for large distance fraction pairs; less bias-prone) or component strategies (for small distance pairs; more bias-prone).

In conclusion, the occurrence of the natural number bias in fraction comparison might depend on problem type (i.e., whether the fractions have common components or not, and on the numerical distance between fractions), and on the strategies that people use to compare the fractions (i.e., holistic or component strategies). Importantly, these two aspects seem to be related: People are more likely to apply component strategies when comparing fractions with common components and when the numerical distance between fractions is small than when comparing fractions without common components and when the numerical distance is large (see Alibali & Sidney, 2015). Recent eye tracking studies corroborate the relation between problems with or without common components and strategy use (Huber, Moeller, & Nuerk, 2014; Ischebeck, Weilharter, & Körner, 2015; Obersteiner & Tumpek, 2015).

The natural number bias in judging the effect of arithmetic operations

There is also evidence that intuitions about the effects of arithmetic operations can be very persistent. This aspect of the bias means that while people readily agree that addition and multiplication make a number larger and subtraction and division make a number smaller, they are more reluctant to agree that addition and multiplication can make a number smaller, and subtraction and division can make a number larger. Van Hoof, Vandewalle, Verschaffel, and Van Dooren (2015) found that school students show a natural number bias not only immediately after they have learned about rational numbers. In paper-pencil tests, there was evidence for the bias in students of grades 8, 10, and 12, which is at the end of secondary school. Using computerized experiments, Vamvakoussi, Van Dooren, and Verschaffel (2013) found the natural number bias with respect to both accuracy and response times in a sample of university students. Siegler and Lortie-Forgues (2015) asked secondary school students, pre-service teachers and university students majoring in mathematics and science to judge the effect of arithmetic operations. While the secondary school students and the pre-service teachers

showed a clear bias, the mathematics and science majors did not, suggesting that high mathematical expertise allowed these students to be unaffected by the natural number bias. Obersteiner, Van Hoof, Verschaffel, and Van Dooren (2015) reported similar evidence: In their study, eighth-grade students and expert mathematicians evaluated the effects that multiplication and division have on fractions. Items were presented in algebraic notation as inequalities. While the eighth-graders showed a clear natural number bias, the expert mathematicians showed no traces of a bias, neither in terms of accuracy nor response times.

To summarize current evidence, it appears that the natural number bias in tasks that address the effects of arithmetic operations might depend on an individual's mathematical expertise. While secondary school students and educated adults without specific mathematical expertise seem to be persistently biased, only adults with high mathematical expertise, such as mathematics majors at the university level and academic mathematicians do not show a bias. As Obersteiner et al. (2015) suggest, mathematically experienced people might use a different solution approach than less experienced people, at least when problems are presented in algebraic notation format. Rather than relying on primitive conceptualizations of arithmetic operations (e.g., addition as putting together, multiplication as repeated addition), they might rely on more structural and abstract conceptualisations of numbers and operations. For example, when asked whether it is possible that multiplying a given positive number a by any number x can result in a number larger than a , they might rely on algebraic rules to solve the inequality $a \times x > a$.

DISCUSSION AND PERSPECTIVES

There is empirical evidence that the natural number bias does not only occur in primary and lower secondary students during the phase of learning of rational numbers, but that it can persist until higher secondary school and adulthood. Moreover, the natural number bias can occur even in individuals with high mathematical knowledge. In conclusion, current evidence suggests that task-related factors and individual factors determine whether and how strongly people experience a natural number bias. It seems more likely to be affected by the natural number bias when the given problem triggers processing natural number magnitudes, such as in fraction comparison with common components, than when processing natural number magnitudes is not sufficient to solve the problem, such as in fraction comparison without common components (which often requires processing the fraction magnitudes). In problems that allow a solution strategy that does not essentially rely on natural number reasoning altogether (e.g., algebraic inequalities that address the effects of arithmetic operations), alternative strategies such as algebraic reasoning can allow an individual to be completely unaffected by the natural number bias.

In view of the evidence that even educated adults and sometimes expert mathematicians show a natural number bias, the negative connotation that is often associated with the term “bias” seems unjustified. Rather, the specific representation

of a task might determine whether and how strongly people engage in natural number processing. Such kind of processing seems to be natural, and—moreover—allow particularly effective processing strategies in some cases (e.g., in fraction comparison with common components).

There are several open questions that should be addressed in future research. First, while current evidence suggests that the natural number bias depends at least partly on perceptual features of the task at hand, the interplay between immediate, perception-based cognitive processes and deeply rooted cognitive dispositions is not clarified yet. This interplay could be disentangled by systematic experimental variations of both mathematical expertise and problem representation formats within the same experiment. Second, as the natural number bias seems to depend on the readiness with which people are able to activate magnitude representations of fractions, it would be interesting to see whether the bias can be completely overcome through intensive practice with selected fractions. As Kallai and Tzelgov (2012) suggested, specific training could lead to more holistic processing of fractions with less interference of the fraction components. Third, as conceptual understanding of natural numbers is both a precondition and an obstacle in learning of rational numbers, future studies could more systematically analyse the relationship between children's natural number understanding and the extent to which they show a natural number bias. Such research could inform educational practices by identifying the ideal point in developmental time at which children should be introduced to rational numbers.

ON NATURAL NUMBERS AND FRACTIONS: A REORGANIZATION (NO BIAS) STANCE

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In this paper I contend that a reorganization stance (Steffe & Olive, 2010) can better explain the relationship between natural numbers and with fractions than a natural number bias (NNB) stance. I focus on three facets of this relationship: (a) the lack of distinction between adults' first order models and children's ways of knowing, (b) participatory and anticipatory stages in reorganizing natural numbers into fractional schemes, and (c) the typical, teaching-born, inadequate conceptual foundation of children's fractional reasoning. These three facets underlie my claim that while the body of empirically sound research, rooted in a bias stance, reveals a valuable phenomenon—it falls short of articulating its cognitive and/or pedagogical sources.

NNB: A MANIFESTATION OF ADULTS' FIRST ORDER MODELS

Much research has focused on student difficulties to properly reason/operate with fractional quantities (Verschaffel, Greer, & DeCorte, 2007). One line of work pointed

out how students' knowledge of natural numbers seems to alter their understanding of and operations with fractions (Behr et al., 1984). Streefland (1991) portrayed the relationship as *interference* of natural numbers knowledge in fraction learning. In recent years, the notions of *whole number bias* (Ni & Zhou, 2005) and *natural number bias* (NNB, Van Hoof et al., 2015) have gained traction.

Van Hoof, Verschaffel, & Van Dooren (this volume) provide an excellent example of meticulous, empirically grounded interpretation of data that, they claim, supports the NNB. They specified three main aspects of NNB (examples taken from their paper). The first, *Size*, refers to ordering of fractions (e.g., determine which is larger, $\frac{3}{9}$ or $\frac{2}{5}$). The second aspect, *Operations*, refers to carrying out and determining results of arithmetical operations (e.g., is $50 \times \frac{3}{2}$ smaller or larger than 50?). The third aspect, *Density*, refers to placing more fractions between any two given fractions (e.g., write a number between $\frac{1}{4}$ and $\frac{3}{4}$). Test items they considered to be congruent (e.g., "Write a number between $\frac{1}{4}$ and $\frac{3}{4}$ ") or incongruent ("Write a number between 3.49 and 3.50") were used to demonstrate how congruency interacted the least with size tasks, somewhat larger in operation tasks, and the most with density tasks.

To me, the notion of NNB (with size, operations, and density as three main aspects) manifests the researchers' *first order model* of the mathematics involved. This claim draws on Steffe's (1995) distinction between two types of mathematical models people may have and use. Contrasted with one's own mathematics as her or his first order model, a *second order model* refers to how one makes sense of someone else's mathematics, that is, of another person's first order model.

Mathematics educators' mature first order models of natural numbers and fractions allow them to make sophisticated distinctions. To researchers versed in mathematics the differences, say, between density of natural and fractional numbers, or between congruent and incongruent test items, seem "painfully obvious." At issue when *researchers interpret* findings about how students solve fractional tasks is not how, from our mature frame of reference, students inappropriately solved such tasks. Rather, our challenge is to articulate how, *from a student's conceptual frame of reference as inferred by the researchers*, her or his solutions do make sense. I contend that, quite often, the notion of 'bias' manifests (a) researcher interpretations based on their first order models and (b) a fundamental lacuna of second order models (SOM) for articulating students' conceptualization of natural and fractional numbers.

SOM: STAGES IN REORGANIZING FRACTIONAL UNITS/OPERATIONS

In this section I briefly describe key components of a framework I find useful for making second order models (SOM) of students' reasoning. The first component is von Glasersfeld's (1995) three-part notion of scheme: (a) an assimilatory *situation* sets one's goal, (b) an activity triggered to accomplish the goal, and (c) a result. In regards to 'result', Simon et al. (2004) proposed the notion of *effect*, to distinguish what a learner notices after carrying out the activity from an effect anticipated before it. For

example, knowing natural numbers (say, ‘3’) entails anticipating the effect of iterating a unit of 1 three times while coordinating it, 1-to-1, with number words.

The second component is Simon et al.’s (2004) elaboration on the notion of *reflective abstraction* to specify how existing schemes are reorganized into new ones, a process termed *Reflection on Activity-Effect Relationship*. This process consists of two types of mental comparisons. *Reflection Type-1* involves comparing between an anticipated and an actual effect of the activity; *Reflection Type-2* involves comparing across records of recurring activity-effect dyads and thus, possibly, constructing a novel, invariant activity-effect relationship. For example, when just beginning to learn about unit fractions ($1/n$), children’s natural numbers entail that to share a given whole (say, a paper strip considered as French Fry) among 5 people each person’s share would be larger than when sharing among 4 people. By iterating one person’s share, however, *they* can soon notice the need to reverse the size relationship, because to fit 5 shares within the same whole (one share = $1/5$) each share has to be smaller than $1/4$ (Tzur & Hunt, 2015). This account clearly acknowledges initial, wrong use of natural number reasoning to solve fractional tasks. Yet, instead of calling it ‘bias’, using natural number reasoning, which *they do know*, is embraced as a starting point to a reorganization process where *their* anticipated effect proves *inadequate to them*.

Within this reflective process, two stages in the reorganization of existing into new schemes were postulated (Simon, Placa, & Avitzur, 2016; Tzur & Simon, 2004). In the first, *participatory stage* a learner’s access to an evolving activity-effect relationship is prompt dependent. Unless prompted, a learner’s only recourse when solving tasks is to access and use schemes established previously as prompt independent. At that second, advanced, *anticipatory stage* a learner is cementing the linkage among the three parts of her new scheme and can thus use it, spontaneously and independently, to appropriately solve tasks. Learners at the participatory stage of constructing a new scheme are bound to *fold back* (Pirie & Kieren, 1994) to using schemes at the anticipatory stage. Rather than ‘bias’ I use folding back, as ‘bias’ connotes preference of one way of reasoning over another that’s also available, while folding back embraces the use of available schemes. Simply put, at the participatory stage of constructing *any* new scheme through reorganizing previous, anticipatory schemes, folding back to the anticipatory scheme is a developmental necessity—not a ‘bias’. Test items not sensitive to measuring the participatory stage would, by default, reflect reasoning based on natural numbers.

The third component of the framework pertains to units and operations that constitute children’s schemes. The foundational unit in students’ reasoning with numbers is composite unit (Steffe, 1992). Initially, a natural number is a unit composed through iteration of 1s. Later, children construct schemes for composing or decomposing numbers into other numbers (e.g., composing 5 as $2+3$; decomposing 6 into 3 units of 2 each). Reorganization of such schemes into fractional units can then build *not on the numerical value of composite units per se*, but on the activity of iteration that gave rise to natural numbers (Steffe & Olive, 2010). To equally partition a unit of 1 into n parts

(say, 5), a child may estimate the size of one part and iterate it 5 times to compose an iterated whole equal in size to the given 1. Thus, instead of thinking of a unit fraction as one-of- n -equal-parts of the whole, this reorganization process fosters thinking of unit fractions as multiplicative relations ($1/5 =$ a unit that 1 is 5 times as much of it). Later, iteration of unit fractions fosters construction of non-unit fractions ($m/n = m \cdot 1/n$) that can be composed/decomposed multiplicatively ($2/7 = 1/7 + 1/7$; $8/7 = 4 \cdot 2/7$). Advanced schemes are then constructed based on a recursive partitioning operation, such as $1/7$ of $1/5$ of One is $1/35$ because the One is 35 times as much of it. This way, inappropriate use of natural number reasoning diminishes as fraction schemes are constructed at the anticipatory stage (Tzur & Hunt, 2015).

TEACHING-BORN, INADEQUATE FRACTION FOUNDATIONS

The last facet of a reorganization stance pertains to typical ways of teaching fractions. Here, I do not focus on the obvious need to reform traditional, show-and-tell practices, but rather on the need to overhaul two prevalent aspects of teaching fractions. The first aspect is the reliance on ‘part-of-whole’ as a chief, often sole meaning for fractions. This reliance, which builds on numerical values of natural numbers, seems to limit students’ reasoning and to underlie much of their folding back to natural number reasoning (e.g., adding $a/n + b/n = (1+b)/(n+n)$, or saying that $1/6 > 1/5$ because $6 > 5$). Instead, instruction should provide foundations for reasoning with fractions multiplicatively. Specifically, to make sense of fraction density, teachers need to develop ways for fostering students’ construction of anticipatory schemes consisting of recursive partitioning activities applied to unit and non-unit fractions (e.g., anticipating that between 3.49 and 3.50 one can locate infinitely many other numbers, because the $1/100$ difference can be recursively partitioned at will). My recent studies on how teaching can foster fractional schemes of unit fractions as multiplicative relations, with students designated as having learning disabilities (Tzur & Hunt, 2015) or with adults/teachers (Tzur, 2015), showed the power of this stance.

The second aspect of my advocacy for a reorganization stance pertains to the way teachers understand and thus teach fractions to their students. Over two decades of work as a mathematics educator with hundreds of teachers, as well as relevant research in our field (Izsák et al., 2012), have convinced me that, by and large, teachers’ reasoning about fractions is limited to meaningless execution of procedures and algorithms. I hold that a teacher’s mathematics serves as an “upper-limit” of what students may learn. Thus, it is no wonder for me that research studies repeatedly demonstrate inappropriate use of all sorts of solution strategies—natural numbers included. Since I first read Erlwanger’s (1973) study of “Benny,” I realized how deep and wide a pedagogical (paradigm) shift is needed for teachers and students alike to construct anticipatory schemes that support robust, multiplicative meanings for fractions. A research project¹ I recently began directing was designed to promote and study such a shift in all grade 3-5 teachers in five schools. If previous, small-scale studies are of indication—I have a good reason to expect this project can demonstrate how such teacher professional development help diminish what some call NNB.

CONCLUDING REMARKS

In this paper I argued for a reorganization stance on the relationship between natural and fractional numbers in place of a ‘bias’ stance. I discussed three main facets of the two stances, including (a) researchers’ use of their first order models to interpret their findings, (b) an alternative framework that can underlie creating and using second order models of students’ ways of reasoning about both number types (including the participatory stage and conceiving of fractions as multiplicative relations), and (c) the crucial role current teaching-learning practices serve in producing inappropriate use of natural numbers reasoning to solve fractional tasks.

Combined, these three facets point to what seems at stake when adhering to NNB. Specifically, a ‘bias’ stance seems to focus on demonstrating *that* inappropriate use of natural number knowledge exists through measuring it in different contexts and situations—but not on *explaining why, when, and for whom* such use occurs. Thus, a ‘bias’ stance seems to also underlie recommendations for making teachers aware of the ‘bias’ (see Van Hoof, Verschaffel, & Van Dooren, this volume). I contend that an alternative, reorganization stance is better slanted for providing specific guidance as to (a) plausible causes for people’s folding back from fractional to natural number reasoning, (b) teaching-learning processes conducive to diminishing such folding back, and (c) how to design professional development efforts that can bring about a momentous pedagogical change rooted in teachers’ deep understanding of how students come to conceptualize fractions.

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CLASSROOM INTERVENTIONS TO OVERCOME THE NATURAL NUMBER BIAS

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Fractions are difficult to learn. One of the main obstacles is the Natural Number Bias (NNB). In this section, I review two classroom interventions designed to improve fraction understanding, and I suggest that, to overcome the NNB, children would benefit from teaching methods that focus on the conceptual understanding of the magnitude of fractions.

INTRODUCTION

Fractions are difficult to understand and children usually dislike learning about them. Developing an understanding of what fractions and what you can do with them is a fundamental stepping stone for learning higher concepts in mathematics, and competency with fractions predicts children's performance in algebra and their general mathematical achievement in later years (Siegler et al., 2012). Conversely, difficulties in learning fractions can lead to mathematics anxiety and affect opportunities for further engagement in mathematics and science. Therefore, it is crucial for children's mathematical development that teaching instils a proper understanding of fractions.

Children's knowledge of natural numbers can interfere with their efforts to learn fractions. However, I share the view of Vamvakoussi and colleagues (2012) that the natural number bias (NNB) itself is not in itself a bad thing; rather, it is the expected result of years of cumulative learning to build knowledge about natural numbers. Nevertheless, the misapplication of natural number rules because of the NNB still poses a road block to the successful understanding of fractions. To mitigate against this, we should pay special attention to the NNB when teaching fractions. This can be done by explicitly drawing pupils' attention to the discrepancies between the properties of natural numbers and fractions, and by being especially explicit about the connections between fractional notations and magnitudes. In this section, I discuss principles that could be used in the classroom to directly address the core difficulties that hold back children's understanding of fractions.

UNDERSTANDING MAGNITUDES

Being able to use fractions requires both procedural and conceptual knowledge. Analyses of current teaching practice revealed a great variety of methods for teaching fractions, often with more focus on procedures than concepts (Charalambous & Pitta-Pantazi, 2007; Gabriel et al., 2013). Further to this, studies have also found that the teaching of fraction concepts was often too narrowly focused on a part-whole interpretation rather than emphasising the fact that fractions are numbers with their own magnitudes (Fuchs et al., 2013). This poses a major problem, as in order to develop an understanding of fractions, students must first appreciate that fractions are not just constructs of two distinct natural numbers but are numbers in and of themselves (Lamon, 1999).

Time and again, evidence from cognitive neuroscience and education has shown that being able to understand the magnitude of fractions is an essential and unavoidable stage in the general understanding of fractions (Siegler, Fazio, Bailey, & Zhou 2013). The level of students' understanding of fraction magnitude is a strong predictor of their competence in algebra (Booth, Newton, & Twiss-Garrity, 2014), and it has been positively related to overall mathematics achievement in countries with cultural and educational practices as diverse as the USA, Belgium and China (Torbeyns, Schneider, Xin, & Siegler, 2015). The appreciation of magnitude is one of the hardest concepts to learn, and can be seen as a consequence of children's failure to understand that natural

numbers and fractions have different properties and characteristics. Indeed, some researchers argue that NNB-related mistakes arise when the mental representations of fraction magnitude are not sufficiently strongly activated or are not precise enough (Alibali & Sydney, 2015); if it is the case, we would expect teaching experiences that help build and strengthen the mental representation of fraction magnitude to lead to improved performance.

CLASSROOM INTERVENTIONS

One way of addressing the NNB-related difficulties that children have when learning fractions is to ensure that they have a solid conceptual understanding of magnitudes. This would give children a firm grasp of what fractions actually mean, a good feel for how procedures on fractions should work, and thus a better ability to catch their own mistakes and to avoid making procedural errors.

This idea has been tested experimentally with children of different age groups in different countries. In the following paragraphs, I will describe two intervention studies, one from our group (Gabriel et al., 2012) and one from Fuchs and colleagues (2013). These studies both introduced adapted teaching regimes that shifted the focus from teaching procedures to teaching the concept of magnitude.

In our study, we designed and conducted an intervention for French-speaking Belgian primary school children, aged between 10 and 11 (Gabriel, et al. 2012). We tested 292 pupils from 4 different schools for 12 weeks. Within each school, we tested four whole classes and randomly assigned two of these classes to the intervention while keeping the other two as controls. The control classes followed their traditional lessons which focused on the part-whole concept and on rote learning of procedures, whereas the intervention group instead followed lessons designed by our research group.

The intervention was based on the use of the concrete-representational-abstract sequence (CRA) and excluded procedural instruction. CRA has been shown to be effective for developing conceptual understanding in several areas of mathematics, including fractions (Butler et al., 2003; Maccini & Hughes, 1997). In this schema, the *concrete* element of the sequence was a set of wooden disks, cut into a range of sizes from halves to twelfths. Pupils used the disks as a reference to manipulate quantities and visualise the sizes of fractions. The *representational* element was a set of cards displaying different representations of fractions (see Figure 1). The *abstract* element was the step of getting children to consider fractions as numbers in and of themselves.

In the process of designing this intervention, a common impression that we received from the children was that they strongly disliked fractions. Play is known to have positive effects on the learning process (Sawyer, 2006), and so in our design, we used play to try to keep the children motivated and engaged by associating fractions with fun.

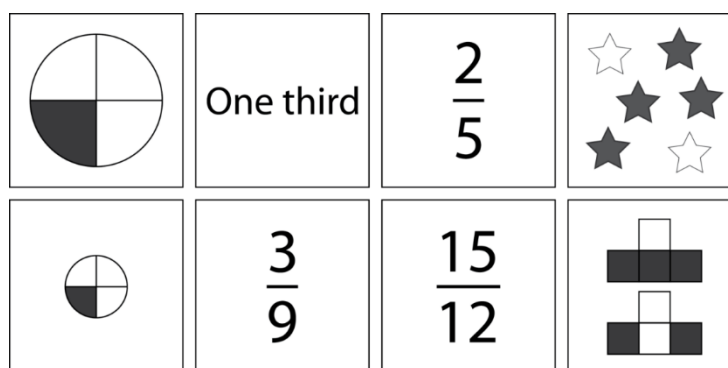


Figure 1: Examples of card faces used in the intervention.

The intervention was conducted jointly by researchers and teachers. There were twenty game sessions of 30 minutes each. Children were split into small groups of 3 to 5 and played different card games (Memory, War, Old Maid, Treasure Hunt, and Blackjack) adapted to use our fraction cards (see Figure 1). These games required the children to estimate, compare, and combine fractions represented either symbolically or as figures. Over the course of the intervention, we gradually introduced more complicated fractions, equivalent fractions, improper fractions and a wider breadth of representations of fractions.

We tested all of the pupils before and after the intervention. From this we were able to measure the change in conceptual and procedural knowledge of fractions, and to assess the impact of the intervention. The intervention led to a 15-20% improvement in conceptual understanding of fractions (i.e. estimating and comparing fractions, and placing fractions on a number line). After the intervention, fewer NNB-related errors were observed in the intervention group compared to the control group. For example, for questions where the pupils had to compare fractions, the intervention group made 16% fewer NNB-related mistakes in the post-test compared to the pre-test, whereas the control group only showed a 6% reduction. The children in the intervention group were able to use their conceptual knowledge to perform simple additions with familiar fractions, but they did not show wide-scale transfer from conceptual to procedural knowledge. Conversely, children in the control group, who received traditional procedure-heavy lessons, improved in procedural skills such as simplification of fractions, but showed no improvement at all in their conceptual understanding.

Fuchs and colleagues (2013) conducted a similar study with 9 and 10 year-old American children. Unlike our study, Fuchs and colleagues (2013) focused their intervention on students with poor mathematics performance. They assigned children to three different groups: an intervention group of poor-performing students ($n=129$); a control group of students of similar performance to the intervention group ($n=130$); and a target group of normal-to-high performing students to gauge the scale of improvement in the intervention group ($n=282$). The intervention and control groups were sampled from children who scored in the bottom 35% of a mathematics

achievement test (Wide Range Achievement Test–4; Wilkinson & Robertson, 2006), and the target group was randomly sampled from the rest of the children.

Similarly to our study, their intervention focused on magnitude (i.e. representing, comparing and placing fractions on number lines), and the control and target groups received instructions which focused on procedures and on the part-whole interpretation of fractions. Their intervention relied on the same theory as ours, but the content and learner activities differed greatly. Children in the intervention group were taken out of their regular classes and put into small groups of three, each group having its own instructor. The intervention was based on CRA, but with far fewer elements of play. The first 22 lessons of the intervention were devoted to teaching concepts with a very heavy emphasis on magnitude, and the remaining 5 lessons were devoted to procedures. Using a very similar battery of pre- and post-tests to ours, they found that their intervention led to increased improvements in both conceptual and procedural knowledge of fractions over the traditional lessons, and they concluded that this was mediated through the improvement in the understanding of magnitude.

These studies show that gains can be made in improving children's conceptual understanding of fraction magnitude, which is a crucial step in reducing NNB-related errors. They also show that it is necessary to include both conceptual and procedural instruction, and the optimum combination is likely to include a greater emphasis on concepts. Further research will be necessary to clarify when and how such interventions can best improve the understanding of the magnitude of fractions, how, in turn, this can reduce the NNB, and how this can impact performance in fraction arithmetic and in mathematics in general.

CONCLUSION

The NNB-related difficulties that children have when learning about fractions can be largely overcome by ensuring that they understand that fractions are numbers that have magnitudes. In the classroom, it is seemingly straightforward to simply increase the emphasis on conceptual knowledge in lessons on fractions, with a focus on magnitude. The challenge here is to balance the need to teach children the procedural knowledge for using fractions with making sure they are instilled with the conceptual knowledge to truly understand how, when and why to use fractions. Teaching methods that incorporate CRA and play appear to be an efficient way to introduce fraction conceptual knowledge to children, pushing them to actively construct knowledge in a way that is meaningful to them.

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TEXTBOOK SIGNATURES: EXPLORING POSSIBILITIES

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AIM AND RATIONALE

Analysing mathematics textbooks can provide a comparison of learning opportunities afforded by the use of textbooks among different countries. As many teachers tend to rely on textbooks as a resource for teaching, textbooks play an important role in shaping what mathematics is taught in schools. Furthermore, as Charalambous, Delaney, Hsu, and Mesa (2010) suggest, textbooks specify how learning experiences may be structured in a mathematics classrooms. A cross-national textbook analysis can thus potentially offer insights into curriculum intents and suggested teaching approaches in the different countries. However, comparing textbooks across different countries poses several challenges, especially when it comes to representing these analyses to surface the patterns or observations of the different textbooks. Following the idea of *lesson signature* as suggested by Hiebert et al. (2003), Charalambous et al. (2010) propose that textbooks within the same country may have a “textbook signature”—“uniform distinctive patterns”—in the textbooks (p. 146).

To test the existence and feasibility of textbook signatures, we proposed a notion of textbook signature and attempted to characterise our analyses of textbooks in gradient (Choy, Lee, & Mizzi, 2015) and fractions (Lee, Choy, & Mizzi, 2016) using it. In our work, we have found the notion of textbook signatures to be useful in characterising the analysed textbooks, and our analysis suggests that textbook signatures are indeed unique in different countries. Despite a promising start in this area of research, we think that the notion of textbook signatures can be further refined, and several important questions posed by Charalambous et al. (2010) remain unanswered. In this discussion group, we aim to further individual and collaborative research efforts by providing a platform for mathematics education researchers who are interested in textbook analysis to:

- suggest refinements to our current notion of textbook signatures;
- explore how research pertaining to textbook signatures may be advanced; and
- connect with other researchers in order to further research on textbook analysis, focusing on the notion of textbook signatures.

KEY QUESTIONS

The discussion group activities will be guided by the following key questions:

- What are the strengths and areas for improvements of our current notion of textbook signatures?
- What criteria can be included in textbook signatures?
- How can we refine our notion of textbook signatures?

- Do analyses of different topics in the same textbook present different textbook signatures? If so, what are the implications for further research related to textbook signatures?
- Do different textbooks used in the same country generate similar textbook signature for the same topic?
- Do textbook signatures influence how mathematics is taught in the classrooms? How are they related to lesson signatures? (Charalambous et al., 2010)
- How do we draw implications for teaching and learning mathematics through textbook signatures?

KEY DISCUSSION GROUP ACTIVITIES

Session	Duration	Description
1	20 min	Presentation: Current notion of textbook signatures
	30 min	Discussion: Questions 1 to 3
	30 min	Presentation by participants
	10 min	Summing up
2	10 min	Setting the agenda: Directions for future research
	30 min	Discussion: Questions 4 to 7
	40 min	Presentation by participants
	10 min	Summing up

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TOOLS, SIGNS, FORMATION OF CONCEPTS AND LEARNING OF MATHEMATICS

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From a Vygotskian perspective, tools and signs are inseparable parts of the teaching and learning of mathematical concepts. The central purpose of this discussion group is to examine this sign/tool-mediated view of learning and concept formation in relation to Vygotsky's notions of:

1. the emergence of learning within the Zone of Proximal Development (ZPD) and
2. the multifaceted relationships among everyday, pseudoconcepts and scientific concepts in the process of learning (Vygotsky, 1978; Vygotsky, 1986).

The ZPD is described by Vygotsky as the distance between the actual developmental level (independent problem solving) and the level of potential development (problem solving under adult guidance or in collaboration with 'more capable peers') (Vygotsky, 1978, p. 69). For Lerman (2014), the zone of proximal development is 'the mechanism through which learning happens' (p. 22). Following Vygotsky's view, in the field of mathematics education the more knowledgeable others are usually conceptualised as agents such as teachers, adults and peers. Roth and Radford's (2010) notion of the participant 'in the know' – the one whom Wertsch & Rupert (1993) called 'the source of authority' – claims that *the more knowledgeable other* arises through collaborative interaction of the participants in which the role of being the more knowledgeable other alternates among them.

Although the Vygotskians' view of learning within the ZPD has been interpreted and used for decades, the attempt of this discussion group is to revisit the issue by tackling some of the fundamental assumptions of the ZPD such as the notion of the more knowledgeable others. We also further expand the ZPD to include the formation of mathematical concepts by looking at the process of the formation of complexes and pseudoconcepts.

To participate actively in the discussions, the attendees will have access to a range of data across several contexts and age ranges. The over arching issue of exploration will be how mathematical concepts are formed within the ZPD and what is the role of more knowledgeable others, in such learning and concept formation. The following two sets of sub-questions will guide our activities and discussions:

- What is a sign/tool-mediated ZPD? In a ZPD who/what is the more knowledgeable other? How might researchers identify this more knowledgeable-ness within interactions in the ZPD? How might the role of the more knowledgeable others, or more knowledgeable tools, alternate? Is the

development of knowledge within a ZPD bi- (Goos, Galbraith & Renshaw, 2002) or multi- directional?

- How can we best distinguish between everyday and scientific mathematics concepts? How do these two types of concepts relate? In particular can we use the notions of everyday and scientific concepts and/or complex thinking and pseudoconcepts to explain the construction of mathematical concepts? Within the complex process of concept formation, how, might we teach a mathematical concept?

We will devote one session to discussions of a wide variety of data to explore and address the first set of questions. We include data from children's interaction with adults, with each other and with the mathematical tools, from different context and different age groups.

For the second session, in addition to our prepared set of data we invited the participants to bring samples from their own data, by working in small groups we explore and address the second sets of questions.

Points arising from both sessions will be fed back to our concluding remarks, in which we combine the discussion from the two sessions to highlight possible further research and actions.

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OBSERVATIONS ON OBSERVING PEDAGOGY: FURTHER DISCUSSION OF RESEARCHING THE UNOBSERVABLE

Coordinator: David A Reid, Universität Bremen and Acadia University

Assistant coordinator: Richard Barwell, University of Ottawa

THEORETICAL BACKGROUND

Many researchers face the challenge of researching the unobservable in mathematics teaching and learning (e.g., reasoning, beliefs, etc.). The Observing Teachers study (<http://www.acadiau.ca/~dreid/OT/index.html>) makes use of an anthropological methodology (Tobin et al. 1989) to address this challenge, in particular to research middle school mathematics “pedagogy” across regions of Canada. Since our first PME discussion group on this topic (Reid & Barwell, 2014), our discussions about pedagogy have shifted level. We have had a range of experiences attempting to observe the unobservable, namely pedagogy. Project team-members’ interactions with data, as well as with each other, while thinking about pedagogy have raised further methodological questions for discussion. Our original use of “pedagogy” to refer to the implicit cultural practices of teachers that guide teaching practice (akin to what Schoenfeld, 2010, calls “orientation”) has interacted in interesting ways with other uses of the same word in the francophone tradition of *didactique*, and the theorising of Max van Manen (1994) and others.

QUESTION AND GOAL

The discussion group will address the question of the multiple meanings of pedagogy, and the methodological implications of different meanings. We will use examples from the Observing Teachers study to provoke discussion of these questions. These examples are derived from our analysis of focus group discussion, in which middle school mathematics teachers observe videos of each other’s teaching, or of the teaching of teachers from other regions of Canada. In our collective examination of these videos, we have noticed differences in our own ways of conducting the focus groups and interpreting the resulting data. We hope participants in the discussion group will experience a little of this sense of difference and will engage in discussion of the implications of such differences for research on mathematics pedagogy.

ACTIVITIES PLANNED

- | | | |
|-------|---|---------|
| Day 1 | a. Introduction to the questions | 10 min. |
| | b. Discussion: Meanings of “pedagogy” in the participants’ contexts. | 20 min. |
| | c. “Pedagogy” in the Observing Teachers research programme | 10 min. |
| | d. Discussion: Which “pedagogies” are observable and how? | 20 min. |
| | e. Example: Observing the observers to research implicit pedagogy. | 10 min. |
| | f. Group activity: Analysis of transcript, looking for implicit pedagogy. | 20 min. |

- Day 2
- a. Group reports and Debrief of group activity 20 min.
 - b. Example: Observing the observers to research pedagogy as 10 min. practice.
 - c. Discussion & closing: Revisiting the question with insights from 60 min. examples

ATTENDEE PARTICIPATION

As outlined above, on the first day the audience will participate by discussing the meaning of “pedagogy” in their own contexts, whether and how different “pedagogies” are observable, and by engaging in analysis of selected pieces of data from the Observing Teachers study (Day 1, activities b, d and f). On Day 2 most of the time will be devoted to attendee participations, thorough debriefing the transcript analysis activity from Day 1 and in a general discussion of the multiple meanings of pedagogy, and the methodological implications of different meanings.

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AESTHETICS IN SCHOOL MATHEMATICS: A HANDS-ON APPROACH

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Beauty plays a central role in the practice of mathematicians. Beauty motivates, guides problem and strategy choice, and shapes solutions (Sinclair, 2004). Not the least of all, it provides an emotional reward for intellectual work. Can beauty play these roles among school children? This discussion group will focus on what beautiful mathematics might mean to a school student and how to investigate this question.

Mathematical aesthetics is a topic that has been of interest since ancient times. Recently there has been increased interest in the topic in several fields, focusing mostly on mathematicians' views of beauty (e.g., Inglis & Aberdeen, 2015; Wells, 1990), the neural correlates of their judgements (Zeki et al, 2014), and the nature of mathematical beauty itself (e.g., Montano, 2014). However, within mathematics education, research related to this area has been scarce (Dreyfus & Eisenberg, 1986). There has been little research dealing with how the aesthetical disposition develops, and how this development might be encouraged in school mathematics. Nathalie Sinclair (2001; 2004) has done some work in this area, showing for instance that positive aesthetic responses are possible among small children. But many questions remain, including whether all children can experience beauty, whether children's aesthetic experiences are comparable to mathematicians, and what kinds of environments are optimal for eliciting positive aesthetic experiences.

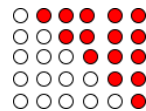
The goal of this discussion group is twofold:

- (1) To gather and discuss examples of beautiful mathematics which are accessible to school age students. Although we, as mathematics researchers and educators might find a specific proof 'beautiful', school students may be too young (mathematically) to appreciate the beauty of a proof. Might we instead discuss with students 'beautiful' explanations?
- (2) To examine methodology for recognizing when students have had a positive aesthetic experience. How to differentiate, for example, between a moment of understanding and a moment of pleasure or excitement that is more purely aesthetic.

The discussion group will take place over two sessions. It is open for researchers with interest in all school grades K-12, though our data will be taken from fifth grade students. The first session will focus on sharing examples of "beautiful" mathematics. We will ask participants to present examples of what they believe are beautiful proofs and beautiful explanations that may be presented to school students of various ages, including primary school students. These examples will be discussed among the discussion group participants debating the positive and/or negative aesthetic merits of

the example, where positive merits may be those mentioned above such as providing emotional rewards and negative aesthetics might refer to explanations which are cumbersome and deflating.

The second session will center on data we have collected from an on-going study of how fifth grade students evaluate different kinds of mathematical explanations. We will briefly describe the project, how the data was collected, and present the tasks. For example, working on triangular numbers, fifth grade students were asked to find out the number of dots in the 100th triangle. After some struggle, the children were shown two explanations. The first was the classic Gaussian proof, which lines up the numbers 1 to n twice, the second line reversing the order, and then adding n pairs of $n + 1$ to get the formula. The second explanation was a picture representation, with two triangles, making up a rectangle:



During the session, we will discuss students' reactions to the two explanations, including some pivotal moments in the data where we observed facial and verbal expressions that may indicate an aesthetic experience taking place. Discussion will focus on the question of whether or not the moment is purely aesthetic, and what (if any) additional type of information might be needed to strengthen the result.

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SEMINAR: REVIEWING FOR THE PME – A PRIMER FOR (NEW) REVIEWERS

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GOAL OF THE SEMINAR

This seminar¹ is intended to provide information about the PME review process and give the opportunity to gain first experiences in providing a high-quality review. The seminar aims especially at the needs of new reviewers², although experienced reviewers are highly welcome in order to facilitate knowledge transition within the PME community. The seminar includes an introduction in the intention and purpose of reviewing from a more general perspective (McKnight et al., 2000; APA, 2009), but also details aspects of the PME review practices. Participants will have opportunities to work with authentic examples from the PME review processes of the last years – provided we find authors that are willing to share their contributions with the review they received. Acknowledging the diversity within the PME community in the review process will be an important aspect of the seminar.

GOALS FOR THE PARTICIPANTS

Having participated in the seminar, the participants will

1. know about reviewing as an aspect of scientific quality management
2. know about the most important differences in reviewing procedures for journals and conferences as well as different types of contributions, especially in the PME context
3. be able to differentiate the specific review categories of PME
4. be able to identify aspects of quality for a review
5. be sensible to aspects of fair, constructive, and inclusive reviews

EXPECTED BENEFIT FOR PME AS A COMMUNITY

PME – as a scientific community – will benefit from the seminar as

- it is expected to improve the knowledge of (new) reviewers about the review process
- it is expected to smoothen (new) reviewers difficulties in composing high-quality reviews

¹ Seminars are intended to provide specific courses for the professional development of PME members.

² PME members with two accepted Research Reports in the past five years or three accepted Research Reports in the past 10 years are eligible as PME reviewer.

METHODS

The seminar will last 90 minutes. It will start with a brief presentation focusing on learning goal 1 and 2. A first group work phase will focus on the specifics of PME reviews and thus contributing to the learning goals 3 and 4. A second group work phase will focus in particular on the aspects of fair, constructive, and inclusive reviews (learning goal 5). Experienced reviewers, who are willing to share their knowledge, are invited to serve as group mentors during the working phase.

APPLICATION

In order to participate at the seminar please indicate your interest via info@igpme.org (administrative manager Bettina Roesken-Winter).

If you are willing to share a former contribution of yourself **TOGETHER** with the reviews you received as authentic examples for the group work phase, please contact Anke Lindmeier at lindmeier@ipn.uni-kiel.de as soon as possible.

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REVIEWING FOR THE PME – A PRIMER FOR (NEW) REVIEWERSAN INTRODUCTION TO ELECTROENCEPHALOGRAPHIC RESEARCH

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The use of neuroscientific methodologies, such as electroencephalography and neuroimaging, is becoming more and more common in mathematics education research. The present seminar aims at providing the PME community with a brief introduction to the technique of electroencephalography (EEG), which consists in recording the electrical activity generated by neuronal populations by means of electrodes located on the scalp. Specific topics to be reviewed include: what kind of neural activity can be measured by EEG, usual restrictions under which EEG recordings are conducted, methods of analysis and graphical representation, and some limitations regarding the conclusions that can be drawn from this technique.

INTRODUCTION

Electroencephalography (EEG), or the measurement of the electrical activity produced by neuronal populations in the brain, was first measured in humans in the 1920's by German neuropsychiatrist Hans Berger (Niedermeyer & Schomer, 2010). Since then, this technique has been developed and used in vast numbers of research studies in neuroscience. It is a technique with a high temporal resolution (of the order of milliseconds) but a fairly poor spatial resolution. Many EEG studies have investigated the brain processes related to the processing of quantities, numbers, and other mathematical objects. This literature is readily available to researchers in mathematics education, but it is often difficult to grasp due to a lack of knowledge about the technique and its possibilities, restrictions, and limitations.

GOALS

This seminar aims at providing the PME community with a brief introduction to EEG, so that mathematics education researchers can approach the EEG literature on numerical and mathematical cognition with an understanding of the technique and the potential, implications, and limitations of such research.

OUTLINE OF CONTENTS AND ACTIVITIES

The seminar will last 90 minutes. It will start with a presentation of the following contents:

- A brief history of EEG, how it works, and what it measures
- Restrictions under which EEG recordings are usually conducted
- Methods of analysis in quantitative EEG (event-related potentials, oscillations)
- Graphical representation of EEG analyses

Subsequently, a round of questions and discussion about the technique will be offered. We will finish by presenting a few EEG studies about numerical processing, discussing their implications and limitations, accompanied by a more general group discussion about what conclusions may be drawn (or not) from EEG analyses.

Participants in the seminar are welcome to propose papers about EEG for the discussion. Proposals should be submitted as soon as possible, to the email address dgomez@ciae.uchile.cl.

EXPECTED BENEFITS FOR THE PME COMMUNITY

Participants of this seminar will become aware of the possibilities and limitations that EEG research may contribute to research projects in mathematics education. By doing this, it is expected that the PME community will be able to better appreciate EEG research and to consider it as a viable tool for certain research questions, as well as to interact with collaborators in neuroscience in a more effective way.

ORGANIZER CREDENTIALS

The organizer obtained a Ph.D. in Neuroscience in the International School for Advanced Studies (SISSA) in Trieste, Italy. He has conducted EEG studies in the context of his doctoral thesis, and participated in several EEG investigations including one on the mental processes underlying fraction comparison (Barraza, Gómez, Oyarzún, & Dartnell, 2014). He currently works at the Neuroscience and Cognition Laboratory of the Center for Advanced Research in Education (CIAE) of the University of Chile, studying the mental and neural processes underlying mathematical concepts.

References and further reading

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THE RITUAL VS. EXPLORATION CONCEPTUAL DYAD – AFFORDANCES AND OPEN QUESTIONS

Coordinator: Einat Heyd-Metzuyanin, Technion

Assistant Coordinator: Talli Nachlieli, Levinsky College

Mellony Graven, Rhodes University; Nadav Ehrenfeld, Ben-Gurion University

Since Sfard and Lavie (2005) introduced the concepts of “ritual” and “exploration” in mathematical learning, there has been increasing use of this conceptual dyad. In this working session, our goal is to examine the affordances and limitations of this dyad, and to explore its connection with other prevalent dyads in the field.

The mathematics education research is replete with conceptual dyads such as “procedural vs. conceptual”, “individual vs. social”, “extrinsic vs. intrinsic motivation” and “mathematical dis/ability”. To these well-known dyads, a relatively new conceptual pair has been introduced: ritual vs. exploration. Ritual, as defined by Sfard and Lavie (2005), is mathematical performance for the sake of connecting with others or “people pleasing” (Heyd-Metzuyanin & Graven, 2015). Exploration, on the other hand, is mathematical activity done for the sake of the activity itself.

Sfard & Lavie (2005) have initially coined the terms “rituals” and “explorations” based on the study of 4-5 year olds learning about numbers. Since then, the conceptual dyad has been found useful for description of Israeli middle-school learners (Heyd-Metzuyanin, 2015), South African elementary school learners (Heyd-Metzuyanin & Graven, 2015) and even instruction of pre-service teachers (Heyd-Metzuyanin, Tabach & Nachlieli, 2015). Moreover, “ritual participation” has been paired with “ritual instruction” connecting learning and teaching practices.

Yet, despite this progress, a central question remains, regarding the relation of the two types of participation as consecutive or parallel. Originally, Sfard and Lavie (2005) suggested that ritual participation is an antecedent of explorations, often an inevitable one that supports learning at the peripheral stage, when participants do not have sufficient conceptual tools to follow the logic of the discourse and must rely on imitation. Despite the theoretical appeal of this conjecture, the evidence collected so far point to an alternative possibility – that students participating ritually advance in a parallel trajectory (often leading to failure) to those who participate exploratively. This alternative is a possibility since, for now, we have not found much evidence for students who *started* ritually and then progressed to explorative participation.

We shall start each part of the Working Session, by presenting one or two studies that made use of the ritual-exploration dyad. The first session will start by introducing the ritual-explorative dyad and connecting it to wider educational ideas outside of mathematics education. We will then present a study of three middle school learners of differing achievements. The low achieving student was consistently ritual, the

deteriorating student was ritual in one domain (fractions), which gradually widened into algebra, and the high achieving student was consistently explorative. The participants will then be invited to engage with analysis of excerpts from the data of these three students according to the definitions of ritual and explorative routines.

The second part of the working session will start by presenting the way in which the ritual-vs.-explorative analysis was employed in a study of two elementary school learners in South Africa. One of them was mostly engaging ritually, while the other was more explorative. This study revealed the importance of the classroom, social and cultural environment of the two learners, pointing to ritual aspects of the classroom instruction and more generally of the culture of learning in South Africa. As a sharp contrast to the South African study, we will present some excerpts from a study of adult Ultra-Orthodox Jews who study algebra for the first time at a pre-college course. These excerpts show the Ultra-Orthodox students' tendency for what may be seen as excessive exploration, evident in their reluctance to follow any rule blindly, even while the teacher signals that it is not the proper occasion.

During the second part of the Working Session, participants will be invited to work on one of two related tasks, according to their choice. The first includes applying the ritual vs. explorative dyad on their own data, and examining whether it opens up new understandings for them. The second compares the use of the ritual vs. exploration dyad to other dyads they may be more familiar with. This task will be scaffolded by excerpts of data from our studies, which will afford participants the opportunity to apply their alternative conceptual dyads on these excerpts.

The presentations will show how the ritual vs. explorative dyad relates to three layers: the individual student, the classroom environment, and the culture of learning. We will therefore ask participants to examine how their alternative concepts relate to these layers. Through this group work, we hope to uncover the unique affordances of the ritual vs. explorative dyad, as well as its interfacing with other major conceptual dyads in our field. In addition, we hope that connecting the ritual vs. explorative framework to existing research within other dyads, will help shed light on the question of parallel or consecutive ritual and explorative participation.

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MATH AND SPECIAL EDUCATION WORKING SESSION

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This working session, active during PME 38 and 39 and PME-NA 34-37, has been developing a research agenda to explore pedagogical approaches for fostering conceptual knowledge of mathematics in students with special learning needs in mathematics. Our work has been rooted in a twofold premise: (a) students with special needs are capable of and need to develop conceptual understanding and mathematical reasoning skills, and (b) special education instruction, assessment, and research needs to transition towards this focus. In this WS, the focus will be on publication of research findings through cross-disciplinary collaboration.

BACKGROUND

About five to ten percent of school-age children have been identified as having mathematics disabilities (Fuchs, Fuchs, & Hollenbeck, 2007) and students whose math performance was ranked at or below the 20 to 35 percentile are often considered at risk for learning disabilities or for having learning difficulties in mathematics (LDM) (Bryant et al., 2011). Students with LDM lag behind their peers beginning in early elementary school and continue to fall further behind as they transition from elementary to secondary school. Research from mathematics education promotes a focus on students' construction of conceptual understanding in mathematics. This includes inquiry-based learning opportunities in which students explain their mathematical reasoning to others, and follow others' reasoning. Yet, according to Baxter et al. (2001), students with LDM had difficulties engaging in class discussions and contributed marginally to the classroom discourse (Baxter et al., 2001). A meta-analysis study (Kroesbergen & Van Luit, 2003) of 58 studies of mathematics interventions for elementary students with special needs indicated that recent changes in mathematics education "do not lead to better performance for students with special needs" (pp. 111-112). A more recent meta-analysis (Gersten et al., 2009) indicated that, although explicit instruction is still one of the dominant instructional approaches (11/41 studies) to teach mathematics to students with special needs, eight studies have explored the effect of "student verbalizations of their mathematical reasoning" (pp. 1210) on mathematics performance. The tension between general and special educators over the two different pedagogies in mathematics instruction involving students with special needs is far from being resolved.

Since 2008, members of this working session have been collaborating on research projects that integrated research-based practices from mathematics education and special education. This proposed working session intends to continue promoting this collaborative, productive, cross-disciplinary work on the topic, particularly by strengthening understanding between international researchers and practitioners of

mathematics education and special education. We will also continue working on relating various research paradigms concerning mathematics instruction of students with special learning needs.

PLAN FOR WORKING SESSION

During the previous two meetings, we have discussed our intention to further explore possibilities for collaborative work in this topic, including proposing and planning for an edited book on mathematics and special education. Table 1 presents a work plan to continue this collaborative publication effort.

Session 1	Session 2
a) Introductions	e) Identify what elements need to be in a book proposal and potential publishing outputs
b) Discussion of commentary from group's previous project to ground constructs of "MLD" and "conceptual knowledge" (Contact authors for copies).	f) Link attendees' work pertinent to the proposed publication
c) Share attendees' work pertinent to mathematics education and special education	g) Begin writing a book proposal and assign tentative chapters
d) Briefly share history of the working session and past work/discussions	h) Identify the tasks for the members of the working session to achieve (i.e., one year) and communication of progress

Table 1: Goals and activities for the working session

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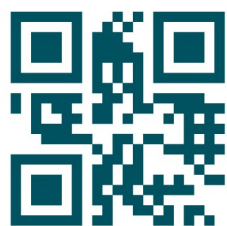
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