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MATHEMATICS
EDUCATION

How to solve it?



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EDITORS: CSABA CSÍKOS • ATTILA RAUSCH • JUDIT SZITÁNYI

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International Group for the Psychology of Mathematics Education



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Editors

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ALTERNATIVE CONCEPTIONS OF LIMIT OF FUNCTION HELD BY LEBANESE SECONDARY SCHOOL STUDENTS

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A good conceptual understanding of “limits of functions” is essential for students to successfully proceed further with other calculus concepts. However, students usually hold wrong or alternative conceptions of limit. This paper aims to investigate the conceptions of limit held by Lebanese secondary school students (17-18 years old) and their perceptions of difficulties faced while learning it. A questionnaire was administered to 35 students. An “Index of Adoption” was created to identify and rank students’ alternative conceptions. Results showed that different models are held by students, the dominant one being a dynamic symmetrical duality model. Most difficulties expressed by students relate to procedural processing of limits.

The concept of “limit of a function” plays a fundamental key role in the study of calculus. A good conceptual understanding of this concept and its applications is essential in order for students to successfully proceed further with other calculus concepts such as continuity, derivative and integral. Students’ erroneous understandings of limits will affect their whole subsequent learning process in mathematics as well as in other subjects.

Literature is rich with studies that investigated the teaching and learning of limits, whether from a psycho-cognitive perspective (Cottrill et al., 1996), or from epistemological perspectives (Cornu, 1991; Moru, 2008; Sierpinska, 1987). Many studies were also concerned with didactical aspects of teaching and learning of limits. Barbé, Bosch, Espinoza and Cascon (2005) consider that the processes of learning and teaching go hand in hand and that the problems that arise in learning the concept of limit require an understanding of the choices that teachers make and the related content of the curriculum. Huillet (2005) investigated five Mozambican teachers’ professional knowledge of limits of functions and showed that they had weak knowledge. Research clearly showed that the concept of limit creates major difficulties for students and that students face many obstacles while learning it (Cornu, 1991; Sierpinska, 1987; Tall & Vinner, 1981). Some obstacles emerge from students’ intuitive understanding of other foundational concepts such as: infinitesimals, the notion of infinity, and continuity. Pehkonen, Hannula, Majjala, and Soro (2006) conducted a study on students’ understanding of the notion of infinity. They consider infinity as an inspiring but rather difficult concept for both mathematicians and students. Being foundational building blocks to the concept of limits, the notions of infinity and infinitesimals are expected to cause difficulties in students’ learning of limits. Students will probably be prone to build erroneous conceptions of limits of functions. It is therefore important to investigate the possible alternative conceptions that students may develop, as a step

toward elaborating teaching strategies to challenge them. Williams (1991) studied the informal models of limit of function held by 10 college students. He then designed problems aiming to create a change in students' conceptions. He came up with the conclusion that the dynamic aspects – based on graphic models – were extremely resistant to change. Students' previous experiences with graphs of simple functions create obstacles to students' developing of a formal view of limit.

This paper reports preliminary results obtained in the context of a large study targeting the teaching and learning of limits of functions in the Lebanese context. Due to the size limitations of the paper, only a part of the study is considered. The purpose is to investigate the different conceptions of *limit of function* held by Lebanese secondary school students, one year after its introduction in grade 11. Grade 11 (16-17 year-old students) is the second secondary year in the Lebanese educational ladder. The reported study targeted grade-12 students, in their last year of secondary school.

METHOD

The study adopts a qualitative analytical approach, based on text analysis of students' answers to a questionnaire. Participants are 35 students, 17-18 years old, in two grade-12 classes of a Lebanese private, mixed-gender school. These students were introduced to the concept of limits in the previous school year, after which they also worked on continuity and differentiability. To investigate their conceptions of limit a year later to its introduction, students were asked to answer a questionnaire consisting of three questions, designed to make explicit those conceptions, as well as their views of the difficulties they associate to the concept of limit.

FINDINGS

Question 1 of the questionnaire (named Selecting) asks students to choose, among six statements, the three statements that best describe their understanding of limits and to rank them in the order of preference. This is a slight adaptation of a question used by Williams (1991, p. 221) who asked students to decide, for each statement, whether it is true or false, and then select only the one that most describes their idea of limit. Question 2 (Formulating), also adapted from Williams (1991), asks students to express, in their own words, what they understand by limit. Question 3 (Expressing difficulties) focuses on getting students' views on the difficulties that they faced and/or are still facing while learning, and working with, limits.

Question 1: Selecting

“From the following list of statements, choose the three statements that best describe your understanding of limit, and rate them from 1 to 3 in the order of preference.”

- A limit describes how a function moves as x moves toward a certain point (S1).
- A limit is a number or point past which a function cannot go (S2).
- A limit is a number that the y -values of a function can be made arbitrarily close to by assigning specific numbers to the x -values (S3).
- A limit is a number or point the function gets close to but never reaches (S4).

- A limit is an approximation that can be made as accurate as you wish (S5).
- A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached (S6).

Students' answers to this question are in the form of three numbers (1, 2 and 3) assigned to three statements selected by each student among the six given statements, 1 being the highest preference and 3 the lowest. For the analysis, students' answers were mapped in a table with two entries, the six statements horizontally, and the student name-codes vertically. Table cells are filled with the level of preference assigned by the corresponding student to the corresponding statement. 0 was assigned if the statement was not chosen by the student.

The results are then compiled as presented in Table 1. The "Selected by" row presents the number of students who selected each statement, irrespective of preference level. For example, S1 was selected by 28 students and S4 was selected by 18. The next three rows present, respectively, the numbers of students who selected each statement at each level of preference. For example, among the 28 students who selected S1, 20 assigned to it the 1st preference level, seven the 2nd preference level, and one the 3rd preference level.

Statement →	S1	S2	S3	S4	S5	S6
Selected by	28	6	17	18	12	24
1st Preference	20	1	5	8	0	2
2nd Preference	7	2	6	5	6	10
3rd Preference	1	3	6	5	6	12
Index of adoption	2.14 (1)	0.29 (6)	0.95 (4)	1.11 (2)	0.51 (5)	1.09 (3)

Table 1: Indices of adoption of statements 1 to 6

To make sense of those numbers, an "Index of Adoption (IA)" was calculated for each statement to express the extent to which this statement was adopted as an alternative conception by students, on a scale from 0 to 3. To calculate the IA of each statement, the levels of preference were weighted: 3 for the 1st preference, 2 for the 2nd and 1 for the 3rd. The IA was calculated, in each cell corresponding to a statement, as follows:

$IA = [(N_{(1)} \times 3) + (N_{(2)} \times 2) + (N_{(3)} \times 1)] \div 35$, where 35 is the global number of students; $N_{(i)}$ is the number of students who selected the statement at level of preference i ($i=1$ to 3). Therefore, for example, IA of the first statement S1 is 2.14 (see Table 1), calculated as: $((20 \times 3) + (7 \times 2) + (1 \times 1)) \div 35$. The Index of Adoption allows ranking the six different statements in the order of adoption by students. The last row of Table 1 provides IA of each statement, and its rank, which is included between parentheses just next to the IA value.

According to the analysis method explained above, we can conclude that S1 is the most adopted by students, with an IA of 2.14, and S2 is the least adopted, with an IA of 0.29.

The difference between these two extremes (the range) is 1.85 on a 0-to-3 scale. Therefore, a dominating conception of limit of function among the participants is that “A limit describes how a function moves as x moves toward a certain point”. Students thus strongly hold a dynamic conception of both, the variable and the function, being in motion, each on its track and toward a certain value. This conception is distinctively higher than the others. It exceeds the next one (IA = 1.11) by 1.03 on a 0-to-3 scale.

The second- and third-adopted conceptions, i.e. S4 and S6, have close IA values, respectively 1.11 and 1.09 on a 0-to-3 scale. So, students moderately think that “A limit is a number or point the function gets close to but never reaches” (S4), yet they also think, at almost the same level, that “A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached” (S6).

The two least adopted conceptions are S5 (IA=0.5) and S2 (IA=0.28), corresponding respectively to: “A limit is an approximation that can be made as accurate as you wish”, and “A limit is a number or point past which a function cannot go”.

Question 2: Formulating

“Please describe in a few sentences what you understand a limit to be. That is, describe what it means to say that limit of a function f as $x \rightarrow a$ is some number L .”

While question 1 provides students with a limited choice of pre-determined statements to select from, question 2 leaves it open for them to freely formulate their understanding of the notion. A text analysis was conducted on students’ answers, with focus on the sentence structure and vocabulary used, in as much as they reflect their conceptions. The following categories of alternative conceptions were identified. They are briefly explained, within the size limits of this paper:

Dual parallelism between variable and function behaviors

Most of the students’ answers express a kind of “parallelism”, reflecting what can be named a “symmetrical duality” in the behaviors of the variable and the function. As one “moves”, the other “moves”, and as one “gets close” to a value, the other “gets close” to a value. Twenty-four out of the 35 students used such a type of duality. Different verbs were used, such as “moves” (M), “tends to” (T), “approaches” (A), “reaches” (R), “gets closer” (Cl), “comes” (Co), “becomes” (B), “is” (I). Following are examples of students’ answers showing duality:

- As x tends to a number a , y tends to a number L (TT)
- As x moves towards a , f moves towards L (MM)
- As the x reaches the number a , the function reaches the number L (RR)
- x tends to a nb “ a ”, thus, y approaches the number L , but might never reach it (T-A)
- When x tends to a number a the function $f(x)$ tends to reach L (T-TtoR)
- The value that the $f(x)$ approaches as x gets infinitesimally [sic] close to a is L (Cl-A)
- As x tends to a $f(x)$ will move as L moves (T-M)
- As x becomes closer to a , $f(x)$ becomes closer to L regardless if L is reached or not, if $f(a) = L$ or not. $f(a)$ could not be defined (Cl-Cl)

The different verbs used by students reflect different conceptions of functions and limits. For example, the verbs “moves toward”, “approaches”, “gets closer to”, “tends to” provide a dynamic nature to the variable and the function. They also reflect the idea that x and/or $f(x)$ may not get equal to the values they are approaching. Such a conception is different from the one reflected by verbs such as “reaches”, “becomes”, “is”, which reflect a static conception, while at the same time expressing the fact that the limit may be attained. It is worth noting that one of the answers included a sentence, explicitly highlighting the fact that the function does not reach the value L and using the idea of the infinitely small:

- $F(x)$ tends to L as x is a , but isn't L , like 0.000001 but not 0 (I-T)

Graph based conceptions

Some of the students' statements included instances of graphical connections that are distinctively different from the above “duality based”, formal and symbolic notions. However, these graphic based answers included erroneous use of the mathematical language that reflects serious confusions and misconceptions, mostly related to their knowledge of functions, the relationships between functions and their graphs, and confusion of the concepts of variable and function. Following are examples, where some of the parts reflecting confusions are underlined:

- The graph tends to be close to $x = a$, it can be $x \rightarrow a^-$ or $x \rightarrow a^+$, giving same limit= L
- As x tends to a , $f(x)$ will move as L moves
- As the function f approaches $x = a$, the ordinate approaches L ; i.e the function curve approaches $y = L$
- The y or ordinate of the number (a) will be obtained by calc. the limit of it as a tends towards it. It might be a number or ∞
- As the curve of a function moves across a plane closer and closer to $x = a$, it also moves closer to the number L , (it may or may not pass through the point (a, L) depending on the domain of f itself)
- A Limit helps us determine where a function moves to or ends at a differentiable pt

Alternative conceptions related to whether the limit can be reached or not

The dilemma about whether a limit is reached or not, or even about whether a limit may be reached or not, is clear in students' answers. Some students consider that the limit of a function is the value of the function at the point, as in “ $f(x) = L$ ”, or “the function reaches its limit”. Others use terms that reflect a notion that interferes with the common language use of the word “limit”, that is the function cannot go beyond the limit; e.g. “the function ends at the point”, or “the function reaches its limit”. Other students' statements, on the other hand, explicitly emphasize the fact that the limit either cannot be reached, or might not be reached. The following are some examples:

- As the function gets close to $x = a$ it reaches its limit and y is approximately equal to L and very close to it
- x tends to a nb “ a ”, thus, y approaches the number L , but might never reach it
- Closest value of y at a certain value x

- When x tends to a number (which is a here) the function $f(x)$ tends to reach L

A thorough text analysis reveals also other various alternative meanings attributed to the concept of limit, related to whether the function would be defined, or undefined at the point, or would be continuous / differentiable at a point:

- As f reaches $x=0$ the function reaches the point $y=L$; fct may not be defined on $f(a)$
- Limit is the number that the function cannot reach, due to the function being undefined at this point. L represents the number that the function would have reached if it is defined at a

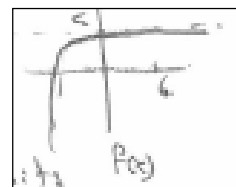
Procedural understanding of limit

Some of the students' answers reflect their permanent concern about solving exercises involving limits of functions, rather than a more conceptual meaning. Their statements either recite rules to use in solving limits as they memorized them, or they present the limit as a way for calculating other values, or for finding some characteristics of the function in certain conditions:

- $\lim_{x \rightarrow a} f(x) = \frac{f(x)-f(a)}{x-a} = L =$ slope of the tangent to the curve at $x = 0$
- If $\lim_{x \rightarrow a} f(x) = \infty \Rightarrow x = a$ is a vertical asymptote
- The y or ordinate of the number (a) will be obtained by calculating the limit of it as x tends towards it. It might be a number or ∞

Finally, the following example presents the limit as a solution for finding an approximate value of the function when it is not possible to calculate the exact value.

- Consider the function $f(x)$. If we want to find the value of $f(6)$, we cannot know the exact value of it. So we use limits so $\lim_{x \rightarrow 6} f(x) = 5$ when x tends to 6. This doesn't mean that $f(6) = 5$ but it means that $f(6)$ is a number very close to 5, might be 4.9999918999....



Question 3: Perceptions of difficulties

“Please describe the main difficulties that you faced while learning limits.”

This question aims to explore the way students perceive the difficulties that they faced and/or they are facing while learning and working with limits. The analysis of students' answers allowed a classification of their views of difficulties in several categories. While four students did not answer this question, and three responded that they did not face any difficulties, the following categories were identified:

Operations and calculations to find a limit

The types of difficulties most expressed by students (18 out of 35) relate to calculations for finding limits. This may be interpreted by the emphasis that the curriculum places on procedural knowledge rather than conceptual understanding (Osta, 2003). Seven of these 18 answers relate to the Indeterminate Forms (IF) and the ways to deal with them, and two to finding asymptotes and differentiating between horizontal and vertical ones.

- Making an undetermined [sic] limit determined [sic]
- If it is an IF and I can't find the proper method to make the limit work
- Dealing with the indeterminant [sic] and the function where the limit doesn't exist
- Multiplying by the conjugate in the denominator
- How we can work out some limits
- The problem was finding limits to relatively hard functions
- The properties of limits (adding, dividing, multiplying)
- The idea of calculating a limit whether to factorize, divide, or plug in numbers
- I never fully understood l'Hopital's rule. I know how to utilize it while solving limits but I don't know why it's there.....or I just don't remember

Continuity and differentiability as related to limits

The confusion between limits, continuity and differentiability comes next in the list of difficulties, expressed by 7 students out of 35.

- To know the difference between continuous, differentiable
- The idea of differentiable, has limit, continuous
- Trying to understand the concept of a function being differentiable at a point and studying it's [sic] limit was tough

Meaning and purpose of limits

Six students expressed their confusion about the definition and meaning of limit and the purpose of its use, "the main concept", "what is limit and why it is used?", "difficulties about the definition", etc.

Metaphysical aspect of limit

Many students expressed difficulties related to aspects of limits that can be related to its "metaphysical" nature, and to the fact that they "had not seen anything similar before". They considered the concept of limit to be a "new idea" and that it is "hard to understand its usage and importance", that "we can't directly understand what we are working with". One student wrote that he could not relate the concept with examples, another student could not visualize the concept in his/her head, a third could not deal with non-existing limit, and another one calculated the limits "without thinking of what are we [sic] finding, whether graphically or logically".

CONCLUSION

The analysis of data from questions 1 and 2 concurred to show that the participating students hold different conceptions of limit of function, some of which are not in line with the formal definition. It also showed that the dominating conception is that of a dynamic, symmetrical-duality model, whereby both, the variable and the function move, each on its track, toward certain values. This result concurs with Williams' findings (1991). Graphical models, expressed with conceptual and language confusions, are moderately held by students. As to the students' perceptions of their difficulties, they are mostly of a procedural nature, related to the calculation of limits or other related entities, together with difficulties about their "metaphysical" nature.

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PROVING ACTIVITIES IN INQUIRIES USING THE INTERNET

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This paper elucidates the nature of proving activities required in the inquiry-based learning of mathematics using the Internet, wherein the didactic contract is different from that in the ordinary mathematics classroom. Based on the anthropological theory of didactics, proving activities conducted in the study and research paths are explored in the context of Japanese pre-service mathematics teacher education. We design and implement situations for finding the cube root of a given number by using a simple pocket calculator. The analysis of the realised situations shows that inquiries using the Internet generate, in a way adidactic, students' different activities related to the proof, such as reading proofs, posing new why-questions, proving by themselves to understand the information obtained on the Internet and the method of calculation.

INTRODUCTION

The difficulties of learning proof and proving are well known, and this has been the subject of a significant body of research (cf. Mariotti, 2006). One difficulty which is often discussed, especially in the authors' country, is the necessity of proofs (MEXT, 2009). Students do not feel the necessity of proving a statement, particularly statements already known as true since elementary school (e.g. properties of a parallelogram). However, teachers also face difficulties in creating learning situations in which proofs are required to solve a problem, that is, situations wherein students feel the necessity of proving. We consider that this difficulty is, to some extent, due to the *didactic contract* (Brousseau, 1997) which is created in ordinary teaching and learning situations in mathematics classrooms, and due to the *paramathematical* nature of a proof (Chevallard, 1985/1991): proof is a tool for studying mathematics rather than a mathematical object to be studied (except in mathematical logic). Since a proof is a paramathematical object, its teaching cannot be dissociated from other mathematical knowledge to be taught. In the classroom, what is justified by the proof is the statement related to this knowledge, and this statement to be proven is always true because what is taught in school is the set of organized objects which are known to be true. There is a contract, that the teacher teaches or education generally provides 'true' knowledge to students. Students know that the statement to be proven is true before proving it, since it is given by the teacher and it is a piece of knowledge which students have to know.

What if the didactic contract differs from that found in the ordinary mathematics classroom? What kinds of proving activities would be conducted? Further, is it possible to radically change such a didactic contract? In a recent study, a 'new' way to conceive mathematics teaching is proposed, and the didactic contract created in such teaching seems very different from the ordinary didactic contract. It is a sequence of activities

called *Study and Research Paths* (SRP hereafter: Chevallard, 2006; 2015) within the *Anthropological Theory of the Didactic* (ATD) developed by Chevallard. SRP is based on the didactic paradigm called *questioning the world* (Chevallard, 2015), in which the learning is aimed at nurturing scientists' attitudes in the process of elaborating an answer to a question. Students investigate a question by means of any tool available (e.g. calculator, computer, Internet, any books), and mathematical knowledge is learnt through a process when necessary. Unlike teaching based on the 'old' paradigm wherein *raison d'être* or rationale as to why students should learn it is often implicit, mathematical knowledge to be taught is not organized in a sequence to be learnt one-by-one, and it is accompanied by a *raison d'être*. Additionally, it might be the teacher who proposes the initial question, but there is no specific expected answer and no specific mathematical knowledge to be taught. The teacher's role is that of a supervisor of scientific research. The didactic contract is thus very different from the ordinary mathematics classroom.

In such inquiries, what kinds of proving activities would be required and conducted especially in the case of inquiries using the Internet? We investigate this question by designing and implementing situations based on the idea of SRP in the context of Japanese pre-service mathematics teacher education. Through an analysis of the realised situations, we try to identify the nature of proving activities in such situations. We expect that different activities related to the proof, difficult to conduct in ordinary teaching, will be identified while the students elaborate an answer to the question.

THEORETICAL FRAMEWORK

In what follows, we briefly introduce the notion of SRP, which plays a crucial role in this study. It is used as a conceptual tool to develop the learning situations to be realised and as an analytical tool to clarify the nature of students' activities conducted in the situations realised in the teaching experiment. In ATD, inquiries in mathematics and other fields are characterised by the notion of SRP (cf. Barquero & Bosch, 2015). SRP expresses dialectic processes between questions and answers, where an inquirer starts from an initial question Q_0 and arrives at a final answer A^\heartsuit . The simplest SRP is modelled as ' $Q_0 \rightarrow A^\heartsuit$ '. However, the process of finding an answer includes other steps. The inquirer usually encounters another various questions Q_k derived from the initial question or others, and finds answers A_k to them. Some answers could have already been produced by the predecessors: those are labelled as A_i^\diamond . This process is modelled, for example, as $Q_0 \rightarrow Q_1 \rightarrow A_1 \rightarrow Q_2 \rightarrow A_2^\diamond \rightarrow Q_3 \rightarrow A^\heartsuit$. However, most study processes cannot be formulated by a linear diagram but by a tree diagram, because a question often leads to multiple questions.

Further, the process of the elaboration of an answer is characterised in ATD by the media-milieu dialectic. Similar to its use in the Theory of Didactic Situations (TDS), a milieu refers to a system without didactic intention, acting as a fragment of 'nature', with which the inquirer interacts during the study process (cf. Chevallard, 2004; Artigue et al., 2010; Kidron et al., 2014). In contrast, the media refers to any system

with the intention of supplying information about the world or a part of it to a certain type of audience (cf. Chevallard, 2004; Artigue et al., 2010; Kidron et al., 2014). In order to get an answer to a question, the inquirer looks for and obtains information from media, and elaborates an answer by interacting with the milieu including such information. SRP based on the *questioning the world* presupposes the use of media as in scientists' activities, restricted in ordinary teaching based on the 'old' paradigm.

METHODOLOGY

In this study, we design and implement learning situations based on the idea of SRP and analyse the data collected in the experiment in order to clarify the nature of proving activities in inquiries. We adopt as a methodology *didactic engineering within ATD*, which includes four phases of the analysis and design of didactic phenomena: preliminary analysis; conception and *a priori* analysis; experimentation and *in vivo* analysis; and *a posteriori* analysis (cf. Barquero & Bosch, 2015). In this paper, we report some parts of these analyses.

As we mentioned above, the notion of SRP is used as a conceptual tool to design learning situations. It allows us not only to design tools for students to use in class (e.g., Internet), but also to consider the nature of the initial question Q_0 proposed to them: Q_0 should be an *alive question*, so that it is connected with various mathematical or other activities; Q_0 should have *generative power*, so that many other questions Q_k are derived. We looked for such an initial question and designed a sequence of situations in the context of pre-service mathematics teacher education. The details of the design are revealed in the next section.

In the experiment, we collected students' worksheets, PC screen views which show the history of pages visited on the Internet, and the video and audio data for the entire lessons and the activities of each group which were translated later. In the analysis, the SRP is now used as an analytical tool. The *tree structure of questions and answers in SRP* allows us to model the dynamics and process of inquiry, and the media-milieu dialectic allows us to model the dynamics of mathematical activities. Specifically, in the *in vivo* analysis, we first identify various questions Q_k posed by students, answers obtained from the media A_i^\diamond , and temporary or final answers elaborated A_k or A^\heartsuit , from which are constructed a diagram representing a tree structure of SRP. Further, we describe, by means of the media-milieu dialectic, students' activities related to these questions and answers, in particular those concerning proving. Then we discuss, as an *a posteriori* analysis, the nature of the proving activities required in SRP, based on the results of the *in vivo* analysis.

MATHEMATICAL AND DIDACTIC DESIGN: A *PRIORI* ANALYSIS

We design situations in the context of pre-service mathematics teacher education in a university dedicated to elementary-school teacher training. Target students are third-year undergraduate students enrolled in a program to obtain a secondary-school mathematics teacher certificate, in addition to the elementary-school teacher certificate. In general, students in this university are not very competent in mathematics.

A_{0-1}	A_{0-2}
1. $[a] [\sqrt{ }] [\sqrt{ }]$	1. $[a] [\sqrt{ }] [\sqrt{ }] [\times]$
2. $[\times] [a] [=] [\sqrt{ }] [\sqrt{ }]$	2. $[a] [\sqrt{ }] [\sqrt{ }] [\sqrt{ }] [\sqrt{ }] [\times]$
3. $[\times] [a] [=] [\sqrt{ }] [\sqrt{ }]$	3. $[a] [\sqrt{ }] [\sqrt{ }] [\sqrt{ }] [\sqrt{ }] [\sqrt{ }] [\sqrt{ }] [\times]$
4. ...	4. ...

Fig. 1. Two answers to the initial question Q_0 . (a is a given number)

The initial question Q_0 we chose is: how to calculate the cube root of a given number by using a simple pocket calculator? The calculator has the function of calculating a square root, in addition to the four basic operations (+, −, × and ÷), but nothing other than these functions. This question is generally well known in Japan, and one may find different websites related to it on the Internet. The question is closed and its answer could be easily found in the media. However, starting from this question, students might ask other different questions that lead to the various mathematical concepts. In this sense, we consider that Q_0 is an *alive question* which has *generative power*.

In search for the answer to Q_0 , one may find two methods of calculation A_{0-1} and A_{0-2} given in Fig. 1. The naïve question derived from these answers, for the students of the university, is the question Q_1 : why does such a method allow the calculation of the cube root? The answer to this question A_1 could be found in the media (websites) or through interacting with a milieu. For example, the operations on the calculator could be translated into an infinite series on the exponent part which converges to $1/3$ (the operations of A_{0-1} to the first line of Fig. 2 and the operations of A_{0-2} to the second line). At this point, students are exposed to mathematical works on infinite series, such as the limit of series and the recurrence relation, and are required to read and understand the proof obtained from the media, which is A^\diamond , or to prove by themselves. Further, the question of calculating the cube root of a given number would also derive questions related to the calculation of the n th root, such as the 5th root and 7th root. Developing an answer to such a question allows students to encounter other mathematical works such as those related to the Mersenne numbers $2^k - 1$ (appearing when solving a recurrence relation such as $x_{n+1} = (x_n a^p)^{(1/2)^q}$), binary numbers (converting $1/n$ to a binary representation provides an infinite series like the second line of Fig. 2), etc.

In the class, students will be asked to conduct the inquiry based on their own interests. While some questions will be provided by the teacher, the derived questions might or might not be the ones we anticipated above. Students deal with the questions they pose on their own. There is no specific mathematical knowledge expected for the students to acquire (open SRP). The objective of the class is to nurture scientists' attitude and to develop students' views on mathematical activities (SRP)

$$\begin{aligned}
 & ((((((\frac{1}{4} + 1)\frac{1}{4} + 1)\frac{1}{4} + 1)\frac{1}{4} + 1)\frac{1}{4} + 1)\frac{1}{4} + 1)\frac{1}{4} \dots \\
 &= \frac{1}{4} + (\frac{1}{4})^2 + (\frac{1}{4})^3 + (\frac{1}{4})^4 + \dots + (\frac{1}{4})^n + \dots \\
 &= \frac{1}{4} \lim_{n \rightarrow \infty} \frac{1 - (\frac{1}{4})^n}{1 - \frac{1}{4}} = \frac{1}{4} \cdot \frac{4}{3} \cdot 1 = \frac{1}{3}
 \end{aligned}$$

Fig. 2. The series on the exponents converges to $1/3$

for teacher training). In such situations, the didactic contract should be different from that in ordinary situations.

EXPERIMENTATION: *IN VIVO* ANALYSIS

The third author of this paper taught a class based on the situations designed in the *a priori* phase. This class includes three teaching periods of 90 minutes (one period per week), allocated to the inquiries using the Internet, and one period for the presentation of the results of the inquiries. Nine students assisted in this class. The inquiry was conducted in a group of three students. Thus, three groups were created. A pocket calculator was provided to each student, and a laptop PC connected to Wi-Fi was provided to each group. At the beginning of the first period, in addition to providing the initial question Q_0 , the teacher explained the objective of the class and the modality of the inquiry. The objective is for students to experience and know the ‘authentic’ mathematical activities that mathematicians conduct in their research. The students may use any tools (media) to advance their inquiries; there is no final goal expected by the teacher and the inquiries may follow any direction, depending on the students’ interests and their new questions. The teacher’s role is to support their inquiries. In the last period, they should present the products of their inquiries. At this stage, the teacher tried to devolve the situations so that the students and the teacher could create a didactic contract which is specific to the inquiries.

Overall, each group worked sincerely during the three teaching periods and also during the time-out period of the class, for preparing a presentation. In the first period, the inquiry is conducted especially for identifying the method to calculate the cube root of a given number and to understand why such a method works. From the second to third periods, each group inquires into its own question and proceeds towards different directions: the first group proceeded to the calculation of the n th root, the second group to another justification of the calculation method by using a graphic representation of the convergence, and the third group to the speed of convergence.

We describe here the process of inquiry through an analysis of students’ activities from the theoretical perspective of ATD, particularly SRP and the media-milieu dialectic. In the *in vivo* and *a posteriori* phases, we focus on SRP of the first period in the second group (Group 2 hereafter). In the beginning of the inquiry for an answer to Q_0 , Group 2 immediately reached a first webpage, ‘calculation of cube roots using a calculator’ (http://www004.upp.so-net.ne.jp/s_honma/urawaza/root.htm). This page introduces a method of calculation by a simple calculator. The explanation starts with the recurrence relation of exponents ‘ $a_1 = a, 4a_{n+1} = a_n + 1$ ’, and then introduces the method ‘ $[a] [\times] [N] [=] [\sqrt{}] [\sqrt{}]; [\times] [N] [=] [\sqrt{}] [\sqrt{}]; [\times] [N] [=] [\sqrt{}] [\sqrt{}] \dots$ ’ in the case of ‘ $a = 2, N = 2$ ’. The explanation justifying the method is given in a way ‘mathematical’. The recurrent relation is given at first without *raison d’être*, and then the formula corresponding to the method ($X_{n+1} = \sqrt[4]{X_n \times N}$) is deduced. Group 2 firstly regarded the given solution as A_{0-1}^\diamond . This answer generated a new question Q_1 : why should we consider ‘ $4a_{n+1} = a_n + 1$ ’? The students was trying to determine the general term a_n by themselves in

interacting with the milieu. However, at that moment, the teacher intervened and asked again Q_0 about the method of calculation. Indeed, the students read and follow the proof for a method, although they did not know the method itself. This teacher's intervention lead the students to focus on the method given in the first website A_{0-1}^\diamond . Further, this information from the media prompted the use of calculators as a part of their milieu. The students worked back and forth between reading the proof on the webpage and calculating using a calculator and found that this method works after checking it with different numbers. The method they verified became their own answer A_0 . However, two new questions were produced successively: 'why does such method works?' (Q_2) and 'why could the first number a be arbitrary?' (Q_3). Related to these questions, the small questions and answers could be identified. For example, they asked about the operations of calculator like 'why are there so many repetitions?' In fact, they did not even realised at the first moment that the repetitive operations and its convergent value correspond respectively to the recurrent relation and the limits of a series. After a short moment, they found an answer related to the limit of a series. For Q_3 , they asked by themselves the meaning of 'arbitrary' and were searching an answer on the Internet. They found some explanation on the websites, but they understood rather in the second website (A_{0-2}^\diamond) about the method of calculating cube roots, wherein the page explains the same method as that of the first website and writes the first number can be any number such as 1, 2, 3 (http://www.nishnet.ne.jp/~math/mr_boo/DENTAKU1.HTM).

In searching for the answers to Q_2 , the students found the third webpage (A_{0-3}^\diamond : <http://blog.livedoor.jp/ddrerizayoi/archives/26225078.html>). This page provides the same method as before in the case of ' $a = 1, N = 7$ ', and also a justification with the recurrence relation of exponents. In contrast to the first and second webpages, the third one explicitly describes the process of exponential changes in each operation: $0 \rightarrow 1 \rightarrow 1/4 \rightarrow (1/4) + 1 \rightarrow (1/4)((1/4) + 1) \dots$. The students interacted with this information as a part of milieu and advanced their inquiry. They first realized the relationship between the operation on the calculator and the number of exponent and also how the recurrent relation given in A_{0-3}^\diamond ($a_1 = a, a_{n+1} = 1/4(a_n + 1)$) relates to the operations. In

Handwritten mathematical proof for the limit of a sequence defined by a recurrence relation. The proof shows the derivation of a closed-form expression for a_n and its limit as n approaches infinity.

$$a_1 = 0, a_{n+1} = \frac{1}{4}(a_n + 1)$$

$$a_{n+1} - \frac{1}{3} = \frac{1}{4}(a_n - \frac{1}{3})$$

$$\{a_n - \frac{1}{3}\} \text{ is a geometric sequence with first term } -\frac{1}{3} \text{ and ratio } \frac{1}{4}.$$

$$a_n - \frac{1}{3} = -\frac{1}{3} \left(\frac{1}{4}\right)^{n-1}$$

$$a_n = -\frac{1}{3} \left(\frac{1}{4}\right)^{n-1} + \frac{1}{3} \quad (n \rightarrow \infty)$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left\{ -\frac{1}{3} \left(\frac{1}{4}\right)^{n-1} + \frac{1}{3} \right\} = \frac{1}{3}$$

よって、 x の3乗根を電卓で求める手順、
 $1 \times 2 = \sqrt{2}, x \times 2 = \sqrt{2}, x \times 2 = \sqrt{2} \dots$
 を n 回くり返すことで $\sqrt[3]{2}$ の値に近づく。

Fig. 3. A_2 : the proof written by a member of Group 2

this website, while the limit of the series '1/3' is given, its proof was not given. Reading this page, the students found the general term a_n by themselves give a proof like Fig. 3. This is thus their devised answer A_2 to Q_2 .

After getting A_2 , the question Q_3 became one of main questions the students of Group 2 tackled in the last part of the first teaching period. We could not provide details here.

However, they carried out different activities such as observing the behavior of convergence when changing the initial number a in the spreadsheet. These processes of inquiry are summarised as Fig. 4.

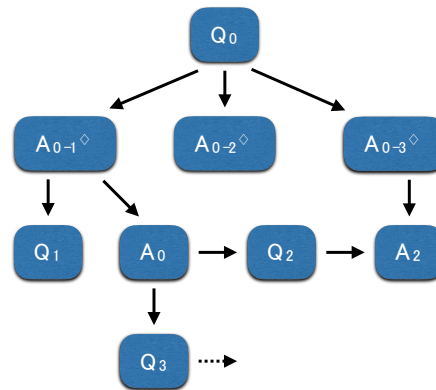


Fig. 4. Tree structure of SRP of Group 2

DISCUSSION: A *POSTERIORI* ANALYSIS

In SRP of Fig. 4, three questions Q_1 , Q_2 and Q_3 emerged not from the teacher but from the students through the media-milieu dialectics. For example, Q_2 and Q_3 were generated, while they were reading the proof given in the first webpage (A_{0-1}), that is to say, Q_2 and Q_3 were produced as a result of the interaction with the milieu including A_{0-1} obtained from the media. What is interesting here is that these questions require a kind of proving activities, while Q_0 asked by the teacher requires just providing a method which could be easily found on the Internet. Further, Q_3 was not expected by the teacher while Q_2 was. In ordinary teaching situations, the question asked by students would not be dealt with as a main issue, because they are based on a didactic contract that the teacher has exclusively legitimacy about questioning (e.g. Chevallard, 2015). In addition, the teacher has a difficulty of creating a situation wherein students ask by themselves why-questions and elaborate their justification to them, as we have discussed earlier. However, in the situations of SRP, such activities could be easily observed.

On the other hand, a written mathematical proof was given only for Q_2 , and Q_3 was investigated empirically at least in this teaching period. Nevertheless, the students validated the method A_0 on their own by interacting with their milieu, and made their own answer A_2 to the question Q_2 by proving a statement. In this step, the students constructed a proof in order to understand the method of calculation and the answer A_{0-3} obtained from the media. The proving for understanding is unfortunately infrequent in ordinary class, although Hanna pointed out that 'proof can make its greatest contribution in the classroom only when the teacher is able to use proofs that convey understanding' (2000, p. 7). The mathematical understanding should be a principal role of proof. However, to what extent does the proving activities carried out by secondary students in mathematics classroom really lead the mathematical understanding? We

consider that the inquiries using the Internet like the SRP have a possibility for overcoming this problem.

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CONTENT AND FORM – ALL THE SAME OR DIFFERENT QUALITIES OF MATHEMATICAL ARGUMENTS?

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Students entering academic mathematics programmes struggle with various challenges in their transition from secondary school to tertiary education. One challenge is the strong focus on formal-deductive argumentation and proof in university mathematics. Producing acceptable mathematical arguments requires both, the ability to find deductive lines of arguments as well as skills to communicate these arguments with precision. We present a study with N=159 students at the transition from secondary to tertiary education that examines how the quality of mathematical arguments and of different formal aspects of their presentation are interrelated. We discuss implications for research as well as for support of students at the beginning of their mathematics study.

INTRODUCTION

A substantial amount of students give up studying mathematics during their first year at university (Heublein, 2014). Possible reasons for the high drop-out rate might be that the character of mathematics as a scientific discipline changes dramatically in the transition from school to university. This is not primarily a change of topics, but there is a shift toward an increased depth in the subject, with respect to the understanding and use of formal mathematics (Clark & Lovric, 2008). In tertiary mathematics courses, abstract concepts, formally presented arguments and proofs play a central role. Students are exposed to the emphasis on multiple representations of mathematical objects and on the precision of mathematical language (Clark & Lovric, 2008). Our study is situated in the transition phase from secondary to tertiary education with a specific focus on mathematical argumentation and proving, and the use of formal representations to communicate mathematical arguments.

Mathematical argumentation, i.e. to generate arguments for or against a mathematical conjecture and to convince oneself as well as the mathematical community about their validity, comprises empirical exploration (e.g., Koedinger, 1998), logical deductions and the ability to deal consciously with formal-symbolic representations and mathematical language (Epp, 2003). Several studies indicated that students at all levels have great difficulty with the task of proof construction (e.g., Healy & Hoyles, 1998; Ufer, Reiss, & Heinze, 2008). Even students who want to pursue undergraduate courses in mathematics at university often show poor proof-writing attempts, which may consist of little more than a few disconnected calculations or are characterised by an imprecise or incorrect use of mathematical words or phrases (Epp, 2003). There has been much research pointing to reasons for these deficiencies (e.g., Selden & Selden, 2011). Models of the proving process suggest to differentiate two idealized sub-

processes of proving when searching for explanations (Selden & Selden, 2009): Firstly, students have to find adequate arguments and organize them into a deductive chain mentally. Secondly, they have to communicate their arguments and proofs in a formally correct way according to mathematical standards.

The content of mathematical arguments

Identifying a conclusive chain of mathematical arguments is a complex problem solving process that relies on several individual prerequisites, like knowledge of heuristic strategies (Schoenfeld, 1985) and conceptual mathematical knowledge (Ufer et al., 2008). Moreover, methodological knowledge on the nature of proofs (e.g., Healy & Hoyles, 1998) is necessary to direct this search process. For example, evaluating the truth or falsity of mathematical statements requires knowledge about the role of examples and counterexamples (Koedinger, 1998). During the proof construction process, students have to identify relations between mathematical concepts, and select those for which they see a chance to support them by acceptable mathematical arguments and organize them in a conclusive deductive chain.

When analysing students' proof skills, research has often focused on the *content of students' arguments* that become visible in students' work, deliberately disregarding the *formal quality* of the presentation of these arguments (e.g., Healy & Hoyles, 1998; Reichersdorfer, Vogel, Fischer, Kollar, Reiss, & Ufer, 2012). Even though this is a reasonable choice when viewing proof from a problem solving perspective, the adequate presentation of arguments is also a relevant goal of most university mathematics programmes (Epp, 2003).

The form of mathematical arguments

Engelbrecht (2010) points out that students have to be able to communicate their arguments in a "subject-specific, scientific language". When thinking about the quality of a specific mathematical argument, however, the use of a specific formal notation or corresponding mathematical language constructs (like "Let x be...", "For all y ...") is certainly not a necessary feature for the validity and acceptability of a proof, even if this feature occurs in many mathematical texts. On the other hand and more generally, the precise communication of mathematical ideas is a decisive criterion. This means that, if a specific formal notation or specific mathematical language is used, it must be used in a precise and correct way.

However, there is a wide basis of research documenting that students have problems to use formal notations and specific mathematical language in a correct way: (1) Students' difficulties in *using logical symbols* correctly are well documented (Epp, 2003). One reason for this might be that logical statements can be interpreted differently in formal and informal settings. For instance, in informal settings, the statement "Some A are B." is taken to imply that "Some A are not B.", but in mathematics, this implication is not valid (Epp, 2003). (2) Clement (1982) reported that a large proportion of university engineering students have problems translating relationships expressed in spoken language into corresponding *mathematical*

expressions, and vice versa. A famous example is the statement “There are 8 times as many people in China as there are in England”. Some students seem to treat variables as symbols for objects or persons, writing $8C=E$ in this case. Comparable problems might be identified in symbolizing relationships like the divisibility of two integers. (3) Connected to this, students often have trouble with using variable symbols correctly. For example, they fail to understand that the value of a variable can be arbitrary, but fixed and does not change its value within one algebraic expression. Some also fail to introduce the meaning of the variable symbols they use. Epp (2011) noted that, alongside the emphasis on mechanical procedures at school, the meaning of variables as unknown quantities with specific properties, such as in functions or as expression for universal statements may be obscured. (4) Students’ problems with quantifiers are also well-documented (e.g., Dubinsky & Yiparaki, 2000; Epp, 2003; Selden & Selden 2011). It seems to be a challenge for students to understand that the meaning of a statement is influenced by the order of the quantifiers, or to know the scope of a quantifier. Selden and Selden (1995) see students’ difficulties in interpreting implicit quantifiers (i.e. expressed in words, not symbols) as a significant barrier for proof construction.

Even though newer studies take into account the content of students’ arguments as well as their formal quality (e.g., Selden & Selden, 2009, 2011), the relation between the two has rarely been studied. In some works, the two quality aspects seem to be treated as fairly separated, as if skills in the formal presentation of arguments are something that is necessary primarily *after* a conclusive chain of arguments is found (e.g., Engelbrecht, 2010). While the skill to use some formal aspects might – in this sense – be fairly independent of students’ skills to find conclusive chains of arguments, this needs not to be held for all formal aspects. Some works emphasize a stronger connection between, for example, understanding the language of logic (as different from everyday language) and logical notation, and the understanding of logical structures themselves (e.g., Epp, 2003). This is in line with theories that emphasize an epistemic function of language use (Sfard, 2008), which assumes that (mathematical) thinking is at least partly structured by the mental use of language. Following this line of argument, not being able to use formal language, notations or representations correctly might reflect and also cause a deficient understanding of the arguments that are constructed and presented in a proving or argumentation process. Thus, it remains an open question, which aspects of formal quality of students’ arguments are connected to the content quality of these arguments, and which are less related to it.

GOALS OF THE STUDY AND RESEARCH QUESTIONS

Although undergraduate students’ problems in constructing mathematical proofs and generating rigorous mathematical argumentations have been reported in many studies (e.g., Selden & Selden, 2011), there have been little attempts to study how the content quality of mathematical arguments and their formal quality are interrelated. To fill this gap, the present study addresses the following questions: (1) Which difficulties of mathematical argumentation regarding content and formal quality can be identified?

(2) Do content quality and different dimensions of formal quality of students' arguments form a one-dimensional construct, or is it necessary to differentiate multiple quality dimensions of students' mathematical arguments?

DESIGN AND METHODS

N=159 incoming students (72 female) from a regular mathematics programme, financial mathematics programme and a mathematics teacher education programme with an average age of 19.67 years ($SD = 3.18$) from two German universities took part in our study, which was embedded in a voluntary two-week preparatory course for university mathematics. Daily lectures and tutorials about elementary number theory as well as about other basic topics such as sets, functions and relations were included in this course. On day four, students worked for 45 minutes on mathematical argumentation problems from elementary number theory on their own adapted from Reichersdorfer et al. (2012). These comprised technical proof skills (e.g., "Show that for all natural numbers, a and b the following statement is true: If 15 divides $(10a-5b)$ then 3 divides $(2a-b)$.", 5 items), flexible proof skills (e.g. "Prove the following statement: The product of three consecutive even numbers is divisible by three.", 4 items) and conjecturing skills (e.g. "Prove or refute the following statement: If the sum of two natural numbers is even, then the product of these two numbers is always even.", 4 items with correct and false statements, each).

To score the *content quality* of students' argumentations a four-level coding was applied. For this, we analysed the mathematical ideas visible in the students' solution, disregarding their formal presentation as much as possible. We scored no or irrelevant trials with score zero, partially correct solutions including less than half of all central arguments required with score one, solutions including more than half of all central arguments but with small methodological errors (like an incorrect proof structure) with score two and completely correct solutions with score three.

Coding schemes for different aspects of formal quality were developed based on data from prior studies: *Symbolizing divisibility* (e.g., use of the symbol \mid) was coded on two levels (0: incorrect, 1: correct). A three-level coding was applied to score the *use of logical symbols* (e.g., \Leftrightarrow or \Rightarrow ; 0: using logical notations, although no logical statement is made, 1: use of incorrect logical symbols for logical statements, 2: correct), *symbolizing definitions* ("Let x be 3...", $=$, $:$, $:=$, $:$ \Leftrightarrow) (0: not symbolizing of definitions, although necessary, 1: incorrect, 2: correct), and the *use of variables* (0: inconsistent or incorrect, 1: correct and consistent, but without systematic introduction, 2: completely correct). The *use of quantifiers* (universal quantifiers and existential quantifiers) was coded on four levels (0: no use of quantifier, although necessary, 1: incorrect use of a single quantifier, 2: correct use of single quantifiers, but problems with the use of consecutive quantifiers, 3: correct). If a certain formal notation or corresponding language constructs were not used in a student solution, the respective value was coded as missing value. The only exception was if the corresponding aspect would have been required to communicate the argument according to the mathematical

standards of the course. If this was the case and the corresponding aspect did not occur, this was coded with the lowest score (0). All arguments were coded by two independent raters and interrater reliability for each part of the test was found to be good (Mean of ICC=.86, SD =.08).

RESULTS

Descriptive results for the content quality of arguments can be found in Table 1.

	Technical proof skills	Flexible proof skills	Conjecturing skills (true)	Conjecturing skills (false)
Mean quality score	1.37 (.75)	1.24 (.80)	1.30 (.73)	1.68 (.85)

Table 1: Means (and standard deviations) of the content quality of arguments

On average, less than half of all arguments required to completely solve the items were present. The findings further support prior results (Reichersdorfer et al., 2012), that students have less trouble with refuting false statements than to solve technical proof tasks, tasks that require flexible proof skills, or conjecturing tasks for true statements. As regards our research here, we see substantial variation in students' proof skills. For space restrictions, we will not differentiate the different task types in the further analysis, even though this might be an interesting direction to pursue.

Table 2 presents how often formal quality aspects were coded in students' solutions, as well as presents means and standard deviations of the standardized quality scores for the different aspects of formal quality. As might be expected from the type of tasks, symbols for defining mathematical objects occurred comparably rarely (24.8%), while variables were used in 84.4% of the solutions. It was, nevertheless, possible to write arguments of high content quality without using variables. We would like to repeat that not using a certain formal notation or corresponding language construct did only result in coding as "incorrect (0)", if the corresponding formal aspect would have been necessary to communicate the students' solution according to the norms of the course.

	Symbolizing divisibility	Use of logical symbols	Symbolizing definitions	Use of variables	Use of quantifiers
Cases	63.5%	59.3%	24.8%	84.4%	40.9%
Mean score	.85 (.36)	.77 (.41)	.53 (.25)	.71 (.32)	.53 (.46)

Table 2: Number of cases coded, means (and standard deviations) of the standardized quality scores of the use of symbolic notations and formal representations

Results indicate that symbolizing definitions and the use of quantifiers caused the most problems, followed by the use of variables and the use of logical symbols. We identified the following difficulties in the use of symbolic notations and formal representations: In 9.4% of all solutions, an incorrect symbolizing of divisibility could be observed. Students showed an incorrect order of symbols or wrote " a/b " even though a did not divide b . In 12.1%, students applied logical symbols invalidly. For

example, they used the implication symbol to delineate different statements, even though no valid implication could be established between the two statements. They marked valid logical relations by the use of an incorrect symbol in 3.4% of all solutions. In 2.6 % of all solutions, definitions were not made explicit at all, although the meaning of a symbol had been changed. In 18.4%, definitions were made explicit, but using a wrong symbol. For instance, some students marked a definition only by using the usual equal sign. In 6.4% of all solutions, variables were used inconsistently, for instance, representing the sum of consecutive even numbers by $(2k) + (2m)$. In 36.1%, variables were used without a systematic introduction that explained what they stood for. We found that in 14.5% of all solutions, students did not use quantifiers or verbal quantifications, even it would have been necessary. In 6.7%, single quantifiers or verbal quantifications were used incorrectly, for example introducing a variable x , with a statement like “ $\exists x...$ ” instead of “ $\forall x...$ ”. In less than 1% of all solutions, students used single quantifiers correctly, but still showed problems with the order of consecutive quantifiers.

	Factor 1	Factor 2
<i>Quality of arguments</i>	.445*	.142
<i>Symbolizing divisibility</i>	.706*	-.037
<i>Use of logical symbols</i>	.676*	.010
<i>Symbolizing definitions</i>	.091	.178*
<i>Use of variables</i>	.004	.582*
<i>Use of quantifiers</i>	-.037	.352*

Table 3: Geomin rotated factor loadings

To analyse how the quality of arguments and the quality of different formal aspects of their representations are interrelated, we used exploratory factor analysis. Missing values in codings of formal quality were accounted for using the Full Information Maximum Likelihood (FIML) method. Each single task solution represented one case. The resulting hierarchical structure of the data (solutions nested in students) was also accounted for statistically analysis. Principal components analysis was used because the primary purpose of this study was to identify and later compute composite scores for the factors. Initial eigenvalues indicated that the first two factors explained 32.67% and 18.5% of the variance in all quality codings. The two factor solution was preferred because of our previous theoretical considerations and because it showed a significantly better model fit than the one factor solution ($\chi^2(9) = 40.946$, $p < .001$). Table 3 contains the Geomin rotated factor loadings for all quality criteria. The two factors were correlated significantly ($r = .40$, $p < .01$).

DISCUSSION

The goal of this study was to identify students' difficulties of mathematical argumentation and proving, and to analyse how the quality of the content of students'

arguments and the formal quality of their presentation are interrelated. Firstly, our study replicates results that finding adequate arguments and communicating arguments with formal precision is a great challenge for students at the secondary-tertiary transition in mathematics (e.g., Clark & Lovric, 2008; Selden & Selden, 2009). In particular, when longer arguments have to be produced, students at the transition show similar problems as were reported for secondary students (Ufer et al., 2008) to find and describe conclusive chains of multiple deductive arguments. Regarding the formal quality of students' arguments, our sample shows evidence of all those problems that are documented in the literature, e.g., use of mathematical symbols, use of variables and quantifiers, and explicating definitions (e.g., Epp, 2003; Selden & Selden, 2011).

Apart from this, our study is to our knowledge the first that systematically studies relations between the content of students' arguments and their formal presentation. There are good theoretical arguments to assume that some of the formal aspects are quite unrelated to the content quality of an argument (Engelbrecht, 2010). Nevertheless, there are also theoretical reasons to assume that some formal aspects might be connected to the content quality of an argument (Epp, 2003; Sfard, 2008). We took an explorative approach to study these relations, and our analyses indicate that two dimensions of argument quality can be distinguished in our sample. One of these dimensions is substantially related to the content quality of students' arguments, but also to higher scores on *symbolizing divisibility* and *using logical symbols* for the respective arguments. Both of these formal aspects address relations between mathematical ideas (numbers and statements). The other dimension, largely unrelated to content quality, described the *use of variables* and *quantifiers* and – less pronounced – *symbolizing definitions*. These formal aspects seem to be more relevant to clarify the meaning of the mathematical objects used in an argument.

Of course our study was restricted to a specific educational setting and mathematical content. Nevertheless, our results indicate that not all, but some aspects of formal argument quality go along with the quality of the argument to be presented itself. If these results can be sustained, they might offer fruitful information to conceptualize student support in the learning of mathematical argumentation and proof. In particular, it might be possible to address some aspects (e.g., variables, quantifiers) separately in form of general behavioural schemata (Selden & Selden, 2009), while for others (e.g., logical symbols) a deeper connection to the underlying argument content will be necessary.

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WHAT HAPPENS WHEN ENTREPRENEURSHIP ENTERS MATHEMATICS LESSONS?

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According to the Swedish curriculum, entrepreneurship is to permeate all teaching in primary school. However, little is known about how entrepreneurship influences the teaching of different subjects. This paper reports on an educational design research study investigating the potential in combining entrepreneurship and mathematics in primary school. Two examples are given of how mathematics teaching changes when entrepreneurship enters mathematics lessons. The results indicate that there may be a win-win situation between mathematical and entrepreneurial competences, at least when it comes to creativity and tolerance for ambiguity.

INTRODUCTION

Entrepreneurial and mathematical competences are two of the key competences the European Community stresses as important in a society of lifelong learning (EU, 2007). On the basis of this, entrepreneurship is getting increased interest in educational settings around the world, not necessarily in the sense of starting companies but rather as an approach to education that gives children opportunities to develop abilities that characterize entrepreneurs. It is believed that entrepreneurial competences, like mathematical competences, will contribute to individuals' future success in society, no matter what kind of work they do.

This paper reports on an educational design research study exploring the potential in combining entrepreneurship and mathematics in Swedish primary schools. It seems to be generally assumed that entrepreneurship is necessarily something positive, but there are very few studies on entrepreneurial competences in subjects in general and in primary school in particular. In the study presented here, instead of taking an unconsidered stance, we try to investigate both possibilities and reservations regarding this combination. The research question we ask is: What happens when entrepreneurship enters mathematics lessons?

ENTREPRENEURIAL COMPETENCES

When stressing entrepreneurship as important in a society of lifelong learning, the European Community refers to the ability to turn ideas into action, which involves such competences as creativity, risk taking, innovation, and managing projects (EU, 2007). As mentioned, the European Community's emphasis on the importance of entrepreneurial competence has increased the attention directed towards entrepreneurship in educational settings and, according to the Swedish curriculum, entrepreneurship is to permeate all teaching in primary school (National Agency for Education, 2011).

The school should stimulate pupils' creativity, curiosity and self-confidence, as well as their desire to explore their own ideas and solve problems. Pupils should have the opportunity to take initiatives and responsibility, and develop their ability to work both independently and together with others. The school in doing this should contribute to pupils developing attitudes that promote entrepreneurship. (National Agency for Education, 2011, p. 11)

Based on the European Community, the national curriculum, and research literature on entrepreneurship (Leffler & Svedberg, 2010; Sarasvathy & Venkataraman, 2011), this study focused on the following six entrepreneurial competences: creativity, tolerance for ambiguity, courage, ability to take initiative, ability to collaborate, and ability to take responsibility. Creativity is about finding new, for the individual, solutions to new and old problems. Tolerance for ambiguity is about solving a task even when a situation is ambiguous and not fully understood, and courage is about stepping out of the comfort zone into situations the individual is not fully comfortable with. Ability to take initiative is about being proactive. The ability to collaborate involves both sharing and absorbing thoughts and knowledge, and the ability to take responsibility involves responsibility for both oneself and others.

MATHEMATICAL COMPETENCES

When stressing mathematical competences as important in a society of lifelong learning, the European Community emphasizes the ability to solve problems in everyday situations (EU, 2007). In the Swedish national curriculum mathematics is described as a “creative, reflective, problem-solving activity” (National Agency for Education, 2011, p. 62). On the basis of these documents, problem solving in mathematics was especially emphasized in the study. In line with research (Cai, 2010; Lesh & Zawojewski, 2007), problem solving is described in the national curriculum both as a purpose (an ability to formulate and solve problems) and a strategy (a way to acquire mathematical knowledge). The study focused on both of these; students worked with problem-solving tasks they did not know in advance how to solve, and they therefore had to develop new (for them) strategies, methods, and/or models to solve the tasks.

EDUCATIONAL DESIGN RESEARCH

The study was conducted through educational design research, which is not a fixed method but a genre of inquiry. Common in educational design research is the iterative development of solutions to practical and complex educational “problems” where the context for the empirical investigation is the educational arena (McKenney & Reeves, 2012). The intention of the methodology is to enable impact and transfer of research into school practice by building theories that “guide, inform, and improve both practice and research” (Anderson & Shattuck, 2012, p. 16). Since collaboration with practitioners improves understanding of the “problems,” educational design research is conducted in collaboration with, not solely for or on, practice (McKenney & Reeves, 2012).

The complex educational “problem” to be explored in the study presented in this paper was “what happens when entrepreneurship enters mathematics lessons”? Each iterative design cycle included (a) preparations for a mathematics lesson into which entrepreneurial competences were merged, (b) implementation of this lesson, and (c) retrospective analysis of the lesson. The goal in educational design research is to – through the iterative design cycles – develop design propositions, which refer to further specifications of what the design should look like to reach a desired situation. However, this study was more explorative since the desired situation was not known in advance. When the study was initiated it was not known whether bringing entrepreneurship into mathematics lessons was something desirable or not; that is what was to be investigated. Problem solving and the six entrepreneurial competences presented in the previous section framed the design of the lessons; that is, creativity, tolerance for ambiguity, courage, ability to take initiative, ability to collaborate, and ability to take responsibility. Each iterative design cycle was conducted in collaboration between teachers and researchers.

THE STUDY

The study was conducted following the above-described educational design research. Nine researchers from mathematics education and entrepreneurship as well as approximately 30 teachers from eight primary schools were involved. These eight primary schools were selected based on the teachers’ interest in being involved in the research project. In Sweden, as in other countries around the world (Tatto, Lerman & Novotná, 2009), most primary school teachers are educated as generalists, teaching several subjects, one of which is mathematics.

This paper will focus on one of the involved schools where the author was the researcher in charge. Ten teachers from preschool class (six-year-olds) up to grade five (eleven-year-olds) chose to be part of the study. In the previous school year the teachers from this school had been involved in a national professional development program named Boost for Mathematics. This program was initiated by the government in 2012 with the aim of improving mathematics teaching and thereby students’ learning. The program is organized around teacher collaboration, where teachers work in groups with external tutors. Within this program these teachers had focused especially on problem solving in mathematics. Thus, based on Boost for Mathematics, they were experienced with problem solving in mathematics, both theoretically and practically. They were also used to collaborating with external participants, so what was “new” for them with this study was mainly the entrepreneurial competences.

Before initiating the iterative design cycle the teachers and their students were interviewed about their experiences with mathematics and entrepreneurial competences. In addition, the researcher visited each class to get an idea of the ongoing teaching and to get to know the teachers and students better. All requirements for information, approval, confidentiality, and appliance advocated by the Swedish Research Council (2008) were followed. After the interviews and the visits, the

researcher and the teachers met to plan the first mathematics lesson into which entrepreneurial competences were to be merged.

In line with educational design research, the planning of the lessons was conducted in collaboration between the researcher and the teachers. The teachers were told that they could either modify tasks they had used before or choose tasks that were new to them. The teachers decided to work with tasks where the mathematical idea would be the same from preschool class up to grade five but with adapted levels of difficulty. The teachers also decided to focus on one of the entrepreneurial competences at a time. The researcher was present during the lessons, taking notes. After each lesson was conducted in all classes, the researcher and the teachers met for an evaluation. This was made on an evaluation form that focused on both entrepreneurial and mathematical competences as well as on possible connections between them. After the evaluation, the next lesson was planned, and the iterative process continued in this manner throughout one school year. As space is limited, this paper will present only the first two design cycles, which introduced creativity and tolerance for ambiguity.

RESULTS

As mentioned, the teachers chose to introduce one of the entrepreneurial competences at the time, and below is a description of how they included creativity and tolerance for ambiguity in their mathematics lessons. As also mentioned, each iterative design cycle included (a) preparations for a mathematics lesson into which entrepreneurial competences were merged, (b) implementation of this lesson, and (c) retrospective analysis of the lesson.

Creativity – The tower task

(a) The teachers chose to start with creativity. In relation to mathematics, the teachers translated creativity as being able to solve tasks without being told which strategy to use beforehand. The teachers chose to modify a task they were familiar with from the Boost for Mathematics program. In the task, the students are shown a picture of a tower and asked how many blocks were used to build it. The teachers considered the task creative since it can be solved by using different strategies; for example, students can build with blocks, draw, make patterns, and/or count. The level of difficulty could be adapted by selecting different towers for students of different ages. No students were to work with towers they had worked with beforehand.

(b) When the teachers worked with “tower tasks” during the Boost for Mathematics program, they introduced strategies for solutions at the start of the lesson. They did not do so during this study, however, in order to promote creativity; instead, they let the students present strategies for solutions at the end of the lesson. The teachers were surprised by the solutions the students came up with, which were more numerous and sometimes more innovative than they had expected. The students thought up more strategies than the teachers normally would have introduced at the start of a similar lesson.

(c) When planning the “tower task” the teachers treated creativity as something “new” to be added into the mathematics lesson. They chose a task they were already familiar with, but to promote creativity they carried it out differently than before. The difference was mainly in the instructions – or rather in the lack of instructions – given in the introduction of the lesson. One could argue that creativity is not something new in mathematics and/or problem solving, and as noted earlier, mathematics is described as a “creative, reflective, problem-solving activity” in the national curriculum (National Agency for Education, 2011, p. 62). However, in this lesson creativity became a goal in itself, not just one notion among others in the curriculum. Studies have shown that there are often big differences in the potentials of a problem-solving task and the opportunities for learning when the task is used in the classroom. This is because many teachers tend to give too many instructions for solutions when introducing lessons, leaving no space for the students to understand the task or figure out its solution by themselves (Mason & Johnston-Wilder, 2006). This seems to have been the case for the teachers in this study also, but this changed when creativity was introduced as a goal in itself. In the evaluation it became apparent that creativity could be a valuable competence for students when working with mathematics, but also that mathematics tasks could be used to promote students’ creativity. Thus, there seems to be a win-win situation between entrepreneurship and mathematics when it comes to creativity, as illustrated in figure 1. The design proposition derived from this design cycle became “say less beforehand.”

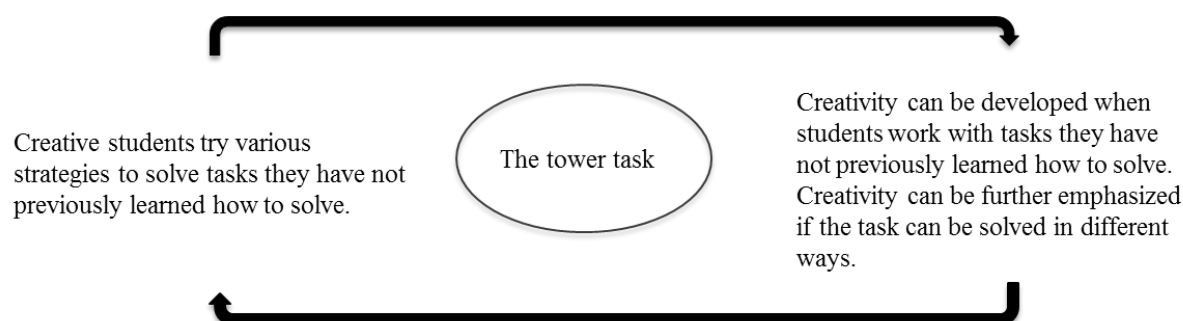


Figure 1: A win-win situation between entrepreneurship and mathematics when it comes to creativity.

Tolerance for ambiguity – The Fermi problems

(a) In the next design cycle the teachers chose to work with tolerance for ambiguity. To promote tolerance for ambiguity they decided to work with Fermi problems which were new for them. These are open problems where exact answers are difficult or impossible to arrive at, so estimates must be made instead, based mainly on known facts or facts that can be easily found (Flognman, 2011). Requiring students to make and justify their estimates without having a fixed answer to check against presents them with a situation of ambiguity. Fermi problems can be solved by using different strategies. As with the “tower task,” the level of difficulty was adapted for students of different ages. Both the context and the content of the tasks chosen for this design cycle

were unfamiliar to the students, and based on the positive results from the first design cycle focused on creativity, “say less beforehand” was used as a starting point for these lessons.

(b) The original plan was to have each class work with one Fermi problem, but because the experiences were so positive, the teachers continued to work with these tasks in one or several of their mathematics lessons for several weeks. Examples of tasks used were “How much popcorn would be needed to fill our classroom?” and “Will I manage to bike to the ice-skating day if I start at half past seven?” The popcorn problem was given to students who had not learned the formula for calculating volume, and the bike problem was given to students who were going on an ice-skating day who not had previous experience calculating speed. Several students became quite troubled when the teachers introduced the Fermi problems. They said things like “That is not possible to find out” or “Do you really know the answer?” However, most of the students became very involved in solving the tasks. To the teachers’ great surprise, it was the students who they considered to be the more talented and interested in mathematics who most strenuously resisted working with the Fermi problems. These students continued to argue that the tasks were impossible to solve since there were no “real answers” and thus these were not proper mathematics tasks. Further, the teachers were surprised by the mathematics that the students used, which involved more advanced calculations than the students had normally worked with, and during the presentations at the end of the lessons, the formulas for calculating volume and speed were presented by the students.

(c) This time the teachers had chosen to work with tasks of a kind that neither they nor their students were already familiar with. Fermi problems are not new in mathematics education, but it was not until tolerance for ambiguity was introduced as an entrepreneurial competence that the teachers at this school became interested in these kinds of tasks. The context of the tasks was new and the students who had been considered particularly talented and interested in mathematics had the hardest time coping with this change in approach. Maybe dealing with tolerance for ambiguity was harder for these students as they were normally quite sure of what to do in the mathematics lessons. Even though the teachers had not specifically worked with tolerance for ambiguity previously, several students might experience tolerance for ambiguity during “everyday” mathematics lessons. In the evaluation it became visible how tolerance for ambiguity can be a valuable competence for students when working with mathematics, and that mathematics can also be used to promote tolerance for ambiguity. Thus, also regarding tolerance for ambiguity, there seems to be a win-win situation between entrepreneurship and mathematics, as illustrated in figure 2.

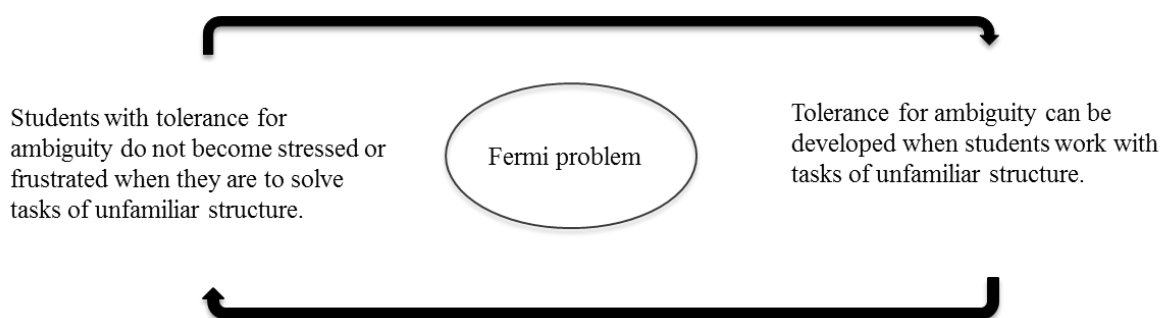


Figure 2: A win-win situation between entrepreneurship and mathematics when it comes to tolerance for ambiguity.

CONCLUSION

In this paper only examples from the first two design cycles from one school are presented and of course the local conditions make generalization a working hypothesis, not a conclusion. Further, a definition of entrepreneurial competences different from the one used in this study probably would have led to different results. However, despite these local conditions there are some interesting issues to consider.

The teachers in the study had experience working with problem solving in mathematics from a national professional development program. Thus, they were familiar with problem solving and they had been teaching problem solving in their classrooms. What was new for them was the entrepreneurial competences. The results indicate that this new focus on entrepreneurial competences actually did strengthen the mathematics teaching in their classrooms. Even though problem solving has a significant role in the syllabi in many countries (Lesh and Zawojewski 2007), teaching mathematics through problem solving has not been substantially implemented in many classrooms (Cai, 2010; Lesh & Zawojewski 2007). Maybe the positive connections between mathematical and entrepreneurial competences as presented in this paper can make a difference there. Although it can be argued that both tasks like the tower task and Fermi problems have been known and promoted in mathematics education for a long time, they were not emphasized in the mathematics lessons of these teachers until creativity and tolerance for ambiguity were introduced as important competences in themselves. Thus, the results indicate that there may be a win-win situation between mathematical and entrepreneurial competences, at least when it comes to creativity and tolerance for ambiguity. The entrepreneurial competences creativity and tolerance for ambiguity are of positive value when students are to learn mathematics in general and problem solving in particular, but at the same time the mathematics teaching can be organized in a way where students develop both mathematical and entrepreneurial competences. Thus, mathematics education can be a tool for working with students' entrepreneurial competences in primary school.

Acknowledgement

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MEANINGS AROUND ANGLE WITH DIGITAL MEDIA DESIGNED TO SUPPORT CREATIVE MATHEMATICAL THINKING

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In this study two groups of Grade-8 students interact with a new expressive digital medium, experimenting with the concept of angles through a “tool-shaping” process. The medium, designed to foster students’ Creative Mathematical Thinking (CMT), provides a novel set of affordances that are studied under the focus of a meaning-generation process. The study indicates that the students can arrive to mathematical meaning that enriches the more abstract understanding of angles, while at the same time improving upon certain aspects of CMT.

INTRODUCTION

Creative Mathematical Thinking (CMT) possesses a central role in the research in mathematics education. However, there is no consensus between the researchers as far as its definition is concerned. It can be seen as a product or process, general or domain-specific ability, situated within the ‘genius’ approach, or the problem solving and posing approach, or lately, in the so-called approach of ‘techno-mathematical literacies’ (Noss & Hoyles, 2013). Only the last one addresses the use and role of digital media for CMT. But, as Healy and Kynigos (2010) noticed, the development of CMT with the use of exploratory and expressive digital media has rarely been centrally addressed in providing users with an access to and a potential for creative engagements in meaning-generation activities. According to Ruthven (2008) the uses of these media is mainly instrumented towards contexts of traditional lecturing and demonstration of exercise solutions, which may not be characterized as learning environments that provoke exploration and dense construction of mathematical meanings by students. Consequently students are not supported to develop CMT. Our paper studies the impact of such media in the development of students’ CMT in conjunction with the generation of mathematical meaning by the students. We call this new kind of mediation ‘c-book’, (‘c’ for creativity) which is designed to afford CMT to the end users. The paper therefore focuses on the question: To what extent does this medium foster the meaning-making process? Are there indicators that aspects of CMT emerge during the meaning-making process?

THEORETICAL FRAMEWORK

Given the diverse approaches to CMT outlined above, and the relative lack of connection to math activity with expressive digital media, the concept remains fuzzy in the literature. In this study, which is part of a broader one (Papadopoulos et al, 2015),

CMT is matched with: (i) ‘construction’ of math ideas or objects which is in accordance to the constructionism that sees CMT being expressed through exploration, modification and creation of digital artefacts (Daskolia & Kynigos, 2012), (ii) Fluency (as many answers as possible) and Flexibility (different solutions/strategies for the same problem) (both seen as characteristics of a creative mathematical process in the literature, see for example Leikin & Lev, 2007), (iii) novelty/originality (Liljedahl & Sriraman, 2006) which is related with new/unusual/unexpected ways of applying mathematical knowledge in posing and solving problems, not easily met in students’ solutions (Vale et al., 2012), and (iv) usability/applicability (Stenberg & Lubart, 2000) through associations between different mathematical areas or between mathematics and other scientific fields and through elaboration which extends the (personal) body of knowledge via formulating new questions, making and checking conjectures, generalizing mathematical content, and reflecting on the mathematical work that takes place.

On the other hand, students’ engagement with expressive media provides rich opportunity for making appropriate mathematical meaning (Kynigos & Psycharis, 2003). Microworlds are such environments, allowing at the same time creativity –in our case CMT, customization and personal construction of tools (Healy & Kynigos, 2010). C-books exploit half-baked microworlds (Kynigos, 2007) which are incomplete by design, challenging students to explore the reason for the buggy behavior they show, and foster learning through tinkering. To understand the process of making meaning in this context, the instrumental approach (Verillon & Rabardel, 1995) as a ‘tool shaping’ procedure seems a useful theoretical tool which refers to how the affordances of an artefact, are adjusted by the student in order to be used as a tool for specific reasons.

THE DIGITAL MEDIUM

The C-book technology

C-book is a new expressive medium that affords the design and use of modules named c-book units. Each c-book-unit includes diverse “widgets” into the text and between the lines of the narrative, which a student can browse through, explore, experiment with, reconstruct and be actively involved in tasks and problems designed to promote CMT (Kynigos, 2015). The term ‘widget’ refers to objects, other than text, such as hyperlinks, videos and most importantly instances, or activities, from a broad range of digital tools in mathematics education such as Geogebra and MaLT2, a web-based Turtle Geometry environment which integrates Logo-mathematics symbolic notation with dynamic manipulation of 3D geometrical objects using sliders as variation tools. C-book also includes the “Workspace”, an asynchronous tool providing the interface for discussions organized in ‘trees’ (Fig.1).

The Don Quixote c-book unit

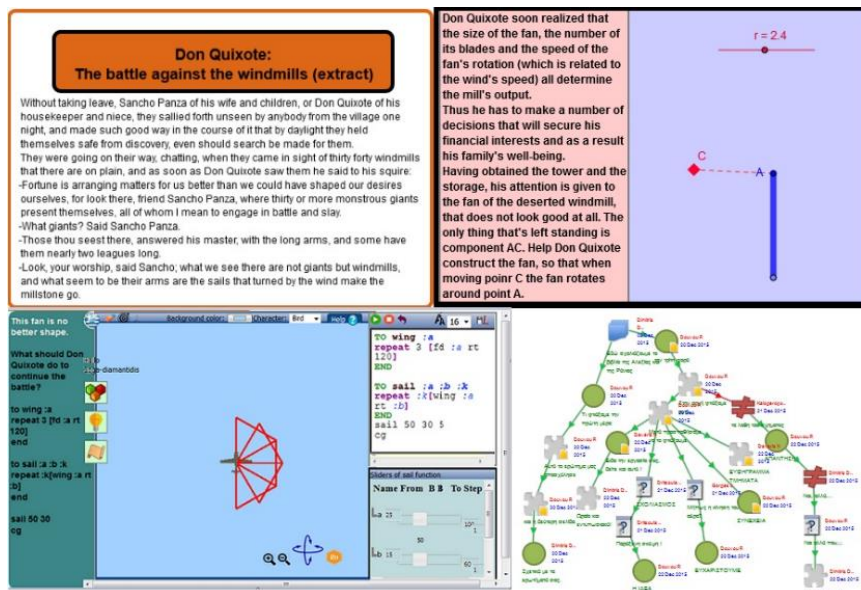


Figure 1: The C-book environment

The c-book unit used in this study presents a different twist of Don Quixote's story, agglomerated with a series of half-baked microworlds and other challenging tasks mostly in MaLT2 and Geogebra, in relation to the storyline. Its design aimed to provoke students to tinker and reconstruct windmills buggy by design, with many functionality issues. Even though the c-book technology affords a non-linear browsing of the c-book unit and engagement in any activity that the students find interesting it makes more sense to read and interact with the c-book unit starting from the beginning because of its narrative.

THE DESIGN OF THE STUDY

In the present study the methodological tool of “design experiments” (Collins et al., 2004) was used, designing and implementing an educational intervention in classroom and searching for relationships between the learning process and the use of digital media used by the students during the implementation phase.

Eighteen Grade-8 and six Grade-9 students of a public Experimental School in Athens, as well as two mathematics teachers and two researchers participated. The study took place in the school's pc lab during after-class Math Club activities (four sessions of two teaching hours each in approximately one month period). The students were divided into eleven groups of two and most of them were familiar with the usage of 2D E-slate Turtleworlds. Researchers took the role of ‘participant observers’ searching for students' interactions with the digital medium. The teachers' main role was to offer assistance in technical issues when required. Conversations between students or groups of students and their constructions on the screen constituted our data. This is why we used voice recorders and a screen-capture software (HyperCam2) to record students' interactions with the c-book unit tools. The data corpus was completed by the researchers' field notes.

The process of meaning-generation around angle of a group of students while engaged in the tinkering of two diverse widget instances of the c-book unit is presented. For the first one (Fig. 2) Logo code was developed in MaLT2 producing a quite abstract representation of an unfinished squared paper. Through the narrative, students were prompted to use it as a guide, in order to make a fan of a windmill like an origami made construction. The second one (Fig. 3) was about another half-baked logo code in MaLT2, where the stake was to shape up a windmill's fan by finding and fixing the bug.

Our hypothesis is that in a creative process (in terms of CMT) the students move from a static conceptualization to a more dynamic one, linking different angle aspects, through consecutively tinkering three diverse challenges of the same c-book unit.

RESULTS

A step towards conceptualization of angle, through elaboration of an artefact

Following the narrative linearly, the students initially had to address the first challenge, creating the fan of a windmill. It was easy for them to create the horizontal and vertical parts of the fan but they needed effort and systematic approach to overcome the difficulty of creating the oblique parts (in terms of angle and length) (Fig. 2).

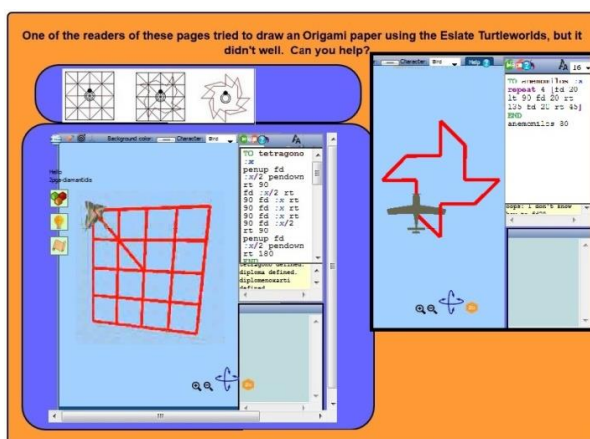


Figure 2: The ‘squared paper’ and the fan students constructed

To achieve their goal they constructed several angles in a more static context, where both arms of each angle were visible. Then they proceeded to the second task trying to shape up a windmill's fan by fixing the bug (Fig 3, left). This fan is much different not only as a geometrical figure, but as an abstract representation of a windmill as well. A line segment stands for the windmill's tower and the turn is represented by a variable in the code. By dragging a slider the user can make the fan rotating around a point. As students were observing the rotating figure, an original and unexpected idea came up. Instead of trying to fix the Logo code -as it was implicitly suggested in this task- they preferred to use the origami-made fan of the previous task.

- Student1: What we want our fan to look like? How many blades should they be?
- Student2: Let me go back... here look at the Origami code.
- Student1: Let's use this figure as a fan.

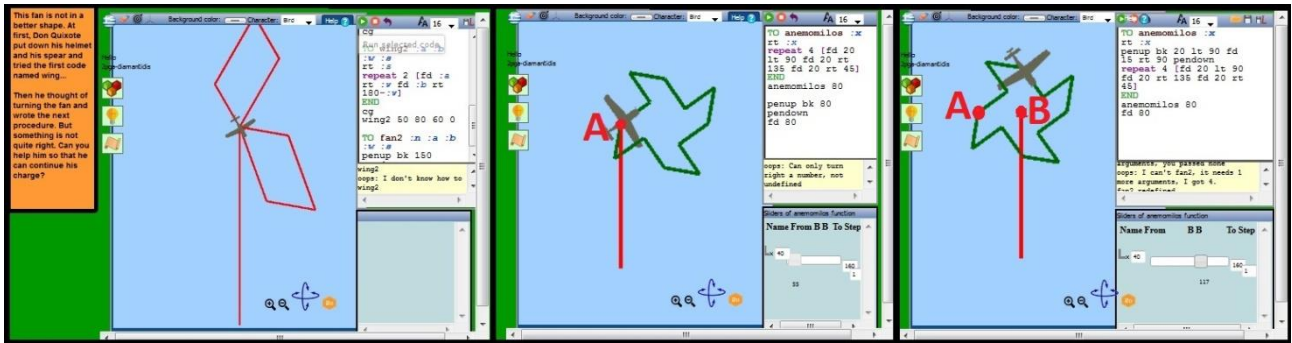


Figure 3: From the original (left) to their own elaborated artefact

This decision raised two extra issues. They had to not only reproduce the previous shape but at the same time to draw it exactly on the top of the ‘tower’, as well. However, by using their own code meant that the affordance of the slider for fan’s rotation was no longer valid.

Student1: The ‘name’ of the slider that causes the rotation in the initial fan is x. So we have to use the variable x, to make our fan rotating.

So the students transferred the command line ‘rt :x’ (turning x degrees to the right), to the new code, just because of its usability. Thus, they changed the affordances of their original artefact, being able now to make it turn around a point, using a slider. However, their initial efforts resulted in a fan that turned around point A, instead of B (Fig.3, middle, right). This made them to focus more on the mathematical aspect of the fan:

Student2: It doesn’t turn well. It should turn around this point! [*The point B*]

Student1: So we must identify the centre of the shape.

Student2: It is turning around this point [A], because A is the starting point [*She moves the slider x*]... Instead of going straight forward vertically, it turns right and then goes forward, drawing this line. As we move the slider it turns right by x degrees.

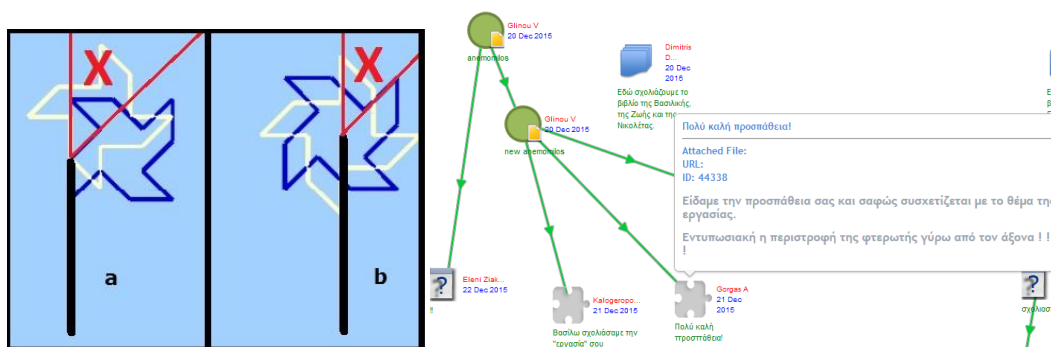


Figure 4: Constructing the new fan (left), comments on Workspace (right)

Actually, Student2 refers to an angle partly shaped. Only one arm of this angle is visible (Fig 4, left-a). The process of visualization is supported by dynamic manipulation—the dragging of the slider, while this angle is formed between the initial and the ending

position of fan's same side. A development of angle's conceptual image takes place from a more physical representation (two arms visible) to a more abstract representation (a dynamic one, with only one arm visible) under the same context (the elaborated Logo-code of a fan used as a tool with more affordances). In the next phase this conceptualization moves another step forward.

The development of a more abstract concept

As they did not achieve their goal yet, students stressed their efforts in making the fan turn around point B (Fig. 3, right). Through tinkering with the Logo-code to change the position of the fan, they decided to add an extra movement of the aeroplane (character in MaLT2) right after executing the command 'right x' to ensure a new starting point A for the fan.

Student2: Can we move the plane without leaving its trace? [*Addressing to the teacher*]

Teacher1: Yes, use the command 'penup'.

Student1: Ok, so we can move the plane to start drawing from another point, in order to turn around the centre of the shape.

Their investigation resulted in a set of moving and turning commands to make the fan turn around point B. Both arms of angle x were now invisible (Fig. 4, left-b). However students refer to this angle to describe their construction.

Student1: Now the fan rotates around its centre.

Student2: Is it right? Is it really the centre?

Student1: Yes, it is. At least the angle's vertex is on the top of the windmill's tower.

This is indicative of a more abstract conceptualization of the angle, in the same context, reflecting on and elaborating the same artefact, to make it usable and appropriate to address these challenges of the c-book unit.

This rotating Origami-inspired fan was then posted in the 'Workspace' and commented by another group as '*An impressive rotation of the fan around its axis!*' (Fig. 4, right), since it was different from the work done by the other groups who preferred to work with the code suggested by the c-book unit.

DISCUSSION

The whole work done by the students can be seen through the two lenses of the meaning-generation process and aspects of CMT. In terms of meaning generation two levels in the students' actions can be identified. The first one is related to the instrumental aspect of their actions whereas the second highlights the evolution in the students' knowledge base about the concept of angle. The students seemed to 'carry' along with them this artefact, as a tool under consideration and development. As an artefact, 'windmills' constitute a central object of this c-book unit, something that motivated students to shift their viewpoint from a pure mathematical context to a more generic framework, using this artefacts again and again as a tool to address other tasks that refer to windmills. The progressive reflection on their constructions resulted to

usable and appropriate tools that not only addressed the tasks in an effective way but also had a crucial impact on the process of meaning making. The students moved from a concrete static image about angle to a more abstract and dynamic one since they started talking about a specific angle given that they were not able to see its sides. In parallel with the meaning generation process, aspects of CMT were also apparent in the students' engagement with the c-book technology. The students' decision to abort the suggested fan and make their own is a construction inspired by the affordances offered by the c-book environment. Within this environment they also were supported by the available technology to exhibit an alternative solution (flexibility) to the problem of the design of the specific fan in a way that indicates originality. Originality can be judged on the basis of the low frequency since this was the less preferred approach (actually the only one) and the same time it was acknowledged as such by the class community (see the peer's comment in the Workspace).

So, on the one hand there is a new medium that provides a context and within this context the students work with activities that contain half-baked microworlds, change and/or fix them, connect the narrative with mathematics, make connections between the tasks. On the other hand, the combination of Constructionism, Flexibility and Originality, despite they come from different theoretical frameworks, constitutes a conceptual tool enabling researchers to better understand the students' CMT.

CONCLUSIONS

Given the lack of consensus about CMT in the Mathematics Education community it is important to look for conceptual tools that would enable us to understand CMT. In this paper it was evident that the affordances of the specific medium enabled processes of meaning generation and the presence of some aspects of CMT. In order to identify and understand the students' CMT a combination of three different theoretical constructs (constructionism, flexibility, and originality) was used. The decision to combine different theoretical frameworks seems to be a conceptual tool that contributes to our understanding of CMT and lessens the fuzziness around it. However, we need further research to develop more precise tools that will enable us to obtain a deeper understanding of CMT.

Acknowledgement

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USING MOBILE PUZZLES TO DEVELOPE ALGEBRAIC THINKING

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In this paper the potential contribution of mobile puzzles in the development of algebraic thinking in 6th graders is examined. The findings give evidence that the students started developing certain types of thinking supporting thus certain algebraic habits of mind. They did not follow arbitrary rules imposed by an authority but they induced them trying to maintain the balance of the mobiles. This was accompanied by an intuitive sense of certain properties of the operations that will later be introduced formally to them.

INTRODUCTION

Young students have natural algebraic ideas that can be used in order to develop certain mathematical habits of mind. These habits must take precedence over rules, formulas, procedures that do not derive from the students' logic (Goldenberg, Mark & Cuoco, 2010). If the foundations for their learning are based on their logic then they will have the tools not only to memorize but to understand. Goldenberg and his colleagues (2015) gave emphasis on developing algebraic habits of mind to the students initially through a series of mathematical textbook and recently by publishing a book entitled 'Making sense of algebra'. In this book a series of logico-mathematical puzzles are suggested in order to foster certain habits of mind. In this paper we focus on the usage of mobiles, one of the suggested puzzles, that can be connected with certain habits of mind such as "Puzzling and Persevering", "Seeking and Using Structure", and "Communicating with Precision". The aim is to look for evidence on whether the specific puzzle environment can foster the development of algebraic thinking and this actually constitutes our research question.

EARLY ALGEBRAIC THINKING AND MOBILE PUZZLES

Early algebraic thinking can occur in several forms in the classroom. Despite some differences it seems that the researchers (see for example Blanton & Kaput, 2005; Usiskin, 1988) agree that these forms can be met through arithmetic generalization, the study of functions and patterns, problem solving and the study of structures. The study of structures mainly refers to recognizing the structure of a simple pattern (Papic et al, 2011). Moss and McNab (2011) found that with appropriate instruction studying patterns support students to improve their algebraic thinking. Additionally, the study of structures very often concerns among others generalizing arithmetic structures (Blandon & Kaput, 2005), and the structure of equivalent number sentences (Mulligan, Cavanagh & Keanan-Brown, 2012). In this study it was decided to use mobile puzzles in accordance to Waren's (2003) (p. 123) claim that mathematical structure is concerned with the (i) relationships between quantities; (ii) group properties of

operations (associative and/or commutative operation); (iii) relationships between the operations (does one operation distribute over the other?); and (iv) relationships across the quantities (transitivity of equality and inequality). This decision raises two questions: Why puzzles? Why mobile puzzles? For the first question Goldenberg and colleagues (2015) claim that puzzles are a safe environment since you do not have to worry in case you are not able to know where to start. Moreover, puzzles allow students to take their time to think. They counterbalance the students' belief that doing mathematics is to learn a collection of facts and rules. They can support differentiate learning since they can be adapted to meet the needs and skills of the students through the control of the level of cognitive demand and the required mathematical knowledge. They can help students to develop mathematical habits useful in making sense of algebraic topics such as modeling with equations, solving equations and systems of equations, seeking and using algebraic structure. In order now to answer the second question it is necessary to present this kind of puzzle.

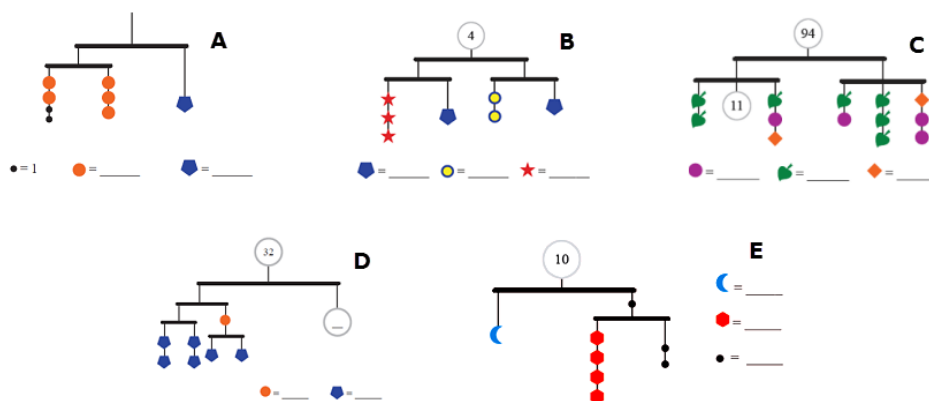


Figure 1. Mobiles A, B, C (first row) D, E (second row)

A mobile puzzle presents multiple balanced collections of objects (Fig. 1). The horizontal beams are always suspended at the middle by strings and for that reason the two ends of each beam have the same amount of weight on them. Beams and strings weight nothing and identical shapes represent the same weight whereas different shapes may have the same or different weights. The puzzler is asked to determine the unknown weight. Actually, the mobile puzzle presents a system of equations in the form of a picture which highlights the underlying structure. These puzzles are focused on the equality of expressions and students use their imagery to build the logic of balancing equations while at the same time they do not need rules to solve them. For one who is fluent with algebra it is an easy task to solve the system of equations. But for one who is neither novice nor expert this becomes a “fun” challenge. While solving the puzzle, the students gradually grasp the concept and role of variable as well as the logic of algebraic manipulation. They start intuitively to use substitution and develop personal strategies that will be later connected with standard algebraic “moves” involved in solving equations (Goldenberg et al., 2015). One step towards all these aspects of algebraic thinking is the translation of the information presented in a mobile

into algebraic notation and making the logic explicit. The lack of relevant studies examining the role of using this kind of puzzles to the development of algebraic thinking became the motivation of this study.

DESIGN OF THE STUDY

We worked with 102 grade-6 students (11-12 year old). They had not yet been taught the concept of variable and it was the first time they faced this kind of puzzles. According to the official curriculum by the end of the year they would be able to solve equations with one variable having no powers (first-degree equations) that are in the form of $a+x=b$, $a-x=b$, $x-a=b$, $ax=b$, $a/x=b$, $x/a=b$. During six weeks, on a regular basis, the students were given in total 16 mobiles to solve that can be organized into three groups. For the first group the total weight or the value of a specific shape is given and students are asked to find the value of the unknown shape as well as to explain how they managed to find the solution (Fig. 1). For the second, the students are asked to decide whether a mobile balances (always, sometimes, never) based on some given information as well as to justify their answers (Fig. 2). For the third one, students are asked to create their own mobile that always balances.

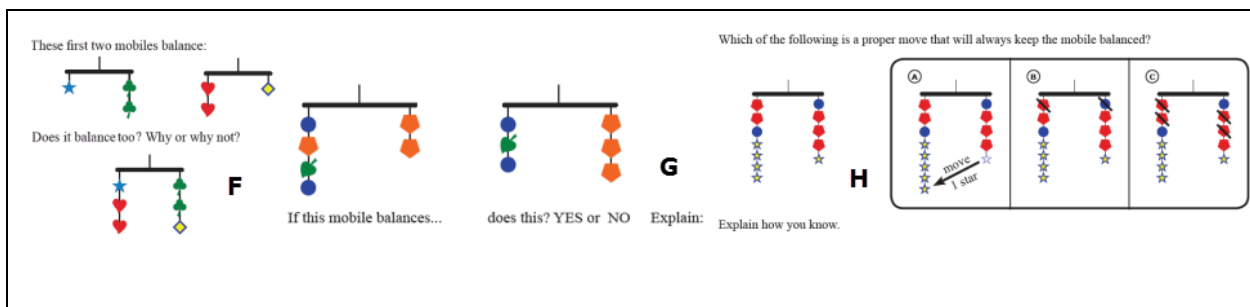


Figure 2. Mobiles F, G, H

The students' worksheets constituted the data of this study. These data were examined in order to identify some evidence of algebraic understanding in the students' answers in conjunction to the mathematical ideas that are implicitly present in these answers. In the context of qualitative content analysis we used inductive category development to determine the various categories which might show a progress in the students' way of thinking in terms of algebra.

RESULTS AND DISCUSSION

The whole effort of the students can be seen in two levels: (a) To describe what they know, and (b) To derive what they do not know (Goldenberg, Mark & Cuoco, 2010). The first level refers mainly to the language used for describing the structure of the mobile and the relations among the involved quantities. The analysis of the data obtained from the first type of puzzles allowed the identification of a progress of five types of thinking that show algebraic understanding. Not all the students used all these types. This is why we chose to present the work of four students as a representative sample of the total population. The five types of thinking are: (i) translating the picture to equality expressions using the shapes of the mobiles, (ii) using words to show the

relationship between shapes in a mobile, (iii) using symbolic language (instead of words) to show the relationship between shapes, (iv) combination of more than one of the previous types indicative of more advanced understanding, and (v) using question mark or an empty box to denote the unknown number. The last type was not used by the specific four students.

Type-1. Translating the picture to equality expressions

It is worth mentioning here that this type incorporates sometimes a thinking that goes beyond the mere translation since it includes properties of the operations or extra components of the mobile structure as it will be shown immediately. Student-1 (S1)

working on mobile-A (Fig.1) wrote $0+0+1+1=6$ and $1+0+0+0=6$. Both expressions describe the situation presented in the mobile-A. However, for mobile-B, instead of the exact translation of the three stars as addition (i.e., $\star+\star+\star=1$) he used

the equivalent expression of multiplication $\star \times 3 = 1$. It seems also that he was able to transfer this knowledge (type-1) to more complex mobiles like the mobile-C. This time a third string had been added to the mobile. Students S1 translated the structure of the

mobile using the expression $13+0+3+13+10+10=47$ for the right part of the mobile. It can be seen that the students do not have to talk about variables or use letters instead of numbers. They merely describe what they know about the mobile and write it simply as they can.

Type-2. Using words to show relationships

In this type students made a step further. They did not merely translate the picture to equality expressions but they noticed certain relationships between the (known and/or unknown) quantities of the mobile. This goes further than the previous type.

For mobile-D (Fig.1), student S2 wrote in his worksheet:

S2:For the left part of the mobile... there are four pentagons on the left and one circle plus two pentagons on the right...which means that one circle equals two pentagons...!

It is obvious here that the student intuitively followed the formal rules in an algebraic context. If we denote by p the pentagon and by c the circle then the left part of the mobile can be written as $4p=c+2p$. Subtracting $2p$ from both sides we obtain $2p=c$. This indicates that the student managed to see a relationship between these two objects which is not obvious when one initially sees the mobile. Moreover, the identification of this relation leads faster to the solution. In a similar manner, student S4 working with mobile-B (Fig.1) wrote that:

S4:three stars equal one pentagon...!

Type-3: Using 'symbolic' language

Students used this type to express again the relationships between the objects. But, this time they used a kind of symbolic language instead of using only words. Since the

concept of the variable is not known, they used the icon of the shape instead of letters. Thus, for mobile-A (Fig.1) student S3 wrote $2 \bullet = 1 \bullet$ to show that two dots equal one pentagon. For mobile-E (Fig.1), student S4 wrote $\bullet = \text{hexagon} \times 2$ to show that a dot equals two hexagons. When compared with type-2 answers this can be considered as a movement towards abstraction. The students intuitively used the icons not as abbreviations of a word but actually as a variable and they were able to see the arithmetical relationships between different variables as they are derived from the picture of the mobile. It can be said that this reflects deeper mathematical insight and as Blanton and Kaput (2011) claim this transition to symbolic representations can be achieved by early schooling and this is why it is important to give children opportunities to begin using it.

Type-4: Combinations of the previous types

This type was met in the second group of mobiles that asked the students to decide whether a mobile balances (always, never, sometimes), demanding at the same time a justification for their decision. The justifications given by the students included a combination of the three previous types. However, due to the limitation of the length of the paper just one example is given showing the co-existence of these types in the same answer. For mobile-G (Fig.2), student S4 decided that there will be no balance and the explanation is:











S4: No! The 2nd mobile does not balance because the 1st mobile has 2  on its right side and 1 , 2  and 1  on its left side. Therefore, 2  and 1  equal 1 . In the 2nd mobile, there are 3  on its right side but actually only 1  on its left side because we know that $00\text{B} = \text{pentagon}$. So, we can draw the second mobile (Fig.3) to be like



Figure 3. S4 explaining his decision

What can be seen in the student's answer is that through the equality expression for the left mobile and by subtracting the same quantity (1 ) from both sides of the mobile he obtained the equation $00\text{B} = \text{pentagon}$. Then he substituted this to the left side of the 2nd mobile and obtained the expression $\text{pentagon} = \text{pentagon} + \text{pentagon} + \text{pentagon}$ which is false and this proved that his argument was correct.

This example makes evident that acting on the mobiles allows the students to induce the formal rules for solving such equations. In the specific example the student applied two such rules by subtracting the same quantity from both sides and substituting a quantity for its equal. This idea of subtracting the same quantity from both sides was also used in mobile-H (Fig.2) by student S3 (Fig. 4).



S3: The correct answer is (c) because we can remove two polygons from each side and this does not influence the balance of the mobile

Figure 4. Intuitive idea of formal rules

Moreover, student S3 went one step further. After subtracting the two pentagons she noticed that she can apply again the same rule and this is why she put the equal sign between the two circles (above) and the two stars (below). So, by subtracting one circle and one star from both sides she resulted to the relationship between the pentagon and the star (Fig.4).

The next intuitive rule is connected with the solution of a system of equations. Student S2 working on mobile-F (Fig.2) initially translated both balanced mobiles into two equations (Fig.5, left). Then he actually proposed to add both equations (Fig.5, right) in order to show that the new mobile is also in balance.



Figure 5. Adding equations

The last task for the students was to create their own mobile that will always balance. It is interesting that all the above presented aspects of algebraic thinking were present in their creations. Three of the mobiles created by the students S3, S4 and S1 are presented in Figure 6. Student S3 explained that given that a pentagon (p) equals 2

triangles (t) $p = 2t$, his mobile will always balance. The substitution of the pentagon on the right side by its equal results to the reflexive property ensuring thus the balance: $c+t+t = c+p = c+t+t$ (c for the cycle). Having the same objects on both sides the total weight for each side is the same. Student S4 chose the star as the basic unit and then used arrows to express the weight of each object in terms of the basic unit: $\star = 1$, $\circ = 2 = \star\star$, $\diamond = 3 = \star\star\star$. Then the strategy was to use on the left any combination of the objects and on the right the substitution of these objects by their equivalent number of units. So, given that a circle (c) equals 2 stars (s) and a pentagon (p) with 3 stars then the mobile will always balance because $4c+s+p = (2s+2s+2s+2s)+s+3s=12s$. Finally, student S1 designed his mobile on the basis of the following ideas: If each string on the right part of the mobile weighs as one bucket and

if $2d = 1 \text{ cookie}$ (i.e., 2 drops(d) must be equal 1 cookie(c)) then the mobile will always balance since as he explained he can “ignore” the stars (s) given that there is one of them on each string. This means that he starts with a relationship between the objects ($2d=1c$), then he applies the knowledge that the same quantity (one star) can be added in both sides of the equality ($2d+s=c+s$). This is the right part of the mobile.

Given that each bucket(b) has the same weight with each one of the strings then he creates a more complex equation $2b=(2d+s)+(c+s)$.

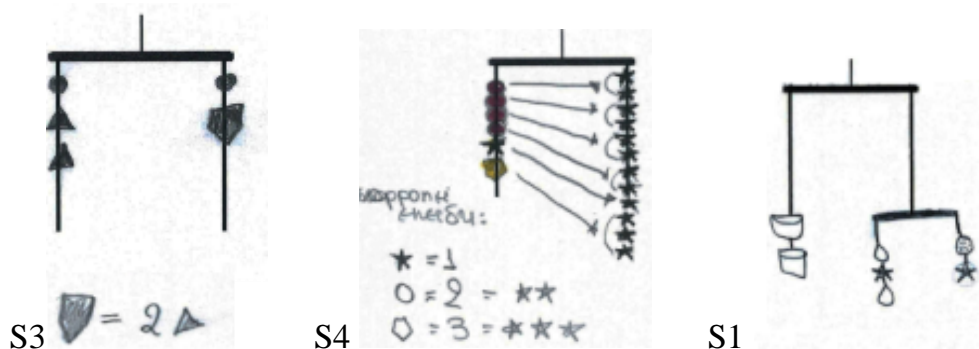


Figure 6. Creating their own mobiles

CONCLUSIONS

The findings of this research study give evidence that the usage of mobiles can smooth the transition from arithmetic to algebra. The types of thinking they used show a progressive movement towards algebraic thinking. The students began with a mere translation of the picture to equation expressions. Then they started using words to express relationships between objects which later was substituted by a kind of symbolic language. The next step was to combine more than one of these types of thinking. The students did not have to think in terms of following certain rules imposed by their teacher. However, they actually induced these rules (isolate variables, add or remove the same amount from both sides, substitute weights that are known to be equal) trying to maintain the balance and make sense of the mobiles. This process included an intuitive sense of certain properties of the operations that will later be introduced formally as commutative, associative or reflexive properties. This does not mean that mobiles are suggested as a substitute for algebra but rather as a tool for thinking about solving equations and grasping the logic that is behind the solution. Moreover, prompting the students to create their own mobiles opens a window to their understanding on what it means for a mobile to balance. Finally, working with mobile puzzles students focused on the following algebraic habits of mind: (i) Puzzling and persevering: It is very important for the solver to figure out where to start and what to do next while solving a problem. Mobile puzzles put emphasis on this particular skill since the students must consider the most effective place to start and the most useful next steps. So, instead of seeing mathematics as a collection of rules to know and follow, mobiles support mathematical ways of essential thinking in algebra, (ii) Seeking and using structure: The students paid attention to the structure of the mobiles, identifying relationships between shapes and thinking on how the mobiles can be translated into equations. For doing this they represented quantities with shapes, words and an early algebraic language, and (iii) Communicating with precision: The issue of language for reasoning about mathematical ideas is crucial in Algebra. As students learn to use symbolic language while dealing with mobiles, in essence they learn to

develop their ability to use language for mathematical discussion, to justify their answers and explain their steps for solving mobiles.

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AN INVESTIGATION OF MIDDLE SCHOOL STUDENTS' PROBLEM SOLVING STRATEGIES ON INVERSE PROPORTIONAL PROBLEMS

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This research was conducted to investigate middle school students' problem solving strategies on inverse proportional problems and whether these strategies change with different number structures. 23 eighth grade students participated in this study. A problem test which contains four inverse proportional missing value word problems with four different number structures was used as a data collecting tool for the research. Data were analyzed by descriptive analysis. Analysis has shown that eighth grade students used six different strategies on solving inverse proportional problems. The findings of the study also indicate that number structure affects the strategies used and the difficulty level of the problems.

THEORITICAL BACKGROUND

Studies on proportional reasoning have shown that additive strategy is the most frequently used erroneous strategy while students solve proportional problems (Karplus, Pulos, Stage, 1983; Tourniaire, 1986; Misailidou & Williams, 2003). Similarly, students give proportional responses to non-proportional problems (De Bock, Van Dooren, Janssens, Verschaffel, 2002; De Bock, De Bolle, Van Dooren, Janssens, Verschaffel, 2003; Van Dooren, De Bock, Evers, Verschaffel, 2006; Van Dooren, De Bock, Vleugels, Verschaffel, 2010; Van Dooren, De Bock, Verschaffel, 2010). Numerous studies have focused on direct proportional and additive problems as measuring and evaluating proportional reasoning. However, the middle school mathematics education programme includes inverse proportional relations along with direct proportional and additive relations in Turkey (MEB, 2013). In related literature, inverse proportional relations investigated to a certain extend (Singh, 2000; Hilton, Hilton, Dole, Goos, O'Brien, 2012; Tjoe & Torre, 2014). The available literature on the factors that have effects on inverse proportional problems is limited and not highlighted as much as direct proportional and additive problems. In this sense, inverse proportional problems placed in the central focus of this study. It was considered to be beneficial to study inverse proportional problems to gain broader aspect about this type of problems in specific and proportional reasoning in general. It was intended to elaborate what kind of strategies students can apply on this kind of problems with different number structures. The present study investigates eighth grade students' problem solving strategies on inverse proportional word problems with different number structures.

Number structure refers to the multiplicative relationships within and between ratios. A 'within' relationship is the multiplicative relationship between elements in the same ratio, whereas a 'between' relationship is the multiplicative relationship between the corresponding parts of different ratios (Steinhorsdottir & Sriraman, 2009). Researchers have identified that the number structures of the problems have various effects on proportional reasoning ability. Van Dooren et al., (2010) stated that the strategies used by students during solving the problems are affected with the number structure of the problems. Steinhorsdottir (2006) stated that number structure influence problem difficulty level. Several studies have shown that students have tendency to use multiplicative strategies when the presence of integer ratios and use additive strategies when the absence of integer ratios no matter of proportional or non-proportional situations (Degrande, Verschaffel, Van Dooren, 2014; Tourniaire & Pulos, 1985; Cramer & Post, 1993; Karplus et al., 1983; Steinhorsdottir, 2006; Van Dooren et al., 2010;). In the current study, besides the effects of the number structures on direct proportional and additive problems, it was also intended to investigate the effects of number structures on inverse proportional problems and whether the difficulty level or strategies used change with different numbers structures.

RESEARCH QUESTIONS

1. What kind of strategies does eight grade students use while solving inverse proportional word problems?
2. Do the difficulty level and strategies used affect by the number structures of the inverse proportional word problems?

METHODOLOGY

The subjects of this study are the 23 (13 girls, 10 boys) eighth-grade students from a public school in a southern province of Turkey. A problem test which contains direct proportional, inverse proportional and additive word problems was designed as a data collecting tool for the research. In this study, merely the findings and results of the inverse proportional problems are presented since this study is a part of an ongoing research. Number structures which involve within integer (WI), between integer (BI), both within and between integer (WBI) and non-integer (NI) relations considered in the problem test. Problems used in this study consisted of four open ended items and these items were developed in parallel with the objectives of renewed elementary mathematics curriculum (MEB, 2013). All students solved four experimental word problems. The number structures and the statements of these word problems are illustrated in Table 1.

Number Structure	Title	Statement
Between Integer Relation	Pool Problem	4 pipes which all of them pour same amount of water fill an empty pool in 12 hours. In how many hours 6 pipes which all of them pour same amount of water fill the same empty pool?
Within Integer Relation	Detergent Problem	A package of detergent finishes in 3 weeks when laundry takes places 4 times a week. When the laundry takes places 2 times a week, in how many weeks the same package of detergent finishes?
Both Between and Within Integer Relation	Sweater Problem	Emel finishes a sweater in 24 days by hand-knitting 2 hours in a day. If Emel hand-knits 4 hours in a day, in how many days can she finish the same sweater?
Non-Integer Relation	Ice-Cream Problem	Irem can take 9 ice-creams which is 2 euro apiece with the money in her pocket. How many ice-creams can Irem take with the money in her pocket which 3 euro apiece?

Table 1. Experimental Items

Data were analysed by descriptive analysis. The strategies used in solving problems with different number structures were identified by evaluating the students' answers on problems and comparisons among the different categories were made. Strategy examples from the students' solutions are presented below.

4.) İrem cebindeki para ile tanesi 2 lira olan dondurmalarından 9 tane alabiliyor. İrem cebindeki para ile tanesi 3 lira olan dondurmalarından kaç tane alabilir?

$$2 \times 9 = 18$$

$$18 \div 3 = 6$$

Tanisi: 2 Lira olan dondurmaların 9 tane alması 2 ile 9'u çarpım 18 çıktı. 18'ide 3 bölünce cevap 6 çıkar.

Figure 1. Example of unit – total strategy in ice-cream problem

In figure 1, student reached 18 by multiplying 2 and 9. Student obtained the total money with this multiplication (total). Then student divided the total 18 to 3 and gained 6 as a result (unit).

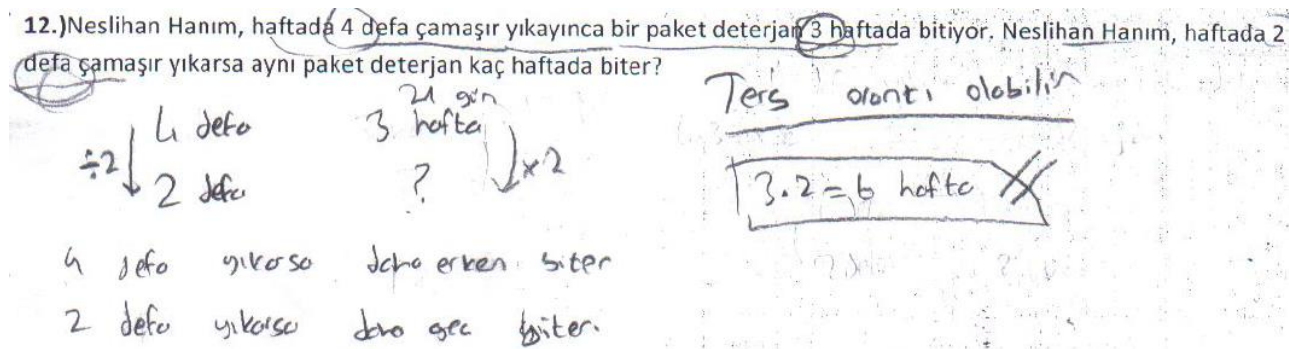


Figure 2. Example of factor – multiple strategy in detergent problem

In figure 2, student realized the inverse proportional relations of the problem then by using the within integer relation $4:2=2$ (factor) student used this factor as a multiple for the unknown in the other ratio $3 \times 2=6$ (multiple).

After the students' solution for each problem examined, clinical interviews were used to elaborate students' judgments and make inferences about their cognitive processes. One student was selected (selection criteria explained in results section) for the clinical interview in order to comprehend whether number structures affect problem difficulty and strategy choices.

RESULTS

Table 2 shows the strategies used by students to inverse proportional problems with different number structures. Analysis of the responses showed that students used six distinct solution strategies in inverse proportional problems. The findings of the study indicate that number structure of problems affect strategies used by students. Students did not use any unit – total strategy in both between and within integer relation (BWI) problem whereas they did not use any factor – multiple strategy in between integer relation (BI) and non-integer relation (NI) problems. Analysis of the students' solutions also showed that students have tendency to use multiplicative strategies when the presence of integer ratios. In these terms, the results are in accord with previous studies (Steinthorsdottir, 2006; Tourniaire & Pulos, 1985; Van Dooren et al., 2010; Cramer & Post, 1993).

Strategies	Number Structure			
	BI	WI	BWI	NI
Unit – Total	3	2	-	19
Factor – Multiple	-	4	8	-
Inverse Proportion Algorithm	-	-	-	1
Evidence of Inverse Proportion	4	2	-	-

Multiplicative	8	3	1	-
Additive	2	1	-	1
Using numbers randomly	-	-	4	1
No explanation (correct answer)	1	7	8	-
No explanation (wrong answer)	4	3	1	1
Empty	-	1	1	-
Total	23	23	23	23

Table 2. Strategies used in inverse proportional problems

Table 3 shows the mean scores on inverse proportional problems. If there is a correct answer for each problem, that problem was scored as “1”. If the solution is wrong, that problem was scored as “0”. Thus, the mean score for each problem is between 0 and 1. Analysis of the mean scores showed that students showed the best performance on non-integer (NI) problems while the worst performance on solving between integer (BI) problems.

Number Structure	BI	WI	BWI	NI
Means	0,17	0,52	0,70	0,74

Table 3. Mean scores on inverse proportional problems

In order to understand the reason why the worst performance occurred on solving between integer relation (BI) problem (Table 1, pool problem), a clinical interview was carried out with a student who could solve all problems but the pool problem. The solution of this student is presented in Figure 3.

2.) Aynı miktarda su akıtan 4 musluk boş bir havuzu 12 saatte dolduruyor. Aynı miktarda su akıtan 6 musluk bu havuzu kaç saatte doldurur? 8

Figure 3. Student's solution for the pool problem

When the solution in figure 3 examined, it is seen that this student did not use any particular solution strategy to pool problem; he only gave a wrong numerical value to the problem. In order to find out why this student could not solve the problem, he was asked to explain how he obtained the result as 8.

Interviewer: So how did you obtain the result as 8?

Student: Since 4 pipes fill in 12 hours, 6 pipes should fill in less amount of time.

Interviewer: Why 6 pipes should fill in less amount of time?

Student: 6 t pipes is 2 more than 4 pipes. When there are more pipes, they will pour more water thus the pool will fill in less time.

This dialogue can be interpreted as that the student is aware of the mathematical structure (inverse proportional) of the pool problem. In this sense, in order to understand whether the number structure of the problem affect the difficulty of the pool problem, problem is asked again by changing number structure while maintaining the content as follows: *“4 pipes which all of them pour same amount of water fill an empty pool in 12 hours. In how many hours 6 pipes which all of them pour same amount of water fill the same empty pool?”* In this case, student could solve the problem and obtain a correct answer. This finding can be interpreted as number structure of the problem affects the difficulty level of the problem.

CONCLUSION AND DISCUSSION

This study focused on the strategies used while solving inverse proportional problems and whether difficulty levels and strategy choices of inverse proportional problems change with different number structures. In the current study, the findings have shown that students used six different strategies on inverse proportional problems' solutions. Furthermore, students' strategy choices were flexible with respect to different number structures. Students could adjust their problem solving strategies to alternating different number structures situations.

Among the four inverse proportional problems, between integer relation (BI) problem (pool problem) had the lowest mean score. In order to understand the reason of this situation, a clinical interview with a student (who could solve all but the pool problem) carried out. Analysis of clinical interview has shown that number structure affects the difficulty level and strategy choice of inverse proportional problems. Similarly the literature (Degrande et al., 2014; Steinthorsdottir, 2006; Tourniaire & Pulos, 1985; Van Dooren et al., 2010) has shown that number structure affects difficulty level and strategy choice of direct proportional and additive problems.

The results of this have some implications for instruction. Educators should consider the number structures when students engage with inverse proportional problem situations. For further studies, it can be suggested to investigate the strategies used while solving inverse proportional problems and the effects of different number structures on inverse proportional problems with larger participants from different grade levels.

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RECOGNISING WHAT MATTERS: IDENTIFYING COMPETENCY DEMANDS IN MATHEMATICAL TASKS

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The aim of this study was to investigate how an item analysis scheme could be utilised by a group of five teachers and prospective teachers to identify the level of competency demands in mathematical tasks. The fairly high overall agreement on such competency demands indicates that the scheme might be used to promote discussions and reflections about the demands of mathematical tasks and, as such, support mathematics teaching. While the assessment output demonstrates that many of the tasks were challenging to the students, the teachers viewed most of the tasks as having a low competency demand. These two findings might stem from different interpretations of the competency descriptors, such as the words 'simple' versus 'complex' or the term 'model'.

INTRODUCTION

For decades, the term 'competence' has been widely used in mathematics education research, and this has influenced what is conceived as the goal of mathematics education (Kilpatrick, 2014). Currently, curricula often focus on competencies and include aspects of mathematical literacy (Burkhardt, 2014). Although several different competency frameworks exist (Kilpatrick, 2014), a common factor is that mathematical competence extends conceptual and procedural knowledge. Traditionally, with a strong focus on procedural knowledge, tasks have played an important role in instruction, offering opportunities for practising skills. Even with the recent development toward mathematical competence, mathematical tasks are still an important tool for teachers. Consequently, it is vital that teachers can judge the tasks that they consider to use in their classrooms, to be confident that these tasks can stimulate learning of mathematical competencies. However, recent research has demonstrated that both textbooks and teacher-made tests to a large extent utilise algorithmic tasks (Palm, Boesen, & Lithner, 2011). A plausible interpretation might be that it is easier for teachers to recognise students' factual knowledge and calculation skills than other aspects of mathematical competence, such as communication or problem solving abilities. In our study, we aimed to investigate to what extent teachers could utilise an item analysis scheme developed by the PISA Mathematics Expert Group (MEG) to identify the competency demands of mathematical tasks.

MATHEMATICAL COMPETENCE AND TASKS

Hiebert (1986) argued that for students to be fully competent in mathematics, they need both conceptual and procedural knowledge and to understand the link and relationship between the two. Hiebert (1986) was commenting on the long-standing tradition in mathematics education of viewing conceptual and procedural knowledge as separate

entities. More recently, a rich view on mathematics and mathematics education has evolved. By the early 2000s, several frameworks had emerged emphasising not only the interaction between conceptual and procedural knowledge but also the importance of abilities such as communication, modeling and mathematical thinking (Kilpatrick, 2014; Niss & Højgaard, 2011). Some (e.g. Kilpatrick, Swafford, & Findell, 2001) have even emphasised the importance of positive beliefs and attitudes toward mathematics. Coinciding with this changing view on mathematics teaching and learning has been an increased focus on the acquisition of competencies in education, which has been embraced as ‘a new standard for curriculum design’ (Westera, 2001, p. 75). Competence can be related to a variety of cognitive abilities, and a lack of a common definition and understanding of the term ‘competence’ poses challenges when developing competence-based curricula (Westera, 2001).

A framework for mathematical competence that has influenced curricula and assessment reforms in several European countries (see Turner, Dossey, Blum, & Niss, 2013) is found in a report from the Danish KOM project (Niss & Højgaard, 2011). This framework comprises eight mathematical competencies that, as a whole, ‘encapsulate the essence of mathematical competence’ (Niss & Højgaard, 2011, p. 50). According to Niss and Højgaard (2011), activities must be orchestrated ‘with the explicit aim of developing the mathematical competencies of the individual’ (p. 31) to offer opportunities for students to develop these mathematical competencies.

Mathematical tasks¹ are regarded as key to mathematics education as a learning resource (Wiliam, 2007), and much of the teaching and learning in mathematics classrooms is situated around solving mathematical tasks. For instance, several studies have indicated that cognitively demanding problems promote higher learning outcomes (Boaler & Staples, 2008; Stein & Lane, 1996). If the students are to develop the mathematical competence described in the curriculum, they need to engage in tasks that stimulate and activate these competencies. To develop or select appropriate tasks, teachers must be able to recognise the competency demands of the tasks. Prior research has shown that this can be challenging for teachers (Wiliam, 2007). Yet some studies have shown how training teachers in analysing task demands might be fruitful. For instance, Arbaugh and Brown (2005) observed that engaging teachers in critically examining mathematical tasks made them consider more deeply the opportunities embedded in the tasks, and also changed the types of tasks they chose for their classes.

Building on the mathematical competence framework developed in the KOM project (Turner et al., 2013), since 2003, the PISA MEG (Mathematics Expert Group) has been continuously developing and refining an item analysis scheme for identifying the mathematical competencies needed to solve mathematical problems (Turner, Blum, & Niss, 2015). Applying the scheme to analyse 48 mathematics items used in both the

¹ In this paper, the term ‘task’ comprises various types of mathematical problems and questions including routine and non-routine, complex and simple problems and assessment tasks. ‘Tasks’ is used interchangeably with the term ‘item’.

PISA 2003 and PISA 2006 surveys, Turner et al. (2013) found that it could be used by experts to effectively identify the competency requirements of items. Further, Turner et al. (2015) proposed that the scheme could be used by teachers to devise assessment items, and that mathematics teaching and learning should focus on developing these mathematical competencies among students.

METHODOLOGY

The aim of this study was to investigate how the item analysis scheme would be utilised by a group of teachers for identifying the competency demands of mathematical tasks. After initial training in applying the scheme, the teachers individually analysed items from the PISA 2012 paper-based assessment ($N_p = 85$) and the 2014 Norwegian grade 10 national exam ($N_e = 56$). For each item, the teachers rated the cognitive demand of six mathematical competencies. The consistency and distribution of the teachers' ratings were analysed to identify possibilities, challenges and limitations connected to the implementation of the scheme.

Participants

Mathematics teachers, prospective teachers and university employees were approached for recruitment. The inclusion criteria were (1) experience teaching mathematics in secondary school and (2) having a degree in or being enrolled in a master's programme in secondary school mathematics teacher education. Two trained teachers and three prospective teachers in their final year of the teacher education programme were recruited, in the following referred to as teachers.

Material

The item analysis scheme (Turner et al., 2015) consisted of operational definitions of six mathematical competencies: Communication (C), Devising Strategies (DS), Mathematising (M), Representation (R), Using Symbols, Operations and Formal Language (SF—referred to as Symbols and Formalism) and Reasoning and Argument (RA). In addition, four levels of demand (0–3) were described for each competency. When analysing an item, the level that best fit the demand of the item was identified for each of the six competencies, with a higher level indicating higher cognitive demand. A competency rated at level 0 implied that the item did not demand the activation of this competency (or at a minimal use), while level 3 implied an advanced or complex level of demand for this competency.

All teachers were provided with an English version of the MEG item analysis scheme in addition to a user guide presenting and explaining the scheme. They were also given examples and an explanation of item analysis performed by the MEG members. The teachers analysed two sets of mathematics assessment items: (1) 85 items from the paper-based PISA 2012 survey and (2) 56 items from the 2014 Norwegian grade 10 national exam, consisting of part one (33 items) mainly comprising traditional tasks focused on procedures and part two (23 items) emphasising problem solving. Both assessments were targeted at 15-year-old students.

Training teachers in item analysis

The five teachers were requested to spend two hours familiarising themselves with the material before attending a one-day training session on understanding and applying the item analysis scheme. The training mainly consisted of individually analysing PISA items from previous cycles, rating the demand of one competency at a time, and then comparing and discussing the given ratings as a group. The aim of the discussions was to reach an agreement on which competency ratings best suited an item, using the analysis scheme and the MEG's explanations and examples as guidelines, and to promote a mutual comprehension of the scheme. Following the training session, the teachers individually analysed the items using the item analysis scheme.

The training and discussions of the ratings were audio recorded and used to further inform the investigation of the teachers' utilisation of the item analysis scheme.

Data analysis procedures

Several approaches were used to investigate how the teachers recognised the competence demand of the assessment items and their utilisation of the item analysis scheme. As high consistency between the teachers' ratings would indicate that the teachers had interpreted and used the scheme similarly, the interrater agreement was examined through an intraclass correlation coefficient (ICC), following the guidelines of Shrout and Fleiss (1979). The distribution of the teachers' ratings for the different competencies was calculated to provide information about the degree to which the teachers employed all four demand levels in the item analysis. In addition, other descriptive statistics were calculated to gain further insight into the utilisation of the item analysis scheme.

RESULTS

In a perfect world, teachers would identify and rate the 'true' competency demands of tasks with perfect agreement. However, in the real world, this cannot be the case, as ratings are influenced by various forms of bias, such as different uses and interpretations of the rating scale (Hoyt & Kerns, 1999). When measuring the agreement using ICCs, both average and single measurements can be calculated (McGraw & Wong, 1996). The *average measure* indicates the trustworthiness of the average ratings of the five teachers, while the *single measure* indicates the extent to which we might rely on the ratings of a single teacher to represent the true competency demands of the item.

Table 1 displays the single and average ICC measures for all items in total and for the PISA items, exam items, and each competency separately. Looking across all items, the ICCs in Table 1 indicate a very good agreement for the teachers' average ratings of all six competencies, with values ranging from 0.80 (Mathematising) to .88 (Devising Strategies). This means that the teachers as a group rated the competency demands of the items rather equally, and it indicates a similar interpretation and use of the item analysis scheme.

Table 1. Agreement measures (ICCs) of the teachers' average ratings of the items for each competency individually. The 'single teacher' measure is given in parentheses.

	C	DS	M	R	SF	RA
All items	.86 (.55)	.88 (.61)	.80 (.44)	.84 (.51)	.85 (.52)	.84 (.51)
PISA items	.77 (.40)	.86 (.54)	.77 (.39)	.84 (.50)	.83 (.49)	.74 (.36)
Exam items	.89 (.61)	.92 (.69)	.84 (.50)	.79 (.43)	.82 (.47)	.89 (.61)
Exam part 1	.86 (.54)	.92 (.69)	.89 (.62)	.86 (.54)	.85 (.52)	.78 (.42)
Exam part 2	.79 (.43)	.88 (.60)	.71 (.34)	.67 (.29)	.79 (.42)	.84 (.52)

Note: C = Communication; DS = Devising Strategies; M = Mathematising; R = Representation; SF = Symbols and Formalism; RA = Reasoning and Argument.

When looking at the agreement for the PISA and exam items separately, we observe lower values for the PISA items for some of the competencies, with Reasoning and Argument having the lowest agreement, with an ICC of .74. Yet this value also indicates a good agreement. Table 1 also shows that the ICCs for single measures are much lower than for the average measures. Looking at all items, the values range from .44 (Mathematising) to .61 (Devising strategies), which can be regarded as moderate agreement. Thus, if we want to have 'reliable' information about the competency demand of the items, the ratings from one teacher would be insufficient, as they would vary significantly depending on the choice of teacher.

When looking at the ICCs of the PISA and exam items separately, we observe higher agreement for the exam items than for the PISA items for all but two of the competencies (Representation and Symbols and Formalism). For the exam items the average agreement across all competencies is .86, while the corresponding agreement for the PISA items is .80. One hypothesis to explain this difference in agreement is that the more complex items (i.e. items that demand a higher number of competencies) are more difficult to analyse. By calculating the average number of competencies demanded (i.e. rated above level 0) per item for the different assessments, we find that the PISA items have a higher average number of competencies demanded per item (3.97) than the exam items (2.96) do and can thus be regarded as more complex. This pattern is even more distinct if we look at the items in exam parts 1 and 2 separately. The agreement measures in Table 1 show a higher agreement in part 1 for five of the six competencies, and also a considerably higher average agreement across all competencies (.86 for part 1, compared to .78 for part 2). At the same time, the average number of competencies demanded per item is almost twice as large for part 2 (4.13) relative to part 1 (2.15), which would explain this pattern.

Figure 1 displays the distribution of ratings across the four levels for all 141 items given by the 5 teachers for each of the competencies.

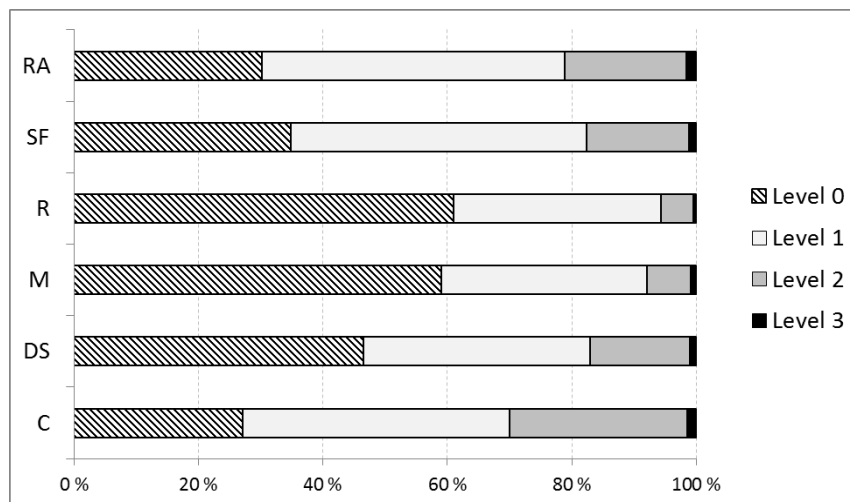


Figure 1. Distribution of the total number of ratings across the four levels for all items.

As can be observed from Figure 1, the majority of the ratings are at levels 0 and 1. Level 3 is rarely used. For the Mathematising and Representation competency a similar pattern is observed for level 2, with less than 5% ratings at this level. There are several plausible reasons for the somewhat surprising pattern, with most ratings at levels 0 and 1. One reason could simply be that the items mostly demand a low level of cognitive demand. However, the students' scores on the two assessments show that both assessments comprise several items that are very challenging to the students (in total, 21% of the items were solved correctly by less than 20% of the students). In addition, when comparing the teachers' ratings to those of the PISA MEG on 48 PISA items (Turner et al., 2013), partly overlapping with the items used in this study, the results indicate a higher portion of ratings at levels 2 and 3. Another plausible explanation might be that the level descriptions are inadequate operationalisations of the actual competency demands, and when asked to analyse the items, the teachers struggled to understand and differentiate between the higher levels. To be able to understand if the scheme or the teachers contributed to the observed pattern in Figure 1, we propose that the two be seen in relation to each other. The teachers' discussions during the training session indicated a somewhat ambiguous understanding of some of the competency definitions and level descriptions. For instance, during the training, some teachers expressed that the Mathematising competency was hard to understand. One reason for this could be the use of the term 'model' in the level descriptions without a proper explanation, as it seemed to be interpreted differently by the teachers. Table 1 shows that Mathematising is the competency where the teachers have the lowest agreement, and at the same time Figure 1 shows that this competency has less than 10% of the ratings at levels 2 and 3. Another issue is the use of relative words in the level descriptions, for instance, 'simple' and 'complex'—words that tend to have different meanings for different people (Turner et al., 2015). During the revisions of this analysis scheme, Turner et al.

(2015) attempted to minimise the use of such terms. Still, when examining the wording of levels 2 and 3, we find that the term ‘complex’ (e.g. representation competency) is frequently used in the descriptions.

CONCLUSION

This study sought to explore whether an item analysis scheme could be applied by teachers to identify the competency demands of mathematics items, and thus be used to support mathematics teaching. The scheme was previously used by Turner et al. (2013) to analyse assessment items and predict the item difficulty, indicating that the scheme potentially is a valuable tool for test developers and item writers.

The rather high agreement measures for the teachers’ ratings indicate that the teachers as a group are fairly consistent when identifying the competency demands of the items. To meet the demands of a competence-based curriculum, teachers should be able to understand and recognise the competencies embedded in mathematical tasks and activities (Niss & Højgaard, 2011). However, the moderate single measures indicate that the identified competency demands of a single teacher are not similarly trustworthy. This implies that for teachers as a group, the item analysis scheme can be a valuable tool for promoting discussions and reflections about mathematical tasks and the competencies students need to activate when solving them. According to Arbaugh and Brown (2005), this type of critical examination of mathematical tasks can support growth in pedagogical content knowledge and change teachers’ practice.

In addition, the agreement observed for the different assessments could indicate that the more complex items were more challenging for the teachers to analyse. One reason for this might be that teachers mainly are exposed to tasks from textbooks that focus on applying algorithms (Palm et al., 2011), and they are not used to examining the demands of complex tasks requiring multiple competencies. This might be an issue, as cognitively demanding tasks seem to promote higher learning outcomes (Boaler & Staples, 2008; Stein & Lane, 1996), and thus should play a considerable role in mathematics education. Even though the teachers seemed to have a similar understanding of the competencies and were able to recognise when an item demanded the activation of a competency, the distribution of the teachers’ ratings shows that they judged only a small proportion of the competency demand to be at the higher levels 2 and 3. This uneven distribution of ratings is most likely not due to low competency demand in the items. Rather, the observed patterns stem from the level descriptions in the item analysis scheme and the teachers’ interpretations of these. Thus, further revisions of the scheme may be needed for teachers to be able to use it to distinguish between the different levels of demand for each competency.

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MAKING SENSE OF DYNAMICALLY LINKED MULTIPLE REPRESENTATIONS OF FUNCTIONS

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The dynamisation of multiple representations of parametrized functions add to the variety of how the effects of a parameter on graph and table are perceived, sometimes in a way that seems specific to a dynamic environment. While these perceptualizations seem perfectly valid within the geometric or numeric representational system alone, they contradict to how the multiple representation environment should be read as a whole. For building a coherent mental model of a dynamic multiple representation of a parametrized function, this paper proposes to identify the parameter as an invariant within and between representational systems. This mainly normative position is further examined in the light of two theories of knowledge construction by perception, and by abstraction.

INTRODUCTION

Perceptual bias with dynamic representations of function

Open a dynamic multi-representational software, add a glider that controls a parameter a , then plot the graph of the function $f(x) = x^2 + a$. Increase the value of the parameter and watch closely how the graph of f changes. It does appear to move upwards, yes, but doesn't it give the impression of becoming narrower, too? This was what a teacher student at the University of Education in Heidelberg pointed out when she was asked to describe the effects of the parameter on the representations of f , using a dynamic multi-representation environment (fig. 1, cf. Pinkernell 2015). She even knew that her perceptualization of how the graph changed contradicted to what she knew from school: The “width” of a parabola is controlled by a parameter in front of the quadratic term, she recalled. But there wasn't one.

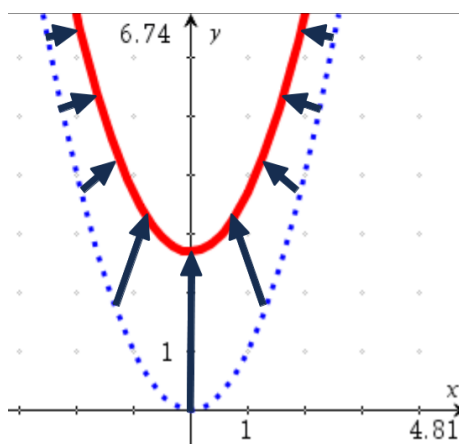


Fig. 1: Moving upwards, and getting narrower, too?

Following the movement of graphs across a dynamic coordinate plane which is restricted by window boundaries seems to invite other perceptualizations than those associated with static material. For another example, plot the graph of the function f with $f(x) = x + a$, then increase the parameter value. Do you actually see the graph moving upwards? Or wouldn't you agree that it rather moves to the upper left corner of the graph window? It is obvious that our instantaneous perception of the movements on the screen is biased by irrelevant or ambiguous properties of the medium. By which they interfere with building an adequate concept of the mathematical notion represented in the dynamic learning material.

The parameter as an invariant in the dynamic multirepresentation of functions

To understand the effects of the parameter on the three representations of $f(x) + a$, it helps to recall that the parameter value a must be present in all three representations, in some form or another. In the algebraic representation the parameter can be identified as the operator $+a$ that increases the function value $f(x)$ by a , in the numeric representation the actual parameter value can be identified as the constant difference between neighbouring cells of $f(x)$ and $f(x)+a$ in each line of the table, and in the geometric representation, the actual parameter value can be identified as the constant vertical distance between the corresponding points $(x, f(x))$ and $(x, f(x)+a)$ of the two graphs. Thus, the parameter is characterised as an invariant within each and between all three standard representations of a function. One could visualize it as an arrow of the same direction and of the same length, placed into appropriate places in table, graph, and equation (fig. 2). Hence, the effect of the parameter change on the graph of $x^2 + a$, in particular, must be described as a vertical translation.

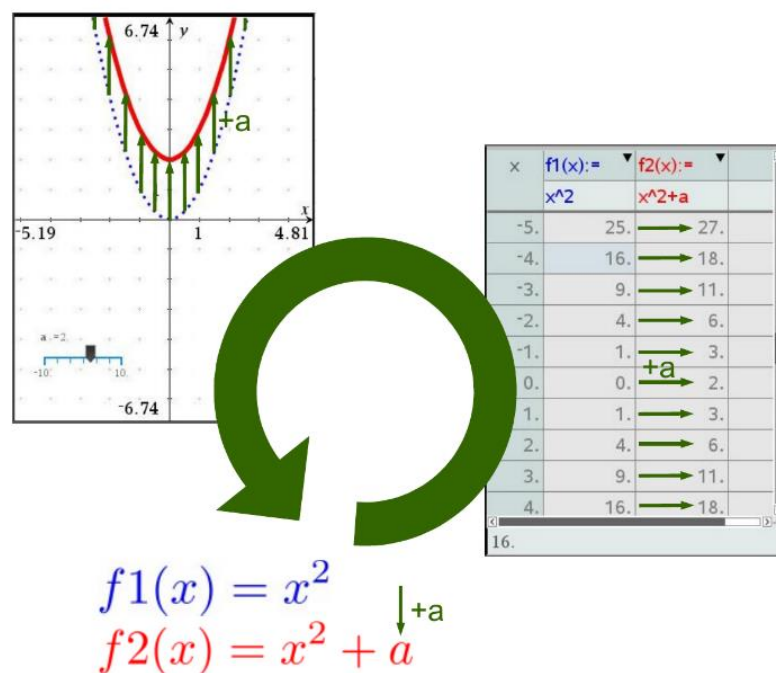


Fig. 2: The parameter, visualized as an invariant operator in all three representational systems

However, one could argue that simple mnemonic phrases should be sufficient to know, eventually, to be able to translate between function representations, e. g. “the parameter a in $f(x) = x^2 + a$ results in a vertical translation of the parabola by a units,” as they are generally found in textbooks (cf. Hußmann & Laakman 2011). Yet the student knew these rules from school. She simply could not apply them when the familiar static representations of functions suddenly became “alive” on the computer screen.

To ask for identifying the parameter as an invariant within and between the dynamic function representations is a normative heuristic. It derives from considering the mathematics behind function representation. In the following we will examine how this position integrates into other theoretical perspectives on the learning with representations. First with a psychological focus on the processes of knowledge construction by perception, then with a domain specific focus on construction of mathematical concepts by abstraction.

MAKING SENSE BY PERCEPTION AND ABSTRACTION

Knowledge Construction by Perception

In his analysis of information processing of pictures, Palmer (1975) differs between parametric (colour, size, etc.) and structural information (figure-ground, relations between elements, etc.). When an individual perceives a change of colours, then these must have changed in the stimulus. When he or she perceives a change of relations of picture elements, the picture itself must not have changed at all. The Necker cube is a well-known example (fig. 3).

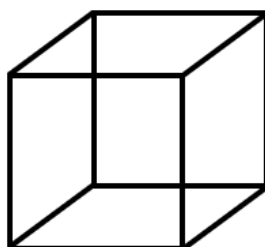


Fig. 3: A transparent cube – as seen from above or from below?

To tell whether the graph of $x^2 + a$ follows a vertical translation only or whether it is getting narrower, too, means to decide on a specific interpretation of the visuo-spatial relations of the movements on the screen. Since depictive information about visuo-spatial relations is ambivalent, both perceptualizations are perfectly valid within the geometric representation of the effects of a on $x^2 + a$. So further information is needed to decide on how the movement of the graph should be seen within the whole of the multiple representation environment.

In their theory of knowledge construction from multiple representations, Schnotz & Bannert (2003) describe the mental model of the given external information as the one cognitive instance that first “makes sense” during information processing (cf. Vogel

2007). While the mental model is based on the perceived properties of the external information, it is also based on cognitive schemata that contain propositional instructions relevant for processing information from the given types of depictive material. For understanding realistic pictures the individual can use cognitive schemata of everyday perception, for understanding logical pictures, e. g. technical graphs or diagrams, so-called graphic schemata are needed for constructing adequate mental models. From a perceptual-cognitive perspective, the heuristic of identifying the parameter as an invariant throughout the dynamic material seems an adequate base for developing cognitive schemata which are suitable for building an appropriate mental model of the dynamic multiple representation of a parametrized function. However, there is evidence that only experts or individuals with higher learning abilities are able to develop graphic schemata for reading information from elaborated graphs or diagrams (Lowe 1999, Tversky, Morrison & Betrancourt 2002).

Knowledge Construction by Theoretical Abstraction

A mathematical concept is, essentially, abstract. There are no real objects called “functions” from which to learn what a function is. To access the mathematical concept of a function means to analyse its representations. Following Duval (1999, 2006), understanding the concept of function by its representations is the ability of modifying representations within the same representational system according to its specific rules, and of translating coherently between representational systems.

All three representational systems are fundamentally different semiotic systems, each with a specific syntax and set of symbols. To identify the parameter within each representational system means to find a form that is specific to each system: In algebra it is the operator $+a$, in the numeric representation it is the constant difference between neighbouring table cells, and in the geometric representation the constant vertical distance between corresponding points of the graphs of x^2 and x^2+a . To identify all these different forms of appearance as referring to the same quantity means to identify structural analogies between the different forms of appearance. What in one representation is a numeric difference between neighbouring values in the table is, in another representation, a geometric difference between corresponding points in the coordinate system. To condense system specific information about relations between cell values or coordinates down to pure structural information about a constant difference between parametrized function values can be characterized as abstraction.

The nature of constructing mathematical knowledge by abstraction has been discussed controversially (cf. Mitchelmore & White 2007), ranging from developing context dependent yet transferable knowledge (Noss, Hoyles & Pozzi 2005) to forming decontextualized mental entities of knowledge (Sfard 1994). In this paper, the term abstraction follows what Mitchelmore & White (2007) call theoretical abstraction. Generally speaking, theoretical abstraction means to create “concepts to fit into some theory” (Mitchelmore & White, p. 4), i. e. theoretical thought is providing a base for

deciding which properties need to be considered and which aspects are irrelevant for constructing a new concept. Moreover, theoretical thought allows identifying objects as relevant which, superficially, seem to have nothing in common: “A theoretical idea or concept should bring together things that are dissimilar, different, multifaceted, and not coincident, and should indicate their proportion in the whole. ... Such a concept, in contrast to an empirical one, does not find something identical in every particular object in a class, but traces the interconnection of particular objects within the whole, within the system in its formation” (Davydov 1990, p. 255). A model of learning by abstraction that describes, to put it simply, how to compare the incomparable seems a suitable framework for describing knowledge construction in a learning environment that consists of different representational systems.

So within the framework of theoretical abstraction, theoretical thought is a base for the construction of abstract knowledge. An initial theoretical base could derive from activating a priori abstract knowledge, it also could derive from a close analysis of the material at hand (Ohlsson & Lehtinen 1997). Material that allows change and variation facilitates forming initial abstractions (Giest 2011). Both activating a priori abstract knowledge about function representations and forming initial abstractions by analysing changes in a multirepresentational environment underlines the pertinency of the heuristic proposed in this paper, i. e. to identify the parameter as an invariant in the dynamic multirepresentational material as proposed above.

SUMMARY AND DISCUSSION

The dynamisation of multiple representations of parametrized functions allows for unusual, if not mathematically incorrect perceptions of changes on the screen. With visuo-spatial information being ambiguous in depictive material, a viewer can form contradicting perceptions of the movements of a function graph across the computer screen, even when he or she knows better.

To form a coherent mental model of dynamic multirepresentational information, it helps to identify the parameter as an invariant within and between all three standard representations. This is mainly a normative view that derives from theoretical considerations of the mathematics involved in constructing representations of functions. It is also a view that finds justification in psychological theories of knowledge construction by perception, where the mental processing of depictive information is guided by cognitive schemata, which helps to form an adequate mental model of the multirepresentational information. It also finds justification in mathematical theories of knowledge construction by abstraction which emphasize the need for a theoretical base for analysing multirepresentational learning material that allows identifying the connecting beyond surface properties of each representational system.

Several questions arise:

- 1 Processing information from logical pictures like graphs or diagrams are highly demanding (Tversky, Morrison & Betrancourt 2002, Lowe 1996). So are pupils or students actually able to give a coherent explanation of dynamically linked multirepresentation of a parametrized function? What kinds of explanations appear at all, which of those refer to invariants? What are the learning preconditions on which this ability relies?
- 2 The potential of dynamic material for the learning of mathematical concepts has been described as allowing search of invariants that helps to identify the irrelevant properties (Ainsworth 2006). A search of invariants yield those initial abstractions that provide the first theoretical base for forming a new concept (Ohlsson & Lehtinen 1997, Giest 2007). For analysing theory-based processes of mathematical knowledge construction, the AiC model of abstract knowledge construction by epistemic actions seems appropriate (Hershkowitz, Schwarz & Dreyfus 2001, Dreyfus 2012). So how does this model apply to the analysis of learning processes that are initiated by dynamic material? Esp., is it possibly to identify instances of initial abstraction with learners?
- 3 A search for invariants starts with a close analysis of the structure of the algebraic expression. The parameters in $b \cdot f(x)$ and $f(x+c)$ obviously result in a vertical dilation or a horizontal translation of the graph of f , resp. Yet with $f(x) = e^x$, the graph's movement across the coordinate plane appears virtually identical in both cases. To be able to decide which interpretation is coherent with the whole multiple representation environment, basic abilities in algebra structure sense (Hoch & Dreyfus 2006) seem indispensable. So is it possible to confirm a correlation between structure sense and the ability to give a coherent explanation of dynamically linked multirepresentation of a parametrized function?
- 4 Knowledge based on static material needs to prove itself in a dynamic environment too. The validity of mathematical knowledge does not depend on how it is represented. So what is the potential of dynamic material for building decontextualized and resilient knowledge? Addressing awareness of perceptual bias adds to its potential. What kind of misperceptions are possible, which do appear? How to develop appropriate dynamic learning material that can unfold its “semiotic potential” (Mariotti 2009)?

Questions 1 and 3 are presently subject of research within the DiaLeCo project at the Pädagogische Hochschule in Heidelberg. A first qualitative content analysis from standardized interviews shows that, regarding questions 1, students were in fact able to refer to structural analogies. One particular neat response involved a ruler that moves vertically across the coordinate plane while measuring a constant distance of the graphs

of $f(x)$ and $f(x) + a$ (Pinkernell 2015). Concerning the third set of questions, a test on algebra is presently being developed with a particular focus on structure sense and representational flexibility.

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A COMPARATIVE ANALYSIS OF WORD PROBLEMS IN SELECTED THAI AND FINNISH TEXTBOOKS

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The purpose of this study is to compare the characteristics of word problems used in a selection of Thai and Finnish mathematics textbooks. A total of 1,565 word problems from a series of 2nd grade to 4th grade Thai and Finnish mathematics textbooks were analysed. The results show that the characteristics of word problems used in Thai textbooks differ from Finnish textbooks in many aspects. A majority of word problems in Finnish textbooks are multi-step word problems, while in Thai textbooks, one-step word problems are more prominent. Finnish textbooks have a higher percentage of repetitive sections (ones that include only the same type of problems) than Thai textbooks. In both countries, word problems requiring the use of realistic considerations are infrequent, making up less than 5 percent of the total.

INTRODUCTION

A word problem is defined as a text which describes a situation with question(s) to be answered by applying mathematical operation(s) based on a provided set of descriptions (Verschaffel, Greer, & De Corte, 2000). However, in early-grade textbooks, instead of using only text, word problems often include graphical representation (e.g., pictures, graphs, tables) to describe situations and provide meaningful numerical data (Pongsakdi, Brezovszky, Hannula-Sormunen, Lehtinen, 2013). Therefore, in this study, a word problem is not only a text, but can also be a combination of text and picture(s) that describes a situation, provides meaningful data and requires applying mathematical operation(s) for the question(s) to be answered. Word problems are intended to provide a connection between classroom mathematics and mathematics in the real world. It is believed that through practicing with word problems, students could learn not only mathematical skills, but also how to apply these skills effectively, which in turn would allow them to solve math problems that they encounter in everyday life (Verschaffel et al., 2000).

For this to be realized, the word problems presented to students need to resemble math problem situations that occur in everyday life. Students also need to understand the situations described in word problem texts and use realistic considerations when solving the problems. Unfortunately these two requirements are rarely met. For instance, several studies indicated that many students do not develop an adequate understanding of the situations described in word problem texts and only apply superficial strategies, such as a keyword approach (looking for the individual word that indicates which calculation to perform, e.g. “altogether” = addition) (Van Dooren, De Bock, Vleugels, & Verschaffel, 2010). Even those students who do use more

comprehensive strategies often do not use realistic considerations when solving the word problems (for an overview, see Verschaffel et al., 2000).

The reason that students apply superficial strategies and exclude the use of realistic considerations in the modelling process may originate from the nature of the word problems and the way they are presented in mathematics textbooks. First, if none of the word problems presented to students resemble math problem situations that occur in everyday life, one can hardly expect students to use realistic considerations. Second, if word problems are sequenced in a way that allows students to determine the solution method and the operation needed without reading the text (e.g., providing students with whole pages of the same type of word problems) (Jonsson, Norqvist, Liljekvist, & Lithner, 2014), this can be expected to trigger *einstellung* (Luchins, 1942) rather than comprehensive strategies that would lead to a proper understanding of the situation presented in the problems. Jonsson and colleagues (2014) explained that when problems are presented in such a way, students do not use conceptual understanding and proper reasoning skills. They only practice computation skills by recalling facts and imitating a solution procedure illustrated in the textbooks. Lastly, some word problems include graphical representations to describe the situation of that word problem, for instance, using pictures to illustrate how 15 candies can be divided equally into 3 boxes. By using graphical representations in this manner, it is already clear to students what they should do, since a solution procedure is explained within the pictures.

Traditional word problems have been described as too simple or straightforward, and solved easily by using superficial strategies (Wyndhamn & Säljö, 1997). They mostly ask for a precise numerical answer, which leaves little room for realistic considerations to be integrated into the solution process (Freudenthal, 1991). Gkoria and colleagues (2013) presented evidence to support this claim. Their studies revealed that around 90 percent of word problems in old and new 5th grade Greek mathematics textbooks can be solved by a direct translation of the problem texts into mathematical operations without the need for any realistic considerations. Joutsenlahti and Vainionpää (2008) obtained similar results, finding that around 94 percent of word problems in 5th grade Finnish mathematics textbooks are word problems that include a simple objective and always have only one correct answer, suggesting a lack of word problems requiring the use of realistic considerations.

Most of the studies concerning the nature of word problems used in textbooks and mathematics education have been made in Western cultures and there have been very few studies in other cultural and educational contexts (e.g., Chan & Mousley, 2005). The purpose of the present study is to explore whether these issues also exist in highly regarded mathematical textbooks (2nd grade to 4th grade), and to compare Thai and Finnish mathematics textbooks from that perspective. Specifically, the present study attempted to answer these four research questions: 1) How do the types of word problems differ between Thai and Finnish math textbooks? 2) How do Thai and Finnish textbooks differ in the number of repetitive sections that contain only the same type of

word problems? 3) How do Thai and Finnish textbooks differ in the graphical representations included with word problems? and 4) How do Thai and Finnish textbooks differ in the number of word problems requiring the use of realistic considerations?

METHODS

Selection of Textbooks

Most studies point out that regular mathematics textbooks mainly include word problems that have a simple goal and do not require students to use realistic considerations in the modelling process. However, it is not clear whether the same problems exist in the textbooks that are considered to be one of the most high quality mathematics textbooks in that country. Therefore, unlike typical textbook studies, this study selected only textbooks that are highly regarded, drawing on the opinions of experienced teachers. A series of 2nd grade to 4th grade mathematics textbooks, used in spring term, were selected for the purpose of this study. A total of 1,565 word problems were analysed.

Grade	Thai textbook		Finnish textbook	
	No. of word problems	No. of sections	No. of word problems	No. of sections
2	81	13	323	64
3	164	28	314	74
4	324	45	359	75
Total	569	86	996	226

Table1: Number of word problems and sections in Thai and Finnish textbooks expressed by grade level.

Analytical Framework

The framework for analysis of word problems consists of four main coding schemes: 1) classification of word problem types, 2) repetitiveness of word problem sequences, 3) graphical representations, and 4) the use of realistic considerations.

Classification of word problem types

The coding scheme for word problem types was constructed based on the classification schemes from Greer (1987). Each word problem in the textbooks was classified as belonging to either one-step addition and subtraction word problem types (21 different types of Change, Combine, Compare, and Equalize word problems), one-step multiplication and division word problem types (18 different types of Multiple group, Iteration of measure, Rate, Measure conversion, Rectangular array, Combinations, and Area), one-step word problems that do not belong to any category (e.g., Metinee finished her homework at 11.25. She spent 1 hour 20 minutes doing it. When did she

start to do the homework?), or multi-step word problem. The inter-rater agreement for word problem types between two independent coders was high ($\kappa = .81$).

Repetitiveness of word problem sequences

Repetitiveness of word problems was investigated by determining the type of word problems used in a section of word problems. A section was considered repetitive if it contained only one type of word problem. For sections that included only multi-step word problems, it was investigated whether those multi-step word problems could be solved in the same way (even if the given numbers were different). Sections in which all multi-step word problems could be solved in the same way were also considered to be repetitive.

Graphical representations

Graphical representations used in word problems were classified according to the coding scheme presented in Table 2. The inter-rater reliability between two independent coders was excellent ($\kappa = .93$).

Types	Description	Code
No graphical representation	There is no graphical representation used in the word problem.	0
Picture containing numerical data	The main purpose of using the picture is to provide numerical data.	1
Picture describing the situation	The main purpose of using the picture is to illustrate the situation of the word problem. Although the picture may contain the numerical data, students do not need to use them since all data already are provided in word problem.	2
Picture representing the object	The main purpose of using the picture is to represent the objects mentioned in that word problem. For example, there are 20 🍌 in the basket.	3
Picture for decorative purposes	The picture is related to the word problem but it is used only for decorative purposes.	4
Chart, graph, table	The data were represented in the chart, graph and table format.	5

Table 2: Classification of graphical representations used in the textbooks.

The use of realistic considerations

This coding scheme for the use of realistic considerations was adopted from Gkoria et al. (2003). If word problems are constructed in a way that requires the use of non-direct translation of the word problem texts on the basis of real-world knowledge and assumptions into the mathematical model, then they are coded as 1; those word

problems that can be answered by direct translation of the word problem texts are coded as 0. For example, the bus problem “304 students must be bused to their camping area. Each bus can hold 32 students. How many buses are needed?” Instead of the answer “9.5 buses”, which derives from a mathematical model translated directly from the problem’s statement ($304 \div 32$), students need to consider whether their answer is appropriate for the situation being modeled, and provide an alternate more suitable answer (10 buses). Therefore, this word problem was coded as 1. The inter-rater agreement between two independent coders was excellent ($\kappa = .91$).

RESULTS

Type of word problems included in Thai and Finnish textbooks

Figure 1 displays the number (and percentage) of word problems by problem types in the Thai and Finnish mathematics textbooks. Overall results showed that a majority of word problems included in the 2nd grade to 4th grade Finnish textbooks were multi-step word problems, while most word problems used in the 2nd grade Thai textbook were one-step multiplication and division word problems. In the 3rd grade Thai textbook, a majority of word problems were one-step addition and subtraction and multi-step word problems, while in the 4th grade Thai textbook, multi-step word problems were more prominent.

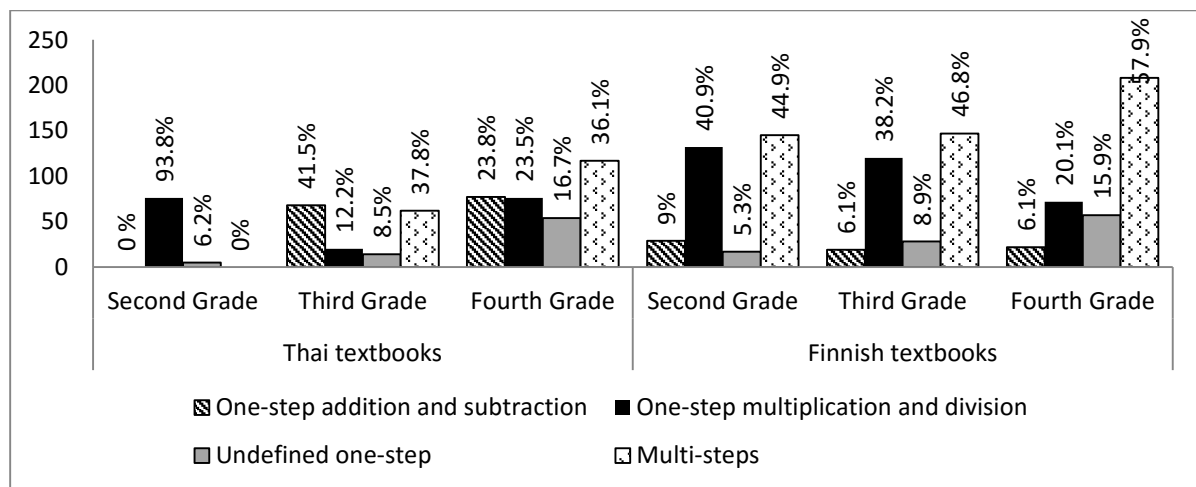


Figure 1: Number (and percentage) of word problems by problem types in Thai and Finnish textbooks.

Repetitiveness of word problem sequences

The repetitiveness of word problem sequence was investigated. Surprisingly, more than half (60.9%) of sections in the 2nd grade Finnish textbook were repetitive. These sections included either the same type of one-step word problems or multi-step word problems that could be solved in the same way. However, the number of sections with the same word problem types was lower in the 3rd (54.1%) and 4th grade textbooks (45.3%). In Thai textbooks, the percentage of sections with the same word problem types was around 38.5% in the 2nd grade, but it decreased in the 3rd (14.3%) and 4th grade textbooks (17.8%).

Graphical representations

The use of graphical representations in all word problems was investigated. A majority of word problems in the 2nd grade Thai textbook used pictures to represent objects and describe the situation, while a plurality of word problems in the 3rd and 4th grade Thai textbooks did not include any graphical representations. In the 2nd grade to 4th grade Finnish mathematics textbooks, a majority of word problems used pictures that contained numerical data.

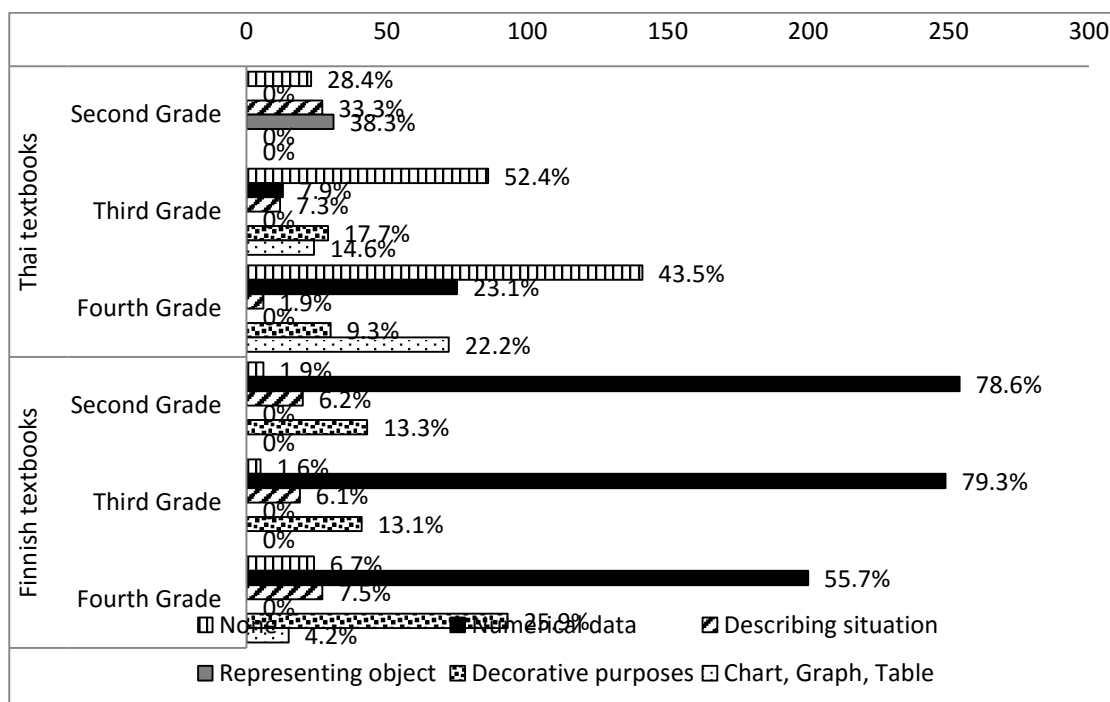


Figure 2: Number (and percentage) of word problems by types of graphical representation used in Thai and Finnish textbooks.

The use of realistic considerations

All word problems were examined for including realistic considerations. The results revealed that there were no word problems in the 2nd and 3rd grade Thai textbooks requiring the use of realistic considerations, while in the 4th grade Thai textbook, the percentage of word problems requiring the use of realistic considerations was just 1.9%. Similar to Thai textbooks, the 2nd grade Finnish textbook contained no word problems requiring students to use realistic considerations. The percentage of word problems requiring the use of realistic considerations in the 3rd and 4th grade Finnish textbooks was 3.2% and 4.5%, respectively.

DISCUSSION

The present study investigated characteristics of word problems from a series of 2nd grade to 4th grade Thai and Finnish mathematics textbooks used in spring term. Although the textbooks used in this study had a good reputation in Thailand and Finland, the results are in agreement with previous studies that most word problems used in textbooks usually include a simple goal without the need for any realistic

considerations (Gkoria et al., 2013; Joutsenlahti & Vainionpää, 2008). The results indicate that the main findings concerning the realistic considerations made in Western educational systems also characterized the use of word problems in Thailand. However, the characteristics of word problems used in Thai textbooks differed from Finnish textbooks in many other aspects. Thai textbooks had a traditional way of introducing word problems to the students. For instance, in the 2nd grade Thai textbook, a majority of word problems were simple one-step multiplication and division problems. This might be due to the Thai curriculum, in which students must learn multiplication and division in the spring term, and results might have differed if the sampled textbooks covered the whole year. Further, multi-step word problems were not yet included in the 2nd grade Thai textbook, although they began to be used in the 3rd and 4th grade Thai textbooks. In contrast, multi-step word problems were already emphasized in the 2nd grade in Finnish textbook, and this trend continued across grade levels. Notably, Finnish textbooks had much more word problems, particularly in the 2nd and 3rd grade, than the Thai textbooks.

Many multi-step word problems, particularly in the 2nd and 3rd grade Finnish textbooks, did not use long sentences to describe the situation of the problems. This might be due to concerns with the reading comprehension skills of young students. Instead of using long texts, graphical representations were utilised to provide meaningful information such as numerical data. For example, a word problem included a picture of several banknotes, and it asked how much money a boy would have left after he bought a ticket which cost 22 Euros. Originally, this word problem was a simple one-step subtraction problem (change problem), but because of the use of graphical representations, students first needed to calculate the total amount of money that the boy had and then subtract 22 Euros from this total. With this additional function of graphical representation, the word problem would be considered a multi-step word problem. In contrast, in the 2nd grade Thai textbooks, a majority of graphical representations were used only to represent objects and to describe the situation. In the 3rd and 4th grade Thai textbooks, word problems hardly included any graphical representations. One possible reason why many word problems in the 2nd grade Thai textbooks included graphical representations to describe the situation is that these pictures might assist students to understand difficult mathematical concepts, such as multiplication and division. However, this type of graphical representations also trivialises the purpose of using word problems. Students already knew the answer to the problem in advance, since a solution procedure is presented within the pictures. Furthermore, the repetitiveness of word problems was investigated. The results revealed that Thai textbooks had a smaller percentage of repetitive sections than Finnish textbooks. Although Thai textbooks used more one-step word problems than the Finnish textbooks, they included a greater variety of types of word problems in each section. Presenting word problems in this manner requires students to read the problems more carefully. It may also prevent students from using the same solution procedure repeatedly without thinking, which can easily occur when the section of problems is repetitive.

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THE EFFECT OF THE EXPLICIT TEACHING METHOD ON LEARNING THE WORKING BACKWARDS STRATEGY

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It has been shown that children who control strategies are able to direct their own learning and knowledge. Seeking for an effective teaching method to achieve this goal, we experimented with the Explicit Teaching method vs. a traditional school one, using both to teach the working backwards strategy. A mixed method analysis showed that explicit teaching showed better results on students' ability to use the strategy. In addition, we found that the teaching method did not affect the students' ability to recognize the strategy. This indicates that young students can understand when to use this powerful tool and, with further guidance, can master their ability to use mathematical strategies.

Theoretical Framework

Children that behave strategically are able to direct their own learning and acquire knowledge of a specific domain. Often, the use of strategies in problem solving will help the child to understand how the strategy works, why it works, and why it is the most efficient way to solve the problem (English, 1993; Portnov-Neeman & Amit, 2015). Students who control many strategies will become faster, more effective and more intelligent problem solvers (Polya, 1957). Tishmen, Perkins & Jay (1996) claimed that most students and adults will not tend to think and behave strategically without proper instruction, guidance and encouragement. Researchers and teachers are constantly learning how best to teach strategies so as to increase students' ability to control them. However, there is a concern among teachers and instructors that teaching mathematical strategies will be difficult to implement and understand (Zbiek & Larson, 2015). The current study will address those concerns by demonstrating the effect of a specific teaching method called *Explicit Teaching* on the learning process of the *Working Backward Strategy*.

Explicit Teaching - Definition and model

Explicit teaching is a systematic methodology of teaching used mainly in areas of reading and mathematics (Anhalt & Cortez, 2015; Archer & Hughes, 2011; Edwards-Groves, 2002). This method is “highly organized and structured, teacher-directed, and task-oriented” (Ellis, 2005). There is a mediation process between the teacher and the learner during all stages of learning (Tetzlaff, 2009), and the teacher is responsible for transmitting an external understanding of information to the learner, who is then

responsible for processing that pre-determined understanding (Olson, 2003). Using explicit teaching does not necessarily predetermine or confine the learners' way thinking; on the contrary, it can help to become more active solvers and foster independent thinking (Portnov-Neeman & Amit, 2015). Tetzlaff (2009) summarized this method into a five step model:

Orientation

Each lesson begins with a clear instruction about the purpose of the lesson. Learners need to understand what they going to learn and how it connects to previous lessons.

Presentation

The lesson material divides into small units that fit the learners' cognitive abilities. The teacher uses a model or schema to guide them through their problem solving process.

Structured Practice

The instructor gives a direct and detailed explanation of the problem solving using a model or schema that was presented in the previous step. During this phase, it is critical that the instructor asks learners questions to check and assess their understanding of the material and clarify any confusion.

Guided Practice

In this practice the instructor addresses individuals' questions and misconceptions one-on-one, and tailors responses to meet the individual needs of each learner.

Independent Practice

In this step, learners are asked to complete an assignment on their own and without assistance. They are not expected to have a flawless understanding of the lesson, but they must understand the steps involved in the process. This step should continue till learners gain full independent proficiency with the materials.

Working Backwards Strategy

The working backwards strategy is a useful and efficient strategy in many aspects of our lives (Newell & Simons, 1972; Portnov-Neeman & Amit, 2015). Sometimes, the achievable outcome is known, but we have not yet determined the path towards achieving it. When dealing with word problems, the information given in a problem can appear like a complex list of facts. In problems such as these, it is sometimes helpful to begin with the last detail given (Wright, 2010). To apply this strategy, the following steps must be followed:

- 1) Read the problem from beginning to end and identify all components and steps that involved in the problem.
- 2) Check the final outcome of the problem.

- 3) From the final outcome, start reversing each mathematical operation in each step until reaching the beginning of the problem.
- 4) Resolve the initial state.
- 5) Check the answer by starting from the initial state and working through the steps to see if the final outcome is achieved (Amit, Heifets & Samovol, 2007).

Methodology

The current study examined the effect of using the explicit teaching method to learn a new strategy, specifically the working backwards strategy for mathematical problem solving. The research questions examined to what extent explicit teaching affects:

- a) The ability to solve working backwards problems.
- b) The ability to recognize the working backwards strategy.

Research Setting

Subject

The study was conducted in the framework of the "Kidumatica" program. Kidumatica is targeted at talented students from the 5th to the 11th grades who are interested in mathematics, but require further tools to reach their full potential (Amit, 2009). Fifty-seven ($N = 57$) 6th grade students were divided in two groups: an experimental group (EG = 30 students) and a control group (CG = 27 students). Over six months, the students studied different mathematical strategies, including the working backwards strategy. The EG studied via the explicit teaching method while the CG studied via the traditional school one. None of them had served as research subjects in previous studies involving the working backward strategy and they had not learned it before. Both groups were taught by the same teacher, who was trained in the delivery of the intervention and was mindful of the possibility of contamination between the different methods employed by the experiment and control group. The fidelity of the teacher to the delivery of the intervention was checked through classroom observations by the program supervisor.

Experimental Group – Explicit Teaching

Students in this group studied the working backwards strategy using the explicit teaching method. The strategy was taught for four weeks and the learning process was based on the explicit teaching model. Each lesson started with an explanation about strategy, including its importance as well as where and how it should be implemented. The teacher demonstrated the model of the strategy and explained the role of each step in the solution process. Afterwards, the teacher demonstrated the strategy on one problem and started a discussion based on students' questions. The following lessons

were dedicated to structural, guided and independent practice, with many complex problems being presented during the lessons.

Control Group- Traditional Teaching

Students in this group studied using a more traditional school approach. The working backwards strategy was taught for the same period of time as the EG. The first lesson started with brief explanation about strategies and their use. Then the teacher demonstrated how to solve several problems based on the working backwards strategy. The teacher did not name the strategy and did not show the model of the strategy. The students then had to solve similar problems by themselves. In the following lessons the teacher presented how to solve more complex problems (similar to the EG) and gave the students some more practice time. The nature of the practice was mainly independent and the teacher gave guidance or explanations only when needed.

Data collection and analysis

Data was collected from pre-post questionnaire tests based on working backwards problems. The tests were conducted at the beginning and the end of the learning process. In the pre-test, students received a worksheet that included 3-5 problems based on one strategy. This paper will address two of these (figure 1). The post-test included six problems, two of which were based on the working backward strategy (figure 2). At the end of each test, the students were asked to write what method helped them to solve the problems. The purpose of the pre-test was to examine students' ability to solve different working backwards problems, and to determine the homogeneity between the two groups. The post-test examined the effect of the teaching methods at the end of the learning process. We used a mixed method to analyze students' answers in both tests.

Card Problem: "Yael Danny and Michael played cards. In the beginning of the game each one had a different amount of cards. Yael gave Danny 12 cards. Danny gave Michael 10 cards and Michael passed Yael 4 cards. At the end each one of them had 20 cards. How many cards did Yael, Danny and Michael have in the beginning?"

The Mangoes Problem: "One night the King couldn't sleep, so he went down into the royal kitchen, where he found a bowl full of mangoes. Being hungry, he took $\frac{1}{6}$ of the mangoes. Later that same night, the Queen was hungry and couldn't sleep. She, too, found the mangoes and took $\frac{1}{5}$ of what the King had left. Still later, the first Prince awoke, went to the kitchen, and ate $\frac{1}{4}$ of the remaining mangoes. Even later, his brother, the second Prince, ate $\frac{1}{3}$ of what was then left. Finally, the third Prince ate $\frac{1}{2}$ of what was left, leaving only three mangoes for the servants. How many mangoes were originally in the bowl?"

Figure 1: Problems from the pre-test.

Weight Problem: “Four students in the class weighed themselves. Cobi was 15 kilograms lighter than Adi. Gaby was twice as heavy as Cobi and Jenya was seven kilograms heavier than Gaby. If Jenya weighed 71 kilograms what was Adi’s weight?”

Basketball Problem: “The Wolverines baseball team opened a new box of baseballs for today’s game. They sent $\frac{1}{3}$ of their baseballs to be rubbed with special mud to take the gloss off. They gave 15 baseballs to their star outfielder to autograph. The batboy took 20 baseballs for batting practice. They had only 15 baseballs left. How many baseballs were in the box at the start?”

Figure 2: Problems from the pre-test

We used a 5 point scale to rank the answers (5 points - full and correct answer, 0 points - no answer). For example in the “Weight problem” (figure 2) there were three steps: (1) Jenya was seven kilograms heavier than Gaby; (2) Gaby was twice as heavy as Cobi; (3) Cobi was 15 kilograms lighter than Adi. If students identified all the steps, calculated each one by doing the opposite mathematical calculation and wrote the final answer correctly, they received 5 points. They got 4 points if they had one calculation mistake but used the strategy correctly. 3 points were given if they failed to reverse one step, 2 points if they did not reverse two steps, 1 point if they did not reverse any step at all, and 0 points they did not solve the problem. Figure 3 shows a five point solution to this problem. The student wrote all the steps and calculated each step correctly. He found the initial weight and wrote the answer. Figure 4 shows an example of a 2 point solution, where the student calculated the first step correctly but did not reverse the next two steps.

	Final phase	Operation	
Adi	47	Cobi was 15 kilograms lighter than Adi	$32+15=47$
Cobi	32	Gaby was twice as heavy as Cobi	$64:2=32$
Gabi	64	Jenya was seven kilograms heavier than Gaby	$71-7=64$
Jenya	71	Jenya weighed 71	

Answer: Adi weights 47 kilograms.

Figure 3: Example of a five point answer to the weight problem

Jenya	Cobi	Adi	Gabi	Categories
1 71	2 $64 \times 2 = 128$	4 $64 + 15 = 79$	3 $71 - 7 = 64$	Weight
71	128	79	64	Answer

Adi weight 79 kg

Figure 4: Example to a two point answer to the weight problem

Findings

The findings from the pre-post tests are summarized in table 1 and figure 5. We can see that both problems in the pre-test showed no significant difference between the groups, which indicates that both groups had the same level of homogeneity. After six months of learning strategies, the average scores in the post-test in both problems was higher among the EG than the CG. We can see a significant difference in the post-test between the two groups in both problems. Figure 5 shows us that students' ability to recognize the strategy improved after the learning process. Both groups had similar results in the pre and post-test.

Test	Problem	Group	N	Mean	SD	Sig
Pre test	Card Problem	EG	30	3.50	1.676	$P = 0.68$
		CG	27	3.30	2.035	
	Mango Problem	EG	30	1.77	1.977	$P = 0.96$
		CG	27	1.74	2.141	
Post-test	Weight Problem	EG	30	4.97	.183	$P = 0.00^*$
		CG	27	3.56	1.987	
	Basketball Problem	EG	30	3.77	1.547	$P = 0.03^*$
		CG	27	2.89	1.553	

* $P < 0.05$

Table 1: Results from pre- and post-test in the EG and the CG.

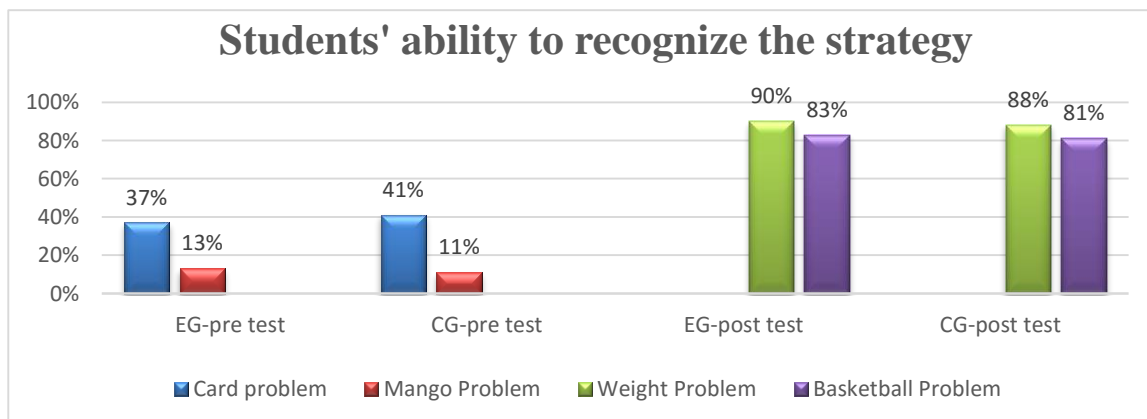


Figure 5: Amount of students from the EG and CG that recognized the working backwards strategy in pre- and post-tests.

Discussion and limitations

There is no doubt that strategies are an important tool for goal-directed procedures in problem solving. Introducing them at a younger age can improve learners' math ability (Polya, 1957) and promote their understanding and thinking (English, 1993). To achieve this goal, it is important to use a specific teaching approach (Tishmen, Perkins

& Jay, 1996). In this study, that approach is the explicit teaching method, which we employed in order to introduce the working backwards strategy. The study examined the effect of this method on students' ability to solve and recognize a working backwards problem. Fifty seven six graders were divided into two groups, an experiment group (EG) that studied with the explicit teaching method and a control group (CG) that studied with a traditional school one. The strategy was unfamiliar to both groups and the findings from the pre-test showed that both groups had a similar starting point.

At the end of the learning process, both groups showed significant improvement, which indicates that young students are capable of using mathematical strategies for problem solving (Tishman, Perkins & Jay, 1996). The group that studied explicitly showed higher results than students that studied with the traditional way, which indicates that the structural and systematic method of explicit teaching proved to be a suitable framework for teaching complex concepts (Anhalt & Cortez, 2015). Previous research has shown that teaching explicitly can help students become active learners and foster their independent thinking (Portnov- Neeman & Amit, 2015). Our results showed that learning explicitly does not necessarily fix students' way of solving a problem and thinking. On the contrary, students understood the principle of the working backwards strategy and applied it in a way that they deemed fit. Though the CG had lower scores in the post test, both groups had similar levels of higher percentage in their ability to recognize a working backwards problem. This finding is very encouraging, since it may indicate that the teaching method did not affect students' ability to recognize strategies. With additional practice, students could master strategies and develop their understanding and their strategic approach towards problem solving. Alongside with those findings, we should take into account that the study had certain limitations. In further research, there is a need to examine regular students and not only talented ones. In addition, there is a need to investigate other mathematical strategies and examine the effect of explicit teaching on larger population.

Conclusion

As educators, our goal is to find the best way to teach specific math concepts. Our concern, however, is that mathematical strategies are difficult to teach and to understand. In order to deal with this concern, we used the explicit teaching method. We found that students who studied with this approach had higher scores than students who studied with a traditional school approach. The use of explicit teaching improved the students' understanding and ability to use the working backwards strategy. As teachers we do not have to be afraid of introducing this subject to young children. The process of introducing mathematical strategies will benefit them and help them evolve into better thinkers and solvers. We can see that students' ability to recognize the time and place where a certain strategy should be used, was not much affected by the method of teaching. This can show teachers that young students are capable of understanding when to use this powerful tool, and that with further guidance and instruction students can master their ability to use mathematical strategies.

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MATHEMATICS AND SCIENCE TEACHERS' COLLABORATION: SEARCHING FOR COMMON GROUNDS

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This paper focuses on the collaboration between one mathematics teacher and three science teachers during a school year in a professional development context supporting inquiry-oriented approaches and connections with the world of work. Through an Activity Theory perspective it addresses contradictions and convergences that emerged in this collaboration as well as interactions between the activity systems of mathematics and science teaching when the mediating tool is the notion of function and its graphical representations. The results indicate the development of shared understandings for the different perspectives that function and graphs are viewed in mathematics and science teaching and shifts in the teaching activity of the teachers in the direction of connecting meaningfully mathematics and science.

INTRODUCTION

The issue of communication between science and mathematics in school classrooms has been acknowledged as crucial for a deeper understanding of common or related conceptual domains. Consequences of the lack of this communication are spread in different directions as on the textbooks' rationale (Triantafillou, Spiliotopoulou & Potari, 2015); students' understanding (Planinic, Susac & Ivanjek, 2012); teachers' classroom discourse and activities (Shirley et al., 2011). The need to provide teachers opportunities to build connections between mathematics and science teaching into their classrooms is more than evident. For example, Berlin and White (1995) argue that this collaboration provides opportunities for students to have less fragmented, and more learning stimulating experiences. However, the undertaken research needs to be strengthened, while more evidence on the actual context of mathematics and science teachers' collaboration could be emerged. Frykholm and Glasson (2005) suggest that authentic contexts could provide fertile ground for this collaboration, while King, Newmann and Carmichael (2009) introduce the idea of 'rich tasks' that involve inquiry-oriented activities in the context of real world scenarios.

This paper refers to a study that took place in the context of a European project, Mascil (see: www.Mascil-project.eu), that aims to promote the integration of inquiry-based learning (IBL) and the world of work (WoW) in the teaching and learning of mathematics and science. To achieve these goals, teacher education and professional development activities have been designed where science and mathematics teachers collaborate in groups to design, implement and analyse lessons in the spirit of lesson study approaches (Hart, Alston & Murata, 2011). A critical issue to consider is in what ways these collaborative activities challenge teachers from different disciplines to

explore the integration of mathematics and science into their teaching and recognize epistemological and didactical issues related to these different practices. To address this issue, we adopt an Activity Theory (AT) perspective to focus on the teaching activity of mathematics and science teachers and on its development in the context of collaboration. We focus on the notion of function and its graphical representation that appeared to be central in the teachers' collaborative activities and we address the following research question: How do mathematics and science teachers' collaborative efforts enrich their teaching activity and enhance connections between the different epistemological and didactical issues on functions and graphs?

THEORETICAL FRAMEWORK

We adopt Engeström's (2001) approach to investigate the process of mathematics and science teachers' professional learning when they are challenged to integrate IBL and the WoW into their teaching. We consider two activity systems, the activity of teaching mathematics and the activity of teaching science, to study the contradictions and convergences that emerged between the two systems, when teachers attribute meaning to the notion of function and its graphical representation. Functions and their graphs are approached from different perspectives in the two disciplines. The teaching of function in school mathematics is mainly formal and the focus is on its definition and its properties. Function is a multifaceted object playing a central role in the development of other mathematical ideas. In science teaching, functions are formulated on the basis of experimental data and are tools for describing, explaining, and predicting real world phenomena (Michelsen, 2006). As regards graphs, making sense of a graph in mathematics means "gaining meaning about the relationship between the two variables and, in particular, of their pattern of co-variation" (Leinhardt, Zaslavsky & Stein, 1990, p.11). In physics, the role of the context in which a graph is used takes a significant role in its meaning (Roth & McGinn, 1997). Below, we provide some main theoretical concepts related to our AT perspective.

The "activity system" is a basic concept of AT in the way that is approached by

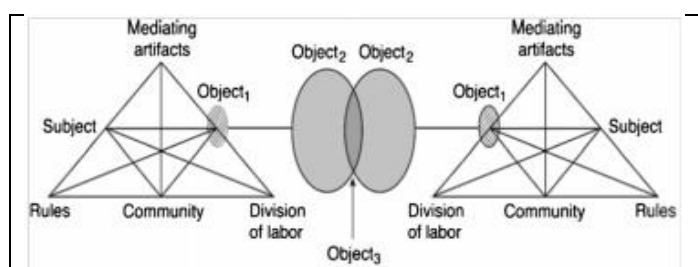


Fig. 1. Interacting activity systems
(Engeström 2001, p. 136)

Engeström (2001). It is collective, tool-mediated and it needs a motive and an object. Individual and group actions are studied and interpreted against the background of entire activity systems. Activity systems are transformed over lengthy periods of time when the object and the motive of the activity are reconceptualized to embrace a radically wider horizon of

possibilities than in the previous mode of the activity. Central to the process of transformation are contradictions within and between activity systems emerging when a new element comes from the outside. The idea of movement across borders appear in what Engeström (2001) describes as third generation of AT. Figure 1 shows a

representation of a third generation activity in the form of two interacting activity systems represented by an extended mediational triangle. The two triangles indicate the basic dimensions of the second generation AT with elements the subject and the object of the activity that is constructed through the mediation of tools, but it is also framed by the community in which the subject participates, its rules and the division of labor. Object 1 moves from an un-reflected and situationally given goal to a collectively meaningful object constructed by the activity system (object 2) and to a potentially shared or jointly constructed object (object 3). By studying contradictions and convergences between the two activity systems, we examine how the notion of function and its graph (mediating artifacts) mediate the teaching actions of mathematics and science teachers (subjects) to form shared meanings and goals (object 3).

METHODOLOGY

In mascil implementation, 12 groups of in-service secondary teachers from mathematics, science and technology have been established. Each group, supported by a teacher educator, participated in two or three cycles of designing, implementing and reflecting during a period of a school year. Before and after each implementation of the designed lessons professional development (PD) meetings took place. During PD meetings teachers collaborated in designing together inquiry-based tasks, shared their experiences from the implementations and discussed emerging issues. Besides, interviews were arranged with a number of participants from each group in order to further address the impact of the PD experience on their professional learning.

In this study, we focus on three science teachers (sctA, sctB, sctC) and one mathematics teacher (mtA) who worked in the same upper secondary school and were members of the same mascil group (7 teachers). These teachers collaborated in the design and implementation of three tasks (*Elasticity of Ropes*, *Biodiesel* and *Drug Concentration*) integrating mathematics and science in the context of three cycles of designing-implementing-reflecting. Here, we analyze data from the first four out of six PD meetings, the classroom implementation of the first task and the teachers' interviews. In these PD meetings, the teachers started to exchange ideas about co- designing, discussed about the design of the first task and reflected on its implementation. The classroom implementation of this task lasted 3 teaching sessions (45 minutes each) and it involved: introduction to the task through short videos of situations where ropes broke; discussion about the importance of exploring these phenomena; experimentation with weights and springs to conjecture Hook's law; experimentation with weights and wires in non-linear situations where the elasticity is destroyed and the material breaks; construction of graphs of the Hook's law by the students based on their measurements; comparison of weight-elongation graphs for different materials (e.g., glass, rubber); classroom discussion on emerging issues about the elasticity of materials and the functional relations used to model the relevant phenomena. The science teachers orchestrated mainly the experimentation phases while the mathematics teacher had the responsibility to manage the classroom activity related to functions and graphs.

The data were audio-recorded and transcribed. Under a grounded theory approach (Charmaz, 2006) we analysed the different sources of data related to the three phases of the first cycle and to the teachers' interviews. Initially, we identified parts of the data concerning the activity of mathematics teaching and the activity of science teaching. Then we focused on common themes that cross the two practices (e.g., inquiry in mathematics and science teaching, the role of context in mathematics and science teaching, the nature of concepts and processes in mathematics and science). Episodes indicating contradictions and convergences were selected within and across the common themes. Finally, we identified interactions between the two activity systems (mathematics and science teaching). In this paper, the steps of the analysis described above concern the notion of function and its graphical representations that appeared to be central in mathematics and science teachers' interactions.

RESULTS

The meaning of contradictions is related to the elements of the AT triangles across the two activity systems. Convergences appear as common actions and goals that indicate an integration of the objects of these systems. Below, we address epistemological and didactical issues around the design and implementation of tasks integrating mathematics and science that emerged in different phases of teacher activity. A central theme of discussion throughout the PD meetings was the notion of function and the different ways by which it is approached in science and mathematics teaching.

Searching for tasks and concepts to integrate science and mathematics

In the first PD meeting, the teachers were introduced to the mascil philosophy through the analysis of existing mascil tasks and they were encouraged to collaborate in co-designing lessons based on these tasks or new ones developed by them. In the second PD meeting, the teachers brought their own ideas for tasks and started to discuss possible links between mathematics, physics and chemistry. The mathematics teacher (mtA) made explicit his willingness to work together with the science teachers by recognising that science teachers could provide ideas for contextual tasks where mathematics is embedded: "I see that you have the knowledge of the contexts that we can use to design lessons together". He also provided specific suggestions promoting their collaboration: "We can co-teach for four hours in the an 11th grade class". The science teachers provided different contexts for potential tasks (e.g., heat engines, biodiesel, elasticity of ropes) and with the encouragement of the teacher educator they suggested possible mathematical ideas related to these contexts. The physics teacher (sctA) suggested as a mathematical idea the concept of function appearing in the transformation of thermal energy in heat engines. He mentioned that graphs of functions such as straight lines, hyperbolas and exponentials used in this context can provide a bridge to mathematics. However, he recognized divergences between how mathematics and science approach functions:

"In science, you first take measurements in an experiment and then you want to see what function is behind. Are you interested in this in mathematics? The function may be a

familiar one, a polynomial. What we actually do as physicists is to do the measurements and insert them in a software that gives us the corresponding function that can be a known one or not”. (sctA, 2nd PD meeting)

In the realm of the discussion, mtA indicated that functions and graphs constitute objects of study in mathematics, but usually in context-free situations. The contradiction that appears here concerns epistemological issues on how function is considered in science and in pure mathematics. The view that sctA expresses is closer to how function is used in modelling, which is not emphasised in school mathematics. In the third PD meeting, the notion of function and its graph emerged when the two physics teachers proposed the *Elasticity of Ropes*. The discussion that followed was around didactical issues such as: students’ tendency to consider all relations as linear; students’ difficulty to connect graphs with physical phenomena; the meaning of inquiry in the task; the connection of the tasks to mathematics curriculum. Initially, mtA found the mathematics involved in the task rather trivial for high school students: “I cannot see how to contribute here. The law is too simple from a mathematical point of view ... It is too experimental”. Later on, the discussion moved in graphs for non-linear relations when the elasticity is destroyed and the students were asked to interpret the graph in relation to the behavior of materials. mtA at this point seemed to overcome his initial doubts and recognized the potential of the task to indicate the distance between real world phenomena and mathematical models: “A law models a situation under certain conditions. And this is important in mathematics as well”.

Implementing the designed tasks

The students have already made the experiment with the springs and have collected their measurements. The two physics teachers had also performed the experiment for testing the elasticity of wires by using weights. At this phase, the mathematics teacher took over the management of the lesson by asking the students to draw a graph of the relation weight–displacement based on their measurements. The notion of function and its main properties again is the common tool pertaining mathematics and science teaching. In the classroom discussion, the teachers took the opportunity to make explicit to the students the different conventions and rules followed in mathematics and science as it appears in the following extract:

sctA: Let’s see how we use graphs in science. In science, we are not allowed to put numbers in the two axes as well as on the graphs. Is this common in mathematics?

mtA: This is not a problem for us.

sctA: The criterion for selecting scale is to find the extreme measurements and their difference. I do not know what mathematics teachers do in the classroom.

mtA: We try to have the same scale in the two axes as we usually draw graph functions with a known formula.

sctA: This is very interesting. We never do this. And we do not have any problem with the origin of the axes.

mtA: And the way we treat the slope is different.

sctA: We discuss it because we realise that we say to our students different things. In science, the slope has always units of measurement, but not in mathematics.

The teachers also started to make connections between the function as a mathematical tool and the physical phenomenon. For instance, mtA working in the context of the physical phenomenon (the specific measurements of the spring displacements for different weights) used the notion of function as a tool for interpreting this phenomenon, an approach that is not common in mathematics teaching. In particular, he challenged students to make connections between the properties of function as a mathematical object with the experiment. Below, we list questions he posed to the students to illustrate his attempts: “I would like you to explain why two successive measurements as points in the graph can be connected only with a straight line”; “If the graph is a straight line what does it mean as regards the relation between the weight and the displacement?”; “Can you make predictions for different values of displacements and weights?”; “What is the meaning of slope in this experiment?”; “How do you interpret the tangent of the angle in the Hook’s law?”. sctA extended the discussion by pointing out that in physics the function that describes a phenomenon is a dynamic object depending on the variability of the measurements: “Why some of your measurements are not on the straight line? It is not needed to connect all the points in one straight line; we draw the best line fit”.

Reflecting on the experience

Reflecting on the implementation in the fourth PD meeting, the teachers discussed about what the students gained from this lesson and what they themselves learned. They made explicit the epistemological divergences underlying the notion of function and they became aware of the fragmented way that this notion is approached in the teaching of mathematics and science in school. In the following two extracts we illustrate mtA’s and sctA’s development of awareness of these epistemological and didactical divergences:

Through observing and interpreting weight-elongation graphs for different materials, the students recognized that the elasticity and the stiffness of the materials are related to the slope of the graph. (mtA’s reflection, 4th PD meeting)

The students managed to connect a mathematical tool, the slope of a line, to the elasticity of materials. They had the opportunity to interpret the slope as we conceive it in physics, as a required force that can cause a unit of change in the length of a spring. They also realised that a graph in physics is beyond the formal way is taught in mathematics. It is a tool that helps them to interpret a physical phenomenon and also make predictions... Teaching mathematics and science together made us realise that we teach the same thing with completely different ways... (sctA’s reflection, 4th PD meeting)

In the interviews, teachers appreciated the collaboration and seemed to become aware of the different epistemological and didactical perspectives that mathematics and science teachers adopt in teaching. They also recognised that students’ learning was

rather fragmented and their own difficulty to bridge the distance between mathematics and science in the actual classroom:

In a school day, I teach my part and the mathematics teacher his own. However, the students can listen to many different things. It is a problem not to have an idea of other teachers' work. Through mascil we realised the diversity of our teaching approaches and the emerging problems. Before, we were not aware of it. (sctA's reflection, interview)

Maybe more time was needed to integrate the actual teaching of the two subjects. In the reality, we distributed our responsibilities, I do this, you do that. Each of us remained in his own space. (mtA's reflection, interview)

The transformation of teaching

By analysing the mathematics and science teachers' collaborative activities we could trace developments and changes in teachers' perspectives as regards epistemological and didactical issues on functions and graphs. Particularly, mtA overcame his initial doubts and recognized the potential challenges of the *Elasticity of Ropes* task for his students. During their collaborative activity, they utilized common tools (e.g., the same worksheet) and transformed their initial goals (individual teaching goals) into shared goals and teaching practices. Their joint activity made them realize divergences in the meaning they attribute to the function concept and the representational conventions and rules they follow in their communities. They also developed awareness on the fragmental way of teaching the notion of function which could have an effect on students' understanding. Besides, we could identify mtA's shifts in his teaching practice of function when he posed context-specific questions to his students (e.g., what is the meaning of slope for this experiment?), or he used the notion of function as a prediction tool for the physical phenomenon under consideration. Finally, in their reflections all teachers appreciated the existed difficulties in achieving the fusion of mathematics and science teaching practices.

CONCLUDING REMARKS

The short analysis supported also by other evidence emerged during the project reveals the strength of collaborative work between science and mathematics teachers and the value of sharing practices in actual science and mathematics teaching. The process of developing a shared understanding of common concepts and the meaning of their teaching for students appears to be rather demanding. However, it evolved through teachers' engagement in discussing connections, discerning epistemological aspects, finding complementary elements and sharing classroom experiences. As King et al. (2009) also argue, inquiry-oriented activities in the context of real world scenarios offered opportunities for science and mathematics teachers to integrate mathematical and scientific ideas and processes into their teaching.

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DISTINGUISHING ENACTIVISM FROM CONSTRUCTIVISM: ENGAGING WITH NEW POSSIBILITIES

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In this paper, we continue the conversation on distinctions between an enactivist theory of cognition and constructivism. These distinctions are not raised to create oppositions or argue against constructivism in favour of enactivism, but to engage in explorations of what these distinctions can allow as possibilities for mathematics education research. We engage with an example to weave together our claims.

CONTEXT OF THE PAPER

Consider this episode taken from a research session in which a group of 12 undergraduates had 20 seconds to mentally solve $x^2-4=5$ for x . Among the many strategies presented, one person's strategy was to depict the equation as the comparison of two equations in a system of equations ($y=x^2-4$ and $y=5$) in order to find the intersection point of those two equations by imagining the graphs. In other words, he thought of the equation as a comparison between two (other) equations in order to find the common value of x , and used the positive and negative values of x to find a second quadratic with a common y . To do so, the student pictured the line $y=5$ in the graph and also superimposed on the same axes, $y=x^2-4$. The latter was referenced to the quadratic function $y=x^2$, which crosses $y=5$ at $x=\sqrt{5}$. In the case of $y=x^2-4$, the function is translated of 4 units lower in on the axes, and the 5 of the line $y=5$ became a 9 in terms of distances. Hence, how to obtain an image of 9 with the function $y=x^2$? With an $x=3$ or $x=-3$. For these, the function $y=x^2-4$ cuts the line $y=5$. Figure 1 illustrates what the student explained having solved mentally.

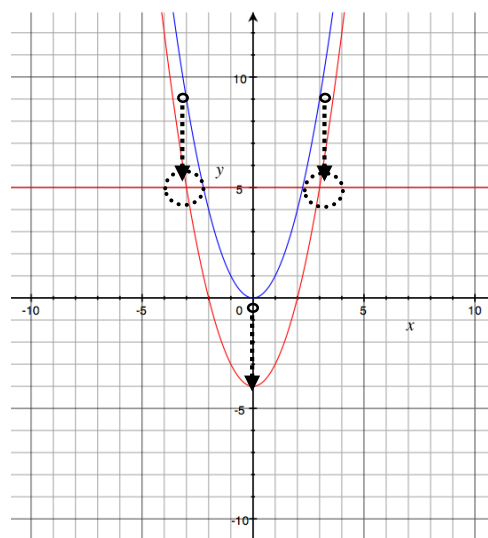


Figure 1. Image of a person's explanation

Our research is about trying to make sense of what we observe people do when they are given a prompt we anticipate will trigger mathematical behaviour. That is, what sense do we make of solver's engagement, as is done in the above example? Our work is not much different than colleagues who for decades have been researching mathematical understanding (e.g. Tom Kieren) and children's mathematics (e.g. Les Steffe). However, we claim differences. To begin, we are mathematics education researchers inspired by an enactivist theory of cognition (e.g. Maturana & Varela, 1992; Varela, Thompson & Rosch, 1991). Further, we have observed over the last decade recurrent assertions that suggest enactivism is a form of constructivism (see e.g., at PME-33, Ernest, 2009). In spite of sharing some similarities, we claim it is distinct from constructivism, something we have written about in e.g., Research Forum05, PME-33; Proulx and Simmt (2012) and we expand upon here. In this paper, we contribute to the ongoing PME (informal and formal) discussions on the matter, as we raise and discuss a number of distinctions around issues of knowledge, problem solving, strategies, and interpretations. By using the above episode to discuss differences that make a difference (as Bateson might note), we illustrate the possibilities that emerge from these distinctions and the potential they offer.

SOME ASPECTS OF ENACTIVISM

Enactivism is a term given to a theory of cognition that views human knowledge and meaning-making as processes understood from a *biological* standpoint. Such biological perspectives have often been adopted as *metaphors* for thinking about knowledge and learning, as is the case within constructivism (see e.g., Piaget in Piatelli-Palmarini, 1979, or Glasersfeld, 1995, for notion of adaptation and evolution). However, for Maturana and Varela (1992), cognition is a biological phenomenon, implying that knowing is *literally* biological. Enactivism considers all living organisms as cognitive: a spider knitting its web, a plant orienting itself toward the sun, a student answering mathematical questions, etc., all act in ways that enable them to continue to evolve, to live, to express knowledge; to maintain their structural coupling with/in the environment (see below). By adopting a biological perspective on knowing, enactivism considers the organism both part of and in an environment. They explain that organism and environment adapt to each other, impacting the other in their courses of evolution. For those of us interested in mathematics knowing, the knower and the problem co-evolve through the process of and with the product of solving. This co-evolution is what Maturana and Varela call *structural coupling*, where environment and organism interact/experience mutual histories of evolutionary transformation, resulting in their adaptability and compatibility to each other.

Every ontogeny occurs within an environment [...] the interactions (as long as they are recurrent) between [organism] and environment will consists of reciprocal perturbations. [...] The results will be a history of mutual congruent structural changes as long as the [organism] and its containing environment do not disintegrate: there will be a structural coupling. (Maturana and Varela, 1992, p. 75)

It follows that environment and organism are mutual “triggers” for the evolution of each other; changes are occasioned by the environment, but determined by the organism’s structure and vice-versa, what they call *structural determinism*:

the changes that result from the interaction between the living being and its environment are brought about by the disturbing agent but determined by the structure of the disturbed system. The same holds true for the environment: the living being is a source of perturbations and not of instructions. (Maturana and Varela, 1992, p. 96)

The structure of an organism is understood as its biological constitution, hence not static and in a constant flux of interaction with the environment, in continual structural coupling with it. This (recursively-dynamically-evolving) structure is more than physical, as it is realized with/in experience, and through its histories of interactions. Enactivism thus deals more with experiential subjects that (en)acts, in the recursive flux of action; and less so with cognitive subjects that build or take things in. Experiences shape one’s structure. In the course of living, an organism integrates experiences in its structure which in turn recursively enables the enactment of (re)actions in specific conditions. It is the structure of the organism that allows for changes to occur, triggered by the interaction of the organism with/in its environment. Maturana and Varela (1992) give the example of a car being destroyed by colliding with a tree and contrast it with an unaffected army tank that collides with the same tree; note that the environment, the tree, is also affected by this interaction. Hence, the interaction is relative to the structures of car, tank and tree. These notions are key to enactivism, to which we raise distinctions with constructivism.

FROM A FOCUS ON KNOWLEDGE TO A FOCUS ON DOING

Several mathematics education scholars have drawn on enactivist ideas to rethink what it means to know mathematically and to reflect on mathematics knowledge (see ZDM, 2015). Focusing on emergence, adaptation and co-specification of knowers and their environments, mathematical cognition has been defined as a dynamic process that emerges in people’s interaction with the environment (Pirie & Kieren, 1994) rather than as mental representations of phenomena from the environment that individuals construct in their minds, as Glasersfeld (1995) expresses:

[Radical Constructivism] starts from the assumption that knowledge, no matter how it be defined, is in the heads of persons, and that the thinking subject has no alternative but to construct what he or she knows on the basis of his or her own experience. (p. 1).

Radford and Sabena (2015) explain that there are two primary traditions that have inspired Western philosophies on knowledge. A first one is the rationalist tradition, where “knowledge is considered to be the result of the doings and meditations of a subject whose mind obeys logical drives” (p. 162). A second tradition is the dialectic-materialist one, where knowledge is “the result of individuals’ sensuous reflections and material deeds in cultural, historical, and political contexts” and is seen as dynamical and cannot be represented because it is conceived as “pure possibility,” as source for action (p. 163). Enactivism aligns itself with the latter, but breaks from it on a specific

matter: knowledge is not the “result of” or a “source of” action, it *is* the emergent action. Mathematical *knowing* becomes inseparable from mathematical *doing* (Davis, 1996), emerging in the interaction with the task. Mathematical strategies brought forth by knowers to solve problems (their emergent adapted responses) are not illustrations of their knowing, but *are* their knowing: the process of knowing and its product are one and the same thing (Pirie & Kieren, 1994). The adaptation process required to engage in a problem is not a representation of one’s capacity for knowing, but *is* one’s knowing: adaptation and action *are* knowledge. There is no separation between knowledge and action, where “all doing is knowing, and all knowing is doing” (Maturana & Varela, 1992, p. 17). In the episode we read above, a constructivist might explain that the person’s knowledge was the source for the strategy, or that the strategy represented the person’s knowledge, but an enactivist would claim that the strategy *is* the person’s knowledge.

For enactivists, knowledge is not in the subject nor in the environment, but emerges in the dynamics of interaction between each. In short, *no interaction no cognition!* Hence, knowledge is not conceived as a possession or a thing one has, but rather an enactment, an emergence in moment-to-moment living. This enactment forms the basis of Maturana’s view of knowledge as *adequate action*:

I am saying that knowledge is never about something. I am saying that knowledge is adequate action in a domain of existence, that knowledge is a manner of being, that knowledge has no content because knowledge *is* being. (Maturana, in Simon, 1985, p.37)

Thus, *if someone claims to know algebra* – that is, to be an algebraist – we demand of him or her to perform in the domain of what we consider algebra to be, and if according to us she or he *performs adequately* in that domain, *we accept the claim*. (1988, pp. 4-5)

Conceiving of knowledge as “adequate action in a domain specified by a questioner” (Maturana, in Simon, 1985, p. 37) insists that this adequacy is not judged on the basis of some allegedly external objective criteria, but in relation the observer who assesses and judges the knowledge on the basis of his/her own reference criteria of what he/she conceived to be adequate in his/her understanding of this domain. In short, the observer matters. In the case of the person’s response to being asked to solve $x^2-4=5$, an observer can claim that this person knows how to solve the algebraic equation, since that observer assesses as adequate this system of equations solution to the task.

FROM INTERPRETATIONS OF THE WORLD TO BRINGING IT FORTH

Issues related to “interpretation” of reality differ between enactivism and constructivism. Constructivists substituted realists’ notions of truth and existence with that of viability, a concept closely aligned with the Darwinian notion of “fit”.

Constructivism goes back to Vico, who considered human knowledge a human construction that was to be evaluated according to its coherence and its fit with the world of human experience, and not as a representation of God’s world as it might be beyond the interface of human experience. (Glaserfeld, 1992, p. 3)

This suggests that there exist a number of viable interpretations of the world, each

knower developing one that fits within his/her functioning of the world. In constructivist thought, interpretations are not said to be made in an arbitrary fashion, but on the basis of invariants or constants one finds in the world and attempts to make sense of. In the case of enactivism, coming to know in a situation is not about the invariants within the environment, but about the coordination of the knower and the environment. The focus on invariants and what the knower can construe from the environment sets these discourses apart. In enactivism, both knower and known, and organism and environment, co-evolve in a constant process of becoming. There is no fixed state for the interpreter to interpret, no invariants or constants that are *a priori*, since both interpreter and environment are in flux, influencing each other in the ongoing process of living: knower and known co-emerge with and in the interaction.

The actions of an animal and the world in which it performs these actions are inseparably connected. [...] What is perceived appears inseparably connected with the actions and the way of life of an organism: cognition is, as I would claim, the *bringing forth of a world*, it is embodied action. (Varela & Poerksen, 2004, p. 87)

In enactivism, the notion of viability of interpretations gives way to a notion of the knower being brought forth as he/she brings forth a world. Rather than learners interpreting the world in multiple ways, enactivists understand them as bringing forth distinct worlds of significance through their knowing. Maturana explains:

Systems theory first enabled us to recognize that all the different views presented by the different members of a family has some validity, but systems theory implied that there were different views of the same system. What I am saying is different. I am *not* saying that the different descriptions that the members of a family make are different views of the *same* system. I am saying that there is no one way which the system is; that there is no absolute, objective family. I am saying that for each member there is a different family, and that each of these is absolutely valid. (Maturana, in Simon, 1985, p. 36)

In the example of $x^2-4=5$, a variety of strategies emerged and were discussed among solvers in the session (for details, see Proulx, 2013). A constructivist could see these as various interpretations or ways of solving the equation $x^2-4=5$. For enactivists, the notion is less about how the problem is interpreted and solved, than about *what* problem was being brought into being and solved. The nuance resides in seeing knowers acting in a multi-verse, bringing forth worlds, rather than interpreting the uni-verse in multiple ways. Hence, an enactivist would say that for each person there were different mathematics being (adequately) solved; and simultaneously knowers (as structurally determined organisms) continue to be brought into being, evolving with their knowing. Solvers brought forth distinct worlds of significance, what Varela addresses through the issue of problem-posing, which we turn to in the next section.

This said, note that this issue is not to be misinterpreted as a relativism *nouveau genre*, since enactivism decries both positivist's top-down view of objective/external knowledge and post-positivist assertions of subjective knowledge emerging from the bottom-up. The enactivist position (e.g. Thompson & Varela, 2001; Varela & Poerksen, 2004) cuts across both views, being on the razor edge, conceiving of

knowledge as a continuously emergent bottom-up phenomena that subsequently imposes itself in a top-down fashion, through a never-ending recursive loop. In enactivism, knowledge is part of the multi-verse, thus is viewed as an ontological question, as well as an epistemological one. In Varela's (1996, p. 99) words, "knower and known, subject and object, are reciprocal and simultaneous specifications of each other. In philosophical terms: knowledge is *ontological*."

FROM PROBLEM-SOLVING TO PROBLEM-POSING

For Varela (1996; Varela et al., 1991), the notion of problem solving implies that problems are already in the world, lying "out there" somewhere, independent of us as knowers, waiting to be solved. Varela explains that because of what we are biologically, historically, socially, culturally, etc., because we are coupled with the environment, and because we and our world co-dependently arise, we do not find problems readymade in our environment but rather we specify the problems in our day to day living through the meanings we make of the world.

The most important ability of all living cognition is precisely, to a large extent, to pose the relevant questions that emerge at each moment of our life. They are not predefined but *enacted*, we *bring them forth* against a background, and the relevance criteria are oriented by our common sense, always in a contextualized fashion. (Varela, 1996, p. 91)

The problems that we encounter, and the questions we ask, are thus as much a part of us as they are a part of our environment since they emerge from our interaction with it. We do not act on pre-existing situations because the pre-existing situation does not arise until we bring it forth. The problems that we solve are relevant for us, they emerge for us as our structure couples with the environment. The effects of the environment are not in the environment, but made possible through structural coupling. The example shows this process unfolding in the solving of $x^2-4=5$. The person described how he posed the problem as a system of equations problem (it became a task about this context) and how he solved it (about this specific context).

As we claimed at last year's PME (Proulx, 2015), reactions to a prompt do not reside inside either the knower or the prompt (as they do in constructivism): they emerge from the knower's *interaction* with the prompt, through posing what is relevant in the moment. If one adheres to this perspective for mathematics education, one cannot assume, as René de Cotret (1999) notes, that instructional properties are present in the (mathematics) prompts offered and that these properties will determine learners' reactions. Strategies are thus not predetermined either by the task setter (teacher) or the task solver (student), but are continuously generated in the solving of problems (which are also emerging with the solvers' actions/acting). Enactivists, contrary to constructivists, do not conceive the solver as encountering a perturbation which causes a disequilibrium in his/her mental structures requiring either accommodation or assimilation of the new stimuli resulting in the solution to the problem. Nor do they conceive, in cognitivist terms, that a solver reads, interprets and plans his/her problem solving, then selects from his/her toolbox or prior knowledge a strategy to solve the

problem and solves it. In that sense, enactivists are letting go of the concept of (building on) prior knowledge. Further, structure is not to be conflated with (prior) knowledge or cognition, because it is the organism's structure that enables cognition/knowledge to emerge in the interaction with the environment (or with a mathematical task), in a constant recursively-dynamical process of mutual influence.

Thus, for enactivists, strategies for solving emerge in the moment-to-moment interaction and co-evolution of knowers and problems. In that sense, problems given are not problems but prompts for solvers to create problems with: *prompts are offered, not problems* (Simmt, 2000). Problems become problems when knowers engage with them, when they pose them as problems to solve. Thus knowers *transform* prompts into mathematical problems for themselves, making the problems theirs, which can be different from the designer's intentions. In this case, $x^2-4=5$ is not a task, but a prompt, with its own designed (emergently observed) structure, with which the solver engages. The prompt was posed by the solver as a system of equations task, and this posed task brought out the consideration of the graph for solving it (and not, e.g., its algebra). The posed task became a graph/system of equation one (even if no graph was provided), which oriented the kind of solution obtained (e.g., in terms of distances, translations, and so forth). There is thus a mutually influential relation between the solving, which generates a context for solving the task, and the generated context itself, which modifies the solving in return, in a continual loop of mutual influence. It is thus not a static "posing of problem" that would give a fixed task to solve; it is one that continually evolves as the task is solved: the posing and the solving are mutually influential and co-evolving. It is also an illustration of how solver and environment both evolve during the solving: the "task" is not static, it evolves as it gets solved for that solver, and as it gets solved, it transforms the solver as well, who is neither static and reacts differently to the prompt as it is transformed. Both are coupled, both evolve in a fitting fashion through shaping each other in this continuous process.

CONCLUDING REMARKS

Raising these distinctions is not to suggest enactivism is "better" than constructivism, since we have discussed, what are for us, useful distinctions between enactivism and constructivism. As researchers inspired by enactivism, we aim for a dynamical view, where knowledge is not a thing, there is no fixed world to interpret, and problems are not waiting to be solved but are dependant on solvers in a constantly recursive dynamical fashion. Although these theories of cognition are related, their distinctions lead the observer, the mathematics education researcher, to bring forth different worlds of significance. Acknowledging that our enactivist way of knowing collapses the epistemological with the ontological, our concerns turn to being, where we find ourselves complicit in the mathematics knowledge we bring forth as observers.

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ON THE CONSOLIDATION OF DECLARATIVE MATHEMATICAL KNOWLEDGE AT THE TRANSITION TO TERTIARY EDUCATION

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The areas of differential and integral calculus, trigonometry, exponential and polynomial functions, equations and inequalities, analytic geometry, and foundations of algebra are both important in high school mathematics and in entry courses at universities. With a questionnaire on declarative knowledge in these areas, new university students of mathematics, physics, computer science and the high school teacher education programme have been tested in three consecutive years. The results are considered to be a function of the time lag between the high school degree and the start at the university. In a longitudinal comparison with exam results of the first courses, time lag turns out to make a significant difference, and more recently acquainted knowledge was less consolidated.

INTRODUCTION

Dropout rate for science and maths students is still a problem of great importance for study programmes at universities. The challenges involved in the transition from high school to university have not been solved over the last decades despite considerable efforts. One reason why the transition has been an issue for a long time is the plain difference in competencies of students who attend universities for mathematics.

In this study the competencies of students beginning their studies in mathematics, physics, computer science, and pre-service teachers who want to teach mathematics at high school, are investigated. Therefore, a test is described that focuses on the declarative knowledge of the students. For a better understanding of the reasons for the measured differences the influences of three individual variables on the students test performance are studied. The three variables are: the study subject, the time of school duration, and possible gap between leaving school and entering the university.

THEORETICAL BACKGROUND

Consolidation of knowledge

There are different approaches to describe the construction of mathematical knowledge. Wilder (1981) regards mathematics as a socio-historical culture with a certain development. A sociological approach considers mathematics as part of a universal knowledge that appears when mathematicians interact (Heintz, 2000, pp. 177-207). The interrelationship between knowledge and social context has been theoretically shaped by Steinbring (2005) for analysing the role of classroom interaction for the establishment of mathematical knowledge. In all these approaches, it is widely regarded as a human activity.

According to the theory of abstraction in context (Hershkovitz, Schwarz, & Dreyfus, 2001), the construction of mathematical knowledge is followed by its consolidation. This process of consolidation has been described in an empirically based theory (Dreyfus & Tsamir, 2004). In-depth qualitative studies have been carried out, for instance, in the cases of algebra (Tabach, Hershkovitz, & Schwarz, 2006) and analysis (Kidron, 2006). The detailed empirical description of the process of forgetting, for instance according to Wixted (1990), is still far from being understood.

Duration of school attendance

The time students attend school in Germany was reduced by one year, down to 12, in most federal states in Germany, excluding two states, which have had a 12-year school duration for a long time. Nevertheless there exists little empirical research on the effects of this change. But there is some evidence that even a small increase in overall course time in mathematics has an influence in the mathematic competence of students (Trautwein et al., 2010). However, teaching time does not have to be reduced when overall duration of school attendance is reduced.

For the federal state of Saxony-Anhalt, a comparison of the scores achieved on final school examinations for mathematics, German literature, and English language was performed (Büttner & Thompsen, 2010). Students of both groups took the same standardized written exam. This made it possible to compare the two groups directly. Results show that there was a better performance of students attending school for thirteen years compared to students attending school for only twelve years in mathematics. For the other two subjects the change in performance was not that strong: in German literature there was no change to be measured, while for the English language test there were only differences for female students, who performed better when being in school for thirteen years.

In Switzerland, different cantons have been compared from the perspective of their human capital (Skirbekk, Lutz, & Leader, 2006). In this case, different cantons have different duration of school attendance. So not a single canton with a change in school duration was investigated, but different cantons had differences in school duration. The data of Third International Mathematics and Science Survey were compared. In this case no differences in performance regarding mathematics and science literacy being dependent on school duration could be found.

Overall, there is some evidence of lower competencies from students who attend school for one less year. This might be especially true for math competencies as the study of Saxony-Anhalt suggests.

Time lag

Some students start their studies with one or more years of lag after leaving school, e. g., due to civil or military service. These students could be considered as having weaker declarative knowledge than students starting their studies without this time lag. However, not much research exists on this phenomenon. A previous analysis of the test

(Halverscheid & Pustelnik, 2013) based on previous data from 2012 showed that a lag can have a big influence on the results in the test and be a risk for dropping out from university. The data from 2013 through 2015 used here confirms this result.

So there is a strong hint that students without a time gap will perform better than students with this time gap.

TEST CONCEPT

The test is based on a comparison between school curricula and the content of the university courses of the first semester, calculus and linear algebra. As a first step, the school curriculum conceptions and fields of knowledge that students should have after leaving school were identified. So, the following eight fields were found: foundations of algebra, systems of equations and inequalities, polynomial functions, exponential functions, trigonometry, differential and integral calculus and analytic geometry. Then, the given competencies of the school curriculum were compared to the university courses' content and the important competencies were identified. Items on the chosen competencies were formulated.

The formulated items are given in three different formats, namely single choice items, multiple choice items, and numeric items. To analyse the test, the one person Rasch Model (Bond & Fox, 2013) was used. So a person parameter was assigned to every student describing his or her test performance.

Example of an item on equivalence equations:

- Choose the equivalent equations to $y = x + 5$ with x and y being real numbers.

$$\begin{array}{ll} y^2 = x^2 + 10x + 25 & y + 2 = x + 7 \\ \sqrt{y} = \sqrt{x + 5} & 13x = 13y - 65 \end{array}$$

Example of an item on the product rule for derivatives:

- Choose the derivative function of $\dot{f} = x^2 * e^x$.

$$\begin{array}{ll} \dot{f} = x^2 * e^x & \dot{f} = 2x + e^x \\ \dot{f} = (x^2 + 2) * e^x & \dot{f} = 2x * e^x \\ \dot{f} = (x^2 + 2x) * e^x & \dot{f} = \frac{1}{3}x^3 * e^x \end{array}$$

RESEARCH QUESTIONS

The research questions correspond to the three variables described:

- Are there differences between the students of different degree courses? If so, how big are they?
- Which influence do one or more years of delay before tertiary education have on the test results of the first year students? Is a possible influence dependent on the study subject?

- Does one less year of school attendance have a measurable impact on the test results?

SAMPLE

The sample of students consists of university students of mathematics, physics, computer science, and pre-service teachers who want to teach mathematics at high school. Although they do not take the same courses in the first semester, these students take the same prep course one month before their actual studies. Participation in the prep course is not compulsory but students are highly advised to take the course. The test was taken on the first day of the prep course before anything else happened.

Data was taken over the last three years. Overall, N=584 students are part of the analysis. The distribution on the years and the subjects can be seen in the table below.

	Mathematics	Physics	Computer science	Pre-service teachers in Maths	Sum
2013	26	93	28	26	173
2014	30	97	32	29	188
2015	36	92	54	41	223
Sum	92	282	114	96	584

Table 1: Participants distribution by year and degree course

The length of a time gap between leaving school and the start of university studies varies from 0 years to 27 years: 408 students start without a time gap, 112 students start with a gap of one year, and 64 students have a gap of at least two years, without big changes in the distribution over the three years.

Regarding the time of school attendance there are 408 students visiting school for twelve years and 176 students visiting school for 13 years. The percentage of students with 13 years of school attendance was much higher in 2013, with 39%, than it was in the other years, with 26%.

METHODOLOGY

To investigate the influence of the different variables an analysis of variance was performed. The test results of the students were the independent variables and there were three independent variables: degree course, time gap, and duration of school attendance. The time gap is split into three groups: No time gap, one year, and at least two years between school leave and entering university. As a fourth factor the year in which the test was taken was included in the analysis.

RESULTS

The analysis of variance revealed no significant effect of the year in which the test was taken: $F(2,581) = 0.923$; $p = 0.398$. The other three factors had significant influence on the test performance. The degree course had an effect on the test performance: $F(3,580) = 29.817$; $p < 0.001$; $\eta^2 = 0.147$, which was a large effect (Cohen, 1988). Existence of a time gap had an effect of medium size on the test results: $F(2,581) = 4.264$; $p = 0.015$; $\eta^2 = 0.016$, as well as the time of school duration: $F(1,582) = 6.113$; $p = 0.014$; $\eta^2 = 0.012$. Furthermore, none of the interaction effects were found to be significant. Notably, there was no interaction effect between time gap and degree course. The mean values and standard deviations for this interaction can be seen in table 2.

	Mathematics	Physics	Computer science	Pre-service teachers in Maths	Overall
No gap	1.64 (0.87)	1.46 (0.80)	0.69 (0.86)	0.64 (0.81)	1.24 (0.91)
1-year gap	1.46 (0.70)	1.17 (0.89)	0.31 (0.82)	0.58 (0.65)	0.96 (0.88)
2-year gap or more	0.95 (0.82)	1.20 (1.01)	0.06 (0.65)	0.28 (0.69)	0.47 (0.88)
Overall	1.54 (0.86)	1.39 (0.83)	0.49 (0.85)	0.57 (0.76)	1.10 (0.93)

Table 2: Influence of time dependence on the degree course

To further investigate the differences between the three significant factors post-hoc-tests with Bonferroni correction were conducted: Regarding the degree courses two pairs could be found. Mean values of the four groups were: Students of Mathematics: 1.53 (SD=0.86); Students of Physics: 1.40 (SD=0.83); Students of Computer Science: 0.49 (SD=0.85); and pre service teachers: 0.57 (SD=0.76). So the differences between mathematics and physics students on the one hand and the students of computer science and pre-service teachers were significant with a large effect size.

Comparing the three groups of time gaps, the students without a gap had the highest mean value of 1.24 (SD=0.91), students with one year gap had a mean value of 0.96 (SD=0.88), and students with a larger gap had a mean value of 0.47 (SD=0.88), revealing all differences to be significant. To further investigate the influence of the time gap the analysis of variance described above was repeated with the eight different test subscales. The difference of mean values and its standard derivation of the post-hoc-test for the effect of a time gap can be seen in table 3.

	No gap – 1-year gap	No gap – 2-year gap or more	1-year gap – 2-year gap or more
foundations of algebra	0.13 (0.10)	0.34 (0.12) *	0.21 (0.14)
systems of equations and inequalities	0.12 (0.10)	0.39 (0.12) *	0.27 (0.14)
polynomial functions	0.23 (0.15)	0.95 (0.19) *	0.72 (0.22) *
exponential functions	0.24 (0.12)	0.46 (0.15) *	0.22 (0.17)
trigonometry	0.48 (0.13) *	1.07 (0.16) *	0.69 (0.19) *
analytic geometry	0.58 (0.14) *	1.37 (0.18) *	0.78 (0.21) *
differential calculus	0.30 (0.11) *	1.02 (0.13) *	0.72 (0.16) *
integral calculus	0.33 (0.11) *	0.87 (0.14) *	0.54 (0.16) *

Table 2: Influence of time dependence on the subscales (* significant on 5% level)

Concerning the duration of school attendance the group with twelve years had a mean value of 1.24 (SD=0.89), which was bigger than the mean value of students with one more year of school attendance 0.79 (SD=0.97).

DISCUSSION

The first important result is the missing influence of the year in which the test was taken. This lack of significant influence shows that the test can be used in the context of the prep course and measure the competencies of the first year students despite changes in the school system over recent years. So it is possible to compare the results of the different years of first-year students.

The main effect found was the importance of the study subject on the test performance. Thereby, the similar test performance of math and physics students made sense based on the importance of mathematics for physics and corresponding demands for the studies. On the other hand, students of computer science and pre-service teachers would need less mathematics for their studies and later jobs. So, the overall order of mean values made sense based on the expected demands of the different degree courses.

On the other hand, mathematics students and pre-service teachers in mathematics had to attend the same lectures in the first year of their studies, while students of physics and computer science had their own mathematics lectures. This was especially problematic since we knew that the used test could predict success in first semester exams. So the big gap caused tremendous problems for the pre-service teachers that could not be overcome in their first year of studies.

But the rather weak results of pre-service teachers did not only influence their studies but also their later work as teachers since teachers also need knowledge of mathematics and not only the content they teach. It is especially known that knowledge in mathematics is necessary for knowledge in mathematics education. So the results suggest that many pre-service teachers were not aware of the amount of mathematics they would need, leading to high dropout rates.

The second important main effect on the test results was the time gap before entering the university. Results showed that not only did one year of time gap have an influence on initial knowledge but also that one year or more of waiting led to worse test results. Also, this second difference might be increased due to grouping all students with more than one year of time gap, especially because these gaps might be caused by worse performances in school. However, the offer of prep courses seemed to be especially helpful for students having a time gap in refreshing their mathematics knowledge since the disadvantages at the beginning could be made up in the first semester. Furthermore, an interaction effect between time gap and degree course could not be found, showing that forgetting affected different groups in the same way.

It was also interesting to look for the fields of knowledge where differences could be found. While having two years of time gap had a significant effect on each field, there was no significant effect in the fields located at the earlier grades between students only having one year of time gap: foundations of algebra, systems of equations and inequalities, polynomial functions, and exponential functions. So it could be assumed that knowledge on these fields is more consolidated and thus not influenced by a time gap of only one year.

The last significant effect was caused by the time of school duration, showing that students with a shorter time of school attendance have better test results, which is a somewhat unexpected direction. However, prior findings on this variable have shown differences. Also, the time spent taking math lessons was more important than the time of school duration, which did not decrease in all cases. So there seemed to be another effect on students with less time of school attendance that lead to better test results.

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CONCEPT STUDY AND TEACHERS' META-KNOWLEDGE: AN EXPERIENCE WITH RATIONAL NUMBERS

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This paper aims to contribute with the reflection on the mathematical knowledge needed for teaching (Ball, Thames and Phelps, 2008). The study's theoretical framework the notion of Concept Study (Davis, 2010; Davis & Renert, 2014), a collective study model in which groups of teachers share their experiences emergent from practice in order to question and (re)construct their own mathematical knowledge for teaching. More specifically, we address the potentialities of Concept Studies to develop participants' knowledge on elementary mathematics and meta-knowledge.

INTRODUCTION

Concerns about the gaps between pre-service teachers' education and their classroom future practice are not new. More than one century ago, on the compendium *Elementary Mathematics from an Advanced Standpoint* (Klein, 1908, 2010), the German mathematician Felix Klein denounces a rupture between school and university mathematics – which he identifies as a *double discontinuity*: little relation is established between the mathematics future teachers get in touch with as university students, the mathematics they have previously learnt at school, and the mathematics they will deal with in their future classroom practice.

Such concerns have echoes on more recent research literature that addresses the content knowledge needed for teaching, its construction and its relations with practice (e.g. Ball et al. 2009). A main reference is Shulman's work, which presents *pedagogical content knowledge* (PCK) as a kind of knowledge which “goes beyond knowledge of subject matter *per se* to the dimension of subject matter knowledge for teaching [...] the particular form of content knowledge that embodies the aspects of content most germane to its teachability” (Shulman, 1986, p.9). An important feature of PCK is that it rests upon its own epistemic groundings, and cannot be reduced to a subset of general content knowledge *per se*. In the words of Davis & Simmt (2006, p.295): “The subject matter knowledge needed for teaching is not a watered down version of formal mathematics.” Similarly, Noddings remarks:

“Knowledge of mathematics cannot be sufficient to describe the professional knowledge of teachers. What does a mathematics teacher know that similar mathematical preparation does not? What specialized knowledge does teacher have? [...] Research on teacher knowledge is crucial not only for the conduct of teaching itself but also for teacher preparation.” (Noddings, 1992, p.202)

This paper addresses the discussion on how to design activities for in-service teacher preparation that takes into account the specificity of the subject matter knowledge needed for teaching. More specifically, we report results from a collective study with

a group of teachers, with focus on the teaching of rational numbers, designed according to the model of *concept study*, proposed by Davis (Davis, 2010). Our aim is to investigate how and to which extent this model of collective study can contribute with the (re)construction of the participants' mathematical knowledge for teaching and development of meta-knowledge. These research goals are also inspired by Klein's ideas about teachers' knowledge.

KLEIN AND ELEMENTARY MATHEMATICS

A key assumption for Felix Klein's program for teacher's education (Klein, 1908, 2010) is the role the author assigns to the School in the scientific development of Mathematics. For Klein, the School plays a role as important as the University in the production of mathematical knowledge: to establish a cultural terrain upon which new knowledge will be constructed. Thus, school mathematical practices interfere in the ways mathematics as a science will follow. Klein's perspective is opposite to the views that attribute to the School a role of spreading knowledge, which would be produced singly in the University, with no interference in this knowledge.

This perspective is related with Klein's notion of *elementary mathematics*, as the nuclear parts that can support and structure mathematical knowledge within a historical context. The author calls *elementarization* the process of historical shifting through which mathematical ideas are progressively more clearly understood and constitute the groundwork for the production of new knowledge. Thus, for the author, there is no hierarchy or difference of value between elementary and advanced parts of mathematics: he regards such hierarchy as an obstacle to overcome.

Under this perspective, for Klein, mathematical knowledge needed for teaching includes the development of a broader view of concepts and theories, their multiple relations, and their historical evolution – *a view of elementary mathematics from a higher standpoint*. According to Schubring (2014), Klein's perspective stresses the importance of a *meta-knowledge*, that is, teachers' knowledge on their own content knowledge. The notion of meta-knowledge, firstly proposed by Smith (1969), is essentially epistemological: mathematics teacher must not only know concept, but mostly be aware of the scientific nature of such knowledge, and its relevance for teaching.

THEORETICAL FRAMEWORK: CONCEPT STUDY

Davis (2010) describes *concept study* as collective study, focused on the mathematical content, in which groups of teacher share their experiences emergent from practice as a means to question and elaborate their content knowledge towards teaching. This model is grounded on a dynamical perspective of the mathematical knowledge for teaching. For Davis and his colleagues (e.g. Davis 2010; Davis & Renert, 2014), a concept study allows a conceptual (re)construction established upon already existent knowledge. Such process is referred to by the author as "*substruct*":

"*Substructing* is derived from the Latin *sub-*, "under, from below" and *struere*, "pile, assemble" (and the root of *strew* and *construe*, in addition to *structure* and *construct*). To

substruct is to build beneath something. In industry, *substruct* refers to reconstructing a building without demolishing it – and, ideally, without interrupting its use. Likewise, in concept studies, teachers rework mathematical concepts, sometimes radically, while using them almost without interruption in their teaching.” (Davis, 2014, p.43, emphasis on the original)

For Davis and his colleagues a concept study highlights and gives assess to the depth and scope of teachers’ knowledge on mathematical concepts. A concept study uses a topic of the school curriculum, as a starting point. This topic determines the range of questions and themes that emerge during the discussion, through the contribution of the participants, as they share their impressions. The data analysis from a concept study is essentially interpretative and structured as identification of *emphasis*, regarding the group’s reflections (Figure 1).

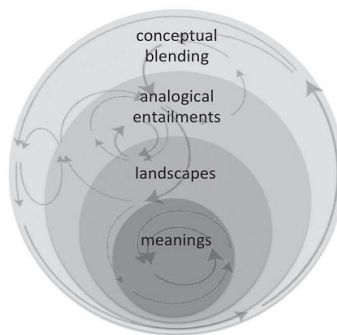


Figure 1: A visual metaphor to the relationships among concept study emphasis (Davis & Renert, 2014, p. 57).

Davis & Renert (2014) stress that only the first emphasis could be described as intentional in any structural sense. The others were emergent – unanticipated, unplanned, arising from shared interests, divergent knowing, and accidental encounters. The first emphasis – *meanings* – is characterized by putting together a list of images, metaphors, impressions that emerge from the collective reflection. The following emphasis are built upon the account of connections between the *meanings* ranked in the first emphasis. Thus, the first emphasis is the only intentional one, from which all the others emerge. Davis (2010, Davis & Renert, 2014) highlights shifts on the perception of Mathematics of the participants as a frequent outcome of Concept Studies. According to the author, this framework allows to identify of developing collective abilities of the teachers, to explore and to reshape their knowledge.

THE STUDY

Aims and Research Questions

The focus of this research is the potential for collaborative studies involving math teacher groups for the construction of mathematical knowledge for teaching. More specifically the central research question that guides this work is: How and to which extend can a collective study, structured in accordance with the model of Concept Study (Davis, 2010; Davis & Renert, 2014), contribute with: (a) *the recognition by the participants of elementary aspects of school mathematics, and its potential influence*

on (re)construction of mathematical knowledge for teaching; and (b) the development of participants' meta-knowledge.

To investigate these questions, the central topic of the concept study was rational numbers. This topic was chosen because it incorporates different aspects (such as representations and operations) commonly recognized by teachers as they involve learning obstacles and difficulties. This research does not intend to map out what teachers *know*, or *do not know*, about rational numbers. The focus is on the investigation of connections and links between various topics of mathematics in a collaborative study aimed at the professional development of mathematics teachers and the (re)construction of their mathematical knowledge for teaching.

Setting and Method

The participants of the collective study were a group of 15 teachers, all working in the public school system, with experience varying from 1 to 20 years, who were taking an in-service training course at the Federal University of Rio de Janeiro. Each session was 4 hour long each, and there were 19 weekly sessions in total. Data collection included audio and video recording (that were fully transcribed), field notes by the researcher, and written registers by the participants during the discussions.

The construction of the list (Figure 2) that characterizes emphasis 1 – *meanings* – were triggered by the question: *What is fundamental when we teach rational numbers at elementary school?* The formulation of this question aims to identification of elementary aspects by the participants.

<i>What is fundamental when we teach rational numbers at elementary school?</i>
<ul style="list-style-type: none"> • To relate the parts with the wholes, in different situations. • To understand the idea of unit. • To operate with fractions. • Equality x equivalence. • Representation on the number line. • Different meanings for fractions – as numbers, as part/whole, as ratio and as division. • To show students that the same rational number has different representations. • To compare rational numbers on the decimal form. • To compare rational numbers on the fraction form. • To give meaning to the rational numbers. • To recognize rational numbers on the percentage form. • To recognize whole numbers as rationals. • Approximations. • Infinite decimal representations – in particular the case of 0,999... • “To know that, given two distinct rational numbers, it is possible to find another one between them.” – density of the rationals.

Figure 2: Meanings.

The subsequent emphases were determined from the dimensions and complexity of the relationships established by the participants between different aspects of the central

topic (rational numbers) and between this and other topics and fields of mathematics. In particular, as these emphases are characterized by the dimensions of the collective discussion, they do not correspond to consecutive and well defined periods of time. Although there is a chronological order between the beginning of each emphasis, thereafter they may overlap and joint nonlinear or stepwise. The dynamic relationship between the emphasis identified in the study can be better perceived from the visual metaphor shown in Figure 1. Therefore, taking into account the complexity of the articulations around the central topic, 3 other emphasis were distinguished: *landscapes*, *entailments* e *inference*.

RESULTS: CONCEPT STUDY EMPHASIS

Emphasis 1 – Meanings

The composition of this emphasis took a long discussion. In general, participants' main reference was their classroom experience, rather than the mathematical relevance of each item. For example, the discussion that led to the inclusion of "understanding the idea of unit" came from the recognition by the group of difficulties students face in solving problems involving units corresponding to sets with more than one element. The only item for which mathematical relevance was explicit (and determinant for the inclusion on the list) was the "*density of rational numbers set*". All the participants agreed that the understand of property that "*given two distinct rational numbers, it is always possible to find another one between them*" was essential for learning, despite some of them did not associate this property with the definition of a dense set.

Emphasis 2 – Landscapes

This emphasis was marked by the recognition by teachers of elementary aspects that form the groundwork for the understanding of rational numbers. A highlight was the notions of equivalence and equality in the context of fractions. From the discussion about the question "*What is right: $\frac{1}{2}$ is equal to or equivalent to $\frac{2}{4}$?*" (brought about by one of the participants), the group reflected upon definitions of equivalence classes and rational number. This question unveiled uncertainties underlying their practices.

This point of the study marked a significant inflexion on the participants' criteria to seek for answers. Until then, they had been using school textbooks as references for their theoretical questions. The uncertainties on equivalence of fractions drove them to seek for answers on academic books. It was clear for them that, despite the formalization of rational numbers is not to be taught at school, its knowledge may provide answers of questions such as the one the group involved with.

Another discussion concerning division with fractions was also prominent. This discussion was triggered by a problem brought by one the participants: "*In a library, all of the books were placed on 6 full shelves. These shelves will be replaced by new ones. Each new shelf fits $\frac{3}{4}$ of the capacity of each of old ones. How many new shelves will be necessary to keep all the books of the library?*" It is not rare that the solution for this problem is based on a strategy that bypasses the division with fractions:

Suppose that the capacity of each of the original shelves is, say, 100 books. Then, each of the new shelves would fit 75 books, and the solution is the result of the division $600 \div 75$. This strategy is certainly correct, however, it avoids the experience with division of fractions. A visual approach for this problem, proposed by one of the teachers, led the participants to associate it with the idea of division as measurement within the context of rational numbers. The discussion raised led them to articulate different elementary aspects of the concept of rational number: the role of the unit; the interpretation of division as measurement; and graphic representations for division.

Emphasis 3 – Entailments

In our analysis, the third emphasis is marked by mathematical connections established, that increased in range and complexity and extended beyond the context of rational numbers. For instance, the discussion reached incommensurability, the notion of infinity and the construction of real numbers. They started to explicitly relate approaches for the elementary school with more advanced mathematical topics.

Emphasis 4 – Inferences

The last emphasis is characterized by a shift on teachers' attitude. A key aspect was that they started to put their warrants of truth at stake. For instance, they were sure about the fact that *every rational number has to representations: as a fraction, and as decimal expansion*. They realized that this was a certainty built throughout their *years as school students, and not on undergraduate courses*. Our analysis suggests that their finding is associated with the double discontinuity point out by Klein (2010).

Moreover, the participants go further and inquire: *What are the warrants for this fact? How should be treated in the classroom?* This suggests that they developed a new perception for the content: *It is not enough to know*, it is also necessary to understand *how this knowledge is constituted, what is its nature and its origin*, as well as *in which sense and to which extend it is relevant to the classroom*. We identified this perspective as a process of meta-knowledge construction. In order to seek for answers for questions that emerged from their practice, it was necessary to recall more advanced knowledge (such as abstract algebra and analysis) and refer to the consistency of formal mathematics. Yet, this perspective did not lead participants to neglect the importance of an approach suitable to elementary school.

Another highlight of this emphasis is the development of a critical attitude by teachers. One of the participants brought forward a problem (Figure 4), because he found out a “flaw” on its formulation. The group noticed that, from a purely mathematical stand point, the choice of interval's borders is irrelevant, and a generic algorithm for the procedure could be established. However, they also noticed that, with data given on the problem formulation, the right answer could be reached *through a wrong strategy*, which they believed to be likely to be done by students: the point B corresponds to the second out of 5 parts in which the segment is divided.

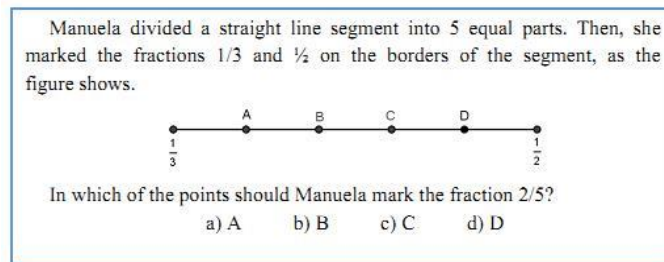


Figure 4: A problem with a flaw.

CONCLUDING REMARKS

Elementary aspects and meta-knowledge

Our analyses suggests that the collective discussion led to the recognition of *elementary* aspects related with rational numbers, and their role in structuring knowledge needed for teaching. This recognition emerged primarily from the reflection on their own experiences as teachers. The discussion included aspect as division, measurement, incommensurability, infinity, and reached the dimension of the nature of their knowledge about these ideas, and relevance for teaching. Therefore, we identify this dimension of discussion as a process of construction of meta-knowledge.

Substruct

The participants engaged on a collective exercise investigating school mathematics, seeking for answers for their questions, nature and relevance of these answers (meta-knowledge). The fact that the participants were actually using these ideas in classroom at the same time they carrying on the concept study was determinant for the dimensions of the discussion. This is opposite to the model of teachers' in-service education that builds on the a priori choices of the tutors, or on the formal structure of mathematical theories. In our case, was built upon the experiences and questions that emerged from the participants' practices. Sharing individual knowledge and experiences triggered the reconstruction of this individual knowledge, and contributed with the development of meta-knowledge, reaching a subjective perspective, beyond the of substantive knowledge (Figure 5).

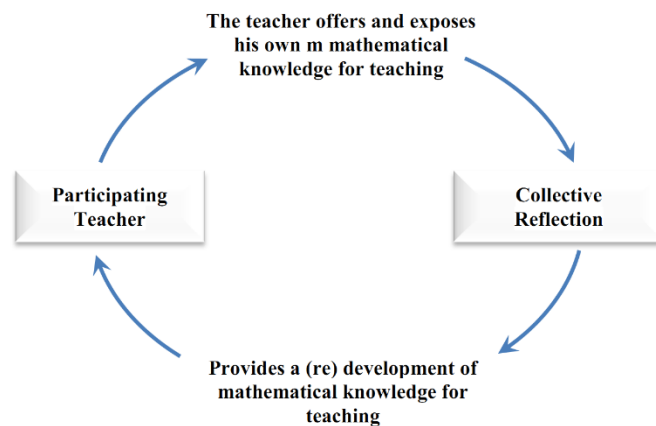


Figure 5: Dynamic of the reflexion process characteristic of the of a study concept

Changes in attitude

Our results indicate a shift to a more inquiring attitude by teachers. The participants engaged on discussion and reflection on ideas that were already familiar to them, and regarded as elementary, without refraining from exposing doubts and uncertainties about these ideas. They exposed their knowledge and beliefs, and expressed the intention to extend the experience with an inquiring attitude to their classrooms. They showed to me effectively more watchful to their students' discourses, reasoning and difficulties. This attitude was clear as the teachers declared to be more confident on dealing with her students' difficulties.

We highlight the potential of the concept study model to raise teachers' awareness of the nature of their knowledge and its relevance to teaching (meta-knowledge), and, mostly, and to equip them with a protagonist role, as role of their own knowledge construction.

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THIRD-GRADERS' BLOCK-BUILDING: HOW DO THEY EXPRESS THEIR KNOWLEDGE OF CUBOIDS AND CUBES?

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The study we report on here intends to detect third-graders' conceptual knowledge on cuboids and cubes, respectively. Avoiding methods which are restricted to commenting verbally or drawing to investigate young children's knowledge on geometrical solids, we used wooden blocks in construction tasks: German and Malaysian children aged 8 to 9 were asked to take wooden cubes, cuboids, prisms or blocks from Froebel's Gifts and to construct cuboids (cubes) by assembling the blocks according to their knowledge and visualization. First observations are interpreted according to the Van Hiele framework. In addition, we have a closer look on the variety of constructions some children produced and raise concluding hypotheses concerning the development of children's conceptual knowledge on geometrical solids.

INTRODUCTION

Geometry education in primary school plays a fundamental role for the development of basic knowledge on geometrical shapes and solids. Thus, classroom activities often focus on naming and sorting shapes. Besides, the primary curriculum has also been extended to activities with hands-on-materials and tasks which have to be solved mentally (Franke & Reinhold, 2016). This includes “working on the composing/decomposing, classifying, comparing and mentally manipulating both two- and three-dimensional figures” (Sinclair & Bruce, 2015, p. 319). Obviously, both sides of the coin – namely visualizing and mentally manipulating and multi-sensory or haptic experiences – facilitate young children's ability of recognizing shapes and foster their acquisition of geometrical knowledge (e. g. Kalenine et al., 2011). As younger children often face difficulties in articulating this knowledge, we consider block building activities to be a meaningful way for them to express their geometrical concepts on solids. Yet, we do not investigate how constructions with tangible blocks foster the development of conceptual knowledge on geometrical solids, in this study.

THEORETICAL FRAMEWORK

Conceptualizing Conceptual Knowledge on Geometrical Solids

The customary conception of a *concept* comprises the “(...) ideal representation of a class of objects, based on their common features” (Fischbein, 1993, p. 139). In this sense, geometrical concepts refer to common features of a class of geometrical shapes or solids which can be visualized or perceived (visually and haptic) when encountering concrete representatives. For example, specific figural properties like the shape of a solid's surfaces or the angles which determine the way the surfaces are interrelated may indicate that a representative is part of a certain class of solids. Based on this

notion, students' conceptual knowledge on geometrical solids reaches beyond the capability of correctly naming concrete representatives or giving a verbal definition, later on at secondary level. It rather comprehends the perception, visualization and identification of distinctive properties which refers to individual mental images students have while thinking of a specific solid (cf. Tall & Vinner, 1981). In addition, Vollrath (1984, p. 9-10) suggests that geometrical concept knowledge can be operationalized by illustrating examples of a certain category of shapes or solids, by assigning the term to a superordinate term, or by solving problems which correspond to the used term and its associated properties.

Development of Conceptual Knowledge on Geometrical Solids

The development of geometrical concept knowledge from primary to secondary has been described by the well-known Van Hiele Model which defines five levels of development which are based on previous level(s) and include specific characteristics: School starters and younger children most often classify shapes according to their holistic appearance which is limited to recognition of resemblance. At this level of *VISUALIZATION* "There is no why, one just sees it." (Van Hiele, 1986, p. 83) Thus, identification of prototypes at this level is fairly easy and enables children to identify other shapes or to visually distinguish different types of four-side figures (e. g. rectangles, parallelograms). Yet, shape recognition is limited to recognition of resemblance and does not pay attention to reasoning on properties or (sub-ordinate) relations between different shapes. In addition, Clements et al. (1999) and others discuss a pre-recognitive level which characterizes young children's abilities before reaching the level of *VISUALIZATION*. Based on this and at the ensuing level of *ANALYSIS*, children are capable of taking a shape's properties into account when they decide upon categorization. Activities of (de)composing, discussing and reflecting upon those activities facilitate children's noticing of properties, but still, they do not realize relationships between properties and are unable to give a concise definition (with necessary and sufficient conditions). Thus, they are usually not able to tell that a cube is a very special cuboid. Only when children are able to cope with questions concerning relationships of shapes and when they start arguing about the impact of various properties on relations among shapes in their definitions, children have reached the level of *ABSTRACTION* (Van Hiele, 1999, p. 311).

Expressing Geometrical Knowledge in Drawings and Constructions

In numerous previous studies, scholars have analyzed children's drawing processes and products to get access to children's understanding and their developmental stages of conceptual knowledge on geometrical shapes. For example, knowledge on the variety of triangles and quadrilaterals in terms of identifying, sorting and comparing representatives was detected by Burger & Shaughnessy (1986). Maier & Benz (2014) stated an immense variety in understanding the concept of triangles according to their analysis of German and English primary children's drawings, too. A significant relationship between children's drawings and their geometric understanding was also

stated by Thom & McGarvey (2015), and Hasegawa (1997) tried to identify stages on the development of an n-gon-concept by using drawing activities and rotations. These and other studies regard student's drawings as a representation of student's geometric concepts (cf. Hasegawa, 1997, p. 177). In line with this research, children's drawing processes and products are widely accepted as individual expressions of spatial abilities (Milbrath & Trautner, 2008) or spatial structuring of two-dimensional shapes (Mulligan et al., 2004; Mulligan & Mitchelmore, 2009). Based on the work of Lewis (1963) who was among the first to investigate how children draw a cube, Mitchelmore (1978) examined how children aged 7 to 15 draw cubes, cuboids, cylinders and four-sided pyramids. Yet, these and following studies have to cope with children's limited drawing skills concerning three-dimensional shapes in primary age. Hence, we derive only very specific information on children's geometrical knowledge on solids when we ask them to draw a solid.

A promising alternative can be found in concrete constructions with blocks: When playing with blocks, even young children deal with geometrical congruence or they distinguish solids according to their properties which is an important aspect of geometrical concept knowledge (see above). Besides, they reflect on spatial relations, orientations or the structure of a three-dimensional array. In Reinhold et al. (2013), we reported on (young) children's difficulties in the (re)construction of cube arrays for purposes of enumeration, but we also found evidence in many ensuing studies¹ that children's fine motor function and their general haptic competence to assemble single blocks or components to three-dimensional arrays is usually entirely developed at the age of 9.

RESEARCH QUESTIONS, DATA COLLECTION AND ANALYSIS

Based on this theoretical framework, we assume that analyses of differences in individual construction processes and products (which may, additionally, be commented verbally) provide deeper insight into children's visualization of solids. This is expected to contribute to a deeper understanding of children's concept knowledge on geometrical solids, while we were interested in exploring to what extent third-graders can articulate their conceptual knowledge on geometrical solids via constructing activities with wooden blocks:

- What kind (and sizes) of cubes and cuboids do third-graders construct and which variations occur?
- Are these constructions in line with their verbal explanations?
- How can we interrelate these results with Van Hiele framework and is there a necessity and supportive data to enrich the framework?

¹ Data was gained in various unpublished Master Theses research studies which reported on part-studies of the project (Y)CUBES at the Universities of Braunschweig and Leipzig, Germany (cf. Reinhold et al, 2013).

Data collection focused on one-on-one-interviews with ten children aged eight to nine in a primary school in one of the larger East-German cities and with twelve nine year-olds in a primary school in a Northern Malaysian city in 2015 (“Grundschule” in Germany and “Malay-medium National School” in Malaysia)². In the beginning, children were asked to explain their ideas and knowledge concerning cubes and cuboids in a short dialogue with the interviewer. Afterwards, a variety of tasks (e. g. “Please, build a cuboid using these blocks.”) invited them to express their knowledge on cubes and cuboids via construction activities with wooden cubes, cuboids, prisms and a collection of different blocks (Froebel’s Gift 6). During their constructions, they were encouraged to describe their proceeding. A manual for all interviews referred to previous research related to the development of geometrical thought (e. g. Crowley, 1987). All interviews were transcribed verbatim and coded with software support by Atlas.ti. A coding guideline was developed mainly according to Grounded Theory Methods (Corbin & Strauss, 2015), trying to detect new facets of articulating conceptual knowledge on geometrical solids and to generate new hypotheses concerning the development of third-graders’ geometrical concepts.

EXCERPTS FROM THE RESULTS

Qualitative analyses of the data reveal a wide variety among either the German or the Malaysian children’s construction activities, and thereby indicate a wide variety in third-graders geometrical concept knowledge on the selected solids.

The range of *PRODUCTS FOR CUBOIDS* (using cubic blocks) included regular cubes (e. g. $2 \times 2 \times 2$ or $3 \times 3 \times 3$), convex constructions with various identical layers (e. g. $3 \times 4 \times 2$), and flat constructions made of only one layer of attached cubes (put as a “lying layer” or as “walls”, e. g. made of $2 \times 5 \times 1$ or $3 \times 1 \times 1$ cubes). Additionally, we observed children who (correctly) identified rows of entirely connected cubes (e. g. $3 \times 1 \times 1$) as cuboids (see first row in figure 1).

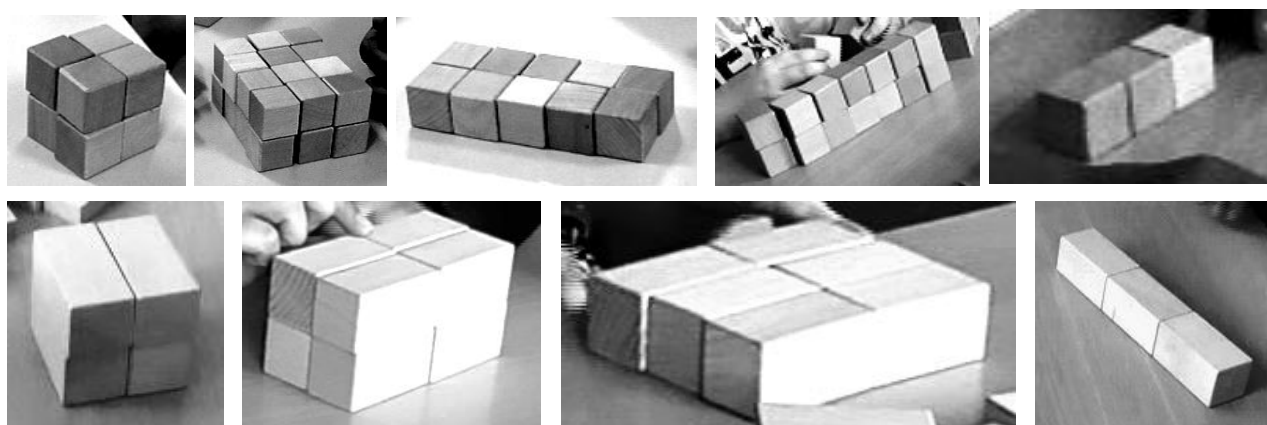


Figure 1: Variety of cuboids constructed by third-graders

² Data collection in Malaysia was supported by the DAAD (Higher Education Dialogue with the Muslime World; Faculty of Education, Leipzig University, Germany and Universiti Sains Malaysia, Penang; “Pupil’s Diversity and Success in Education in Germany and Malaysia”).

Most interestingly, solutions which led to prototypical representatives (convex with various layers or flat lying, e. g. a $2 \times 3 \times 4$ cuboid) were prevailing, whereas constructions resembling “thin and long” objects (with several cubes which are aligned as a row in horizontal position) were rare (see table 1 for a brief overview on types of (correct) representatives for cuboids ten German and twelve Malaysian children constructed with some children finding various solutions). Very similar types of products were constructed when children used cuboid blocks for the construction of bigger cuboids (see second row in figure 1).

type of product	total among German children (using cubes)	total among Malaysian children (using cubes)
Cube	0	1
convex with various layers	7	4
flat lying	12	1
flat wall	1	0
row	5	2

Table 1: Total number of correct representatives of cuboids in constructions

Taking a closer look on the *PRODUCTS FOR CUBES* children constructed during the interviews, we made the general observation that the property of quadratic surfaces is obviously a fairly dominant split of knowledge children express in their constructions. Yet, most children focus on a square base area during their constructions (see figure 2, two examples on the right side). For example, we found that three (out of ten) German children constructed only the quadratic base of the solid and named this building a “cube”. Similarly, three (out of twelve) Malaysian children presented the same kind of construction.

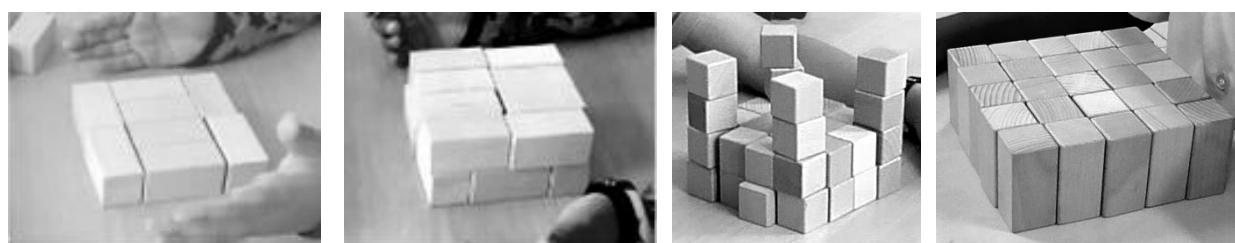


Figure 2: Two “cubes” constructed during the same sequence and further constructions named as “cubes” (with common feature of a quadratic base).

The German third-grader Anna struggles with the demands she has to cope with when constructing a cube, too: Within a longer sequence of the interview she initially constructs a flat lying cuboid with all blocks arranged in a quadratic array. Next, she constructs a second quadratic layer on the lower quadratic layer – naming both constructions a “cube” (see figure 2, first and second picture). Additionally, her focus lies on quadratic arrays as a starting point when building cuboids, as well. She does not identify the thin and long cuboid (from Froebel’s Gift) as a cuboid (“No, this one is not a cuboid, because it is too long.”), but identifies another cuboid (from Froebel’s Gift) with the feature of two quadratic surfaces correctly. These comments and constructions are in line with Anna’s verbal explanation in the beginning of the interview “A cube is quadratic.” and “A cuboid has equal long sides, except for this side (*showing the lateral quadratic surfaces of a block lying on the table.*).” In summary, we can state that Anna is on her way to the level of *ANALYSIS* as she tries to use descriptive mathematical knowledge when giving comments on her construction (e. g. using mathematical terms like “side” or “edge”).

On one hand, these observations obviously reveal problems in developing a sound geometrical concept of “cuboid” and the sub-ordinate concept of “cube”. On the other hand, most German children tried to name properties and offered answers like “because it has equally long edges” when they were asked to explain why they considered their own building to be a cube. Some Malaysian children were capable of arguing in a similar way and offered arguments like “It looks the same from all sides.” or “All surfaces are the same and it’s three-dimensional.”

Another interesting aspect was to observe cognitive conflicts some German children faced when using the material: For example, they said “With cuboid-bricks I can’t build a cube.”, “With this strange bricks (*referring to prisms*) I can’t build a cube or cuboid.” or “With triangles I can’t build a cube.” This reveals that the participating third-graders often *DO* identify at least a limited set of common features of cuboids (and of the sub-ordinate class of cubes) in the sense of Fischbein (1993). Yet, they obviously often have difficulties in considering all relevant features at the same time.

Compared to German children, children’s block constructions in Malaysia revealed a wider distribution on different developmental stages of geometrical concept knowledge (e. g. several children stating “I just know this is a cube.” at the level of *VISUALIZATION*, but only a few children listing properties of the constructed object in detail at the level of *ANALYSIS*). These differences could be due to language peculiarities: In German, the term “Wuerfel” is used in children’s every-day-life. It serves both for dice and cubes and is particularly different from “Quader” (cuboid), whereas there is a significant similarity of the words “cubes” and “cuboids” (which is also obvious in Bahasa Malay some children speak at home: “Bentuk Kiub” or “Bentuk Kubus” for “cube” and “Dadu” for “dice”).

CONCLUSIONS AND OUTLOOK

Aiming at more detailed information on the question how third-graders articulate their geometrical knowledge via constructions with wooden blocks, we found an impressive variety of different types of products and of individual approaches which provided the opportunity to interrelate the constructive activities with the Van Hiele framework. According to our analyses of third-graders' conceptual knowledge on cubes and cuboids, none of the participating German and Malaysian children was in the phase of transition from *ANALYSIS* to *ABSTRACTION* – a result which is basically in line with similar studies (e. g. Szinger, 2008, p. 173). All children faced difficulties in realizing relationships between the geometrical solids cube and cuboid. The more surprising results were the difficulties some children had in constructing *ANY* correct representative of adjacent blocks for either cubes or cuboids or both.

Additionally, the results from our work with children of different cultural backgrounds may serve as an empirically grounded enrichment of the Van Hiele framework – keeping in mind that all data only derived from a fairly small sample ($N = 22$). The results also raise new hypotheses concerning the development of children's conceptual knowledge on geometrical solids: As the variety we detected among third-graders is likely to enlarge in ensuing years of children's development, the individual variety and flexibility in constructing cuboids and cubes and the ability to give comments might extent and change during a longer phase of children's individual development (especially from grade three until grade five). In this sense, the results of our initial study in this field provides the starting point for a longitudinal study we have set up recently. This is encouraged by a particular interest in children's development on geometrical concept knowledge on cuboids and cubes which has not been tracked intensely, so far.

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ARE MATHEMATICAL PROBLEMS BORING? BOREDOM WHILE SOLVING PROBLEMS WITH AND WITHOUT A CONNECTION TO REALITY FROM STUDENTS' AND PRE-SERVICE TEACHERS' PERSPECTIVES

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In this study, we asked 100 ninth graders about their boredom while solving problems with and without a connection to reality. We additionally asked 163 pre-service teachers to judge students' task-specific boredom with respect to the same problems. Our results show that whereas students experienced the same level of boredom for problems with and without a connection to reality, pre-service teachers judged students' boredom as higher for problems without a connection to reality. Moreover, pre-service teachers' judgment accuracy of students' boredom was low for both problem types with huge variability among pre-service teachers.

INTRODUCTION

Emotions are important for mathematics learning and achievement (Hannula, Evans, Philippou, & Zan, 2004). In the mathematics classroom, mathematical tasks can induce emotions in students (McLeod, 1992), and it can be assumed that varying the types of tasks might induce different emotional reactions. For example, a student might enjoy working on a real-world problem but might be bored when solving a purely mathematical problem or vice versa. In order to enhance lesson quality, teachers should be aware of students' task-specific emotions as teachers select problems for their classes. Thus, teachers need to accurately judge students' task-specific emotions. The aim of this study was to investigate students' experiences of boredom as they solved problems with and without a connection to reality and the ability of pre-service teachers to judge students' task-specific boredom.

THEORETICAL BACKGROUND

Problems with and without a connection to reality

Mathematical problems can be divided into problems without a connection to reality and problems with a connection to reality, and the latter can be subdivided into modelling problems and “dressed up” word problems. Examples of all problem types are illustrated in Figure 1. The differences between the problem types arise from the cognitive processes that are necessary to solve the problems (Niss, Blum, & Galbraith, 2007). To solve a modelling problem, the student first has to construct a mental model of the realistic problem situation, which then has to be simplified, structured, and mathematized to construct a mathematical model of the problem. All cognitive processes in modelling are challenging for students, as structuring, for example, can include making assumptions about missing data. After the mathematical model is

constructed, mathematical methods can be applied to compute a mathematical result, which finally has to be interpreted and validated with regard to the real situation. In a “dressed up” word problem, the reality-related cognitive processes are less complex. A simplified situation model is already given and only has to be “undressed” to find the mathematical model. Validation of the real result is limited to checking the mathematics and does not include checking the hypothesized models. Modelling and “dressed up” word problems have in common that they require processes of transferring between reality and mathematics and vice versa. By contrast, in a problem without a connection to reality, the mathematical model is already given. Mathematical methods can be applied directly, and the mathematical result does not have to be interpreted in reality. All problem types are important for students’ learning (Schukajlow et al., 2012). For example, by solving problems without a connection to reality, students can practice mathematical procedures. Solving “dressed up” word problems can introduce students to modelling activities. And finally, by solving modelling problems, students can learn to apply their mathematical knowledge in reality.

Students’ experiences of boredom while solving mathematical problems

Mathematical problems can elicit emotional reactions in students (e.g. boredom; Hannula et al., 2004). Boredom is one of the most frequently experienced emotions in the mathematics classroom (Frenzel, Pekrun, & Goetz, 2007) and can negatively influence students’ thoughts, motivations, and achievements (Schukajlow, accepted; van Tilburg & Igou, 2012). The control-value-theory posits that students’ perceived competence and students’ value appraisals are important sources of students’ boredom (Pekrun, 2006). Students’ *perceived competence* is related to students’ ability to perform a task and depends on the difficulty of the task. As task difficulty can vary within problem types, the impact of task difficulty on students’ boredom should be taken into account in research on students’ task-specific boredom. Students’ *value appraisal* refers to the perceived valences and personal relevance of task activities and outcomes. Accordingly, boredom is elicited by a mathematical problem if the student perceives the activities of solving the problem to be meaningless (van Tilburg & Igou, 2012).

Value appraisals for problems with and without a connection to reality can have different sources. A student might attribute a high value to solving an intra-mathematical problem because he or she perceives that solving the mathematical problem is valuable in its own right (e.g. because the problems helps the student to understand a mathematical idea or to practice mathematical procedures). A student who attributes a high value to a problem with a connection to reality may perceive either solving the real problem or solving the inherent mathematical problem as a meaningful activity. Consequently, the experience of task-specific boredom can differ for problems with and without a connection to reality according to students’ task-specific value appraisal. In mathematics education, it seems to be a common belief that problems with a connection to reality can improve students’ affect in relation to

mathematics (Beswick, 2011). The underlying assumption is that real-world problems make students experience and value the usefulness of mathematics in real life. However, Beswick (2011) argues that there is a lack of evidence for the positive impact of real-world connections on students' affect. For example, previous research did not find a difference in students' enjoyment while solving problems with and without a connection to reality (Schukajlow et al., 2012). However, in other studies on this issue, the impact of task difficulty was not controlled for (Schukajlow & Krug, 2014).

Pre-service teachers' judgments of students' boredom

As solving problems is a central activity in mathematics classrooms (Hiebert et al., 2003), knowledge about students' boredom while solving mathematical problems is important for teaching quality. Teachers have to judge students' task-specific emotions in order to be aware of task-specific effects on students' boredom. The accuracy of judgments of students' cognitive and affective characteristics is regarded as a key aspect of teacher expertise. Previous studies have indicated a deficit in teachers' ability to judge students' affective characteristics (Givvin, Stipek, Salmon, & MacGyvers, 2001; Karing, Dörfler, & Artelt, 2013). As one example, Karing et al. (2013) reported low-to-medium correlations between teachers' judgments and lower secondary students' anxiety in mathematics. Pre-service teachers' ability to judge students' boredom is a concern in teacher education, but it has not been investigated yet.

Research questions

In this study, we examined three research questions:

1. Does students' task-specific boredom differ between problems with and without a connection to reality?
2. Do pre-service teachers' judgments of students' task-specific boredom differ between problems with and without a connection to reality?
3. Do pre-service teachers accurately judge students' task-specific boredom when students solve problems with and without a connection to reality?

METHOD

Procedure and participants

In this study, we asked 100 ninth-grade students (56% female) from two German comprehensive schools to indicate their task-specific enjoyment and boredom on a questionnaire administered after task processing. Students' mean age was $M = 15.97$ years ($SD = 0.93$). We additionally administered an adjusted questionnaire to ask 163 pre-service teachers (86% female) in their first university year to judge ninth graders' task-specific enjoyment and boredom when solving the problems. The pre-service teachers' mean age was $M = 21.01$ years ($SD = 2.51$).

Sample problems

We used eight problems with a connection to reality and four problems without a connection to reality. All problems could be solved by using the Pythagorean theorem.

Figure 1 shows sample problems for both problem types. The problems with a connection to reality could be subdivided into dressed up word problems (e.g. Table tennis) and modelling problems (e.g. Maypole).

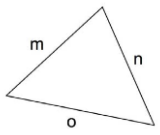
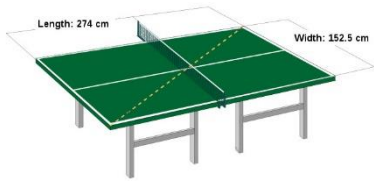

<p>Angle</p> <p>Where does the right angle have to be in the triangle (not drawn true to scale) so that the equation</p> $n^2 - o^2 = m^2$ <p>is satisfied?</p> <p>Draw the right angle into the triangle.</p> 	<p>Table tennis</p> <p>How long is the diagonal (dashed line) of a table tennis table?</p> 	<p>Maypole</p>  <p>Every year on Mayday in Bad Dinkelsdorf, there is a traditional dance around the maypole (a tree trunk approx. 8 m high). During the dance, the participants hold ribbons in their hands, and each ribbon is fixed to the top of the maypole. With these 15-m long ribbons, the participants dance around the maypole, and as the dance progresses, a beautiful pattern is produced on the stem (in the picture, such a pattern can already be seen at the top of the maypole stem).</p> <p>At what distance from the maypole do the dancers stand at the beginning of the dance (the ribbons are tightly stretched)?</p>
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Figure 1: Problem without a connection to reality (Angle) and problems with a connection to reality (Table tennis and Maypole)

Affect scales

To measure task-specific boredom, we adapted well-evaluated scales from previous studies (Schukajlow et al., 2012). In the questionnaires, each problem was followed by statements about students' affect.

In the students' questionnaire, the statement about boredom was "I was bored when working on this problem." Students rated the degree to which they agreed with the statements on a 5-point Likert scale (1=*not true at all*, 5=*completely true*).

In the pre-service teachers' questionnaire, the statement about students' enjoyment was "Students enjoy working on this problem," and the statement about students' boredom was "Students are bored when working on this problem." Pre-service teachers applied a 5-point Likert scale (1=*not true at all*, 5=*completely true*) to rate the degree to which the statements were true for ninth graders from a German comprehensive school.

Task difficulty

In order to exclude the confounding effect from task difficulty on task-specific boredom, we adjusted students' boredom values and pre-service teachers' judgments by the impact of task difficulty.

To adjust students' boredom values, we used students' task performance as an indicator of task difficulty. A code of 0 was given for an incorrect problem solution, and a code

of 1 was given for a correct problem solution. Inter-coder reliabilities for task performance were good ($\kappa > .86$).

To adjust pre-service teachers' judgments of students' boredom, we used pre-service teachers' perceptions of task difficulty, which were assessed in the questionnaire. Pre-service teachers used a 5-point Likert scale to rate the degree to which the statement "This task is too difficult for students" was true for ninth graders.

RESULTS

Preliminary results

In order to control for the impact of task difficulty on boredom, we computed adjusted boredom values. The adjusted values were only slightly different from the unadjusted values (Table 1). However, we used the adjusted values for our further analyses to control for the theoretically justified impact of task difficulty on boredom.

Table 1: Adjusted values for students' boredom and pre-service teachers' judgments

Problem type	Students		Pre-service teachers	
	<i>M</i> (<i>SD</i>)	<i>M_{adj}</i> (<i>SD_{adj}</i>)	<i>M</i> (<i>SD</i>)	<i>M_{adj}</i> (<i>SD_{adj}</i>)
With a connection to reality	2.46 (1.09)	2.49 (1.08)	2.59 (0.45)	2.61 (0.44)
Without a connection to reality	2.48 (1.12)	2.46 (1.12)	3.14 (0.73)	3.11 (0.73)

Students' boredom while solving problems with and without a connection to reality

Students' adjusted mean values on boredom were $M = 2.49$ ($SD = 1.08$) for problems with a connection to reality and $M = 2.46$ ($SD = 1.12$) for problems without a connection to reality (Table 1). Means and standard errors are graphically displayed in Figure 2. A t-test for dependent samples showed that the difference in students' adjusted task-specific boredom was statistically nonsignificant ($t(99) = 0.49$, $p > .05$). This means that students experienced the same level of boredom while solving problems with and without a connection to reality when the impact of task difficulty was controlled for.

Teachers' judgments of students' task-specific boredom

We also asked the pre-service teachers to judge the level of boredom that the students experienced while solving the same problems. When task difficulty was controlled for, pre-service teachers predicted a mean value of $M = 2.61$ ($SD = 0.44$) for problems with a connection to reality and a mean value of $M = 3.11$ ($SD = 0.73$) for problems without a connection to reality. A t-test for dependent samples revealed that the difference in pre-service teachers' judgments was statistically significant ($t(162) = -9.29$, $p < .05$) and that the effect size was large ($d = 0.73$). This means that pre-service teachers believe that students experience more boredom while solving intra-mathematical problems than while solving real-world problems.

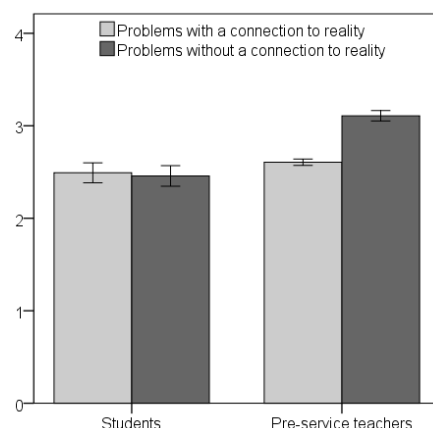


Figure 2: Means for students' boredom and pre-service teachers' judgments for problems with and without a connection to reality (Error bars represent standard errors)

Pre-service teachers' judgment accuracy

To assess pre-service teachers' accuracy in judging students' task-specific boredom, we estimated the level component and the rank component of judgment accuracy (Helmke & Schrader, 1987).

The level component of judgment accuracy relies on difference scores computed between students' boredom values and pre-service teachers' judgments and indicates whether pre-service teachers are able to accurately judge students' absolute levels of boredom. The mean difference scores indicated that pre-service teachers overrated students' boredom for problems with a connection to reality ($M = 0.09$, $SD = 0.40$) and problems without a connection to reality ($M = 0.64$, $SD = 0.74$). Single-sample t-tests showed that difference scores for problems with and without a connection to reality differed significantly from a value of 0, which stands for accurate judgments ($t(162) = 2.88$, $p < .01$, $d = 0.23$ and $t(162) = 11.12$, $p < .01$, $d = 0.86$, respectively).

The rank component of judgment accuracy indicates whether pre-service teachers are able to rank problems according to the level of boredom that the problems induce in students. For students' boredom, the mean correlation was $r = .02$ ($SD = .37$) for problems with a connection to reality and $r = .02$ ($SD = .70$) for problems without a connection to reality. Near-zero correlations and a huge range of correlations indicated that pre-service teachers have trouble judging students' task-specific boredom and that the ability to make accurate judgments differs greatly among pre-service teachers.

DISCUSSION

In this study, we found that students experience the same level of boredom while solving problems with and without a connection to reality when the difficulty of the assessed problems was taken into account. According to the hypothesized relation between feelings of boredom and the subjective values of activities in the control-value-theory (Pekrun, 2006), it can be assumed that students perceive intra-mathematical problems and real-world problems as equally meaningful. This means

that students perceive that solving an intra-mathematical problem (e.g. to understand a mathematical idea) is a valuable activity in its own right and that its value is not necessarily extended by a real-world connection. This result is in line with previous findings on students' task-specific enjoyment (Schukajlow et al., 2012).

In our study, pre-service teachers predicted that students would experience more boredom while solving problems without a connection to reality. This finding might indicate that pre-service teachers believe that students place more value on the use of mathematics to solve problems in the real world than they do on intra-mathematical problem solving—a commonly articulated argument in favor of real-world problems (Beswick, 2011). However, our study shows that students do not perceive intra-mathematical problem solving as particularly boring.

In line with previous research (Karing et al., 2013), our findings on pre-service teachers' judgment accuracy indicate that pre-service teachers have trouble judging students' boredom. Pre-service teachers overrated students' boredom for both problem types and were not able to rank problems according to the level of boredom that students experience while solving the problems. Moreover, our results showed huge variability in judgment accuracy among pre-service teachers. The deficit in pre-service teachers' ability to judge students' emotions should be addressed in teacher education and classroom practice. One method that can be used to improve teachers' knowledge about students is student feedback (Hattie, 2013). Regularly asking students to give feedback on their emotions can help teachers improve their judgment accuracy and enable them to match their teaching to students' learning conditions, which can improve learning.

Limitations and future directions

In this study, we distinguished between problems with and without a connection to reality. However, problems with a connection to reality can be subdivided into modelling problems and dressed up word problems. Although Schukajlow et al. (2012) did not find differences in students' boredom for modelling and dressed up word problems, it remains an open question whether students' experiences of boredom vary for different types of real-world problems when the impact of task difficulty is controlled for.

Conclusion

Are mathematical problems boring? Our results show that students and pre-service teachers answer this question differently. Whereas students report the same level of boredom while solving problems with and without a connection to reality, pre-service teachers judge students' boredom as higher for problems without a connection to reality. This result indicates a deficit in pre-service teachers' ability to judge students' task-specific boredom, which could also be seen in pre-service teachers' trouble in ranking problems according to students' task-specific boredom.

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MATHEMATICAL CRITICAL THINKING: THE CONSTRUCTION AND VALIDATION OF A TEST

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Critical thinking is an important component of general competencies, even though it is rarely mentioned explicitly in curricula. In the psychological literature, critical thinking is generally discussed as domain-general. However, domain-specific conceptualizations of critical thinking have recently gained interest. In this article, the development and validation of a test of mathematics-specific critical thinking is described and reflected upon. For this purpose, the results of three quantitative and one qualitative pilot studies are presented.

MOTIVATION AND RATIONALE

In educational psychology, critical thinking (CT) is framed “as a set of generic thinking and reasoning skills, including a disposition for using them, as well as a commitment to using the outcomes of CT as a basis for decision-making and problem solving.” (Jablonka, 2014, p. 121). In his Delphi Report, Facione (1990, p. 3) understands CT “to be purposeful, self-regulatory judgment which results in interpretation, analysis, evaluation, and inference, as well as explanation of the evidential, conceptual, methodological, criteriological, or contextual considerations upon which that judgment is based. [...]”. Though there are many different conceptualizations of CT (in philosophy, psychology, and education) the following abilities are commonly agreed upon (Lai, 2011, p. 9 f.): analyzing arguments, claims, or evidence; making inferences using inductive or deductive reasoning; judging or evaluation and making decisions; or solving problems.

CT skills are widely accepted as a very important part of student learning in schools as well as in universities (Lai, 2011; Jablonka, 2014). CT has long been supported by educators – and especially mathematics educators –, even though explicit reference to CT is rare in curricula around the world (Jablonka, 2014, p. 122).

CT skills cannot be located within mathematics alone, as Facione (1990, p. 14) emphasizes: Narrowing the range of CT to a single domain would misapprehend its nature and diminish its value. Learning CT can clearly be distinguished from learning domain-specific content. However, there can be domain-specific manifestations of CT and subject contexts play an important role in learning CT (ibid.).

Jablonka (2014, p. 121) stresses the importance of mathematics education for the development of CT skills: “The role assigned to CT in mathematics education includes CT as a by-product of mathematics learning, as an explicit goal of mathematics education, as a condition for mathematical problem solving, as well as critical

engagement with issues of social, political, and environmental relevance by means of mathematical modeling and statistics.”

Because of these relationships between CT and mathematics education, further research is needed that highlights mathematics-specific approaches on CT. However, existing tests that measure CT skills are mostly domain-general and do not consider mathematics-specific features. For example, the Ennis-Weir test of CT uses the context of general argumentation. The participants are presented with a letter that contains complex arguments. They are supposed to write a response to the given letter, defending their judgments with reasons in nine paragraphs. Each paragraph is rated with a score between -1 and 3 on the basis of a coding manual (Ennis & Weir, 1985).

The **research intention** described in this article is, therefore, to construct and validate a test to measure certain aspects of mathematics-specific CT. The test should be applicable for upper secondary and university students. We report on four pilot studies (three quantitative and one qualitative) to document the development of such a test.

THEORETICAL BACKGROUND

In an attempt to measure mathematics-specific components of CT, one cannot include all aspects mentioned in the previous paragraph. Therefore, we focus on a rather basic and implicit dimension of CT that addresses the process of judgment during mathematical problem solving. This can be connected to a cognitive model by adapting and extending *dual process theory* (e.g., Kahneman, 2003). Doing this, Stanovich and Stanovich (2010) propose a tripartite model of thinking in which they locate CT. Similar to dual process theory, they distinguish subconscious (“type 1” or “autonomous thinking”) from conscious thinking (“type 2”). Subconscious thinking is characterized as fast, automatic, and emotional, whereas conscious thinking, which can override subconscious thinking, is characterized as slow, effortful, logical, and calculating. In addition to dual process theory, the tripartite model further differentiates conscious thinking into “algorithmic” and “reflective thinking” (see Fig. 1). For Stanovich and Stanovich (2010, p. 204), this differentiation is necessary as “all hypothetical thinking involves type 2 processing [...] but not all type 2 processing involves hypothetical thinking.”

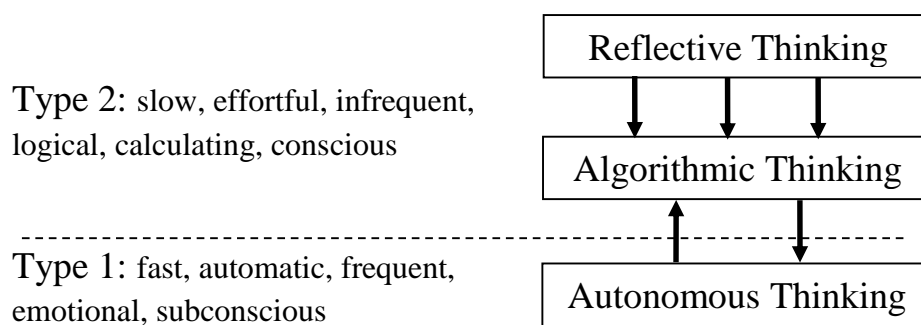


Figure 1: The tripartite model of thinking adapted from Stanovich & Stanovich (2010, p. 210); the broken horizontal line represents the key distinction in dual process theory.

Stanovich and Stanovich illustrate their idea of CT with problems like task 1:

TASK 1: Each of the boxes below represents a card lying on a table. Each one of the cards has a letter on one side and a number on the other side. Here is a rule: If a card has a vowel on its letter side, then it has an even number on its number side. As you can see, two of the cards are letter-side up, and two of the cards are number-side up. Your task is to decide which card or cards must be turned over in order to find out whether the rule is true or false.



Indicate which cards must be turned over.

The most common answer to task 1 is to pick A and 8, whereas A and 5 would have been the correct answer. To answer correctly, type 2 processes are necessary and the problem solvers have to consider what they can learn about the cards by picking two.

Another example for a problem that depends on the problem solver's willingness to use type 2 processes (overwriting type 1 thoughts) and to reflect upon his solution is task 2:

TASK 2: A bat and a ball cost \$1.10 in total. The bat costs \$1 more than the ball. How much does the ball cost?

The spontaneous, autonomously produced answer that most problem solvers come up with is \$0.10. A critical thinker would question this answer and realize that the ball should cost \$0.05, whereas people who do not use CT do not evaluate their first thought and do not adapt their spontaneous solution.

Therefore, when solving mathematical problems, CT can be attributed to those processes that consciously regulate autonomous and algorithmic use of mathematical procedures. Consequently, tasks to measure mathematics-specific CT that reflect this definition should (i) reflect discipline-specific solution processes but should *not* require higher level mathematics, (ii) require a reflective component of reasoning and judgment when solving a task or evaluating the solution, and (iii) reflect an appropriate variation of difficulty within the population.

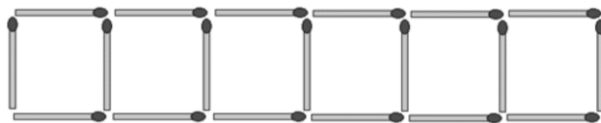
CONSTRUCTING A TEST FOR MATHEMATICAL CRITICAL THINKING

Using the tripartite model of thinking (Fig. 1), CT can be operationalized by situations that demand a critical override of autonomous and algorithmic solutions by reflective and evaluative processes. Such situations can be initiated by tasks as stated above. Additionally, the tasks need to be situated in mathematics to measure domain-specific CT but should be solvable with basic level mathematics.

To compile a CT test, we collected tasks from the according literature – including task 1 (cards with vowels and even numbers) and task 2 (bat-and-ball with varied numbers in the first version: €10.20 for both with the bat costing €10 more than the ball). We also adapted tasks from other contexts and constructed tasks by ourselves. This way, we collected more than 30 tasks to measure mathematical CT.

In this article, we present another three examples from our list of tasks:

TASK 3: A sequence of 6 squares made of matches consists of 19 matches (see the figure). How many matches does a sequence of 30 squares consist of?



Uncritical thinkers might infer from 6 squares to 30, resulting in $19 \cdot 5 = 95$ matches. This solution uses algorithmic thinking without the realization that the correct solution is only 91 matches because of twice counted matches after six squares each.

TASK 4: If the sum of the digits of an integer is divisible by three, then it cannot be a prime number. This statement is

☐ correct ☐ incorrect

Uncritical thinkers might answer “correct” because of the “divisible by three”-rule without realizing that the prime number 3 also has a sum of digits divisible by three.

TASK 5: Write an equation using the variables S and P to represent the following statement: “There are six times as many students as professors at this university.” Use S for the number of students and P for the number of professors.

This task by Kaput and Clement (1979) is famous for its difficulty with most persons wrongly answering “ $P = 6S$ ”, revealing missing reflection.

The list of CT tasks was then used to construct a test of mathematical CT. The first version of this test did not include all tasks from our list but only 22 CT tasks. This was done to keep the time required to carry out the test below 30 minutes.

VALIDATING THE TEST FOR MATHEMATICAL CRITICAL THINKING

To control whether our test is suited to measure CT, we designed and carried out three quantitative and one qualitative pilot studies. In all studies, the tasks were rated dichotomously, 1 point for a correct answer and 0 points for a wrong answer.

Pilot Study 1: CT vs. non-CT items; task formulations

The first pilot study was designed to test the tasks and their formulations. It was also used to explore whether our test actually addresses CT. To investigate on the latter question, we constructed non-CT tasks that can be solved using algorithmic thinking without the need for reflection. We matched those tasks to the CT tasks with a similar context and similar computational difficulty. For example, we used the following task as a non-CT version of the bat-and-ball and matches tasks, respectively:

TASK 2b: You buy eight items for €14.32 altogether. You pay with a 20 Euro note. How much change do you get?

TASK 3b: How many matches does the figure consist of? [The according picture shows 30 squares of matches similar to task 3, arranged in a 5x6 pattern.]

In total, the test of study 1 consisted of 22 CT tasks and 10 non-CT tasks. It was carried out with $n=15$ upper secondary students (grade 11) within 40 minutes in October 2013.

The students correctly solved 80 % of the non-CT tasks but only 58 % of the CT tasks. Therefore, we concluded that our collection of tasks was suited to measure CT. As a result of this study, we removed 8 tasks from our collection due to floor or ceiling effects. Additionally, we improved the wordings of some tasks on which the students orally reported difficulties in understanding the formulations after completing the test.

Pilot Study 2: fatigue and learning effects within the CT test

The second pilot study was designed to test for decreasing concentration and possible learning effects within the 30 min of testing. We used the improved test with 14 CT tasks in two versions. Version A had the tasks 1 – 14 whereas version B had a different order of tasks with tasks 8 – 14 in the first and tasks 1 – 7 in the second half of the test.

In April 2014, this study was carried out with $n=121$ pre-service teachers – students at the University of Education, Freiburg – that attended a lecture on arithmetic. The students were split into four practice groups, with two groups getting version A and the other two groups version B of the test. The results are summarized in table 1. There were no statistical differences between the two groups (multiple t-tests with Bonferroni correction), indicating no fatigue or learning effects within working on the CT test.

	task 1 – 7		task 8 – 14		total	
	M (SD)	min / max	M (SD)	min / max	M (SD)	min / max
A (n=66)	3.50 (1.26)	1 / 6	2.73 (1.32)	1 / 6	6.23 (2.02)	2 / 12
B (n=55)	3.53 (1.32)	0 / 7	2.60 (1.34)	0 / 6	6.13 (2.19)	2 / 11
total (n=121)	3.51 (1.28)	0 / 7	2.67 (1.33)	0 / 6	6.18 (2.09)	2 / 12

Table 1: Results of study 2, mean values (standard deviations), minimum / maximum

Pilot Study 3: differentiation between groups

The third pilot study was conducted to examine whether the CT test is able to discriminate between different groups of students, which were either enrolled to become mathematics teachers for upper secondary schools or to become computer scientists. Some of the students were in the so-called “basic study” (semesters 1 – 4) whereas others were in their main study period (semesters 5 or higher). We used a shortened version (15 min) of the test from study 2 with 11 CT-items. This study was carried out with $n=94$ students at the University of Duisburg-Essen in August 2014.

Our hypothesis was that students with more university experience (i.e. a higher number of semesters) would score better than students with less university experience. Table 2 (left side) shows the results of the students in their basic study and main study period, respectively. A t-test (after testing for normal distribution) confirmed the expected differences in favour of the more experienced students ($p_{1\text{-sided}} = 0.008 < 0.01$).

For the comparison of the students of both study programs, we did not have an assumption which program would prepare its students better for mathematical CT. The

results, however, show a clear advantage for the pre-service mathematics teachers (see table 2, right side, $p_{2\text{-sided}} = 0.038 < 0.05$).

university experience	task 1 – 11	study program	task 1 – 11
semester ≤ 4 (n=28)	4.29 (2.19)	mathematics (n=46)	5.63 (2.50)
semester ≥ 5 (n=66)	5.50 (2.34)	informatics (n=48)	4.67 (2.14)
total (n=94)	5.14 (2.31)	total (n=94)	5.14 (2.31)

Table 2: Results of pilot study 3, mean values (and standard deviations)

Overview of pilot studies 1 – 3

Interestingly, the tasks showed very similar solution rates within all pilot studies despite the considerably different study participants. Table 3 presents these rates for the five tasks selected for this paper for all (sub-) populations (see above).

study		S1	S2				S3				
task	/ n=	15	66	55	121	28	66	46	48	94	
1: Cards (K, A, 8, 5)		0.20	0.20	0.13	0.17	0.14	0.20	0.28	0.08	0.18	
2: Bat-and-ball*		0.67	0.45	0.49	0.47	0.50	0.62	0.59	0.58	0.59	
3: Matches		0.53	0.59	0.58	0.59	0.61	0.61	0.74	0.48	0.61	
4: Digits divisible by 3		0.40	0.41	0.36	0.39	0.54	0.65	0.67	0.56	0.62	
5: Students & professors		0.20	0.06	0.09	0.07	0.14	0.24	0.28	0.15	0.21	

*Using other numbers (€10.20 for both bat and ball) leads to more computational solutions and, thus, higher solution rates in study 1. We therefore used the original version (with €1.10) in later studies.

Table 3: Solution rates of the five selected tasks in all three pilot studies

Pilot Study 4: task-based interviews

The fourth pilot study was designed to better understand the way students worked on the CT tasks. Therefore, task-based interviews with n=5 pre-service mathematics teachers were conducted: three interviews at the University of Education Freiburg and two at the University of Duisburg-Essen in the period from January 2014 to September 2014. These interviews covered all 14 tasks that were used in pilot study 2. Due to space reasons, we can only present a small excerpt of these interviews.

The interviews regarding task 1 (vowels and numbers on cards) revealed that this task rather tested for knowledge (rules of mathematical reasoning) instead of CT. However, one interviewee (that previously did not have the required knowledge) solved this task correctly by reflecting on his choice of cards, showing the importance of CT for task 1.

Working on task 2 (bat-and-ball), four interviewees spontaneously said 10 Cent. However, two of them corrected their solution to 5 Cent shortly afterwards. Both told the interviewer that they found the correct solution because of checking their result.

Therefore, this task is suited to reveal CT. The fifth interviewee did not express a spontaneous solution but used an equation from the beginning. It should be added that both students who checked their solution admitted that this checking was mostly due to the interview situation. This information could lead to further studies revealing situations that trigger the use of CT within students (see future prospects, below).

For task 3 (matches), the interviews showed that the wrong approach (multiplying by 5) seems to be an obvious idea. Three interviewees expressed this idea with two of them correcting their approach after a check. The other two solved this task correctly from the start. Thus, this task is also suited to test for reflective thinking.

Task 4 was solved correctly by all five interviewees with all of them showing signs of CT by expressing thoughts like: “The statement is correct. Wait, does this rule include the number 3 itself? Then it is not correct.”

Task 5 was solved correctly by only one interviewee who knew the task beforehand.

In total, the task-based interviews helped us to reveal tasks that did not require CT and to confirm the use of critical or reflective thinking (in contrast to automatic or algorithmic thinking) with other tasks.

CONCLUSION AND DISCUSSION

Based on the results from the quantitative pilot studies (floor and ceiling effects) and the insight provided by the interviews, we eliminated tasks (e.g., task 5). The final test consists of 14 CT tasks (including tasks 1 – 4) with an average time requirement of 20 minutes in total.

With our approach, we do not intend to include the broad range of aspects and dimensions that are currently discussed under the umbrella term “critical thinking”. We also cannot contribute to the societal, curricular, and philosophical aspects of the topic. However, when one realizes the few efforts to measure CT with respect to mathematics, one could take the studies presented as an approach to pinpoint interindividual differences within CT quantitatively. Furthermore, the connection to *dual process* theory allows for a theoretical interpretation of cognitive processes that contribute to CT. In this context, our instrument can be useful in further studies to elucidate the relations of CT with other aspects such as knowledge, dispositions, and epistemological beliefs (see Rott, Leuders, & Stahl, 2015). Some further theoretical connections to other relevant theories of mathematical thinking, especially to problem solving and the role of metacognition, which could not be addressed in this article, should be explored more deeply on a theoretical and empirical basis.

ADDITIONAL REMARKS AND FUTURE PROSPECTS

In two studies with $n=215$ and $n=463$ mathematics pre-service teachers, a short version of the CT test (11 tasks) was used to measure the students’ CT as a part of their mathematical knowledge base. With the test, we were able to differentiate between students of different study programs (students for upper secondary schools scored better than for primary and lower secondary schools) and different university experience (students

with a higher number of semesters scored better). There were also highly significant correlations between good test scores in the CT test and the ability to justify epistemological beliefs sophisticatedly (for details, see Rott et al., 2015).

Even though this test has been validated and used, it still needs to be further explored. The next step will be a study with two groups. One group will be working on the test without further information, while the other group will receive prompts that encourage them to “think critically” and “check [their] results”. We expect the second group to score significantly better, confirming the sensitivity of the test for reflective thinking.

Additionally, the correlation of our mathematics-specific CT test with general CT tests (e.g., by Enis & Weir, 1985) should be explored as well as its correlation to students’ performance in mathematical problem solving and/or argumentation. Besides improving the test, such studies could also help us to better understand the general conception of CT and its connections to problem solving, argumentation, and meta-cognition as “such association[s] remain under-theorized” (Jablonka, 2014, p. 121).

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THE PING-PONG-PATTERN – USAGE OF NOTES BY DYADS DURING LEARNING WITH ANNOTATED SCRIPTS

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23 pairs of novice students from two German universities learned with video tutorials or verbally annotated scripts in different study settings, either with or without accompanying prompts. This paper focuses on phases in which students sum up and review the notes they have taken while watching the videos or presentations. The reported case study shows in what ways notes influence and structure the communication and interaction processes of dyads that are learning with annotated scripts on descriptive statistics.

NOTE-TAKING

During the last years, mathematical learning with new instructional media like video tutorials, podcasts, or animated worked-out examples has become gradually more influential from primary school to further education. Especially universities make more and more use of such formats to provide first semester students with chances to acquire basic mathematical knowledge for studying successfully (Biehler et al., 2014). Ongoing research analyzes in how far e.g. the social form (learning alone, learning in dyads, learning in groups, etc.) or supporting impulses (prompts, trainings, quizzes, etc.) influence learning outcomes in settings with instructional media (e.g. Lou, Abrami & d'Apollonia, 2001).

In all of these different settings, taking notes can be a helpful strategy. Note-taking is a frequently used and well reported activity consisting of filtering, comprehending, writing down, organizing, restructuring and integrating newly presented information in already existing knowledge (Makany, Kemp, & Dror, 2009, p. 620; Anderson & Armbruster, 1986). There are several strands of research detectable either focusing on the notes themselves (i.e. methods, media, and functions of note-taking) (Lawson, Bodle, and McDonough, 2007; Staub, 2006; Bui, Myerson, and Hale, 2013; Mueller & Oppenheimer, 2014), or analyzing the usefulness of note-taking for different groups of individuals, different learning styles or different aims to be pursued such as solving problems or passing an exam (Kiewra, 1989; Staub, 2006; Makany et al., 2009).

From a cognitive point of view, there are two main functions of note-taking (Kiewra, 1989; Anderson & Armbruster, 1986): (1) The encoding function regards the process of taking notes itself as facilitative for learning (Staub, 2006, p. 61). (2) The storage function emphasizes the preservation of notes for later use, e.g. reviewing information before an exam. To make sure the storing works, Anderson and Armbruster (1986, p. 20) point out the importance of deep instead of shallow processing regarding the review of the notes.

The large majority of studies analyzing notes and note-taking focuses on single learners. Investigating the process of note-taking and working with notes in dyads has been neglected so far. However, analyses of communication processes in different domains have shown the importance of inscriptions and materialities, and their impact on face to face interaction (e.g. Streeck, Goodwin & LeBaron, 2014).

In mathematical contexts in primary school mathematics, Fetzer (2007) and Fetzer, Schreiber & Krummheuer (2004) analyze writing processes and products, and their effect on the accompanying communication processes, and vice versa. The researchers could show that writing processes affect the interacting participants and can lead to 'condensed argumentation processes', that means that learners actively address inconsistencies in argumentations instead of passively acknowledging them (Fetzer 2007). In how far such results can be transferred to learners of other ages remains unclear.

Based on these findings and open questions, our study concentrates on note-taking and its interplay with the corresponding communication between learners in dyads in mathematical contexts at universities. While cognitive functions of notes are identified in various studies, their possible roles in communication processes are investigated sparsely, especially in university context. The main research question is as follows:

How do taken notes influence and structure the communication and interaction processes of dyads that are learning with educational videos or annotated scripts on a mathematical topic in university context?

METHODS

Procedure

Students worked in dyads with annotated presentations or video tutorials. Half of the dyads received prompts (fig. 1). Those who did not receive prompts were asked to learn the mathematics shown in the instructional material in order to pass a post-test on the topic. The learning phases with the instructional material lasted about 75 minutes on average. The computer screen was captured, the sound and the image of the two learners were videotaped. All notes taken by the learners were scanned afterwards.

Sample & Learning Material

The 46 students who worked in dyads during the media-intervention-period are in their first semester at two German universities: 22 students (3 female, 19 male) from the University of Applied Sciences in Offenburg were enrolled in a statistics course in business studies, 24 students (21 female, 3 male) from Bielefeld University were enrolled in a statistics course in psychology.

Students in Offenburg learned with two educational videos, both about 15 minutes long. The first one focuses on measures of center (i.e. arithmetic mean, median, harmonic mean), the second one on measures of spread (i.e. variance, standard deviation). The video tutorials explain statistical terms and concepts with the help of short stories that deal with realistic contexts relevant for students of business studies.

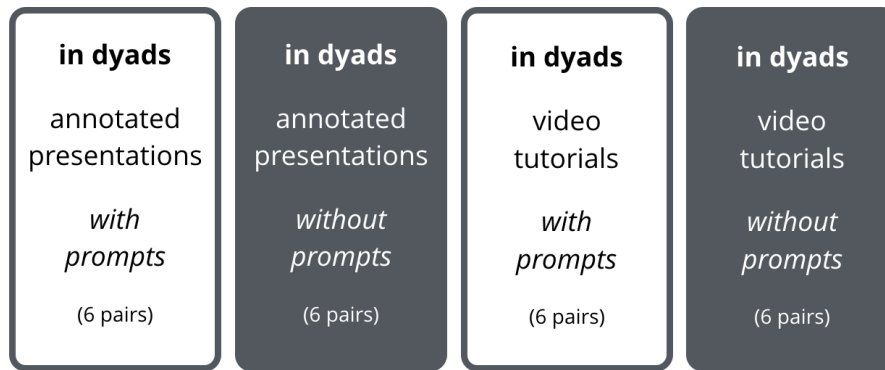


Figure 2: The different conditions with dyads in the pilot study.

Students in Bielefeld learned with two annotated scripts addressing the same topics. The scripts encompassed 8-10 slides which the students could rewind, forward or play again. The slides resemble a usual lecture script with formulas, definitions and examples with a psychological context (fig. 2), accompanied by verbal annotation. The oral comments stay close to the written words.

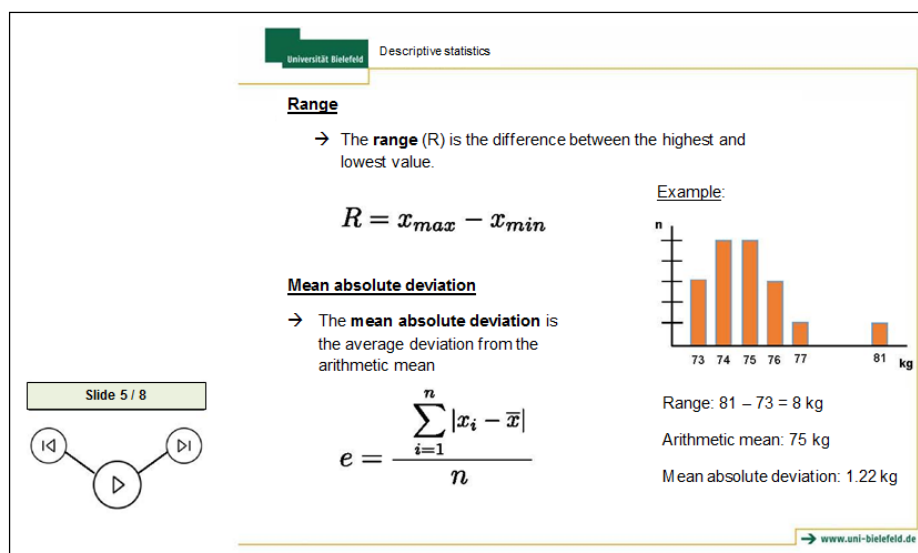


Figure 2: Screenshot of an annotated script on measures of center and spread.

Analysis

The learner interaction and the computer screen were captured and analyzed qualitatively using video recordings as well as the students' individual notes generated in this phase.

The analysis follows the triangulation method by Fetzer (2007): First, the video data is interpreted, second, the notes are analyzed, and third, both documents are synchronized and interpreted together. In the analysis we focus on phases in which students sum up and review the notes they have taken while watching the videos (*review phases*, Anderson & Armbruster, 1986). These review phases can be identified in nearly all videos; some pairs make extensive use of these phases, others have repeatedly short phases between the videos or in short pauses during video learning.

In this paper, due to the limited space, an excerpt of one case of the condition “annotated presentations without prompts” is depicted. This case was chosen for its passages that illustrate central findings and patterns during review phases.

RESULTS – THE CASE OF LISA AND RANA

Lisa (L) and Rana (R) watched the presentations thoroughly, paused them several times and copied many statements from the slides into their notes. Both students wrote down a lot of similar facts and aspects from the annotated script. Their notes show typical attributes compared to the notes of the other pairs that worked without prompts, such as many verbatim statements, mainly technical terms accompanied by explaining statements, and a contentual structure similar to the presentations. Furthermore, the notes of Lisa and Rana embody the highlighting and underlining of terms.

Having finished the second presentation, Lisa and Rana agree to review their notes before dealing with the post-test: “Let’s go through it again.” The scene starts at approximately 1:30 of this review phase, when they reach the paragraph “measures of dispersion” in their notes (fig. 4).

- 01 **L:** (*looks at her notes*) And, uhm, the measures of dispersion, uhm, gives the spread
- 02 **R:** (*looks at her notes*) Exactly, how do the characteristic attributes differ...
- 03 **L:** Yes (*circles the words “spread” and “measures of dispersion”, then*
- 04 *circles “measures of centre” and “measures of dispersion” various times, [fig. 4]*)
- 05 **R:** (*looks at Lisa’s circling, laughs*)
- 06 **L:** (*laughs*) Let’s see what this does. Ok, uhm, yes .. how they differ...
- 07 (*underlines “differ”*)
- 08 **R:** (*takes her pencil*) Then we had the difference between nominal scaled
- 09 and ordinal scaled variables (*circles “nominal scaled variables”*)

The scene starts with Lisa’s statement addressing the term “measures of dispersion” that “gives the spread”, while she is looking at her notes (01). Rana looks at her notes, too, agrees and adds, that the spread describes “how the characteristic attributes differ” (02). Lisa agrees and draws circles around the terms “spread” and “measures of dispersion” (03), then she intensifies the circle around “measures of dispersion” and a former drawn circle around “measures of centre” (04).

Rana recognizes Lisa’s drawing and laughs (05, fig.4). Lisa returns this laughter and utters the hope that her strategy may be helpful (06). Then she repeats the words “how they differ” Lisa already said and underlines “differ” in her notes (06-07). Rana looks back at her notes, grabs her pencil and begins to address the difference between two variables, which is the next paragraph on her notepad (08-09). She now circles the words “nominal scaled variables” in her notes.

Building blocks of Lisa's and Rana's review phase: The ping-pong-pattern

In this short scene, an interaction pattern can be observed. An *interaction pattern* is defined as a structure of interaction of two or more subjects, if a) with that structure a specific social and contentual regularity is reconstructed, b) the structure is formed by the actions of at least two interacting subjects, c) the structure can not be explained with the compliance with given rules and d) the interacting subjects neither reflect the regularity nor create it consciously but routine (Voigt, 1984). We called the observed pattern the *ping-pong-pattern*:

- i. *Opening*: One of the two students begins with paraphrasing or reading out a statement based on his or her notes. This statement is the beginning of a subtopic in the annotated video presentation, marked on the notepad with a headline or emphasized with a keyword /technical term in the presentation itself (see fig. 4 for examples of subtopic paragraphs in notes).
- ii. *Approval and additional statement #1*: The second student agrees on the opening by uttering "Yes" or "Ok". The same student continues paraphrasing or reading out a statement based on his or her notes.
- iii. *Approval and additional statement #2 – #n*: The first student confirms the previous statement with an approval, because it seems to be in line with his or her notes (identification of a similar aspect/statement in his or her own notes). The first student now continues by paraphrasing or reading out a statement based on his or her notes. Then, the speaker switches again.
- iv. *Finish/Closure*: This alternating procedure regularly ends, when the corresponding paragraph in the notes has been worked through completely.

The majority of Lisa's and Rana's review phase consists of such "switch-overs" between the speakers (fig. 3). The number of switch-overs depends on the length of the noted paragraph, the resemblance of the notes and the occurrence of irritation by the learners. Especially the similarity of the noted aspects plays an important role for the emergence of the ping-pong-pattern. In the presented case, three alternative progressions of the pattern could be observed:

- a) The switch-overs cover all noted aspects of a subtopic paragraph in a linear way and come to an end without irritation.
- b) The switch-overs are interrupted or paused by a question, some comments, a phase of memorization without notes, comparison and alteration of notes or metacognitive statements, etc. Afterwards, further additional statements addressing this subtopic paragraph are thematized and the switch-overs continue.
- c) The switch-overs based on a subtopic are cancelled and the rest of the notes concerning this paragraph is skipped. Cancellations occur when the content is perceived as too difficult or not important with regard to the post-test. Oftentimes, the next jotted down subtopic is chosen.

Figure 3 shows that Lisa's and Rana's review phase is structured in large parts by ping-pong-patterns. Only between 8:30 and 9:10 a dialogue detached from verbatim notes and without the characteristic switch-overs could be observed.

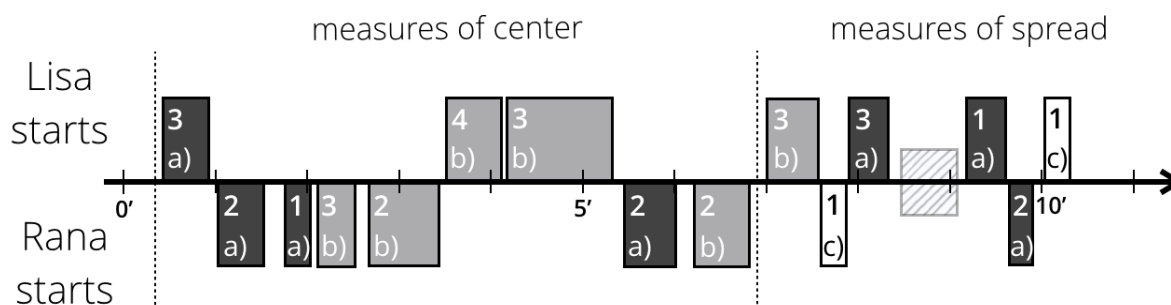


Figure 3: Occurrence of the ping-pong-pattern during the review phase that lasts approximately 10:30 minutes. *(The boxes above and below the time line stand for dialogues that follow the described pattern about a subtopic paragraph depicted in the notes. The position of the boxes above and below the time line indicates which student “opens” the sequence. The number in a box gives the total number of additional statements. The color of a box and the letter below the number indicate, whether a) (black) the sequence of switch-overs ends without irritation, b) (grey) the switch-overs are interrupted, or c) (white) the switch-overs are cancelled before all aspects from a paragraph in the notes have been mentioned. The hatched box located directly on the time line at 9:30 stands for a dialogue that does not follow the described pattern).*

Highlighting during interaction

Lisa starts highlighting (circling and underlining) various words during the review phase with a different color. Shortly after her first circling (03-04), she intensifies the circles around the terms “measures of dispersion” and “measures of centre”. By this text markup, she adds a typical layout characteristic of mathematic textbooks to her notes: She highlights the term that is specified by a definition. With further lines she emphasizes words in the defining text (“differ”).

Lisa's first act of highlighting takes place after Rana's approval, the second one takes up Rana's additional statement “how do the characteristics differ”. This can be interpreted in two ways. On one hand, Lisa's choice could be influenced by Rana's approval and her additional statement. Lisa agrees with Rana, that “differ” is an important term in this context. On the other hand, Lisa's circles and lines could be a kind of check off. Topics and terms that were addressed during the review phase are marked to document that they have already been discussed. Both interpretations show how the interaction between the learners can influence note-taking processes. In later scenes, Lisa and Rana supplement their notes with further comments based on their interaction.

Additionally, Rana adapts Lisa's highlighting strategy (08): She grabs her pencil and structures her notes with circles like Lisa did. This adaption of an activity hints at further factors that influence notes and the note-taking process.

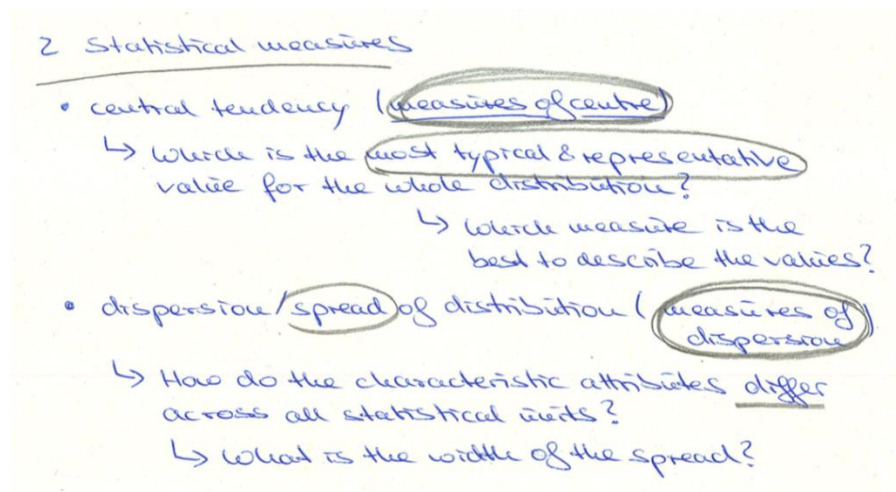


Figure 4: Cut-out of Lisa's notes. Depicted are two subtopic paragraphs: "measures of centre" (first bullet point) and "measures of dispersion" (second bullet point).

Conclusion & Perspectives

The reported case shows how the review phase can be influenced by notes that students take when learning with annotated scripts. Although the students could draw on the annotated script again, they rely solely on their written notes. During the work with these similar notes, the ping-pong-pattern could be identified as a constituting interaction pattern of Lisa's and Rana's review phase. This pattern could also be identified in parts of the communication processes of other dyads.

Vice versa, communication processes affect the process of note-taking and therefore the notes themselves. Interactive alteration of taken notes (adding or deleting sentences or terms, highlighting) and adaptations of learning strategies could be observed in the learning processes of various pairs.

A deeper analysis of the interruptions of the ping-pong-patterns (boxes with "b" in fig. 3) could reveal in how far the alternating statements may initiate dialogues that lead to integration of prior knowledge, cross-linking of mathematical aspects and other meaningful learning activities (e.g. self-explanations, Chiu & Chi, 2014).

A closer look to the review phases may help to identify functions of notes in communication processes that supplement cognitive functions (cf. Anderson & Armbruster, 1986). Furthermore, the analyses may help foster learning strategies and argumentative activities with taken notes in cooperative settings (cf. Fetzer, 2007).

The presented findings have to be regarded as a partial result of a pilot study in which we explore learning processes with annotated scripts and video tutorials. In the main study we hope to characterize typical interactive activities, communication patterns and learning strategies concerning the use of notes through a comparative analysis with

more cases and further types of instructional material (scripts without comments, animated screencasts).

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IMAGES OF ABSTRACTION IN MATHEMATICS EDUCATION: CONTRADICTIONS, CONTROVERSIES, AND CONVERGENCES

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In this paper we offer a critical reflection of the mathematics education literature on abstraction. We explore several explicit or implicit basic orientations, or what we call images, about abstraction in knowing and learning mathematics. Our reflection is intended to provide readers with an organized way to discern the contradictions, controversies, and convergences concerning the many images of abstraction. Given the complexity and multidimensionality of the notion of abstraction, we argue that seemingly conflicting views become alternatives to be explored rather than competitors to be eliminated. We suggest considering abstraction as a constructive process that characterizes the development of mathematical thinking and learning and accounts for the contextuality of students' ideas by acknowledging knowledge as a complex system.

INTRODUCTION

Several scholars in the psychology of mathematics education have recognized abstraction to be one of the key traits in mathematics learning and thinking (e.g., Boero et al., 2002). The literature acknowledges a variety of forms of abstraction (Dreyfus, 2014) that take place at different levels of mathematical learning (Mitchelmore & White, 2012) or in different worlds of mathematics (Tall, 2013), and underlie different ways of constructing mathematical concepts compatible with various sense-making strategies (Scheiner, 2016). While the complexity and multi-dimensionality of abstraction is widely documented (e.g., Boero et al., 2002; Dreyfus, 1991), the literature lacks a discourse on – conflicting, controversial, and converging – images of abstraction in mathematics education.

In this article, we offer a reflection on the literature on abstraction in mathematics learning that is somewhat at variance with other reflections and overviews. We explicitly focus on what key writings in this realm assert, assume, and imply about the nature of abstraction in mathematics education. Much of the literature is concerned with a discussion about the multiplicity and diversity of approaches and with frameworks of abstraction; however, what is missing is an articulation of basic orientations or images of abstraction. Our reflection is intended to provide readers with an organized way to discern the controversies, contradictions, and convergences of the many images of abstraction that are explicit or implicit in the literature.

The three following sections consider each of the above facets (contradictions, controversies, and convergences), and relate our reflections on the literature regarding abstraction in mathematics education. We approach each of them by presenting issues that in our view are central to the debate. We conclude with some remarks on viewing

knowledge as a complex dynamic system that acknowledges abstraction in terms of levels of complexity and increases in context-sensitivity.

SOME CONTRADICTING IMAGES OF ABSTRACTION

We take the following description of abstraction by Fuchs et al. (2003) as a starting point for discussing the main contradicting images of abstraction still present in the literature:

“To abstract a principle is to identify a generic quality or pattern across instances of the principle. In formulating an abstraction, an individual deletes details across exemplars, which are irrelevant to the abstract category [...]. These abstractions [...] avoid contextual specificity so they can be applied to other instances or across situations.” (Fuchs et al., 2003, p. 294)

The contradicting image of abstraction as generalization

The description of abstraction given by Fuchs et al. (2003) focuses on the generality, or, rather, on the generic quality of a concept. Here abstraction is identified with generalization. Generalization of a concept implies taking away a certain number of attributes from a specific concept. For example, taking away the attribute ‘to have orthogonal sides’ from the concept of rectangle leads to the concept of parallelogram. This operation implies an extension of the scope of the concept and forms a more general concept.

Abstraction, in contrast, does not mean taking away but *extracting* and *attributing* certain meaningful components. In considering forms of abstraction on the background of students’ sense-making, Scheiner (2016) argued that ‘abstractions from actions’ approaches (e.g., reflective abstraction) are compatible with students’ sense-making strategy of ‘extracting meaning’ and ‘abstractions from objects’ approaches (e.g., structural abstraction) are compatible with students’ sense-making strategy of ‘giving meaning’ – two prototypical sense-making strategies identified by Pinto (1998). From this perspective, in attributing meaningful components, one’s concept image becomes richer in content.

Thus, the image of abstraction as generalization seems inadequate when knowledge is considered as construction. The image of abstraction as generalization is elusive about abstraction as a constructive process and overlooks abstraction that takes account of an individual’s cognitive development.

The contradicting image of abstraction as decontextualization

The above quoted description of abstraction by Fuchs et al. (2003) implies that abstraction is concerned with a certain degree of decontextualization. This is not surprising, given the confusion of abstraction with generalization as “generalization and decontextualization [often] act as two sides of the same coin” (Ferrari, 2003, p. 1226). Fuchs et al. (2003) suggested getting away from contextual specificities so that “abstractions [...] can be applied to other instances or across situations” (p. 294). Furthermore, the meaning *abstract-general* of the term ‘abstract’ (Mitchelmore &

White, 1995), refers to ideas which are general to a wide variety of contexts, and this may cause such confusions.

The consideration of abstraction as decontextualization contradicts the recent advances in understanding knowledge as situated and context sensitive (e.g., Brown, Collins, & Duguid, 1989; Cobb & Bowers, 1999). Several scholars in mathematics education have argued against the decontextualization view of abstraction. For example, Noss and Hoyles' (1996) *situated abstraction* approach and HersHKowitz, Schwarz, and Dreyfus' (2001) *abstraction in context* framework have foregrounded the significance of context for abstraction processes in mathematics learning and thinking. These contributions go beyond purely cognitive approaches and frameworks of abstraction in mathematics education and take account of the situated nature and context-sensitivity of knowledge, as articulated by the situated cognition (or situated learning) paradigm. van Oers (1998) focussed on this aspect in arguing that abstraction is a kind of *recontextualization* rather than a *decontextualization*. From his perspective, removing context will impoverish a concept rather than enrich it. Scheiner and Pinto (2014) presented a case study in which a student integrated diverse elements of representing the limit concept of a sequence into a single representation that the student used generically to construct and reconstruct the limit concept in multiple contexts. Their analysis indicated that the representation (that the student constructed) supported his actions through its complex sensitivity to the contextual differences he encountered.

Thus, from our point of view, we acknowledge abstraction as a process of increasing context-sensitivity rather than considering abstraction as simply decontextualization.

SOME CONTROVERSIAL IMAGES OF ABSTRACTION

The controversial image of abstraction on structures: similarity or diversity?

Theoretical research in learning mathematics has long moved beyond categorization or classification, that is, beyond collecting together objects on the basis of similarities of their superficial characteristics. As diSessa and Sherin (1998) reminded us, though abstraction as derived from the recognition of commonalities of properties works well for 'category-like concepts', empirical approaches limited to the perceptual characteristics of objects do not provide fertile insights into cognitive processes underlying concept construction in mathematics. Skemp's (1986) idea of abstraction, that is, of studying the underlying structure rather than superficial characteristics moved the field in new directions. Further, Mitchelmore and White (2000), in drawing on Skemp's conception of abstraction, developed an empirical abstraction approach for learning elementary mathematics.

Though the literature portrays a mutual understanding that abstraction in mathematics is concerned with the underlying (rather than the superficial) structures of a concept, there is a controversy as to whether abstraction means the consideration of similarities of structures or of their diversity. While Skemp (1986) focused on similarities in structures, Vygotsky (1934/1987) considered the formation of scientific concepts along differences.

A theoretical idea or concept should bring together things that are dissimilar, different, multifaceted, and not coincident, and should indicate their proportion in the whole. [...] Such a concept [...] traces the interconnection of particular objects within the whole, within the system in its formation. (Vygotsky, 1934/1987, p. 255)

Scheiner (2016) proposed a framework for structural abstraction, a kind of abstraction, already introduced by Tall (2013), that takes account of abstraction as a process of complementarizing meaningful components. From this perspective, the meaning of mathematical concepts is constructed by complementarizing diverse meaningful components of a variety of specific objects that have been contextualized and recontextualized in multiple situations.

Thus, it is still debated whether the meaning of a mathematical concept relies on the commonality of elements or on the interrelatedness of diverse elements – or, to put it in other words, whether the core of abstraction is similarity or complementarity.

The controversial image of abstraction as the ascending of abstractness or complexity

Scholars seem to agree in distinguishing between concrete and abstract objects, yet not between concrete and abstract concepts since every concept is an abstraction. In fact, scholars differ with regard to their understanding of the notions of ‘concrete’ and ‘abstract’. According to Skemp (1986), the initial forms of cognition are perceptions of concrete objects; the abstractions from concrete objects are called percepts. These percepts are considered primary concepts and serve as building blocks for secondary concepts; the latter are concepts that do not have to correspond to any concrete object. Taking this perspective, it is not surprising that concreteness and abstractness are often considered as properties of an object. In contrast, Wilensky (1991) considered concreteness and abstractness rather as properties of an individual’s relatedness to an object in the sense of the richness of an individual’s re-presentations, interactions, and connections with the object. This view leads to allowing objects not mediated by the senses, objects which are usually considered abstract (such as mathematical objects) to be concrete; as long as that the individual has multiple modes of interaction and connection with them and a sufficiently rich collection of representations to denote them.

Skemp viewed abstraction as a movement from the concrete to the abstract, while, according to Wilensky, individuals begin their understanding of scientific mathematical concepts with the abstract. This ascending from the abstract to the concrete is the main principle in Davydov’s (1972/1990) theory and has been taken as a reference point for the development of other frameworks of abstraction (e.g., Hershkowitz, Dreyfus, & Schwarz, 2001; Scheiner, 2016).

On the other hand, Noss and Hoyles (1996) adopted a situated cognition perspective to investigate mathematical activities within computational environments. These environments are specially built to provide learners an opportunity for new intellectual connections. The authors’ concern is “to develop a conscious appreciation of

mathematical abstraction as a process which builds upon layers of intuitions and meanings” (Noss & Hoyles, 1996, p. 105).

Thus, in taking the understanding of the concrete and the abstract as properties of objects, scholars could consider abstraction as levels of abstractness; while, in taking the understanding of concreteness and abstractness as properties of an individual’s view of objects, scholars could view abstraction as levels of complexity, as Scheiner and Pinto’s (2014) recent contribution indicated.

SOME CONVERGING IMAGES OF ABSTRACTION

Piaget (1977/2001) made a distinction between cognitive approaches to abstraction: dichotomizing ‘abstraction from actions’ and ‘abstraction from objects’. Research in mathematics education has mostly considered the first of these approaches to abstraction. In referring to the latter, Piaget (1977/2001) limited his attention to empirical abstraction, that is, to drawing out common features of objects, “recording the most obvious information from objects” (p. 319). Supported by Skemp’s view on abstraction, Mitchelmore and White (2000), and later Scheiner and Pinto (2014), considered objects as starting points for abstraction processes, and, in doing so, took account of ‘abstraction from objects’. Scheiner (2016) blended the abstraction from actions and the abstraction from objects frameworks to provide an account for a dialectic between reflective and structural abstraction. In the following, we provide convergent images of these various notions of abstraction, as we see them.

The converging image of abstraction as a process of knowledge compression

Here we understand compression of knowledge as “taking complicated phenomena, focusing on essential aspects of interest to conceive of them as whole to make them available as an entity to think about” (Gray & Tall, 2007, p. 24). Or, to put it in Thurston’s (1990) words, knowledge is compressed if “you can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process” (p. 847).

Dubinsky and his colleagues’ (Dubinsky, 1991; Cottrill et al., 1996) APOS framework, which seems to refer mostly to ‘abstraction from actions’, proposed the notion of encapsulation of processes into an object through what Piaget called reflective abstraction. The single encapsulated object may be understood as a compression in a sense that encapsulation results in an entity to think about. The same holds for Sfard and Linchevski’s (1994) framework of reification, a process that results in a structural conception of an object. In the same strand, Gray and Tall (1994) considered some mathematical symbols as an amalgam of processes and related objects; thus, compressing knowledge into a symbol which is conveniently understood as a process to compute or manipulate, or as a concept to think about. They proposed that “the natural process of abstraction through compression of knowledge into more sophisticated thinkable concepts is the key to developing increasingly powerful thinking” (Gray & Tall, 2007, p. 14).

Researchers working within the ‘abstraction from objects’ strand (Mitchelmore & White, 2000; Scheiner & Pinto, 2014) are guided by the assumption that learners acquire mathematical concepts initially based on their backgrounds of existing domain-specific conceptual knowledge – considering abstraction as the progressive integration of previous concept images and/or the insertion of a new discourse alongside existing mathematical experiences. For instance, the cognitive function of structural abstraction is to provide an assembly of such various experiences into more complex and compressed knowledge structures (Scheiner & Pinto, 2014).

Thus, both ‘abstraction from actions’ and ‘abstraction from objects’ approaches seem to share the image of abstraction as a process of knowledge compression.

The converging image of abstraction as a complex dynamic constructive process

One may argue that researchers who see abstraction as decontextualization propose the result of an abstraction process as a stable stage. Once decontextualized, the product of an abstraction – the concept – appears as standing still. An understanding of the entire process as a recontextualization considers abstraction to be a dynamic constructive process, which could evolve in a movement through levels of complexity. In fact, concepts can be continuously revised and enriched while placed in new contexts. This seems to agree with the understanding of Noss and Hoyles (1996) and of Hershkowitz, Schwarz and Dreyfus (2001). In the case of Scheiner and Pinto (2014), the underlying cognitive processes support a specific use of the concept image while building mathematical knowledge. Models of partial constructions are gradually built through these processes and are used as generic representations. In other words, a model of an evolving concept is built and used for generating meaningful components as needed, while inducing a restructuring of one’s knowledge system. From this perspective, an individual’s restructuring of the knowledge system aims for stability of the knowledge pieces and structures. Such dynamic constructive processes emphasize a gradually developing process of knowledge construction.

Thus, rather than considering knowledge as an abstract, stable system, we consider knowledge as a complex dynamic system of various types of knowledge elements and structures.

FINAL REMARKS

This brief discussion underlines the many images of abstraction in mathematics learning and thinking. If abstraction is regarded from the viewpoint of knowledge as a static system, then abstraction refers to meanings that are ‘abstracted’ from situations or events. By taking this view, abstraction is considered as a highly hierarchized process, whereby abstractions of higher order are built upon abstractions of lower order. However, if we consider knowledge as a complex system, it is possible to acknowledge abstraction in terms of levels of complexity and increases in context-sensitivity. In viewing knowledge as a complex dynamic system rather than a static system, seemingly conflicting views become alternatives to be explored rather than competitors to be eliminated. The central assertion of all approaches and frameworks

should be to consider abstraction as a constructive process that characterizes the development of mathematical thinking and learning and accounts for the contextuality of students' ideas.

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CREATIVITY IN THE EYE OF THE STUDENT. REFINING INVESTIGATIONS OF MATHEMATICAL CREATIVITY USING EYE-TRACKING GOGGLES

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Mathematical creativity is increasingly important for improved innovation and problem-solving. In this paper, we address the question of how to best investigate mathematical creativity and critically discuss dichotomous creativity scoring schemes. In order to gain deeper insights into creative problem-solving processes, we suggest the use of mobile, unobtrusive eye-trackers for evaluating students' creativity in the context of Multiple Solution Tasks (MSTs). We present first results with inexpensive eye-tracking goggles that reveal the added value of evaluating students' eye movements when investigating mathematical creativity—compared to an analysis of written/drawn solutions as well as compared to an analysis of simple videos.

INTRODUCTION

Creativity as an ability is crucial whenever novelties are generated—this concerns problem solving situations in educational learning contexts as well as everyday life problems. In particular, mathematical creativity is significant for improved innovation and problem-solving processes within all STEM areas (science, technology, engineering, and mathematics) in our increasingly interconnected high-technology based society and economy. All students have the potential to be mathematically creative (Mann, 2005). However, research findings indicate that their creativity differs and that—as a trend over time—students tend to be less creative than they were in the past (Kim, 2011). Therefore, it is adequate that research increasingly focuses on mathematical creativity (e.g., Leikin & Pitta-Pantazi, 2013; Sheffield, 2013).

For investigating how mathematical creativity can be best fostered, it is important to address the question of how creativity can be best investigated. The methods of investigation have gained special interest within research (Sriraman, Haavold, & Lee, 2014; Joklitschke, Rott & Schindler, 2016). Different approaches for assessing or rather measuring mathematical creativity have been developed and established (e.g., Kattou et al., 2013; Leikin & Lev, 2013). However, these approaches have certain restrictions: The aim to find measurement tools has led to a product-view on students' solutions, in which creative problem-solving *processes* are neglected. However, research is needed that investigates how creative solutions emerge in students. Additionally, dichotomous scoring schemes have led to an analysis which excludes approaches that are not complete or not completely correct. All in all, the question arose whether this assessment of creativity is valid and how mathematical creativity can be investigated more adequately (Joklitschke et al., 2016).

This paper contributes to research on mathematical creativity. It relates to the question which additional value a process-oriented analysis of students' mathematical creativity offers—compared to an analysis of only products. In an empirical investigation, we analyzed to what extent eye-tracking can contribute to studying students' mathematical creativity in a process-view. We compared the findings from three analyses of the same data: First, the “common” analysis of the written solutions (product-view), (b) a video analysis of students' creative problem-solving (process-view), and (c) an analysis of eye-tracking videos (process-view). Our results show benefits for addressing mathematical creativity arising from the eye-tracking data, which, for example, allow to disambiguate alternative interpretations of the product-view and to discover creative processes that are not even observable in a video analysis.

THEORETICAL BACKGROUND

Mathematical Creativity and its investigation

The concept of creativity is derived from research in psychology. Here, creativity was originally seen as one dimension of intelligence (Guilford, 1967). Creativity is furthermore characterized as a key component of the ability to find unique and manifold ideas, called *divergent thinking* (Guilford, 1967). Four aspects are differentiated with respect to divergent thinking; these are *fluency*, addressing the number of solutions; *flexibility*, addressing the diversity of produced solutions; *originality*, addressing the uniqueness of produced solutions; and *elaboration*, addressing the level of detail.

Within mathematics education research, the psychology approach to creativity has been taken up and adapted. Here, tests have been developed for quantifying mathematical creativity (e.g., Kattou et al., 2013; Leikin & Lev, 2013). These tests draw on mathematical problems that can be solved in diverse ways—so called Multiple Solution Tasks (MSTs). In these tests, students are supposed to solve the MSTs in as many ways as possible—based on the theoretical assumption that “solving mathematical problems in multiple ways is closely related to personal mathematical creativity” (Leikin & Lev, 2013, p. 185). This way of testing mathematical creativity is accepted and appreciated within educational research (e.g., Muldner & Burleston, 2015). For measuring students' mathematical creativity, the tests draw on Guilford's categories of fluency, flexibility, and originality which are counted in a dichotomous scoring: only mathematically entirely correct and complete solutions are considered. However, research indicates validity concerns of this approach investigating students' mathematical creativity: The analysis of students' written solutions revealed that students provided solutions that were partially not completed or not entirely correct (Joklitschke et al., 2016). Even though these incomplete approaches of the students were not counted in the scoring schemes, they indicate creative processes; which are then, however, not appreciated in the existing methods. Joklitschke et al. accordingly suggest to improve the methods for investigating mathematical creativity—particularly to focus on the *processes* in which students solve problems.

Eye-Tracking

Eye-tracking technologies with which students' eye-movements are investigated are increasingly used in educational research (e.g., Scheiter & van Gog, 2009), particularly in mathematics education research (e.g., Andr   et al., 2015; Epelboim & Suppes, 2001) and in research on mathematical creativity (Muldner & Burleston, 2015). Even though the use of eye-tracking in educational research is still in the early stages of its development, existing findings show its remarkable potential: Following Andr   et al. (2015, p. 241), we assume that "the merit from a didactic perspective is that we can examine how and which information students are attending to". Based on findings in neurosciences, research on eye-tracking has shown that what students look at correlates with what they pay attention to (e.g., Andr   et al., 2015; Rayner, 1998). Eye-tracking helps us to understand what students focus on when working on a problem. It is used and perceived as especially beneficial in geometrical settings (e.g., Muldner & Burleston, 2015; Schimpf & Spannagel, 2011; Epelboim & Suppes, 2001).

Our research connects to Muldner and Burleston (2015) who investigated eye movements on subjects who dealt with mathematical MSTs that addressed proof in geometry (see also Levav-Waynberg & Leikin, 2012). This study showed that and why eye-tracking is feasible for investigations with these kinds of problems: As MSTs are rich, allow different ways to solve them and do not require extensive background knowledge, the analysis of eye-movements holds an enormous potential. Muldner and Burleston's (2015) purpose was to find "reliable differences in sensor features characterizing low vs. high creativity students" (p. 127). By comparing for instance students' saccade lengths and saccade speed with EEG data, they characterized groups of students with their data. However, eye-tracking research is needed that rather investigates mathematical creativity and can contribute to rethinking the investigations of mathematical creativity in order to appreciate students' creative approaches more adequately from a process-oriented view (Joklitschke et al., 2016). Based on findings that show that cognitive processing correlates with fixations (see Andr   et al., 2015), we assume that the analysis of students' eye-movements can contribute to understanding what students focus on when creatively solving MSTs. Thus, eye-tracking technology can be used to better understand how students "think" in terms of what they pay attention to when figuring out ways to solve a MST. Accordingly, we ask the research question: *To what extent does the analysis of students' eye-movements contribute to understanding their creative problem-solving processes, and hence, mathematical creativity?*

METHOD

Setting the scene

In order to answer the research question, we used data of four upper secondary school students in the Swedish research project KMT ("kreativa mattetr  ffar"). In the project, which takes place at   rebro University, the students meet every second week for working on multifaceted mathematical problems. The idea of the project is to foster

the abilities of interested students and especially their mathematical creativity. Therefore, they are, for example, often involved in collaborative, inquiry-based group work. Furthermore, they work on MSTs; first, they work on the problems individually; afterwards, they work in groups, discussing their solutions. The MSTs are mostly derived from scientific publications (e.g., Novotná, 2015; Joklitschke et al., 2016). The MSTs undergo an *a priori* analysis, in which the potential for finding both unique and manifold solutions is assessed. The investigation presented in this paper took place when the students were already used to dealing with MSTs.

The Multiple Solution Task (MST)

The investigation presented in this paper focuses on students' creative problem-solving when dealing with the MST shown in Figure 1. It was chosen because it addresses proof in geometry, which research had shown to be a suitable context for applying eye-tracking (Muldner & Burleston, 2015). We used this particular problem because it had revealed itself rich and suitable for addressing mathematical creativity in prior work (Joklitschke et al., 2016), motivating students to find manifold approaches. Second, the problem does not require extensive subject-matter knowledge of the students; which is desirable as we wanted to address creativity rather than assessing prior mathematical knowledge and achievement. For this purpose, we added the information that all angles in an equilateral hexagon are 120° .

Task: Solve the following problem. Can you find different ways to solve the problem? Show as many ways as you can find.

Problem: This figure is an equilateral hexagon: How big is the angle ε ?
Remember: In an equilateral hexagon, all sides have the same length and all angles have the same size, which is 120° .

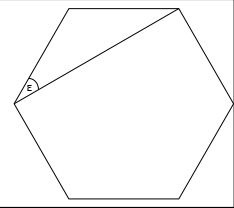


Figure 1: The hexagon-problem (Multiple Solution Task)

Eye-Tracking

The four participating students worked on the hexagon-MST in turns wearing eye-tracking goggles (see Figure 2(1)), which allow to record gaze point sequences, projected on the scene view from the perspective of the student (see Figure 2(3,4)). The time to work on the MST was 15 minutes and we asked the students to change pen colors for every new approach. Apart from the calibration routine at the beginning of each session and the placement of the MST on a reading stand for improved eye-tracking (see Figure 2(2)), no further adjustments were necessary. In this work, we recorded gaze point sequences and analyzed them manually.

Even though stand-alone eye-trackers measuring eye-movements on a computer screen can be advantageous in terms of accuracy (see Muldner & Burleston, 2015; Epelboim & Suppes, 2001), we propose to use *eye-tracking goggles* for purposes as ours. In this study, we used the headset Pupil Pro (Kassner, Patera & Bulling, 2014; see Figure 2(1)), which has a number of advantages for our purpose: First, goggles allow for mobile eye-tracking and are easy to set-up. Thus, they can be used straightforwardly in a room students are familiar with, avoiding biases through an artificial surrounding.

The students in our project usually solved MSTs with paper and pen and we wanted to provide the same possibility to draw with pens, to use a ruler, etc., and keep the setting as familiar as possible. Second, eye-tracking with goggles is unobtrusive. We observed that the students soon “forget” they are wearing the eye-tracking device and thus act naturally when working on their tasks. Third, this type of eye-tracker is more affordable than traditional eye-tracking devices (we purchased the Pupil Pro for approx. 2,000\$). It is thus possible to acquire and use several of them for research studies and it is also conceivable that similar, less expensive eye-tracking headsets could be routinely used in educational contexts in the future.

Data analysis

In a first step, we evaluated—similar to previous researchers (Kattou et al., 2013; Leikin & Lev, 2013; Joklitschke et al., 2016)—students’ mathematical creativity using their *solutions drawn/written on paper*. Two different researchers independently analyzed the documents and then compared their analyses. In a second step, we evaluated *simple videos*. These videos from the eye-tracking goggles show the view of the students. We used them without the eye-tracking overlay in order to be able to investigate the additional value of gaze point sequences later on. Using the simple videos, we investigated how students proceeded. Therefore, we focused on their drawings, writings, and gestures. In a third step, we evaluated the *eye-tracking videos*. These were derived from overlaying the simple videos with gaze point sequences (Figure 2(4)) indicated by green dots connected by magenta lines. Following André et al. (2015), we conceptualize students’ focuses and eye-movements as indicating their area of interest. In a *micro-level analysis*, we evaluated students’ particular focuses of attention in order to investigate creative approaches in detail (André et al., 2015).

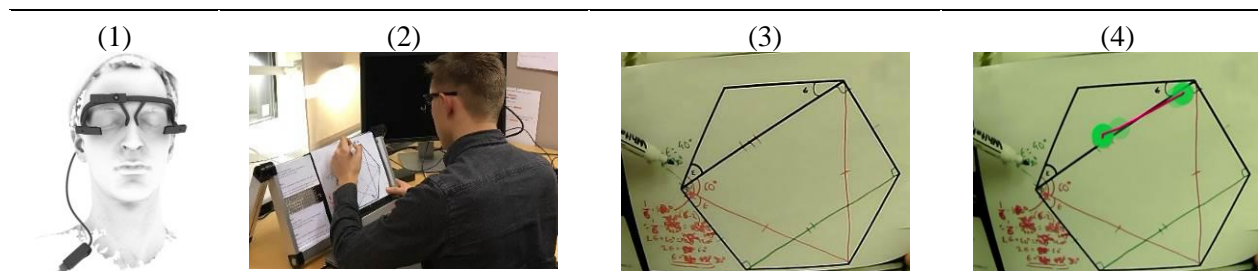


Figure 2: (1) Pupil Pro eye tracking headset used in this study (<https://pupil-labs.com/>); (2) Student working on the hexagon MST; (3) screenshot of simple video and (4) screenshot of video with eye-tracking overlay

FIRST RESULTS

In the following, we present first results and illustrate those using data from David, an 18 year old student.

In the *analysis of students’ drawings/writings*, we were able to get a first account on students’ approaches. In David’s case (see Figure 3), both researchers independently found four approaches, and named them after the colors used (red, green, blue 1 (upper left corner), blue 2 (lower right corner)). The interpretation of three of the approaches

was very similar or identical; one approach was interpreted differently (blue 1). Here, the writing/drawing did not allow to clearly reconstruct how the approach emerged. One approach (red) was evaluated as correct and complete by both researchers; another one (green) by only one. In a dichotomous scoring, as used in the creativity-test offered by Leikin and Lev (2013), one or respectively two solutions would accordingly count for evaluating fluency, flexibility, and originality. In sum, we found that this analysis did not suffice for reconstructing how students came up with their creative ideas, how these developed, and how they built on one another (incl. their order).

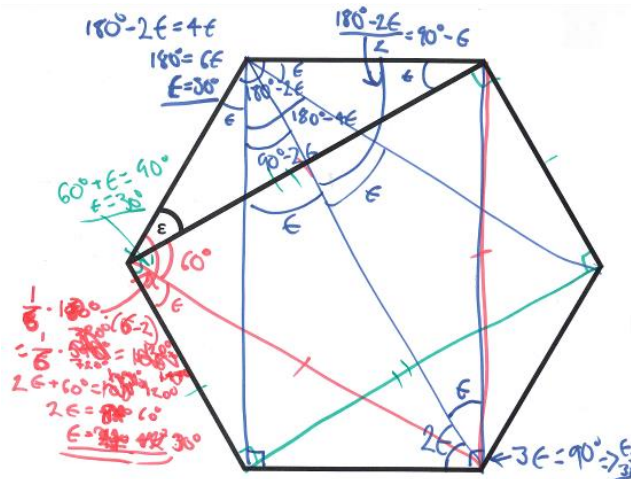


Figure 3: David's written/drawn solution

The *evaluation of simple videos* (without eye-tracking overlay) revealed in which order the approaches emerged. In David's case, this analysis showed that he started with the red approach, went on with the green one, interrupted for correcting the red one, and then continued with the green one. Later on, he went on with "blue 1", then intermediately worked on "blue 2", and finally finished approach "blue 1". However, using the analysis of simple videos does rarely reveal what student focus on when, for instance, switching approaches, and therefore sheds little light on what reasons they have for rethinking or for interrupting. Here, we expected the analysis of eye-tracking videos to be advantageous for evaluating students' focuses of attention. Also, it appeared that students, such as David, interrupt their proceeding for over 20s in which they did not write, draw or point at something. We assumed that eye-tracking videos offer information on what the students are paying attention to in these episodes.

The *evaluation of eye-tracking videos* offered, indeed, a more fine-grained access to what students were paying attention to and focusing on. Regarding the evaluation of mathematical creativity, it especially contributed to reconstructing how new, creative ideas evolved, to reconstructing approaches that were complex and whose written/drawn descriptions did not allow to clearly reconstruct them, and to evaluating the degree of elaboration of students' approaches. In David's case, it shed light on how he proceeded in detail and what he focused on for instance in the approach "blue 1". Here, he started with paying attention to the symmetry of the upper triangle: He focused on the equal-sized angles in the two lower corners (ϵ and the symmetrical angle, see

Figure 4): He first focused on the right-handed angle, second, looked towards the left-handed angle ε and back, then marked the right-handed angle as ε , and finally looked back and forth between the two angles, probably checking his idea. Following his eye-movement, we found that his approach was much more elaborated than we had expected. In fact, it revealed that his two blue approaches complement each other and that he—contrary to our prior analysis—inferred them correctly in an approach that was complete. The eye-tracking was crucial for revealing his entire creative effort and for finding that his approach was correct.

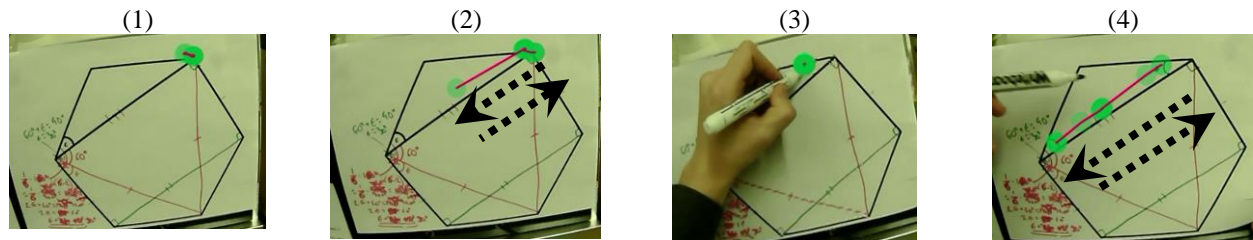


Figure 4: David's eye-movement marking ε in the upper triangle

CONCLUSION AND OUTLOOK

Our results support the findings of Joklitschke et al. (2016) that a more sophisticated evaluation is valuable for understanding students' mathematical creativity. Drawing on the idea to study mathematical creativity using MSTs (Leikin & Lev, 2013), we investigated to what extent the required deeper insight into creative problem-solving processes can be achieved using mobile, unobtrusive eye-trackers that do not require substantial adjustments of standard problem-solving settings. We presented first results obtained with eye-tracking goggles, which are representative of a new generation of mobile, inexpensive eye-tracking devices, and observed the remarkable potential of these novel devices for creativity research: Using the data we were able to shed light on how new, creative ideas evolved and how students inferred them. In particular, analyzing eye movements enables us to evaluate the degree of elaboration, which is not yet sufficiently addressed in research on mathematical creativity (Joklitschke et al., 2016). We were able to reconstruct approaches that the analysis of written/drawn solutions and simple videos of the scene as looked at by students had not clarified. Through the improved capability to reconstruct students' approaches, we are able to better evaluate their mathematical creativity. As the value of the analysis of eye movements was persuasive in our study, it is inevitable to use data from eye-tracking goggles in future work on mathematical creativity. We will investigate more extensively how creativity maps to gaze sequences and investigate how to partially automate the analysis of gaze sequences in research on mathematical creativity.

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FACILITATING MATHEMATICS TEACHERS' SHARING OF LESSON PLANS

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This paper presents results of 3 preliminary research and development studies, each aimed at examining some deliberation about shaping up our initiative towards facilitating mathematics teachers' collaboration in lesson planning and in sharing lesson plans with one another. These studies involve, among others, the development of software that supports the accumulation, preservation and on-going modification of teachers' lesson plans. Additionally, some of the open questions we are still struggling with are described.

INTRODUCTION

Almost every country in the world has assumed some form of educational reform during the past two decades, but “very few have succeeded in improving their systems from poor to fair to good to great to excellent” (McKinsey report, Mourshed, Chijioke & Barber, 2010, p. 10). This report examined school systems' improvement by analysing the experiences of 20 education systems around the world that achieved significant and sustained student outcome gains, as measured by national and international standards of assessment in recent years. Although there is no conclusive answer to why a certain reform fails while another one succeeds, clearly, as stated in their previous McKinsey report (2007), “*The quality of an education system cannot exceed the quality of its teachers*” and “*The only way to improve outcome is to improve instruction*” (ibid p. 43). One of the eight factors identified as contributing to success of a reform was nurturing teacher cooperation, and cultivating the next generation of system leaders to ensure a long-term continuity in achieving the reform goals.

Inspired by these reports our deliberation focused on appropriate manageable ways for nurturing teacher cooperation and cultivating system leaders. Stimulated by wiki-based software such as Wikipedia, which enable sharing common knowledge, its accumulation and preservation, we considered ways to support such processes by adapting existing powerful technology to this initiative.

In this paper we describe an educational R&D project, i.e. it can be characterised as “applied research that seeks solutions to practical questions in education, with less emphasis on developing, testing and advancing theory” (OECD, 2004, p.8). We present our dilemmas, and results of three preliminary studies that assisted us in shaping up our initiative.

THEORETICAL BACKGROUND

In this section, we briefly present some of the theoretical background related to teachers' community of practice (TCoP) and lesson plans (LPs) as the community's central resource.

Teacher professional development. Mourshed et al.'s (2010) report points at the central role of investing in teachers' professionalization in the success of a reform. Frameworks aimed at supporting teachers' professional development can take various forms, have diverse goals, different duration, etc. Nevertheless, teachers' participation in such programs, per se, does not guarantee their development (Guskey, 2002). Moreover, the effect of teachers' professional development program, and as a result its effect on students' outcomes, often fade out quite quickly soon after the program ends. One of the reasons for this phenomenon is that, generally speaking, teachers' professional development programs are led by off-school factors (e.g. academic institutions, Ministry of Education) and do not support the formation of an autonomous professional TCoP (Movshovitz-Hadar, Shriki & Zohar, 2014).

Teacher community of practice. The last two decades have brought educators to acknowledge the need for teachers to "abandon" their typical isolation for the benefit of joining forces and sharing knowledge. Managing shared knowledge might be best achieved by nurturing professional CoPs (Levine, 2010). The origin of the concept "community of practice" is rooted in learning theory, and was coined by Lave & Wenger (1991) while studying apprenticeship as a model for learning a profession. CoPs are formed by people who engage in a process of collective learning in a common domain, share a concern or a passion for something they do, and learn how to do it better as they interact with one another on a regular basis. In this view, becoming a professional is not seen as the individual's acquisition of knowledge, but rather as a social process of participation in a learning community. In order for a community to be recognized as a CoP, a combination of three characteristics should be fostered simultaneously (Wenger, 1998): (1) The domain: A CoP must have an identity defined by a shared domain of interest; (2) The community: Members engage in joint activities and discussions, help each other, share information, and build relationships that enable them to learn from each other. They do not, however, necessarily work together on a daily basis; (3) The practice: Members of a CoP are practitioners. They develop a shared repertoire of resources, such as experiences, stories, tools, and ways of addressing recurring problems, thus learn with and from each other. Such communities develop their practice through a variety of activities, among them: documenting projects and ideas, assisting each other in finding information, sharing resources, discussing developments, solving professional problems collectively, mapping knowledge, and more. In general, national mathematics TCoP conform to Wenger's first two characteristics: they share an interest in mathematics, its teaching and learning, they meet in professional conferences, read professional journals, and share a professional terminology. However, the third characteristic, to a large extent, is still absent in many national mathematics TCoP (Shriki & Movshovitz-Hadar, 2011).

This raises questions related to steps needed for enabling the development of an autonomous TCoP. In this context, we were mainly concerned about the meaning of “resources” and the technology through which they can be preserved, shared, accumulated, discussed, and continuously improved through a collaborative effort, keeping in mind the aim of improving students' outcomes and attainments. This led us to deepen our understanding in sharing LPs and the role of joint lesson planning.

Joint lesson planning. Teaching is an extremely complex profession. Teachers need to possess a wide range of skills and various types of knowledge, e.g. pedagogical knowledge that relates to teaching materials and methods, knowledge about students' learning, capability of analyzing reflectively their actions and impact, and more (Shriki & Lavy, 2012). But above all, they should be able to integrate these skills and knowledge and translate them into LPs. In fact, designing LPs are at the heart of teachers' professional work. However, in most cases teachers prepare their LPs “in mind”, and the preparation of a detailed LP is considered to be an “unnecessary burden” required only in pre-service teacher education. Even after teaching a certain lesson, LP is not recorded and, at best, notes are written in the textbook for future reference. As a result, at the individual level, drawing conclusions is limited, and at the community level there is a lack of sharing practical knowledge with colleagues (Movshovitz-Hadar et al., 2014). This stands in a stark contrast to the recognized benefits of sharing knowledge through joint lesson planning: *“we discovered the magic of effective joint lesson planning... Joint lesson planning has become a cornerstone of...collaborative practice...The expectation of teachers is not only that they should develop and employ effective practices in the classroom, but that they should share them throughout the whole system. Best practice therefore quickly becomes standard practice, adding to the pedagogy”* (Mourshed et al., 2010, p. 77).

FROM THEORY TO PRACTICE

With respect to writing and sharing LPs, and to the role of sharing LPs and providing/receiving feedback in the process of becoming TCoP we looked for answers to the four questions: (1) Are there ongoing voluntary processes of sharing knowledge among teachers? If so, what motivates these processes? If not, why? (2) What are the processes involved in designing LPs for sharing with colleagues, as compared with processes of designing LPs for one's own use? (3) What kind of interaction occurs in the process of joint preparation of LP? (4) In providing and receiving feedback to peers' LPs: To what extent are teachers ready to provide feedback to peers' LPs, to receive feedback from peers to their LPs, to reflect on peers' feedback and to accept it?

In order to receive initial answers, we conducted three preliminary R&D studies.

R&D study 1 - My favorite math LP. In this study, we examined teachers' willing to respond to an e-mail call for sharing their LPs voluntarily. We approached about 400 high school mathematics teachers, asking them to send their favorite LP, written according to specific guidelines provided. They were asked to approve uploading their LPs into a designated open web site. To encourage the teachers to share their LPs, we

announced three “raffle prizes” of \$250 (in local currency) to be held at the annual National Conference of High School Mathematics Teachers. It should be emphasized that there was no judgmental process as to the quality of the LPs, since we believe that relevant standards should be determined and take shape by the TCoP itself. Four rounds of raffles took place in 6 months intervals. Only 10-15 LPs were sent to each round. This first step left us not only disappointed but with many open questions related to teachers’ responsiveness and motivation to share their LPs with their colleagues. However, a large number of teachers were curious to see other teachers’ LPs as indicated by the number of entries to the website where these LPs appeared (<http://ramzor.technion.ac.il>).

R&D study 2 - Joint lesson planning on MediaWiki system. Eleven graduate students, experienced mathematics high school teachers, participated in a semester long activity in which they collaboratively designed LPs on a MediaWiki system. At the time this experiment was carried out, MediaWiki seemed to us as the best available platform for facilitating collaborative group work aimed at developing a dynamic repository of LPs and discussing educational ideas. Results of a study that followed the teachers’ experience (Shriki & Movshovitz-Hadar, 2011) indicated that the process of joint lesson planning supported the development of the participants as a small TCoP that interact on a daily basis, discuss ideas, and share LPs and other professional resources. The results also pointed at many concerns of the participants, categorised as social and technical ones. The social concerns were associated with participants’ contemplating about ways to provide and receive feedback, and fears of losing ownership over their creative work as authors of LPs. The technical concerns were linked to difficulties the teachers faced while writing in Wiki syntax.

These results led us to recognize the need for teachers to arrive at agreed upon social norms for managing a shared repository of learning and teaching resources as a preliminary necessary condition for nurturing TCoP. It should be remembered that unlike Wikipedia, which is mainly an encyclopedic or consensus-based reference repository, teachers’ LPs repository is a creative design work, experience-based, that expresses personal endeavors. Thus, as part of becoming TCoP, teachers should decide how to carry out a productive discourse and successful collaboration, what is the meaning of “constructive feedback”, how to consider provided feedback, how to keep ownership, and more. There are also questions related to the proper ways for reaching agreement on each issue. Subsequent to this experience we also realized that the technical concerns related to WikiMedia make it an inappropriate platform for accumulating, preserving, and improving mathematical LPs. To develop more appropriate software we approached Omnisol Information Systems Company, and started the development of RAMZOR software. The term “RAMZOR” means “traffic light” in Hebrew. This term was chosen to metaphorically signal: Red light - Stop to search and ponder about your next lesson; Yellow light - Get Ready by looking for various LPs in your desired topic and/or prepare your own LP; Green light – Go well prepared to your class and possibly afterwards upload your experience results.

R&D study 3 - A 3-day summer school for joint lesson planning. With the insight gained from R&D study 2, we organized a 3-day summer school for two consecutive years in which two groups of selected teachers designed LPs (individually or in pairs/small groups), provided feedback to peers' LPs (orally or in writing), improved their own LPs subsequent to receiving feedback, and reflected upon the entire process. The teachers participated in the summer school (18 in the first one and 20 in the second one) on a voluntary basis subsequent to an invitation that was e-mailed to them following recommendations received from the superintendent of school mathematics and the school principals. Data was gained through questionnaires, interviews, transcripts of small groups discussion and whole group ones, and content analysis of the LPs and feedbacks (Movshovitz-Hadar et al., 2014). Our findings indicated that ongoing processes of collaboration and sharing are rare at schools. According to the teachers, this situation is a result of several causes, among them: (1) Heavy workload that leaves no time for interaction (*"We work very intensively, and fail to find a suitable time to sit and think together beyond planning exams"*); (2) Mathematics teachers' tendency not to consult their colleagues for fear of being perceived as having insufficient mathematical knowledge (*"Math teachers do not ask each other questions about how to solve a specific problem, or how to teach a certain topic. I know it is something typical for math teachers. Perhaps we are afraid to be seen as someone who does not know enough math"*); (3) In small schools there is often only one mathematics teacher or one mathematics teacher for certain grades/levels of teaching, and therefore has no colleague to consult with (*"In my school I am the only one who teaches high level math, so I have no one to learn from or exchange ideas with"*); (4) A lack of awareness to the benefits of cooperating and sharing knowledge (*"I have been teaching math for 13 years now. I don't believe other teachers can tell me something I don't know yet"*).

In the framework of the summer schools we mainly focused on bringing teachers to acknowledge the benefits of collaborating in planning detailed LPs in writing and of sharing knowledge, as well as the limitations and the affective aspects that are involved in such processes. The LPs were written using the initial version of RAMZOR software. This enabled participants to relate to the LPs, and enabled us to witness shortcomings of the software, thus to extend our R&D efforts towards improving the suitability of RAMZOR as a tool for managing professional knowledge.

The teachers' reflections (verbally and in writing) indicated that they had developed awareness of their personal gains from writing LPs, and from receiving peers' feedback. The teachers also pointed out that writing LPs and sharing knowledge strengthened their self-efficacy and contributed to empowering them as members of the TCoP. As for the personal gains, teachers realized that a detailed design of LPs *"enables to verify what you intend to achieve in the lesson, and make sure that what you are going to teach corresponds to your goals"*; *"A detailed plan of the 45 minutes class lesson by units of 5-10 minutes increases the likelihood that the time will be used optimally"*. Teachers also realized that writing LPs helped them focus on learning

processes: *“It compels you to think about how your teaching will affect students’ learning”*; *“In writing the LPs, I had to think about students’ difficulties, how to prepare for it, what examples to present, what questions to ask, and how to phrase them”*; *“A detailed pre-planning assures an interesting and challenging lesson that the students will always remember”*. Furthermore, teachers had discovered that writing LPs has implications for deepening their mathematics knowledge: *“I found myself probing in specific aspect of derivations which I had never thought about before. No doubt that writing LPs contributes to our understanding of math”*. The majority of the teachers concluded by saying something similar to: *“I’m leaving this summer school feeling that I had become more professional...I understand now that we have to be more accountable to our teaching in each lesson”*. Nonetheless, 4 teachers (about 10%) said that *“writing LPs is exhausting. I believe in most cases it is enough to write only the numbers of the exercises one is going to give, while a detailed LP should be written only in special cases”*; *“I’m not sure I’ll actually teach exactly the way I planned it, so it makes me think about ‘cost-effectiveness’ issues of the investment in detailed writing of LPs”*. Receiving peers’ feedback had a meaningful effect on teachers. All the teachers admitted that *“it was the first time I had the opportunity to share my thoughts about lesson ideas”*. In fact, *“Just the knowing that the other teachers are going to give feedback to my LPs, motivated me to think more deeply about all possible aspects of the lesson and improve it”*. This stemmed from two main motives: *“I have to write the best LP I can in order to leave a good impression, and also, I definitely want my colleagues to try out my LP in their classes”*.

The mutual feedback was provided in various phases of writing the LPs (from a consultation regarding not-detailed LP outlines, to comments on a complete detailed LP), and in three main modes (small group talk, a whole class discussion, and written feedback through the software). Most teachers believed that receiving feedback from their colleagues is beneficial at every phase of designing the LP, since *“At any point you are in a different state of ‘maturity’, so at every phase you have different gains”*. Teachers felt their main benefits from receiving feedback were related to gaining *“new ideas and fresh viewpoints”* and *“insights regarding the weak points of the LP”*. But above all, *“I figured out that there is no substitute for consultation with colleagues”*, and *“the feedback I received changed my entire thinking about teaching students’ learning”*. As for the mode of receiving feedback, while at the beginning all the teachers thought that *“feedback given face-to-face is more effective, because one may ask clarifying questions”*, soon after they experienced the process of receiving feedback through the software many of them admitted that *“such feedback is no less efficient. I could learn a lot from the written comments”*. In particular, most teachers realized that *“in ‘real life’ it makes more sense to expect a written feedback, because one can write it in his or her available time, and there is no need for scheduling face-to-face meetings”*.

Whereas the teachers’ responses that relate to the benefit of writing LPs and receiving feedback did not surprise us, we could not anticipate the effect of this process on

strengthening their self-efficacy and on sensing that the process contributed to their empowerment as members of TCoP: *“I realized that our work as teachers is somewhat ‘amateurish’. No one inspect our work (except for maybe the grades of our students in the matriculation exams). Each teacher works at his or her discretion, no supervision, no setting goals. In contrast, working together, ‘regulating’ each other, and collaboratively setting the standards that would be expressed in our LPs, no doubt would lead us to do our job much more professionally”; “The unique thing was that we had a chance to learn about lesson planning from each other, and not from some academic figure. I learnt the strength of learning from peers, learning from equals. It is much more meaningful than any other kind of learning”; “What we did here was the start of a social revolution. I felt that everything is in our hands, the teachers’ hands. This is the first time that I feel trusted as a teacher. It really made me proud!”*. In this regard, most teachers specifically related to the central role of repository of LPs and a media through which they can interact and share knowledge: *“Our community needs to change the traditional approach of adhering exclusively to textbooks. Only a joint effort of all members of our community to generate a database of LPs will make a change in our profession”; “This repository of LPs on RAMZOR is the only way to preserve the community knowledge for the benefit of all, new as well as veteran”; “This software is an amazing tool. It allows teachers to see they are not alone, they are part of a community. They can see how others teach and learn from it. Networking with colleagues allows to maintain fruitful discussions and improve the teaching”*.

CONCLUDING REMARKS, MOVING TO R&D STUDY 4

To summarize our three preliminary studies, one important observation is that although teachers recognize the major role of planning their lessons in details, they do not rush into the opportunity to share LPs, and they refrain from sharing their LPs with others unless they are put in a framework that makes them do it. Another observation is that once provided with software that enables lesson planning they become aware of the impact of writing detailed LPs on the quality of their lessons. In addition receiving and giving feedback on LPs are processes which are highly demanding, and teachers gradually become appreciative of their potential to improve their work.

Towards the next step we also considered the short duration of each preliminary R&D study which did not allow us to examine a long-term effect of writing detailed LPs through RAMZOR software and sharing them on teachers’ professional development, or long-term processes of the evolvement of an independent TCoP. Furthermore, the small samples and the fact that the studies were carried out under “laboratory conditions”, do not allow us to draw conclusions about the impact of lesson planning via RAMZOR on the professional development of various mathematics teachers.

As typical to an R&D project, we put less emphasis on developing, testing and advancing theory (OECD, 2004, p. 8). Our emphasis is rather the design of a long-term study, situated in the real-life school settings, which involves a larger sample of mathematics teachers using RAMZOR for planning their work and sharing their

experiences. Following the three R&D preliminary studies and 2 years of RAMZOR software development, R&D study 4 started in the school year 2014/15 in 19 high schools spread about Israel Northern District. This is a three-year project which enables data collection aimed at finding answers to the yet open questions.

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DIFFERENT GENERALITY LEVELS IN THE PRODUCT OF A MODELLING ACTIVITY

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The current study examines features of modelling processes and the competence of groups that elicit models with different generality levels while working on modelling activity. To this end, 34 practicing teachers and 72 prospective teachers engaged in a modelling activity in 23 groups. Data were collected from reports, worksheets and video recordings. The findings indicate that the models elicited by the 23 groups can be divided into two main generality levels: 74% of the models were symbolic-general while 26% were numerical. Analyses of the modelling processes of six groups indicate that the general and numerical groups went through the entire modelling cycle, including all the phases and actions. However, the modelling route was different, and some of the modelling competence was lacking in the numerical groups.

INTRODUCTION

While the product of a modelling process is a model (Sriraman, 2005), modelling perspectives tend to emphasize the process over the product (Ang, 2001). The importance of modelling processes led researchers (Stillman, Galbraith, Brown & Edwards, 2007; Borromeo Ferri, 2006) to focus only on the process itself, with little attention devoted to the relations between the modelling process and the final models. We believe that monitoring and comparing the modelling processes of groups whose final product models differ in level of generality may shed light on the competencies needed for eliciting models that are more general. In this study, we focus on practicing and prospective teachers because they play a pivotal role in guiding student learning in mathematical modelling activities (Borromeo Ferri & Blum, 2010) and they consider modelling to be difficult (Blum & Borromeo Ferri, 2009). The current study attempts to shed light on the competencies of the participating teachers, which may result in differences in the generality levels of their models.

THEORETICAL BACKGROUND

Modelling

Mathematical modelling is considered to be the two-way process of translating between the real world and mathematics (Blum & Borromeo Ferri, 2009). The modelling approach emphasizes the effectiveness of mathematics in real life (Vorhölter, Kaiser, & Borromeo Ferri, 2014). Modelling activities begin with incomplete, ambiguous or undefined information about a situation, and learners are required to mathematize this information in meaningful ways while working in small groups (Doerr & English, 2003).

Modelling processes of authentic/real world problems are described as cycles that translate between the real world and mathematics in both directions through a series of steps or phases (Blum & Borromeo Ferri, 2009). We have adapted the modelling cycle of Blum and Leib (2005), who organized the modelling process into six actions and four phases. The actions consist of (1) understanding the problem and simplifying a situation model; (2) presenting a real model; (3) mathematizing, which leads to constructing a mathematical model; (4) applying the mathematical model that elicits mathematical results; (5) interpreting these mathematical results while considering the real-world situation; and (6) validating these results according to the original situation. These actions lead to the modelling phases, which include (a) a real model; (b) a mathematical model; (c) mathematical results; and (d) realistic results. If the results are unacceptable, the cycle starts again.

These actions describe the transitions between the modelling phases and include several modelling competencies. *Modelling competencies* include “skills and abilities to perform modelling processes appropriately and goal-oriented as well as the willingness to put these into action” (Maaß, 2006. P. 117). Modelling competencies are needed in order to complete modelling activities successfully (Stillman et al, 2007). Researchers (Maaß, 2006; Stillman et al, 2007) defined lists of modelling competencies in each transition between the modelling phases. These include: (i) to make assumptions about the problem and simplify the situation; (ii) to recognize relevant variables and to mathematize them; (iii) to mathematize relevant quantities and their relations; (iv) to use mathematical knowledge to solve the problem; (v) to select and apply appropriate formulae; (vi) to generalize or extend the solution; (vii) to critically check results with the real situation; and (viii) to consider implications of decisions and results.

Models are the product of the modelling process (Sriraman, 2005). They represent the phase in which the learner makes external representations on a mathematical level (Borromeo Ferri, 2006) or abstractions of a complex real situation into a mathematical form (Ang, 2001).

RESEARCH QUESTIONS

1. What are the differences in the modelling cycles of groups that elicit models with different levels of generality?
2. What are the differences between the modelling competencies of groups that elicited models with different levels of generality?

METHOD

The current study included 106 participants, 34 of them practicing teachers (primary and middle schools) that took a problem-solving course as part of their master’s degree studies at a college of education. The other participants included 72 prospective teachers taking a problem-solving course at a different college.

As an introduction to the activity (Figure 1), the first author presented some historical facts on the development of toothpaste production. Then, the participants received the activity. They worked in groups and were asked to submit reports that included an explanation about the change in toothpaste consumption.

Data were collected from the reports of 23 groups in the form of worksheets and notes. In addition, the work of six groups was video-recorded and transcribed verbatim.

To analyze the modelling products, we categorized each model in the reports according to mathematical operations, relations and processes emerging in the mathematical models. The worksheets and notes served as a source for triangulating our interpretation of the mathematical models.

We used an iterative process of reading the transcripts and watching the video to analyze the modelling of the six recorded groups. We analyzed the participants' discussions in each group according to the modelling cycle of Blum and Leib (2005). The researchers identified and distinguished the modelling process (phases and actions) of each group and presented their analyses visually (see next section). The modelling competencies were analyzed according to definitions of modelling competencies by Stillman et al. (2007) and Maaß (2006).

A student went to the general manager of the Colgate corporation and suggested an idea that would increase company profits without any effort. The student said, "I would be happy to share my idea with you, but you must pay a million dollars in case you decide to use the idea." The general manager accepted the condition, and the young student suggested enlarging the opening of the toothpaste tube.

The opening of your toothpaste tube has been enlarged. Write a letter that includes a description of the change in your consumption compared to the original toothpaste tube.

Figure 1: The toothpaste activity

FINDINGS

Categorization of the elicited models

The 23 models were categorized according to mathematical operations and processes into two categories: (1) general algebraic models - 74% of the groups; and (2) numerical models - 26% of the groups. The features of the general model used general algebraic expressions, making the model appropriate to various situations. The algebraic expressions differed. Some expressed the variables of the situation as additive relations or multiplicative relations and some used ratios. The features of the numerical model referred to specified numbers, making the model relevant to a single situation. Examples of one model from each category follow.

General model: "The radius of the opening of the old toothpaste was x : $R=x$. We enlarged the radius by y , so that R_* is the new radius: $R_*=x \cdot y$. The volume of the amount of toothpaste that comes out of the original tube is $x^2\pi h$, where h = the height of the cylinder. The volume of the toothpaste that comes out of the new tube is $R_*^2\pi h=(xy)^2\pi h$. The rate of flow is $(xy)^2\pi h / x^2\pi h = y^2$, so that consumption is increased by y^2 ."

Numerical model: “If the original r is 0.5 and the new r is 0.7 where $h=2$, the original volume is $\pi 0.5^2 \cdot 2 = 1.57$, and the new volume is $\pi 0.7^2 \cdot 2 = 3.07$. If the volume of the tube is 120, each toothpaste consumption with the old tube is $120 \setminus 1.57 = 76$. With the new tube each toothpaste consumption is $120 \setminus 3.07 = 39$. Therefore, the rate of consumption with the new tube is about two times greater.”

More numerical models were observed among the practicing teachers than among the prospective teachers. Further discussion of this finding is beyond the scope of this report. Of the six recorded groups, four groups created a general model while the other two groups created models at the numerical level.

Modelling cycles

Analyses of the modelling processes of the six groups indicate that all the groups went through the entire modelling cycle, including whole phases and actions, but the modelling routes differed. The modelling routes of the numerical groups went through the modelling phases sequentially, but among the general groups we observe skipping of some modelling phases. Next, we describe the modelling cycles of two groups general group (Figure 2) and numerical group (Figure 3). In Table 1, we detailed the phases and actions of the general group.

Table 1: the Modelling cycles of a group with a general model

Modelling cycle	Phase\ action	Explanation
The first cycle	C.1	Understanding the situation, simplifying and identifying the important variable
	C1.A1	Real model: depending upon the amount of toothpaste that comes out
	C1.2	Mathematization: assuming variables for the dimensions of the cylinder and the ball
	C1.B1	Initial mathematical models: $\pi(r+x)^2h \setminus \pi r^2h$ and $\frac{4}{3} \pi(r+x)^3 \setminus \frac{4}{3} \pi r^3$
	C1.B2	General mathematical models: $(r+x)^2 \setminus r^2$ and $(r+x)^3 \setminus r^3$
The second cycle	C2.1	Return to the situation, assuming data.
	C2.B1	Initial mathematical model
	C2.2	Applying the data from the Initial mathematical models
	C2.C2	Mathematical results
	C2.3	Interpreting to reality
	C2.D2	Realistic results
	C2.4	Validating the results.
The third cycle	C3.1)	Return to the situation, assuming data
	C3.B2	General models
	C3.2	Applying the general models
	C3.C3	Mathematical results
	C3.3	Interpreting to reality
	C3.D3	Realistic results the consumer increased in this w
	C3.4	Validating the results in the situation

(Due to space limitation, we only bring the work description of one group)

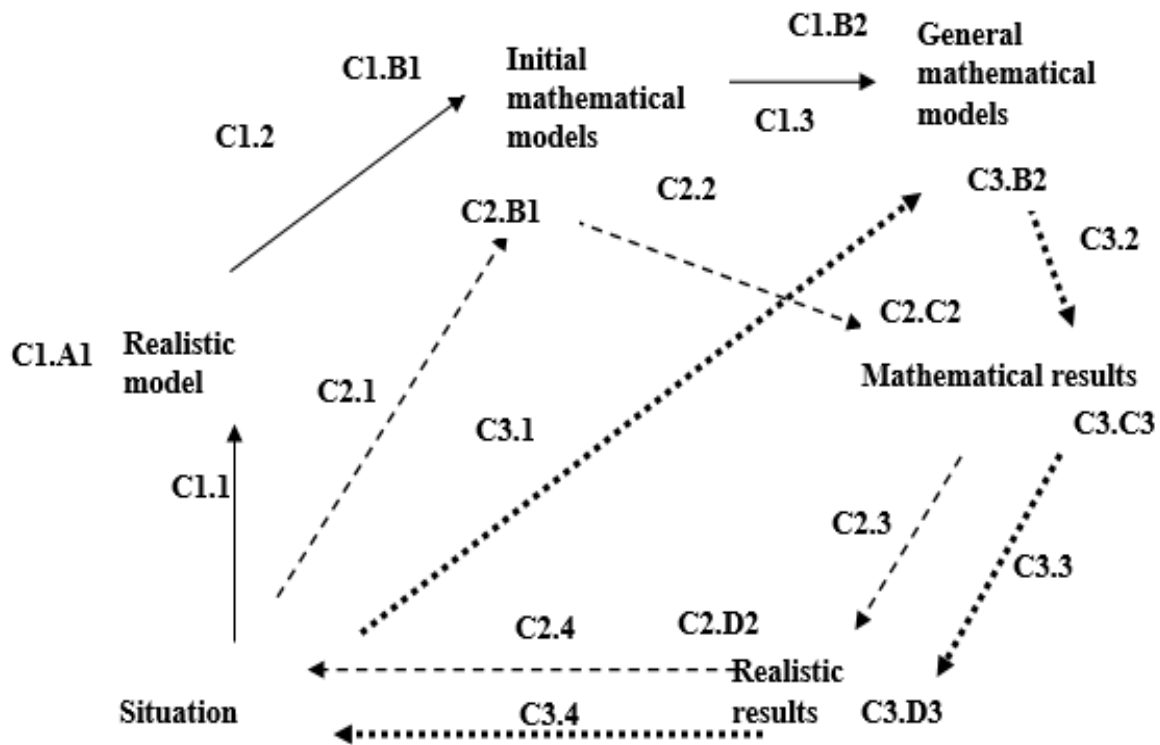


Figure 2: The modelling cycle of general group

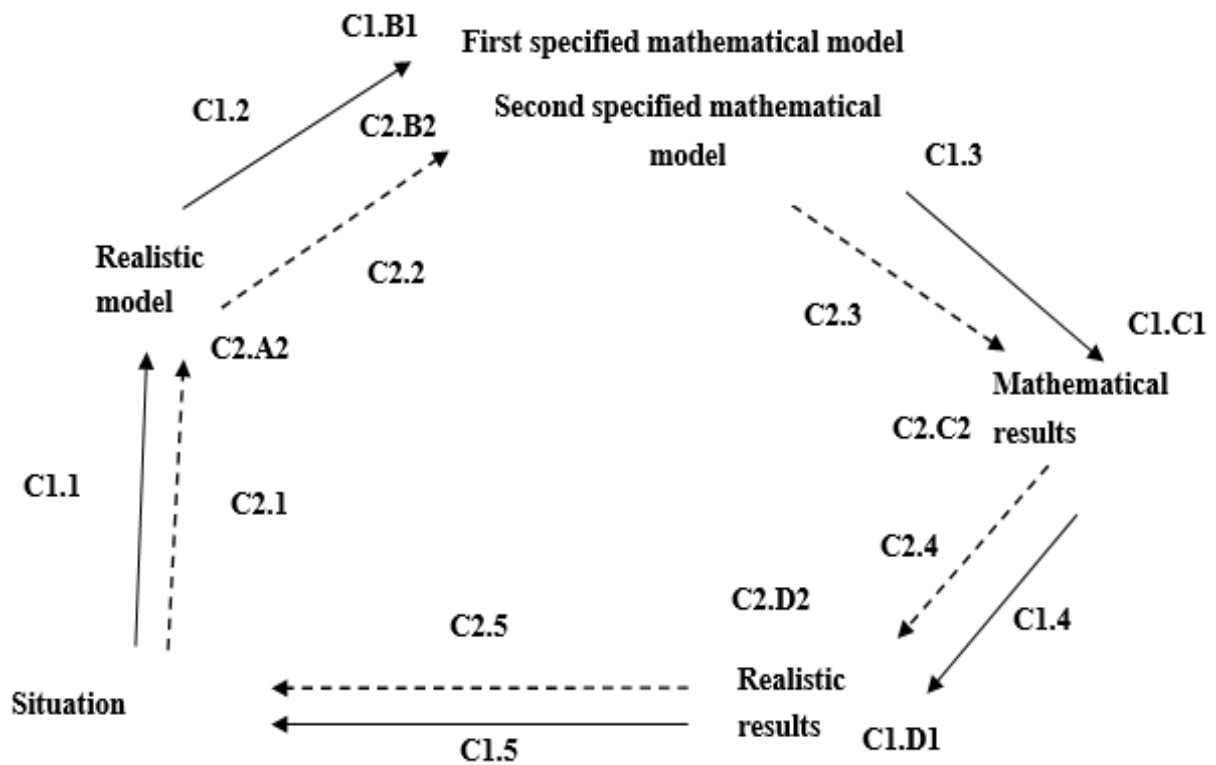


Figure3: the modelling cycle of numerical group

Modelling competences

Closer analyses of the discussions of the six groups allowed us to identify differences in modelling competencies through the transition from the situation to the real model and through the transition from the real model to the mathematical model. Table 2 shows the main differences in modelling competencies. However, the modelling competencies in the transitions between the mathematical model and the mathematical results and between the mathematical results and the realistic results, and the validating process of the realistic results were similar between the numerical groups and the general groups.

Table 2: Different modelling competencies identified in the discussions of six groups and examples of student discourse. (The numbers in the examples indicate numbered line in the transcript).

Modelling competencies in the transitions	Numerical groups	General groups
<i>From situation to real model:</i> - To simplify the situation.	They use specific examples in order to simplify the situations. Ex. [2]Amani: What will happen to the consumption? [3]Manal: It will change. [6]Manal: For example, we have a tube with volume of 150ml. [7]Rana: 100 or 150. [14]Manal: Ok, let's take 100. We have to organize a table that shows the new and old consumption.	They use general terms through simplifying the situation. Ex. [2]Muhammed: It was like this and now it changes (drawing two cycles). [12]Areej: It means how much toothpaste comes out now and how much came out with the old opening. [14]Muhammed: We must look at the ratio by which the use increased.
- To identify dependent and independent variables	They did not identify dependent and independent variables. Ex. [28] Manal: How many times does a person brush his teeth? [31] Rana: Let's say twice.	They identified the relevant variables. Ex. [9]Fatmeh: It is related to the opening. [43]Fatmeh: How long a person brushes his teeth does not matter.

<p><i>From real model to mathematical model:</i></p> <ul style="list-style-type: none"> -To choose appropriate mathematical notations -To mathematize relevant quantities and their relations. -To select and apply appropriate formulae 	<p>They assume numbers through the mathematization process. They did not use algebraic notions and did not suggest formulas.</p> <p>Ex.</p> <p>[6]Manal: The opening is 2 mm, the length is 1.5 cm.</p> <p>[23]Rana: The new opening is 4.</p> <p>[24]Manal: Maybe 3, we can use different numbers each time.</p> <p>[60]Manal: The radius is 0.5, the old one is 0.25, the length is 1.5.</p> <p>[61] Rana: We first compute the area of the base and then multiply.</p>	<p>They assume variables through the mathematization process. They use variables to present the mathematical model. They use appropriate formulae.</p> <p>Ex.</p> <p>[56] Areej: We assume the old radius is r. The length of the brush is h. The amount is $\pi r^2 h$.</p> <p>[62]Areej: If this is like pea, we need to compute the volume of the ball.</p> <p>[75]Areej: If we expand the opening by x.</p> <p>[91]Fatmeh: the ratio will be $\frac{4}{3} \pi (r+x)^3 \setminus \frac{4}{3} \pi r^3$.</p> <p>[92]Areej : The ratio is $(r+x)^3 \setminus r^3$</p>
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DISCUSSION

The main finding of the current study is that there is no relation between going through the entire modelling cycle and the generality level of the models. Working through the entire modelling cycle as was defined in different studies (Blum and Leib, 2005; Stillman et al., 2007) does not necessarily lead to sophisticated models. The numerical-model groups went through all the phases and actions in the modelling cycle in a manner similar to that of the general-model groups. Yet, the elicited models of the two groups differed in their generality level. The differences between the groups are similar to the differences between beginners and expert modellers according to Kaiser (2007). She explained that beginners tend to produce assumptions for modelling without any plan and without regard for the involved complexity of the models. Experts, on the other hand, control their solving strategies and therefore achieve their aim faster.

Finer analyses of the modelling processes indicate that the differences between the numerical and general groups were found in some of the modelling competencies. The numerical groups lacked competencies, such as recognizing relevant and irrelevant variables, choosing appropriate mathematical notations, generalizing or extending solutions. Lacking modelling competencies is considered a barrier to successful completion of modelling activities (Stillman et al, 2007). However, the findings obtained from the analyses of the modelling processes of the numerical and general groups indicate that the validating process did not play a role in distinguishing the

generality level of the elicited models. As a result, specified mathematical models were accepted yet did not meet the demands of the situation.

We recommend expanding the current study with several modelling activities and examining the differentiation between specified mathematical models and general models. This may also lead to expanding the mathematical model phase in the modelling cycle to provide a tool for distinguishing the modelling route of models with different generality levels.

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TRANSFORMATION OF STUDENTS' VALUES IN THE PROCESS OF SOLVING SOCIALLY OPEN-ENDED PROBLEMS (2): FOCUSING ON LONG-TERM TRANSFORMATION

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Bishop (1991) pointed out the importance of research on values in mathematics education. Based on this idea, Shimada and Baba (2012) developed three “socially open-ended” problems. They gave each of them to fourth graders, and identified four characteristics. In our subsequent research (Shimada & Baba, 2015), we researched the transformation of students’ social values and mathematical models emerging within a lesson. However, the issue of the long-term transformation of their values and models remained. The aim of this paper is to study this issue. To attain this aim, the current study employs a comparison of students in the sixth grade with those in the fourth grade. As a result of our analysis, we identified three characteristics such as transformation of values, re-existence of implicit values, and change of models.

RESEARCH BACKGROUND

In certain “socially open-ended” problems¹ (Baba 2010), it has been pointed out that values are expressed with mathematical solutions in the process of problem solving (Iida et al., 1995). We believe that the values described in this paper exist within the reasoning provided for the mathematical solutions. For example, in the problem of division of a cake, when we divide it equally for reasons of fairness, we judge the equal division as the mathematical solution and the fairness as the value. It is important for students to associate mathematical solutions and values, in order to develop problem-solving abilities related to issues such as environmental problems, which may produce the different value judgments that are seen in modern society. According to the current Japanese course of study, teachers make much of cultivating judgment using mathematics. Therefore, there is a demand for teachers to think about different mathematical solutions together with the reasons in the background. Shimada and Baba (2012, 2015) conducted teaching experiments to discuss these mathematical solutions and values at the same time in the classroom. Through such discussions, the students actively expressed their ideas regarding their mathematical solutions and the reasons for them, and refined their mathematical solutions by listening to the mathematical solutions and reasons that other students expressed, and thus transformed their values (Shimada & Baba, 2012, 2015). In these papers, we mainly examined a transformation of values and mathematical models between the beginning and end of a class. We also pointed out that next our research would be to confirm the influence such teaching has on students in the long term. Therefore, in this study, we hope to work on this issue.

RESEARCH OBJECTIVE AND METHODOLOGY

Research Objective

The objective of this paper is to study the long-term transformation of students’ social values and mathematical models, which occur through problem-solving.

¹ A socially open-ended problem is a particular type of problem (Baba, 2010) which has been developed to elicit students’ values by extending the traditional open-ended approach (Shimada, 1977).

Research Methodology

Overview of the class: Here we will explain the first intervention. The first author carried out a problem-solving lesson using the socially open-ended problem “Hitting the target” with fourth graders in a private elementary school in Tokyo on March 12, 2013. The problem is shown in Figure 1.

“Hitting the target:” At a school cultural festival, your class offers a game of hitting a target with three balls. If the total score is more than 13 points, you can choose three favorite gifts. If you score 10 to 12 points, you get two prizes, and if you score 3 to 9 points, you get only one prize. A first grader threw a ball three times and hit the target in the 5-point area, the 3-point area, and on the border between the 3-point and 1-point areas. How will you assign a score to the student?

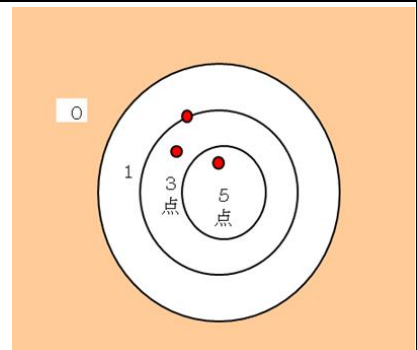


Figure 1: Problem-solving task

There were 38 students, comprising 19 boys and 19 girls. The first author was a teacher who specialized in mathematics education, with 40 years of teaching experience. The lesson follows the sequence of provision of a problem, individual solutions, presentation and discussion of the mathematical models and reasons, and finally collective selection of one model with its reason at the end. There are two groups of reasons such as “kindness to the first grader” and “fairness and equality”. The former tends to give more points to the player and thus develops the model like “ $5+3+3+1=12$ ”. The latter gave emphasis on the fairness by giving sensible points by considering all members.

The research method on the transformation of the students’ values

Seah, one of the leaders of the Third Wave international research project on values, stated the following in an overview of research on values:

The researching of values in the mathematics classroom has traditionally been approached using the research methods of questionnaires, observation, and/or interviews. ... By the late 2000s, values were also identified through content analyses of artefacts such as photographs and drawing, often followed by participant interviews which served to clarify initial findings or questions. (Seah, 2012, pp. 2–3)

In this paper, we document and research the transformation of students’ values as they appear in the problem-solving process. The research method on the long-term transformation of the students’ values and mathematical models involves not intervening using socially open-ended problems in regular classes for two years, and giving the same problem “Hitting the target” that was solved in the fourth grade to the students as sixth graders who have finished all the mathematics content of the elementary school. We also adopt a method of comparing values and mathematical models among students in the sixth grade and in the fourth grade. The aims of this investigation are to clarify the following. (1) How do the students transform their values at the time of graduation as sixth graders after two year-non-intervention period? (2) Is the students’ consciousness of social values maintained after two years? (3) How do students transform the mathematical model at the time of graduation as sixth graders who have finish learning all the mathematics content of the elementary school? We think that

clarifying these three issues will lead to the creation of basic documents when we perform active intervention for mathematics teaching in the future. Therefore, we clarify some characteristics of the long-term transformation using the same method for analysis we previously reported (Shimada & Baba 2015), in which we used quantitative analysis and qualitative analysis.

ANALYSIS OF STUDENTS' DATA

The comparative analysis of students' values and mathematical models on a worksheet both in the sixth grade and at the final selection time in the fourth grade reveals three characteristics of students' long-term transformation of values and mathematical models, which are noted below.

Some students transform their values from the fourth grade to the sixth grade

The first characteristic regards the existence of both students who transform their values from the fourth grade to sixth grade, and those who do not. Table 1 is a cross-tabulation table showing the relationship between values in the fourth grade and in the sixth grade. All numbers are percentages except those in parenthesis. The fractions in parenthesis show the number of students who expressed the values in both the fourth grade and in the sixth grade over the number of all students in the class. Table 1 below shows the values by types, both for the fourth grade and the sixth grade. For example, the percentage of students who selected the values "fairness and equality" in the fourth grade and selected the value "kindness to the first grader" in the sixth grade is 21.1%. Overall, the percentage of students who selected different values in both grades is 50.0% ($21.1 + 28.9 = 50.0$). We identified that half of the students transformed their values after 2 years.

		Values in the sixth grade		
	Values	Fairness and equality	Kindness to the first grader	Total
Values in the fourth grade	Fairness and equality	31.5 (12/38)	21.1 (8/38)	52.6 (20/38)
	Kindness to the first grader	28.9 (11/38)	18.4 (7/38)	47.4 (18/38)
	Total	60.5 (23/38)	39.5 (15/38)	100.0 (38/38)

Table 1: The Values in the Fourth Grade and in the Sixth Grade (n = 38)

Furthermore, we understood the following from Table 1. In the fourth grade, the percentage of fourth graders who select the values "fairness and equality" and "kindness to the first grader" is about 50% each; in contrast, the ratio of the values "fairness and equality" to "kindness to the first grader" in the sixth grade is approximately 3:2. From these data (Table 1), we hypothesized that some students might have transformed from the value of "kindness to the first grader" to the values of "fairness and equality" as they became older. Why did half of the students transform their values? We think that the transformation of students' values was affected by social and cultural experiences accompanying growth.

Some students transform from explicit values to the re-existence of implicit values in the sixth grade

The second characteristic is that there are some students who transformed from explicit values to the re-existence of implicit values in the sixth grade. Table 2 below shows the

transformation of the values “fairness and equality” in the fourth grade and in the sixth grade. The 10 students noted in Table 2 are those who selected the same values “fairness and equality” in both grades, and they were able to express their mathematical models and reasons using words indicating the values “fairness and equality,” such as “fairness,” “fair,” “equality,” “equally,” “equal,” “all people,” and “for the upper graders,” in the fourth grade. However, 8 students, I.K., T.R., H.K., Y.S., K.H., T.H., T.A., and T.J., could not express their reasons using the above value words in the sixth grade, and only 2 students, A.T. and M.H., could express their reasons using these words in the sixth grade. However, almost all of them mentioned that the ball is on the boundary in their reasons. The consciousness of this boundary condition may be polished by their daily and mathematical experience. Table 3 below shows the transformation of the values “kindness to the first grader” in both grades. The 6 students noted in Table 3 were those who selected the same value “kindness to the first grader” in both grades, and were able to express their reasons using words indicating the value “kindness to the first grader,” such as “for the first grader” and “to the first grader,” in the fourth grade. All these students could express their reasons using these value words in the sixth grade. From the above results, in the transformation of the values “fairness and equality,” we were able to understand that some students transformed their values from explicit values to the re-existence of implicit values in the sixth grade. In other words, they could express themselves regarding the values “fairness and equality” in the fourth grade, but they could not express these values in the sixth grade. In contrast, the value “kindness to the first grader” could be expressed in both grades. From these facts, we can learn that some values may become implicit unless we have a continuous intervention using the expression of values and mathematical models.

Name	In the fourth grade		In the sixth grade	
	Mathematical models	Explanation	Mathematical models	Explanation
I.K.	$5+3+1=9$	I selected K’s opinion. Because I thought that it is good for everybody to be equal.	$5+3+2=10$	I gave two points because the ball is on the boundary of 3 points and 1 point.
T.R.	$5+3=8$, $(1+3)\div 2=2$, $8+2=10$	I selected my idea. Because I felt my idea is like sportsmanship and equality. So, my idea was good.	$(3+1)\div 2+5+3=10$	The ball is on the boundary of 3 and 1. It becomes 4 by adding 1 and 3, then it becomes 2 by dividing 4 by 2. It becomes 10 when I add 8 and 2.
H.K.	$5+3+1=9$	I selected K’s opinion. Because nobody complains if I treat all people equally.	$5+3+2=10$	Because the ball was on a line between 1 and 3, I gave two points for the middle.
Y.S.	$5+3+1=9$	I selected K’s opinion. Because K’s opinion is equal for all people.	$5+3+1=9$	Because the ball was very close to one point.

	In the fourth grade		In the sixth grade	
Name	Mathematical models	Explanation	Mathematical models	Explanation
K.H.	$5+3+2=10$	I selected H's idea. Because H's idea is equal for all people.	$5+3+(3-1)=10$	Because the ball was on a line between 1 and 3, I gave two points for the middle.
T.H.	$5+3+2=10$	I selected S's opinion. Because S's opinion is fair and a more clear expression than my expression.	$5+3+2=10$	Because the ball was on a line between 1 and 3, I gave two points for the middle.
T.A.	$1+3+5=9$	I selected K's opinion. Because K's idea is fair for all people.	$5+3+1.5=9.5$	The ball is on the boundary of 3 points and 1 point. I give 1.5 point because the 3-point area of the ball is half of the ball.
T.J.	$1+3+5=9$	I selected K's opinion. Because K's idea is fair for all people.	$3 \div 2 = 1.5$, $1.5+3+5=9.5$	Dividing 3 by 2 gives 1.5, because the 3-point area of the ball is half of the ball.
A.T.	$5+3+1=9$	I selected K's opinion. Because K's idea is fair for all people.	$5+3+1=9$	I do not give three points to the first grader, and gifts are not enough. Besides, if I give three points to a first grader, I should give three points to all people.
M.H.	$5+3=8$, $(1+3) \div 2 = 2$, $8+2=10$	I selected R's opinion. Because R's opinion is the same as my opinion, but R's opinion is more a concise expression than mine. I feel sorry for upper graders when I give three points to small child.	$3-1=2$, $5+3+2=10$	If I give three points and one point to a first grader, it is not fair. So I gave two points. Because the ball was on a line between 1 and 3, I gave two points for the middle. I think that it is nice to give two points because of equality.

Table 2: The Transformation of the Values “Fairness and Equality” in the Fourth Grade and in the Sixth Grade

	In the fourth grade		In the sixth grade	
Name	Mathematical models	Explanation	Mathematical models	Explanation
K.R.	$5+3=8$, $1+3=4$, $5+3=8$, $1+3=4$, $8+4=12$ $12+1=13$	I selected K's idea. Because K's idea is good for the first grader. I more strongly affected by the value of "kindness to the first grader."	$5+3+3=11$	I give three points to the first grader, but the ball is close to the one-point area. The first grader is happy.
S.J.	$5+3+3+1=12$	I selected Ko's idea. My idea is a small service for the first grader. But Ko's idea is just good for the first grader.	$5+3+3=11$	I gave three points to the first grader. It is good for us to give a bonus to the first grader.
T.K.	$3+3+5 = 11$	I selected my idea. Doing something for the first grader is kind and agreeable. The first grader will be happy and come here again.	$5+3+3=11$	I gave three points to the first grader, because a first grader threw a ball.
I.A.	$5+3=8$, $1+3=4$, $8+4=12$, $12+1=13$	I selected K's idea. K's idea is good for the first grader.	$5+3+3=11$	I gave three points to the first grader. It is good for us to be kind to the first grader.
O.N.	$5+3+3=11$	I selected S's idea. Because I think it is good for us to give a bonus to the first grader.	$3+3+5=11$	I gave three points to the first grader. The first grader feels happy.
T.A.	$5+3=8$, $1+3=4$, $8+4=12$, $12+1=13$	I selected K's idea. Because it is good for us to give a bonus to the first grader.	$3+3+5=11$	I give three points to the first grader, but the ball is close to the one-point area. The first grader is happy.

Table 3: The Transformation of the Value "Kindness to the first grader" in the Fourth Grade and in the Sixth Grade

Many students change mathematical models in the sixth grade

The third characteristic is that there are many students who changed mathematical models in the sixth grade. Table 4 is a cross-tabulation table for viewing the relationship between mathematical models in the fourth grade and in the sixth grade. All numbers are percentages except those in parenthesis. The fractions in parenthesis show, for example, in the case of $3/38$, the number of students who expressed the same mathematical models in both grades with respect to the values "fairness and equality" over the number of all students and $9/38$, the number of students, who expressed different mathematical models to the same values. Thus, the percentage of students who selected the values "fairness and equality" in the fourth

grade but changed their mathematical models in the sixth grade is 44.8% ($23.7 + 21.1 = 44.8$). On the other hand, the percentage of students who selected the value “kindness to the first grader” in the fourth grade but changed to a different mathematical model in the sixth grade is 44.7% ($28.9 + 15.8 = 44.7$). Overall, the percentage of students who changed mathematical models is 89.5% ($44.8 + 44.7 = 89.5$). From this fact alone, we understood that about 90% of students changed their mathematical models in the sixth grade.

		Mathematical models in the sixth grade				
		Fairness and equality		Kindness to the first grader		
		Same models	Different models	Same models	Different models	Total
Mathematical models in the fourth grade	Fairness and equality	7.9 (3/38)	23.7 (9/38)	0 0/38	21.1 (8/38)	52.6 (20/38)
	Kindness to the first grader	0 (0/38)	28.9 (11/38)	2.6 (1/38)	15.8 (6/38)	47.4 (18/38)
	Total	7.9 (3/38)	52.6 (20/38)	2.6 (1/38)	36.8 (14/38)	100.0 (38/38)
	Total	60.5 (23/38)		39.5 (15/38)		100.0 (38/38)

Table 4: Mathematical Models in the Fourth Grade and in the Sixth Grade (n = 38)

Table 5 below shows examples of mathematical models in the fourth grade and in the sixth grade. These students did not transform their values but changed mathematical models. T.R.’s mathematical model shows an example of a transformation from three formulae to one formula. T.R. transformed the former expression to a concise expression. K.H.’s model shows an example of a transformation to an expression in which the numerical meaning was clarified. K.H. transformed the former expression to a clear expression. T.J.’s model shows an example of a transformation to a different expression using division. T.J.’s formula is an expression that uses an idea similar to averaging. T.J. transformed the former expression to a different expression using a new idea. This idea was not seen in the fourth grade. Overall, Table 5 summarizes the fact that these students improved their mathematical values.

Name	Mathematical models in the fourth grade	Mathematical models in the sixth grade
T.R.	$5+3=8$, $(1+3)\div 2=2$, $8+2=10$	$(3+1)\div 2+5+3 = 10$
K.H.	$5+3+2=10$	$5+3+(3-1)=10$
T.J.	$1+3+5=9$	$3\div 2=1.5$, $5+3+1.5=9.5$

Table 5: Examples of Mathematical Models in the Fourth Grade and in the Sixth Grade

CONCLUSION AND FUTURE ISSUES

In this paper, we analyzed a long-term transformation of values and mathematical models for 2 years from the fourth grade to the sixth grade, and concluded that the following three

characteristics apply: half of the students transformed their values in the sixth grade; some students transformed from explicit values to the re-existence of implicit values in the sixth grade; and 90% of students changed their mathematical models in the sixth grade. Looking at the second one more closely, we realize that kindness to the first grader has been sustained well. So becoming implicit does not apply equally to all kinds of values. Besides, consciousness of the critical condition in the value “fairness and equality” is also developed, and the reason behind it may be both experience-based and mathematical learning-based. From these results, we hope to suggest the following for performing active intervention in mathematics teaching. (1) Because the students’ consciousness of at least the value “fairness and equality” does not continue, it is necessary to repeat the class using the socially open-ended problems. Generally in the social setting, the judgment can be done by based on not only mathematical models but also the reasons behind the models. (2) An idea that resembled averaging, which is to be learned by fifth graders and sixth graders in Japan, was newly seen when the students became sixth graders, so we understood that various mathematical models were expressed as they learned many kinds of mathematical content. The same values can be represented by more mathematically sophisticated models. (3) As its example, concise expressions and expressions of numerical meanings were seen in the mathematical models of the sixth graders. These are forms representing mathematical values. In this sense, both social and mathematical values are relating to each other. From these three points, we conjectured that this long-term transformation of students’ mathematical models and values was affected by both mathematical learning and social and cultural experiences in daily life. So in order to grasp this transformation, we will follow the process of transformation of students’ mathematical models and values, when the students learn continuously socially open-ended problems for two years, as distinct from the present study. Furthermore, in this process, we are to analyze how implicit values change and what kind of experiences make an impact on this process.

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PROSPECTIVE MATHEMATICS TEACHERS' PROOF COMPREHENSION OF MATHEMATICAL INDUCTION: LEVELS AND DIFFICULTIES

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The purpose of this paper is to characterize the levels of proof reading comprehension specific to proof by mathematical induction in order to provide a broader framework for analysing various difficulties. Especially, we focus on the prospective mathematics teachers' difficulties in understanding of the necessity of the base step and the logical validity of the inductive step. In this study, we pay particular attention to the local level of comprehension rather than the holistic level. Data are collected through the subjects' writing responses to a set of scripted statements and proofs. The results suggest that the essential difficulties of MI are characterized in terms of the gaps between the levels of proof comprehension. Based on the findings, the necessity of "encapsulation" is also discussed.

DIFFICULTIES OF MATHEMATICAL INDUCATION

In general, a proposition " $\forall n \in \mathbb{N}, P(n)$ " can be proven by two steps in MI: the base step, which establishes the base case such as $P(1)$, and inductive step, which proves the implication $P(k) \rightarrow P(k+1)$ for an arbitrary $k \in \mathbb{N}$. Since both the base and inductive steps have been performed, by appealing to the Principle of Mathematical Induction (Peano's fifth axiom for the foundation of natural numbers), the original proposition $P(n)$ holds for all natural numbers. From the logical point of view, by appealing to logical inferences such as *conjunctive inference* ($p, q \rightarrow p \wedge q$) and *modus ponens* ($[p, p \rightarrow q] \rightarrow q$), the structure of proof by MI can be represented as follows (see also, Ernest, 1984; Movshovitz-Hadar, 1993; Shinno & Fujita, 2015):

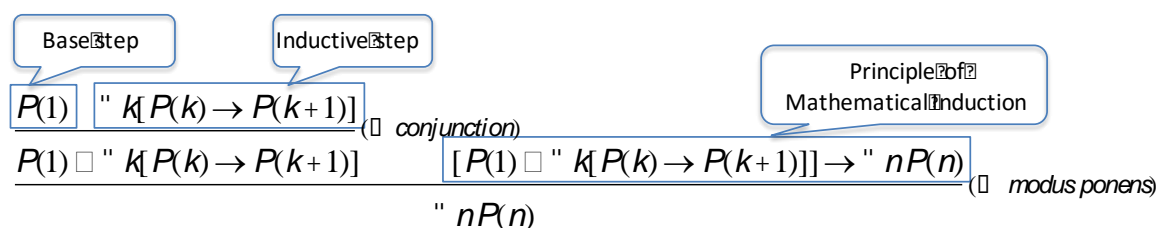


Figure 1: Logical inference form of MI

In many countries, mathematical induction (MI) has been introduced at upper secondary school level, although in some countries it may be intended to be taught at college or university level. A number of previous studies on MI in the field of mathematics education have investigated various difficulties or weak understanding targeting high school students, university students, or prospective teachers. For

example, Stylianides et al. (2007) have reported a weak understanding of the base step as well as a misunderstanding of the implication statement $P(k) \rightarrow P(k+1)$ in the inductive step (see also, Ernest, 1984; Dubinsky & Lewin, 1986). Recently, Palla et al. (2012) mentions that there is a gap between the operational and structural level in understanding MI, as follows:

Operational level is the initial approach to MI, which is also emphasized in most school textbooks. At this level, the structure of the natural numbers is implicit and appears in an intuitive form. The “structural level” is mainly encountered in advanced mathematical studies and refers explicitly to Peano’s fifth axiom of the structure of natural number. (Palla et al., 2012, p. 1025)

When writing or reading proof by MI, one may often pay attention to the operational aspects of MI but rarely recognize the substance of the structural and logical aspects because of its implicit nature. Shinno and Fujita (2015) have provided a more explicit distinction between these aspects in terms of Mathematical Theorem (Mariotti et al., 1997) that is constituted by a system of relations between a *statement*, its *proof*, and the *theory* within which the proof makes sense. Since different difficulties in MI have been reported in different studies, a broader framework may be necessary to synthesise different studies and propose a way to characterize different kinds or levels of difficulties of MI. In this paper we will consider the levels of proof comprehension (Yang & Lin, 2008; Mejia-Ramos et al., 2012) as a theoretical framework for characterising different difficulties of MI. When taking proof comprehension into account, there are few links between previous studies on proof comprehension and on MI. Thus, a research question appears as follows: *How can we apply the theoretical model of proof reading comprehension to proof by mathematical induction?*

THEORETICAL FRAMEWORK

Let us briefly explain the basic tenets of the frameworks. Yang and Lin (2008) proposed a model for proof comprehension, which is called “reading comprehension of geometrical proof (RCGP)”. A model for RCGP consists of four hierarchical levels: *surface*, *recognizing the elements*, *chaining the elements*, and *encapsulation*. The first level involves epistemic understanding of the meaning of mathematical terms, symbols or figures. At the second level, the comprehension involves recognizing the premises, conclusions or properties that may be implicit in the proof. The third level focuses on logical connections between the premises, conclusions and properties that are recognized at the second level respectively. Finally, at the fourth level, the proof may be viewed as a whole, where one reflects on how to apply the proof to other contexts.

Recently, Mejia-Ramos et al. (2012) reconstructed the model for RCGP, by taking the proofs in advanced mathematics into account. In this work, they distinguished the local and holistic proof comprehension in order to consider or assess the more complex proofs that undergraduate students would encounter. The local comprehension consists of three levels, which corresponds to the first three levels of the RCGP model, although Mejia-Ramos et al. (2012) termed 1) *meaning of terms and statements*, 2) *logical status*

of statements and proof framework, and 3) *justification of claims*. Additionally, they introduced the notions of holistic comprehension of the proof, which “must be ascertained by inferring the ideas or methods that motivate a major part of the proof, or the proof in its entirety” (p. 6). Mejia-Ramos et al. (2012) elaborated the notion of encapsulation by Yang and Lin (2008) in terms of four different levels: 4) *summarizing via high-level ideas*, 5) *identifying modular structure*, 6) *transferring the general idea or methods to another context*, and 7) *illustrating with examples*. By using these seven levels, they described ways to assess students’ comprehension of the theorem related to number theory. It implies that these local and holistic levels of proof comprehension can apply to proofs in different mathematical domains other than geometrical proof. In the present study, we attempt to apply mostly the local levels of comprehension made by Mejia-Ramos et al.’s (2012) model, with special attention to the notion of encapsulation by Yang and Lin’s (2008) model, to proof by MI, in order to consider what is specific to the comprehension of MI and its difficulties.

METHOD

Data are collected by a set of questions based on Stylianides et al.’s (2007) item with additional input from the idea of “proof script” (Zazkis & Zazkis, 2015), which involves a scripted proof and a scripted dialogue. We use this method as a tool for engaging prospective teachers in considering particular students’ difficulties as well as for identifying the prospective teachers’ comprehension of the proof.

The figure 2 shows a scripted proof (and a given statement) used in the present study. In this script, the proposed proof is invalid, but there are three points that have to be examined. Firstly, the given statement does not hold for any natural numbers. Secondly, the base step is missing in the given proof. Thirdly, the inductive step is still correctly applied. We, like Stylianides et al. (2007), aimed to see whether the prospective teachers who could realize the absence of the base step would be able to explain why the base step is necessary. We also intended to investigate the prospective teachers’ understanding of the logical validity of the inductive step by reading the proof.

Statement: For every $n \in \mathbf{N}$ the following is true: $1+3+5+\dots+(2n-1)=n^2+3$ (*)
Proof: I assume that (*) is true for $n=k$: $1+3+5+\dots+(2k-1)=k^2+3$
 I check whether (*) is true for $n=k+1$:
 $1+3+5+\dots+(2k-1)+(2k+1)=(k^2+3)+(2k+1)=(k^2+2k+1)+3=(k+1)^2+3$
 True.
 Therefore (*) is true for every $n \in \mathbf{N}$.

Figure 2: A scripted proof (Stylianides et al., 2007, p. 151)

In order to utilize this item, unlike Stylianides et al. (2007), we introduced the questions with the following dialogue (Figure 3). The first dialogue by Alan and Barbara is concerned with the reason way the base step is essential. The second dialogue by Christine and David is related to the logical validity of the inductive step, although David’s suspicion might suggest additional misunderstanding about circular reasoning

(Ernest, 1984). Participants were asked to first read the scripted proof above and then write their thought or rationale regarding four each scripted dialogue.

Alan and **Barbara**, high school students, are having a conversation about the above proof. Read through and answer the following questions.

Alan says: *This proof is not valid. Because its first step is missing.*

Barbara says, followed by Alan: *Why is it necessary to check for $n=1$?*

Christine and **David**, high school students, are having a conversation about the above proof. Read through and answer the following questions.

Christine says: *This proof shows the inductive step, that is, “if it is true for $n=k$, then it is true for $n=k+1$ ”. So, the proof of inductive step is valid.*

David says, followed by Christine: *Mathematical induction is the method in which you assume what you have to prove, and then prove it. So, I have a suspicious likeness to assuming what you have to prove!*

Figure 3: A scripted dialogue

In what follows we present findings from our selected cases of 38 prospective secondary school mathematics teachers both in England (N=19) and Japan (N=19). They were asked to write their thoughts by reading the above scripted proofs and dialogues. The 19 participants from England were trainees on a Post Graduate Certificate of Education in secondary mathematics course. Most of them have majored in mathematics at undergraduate level, although a few majored in physics or engineering. The 19 participants in Japan were third year undergraduate students of mathematics in the faculty of education.

The results will be considered for exemplifying the first three levels of proof comprehension, that is, the local comprehension of MI. Based on these findings, the necessity of encapsulation will be also discussed in the final place of the paper.

RESULTS AND DISCUSSION

The first level: *Meaning of terms and statements*

At the first level of reading comprehension, although it may be not specific to proof by MI, it is important to understand the meaning of mathematical terms included in a given statement. In the case of the scripted proof, it may involve understanding the meaning of the symbol “ $n \in \mathbb{N}$ ” or the given equation, and understanding the fact that the equation does not hold for any natural numbers. Most of participants (89.5%; 34/38) agreed with Alan’s remark by stating, for example, “this proof is not valid. Because its first step is missing”, or “Because the presented statement is not true”. On the other hand, the following responses exemplify weak understanding of the base step. (Note: “J5” represents “participant #5 in Japan, and A (B) represents the response to Alan’s (Barbara’s) remark”) (underline is added):

J5-A: True. Since we show that it holds for all natural number, we need to show that it holds for 1, the minimum value in natural numbers.

J5-B: When it says “for $n=k$ ”, it doesn’t say that k is an arbitrary natural number. But if k is a natural number, I don’t think that it needs to prove the case for $n=1$.

J5-A might be seen as an acceptable explanation, but the same participant went on to remark, in J5-B, that the base step is not necessary.

J6-A: When prove for $n=1$, we see that the statement is not true.

Some participants who were considered as demonstrating the first level of comprehension had difficulties in explaining the validity of the presented proof without the base step. For example, E8 wrote as follows (Note: “E8” represents “participant #8 in England”):

E8-B: It is necessary to check for $n=1$ since without this, the statement may only hold for some n , beginning it a number higher than 1.

E8 also recognized that in the presented proof the inductive step is valid as follows:

E8-C (response to Christine’s remark): The proof shows that if true for $n=k$, true for $n=k+1$ and hence this shows it is true for each consecutive number onward. (But $n=1$ would still have needed proof)

This finding is consistent with the fact, found by Stylianides et al. (2007), that some prospective teachers claimed that the statement is true “in some cases”. In the presented statement and proof, even if the proof of the implication statement “ $P(k) \rightarrow P(k+1)$ ” is valid, the original statement does not hold for any natural numbers. Although it is unclear if s/he actually checked the statement for $n=1$, it suggests the participant’s focus was on the surface or appearance of the presented proof.

The second level: *Logical status of statements and proof framework*

At the second level of reading comprehension, “understanding the status of the different assertions in the proof is necessary to understanding the logic of the proof” (Mejia-Ramos et al., 2012, p. 9). In the case of MI, it is reasonable to say that a reader needs not only to identify the statement to be proven and the proof of the base and inductive step, but also to recognize “previous statements and mathematical principle” used in the two proof steps. When the statement is about the domain of all natural numbers, an initial number should be $n=1$. So, for example, E15-B can be considered as demonstrating the second level of comprehension, in which the essence of the base step $P(1)$ can be explained as follows:

E15-B: Because $n=1$ is the first natural number, so to prove for all $n \in \mathbb{N}$, you need to prove the first step and then use induction to prove for all.

Moreover, concerning the inductive step, in this level, a reader needs to identify a procedure used in the proof of the inductive step. For example, when showing “ $1+3+5+7+\dots+(2k-1)+(2k+1)=(k^2+2k+1)+3=(k+1)^2+3$ ”, a multiplication formula “ $a^2+2ab+b^2=(a+b)^2$ ” is correctly applied. However, J11-C who viewed Christine’s remark as false could read incorrectly the computational aspect of the presented proof the inductive step as follows:

J11-C: Incorrect. For $n=k+1$, the right side k^2+3 should be $(k+1)^2+3$ by substituting $k+1$ for k .

Most participants (92.1%; 35/38) agreed with Christine's remark such that the proof of inductive step is valid, but they had difficulty in explaining the truth of the implication statement. Such participants can be considered to be at the second level, i.e. their reading comprehension is heavily influenced by the existence of the two steps. In other words, since the proof by MI always requires a rigid 2-step format, when the reader sees two steps are stated, they may think this is adequate and not challenge the content of those steps, so the proof of the implication statement may be hidden or out of focus for a reader at this level.

The third level: *Justification of claims*

At the third level of reading comprehension, "the reader needs to infer what previous statements and mathematical principle are used to deduce a new assertion with a proof" (Mejia-Ramos et al., 2012, p. 9). This level, termed *chaining elements* in the RCGP model (Yang & Lin, 2008), deals with relating premises, properties and conclusions in the proof in order to establish logically chaining arguments. In the case of MI, chaining elements in this level are considered as the logical necessity of the base step, and the logical validity of the proof of the implication $P(k) \rightarrow P(k+1)$. Since the base step is associated to the inductive step, the logical necessity of the base step can be explicit as a logical form: i.e., $(P(1) \wedge [P(k) \rightarrow P(k+1)])$ or informally, "P(1), and P(k) implies P(k+1)". Superficially most participants (92.1%; 35/38) agreed with Christine's remark. But this alone does not suggest that they have solid knowledge of the inductive step. For example, eight participants (21.1%), like J8-D, claimed intuitively an incorrect implication rule such that "If P(1) and P(k), and if P(k) implies P(k+1), then for P(n)" in responding Christine and David's dialogue.

J8-D (response to David's remark): So, we need to prove the first number like $n=1$, and show the equation holds. If it is not true for the first number, we should not assume the truth [for $n=k$] in the inductive step. If it is true for the first number, we can assume it because at least it holds for $n=k=1$. By this, if it holds for $n=k=1$, then it holds for the next natural number.

Only one participant gave a good answer and responded explicitly with the logical validity of the proof of the implication statement:

J10-D (response to David's remark): Even if it is true for one number, it doesn't mean that it is true for the next number. The truth of "A" or "B" is different from the truth of " $A \rightarrow B$ ".

As far as Christine and David's dialogue are concerned, like some previous studies (Dubinsky & Lewin, 1986; Stylianides et al., 2007), we also found that some participants (15.8%; 6/38) tend to think that the inductive step proves $P(k+1)$ rather than the implication $P(k) \rightarrow P(k+1)$ (e.g., E12-C), as well as the inductive step proves " $P(k)$ and $P(k+1)$ " rather than " $P(k)$ implies $P(k+1)$ " (e.g., J1-C; J15-C).

E12-C: No: you have to check if it is true for $n=k+1$, otherwise you haven't proved it.

J1-C: We have to assume that antecedent is true, if not, the statement will always be false.

J15-C: [The presented proof is] True. It applies the equation that holds for the assumption $n=k$, then it deduces $n=k+1$.

At the third level, moreover, when concluding that a given statement holds for all natural numbers, the Principle of Mathematical Induction (PMI) is implicitly applied. Therefore, at the third level, it is also important to make the implicit status of PMI explicit. In our study, however, none of the participants clearly referred to PMI.

Necessity of encapsulation: A discussion

The above findings suggest that it is necessary to explore the status of PMI as well as the necessity of encapsulation for the holistic comprehension further. In the study, a considerable number of participants (18.4%; 7/38) stayed at local levels of comprehensions, relying on the procedural or sequential chaining of modus ponens in responding Christine and David's dialogue as follows:

E11-D: You assume it is true for k . Then prove that if it is true for k , then it is true for $k+1$. We check it is true for $n=1$, if it is, then it is also true for 2, so it is also true for 3, etc. \therefore true for all $n \in \mathbb{N}$

J19-D: Mathematical induction requires that at first we show that it holds for $n=1$, then we assume that it holds for $n=k$, then we show that it holds for $n=k+1$. Since we have already shown that it holds for $n=1$, it holds for $n=2$, and if it holds for $n=2$, then it holds for $n=3$, likewise, this proof method proceeds successively by using the truth of the predecessor.

It seems that their comprehension of MI has not yet encapsulated as a fully-fledged structural object. Figures 4 and 5 represent two different forms of modus ponens that are carried out in MI (cf., Dubinsky & Lewin, 1986; Movshovitz-Hadar, 1993); Figure 4 suggests a local or sequential view, and Figure 5 suggests a holistic or static view.

$ \begin{array}{c} P(1) \\ \hline P(1) \rightarrow P(2) \\ \hline P(2) \\ \hline P(2) \rightarrow P(3) \\ \hline P(3) \\ \vdots \\ \hline P(k) \\ \hline P(k) \rightarrow P(k+1) \\ \hline P(k+1) \\ \vdots \\ \hline \forall n, P(n) \end{array} $	$ \begin{array}{c} \left(\begin{array}{l} (i) \ P(1) \\ (ii) \ P(k) \rightarrow P(k+1) \end{array} \right) \\ \hline (i) \wedge (ii) \\ \hline \left(\begin{array}{l} (i) \ P(1) \\ (ii) \ P(k) \rightarrow P(k+1) \end{array} \right) \rightarrow \left(\begin{array}{c} \forall n, P(n) \end{array} \right) \\ \hline \forall n, P(n) \end{array} $
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Figure 4: Local form of modus ponens Figure 5: Holistic form of modus ponens

We think that Figure 4 can represent a specific character of the local comprehension of MI. As far as the proof by MI concerned, we use the term encapsulation to refer to the

progression or transition into holistic comprehension. The holistic level of comprehension of MI relies on viewing the proof method by MI as a whole, like Figure 5, where the status of PMI can be conceptualized as a more explicit object. We think that the transition from local to the holistic comprehension of MI requires encapsulation of an infinite chain of modus ponens as a whole, because “in studying the specific logical details of the proof, one can lose track of the big picture” (Mejia-Ramos et al., 2012, p. 11). In this study, we briefly mentioned the necessity of the encapsulation of MI, as a next step, it should be worthwhile examining in detail the status and process of the encapsulation to proceed to the holistic comprehension level regarding this proof method.

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PATHS OF JUSTIFICATION IN ISRAELI 7TH GRADE MATHEMATICS TEXTBOOKS

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This study examines the paths of justification offered in 7th grade Israeli textbooks. Analysis included the paths formed by instances of justification for 10 mathematical statements, in eight 7th grade Israeli textbooks. The findings suggest that the lengths of the paths of justification varied, for different statements in the same textbook, and for the same statement across textbooks. Many paths included both empirical and deductive types of justification. Three types of justification were prevalent – Experimental demonstration (in most paths), Deduction using a specific case (in paths of algebra statements) and Deduction using a general case (in paths of geometry statements). Experimental demonstration commonly preceded deductive type(s), and Deduction using a specific case usually preceded Deduction using a general case.

INTRODUCTION

Justifying is an important component of doing and learning mathematics. However, the extensive research on students' conceptions of proof and ways of justifying mathematical claims reveals students' difficulties in understanding the need for justification and in distinguishing between deductive and other types of justification (e.g., Harel & Sowder, 2007). Research suggests that the textbooks used in class considerably influence students' opportunities to learn mathematics in general (Haggarty & Pepin, 2002), and to justify in particular (Ayalon & Even, in press). Accordingly, the study of the opportunities to learn to justify offered in mathematics textbooks is increasing in recent years. This research focuses on (1) the justifications for mathematical statements presented in textbooks (e.g., Dolev, 2011; Stacey & Vincent, 2009), and (2) the opportunities for students to justify and explain their own mathematical work (e.g., Dolev & Even, 2013; Stylianides, 2009). This paper belongs to the first mentioned line of research. It examines the justifications to mathematical statements that are offered in Israeli 7th grade mathematics textbooks. The study is part of a larger research program that examines the opportunities to learn to justify mathematical statements offered in mathematics textbooks.

THEORETICAL BACKGROUND

Research shows that the justifications to mathematical statements presented in textbooks are of different kinds (e.g., Stacey & Vincent, 2009; Stylianides, 2009). Grounded in an analysis of Australian 8th grade textbook explanations, Stacey and Vincent (2009) identified seven types of textbook justifications. These types of justification are a refinement of Harel and Sowder's (2007) categories of proof schemes used by students: external, empirical, and deductive, documented in numerous studies of justification in school mathematics. Table 1 presents Stacey and Vincent's seven

types of textbook justifications of mathematical statements, grouped into Harel and Sowder's three categories.

Table 1. Types of justification (adapted from Stacey and Vincent, (2009)).

Type of justification	Description
<u>External</u>	
<i>Appeal to authority</i>	Reliance on external sources of authority.
<i>Qualitative analogy</i>	A surface similarity to non-mathematical situations.
<u>Empirical</u>	
<i>Experimental demonstration</i>	A pattern formed after checking specific examples.
<i>Concordance of a rule with a model</i>	Matching specific results of a rule and a model.
<u>Deductive</u>	
<i>Deduction using a model</i>	A model illustrating a mathematical structure.
<i>Deduction using a specific case</i>	An inference process by using a special case.
<i>Deduction using a general case</i>	An inference process by using a general case.

Using this framework, Stacey and Vincent (2009) analysed the justifications offered for seven mathematical statements in nine 8th grade Australian textbooks. They found that the textbooks employed several types of justification when justifying mathematical statements, and in some cases, textbooks justified a statement using more than one type of justification or one type more than once. Dolev (2011) used this framework to analyse the justifications offered for three mathematical statements in six 7th grade Israeli textbooks (experimental version), and obtained similar results.

This use of several justifications for one mathematical statement could serve a didactic goal of reinforcing and extending students' understanding – as the use of several justifications for one statement is likely to have an additive effect (Sierpinska, 1994). The finding that textbooks present more than one justification for one statement indicates that in addition to examining the types of justification used to justify mathematical statements, it is important to attend also to the “paths of justification”, i.e., to the ways justifications of one statement are arranged and structured – an aspect that receives little attention in the literature. This is the focus of our study. It examines the types and the paths of justification to key mathematical statements in Israeli 7th grade textbooks.

METHODOLOGY

Analysis included all eight approved Israeli 7th grade textbooks (and teacher guides) for Hebrew speakers. Six textbooks (labelled A-F) are of regular/extended scope, and two (labelled G-H) are of limited scope, written for students with low achievements.

Ten key mathematical statements were selected for analysis from the Israeli 7th grade mathematics national curriculum, five in algebra and five in geometry:

- The distributive property: $a(b + c) = ab + ac$ for any three numbers a , b , c .

- Division by zero is undefined.
- Manipulating algebraic expressions using properties of real numbers transforms expressions into equivalent expressions.
- The product of two negative numbers is a positive number.
- Applying operations to both sides of an equation yields an equivalent equation.
- The area of a trapezium with bases a , b and altitude h is $(a + b)h/2$.
- The area of a disc with radius r is πr^2 .
- Vertical angles are equal.
- Corresponding angles between parallel lines are equal.
- The angle sum of a triangle is 180° .

For each statement, data sources included the textbook chapters introducing it, in each textbook (3-49 pages per statement per textbook) – a total of 816 textbook pages. We analysed both the explanatory texts and the related task pools. We also analysed the related teacher guides to help interpret the justifications offered in the textbooks. In addition to the first author, about 70% of the data were analysed and discussed by several members of our research group (1-4), including the second author, until a consensus was achieved; the remaining 30% were analysed by the first author alone.

Analysis comprised four stages:

1. Identifying instances of justification for each statement, in each textbook. Figure 1 illustrates two instances of justification for the area formula for a trapezium, both in textbook B.
2. Coding the type of justification for each instance of justification (following Stacey & Vincent, 2009). For example, the instance of justification in Figure 1(a) was coded as *deduction using a specific case*, and the one in Figure 1(b) as *deduction using a general case*.
3. Constructing paths of justification for each statement, in each textbook (80 paths in total). Figure illustrates paths of justification for the area of a trapezium in two textbooks (B and F). Each step represents either a single instance of justification or the location of the mathematical statement in the path, in order of appearance in the textbook. Each instance of justification is labelled for its type of justification.
4. Comparative analysis of types and paths of justification, by textbook and by mathematical statement.

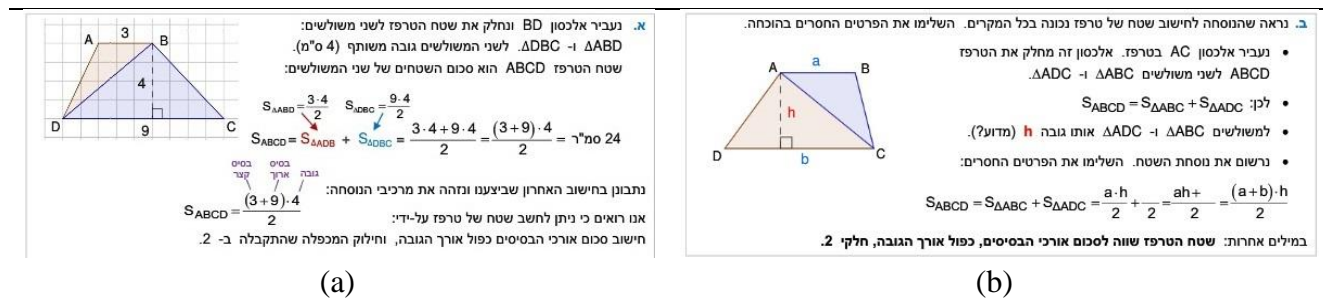


Figure 1. Area of a trapezium – two instances of justification in textbook B

Textbook B	$e \rightarrow s \rightarrow \text{Statement} \rightarrow g \rightarrow g \rightarrow g$
Textbook F	$e \rightarrow e \rightarrow e \rightarrow g \rightarrow g \rightarrow \text{Statement}$
e = experimental demonstration; s/g = deduction using a specific/general case.	

Figure 2. Area of a trapezium – two paths of justification (textbooks B and F)

FINDINGS

Analysis reveals that Israeli 7th grade mathematics textbooks provided justifications for all analysed statements (but one statement in one textbook). A total of 225 instances of justification were found for the ten analysed mathematical statements. They were typically included in the explanatory texts, yet several instances of justification were found in tasks intended for student individual or small-group work.

Six of the seven types of justification in Stacey and Vincent's framework (2009) were identified in the Israeli textbook justifications; all but *concordance of a rule with a model*. In the following we describe the types of justification found, first across textbooks and then across mathematical statements. Then we present initial findings regarding paths of justification.

Types of justification across textbooks

Table 2 presents the frequencies for the types of justification in instances of justification, by textbook. As can be seen, the total number of instances of justification was between 23-35 instances per textbook. The frequencies of most types of justification were similar across the textbooks, except for *Deduction using a specific case*, and *Deduction using a general case*, where there was a noticeable variation. Nevertheless, most of the instances of justification in each textbook were of three types: one empirical – *Experimental demonstration*, and two deductive – *Deduction using a specific case*, and *Deduction using a general case*. The latter two types accounted for about one-half of the instances of justification in seven of the textbooks and almost one-third in one (G). The two external types of justification – *Appeal to authority* and *Qualitative analogy* – were rare, accounting for less than 3% of all instances of justification.

Table 2. Frequencies of types of justification by textbook.

Type of justification	Textbook								Total (%)
	A	B	C	D	E	F	G	H	
<u>External</u>									
<i>Appeal to authority</i>	0	0	0	0	1	0	1	0	2 (1%)
<i>Qualitative analogy</i>	1	1	0	0	2	0	0	0	4 (2%)
<u>Empirical</u>									
<i>Experimental demonstration</i>	9	10	11	11	9	12	12	12	86 (38%)
<u>Deductive</u>									
<i>Deduction using a model</i>	8	3	3	2	3	3	3	3	28 (13%)
<i>Deduction using a specific case</i>	9	8	9	4	10	5	5	9	59 (26%)
<i>Deduction using a general case</i>	8	7	5	7	3	10	2	4	46 (20%)
Total	35	29	28	24	28	30	23	28	225 (100%)

Types of justification across mathematical statements

Table 3 presents the frequencies for the types of justification in instances of justification, by mathematical statement. As can be seen, there was a great variation in the number of instances of justification across the statements, between 9-38 instances per statement. There was also a great variation in the frequencies of all types of justification (but the rarely used ones) across the statements.

Table 3. Frequencies of types of justification by mathematical statement.

Type of Justification	Mathematical statement										Total
	Distributive law	Division by zero	Equivalent expressions	Product of negatives	Balancing equations	Area of trapezium	Area of disc	Vertical angles	Corresponding angles	Angle sum of triangle	
<u>External</u>											
<i>Appeal to authority</i>	0	0	0	1	0	0	1	0	0	0	2 (1%)
<i>Qualitative analogy</i>	0	0	1	2	1	0	0	0	0	0	4 (2%)
<u>Empirical</u>											
<i>Experiment. demonstration</i>	2	3	9	0	24	14	0	4	15	15	86 (38%)
<u>Deductive</u>											
<i>Deduction using a model</i>	13	1	6	2	6	0	0	0	0	0	28 (13%)
<i>Deduction/specific case</i>	0	8	17	12	1	13	0	5	1	2	59 (26%)
<i>Deduction/general case</i>	0	5	0	0	0	11	8	7	2	13	46 (20%)
Total	15	17	33	17	32	38	9	16	18	30	225 (100%)

Furthermore, the two deductive types of justification that were commonly used in all textbooks: *Deduction using a specific case*, and *Deduction using a general case* (see Table 2), were used to different extents in algebra and in geometry statements (see Table 3). *Deduction using a general case* was prevalent almost exclusively in geometry statements whereas *Deduction using a specific case* was common mostly in algebra statements (and in a geometry statement involving the use of algebra: the area formula of a trapezium), and so was *Deduction using a model*. In contrast, *Experimental demonstration*, which was used to a large extent in all textbooks (see Table 2), was used in both algebra and geometry statements.

Paths of justifications

Table 4 presents the paths of justification for each mathematical statement, by textbook. As can be seen, the lengths of the paths varied considerably, between one and seven instances of justification per path. Likewise, the lengths varied for different statements in the same textbook. For instance, in textbook A, the path of justification for the statement *The product of two negative numbers is a positive number* included five instances of justification, but only one for the statement *The area of a disc with radius r is πr^2* . Moreover, the lengths varied for the same statement across textbooks. For example, the paths of justification for the above mentioned statements in textbook F included one and two instances of justification (respectively).

Table 4: Paths of justification by textbook and statement.

Statement	Textbook							
	A	B	C	D	E	F	G	H
Distributive law	m,m	m,m	m	m,m	m,e	m,m	e,m	m
Division by zero	s,m	s	e,s,g	e,s,g	s,g	s,g	s	e,s,g
Equivalent expressions	s,s,m, s,e	m,e, s,s	m,e, s,s	e,e,s	q,e,m, s,s	e,s,s,s	m,e, s,s	m,e, s,s
Product of negatives	m,s,s, q,m	s,s,q	s,s	s,s	s,s	s	a	s
Balancing equations	e,m,e,m	e,e,s,e	e,e,e,e ,m	e,e	e,m,e, e,q,e	e,m,e	e,e,e	e,e,e, m,e
Area of a trapezium	e,g,s, s,s	e,s,g, g,g	e,s,s, s,g	e,e,e, g,g	s,s,s,e	e,e,e, g,g	e,e,g,e	e,s,s, s,g
Area of a disc	g	g	g	g	a	g,g	g	g
Opposite angles	e,e,g	s,g	s,g	g	s,g	e,g	e,s	s,g
Corresponding angles	e,g,e,g	e,e	e,e	e,e	e	e	e,e,e	s,e,e
Angle sum of a triangle	e,g,g,g	e,e,e, g,g	e,e,g	e,g,g	e,s,g	e,g,g,e e,g,g	e,s	e,e,e

a= appeal to authority; **q**= qualitative analogy; **e**= experimental demonstration; **m**= deduction using a model; **s**= deduction using a specific case; **g**= deduction using a general case.

As can be seen in Table 4, most paths of justification included more than one type of justification (64 out of 80 paths); often both empirical and deductive (39 paths). Focus on the most commonly used types of justification suggests that in almost all paths that included both *Experimental demonstration* and *Deduction using a specific case* or *Deduction using a general case*, *Experimental demonstration* preceded the deductive one(s) – 29 out of 32 paths. Similarly, in most paths that included both *Deduction using a specific case* and *Deduction using a general case*, the specific case preceded the general more formal mathematical justification – 13 out of 14 paths.

DISCUSSION

This study examined types and paths of justification offered to 10 key mathematical statements in eight Israeli 7th grade textbooks. Our findings show that almost all instances of justification in the textbooks (97%) were either deductive or empirical; types of justification that are considered desirable in school mathematics (Harel & Sowder, 2007; Stylianides, 2009). This finding is different from the results in Stacey and Vincent (2009), where 17% of the justifications for similar topics in Australian textbooks were neither deductive nor empirical.

Our findings also suggest that the most employed types of justification in all textbooks were three (out of the six types identified): *Experimental demonstration*, *Deduction using a specific case*, and *Deduction using a general case*. Together these types accounted for 84% of the instances of justification. Still, whereas *Experimental demonstration* was used in both algebra and geometry statements, this was not the case with the deductive types. *Deduction using a specific case* was mostly used in algebra statements, and *Deduction using a general case* was prevalent almost solely in geometry statements. Hence, the type of justification closest to a formal proof was used mainly in geometry statements. This might convey to students that proof is part of doing mathematics in the case of geometry but not in algebra, where one could use “softer” ways of justification. Still, *Deduction using a specific case* may allow students who are newcomers to algebra to experience an inference process with a lower risk of ‘getting lost’ in algebraic manipulations.

The similarities among the textbooks in using the aforementioned three types of justification did not imply identical paths of justification in different textbooks for the same mathematical statements. As shown in Table 4, there were cases where some textbooks offered long paths that included an assortment of types of justification – offering students a variety of opportunities that may have an additive effect (Sierpinska, 1994) – whereas other textbooks used rather short paths with limited types of justification for the same statement. This difference implies a great variety in students’ opportunities to learn to justify that were offered in different textbooks.

In spite of these differences, it appears that textbooks that used both empirical and deductive types of justification tended to offer the empirical before the deductive. Similarly, textbooks that used both deduction using specific and general cases, tended to offer the specific case before the more general deductive justification type. This

order appears to reflect a shared view among textbook authors about ways of learning to justify mathematical statements that indeed might be useful for helping students learn to justify. Yet, there is a need to examine whether deductive ways of justifying have unique status in mathematics.

Finally, we would like to emphasize that our study focused on 7th grade textbooks. As Thompson (2014) noted, the similarities and differences identified in this particular grade level among textbooks might change over a textbook series. Additional research is needed to characterize the paths of justification in textbooks intended for higher grades.

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HOW DOES AN ICT-COMPETENT MATHEMATICS TEACHER BENEFIT FROM AN ICT-INTEGRATIVE PROJECT?

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We investigate an ICT-competent mathematics teacher's potentials for professional development as she participates in a sixth-grade statistics project aimed at developing practices that integrate ICTs. This is a critical case study, partly because the teacher is not challenged by the proposed ICTs. We use a theoretical framework for classroom mathematical practices to conceptualise teachers' learning from a participatory perspective. On the one hand, the teacher realises a potential for a more dialogical approach to teaching. On the other hand, she appears to maintain her habits in relation to ICT-use. These contrary tendencies negatively influence the students' learning opportunities. We offer explanations for why the teacher seems to stick with her ICT-habits as well as suggestions for future research- and development projects.

It is generally acknowledged that the teacher plays a critical role in the integration of ICT in teaching. However, only a limited number of research studies have systematically examined teachers' appropriation of ICT into their classroom practices (Healy & Lagrage, 2010). We aim to contribute more insights by focusing on how an ICT-competent teacher develops through her participation in a large Teacher Professional Development (TPD) project in Denmark intended to enhance the integration of ICT in the major school subjects. The teacher, Ea, is not technologically challenged by the ICTs suggested in a sub-project on sixth-grade statistics "Youngsters and ICTs"; rather she perceives the proposals as insufficiently innovative. In this light we aim to explore her developments in relation to how she contributes to the implementations of the sub-project's intended classroom mathematical practices. Our hypothesis is that if in this critical case the teacher does not to some extent implement these practices, then the chances that less ICT-competent teachers will are poor. More precisely, we ask: What potentials for professional development does an ICT-competent teacher realise when participating in "Youngsters and ICTs"?

TEACHER LEARNING AS REGARDS ICT

Generally, research in mathematics education agrees that ICT offers potentials for students to develop fundamental mathematical understandings. After an initial period in the 80s of high optimism that ICT would transform teaching, researchers now regard the integration of ICT as a more complicated and prolonged process (Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010). This view is underpinned by a new OECD-report (2015) concluding that "PISA results show no appreciable improvements in student achievement in reading, mathematics or science in the countries that had invested heavily in ICT for education" (p.17). The report interprets the slight or non-existing correlation between students' learning and accessibility

to/use of computers at school as “we have not yet become good enough at the kind of pedagogies that make the most of technology” (p.3).

Research studies support the assertion that we need to know more about teachers’ appropriation of ICTs and about appropriate ways to bolster it (Grugeon, Lagrange, & Jarvis, 2010; Healy & Lagrange, 2010). Pierce and Stacey (2013) took a two-year Lesson Study TPD approach with *early majority* teachers (teachers with no specific interest in ICT but who accept the necessity of changing). They conclude that the teachers tended to absorb the new technology (mathematics analysis software) “into current practices, more than changing practice” (p.323). They further conclude that the didactical contract (the mutual obligations and expectations between teacher and students) was unchanged with respect to mathematics. Similarly, studies report how teachers use ICTs to absorb or accentuate certain pedagogical priorities in their practices. For example, Ruthven, Hennessy & Deane (2008) use the concept of “interpretative flexibility” to emphasise that teachers align and adopt the use of ICTs according to their own concerns and settings. They argue that interpretative flexibility can explain disparities in the ways teachers use ICT, as well as disparities between designers’ intentions with dynamic geometry and its use in pioneering studies, on the one hand, and its more mainstream use, on the other hand. They characterise this mainstream use as “a marginal amplifier of established practices” (p.315). Drijvers et al. (2010) investigate three teachers’ instrumental orchestrations in a project based on a digital algebraic learning environment with prescribed research-based activities and a teacher guide. They conclude that the teachers’ choice of orchestration is related to their views of mathematics education and the role of ICT herein, especially their conceptions of technological and time constraints as well as issues of control.

Some of the challenges that teachers face include the (too) rapid development of ICT and the complexities involved in learning the appropriate technological skills. On the other hand, ICTs seem to enhance the complexity of teachers’ practice as “practices used in ‘traditional’ settings can no longer be applied in a routine-like manner when technology is available” (Drijvers et al., 2010, p. 214). In keeping with this, Ruthven, Deane & Hennessy (2009) emphasise that using ICT in the classroom requires that key structuring features of classroom practices be adapted. These features include *the working environment* (room location, physical layout, class organisation), *resource system* (a coherent combination of a teacher’s range of tools, ICTs, textbooks, etc.), *activity format* (the framing of the activities and of interactions in the classroom), *curriculum script* (a teacher’s broad mental sketch for teaching a particular topic and a flexible enactment of the sketch) and *time economy* (the time cost of using ICT versus the students’ expected outcome). These two sets of challenges might seem insurmountable for a teacher. Thus, the elaboration of teaching practices that integrate ICT ought to be a shared responsibility among teachers, curriculum developers, teacher educators and researchers, and not a task more or less solely entrusted to teachers.

We use a different theoretical framework than normal in research on ICT. We adopt a participatory perspective on teachers’ learning by using Cobb’s concept of classroom

mathematical practice (2002). This concept is part of a larger theoretical framework that makes an overall division into a social and a psychological perspective. A classroom mathematical practice is established when a procedure or way of doing mathematics is generally accepted in the classroom and is treated as a self-evident mathematical fact. By participating in a practice, students develop idiosyncratic mathematics interpretations while also contributing to the development of the practice at the classroom level. In this way the relationship between practices (the social perspective) and the participating students' interpretations (the psychological perspective) is reflexive (Cobb, 2001). Cobb focuses on how students contribute to practice formation, while we are interested in how teachers initiate, negotiate and guide the establishment of practices. This seems possible within the framework, as the teacher has a special authority to initiate, guide and re-negotiate the establishment of practices regardless of how students contribute to shaping them. We consider changes in a teacher's way of participating in classroom mathematical interactions as a sign of learning. A classroom mathematical practice is characterised by three normative aspects: a normative purpose, normative standards of argumentation and normative ways of reasoning (Cobb, 2002). This research report focuses on the first aspect. By teaching practices we mean actions the teacher takes to support the development of classroom mathematical practices.

“YOUNGSTERS AND ITCs”

The overarching project determines to a great extent the design and assumptions of “Youngsters and ICTs”. To characterise the sub-project, we use a framework developed by Grugeon, Lagrange and Jarvis (2010). The framework consists of three different views: 1) views of the implementation of technology in the classroom, 2) views of changes in the teacher's role, activity and practices, and 3) views of teacher preparation. The first view subdivides into views regarding the contribution of ICT on one axis and modes of supported use on the other axis. To some extent “Youngsters and ICTs” inherits the one extreme of the contribution axis from the overarching project, namely the view that ICTs will necessarily improve learning if teachers implement “Youngsters and ICTs” as prescribed. However, the other extreme view is also present as the researchers are concerned more about the complexity of the suggested teaching practices that integrate ICT and less about the complexity of the ICT itself. The suggested practices are designed in keeping with the reform orientation of mathematics teaching as conveyed by the NCTM (2000), while an inherited design principle was to use well-known or easily accessible ICTs rather than new. These two design principles also constitute our reasons for not adding an artefact perspective to Cobb's framework. Regarding the second axis (mode of use), “Youngsters and ICTs” prepares teachers for classroom use of ICT (and not for communication use) by offering a comprehensive, research-based course consisting of 15 lessons with detailed descriptions of the working environment, resource systems, activity format and part of the curriculum script. The course also includes elaborated classroom cases from teaching in test classes and provides outside support from teacher educators. In relation

to the second view (changes in the role of the teacher, etc.), we place “Youngsters and ICTs” in a “new role for the teacher”, as it focuses on developing new teaching practices, not on integrating new ICT-based activities. As regards the third view, “Youngsters and ICTs” is placed in the short-term end of the time duration axis and in the middle of the professional proficiency axis, as it has suggestions for both statistical content and teacher practices.

When designing “Youngsters and ICTs”, we were inspired by Cobb and McClain’s design-based research, especially the five design principles shown to be critical to students’ development of statistical reasoning (2004). These principles urge a focus on central statistical ideas, the instructional activities, the classroom activity structure, the computer-based tools used by the students and the classroom discourse.

“Youngsters and ITCs” aims to get teachers to use ICTs to initiate, negotiate and establish two overall classroom mathematical practices: to be critical towards the use of statistics and to investigate tendencies and patterns in data sets. To realise these aims, the course frames and prescribes the way that teachers engage students in statistical investigations: Formulate statistical problems; generate, analyse and reason about data; interpret results and disseminate them in/out of the school. The proposed teaching practices integrate, for instance, the use of spreadsheets and MiniTools (ibid) to support students’ data analysis and reasoning processes, electronic surveys to aid data generation processes and Explain Everything (app) to support students’ reasoning and interpretation. One central and general teaching practice is to include and capitalise on student’s mathematical contributions in classroom discussions.

METHODOLOGICAL APPROACH

Ea’s participation in “Youngsters and ICTs” is a critical case study (Flyvbjerg, 2006). Such a case can be identified as either “most likely” or “least likely” cases (p. 231). Our case is “most likely” in the sense that if Ea does not establish the intended classroom mathematical practices, the chances seem poor that less ICT-competent teachers might establish them. The case is identified on the basis of Ea’s eager participation in “Youngsters and ICTs”, her engaged ICT narratives and preoccupation with its potentials.

Over a two-year period we observed 31 classroom lessons (16 from the sub-project and 15 from before or after it (video recorded)) and conducted four semi-structured interviews (audio recorded). We transcribed the interviews and selected excerpts from the observations, corresponding to 15 lessons. Inspired by grounded theory (Charmaz, 2014), we coded the transcriptions (in *Nvivo*) word-by-word and line-by-line. Out of 28 developed coding categories, we selected 10, such as “Learning mathematics with ICT”, “Communication” and “Classroom organisation”, as a basis for our analysis.

The analysis produced a teacher profile and a case in which Ea teaches the first lesson of “Youngsters and ICTs” during her second implementation of the course. The main aim of the profile is to make Ea’s daily teaching practices and priorities visible, thus enabling us to consider the case as having development potential. The case is

representative of Ea's general approach to teaching with ICT within the sub-project and illustrates a potential for professional development, which Ea seems to realise.

EA'S PROFILE AS AN ICT-COMPETENT MATHEMATICS TEACHER

Ea is an experienced mathematics teacher at a large school with a strong ICT profile. Ea says of the school: "They like that we're ahead of it ... it's a prestige project." Ea is an *early adaptor* (Sahin, 2006), as she integrates ICT in her classroom, is selected by the school management to support her colleagues in integrating ITC and participates in ICT-development and research projects. She recounts her experiences from such projects: "I have a naive belief that someone can inspire me and tell me what they do and how... but no-one does." Ea considers "Youngsters and ICTs" as insufficiently innovative: "I do not think there has been enough ICT ... but mathematically it's another way of teaching than I have taught in the past."

Ea's classroom has an interactive whiteboard, and all students obtained iPads last year. Ea's incorporation of ICT into her daily teaching has only superficially changed the working environment, the activity format and her curriculum script. Generally, she only gives short classroom directions followed by individual or small group work with the textbook or iPad, partly because "I fail to tell them anything in twenty minutes ... I think I often supervise more than I teach ... because I do not think they listen, when I stand at the blackboard". Thus, she rarely assembles the whole class and seldom for joint mathematical activities. She primarily uses an e-exercise base for skill practice and a digital platform as a framework for student homework. She describes her teaching with ICTs as: "So that part [the homework] is different ... it is not my teaching that has changed ... still one-man work ... it has really not changed."

Ea conceives teaching mathematical skills as a prerequisite to working with problem solving: "I am a bit old-fashioned. I think it is most important they have skills... I can pose open problems, but if they do not have the skills, then they cannot solve them." Ea's curriculum script is dominated by skills practice and individual or small group work framed by the textbook or iPad exercises. Interpretative flexibility can explain her daily use of ICTs primarily to support skills practice, to structure student work and to control/check it.

CASE OF EA'S TEACHING IN "YOUNGSTERS AND ICTs"

By and large, Ea follows the proposals in the plan as regards the working environment and the activity format. Thus, she has produced a flipped-classroom video introducing the course by using the suggested text. She initiates the lesson by showing the video even though this was homework. She then introduces the course in her own words, emphasising the end product (a newscast about young people's use of ICTs) and what the students are going to do. This is mainly to answer a range of questions, for instance one of the course's principal ones: "Is your own use of ICTs too high?" She says nothing about the statistical content nor establishes an inquiry-based frame. The object appears to be answering questions more or less mechanically. Ea cancels her planned small group work, as only one student has done the homework; instead, she initiates

and maintains a long classroom discussion (20 min) about the students' experience with everyday use of statistics. She structures the discussion by dividing the blackboard into two columns headed "How" and "Where", under which she writes the students' contributions. Generally, Ea accepts the contributions unconditionally without inquiring into associated societal issues, questioning the data generation process or unfolding the emergent statistical potentials. Thus, the classroom discussion degenerates into one of enumerating arbitrary examples. Ea then initiates a classroom dialogue (2 min) about related statistical concepts and methods, in which the students' contributions stand unquestioned and unrelated to the long list of newly produced examples. The emergent purpose of the dialogue also becomes one of enumeration, this time of statistical concepts. Subsequently, the students are to write the examples and the few mentioned concepts into their own digital concept map. Ea states "you have to write it all down" as the purpose of the activity. For nearly half of the activity time (25 min), Ea solves technical problems. As the students finish up, they are told to work with an e-exercise base individually (10 min). There is no common closing. There is a friendly, pleasant atmosphere with good relations between the teacher and the students. Most of the students are active during the lesson and participate eagerly in the discussions.

REALISATION OF POTENTIALS FOR DEVELOPMENT

In the case Ea challenges her normal priorities as regards the communicative aspects of the working environment and activity format, as she initiates and maintains two classroom discussions related to the statistical content. Thus, she prioritises classroom discussion over individual and small group activities and the subject matter over directional information. In interviews, Ea discloses that she does not feel competent in her verbal communication about mathematics, but she emphasises this very aspect as having been valuable to her professional development as well as her students' learning. Especially in this light, Ea's changed participation in classroom interactions as regards her initiative to discuss content-related themes in the class is noteworthy.

An analysis of the classroom discussions confirms that it is not simple to change one's teaching practices. Regardless of Ea's presumed intentions, the emergent purposes of the discussions become superficial enumerations that do not contribute to the students' development of statistical reasoning or understanding. These emergent purposes are in keeping with the norms, routines and conception of mathematical activities generally established in the classroom. As indicated in the case, there are no norms requiring the teacher or the students to explain or argue for the proposed examples or concepts. It also appears that the overall purpose of mathematical activity is to answer closed questions. That is questions with one answer and one way to find this answer.

Like the teachers in Drijvers et al's study (2010), Ea ignores to a large extent the ICT related proposals in "Youngsters and ICTs". The intention of the case lesson is for the teacher to use a digital concept map to initiate, negotiate and guide the establishment of a mathematical practice: to meaningfully relate everyday statistical examples to

statistical concepts and methods. By jointly constructing a digital concept map the teacher and students were supposed to investigate and negotiate connections and relations between examples and statistical concepts/methods. Ea refrains from this opportunity of classroom investigation by deciding that each student makes a concept map on their own iPad. Ea further reduces the students' learning opportunities by requiring a digital re-writing of the examples from the black board, and not requesting the construction of a genuine map. On the one hand, Ea seems to initiate the intended classroom mathematical practice, but then her decisions, regarding ICT-use in particular, prevent her from maintaining the practice, and the activity degenerates into a mechanical re-writing of insignificant examples.

We thus conclude that Ea realises opportunities for professional development with regard to dialogical aspects of the working environment and the activity format. In relation to the resource systems, curriculum script and the ICT-part of the activity format, it appears that Ea retains her habits, thus confirming the tendency to absorb ICT into existing teaching practices (Pierce & Stacey, 2013). This is notably, as she is participating in a TPD-project focusing on changing such practices.

CONCLUSION AND DISCUSSION

Our study's overall results are that the teacher to some extent realises a developmental potential for a more dialogical teaching approach, while she does not realise potentials regarding ICT-use. On the contrary, she maintains her usual ICT-habits. Together these two tendencies negatively influence the students' learning opportunities.

The teacher's maintenance of her habitual ICT-use can partly be traced to her conception of mathematics education and the role of ICT herein (Drijvers et al., 2010). Her skill-based conceptions can be seen as orienting her contributions to classroom interactions in a product-oriented way, focusing on readymade procedures and facts. However, our study suggests a further explanation concerning the conception of her as a highly ICT-competent mathematics teacher, a conception shared by the school management, her colleagues and herself. This constructed image apparently legitimises and promotes her maintenance of ICT-habits, thus preventing the intended developments. As such our initial hypothesis is too limited. The complexities involved in developing teachers' ICT-use seem far more exhaustive. Firstly, teachers' formations of professional identities appear to play a significant role, which point to a need to understand and research teachers' appropriation of ICT from a participatory perspective of learning. Secondly, our study shows that to develop an ICT-competent teachers' further appropriation might entails more than developing comprehensive, research-based teaching material with suggestions for teaching practices that integrate ICT and providing short-term outside support to help teachers develop these practices. Presumably, a more fruitful way would be to (co-)develop such practices in a long-term, onsite collaboration between teachers and teacher educators/researchers.

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PROOF VALIDATION ASPECTS AND COGNITIVE STUDENT PREREQUISITES IN UNDERGRADUATE MATHEMATICS

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Proof validation is an important skill to acquire for university students and is also essential as a monitoring activity during proof construction. Our study analyzes students' difficulties with three aspects of proof validation as well as the influence of domain-specific and domain-general cognitive student prerequisites (CSP) on proof validation skills. Results indicate that students' proof validation skills depend on the type of error in a purported proof and are influenced by conceptual mathematical knowledge and metacognitive awareness. Overall domain-general and -specific CSPs affect performance to roughly the same degree, whereas generative CSPs like problem solving skills have no contribution. These results question the current way of teaching the concept of proof primarily by proof construction exercises.

INTRODUCTION

Proof construction has been a focus of university mathematics for a long time and constitutes a research focus within mathematics education. Yet, students often get in touch with proofs in other ways, e.g. they engage in proof comprehension when reading textbook proofs or in proof validation when reading and judging the correctness of their own or other students' proofs or potentially erroneous lecture notes. Mastering these activities per se is essential for students, but is even more so since proof validation, i.e. the ability to evaluate individual arguments and entire proofs, is a crucial monitoring activity while constructing proofs (Selden & Selden, 2003). Apart from proof construction, research thus increasingly focuses on proof comprehension and proof validation (Healy & Hoyles, 2000; Inglis & Alcock, 2012; Selden & Selden, 2003; Weber & Mejía-Ramos, 2011).

Prior research revealed that even university students have severe problems in validating proofs (Selden & Selden, 2003). Gaining effective means of fostering students' proof validation skills therefore is of utmost importance. A prerequisite to design instruction that is effective for the acquisition of proof validation skills at the university level is a better understanding of different aspects of proof validation skills and of their relation to cognitive student prerequisites (CSP). The present study therefore explores students' proof validation skills in two ways: We analyze which types of errors in purported proofs are easy to detect for students and which pose difficulties. In addition, we explore the dependency of proof validation skills on various domain-specific and domain-general CSPs.

PROOF VALIDATION

“Reading” proofs comprises three main activities, each having different goals (Selden & Selden, 2015). *Proof comprehension* is the activity of reading a proof (e.g. when studying a textbook proof) that is known to be true with the aim of understanding it. *Proof validation* refers to reading a proof and trying to judge its correctness. The third related activity is *proof evaluation*, which is not only aimed at assessing the correctness of a proof, but also at evaluating the proof regarding multiple other properties, e.g. its clarity or convincingness.

Amongst these three skills, proof validation is closest related to mathematical proof construction skills, because validation is essential during proof construction for checking individual inferences as well as the overall structure and conclusiveness of a constructed proof. Due to this status as a monitoring activity, similar activities can be found in many domain-general frameworks for argumentation or problem solving, e.g. as *evidence evaluation* and *drawing conclusions* (Fischer et al., 2014) or *looking back* (Polya, 1945), or in modern self-regulation frameworks (De Corte et al., 2011).

Recent studies underline the importance of proof validation and unveiled clear differences in proof validation behavior, e.g. between experts and novices (Inglis & Alcock, 2012; Weber & Mejía-Ramos, 2011). While novices tend to focus on surface features of proofs and individual inferences (*zooming in*), experts rather focus on the high-level structure (*zooming out*) and skim proofs to grasp the overall structure before zooming in and looking at details.

Individual prerequisites of proof validation

Proof validation requires different knowledge facets and skills; e.g., judging the correctness of a proof is hardly possible without a sufficient mathematical knowledge base. Proof validation can therefore be seen as a complex cognitive skill that depends on several CSPs that can roughly be divided into two parts: domain-specific and domain-general prerequisites (c.f. Figure 1). Prior research on proof construction (Chinnappan, Ekanayake, & Brown, 2011; Schoenfeld, 1985) indicates that both could influence students’ proof validation skills, but their relative impact is still unclear. Yet, in contrast to proof construction, which requires the generation of own, multi-step arguments, we view proof validation as a non-generative, more evaluative activity so that the impact of rather generative CSPs is questionable.

On the domain-specific side, it is assumed that students’ conceptual and procedural mathematical knowledge base as well as mathematical strategic knowledge (Weber, 2001) impact proof construction skills. Transferring this to the non-generative activity of proof validation, at least the influence of procedural knowledge is questionable, but also the approach strategies for mathematical proofs encoded in mathematical strategic knowledge might not be relevant. On the domain-general side, various constructs, including problem solving skills (Chinnappan et al., 2011) and general inferential reasoning skills, likely influence proof construction. Again, the influence of students’ problem solving skills on the non-generative proof validation is questionable. Finally,

prior research (e.g. Yang, 2012) suggests an influence of metacognitive awareness on students' proof validation skills since proof validation can be seen as cognitively demanding, requiring students to reflect the given proof and their proof validation process on various levels.

Apart from the individual relations to proof validation skills, knowledge of the contribution of domain-specific vs. domain-general CSPs can be utilized to effectively foster proof validation skills: E.g., a high impact of domain-general prerequisites would support the inclusion of instructional support for more general skills, while a low impact would support trainings mostly focusing on conceptual knowledge of the corresponding proof content.

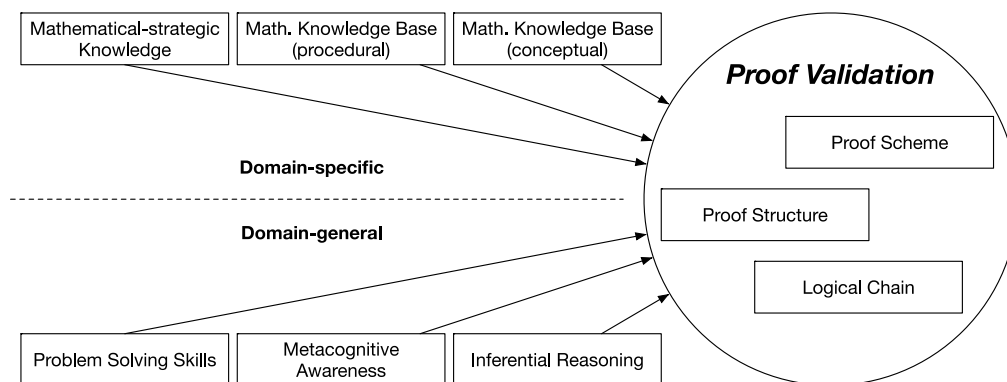


Figure 1: Conceptual framework of CSPs and aspects of *proof validation skills*

Aspects of proof validation

Proof validation is concerned with finding errors in purported proofs. Yet mathematical proofs can contain different kinds of errors (e.g. unsupported inferences, wrong use of definitions or cyclic argumentation) that refer to different aspects of a proof and that are not equally easy to detect (Healy & Hoyles, 2000). Accordingly, these different aspects of proof can be used to examine proof validation skills more closely, i.e. to differentiate between students' proof validation skills to detect specific types of errors. Heinze and Reiss (2003) put forward three aspects of *methodological knowledge* that are inherent to every proof and can be used to structure the different kinds of errors: *Proof scheme* refers to the kinds of reasoning used within each argument of a proof, e.g. inductive and deductive inferences or reference to an authority. *Proof structure* refers to the overall argumentative, logical structure of a proof. For a linear, direct proof, this structure should begin with the given premises and end with the statement that has to be shown. Finally, *logical chain* focuses on individual inferences within a proof. In order to obtain a correct logical chain, the premises for each step have to be proven beforehand or be part of the theoretical basis and no unknown or unproven statements may be used.

AIM AND RESEARCH QUESTIONS

The goal of this study was to identify university students' skills and problems in proof validation regarding different aspects of proofs and to explore the impact of domain-

specific and domain-general CSPs on the proof validation skills. We focused on the following questions:

1. Are there differences in students' proof validation skills regarding the detection of the three different errors types? Are students able to relate the reasoning for their judgment to the (in-)correct aspects of the purported proofs?
2. What is the influence of students' domain-general and domain-specific cognitive prerequisites on their proof validation skills?
3. Does this influence depend on the error type contained in a purported proof?

SAMPLE AND METHOD

66 mathematics university students (24 male, 41 female, 1 NA; $M_{\text{age}} = 21.19$) who had finished their first semester participated in the study, which is part of a larger investigation aimed at fostering students' mathematical proof skills.

To examine students' proof validation skills, they were asked to judge the correctness of four purported proofs of one proposition (closed format) and to explain their decision (open format). To assure an unbiased validation, they were told that fellow students had created the proofs. Students were given liberal but limited time to think about the purported proofs. The proposition was taken from elementary number theory to ensure that a potential lack of advanced mathematical knowledge did not hinder the proof validation. Two days later the students were asked to judge four purported proofs of another proposition in order to validate the initial findings.

Proposition

The product of three arbitrary consecutive integers is divisible by 3.

Martin's proof

Let $a \in \mathbb{Z}$ be an arbitrary integer. Accordingly the two consecutive integers can be written as $a + 1$ and $a + 2$. We are interested in the product

$$a \cdot (a + 1) \cdot (a + 2)$$

Expanding the term yields:

$$a \cdot (a + 1) \cdot (a + 2) = a \cdot (a^2 + 3a + 2) = a^3 + 3a^2 + 2a$$

Since the proposition should hold for an arbitrary integer a , the statement has to hold independently of a . We therefore only look at the coefficients of the term $a^3 + 3a^2 + 2a$. For the sum of these coefficients we get:

$$1 + 3 + 2 = 6$$

Accordingly since $3|6$ holds, it also holds that $3|a^3 + 3a^2 + 2a$ resp. $3|a \cdot (a + 1) \cdot (a + 2)$. Thus the product of three arbitrary consecutive integers is divisible by 3 and we have proven the proposition.

Figure 2: Proposition 1 and the purported proof with an error in the *logical chain*

For both propositions, one proof was correct and each of the three other proofs contained an error corresponding to one of the three error types. A translated version of proposition 1 and the purported proof containing an error in the logical chain are shown in Figure 2. Obviously the conclusion that $a^3 + 3a^2 + 2a$ is divisible by 3 independently of a when the sum of the coefficients $1 + 3 + 2$ is divisible by 3 is both wrong and unwarranted. Nevertheless, the step is deductive in nature since Martin seems to refer to some general rule for this (without stating it explicitly).

To measure their CSPs, students were given paper and pencil tests measuring their mathematical knowledge base (conceptual and procedural), mathematical strategic knowledge, inferential reasoning skills (Inglis & Simpson, 2008), metacognitive awareness (Schraw & Dennison, 1994) and problem solving skills (four non-mathematical problem solving tasks). The tests used closed and open items. Two raters coded the open items following theory-based coding schemes. The interrater reliability was $\kappa > .76$ ($\kappa_{\text{Mean}} = .93$; $SD = .09$). All scales had an acceptable internal consistency with $\alpha_{\text{Mean}} = .70$ ($SD = .10$), only the internal consistency for mathematical strategic knowledge was a bit low with $\alpha = .58$ (4 items).

RESULTS

With a total of 59.1 % correct answers, students' overall performance in judging the correctness of proofs was moderate, yet significantly greater than chance ($t(259) = 3.29$, $p < 0.001$). Comparing the different purported proofs (correct proof and proof with errors in the proof scheme, proof structure or logical chain) a Conchran's Q test determined significant ($\chi^2(3) = 70.97$, $p < .001$) differences between the solution rates (cf. Figure 3, left; dark-grey). Pairwise comparisons between the four purported proofs with a Bonferroni correction yielded significant ($p < .05$) differences for all comparisons except for correct proof vs. proof scheme.

Students were quite accurate in judging the correct proof as correct (81.8 %) and the purported proof with an inductive proof scheme as wrong (86.4 %). On the other hand, students performed about chance on the proof containing an error in the logical chain (45.5 %) and significantly below chance ($t(64) = -5.54$, $p < 0.001$) in the proof containing an error in the proof structure (22.7%). The delayed test with proposition 2 showed similar patterns regarding students' judgments (cf. Figure 3, left; light-grey) as well as for the Conchran's Q test ($\chi^2(3) = 28.88$, $p < .001$).

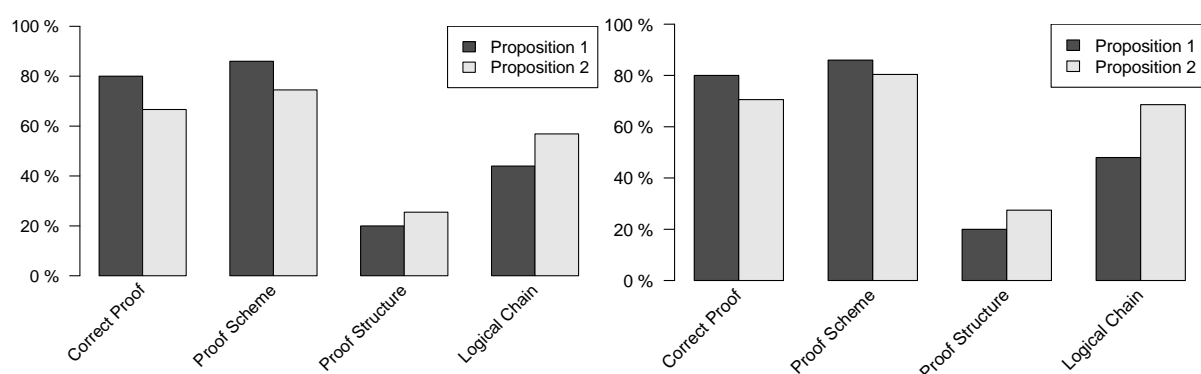


Figure 3: Solution rates without (left) and with consideration of explanations (right)

Students' explanations for their decisions revealed that some students marked faulty proofs as correct, although they had spotted the error or weakness of the proof. A typical statement was "Except for the part with 'only looking at the coefficients' the proof seems to be correct". Counting all answers as correct that showed that the error

had been detected yielded the same result patterns for both propositions although numbers change slightly (cf. Figure 3, right).

The number of students stating an explanation for their decisions varies widely between the purported proofs (43.9 % correct proof, 74.2 % proof scheme, 25.8 % proof structure, 66.7 % logical chain). For the proofs with an error in the proof scheme students were best in finding the correct reason for judging the proof as wrong (75.5 % of the given explanations), for proof structure the worst (29.4 % of the given explanations). The same pattern was observed for proposition 2.

We employed a generalized linear mixed-effects model (GLMM) analysis using the lme4 package (Bates, Mächler, Bolker, & Walker, 2015) to analyze the influence of the CSPs on proof validation. The model includes all six CSPs as well as the aspects of proof validation as fixed effects and the participant as a random effect. The model explained 36.8 % of the variance in students' proof validation performance by the CSPs and the aspects of proof validation. The model shows that, compared to identifying a correct proof, it is much harder to identify errors in proof structure ($b = -2.91, p < .001$) and logical chain ($b = -1.52, p < .001$) but easier to identify errors in the proof scheme ($b = 0.35, p > .05$). Of the CSPs, only the conceptual mathematical knowledge base and metacognitive awareness showed significant relations (c.f. Figure 4; stand. regression weights $\beta = 0.39$ and $\beta = 0.33$ respectively) to students' proof validation skills. Employing the GLMM on data from both propositions yields similar results.

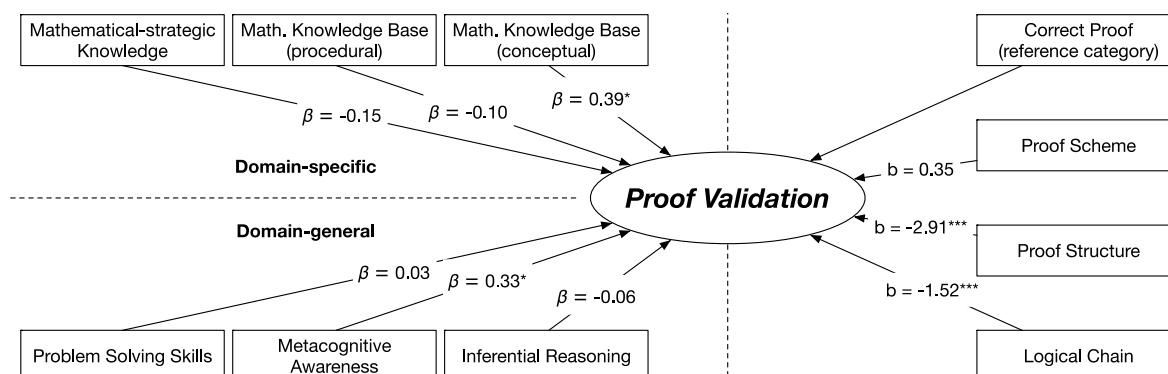


Figure 4: GLMM of CSPs and aspects of proof validation skills

For proposition 1, an analysis of interaction effects revealed that students' conceptual knowledge supports the detection of a wrong proof structure ($p < .05$) more than the identification of the other proofs. On the other hand, conceptual knowledge shows a weaker connection to identifying wrong proof schemes as compared to identifying the other proofs ($p < .05$). Finally, the impact of metacognitive awareness is stronger for detecting errors in the proof scheme as compared to evaluating the other proofs ($p < .05$). Again, the data from the second proposition showed a similar pattern.

DISCUSSION

The results of our study focusing on the aspects of proof validation skills reveal clear differences in students' performance. Students have little problems identifying correct

proofs and refusing inductive proof schemes, but perform poor when confronted with other error types. The low success rate in finding errors in the overall logical structure of the proof can be seen as additional evidence for the results that mathematics students often focus too narrowly on individual inferences (*zooming in*) (Mejía-Ramos & Weber, 2014; Weber & Mejía-Ramos, 2011). Yet, students were also not excellent at finding errors in the *logical chain* that refers to the individual inferences and the *zooming in*. The analysis of students' explanations adds to this: How come students mark proofs as correct although they were able to identify errors? And why do only few students give reasons for their judgments although they were explicitly prompted? One answer might be problems in understanding the proofs and giving suitable reasons. Alternatively, the wrong judgments despite finding the errors could also be a side effect of good proof constructions skills and students' insight that the proof could be tackled with a similar argument. Further evidence on students' thoughts and views is needed here, e.g. from interview or think-aloud studies.

So far, the overall results on the aspects of proof validation resemble those from prior research, e.g. on secondary students in the area of geometry, indicating some generalizability of these results over content area and age groups. Yet a replication with propositions from multiple content areas would be beneficial to further assure the validity and generalizability of the results.

The results regarding the influence of CSPs on proof validation skills indicate complex relations. Both the domain-specific as well as domain-general CSPs showed significant relations to students' proof validation skills of similar magnitude. Therefore, domain-general interventions, e.g. for metacognitive awareness, could have positive effects on students' proof validation skills and, vice versa, interventions on proof validation skills might be expected to transfer to skills in other domains to a certain extent. Yet, the results from the GLMMs show, that according interventions have to be created carefully: Amongst the CSPs, neither the mathematical-strategic knowledge nor the prerequisites referring to generative activities (procedural mathematical knowledge and problem solving) show a significant relation to proof validation skills. This missing relation of generative activities to proof validation is plausible. Beyond that, it indicates that proof validation might offer a better entry into the learning of proof than proof construction activities, since proof validation seems to be dependent on fewer prerequisites, in particular generative skills. This would be an alternative to the current university teaching style of mathematical proof, which is often mostly based on proof construction. Although a potential impact of proof validation on proof construction was not studied here, there are first results showing a significant connection of proof validation and construction skills. Thinking of proof validation as one prerequisite for proof construction also warrants such an approach. Still, more research, in particular from intervention studies would be required to support this strategy.

Overall, our approach using CSPs and aspects of proof validation was applied successfully and yielded several interesting implications for research on as well as for the teaching of mathematical proof. The fact, that proof validation seems to depend

less on some of the CSPs than proof construction (e.g. problem solving) and that proof validation skills are needed for proof construction underlines the idea of “validation before construction”.

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POETIC STRUCTURES AS RESOURCES FOR PROBLEM-SOLVING

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Speakers engaged in conversation typically repeat and modify earlier comments. These repetitions, or poetic structures, are commonplace in mathematical conversations, too. A close analysis of 90 turns of an algebraic problem-solving conversation suggests that poetic structures significantly facilitate the discovery of mathematical relationships. I identify eight types of poetic structures that appear to act as language resources for learning mathematics.

INTRODUCTION

Speakers of all languages repeat each other. Because repetition is pervasive in daily speech (Du Bois, 2014), it may contribute to mathematical problem-solving conversations. This paper highlights ways in which particular types of repetition—poetic structures—facilitate students' mathematical learning. Poetic structures occur when speakers repeat the grammatical structures of phrases spoken before, perhaps changing words or small aspects of grammar.

This close analysis of poetic structures over 90 conversational turns of an algebraic problem-solving session seeks to contribute to research on language as a resource for mathematical learning. This research largely grew from studies of multilingual classrooms (e.g. Barwell, 2015; Planas & Setati-Phakeng, 2014), and is concerned with issues such as code-switching, the influences of educational policy on classroom communicative practice, and language as a resource in formal vs. informal mathematics discourse. Because repetition is so commonplace, its analysis can deepen our understanding of language as a resource for learning in multilingual classrooms, or in monolingual classrooms in any language.

THEORETICAL FOUNDATION

Dialogic syntax, an emerging research focus in linguistics, forms the theoretical foundation for this analysis (Du Bois, 2014; Sakita, 2006). Dialogic syntax recognizes that as speakers repeat prior statements—their own or those of others—they reproduce syntactic arrangements that create meaningful relationships across sentences and across speakers. Hearers decode and respond to the meanings that are created at these structural levels beyond the sentence. For example, in the hexagon task described in this paper, Sheila's *minus 2...times 2* is recast in Joseph's clarifying question:

- 78 S: So number of hexagons would be 4 times 6 minus n minus 2. So 4 times 6 would be 24. Number of hexagons would be 1, 2, 3, 4. 4, uh, times 2.
- 79 J: Times 2 or minus 2?

The verbs *minus* and *times* shift within and across the speakers' comments, while retaining the direct object of 2. Dialogic syntax proposes this coordination as a “new, higher order linguistic structure...the coupled components recontextualize each other, generating new affordances for meaning” (Du Bois, 2014, p. 360).

Du Bois (2014) provides a useful review of the theoretical antecedents of dialogic syntax, which draw from a wide range of fields, including linguistics, anthropology, literary theory, and cognitive science. He identifies four foundational themes, some of which resonate with prior research in mathematics education. The first theme, parallelism, refers to the concrete repetitions within nearby utterances. In the example above, Joseph's *minus* 2 is parallel to his *times* 2, and both are repetitions of the endings of Sheila's sentences. Staats (2007) highlights ways in which these parallel, poetic structures can express both inductive and deductive mathematical reasoning.

Underlying grammatical parallelism is the principal of indexicality, or the capacity of language to refer to or point to other words and to elements of the situational context. Indexical words like *this*, *that*, and variable names like n have been associated with mathematical activities such as generalization and collaborative learning (Barwell, 2014; Radford, 2003). Parallelism occurs when units larger than a word—*times* 2—point to corresponding units like *minus* 2, creating bundles of indexicality.

Du Bois' second theme, analogy, refers to the meanings created through manipulation of similar units. For example, *times* and *minus* are alternatives within the frame of mathematical operations. The third theme, priming, is the experimentally-measured tendency to repeat lexical or syntactic units.

The fourth theme, dialogicality, has received slightly more attention in mathematics education research. Barwell (2015) following Bakhtin (1981), discusses three orientations of dialogicality: multivoicedness, multidiscursivity, and linguistic diversity. The first of these, multivoicedness, recognizes that all speech has a history. Speakers recast words and meanings from their past interactions each time they talk. This paper provides a detailed analysis of multivoicedness in a mathematical problem-solving session. Overall, then, dialogic syntax is a new framework for mathematical education research, but through its interdisciplinary character, it shares theoretical antecedents with research on language as a resource for mathematical learning.

PARTICIPANTS AND TASK

Sheila and Joseph are undergraduate students who had recently completed a university class in precalculus. They participated in a paid problem-solving session outside of class that was audio- and video-recorded. Their task was to find an equation for the perimeter of a string of n adjacent hexagons, arranged so that pairs of interior sides are removed from the perimeter. They worked for about 40 minutes without any teacher intervention; about nine minutes of the conversation are analysed here. The task includes diagrams for hexagon strings for $n = 1$ to $n = 4$ hexagons, and a table of values to be completed for $n = 1$ to $n = 5$ hexagons. A correct answer is $p = 4n + 2$. The task

was based very closely on a proposed measure of readiness for undergraduate study and can be viewed at Wilmot, et al (2011, p. 287).

METHODS FOR IDENTIFYING POETIC STRUCTURES

The first 90 turns of the conversation were coded using a spreadsheet to note the ways in which a phrase formed a poetic structure with a previously spoken phrase. It was necessary to develop a coding protocol, because a phrase can repeat elements of several previous phrases. The coding approach relied on a combination of close attention to poetic structures and grounded theory coding to iteratively improve the choices about what phrases counted as repetitions of prior statements (Charmaz, 2006). The resulting system was comparatively conservative. In Gries (2005), for example, any repetition of syntax counts as repetition, even if all the words change. The phrase *3 times 2* would be considered a repetition of the phrase *4 minus 1*, because both involve a subject-verb-object construction. However, mathematics education audiences are concerned with language that facilitates mathematical learning. To better focus on continuity of mathematical topic, two phrases had to share syntax and at least one word in order to be considered a repetition. When multiple previous utterances could have been the foundation of a repetition, I chose the most recent one. This method undercounts poetic structures in comparison with related linguistics research.

I recorded the most recent previous turn in which the phrase occurred, even if this was within the same speaker's conversational turn; what the earlier phrase was; whether there was a change in speaker; and whether the phrase was a nearly-perfect duplicate of the previous line or a transformation of it.

I separated the conversation into four episodes, each representing a mathematical insight that the students achieved together. In episode 1, turns 1- 28, Sheila and Joseph filled a table of values on the task sheet for $n = 1$ to 5 hexagons and the corresponding perimeter. In episode 2, turns 29-58, they determined that they should calculate perimeter rather than area. In episode 3, turns 59-71, they initiated the idea that the shared interior sides of the hexagon strings required them to subtract two, but they did not resolve how many times to subtract two. In episode 4, turns 72-90, they expressed a correct method, began to check their work, and wrote a formula in which both H and N stand for the number of hexagons, $\#H(6) - 2(N - 1) =$.

RESULTS: TYPES OF POETIC SPEECH

Close analysis of poetic structures over 90 turns at talk suggested eight types of repetitions that contributed to the discovery of the mathematical relationships. There were in addition poetic structures of that didn't fall into a clear type. It is important to note that each type is a discursive move that could easily occur in a non-mathematical conversation. *Contrast* could occur, for example, as *steamed rice or fried rice?* A comment *Mark has some advice for you* could prompt the *Reversal*: *Well, I have some advice for Mark!* Because these poetic structures are all general discursive options, when they occur in mathematics conversation, they help us identify moments when language is a resource for mathematical learning. In the following section, I exemplify

each of these types and I highlight moments when these poetic structure types appear to facilitate mathematical thinking.

Poetic Structure	Definition and Transcript Example
List	Saying a pattern. Turn 14: <i>1, 2, 3, 4, 5, 6. 6.</i>
Echo	Revoicing a short phrase. In turn 14, the second <i>6</i> is an echo.
Comparison	Saying two things are associated. Turn 27: <i>22 for 5.</i>
Contrast	Posing two things as alternatives. Turn 79: <i>Times 2 or minus 2?</i>
Interposed List	Two lists are collated. Turn 24: <i>...1 to 6, 2 to 10, 3 to 14...</i>
Consolidation	Two previous poetic structures are combined. Turn 62: <i>...it would be 6L...it would be...10L...</i> This combines prior repetitions in lines 52.2-52.3.
Expansion	A previous phrase has a clarifying phrase inserted. Line 64.5: <i>total number of sides minus 2, 4, 6.</i> Here, <i>total number of sides</i> had the new phrase <i>2, 4, 6</i> inserted.
Reversal	Subject and direct object switch places. Turn 75: <i>So it'd be like 6 times x per se number of hexagon times ... hexagons</i> is a reversal of Turn 72: <i>hexagons times six</i>

Table 1: Types of repetition in the hexagon conversation

POETIC SPEECH AS RESOURCE FOR MATHEMATICAL LEARNING

Episode 1: Filling the table

Sheila and Joseph found the perimeter of hexagon strings for $n = 1$ to 5 hexagons in order to fill the table of values. I use *List* to refer to Sheila's statement of *1, 2, 3, 4, 5, 6*, as she counts the sides of the $n = 1$ hexagon case. This habit of naming elements of a pattern became one of the most robust discursive moves for this conversation. Lists always form an internal repetition, because the elements 1, 2, 3 suggest the next element will be 4, but in line 15, Joseph's *List* is also a repetition of Sheila's speech, because he follows Sheila's method of noticing a pattern.

14 S: 1, 2, 3, 4, 5, 6. 6.

15 J: So, this would be 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. 10.

Both Sheila and Joseph used *List* to count the perimeter for the $n = 3$ and $n = 4$ cases, and they both used *Echoes*, in which the terminal number in the list is repeated. At turn 24, they used a new repetition type that coordinates all the previous lists, a type I call *Interposed List*, in which the terminal numbers of 5 previous turns are recast in a new list. At 27, Joseph extends the *Interposed List* with a *Comparison* poetics structure.

24 S: Huh, okay, so we're just putting in the 1 to 6, 2 to 10, 3 to 14, uh, 5, 1, 2. Wait.

27 J: 22 for 5.

The mathematical achievement of episode 1, developing and coordinating a data set, was facilitated by four types of poetic structures.

Episode 2: Perimeter or Area?

Joseph suggested that they could draw additional interior sides to create interior triangles. At turn 52, Sheila asserted that they should work on perimeter instead. Here, I separate turn 52 into sublines, and I use indenting to place syntactically similar units above each other. This formatting helps draw attention to the poetic structure.

52.1 S: ...So then we're counting all the sides,

52.2 so it'd be 6L.

52.3 For 2 it'd be 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. 10L.

At 52.3, Sheila has used a *Consolidation* poetic structure, by modifying 52.2 and inserting the *List* from 15. Turn 56 established a new *List* that isolates the perimeters but that references and simplifies the *Interposed List* at 24. At turn 56, the *Interposed List* was shortened into a new *List* that focuses only on the perimeters:

56 S: So this was 10L, 14L, 18L, 22L, right?

Episode 2 demonstrates the way in which new poetic structures can emerge from the interaction of previous ones. Isolating the perimeters into a new list objectifies them and allows speakers to create conjectures about them more easily. This mathematical result was facilitated by the interaction of several poetic structures: *List*, *Consolidation*, and *Interposed List*.

Episode 3: Each intersecting would be a negative 2.

Here, Sheila and Joseph began to consider the interior sides of the hexagon strings. They grappled with the idea that they must perform a subtraction for these interior sides, but they did not yet discover that they must subtract two ($n - 1$) times.

62.1 S: Uh, so this would be 6L. 6.

62.2 And then this would be 10L, minus 2. [*Pencil touches 10L for the $n = 2$ case in the table of values*]

62.3 Minus 2.

62.4 This would be 2, 4 minus 4. [*Pencil touches $n = 3$ diagram*]

- 62.5 This would be 6. 18L. [*Pencil hovers at the $n = 4$ diagram*]
 62.6 So the total number of sides minus 2 on this side.
 62.7 So it'd be, uh, 6. 6, [*Finger touches the $n = 1$ diagram*]
 62.8 and then this would be, uh, 12 minus 2. [*Hovers at the $n = 2$ diagram*]
 62.9 So. [At 63, Joseph responds: Okay.]

There are many poetic structures at play in turn 62. The first mention of *minus* is at 62.2. The interpretation that accounts for the longest stretch of these words is that 62.1 and 62.2 are a *Consolidation* of 52.2-52.3 (*it'd be 6L...it'd be...10L*) with the extension of *minus 2*. Line 62.4 repeats and modifies 62.2, and also inserts 2, 4. Lines 62.2 and 62.4 are the first times that Sheila counted the interior sides. Line 62.4 is a new type of poetic structure, *Expansion*, because 62.2 is expanded with a novel list.

Turn 64 is largely a repetition of 62, with poetic correspondences (64.2,62.1), (64.3, 62.2), (64.4, 62.4), (64.5, 62.7).

- 64.1 S: So that would be like a formula, right?
 64.2 So this would be 6L,
 64.3 and then this one would be, uh, the total number of sides minus 2.
 64.4 And then this one would be the total number of sides minus 4.
 64.5 This would be the total number of sides minus 2, 4, 6.

After 64.2, the specific perimeters are replaced with the more generalized though ambiguous *total number of sides*, and 64.5 is a consolidation of the poetic structures in 64 and lines 62.4 and 62.7. The generalization may be an attempt to move towards the use of variables. This generalization at 64.5 was developed through a sequence of repetitions than spans all three episodes, at turns 62, 52 and 15.

Episode 4: Total number of hexagons minus 1

In turns 72-90, Sheila and Joseph developed a method for correctly calculating the perimeter. In episode 4, a type of poetic structure that occurs several times is *Reversal*. Joseph's comment at turn 75, for example, was a reversal of Sheila's comment at 72:

- 72 S: So, so the total number of hexagons times six.
 75 J: So it'd be like 6 times x per se number of hexagons.

Here, Joseph shifted *hexagons times six* to become *6 times...number of hexagons*. Although Sheila didn't confirm this reversal verbally or in writing at that moment, it was still a significant moment mathematically. Joseph's reversal is a movement towards discursive standardization, because he suggested a standard variable form, x, and he suggested writing the coefficient before the variable.

As Sheila and Joseph worked on the $n = 4$ hexagon case, they knew that there is a two involved and that there is a subtraction. They worked through several approaches to writing the formula. At turn 79 (we saw this above), Joseph repeated Sheila's *times 2* with a *Contrast* poetic structure to clarify her method.

During turn 90, Sheila had written a column of formulas on the paper:

$$4H - N - 2$$

$$4(6) - N - 2$$

$$24 -$$

The last of these was edited and erased several times as both students tried various ways of getting 18 from the $n = 4$ case. At line 90, Sheila first expressed a correct method for calculating the perimeter of 18, and completed her writing at 90.7 with $24 - 2(4 - 1) =$. Her pencil tip shifted between touching the hexagon diagrams and writing an equation.

- 90.1 S: So this would be 2, 4, 6. [*touching interior sides for $n = 4$ diagram*]
 90.2 So that would be 1, 2, 3, 4, 6. [*touching $n = 1$ to 4 diagrams*]
 90.3 So let's see, 1, 2, 3. [*touching interior sides for $n = 4$*]
 90.4 So number of insides, 4,
 90.5 so 4 minus 1 times, uh,
 90.6 4 minus 1,
 90.7 so this would be 2 into 4 minus 1 equals, right?
 90.8 So that would be 3.
 90.9 3 times 2 would be 6.
 90.10 6 from 24 is 18, right?

The internal poetic structures from 90.1 to 90.3 shift attention across different potential variables—the number of interior sides, the number of hexagons, and finally, the number of pairs of interior sides with the new list 1, 2, 3. These three poetic structures seem to facilitate the shift from 4 to 4 *minus* 1 that happens at 90.4 to 90.5. A *Reversal* from 90.5 to 90.7 in which 4 *minus* 1 shifts from subject to predicate position seems to help get the written 2 in front of the written (4 – 1), though there is some ambiguity here due to erasures and the angle of the video recorder. Turn 90 ends with a number of *Echoes*, which may signal a shift into focusing on calculation instead of coordinating variables and deciding on notation. Some of the clear poetic structures in turn 90, like 4 *minus* 1 and 3 *times* 2, were not coded as repetitions for this analysis because they did not meet the criteria of shared syntax and at least one word in common.

CONCLUSION

This analysis identified poetic structure transformations across most-recent pairs of utterances. Thus, it allows repetitions to be traced backwards across turns at talk through many transformations. For example, the list 1, 2, 3, 4 occurred directly in turns 78 and 74. In 74, it was associated with the phrase *number of hexagons*; this phrase was transformed from *number of sides* in 62. This phrase in 62 was a poetic structure transformation of the specific perimeters 6L, 10L and 18L in the same turn, which trace to turns 56 and 52 where the students isolated the perimeters, e.g. 10L, 14L, 18L, 22L.

Lines 56 and 52 had coded repetitions to the interposed list at turn 24, which grew from the *Lists* and *Echoes* at turn 14: 1, 2, 3, 4, 5, 6. 6.

These very small discursive moves like *Lists* and *Echoes* grew, through a series of repetitions and modifications, into a solution for the hexagon task. Mathematical achievements that co-occurred with poetic structures included: development of a data set (*List*, *Echo*); coordination of different levels of data (*Interposed List*, *Comparison*); isolating one data level for further analysis (*List*); generalization (*Consolidation*); transformation of the variable n into $(n - 1)$ (general poetic structures); and moving towards a standard form of mathematical writing (*Reversal*). In each case, poetic structures acted as language resources for discovering mathematical relationships.

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DECISION MAKING IN THE CONTEXT OF ENACTING A NEW CURRICULUM: AN ACTIVITY-THEORETICAL PERSPECTIVE

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In the present paper we study teachers' decision making as response to emerged contradictions and how these decisions are framed in the context of enacting a new set of curriculum materials. Our data come from discussions in teachers' group meetings through one year. We use activity theory to capture the social, temporal, moral and developmental dimensions of decision making and to interpret two teachers' concrete decisions. The social and systemic context appear to frame and influence teachers' decisions of their goals and the undertaken actions.

INTRODUCTION

In the context of curriculum reform efforts, teachers are seen as active agents and designers, whose instructional actions are influenced by curricular materials, but also shape the enacted curriculum alongside their students (Remmilard, 2005). Situating teacher at the centre of the curriculum enactment, highlights the importance of teacher's decision making. Thus, a number of research studies focus more or less explicitly on teachers' decisions. For example, Lloyd (2008) concludes that the participating teacher's perception of students' expectations and his own discomfort associated with using the new curriculum were key factors in his decisions. Stockero & Van Zoest (2012) classify as productive teachers' classroom decisions that extend mathematics, emphasize mathematical meaning and pursue student mathematical thinking. Schoenfeld (2011) uses the notions of resources (knowledge and other material and intellectual resources), goals (conscious or unconscious aims) and orientations (beliefs, values, biases, etc.) to "offer a theoretical account of the decisions that teachers make amid the extraordinary complexity of classroom interactions" (p. 3). Thomas & Yoon (2014) describe a teacher's conflictual goals and use Schoenfeld's framework to interpret his decision to modify these goals in action.

The above studies research in-the-moment teacher decisions, focusing on the classroom context and emphasizing the individual dimension of deciding. Nevertheless, the broader social, temporal and cultural dimensions of a teacher's decisions are not addressed. In our study we seek a better understanding of how decision making process develops drawing on cultural historical activity theory. The study is conducted in two secondary schools in Greece at the time of the introduction of a newly prescribed mathematics curriculum, in years 2012-13. In Stouraitis, Potari & Skott (2015), we have analysed the contradictions emerged in teaching and discussed in reflective group meetings of the schoolteachers. In this paper we study teachers' decisions while dealing with the emerged contradictions in the context of enacting a new set of curriculum materials. In particular, we focus on teachers who decide to make

or not shifts into their teaching and we examine how decision-making is framed and develops considering social and systemic dimensions.

THEORETICAL CONSIDERATIONS

Activity theory (AT) offers a lens that tries to capture the complexity of teaching, by integrating dialectically the individual and the social/collective. The activity is driven by a motive and directed towards an object (Leont'ev, 1978), in our case the motives of students' learning of mathematics and the fulfilment of teachers' other professional obligations. From this perspective, the unit of analysis is the activity system (AS) (Engeström, 2001a) that incorporates social factors (related to the communities, the rules, and the division of labour within these communities) that frame the relations between the subject and the object with the mediation of tools (figure 1). In our case, one of the tools with considerable influence is the new curriculum.

Activity is carried out through actions which are "relatively discrete segments of behaviour oriented toward a goal" (Engeström, 2001b). We conceptualise teaching action as discrete instructional acts or clusters of acts that constitute the teaching activity, e.g. the selection or creation of a task, the enacting of a lesson plan, etc.

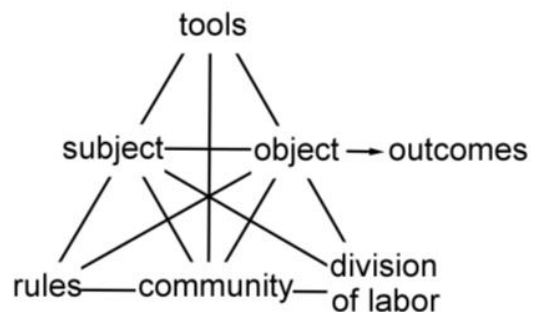


Figure 1. The activity system (adapted from Engeström, 2001)

Every AS is characterised by contradictions. They may emerge when an AS adopts new elements from the outside, such as a new tool or a new rule, causing a conflict with how it functions at present. Contradictions are the driving forces for the development of every dynamic system. They may create learning opportunities for the subject and may broaden the activity, for example leading to reconsideration of the actions and goals (Engeström, 2001a; Potari, 2013). In our study, the introduction and enactment of the new set of curricular materials produced or revealed contradictions in teaching activity that emerged in group discussions (Stouraitis, Potari & Skott, 2015).

Dealing with contradictions involves decisions about the goals and the actions to be undertaken. Of particular importance are decisions related to the "discrete individual violations and innovations" (Cole & Engeström, 1993), that is the search of novel solutions as response to the emerged contradictions. Engeström (2001b) argues that:

Decisions are not made alone, they are indirectly or directly influenced by other participants of the activity. Decisions are typically steps in a temporally distributed chain of interconnected events. Decisions are not purely technical, they have moral and ideological underpinnings with regard to responsibility and power. And the content of decisions is not restricted to the ostensible problem or task at hand; they always also shape the future of the broader activity system within which they are made. (p. 281)

Engeström characterizes the four dimensions of decision making described in the extract respectively as: social-spatial, anticipatory-temporal, moral-ideological and systemic-developmental. These dimensions are used in this study to capture the nature of the decisions that teachers make and the underlying reasons.

METHODOLOGY

A new set of reform-oriented curricular materials was introduced and piloted in a small number of schools in Greece in 2011-12 and 2012-13. The new materials emphasize students' mathematical reasoning and argumentation, connections within and outside mathematics, communication through the use of tools, and students' metacognitive awareness. It also attributes a central role to the teacher in designing instruction. In 2012-13 we collaborated with teachers in three of the lower secondary schools that piloted the new materials. The collaboration took place in group meetings at the respective schools, as the author, who was also a member of the team that developed the curriculum materials, supported the teachers by providing explanations about the rationale of curriculum materials. In these meetings the teachers discussed about their lesson planning and reflected on their experiences from teaching different modules of the designed curriculum. In this paper, we refer to a group of five teachers working in school A that participated in eight 2-hour meetings during a school year.

School A is a Greek experimental school with an innovative spirit. Our focus here is on the teaching decisions of two teachers, Marina and Linda. They both have more than 25 years of teaching experience and additional qualifications beyond their teacher certification, as Marina has a masters' degree in mathematics and Linda has one in mathematics education. Also, they both have experiences with innovative teaching approaches, and they have participated in teacher collaborative groups that develop classroom materials. Further, Marina has written papers for conferences and for journals for mathematics teachers, maintains links to communities dealing with mathematics and is more informed than Linda about the recent activities of the mathematics education community in Greece. Linda has also been involved in producing materials and offering professional development courses for mathematics teachers. Both teachers have strong views about their instructional choices and a critical stance on teaching innovations and materials introduced from various agents. Concerning the new mandated curriculum, Marina says that she considers it a "legitimizing umbrella over my practice"(Marina, 1st interview), a comment with which Linda explicitly agrees.

The data material consists of transcriptions of audiotaped conversations and interviews. The transcriptions were analysed with methods inspired with grounded theory (Charmaz, 2006). The initial open coding resulted in the identification of discussion themes for each meeting, forming thematic units. We used the thematic units to identify situations that teachers experience contradictions and decide to make or not shifts into their teaching. We traced teachers' decisions resulting to shifts through different meetings and interviews to interpret these decisions and the factors influencing them.

This interpretation is inspired by AT and Engestrom's four dimensions of decision making discussed in our theoretical framework.

RESULTS

Below we present two examples in which Marina and Linda become aware of a contradiction but decide to deal with it in contrasting ways. In particular, we describe how decision making evolves in time and is framed in the teachers' interaction.

First example: teaching congruence involving geometrical transformations

Geometrical transformations are introduced as a distinct topic in the new curricular materials with the rationale of supporting students' development of spatial sense and of using transformations when tackling issues of congruence and similarity. Teaching issues of the topic are discussed repeatedly in the reflection groups, as the topic has not been taught before under this new perspective. The use of transformations as a proving tool is an alternative to the Euclidean perspective on school geometry: the intuitive use of the moving figure is seen as incompatible with the rigorous deductive rationale of Euclidean geometry. This issue was highlighted in the discussions in school A. The discussion below is whether geometrical transformations are to have a role in teaching congruence of triangles in grade 9.

In the fourth meeting (A4), Marina refers to her introductory lesson on triangle congruence in grade 9 and to her students response that two triangles are congruent if they "match after translation or reflection or rotation". She considers using tasks with geometrical transformations when teaching the congruence of triangles and she describes her goal saying "I want them [the students] to understand that when we compare angles or segments or generally elements of polygons, we have two tools. One is transformations and the other the criteria of triangle congruence". However, she has not decide, since she is wondering how she can do this, as "there is a need of investigation and inquiry before doing so". Linda listens to Marina and finds her thoughts interesting. But she claims that "every topic has its purpose" and that "there is a purpose to learn how to write [a justification], to observe the shape, to distinguish the given data from the required claims, to make conclusions, and to prove" implying that these goals can be achieved through teaching congruence with a Euclidean perspective, without involving transformations.

In the next meeting (A5) Marina, having made the decision to combine the two approaches, describes how her students in grade 9 work with the congruence of triangles in combination with geometrical transformations to prove the congruence of segments or angles. She notices that this happened regularly in the class she taught last year, but not very often in the one is teaching now. In this meeting, epistemological issues concerning the rigor and the intuition inherent in different approaches are also discussed. Linda follows the discussion, appreciating Marina's approach as a "nice idea" and saying that she likes children working in both ways (triangle congruence and geometrical transformations).

In the sixth meeting (A6) Marina has completed the topic of congruence and reflecting on her use of transformations in the classroom and on students work, explains her decision as creating an "opportunity to change the framework [of proving] in grade 9" and to "get away from Euclidean geometry".

In the 8th meeting (A8) Marina mentions a seminar on transformations she attended three years ago. She also mentions that some students use transformations in other topics, such as trigonometry, indicating that they use them as an operational tool to visualize and prove congruence. In this discussion Linda expresses her decision not to intertwine the different topics saying: "I like transformations per se. I don't like overusing them later in congruence ... I don't find the reason to [do so]".

Examining Marina's approach, as it appears in the discussions, a shift in her teaching of congruence can be traced. She realizes the possibility of combining congruence and geometrical transformations, she decides to do so, and later she selects tasks to highlight the potential of transformations. Her initial goals to highlight the existence of two proving tools, are later enriched with epistemological dimensions "to get away from Euclidean geometry" (A6). These shifts seem to have been facilitated through Marina's work in her classroom and her reflections during the group discussions. Linda acknowledges that geometrical transformations can be used as proving tools for congruence, but she prefers not to combine these two perspectives, pursuing the affordances of Euclidean geometry.

Second example: the use of counters in teaching integer's operations

The new curriculum materials suggest the introduction of operations with positive and negative numbers by using models, like counters, and intuitive approaches, like the movement on the number line. In the year of the study this introduction was condensed in 7th grade with an emphasis on the intuitive basis for the pupils' engagement with concepts and procedures. The use of intuitive models is seen as a way to deal with the contradiction between the concrete context on which operations with integers are based and the abstract (mathematical) definitions of operations of integers. Below we describe the way Linda and Marina cope with this contradiction.

In the 4th meeting the discussion is about teaching integers and their operations. Marina and Linda claim that negative numbers are easily introduced because of the children's experiences and addition of integers is understandable using metaphors such as profit and loss. However, both teachers recognize the difficulties in teaching subtraction, especially when a negative is subtracted from another integer.

Linda describes her use of counters in the form of abstract symbols (\bullet for +1 and o for -1) (A4, 12-15). She explains that for the subtraction $3 - (-2)$, we take 2 o from a set of 3 \bullet , and that she called children to add two zero-pairs (every zero-pair is consisted from one \bullet and one o). In this way they were able to take away the two o . She says that she found this model "somewhere" and she implemented it, recognizing that the curriculum materials suggest a similar model with counters in the form of cards.

Marina states that she "has a problem with these" (A4, 25), because "for some children is very difficult to understand the model" (25) and "it is easier to discuss that subtracting is equivalent to add the opposite" (37). With the counters "you need too much time to teach a model [that students] may never understand" (42), while "it sounds very reasonable to tell them that subtraction is the opposite of addition" (46). Linda, supporting her decision, says that her goal was students' understanding of operations, "why is done this way" (32), to become "convinced" (34), "not to use it [the model] for long period of time, but to understand why subtraction is transformed into addition" (36). When Marina states that her goal in teaching operations is students to manipulate negative numbers making operations correctly, Linda argues that if we want this, we must have convinced them. "Unless we teach them in a completely formal way, that's it and do it. But then, the message you give is that you must do what you are told to do. It isn't right ..." (53). Describing the discussions in the classroom, she says: "we discussed it in two ways. I told them that after all these I'm convinced a little. I didn't tell them that I'm fully convinced". Later, another colleague suggested a model with ice cubes for -1 , and Linda responded "I suggested the bullets [\bullet and \circ for $+1$ and -1] to think abstract [the students]. Because if I start describing ice cubes, they'll be stuck in the ice cubes" (167)

As it appears in the discussions, Linda adopts a model as a tool for teaching operations. She seems to be aware about the affordances and the limitations of similar models and she decides to use this one which is compatible with her goal for students' understanding and for the development of students' abstract thinking. Her decision is in line with the new curriculum materials, but in opposition with the previous ones and with her colleagues' decisions. She also exhibits sensitivity to her students' need to understand and to get involved. Marina prioritizes the goal of quick and error-free execution of operations by students, and she decides not to use such models.

DISCUSSION AND CONCLUSION

The introduction of the new curricular materials created conflicts with the pre-established tools and forms of the teaching activity. The emerged contradictions may provide opportunities for teachers to engage differently in mathematics teaching and learning. The analysis exemplifies these opportunities and the teachers' decisions to make or not shifts into their teaching. Furthermore, teacher's decisions of the goals and the undertaken actions, appear to be socially, historically and systemically influenced. Below we discuss Marina's and Linda's decisions in the aforementioned examples, related to the four dimensions of decision making as formed by Engeström (2001b).

The *social-spatial* dimension is found in the communities influencing the decisions. In the first example, the group discussions in the meetings appeared to be supportive to Marina's gradual formulation of goals and means, while students' predisposition to use geometrical transformations in congruence functioned as trigger for her decision. Linda had not such experience with her students and she did not adopt Marina's goals and decisions despite her involvement in the group discussions. But in the second example,

Linda's classroom experiences were supportive for her decision to use counters. For both teachers, participating in communities before the year of the study seem to influence their decisions. For Marina, her comprehensive experiences with mathematics and her engagement in a learning community specifically committed to discuss geometrical transformations may be important. Respectively for Linda, participating in communities dealing with teaching materials and teachers' guides supported her fluency in adopting tools such as the counters.

The *anticipatory-temporal* dimension can be found in the temporally distributed steps of decisions. Marina's decision to intertwine geometrical transformations with Euclidean geometry in grade 9, came after her decision to teach systematically transformations in grade 8. It is also precursor for using transformations in other topics such as trigonometry in grade 9. Linda's decision to use the model of counters is a link of the chain including realistic situations modelled by positive and negative integers, other models for operations and mathematical reasoning for these operations.

The *moral-ideological* dimension is grounded on issues of power and teacher's responsibility about students' well-being. In the first example, students' positive reactions to Marina's attempts to consider transformations as proving tool, were crucial to her decisions. Similarly in Linda's decision, students' questioning and responding were supportive for her. In both examples, students' involvement, understanding and positive dispositions is the ground for teacher's decisions.

The *systemic-developmental* dimension is found in the possibilities for action based decisions to shape the future of the broader activity. In both examples, if adopted by the collective subject (the community of mathematics teachers), Marina's and Linda's decisions can influence the teaching activity. Using geometrical transformations as alternative proving tool alongside Euclidean geometry and using models and intuitive approaches for teaching operations of integers are decisions that can broaden the horizon of teaching activity, at least in Greek educational context.

Linda and Marina share similar experiences and perspectives with the new set of rules and tools in the form of the new curricular materials. For them significant communities include the school they both work at, and the same reflection group that discusses approaches to teaching according to the new curriculum materials. Both adopt a similar – but not identical – view for students' learning as the object of the activity: they prioritize understanding, mathematical reasoning and connections with reality and within mathematics. Yet, there are significant differences between the goals they are setting, the decisions they make and, consequently, the actions they undertake. This is less striking if one considers these two teachers as having "different positions and histories and thus different angles or perspectives on their shared general object" (Engeström, 2001b, p. 286). Marina appears more fluent with the mathematics of geometrical transformations to use them as a proving tool alternative to Euclidean geometry, while Linda is more informed and familiar with manipulatives and models as teaching tools to exploit them in teaching operations with integers. The apparent

differences may possibly and in part be explained by the different communities they have participated in and the tools mediating the respective activities.

Schoenfeld's framework (2011) may be fruitful for interpretations about the classroom in-the-moment decisions of Marina and Linda. But, "traditional views locate decision making in the heads of individuals at a given point of time in a particular place" (Engeström, 2001b, p.282) and thus, the social, historical and systemic character of decision making are out of search. Searching what makes teachers form goals and what creates the horizon for possible actions under an activity theoretical view contributes to our understanding of teachers' decisions in social, temporal, moral and systemic terms.

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UNDERSTANDING VARIATION IN ELEMENTARY STUDENTS' FUNCTIONAL THINKING

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This research is part of a larger study that used a learning progressions approach to characterize students' algebraic thinking over time in terms of levels of sophistication. In this paper, we report on analyses of two students' interviews over a three-year period and focus on one big idea from our learning progression—functional thinking—to demonstrate how the development of the two students' functional thinking varied. The results of this study lead us to hypothesize that such variation may be due to differences in the development of students' understandings of other core algebraic concepts.

INTRODUCTION

While early algebra researchers have traditionally focused their work on fairly small samples of students, some large scale and/or longitudinal studies in early algebra settings have recently been conducted (e.g. Britt & Irwin, 2008; Schliemann, Carraher, & Brizuela, in press). We have likewise taken a longitudinal approach to early algebra research, using the construct of learning progressions as a tool to frame our work (Fonger et al., 2016). This work has included a focus on functional thinking and the characterization of the development of students' understandings around this concept over time (Stephens, Fonger, Blanton, & Knuth, 2016a). Our focus has been on the identification of shifts in understanding observed across multiple classrooms of students. One unexplained phenomena in this work, however, has been the variation in individual students' progress over time as they develop more sophisticated ways of thinking about early algebra concepts. Our purpose in this paper is to investigate the unexplained phenomena of variation in children's functional thinking as they progress through a three-year early algebra intervention.

BACKGROUND

This research is part of a larger project concerned with the fundamental question of how to support students in elementary grades to be prepared for middle grades algebra and beyond (cf. Blanton et al., 2015). The data described here are situated in the context of an *Early Algebra Learning Progression* [EALP] that involves the coordination of a curricular framework and progression, an instructional sequence, assessments, and

levels of sophistication that characterize student' understandings over time (Fonger et al., 2016; Fonger, Stephens, Blanton & Knuth, 2015).

As Fonger et al. (2016) detail, the EALP's curricular framework guided the development of an early algebra intervention and associated assessments. The early algebra intervention consists of an instructional sequence of lessons for Grades 3–5 (ages 8–11 years). Written and interview assessments included anchor items given at each grade level to track learning over time. We identified *levels of sophistication* by coordinating mathematical perspectives, existing literature on students' understandings of various algebraic concepts, and our analyses of children's responses to anchor items. These levels enabled us to describe trends in students' understandings of core algebraic concepts over time in the context of our curricular framework/progression, instructional sequence, and assessments. Next, we explain the levels of sophistication used to characterize children's developing *functional thinking* (FT) as they participated in our Grades 3–5 early algebra intervention (see Stephens et al., 2016b for further elaboration of these levels).

LEVELS OF SOPHISTICATION

We define levels of sophistication as “benchmarks of complex growth that represent distinct ways of thinking” (Clements & Sarama, 2014, p. 14), capturing patterns in students' reasoning over time. It is not uncommon for children to skip levels, or regress to previous levels of thinking when faced with a new task (Clements & Sarama). See Table 1 for the levels of sophistication describing children's functional thinking and see Stephens et al. (2016b) for elaboration on the research that informed the positing of these levels of sophistication. In this ongoing work, we found that as a group students progress “in order” through the levels. Some students, however, demonstrate variation in their progress over time. In this paper we build on our previous work by seeking to better understand this variation in students' thinking.

Level of sophistication	Description of Level
Other	Student uses alternative or unidentifiable strategy.
L0: Restatement	Student restates the given information.
L1: Recursive-Particular	Student identifies a recursive pattern by referring to particular numbers only. The pattern may be identified as a value for the independent or dependent variable, or both.
L2: Recursive-General	Student identifies a correct recursive pattern. The pattern may be identified for the independent or dependent variable, or both.
L3: Covariation	Student identifies a correct covariational relationship. The two variables need to be coordinated rather than mentioned separately.
L4: Functional-Particular	Student identifies a functional relationship using particular numbers but does not make a general statement relating the variables.

Level of sophistication	Description of Level
L5: Functional-Basic	Student identifies a general relationship between the two variables but does not identify the transformation between them.
L6: Functional-Emergent Variables	Student identifies an incomplete function rule using variables, often describing a transformation on one variable but not explicitly relating it to the other. Student might also write several function rules, indicating an emerging understanding of how to relate two variables.
L7: Functional-Emergent Words	Student identifies an incomplete function rule in words, often describing a transformation on one variable but not explicitly relating it to the other or not clearly identifying one of the variables.
L8: Functional-Condensed Variables	Student identifies a function rule using variables in an equation that describes a generalized relationship between the two variables, including the transformation of one that would produce the second.
L9: Functional-Condensed Words	Student identifies a function rule in words that describes a generalized relationship between the two variables, including the transformation of one that would produce the second.

Table 1: Levels of Sophistication Describing FT (from Stephens et al., 2016b).

METHOD AND DATA SOURCES

Five of the students who participated in our intervention were interviewed at the end of Grades 3, 4, and 5. We selected the five students because their teachers identified them as belonging to the “upper 30%” of the class mathematically and as students who were more likely to discuss their thinking.

In the interviews, students were asked to solve problems similar to those posed in the written assessments so that we could gain further insight into their thinking. Interviews took place several weeks after the year-end written assessments. In the interviews, students were presented one problem at a time on paper. Most chose to write their responses first and then discuss their reasoning with the interviewer. The interviewer asked additional questions to better understand the students’ thinking. Interviews were videotaped and transcribed.

We focus here on results generated from a functional thinking item (see figure 1) that students solved in all three interviews. Students completed parts a-d in the Grade 3 interview and all parts of the item in the Grades 4 and 5 interviews. Although part e was not included in the Grade 3 interview, we are able to compare students’ thinking on part e to their thinking on part c in later grades. Part e is more challenging than part c, because the solution in part e requires another step (adding two). However, like part c, part e assesses the sophistication of students’ representations of functional relationships.

Anne's teacher gives her 5 stickers for every book she reads.

a) Fill in the table below to show how many stickers Anne can earn for reading different numbers of books.

Number of books	Number of stickers
1	
2	
3	
4	
5	
6	
7	

b) Do you see any patterns in the table from part a? If so, describe them.

c) Think about the relationship between the number of books and the number of stickers. Use words to write the rule that describes this relationship.
Use variables (letters) to write the rule that describes this relationship.

d) If Anne read 100 books, how many stickers would she earn? Show how you got your answer.

e) Suppose Anne already had two stickers when she came to school. How does this new information affect the rule you wrote in part c?
Use words to write the rule that describes this relationship.
Use variables (letters) to write the rule that describes this relationship.

Figure 1: Functional Thinking Interview Item.

We analyzed all five students' interview transcripts and written responses generated during the interview. In this paper we present results from two of the five students, whom we call Barry and Meg. We choose to focus on Barry and Meg because their interview responses provide us with the opportunity to illustrate and explore variation in how students progressed through the *levels of sophistication* over time. Responses were coded based on the sophistication of the thinking demonstrated by the student in his or her written and verbal responses per Table 1.

RESULTS

In what follows, we share results from interviews conducted at the end of Grades 3, 4 and 5 for two students, focusing on one interview assessment item (figure 1).

Barry: The influence of one concept on another.

In Barry's Grade 3 interview, he correctly responded to parts a, b and d. When asked to write the function rule using words (part c), Barry stated, "...you multiply the number of books she reads by 5," a general statement describing the transformation without explicitly relating the number of books to the number of stickers (L7). When asked to write the function rule using variables (part c), Barry wrote " $a \times 5 = b$ " and " $b \times 5 = c$," two correct but redundant symbolic representations. When asked why he wrote two equations, Barry explained,

a times 5 would be, since *a* is the first letter of the alphabet, I did *a* for 1, and since *b* is after *a*, well, I don't really know why I put *b*, but I just wanted to put *b*, so *a* times 5 equals

b , which would be 5. And since b is after a , it's 2 times 5, and I just assumed that the b would be counting by 5s. Then c , which is after b , since these variables are counting by 5s, 5, 10. So you would be counting...

Although either of Barry's written equations suggest a L8 understanding, his accompanying explanation indicates that his understanding of how to relate two co-varying quantities using variables is emerging (L6).

In Barry's Grade 4 interview, he correctly responded to parts a, b, c and d. In response to part c, Barry explained, "The relationship was that, uh, every number of books she reads you multiply by 5 and that gives you how many stickers she has," and wrote " $x \times 5 = y$ " (L9 and L8). In part e, when Barry was asked to write a rule using words and variables for a new situation, his responses indicated a lower level of sophistication. The more challenging task revealed weaknesses in Barry's understanding of the equal sign that influenced the sophistication with which he could represent a functional relationship.

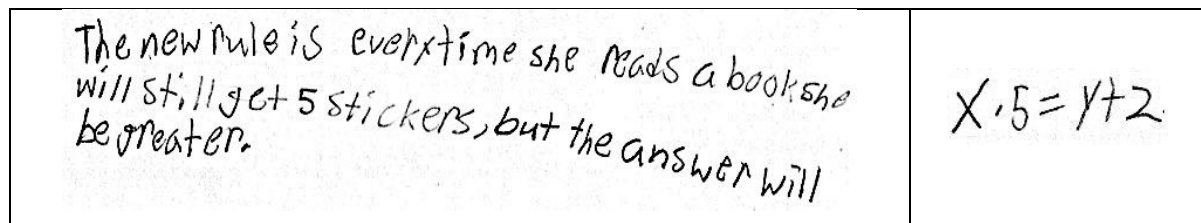


Figure 2: Part e – Barry Demonstrates L7 in Grade 4.

First, Barry used words to represent the new situation by describing a transformation on one variable without explicitly relating the two variables (L7; see figure 2). Then, Barry incorrectly represented the function rule using variables. When he described the relationship he said "the new rule is plus five, uh, if, uh, the old rule was times, was times five the number of books, you'd have to do times five plus two" and wrote " $x \times 5 = y + 2$ " (see figure 2). Barry's response is not characterized by one of our current levels of sophistication, though in our ongoing work we are further examining responses coded as "Other" with the intention of refining our levels. The point we wish to emphasize here is that the sophistication of Barry's response appears to have been dependent on the sophistication of his understanding of the equal sign. We elaborate on this point in the discussion.

In part e of the Grade 5 interview, Barry demonstrated L6 understanding in writing " $x \times 5 + 2$." Although he represented the transformation correctly, he did not explicitly relate the variables in the function rule. The sophistication Barry demonstrated across interviews is summarized in Table 2.

Level	L0	L1	L2	L3	L4	L5	L6	L7	L8	L9	Other
Grade3				•			•	•			
Grade4				•				•	•	•	•
Grade5							•				

Table 2: FT Levels of Sophistication Revealed in Barry's Interviews.

Meg: Co-developing algebraic concepts.

Another student, Meg, demonstrated consistent progress. Like Barry, Meg participated in all three years of the intervention and interviews. However, due to space constraints, we only share results from one of Meg's interviews that provide a contrast to the results generated in Barry's interviews.

When Meg was interviewed at the end of Grade 4, she correctly responded to all parts of the item. Meg described the relationship as “the number of books times 5 equals the # of stickers” (L9), and she wrote “ $x \times 5 = y$ ” to correctly represent the function rule using variables (L8). In part e, Meg wrote “ $x \times 5 + 2 = y$ ” to correctly represent the more complex function rule using variables (L8). When asked how she knew to add the two to the “ $x \times 5$,” Meg explained, “Um, because if you get to the y , $y + 2$, it wouldn't be, like, balanced or equal...” The sophistication of Meg's responses across interviews is summarized in Table 3.

Level	L1	L2	L3	L4	L5	L6	L7	L8	L9	Other
Grade3			•				•	•		
Grade4			•					•••	•	
Grade5								••		

Table 3: FT Levels of Sophistication Revealed in Meg's Interviews.

DISCUSSION

Elsewhere (e.g. Stephens et al., 2016a; Stephens et al., 2016b), we examined broad patterns in students' responses to functional thinking items in order to discern levels of sophistication in their thinking. However, one important underlying assumption of how we take up the notion of levels of sophistication is that not all students' thinking develops in the same way. In this study we elaborated a more nuanced story of the variation in students' thinking within and across individual students. In what follows, we discuss how the co-development of core algebraic concepts may influence the sophistication of a child's functional thinking.

In Grade 4, the sophistication of Barry's thinking is not consistent across tasks. On parts b and c, Barry demonstrates thinking at L8 and L9. However, consistent with Clements and Sarama (2014), when presented with a new situation (part e), Barry regressed to an incorrect representation of the function rule. Barry's explanation and response indicate an operational view of the equal sign (i.e., notion that the equal sign is a direction to compute; Carpenter, Franke & Levi, 2003). We hypothesize that Barry's co-developing understandings explain the varying levels of sophistication he demonstrated.

We wondered whether Barry's operational view of the equal sign was consistent across tasks, so we conducted an ad hoc analysis of Barry's responses to the assessment items that addressed students' understanding of the equal sign. Interestingly, Barry demonstrated a relational understanding of the equal sign (i.e., understanding that the

equal sign indicates an equivalence relation rather than a direction to compute; Carpenter, Franke & Levi, 2003) on two written assessment items and one interview item in Grades 3, 4, and 5. In other words, Barry's relational understanding of the equal sign appears to have not been employed when he was faced with a new and perhaps more complex context.

In Grade 5, Barry demonstrated a more sophisticated level of functional thinking when he wrote " $x \times 5 + 2$." While he now represented the transformation correctly, he did not explicitly relate the variables in the function rule. Perhaps, Barry's understanding of the equal sign in the context of functional thinking was simultaneously emerging and thus hindering the sophistication of his response.

Interestingly, Meg's responses also indicated that her understanding of the equal sign influenced the sophistication of her functional thinking. Meg represented the new situation (part e) correctly. In response, the interviewer asked Meg about her placement of "+2," and revealed that Meg had a relational understanding of the equal sign. We looked at Meg's responses to the assessment items that assessed students' understanding of the equal sign. Not surprisingly, she demonstrated a relational understanding of the equal sign (Carpenter, Franke & Levi, 2003) on each of these items in Grades 3, 4, and 5. Unlike Barry, Meg had a relational understanding of the equal sign that held across contexts.

FUTURE RESEARCH

Moving forward, we suggest that by comparing the levels of sophistication that describe individual students' functional thinking to the levels of sophistication that describe their understanding of the equal sign, we may gain insight about students' co-development of core algebraic concepts. We hypothesize that variation may occur due to factors surrounding the co-development of concepts and suggest that future research should explore this co-development and the links that exist between students' understandings of algebraic concepts.

Additional information

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HOW LONG WILL IT TAKE TO HAVE A 60/40 BALANCE IN MATHEMATICS PHD EDUCATION IN SWEDEN?

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We investigate female participation in PhD education in mathematics. Nine of eleven subject areas for PhD studies in Sweden had reached a 60/40 gender balance in 2010, the exceptions being mathematics and engineering and technology. Using linear regression, we fit a growth model to the increase in the proportion of female PhD students. We show that mathematics has a slower growth rate in female participation than other subjects, and present differences can't be attributed simply to a lower initial female participation. If current trends continue, it will take approximately another 15 years for mathematics to reach a 60/40 gender balance.

INTRODUCTION

In Sweden, at undergraduate level in most subjects, women are in majority. This is true for many other countries as well (OECD, 2015). During the last decades, female participation including advanced higher education has not only increased but also in many areas reached a balance within the 40-60 % span (Lindberg, Riis & Silander, 2011). This balance of 60/40 is the Swedish government's criteria for equality. The increase in female participation is a global trend: for 2012, the OECD average was 47 % female doctoral (or equivalent graduates) and the EU21 average was 48 % (OECD, 2015). The situation for mathematics is different. In Sweden, there are 50 % girls in the most mathematical intense upper secondary school programme, the Natural Science programme, but only one third of the students at undergraduate level in mathematics or other mathematics intensive courses including engineering and teacher education are women (Brandell, 2008). This pattern has been observed in many other western countries as well e.g. in USA (Herzig, 2004; Piatek-Jimenez, 2015) and the UK (Burton, 2004). Moreover, there are less women doing doctoral studies. In 2007, 23% of the doctoral degrees in mathematics in Sweden were completed by females (Lindberg, Riis & Silander, 2011). Mathematics here includes areas such as mathematics statistics, applied mathematics, mathematics history, and mathematics education. Hence, women disappear in mathematics, where the first filter is from upper secondary school to university and the second filter from undergraduate level to PhD. How it this 'disappearance' compared to other subjects? In this paper, we focus on the second filter and pose two research questions: (1) Has the proportion female PhD students in mathematics followed a different growth rate compared to other subjects?; and, (2) If current trend continues, how long will it take to reach a 40-60 balance?

BACKGROUND

We see gender as a social construction, meaning that gender is something more than just a consequence of a biological sex (West & Zimmerman, 1987). Connell (2006) explained gender as:

“a pattern of social relations in which the positions of women and men are defined, the cultural meanings of being a man and a woman are negotiated, and their trajectories through life are mapped out.” (Connell, 2006, p. 839).

The characteristics and culture dependent traits are attributed by the society to men and women. In the long term, these traits create norms and gender could therefor be thought of “as socially constructed differences between men and women and the beliefs and identities that support difference and inequality” (Acker, 2006, p. 444). This is a dynamic process meaning that the attributions, beliefs, identities etc. are not static (Damarin & Erchick, 2010). The concept gender can be divided into different aspects or dimensions. Here, we want to understand structural aspects of gender balance and we use the four different aspects of gender described by Bjerrum Nielsen (2003): structural, symbolic, personal, and interactional gender. Structural gender refers to gender as part of a social structure alongside with other factors e.g. ethnicity and class. An example of structural gender is the percentage that gets an academic profession. In organisations, gender together with class and race create the base for inequality (Acker, 2006). Gender is still a main factor for women participation at work and we find old patterns of gender segregation (Bergström, 2007). The focus in this paper is the number of female PhD students in mathematics compared to other subjects in Sweden, which falls into this aspect of gender.

The second aspect is symbolic gender which appears in the shape of symbols and discourses (Bjerrum Nielsen, 2003). It tells us what is normal and what is deviant such as the idea of mathematics as a male domain (Brandell, Leder & Nyström, 2007). These symbols can have very strong impact. The explanation model for success using the two symbols ‘the hard working female’ (e.g. Hermione Granger) and ‘the male genius’ (e.g. Sherlock Holmes) is considered one of the main reason for gender imbalance at university level (Leslie, Cimpian, Meyer & Freeland, 2015). The third aspect, personal gender, focuses on on how the individual perceive the structure with its symbols (Bjerrum Nielsen, 2003). As stated earlier, this is a dynamic process and the structure and its symbols can influence and change in a constant on-going process which affects personal gender. In her study of female undergraduate students in mathematics, Solomon (2012) concluded that the students were forced to work with their identity, their self-concept as ‘a woman in mathematics’, and this work included how they talked about themselves and their situation. The last aspect described by Bjerrum Nielsen (2003) is interactional gender. These four aspects are inter-related creating gender regimes. An example of this is the case of homosociality (Lipman-Blumen, 1976). This is a pattern where primarily men construct and choose situations dominated by men such as male professors deciding to employ male PhD students similar to themselves, or male students choosing mathematics since it is a ‘good’ environment.

Such patterns, or gender regimes, “provides the context for particular events, relationships, and individual practices.” (Connell, 2006, p. 839). Gender division of labour is then not just a question of glass ceilings but more a question about gendered institutions including relations of power and symbolism (Connell, 2006). One result of gendered institutions could be women leaving mathematics. In previous papers, the reasons why female mathematicians decide to leave academia after their PhD have been investigated (Sumpter, 2014a; 2014b). The main reason was the difficulty getting a job without support which has been reported in previous research (Husu, 2005). Therefore, the number of women is an important factor when wanting to understand why some subjects have more women participants than others. This is particularly central since women in male-dominated professions don’t seem to benefit of the ‘glass escalator’ as men do in female-dominated professions but instead they hit the glass ceiling (Budig, 2002; Hultin, 2003). Another reason why female Swedish mathematicians left the subject was the hostility of the environment (Sumpter, 2014b). In a summary of the theory of gendered organisations developed by Acker/ Williams, we read that “woman does not fit the disembodied category of the ideal worker (Budig, 2002, p. 261)”. If we apply the theory of gendered organisations to female in mathematics with mathematics as a male domain, by default women are not mathematicians.

METHOD

In order to answer the research questions, we downloaded open access data from SCB (Statistics Sweden) that has been provided by UKÄ (Swedish Higher Education Authority). The data had the number of recorded PhD students ordered in research subject (according to national division of subjects), sex (female/male), and percentages of activity (full-time/part-time/ null activity). The data set comprised figures from the second half of the calendar year from 1973 to 2010. Here, we are interested in active students and therefore students recorded with null activity were removed from the data set. Given that we use data over almost four decades allows us to give a historical perspective of the growth rate. Since the data are presented according to the national division of subjects, mathematics at this level of division means mathematical sciences and it is not just restricted to pure mathematics. The other subjects are: veterinary medicine, law, dentistry, medicine, humanities, social sciences, agricultural sciences, engineering and technology, and natural sciences.

For each time series of proportion of female students, we fitted a logistic growth equation, commonly used for describing the spread of ‘innovations’ (Rossman, Chiu & Mol, 2008). We set

$$p(t) = \frac{1/2}{1+\exp(a-rt)} \quad (1)$$

where t is number of years since 1973, $p(t)$ is the proportion of women in each subject, $a = \ln(1/(2p(0)) - 1)$ sets the initial proportion in 1973 ($t=0$), and r determines the rate of increase of female PhD students.

To fit equation (1) and estimate parameters a and r we first transformed the data so we could perform linear regression, i.e.

$$\ln\left(\frac{2p(t)}{1+2p(t)}\right) = -a + rt \quad (2)$$

Equation (1) implies that the maximum proportion of females is 50%. Some subject areas, in particular veterinary medicine, have a greater than 50% female gender balance. In fitting the curves, however, we assume that all data values where $p(t) > \frac{1}{2}$ are set equal to $p(t) = \frac{1}{2}$. This is consistent with our research question concerning the time until parity is reached. We estimated the parameters a and r along with standard error for each value using the linear regression equation. The range for $p(0)$ is then determined by $\frac{1/2}{1+\exp(a \pm s_a)}$ where s_a is the estimated standard error of a . The range of r is the estimated value plus/minus its estimated standard error.

RESULTS

Figure 1 shows the change in female participation in the eleven distinct subjects. Nine of these subjects had, by 2010, reached at least a 40% female PhD students. The two exceptions are mathematics and engineering and technology.

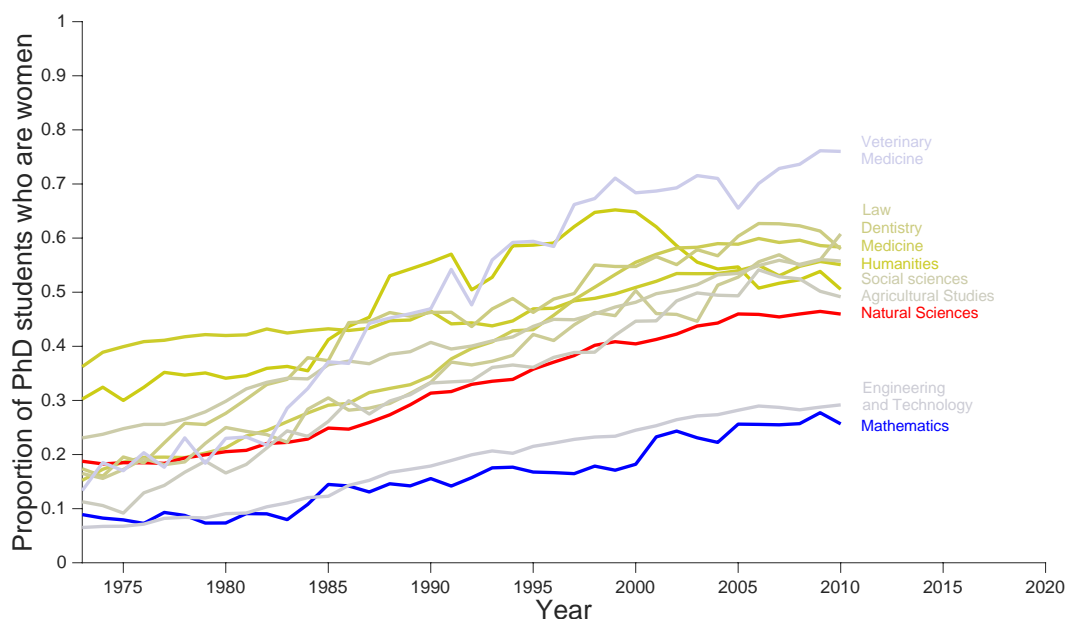


Figure 1: Change in the proportion of female PhD students between 1973-2010 grouped by subject area. From the top: Veterinary medicine, Law, Dentistry, Medicine, humanities, Social science, Agricultural studies, Natural sciences, Engineering and technology, and Mathematics.

Figure 2 shows the fit of the logistic growth equation to the increase in the proportion of women in four different subject areas.

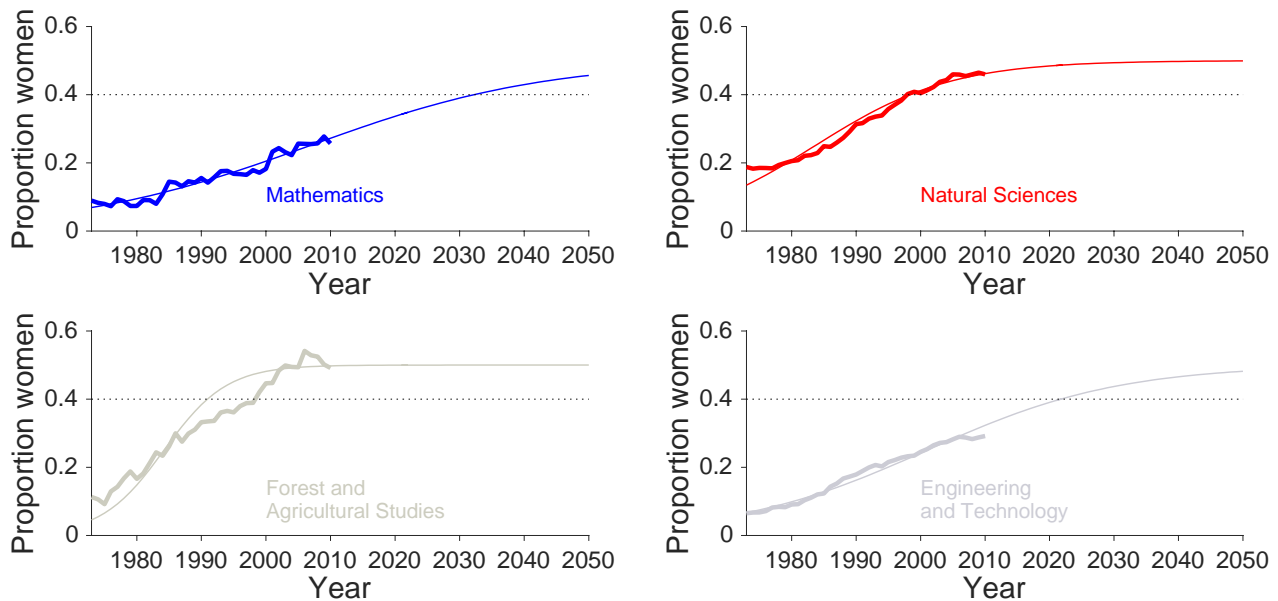


Figure 2: Change of proportion of PhD students between 1973-2010 for four subjects: mathematics (top left), natural sciences (top right), forest and agricultural sciences (bottom left) and engineering and technology (bottom right). Thicker line is data from Figure 1. For parameter estimates see table 1. Dotted line is threshold of 40% women.

Forest and agricultural studies saw a rapid increase in female participation, from initially low levels. Natural sciences also saw relatively rapid increases, but from higher initial levels. Both of these subjects passed the 40% level before 2010. The growth rates of mathematics and engineering and technology are smaller, with mathematics projected to pass the 40% level in 2031 and engineering and technology projected to pass 40% in 2022. Table 1 gives the parameter estimates for growth rate r and initial levels $p(0)$ for all eleven subjects:

Subject area	Initial proportion female (range): $p(0)$	Growth rate: $r \pm (\text{std. error})$
Pharmacology	[0.2071, 0.3522]	0.229 ± 0.022
Humanities	[0.1992, 0.2734]	0.186 ± 0.015
Mathematics	[0.0653, 0.0662]	0.054 ± 0.002
Medicine	[0.0371, 0.0395]	0.266 ± 0.018
Natural Sciences	[0.1234, 0.1274]	0.094 ± 0.003
Dentistry	[0.0779, 0.0850]	0.253 ± 0.012
Law	[0.0445, 0.0481]	0.219 ± 0.019
Social sciences	[0.0763, 0.0857]	0.217 ± 0.016
Forest and Agricultural Studies	[0.0365, 0.0386]	0.206 ± 0.016
Engineering and Technology	[0.0626, 0.0633]	0.067 ± 0.002
Veterinary Medicine	[0.0755, 0.0869]	0.276 ± 0.019

Table 1: Parameter estimates from fitting logistic growth (equation 1) to data.

A useful interpretation of the logistic growth equation can be made in terms of how the rate of increase of female PhD students depends upon the current proportion of female PhD students, i.e. in terms of feedbacks between current levels and further increases. Equation (1) is the solution to the differential equation $dp/dt = rp(1-2p)$. This equation implies that the rate at which females are recruited in an area increases with the number of women already in the subject area, but decreases as equality is reached. The parameter r thus determines the strength of positive feedback between the current proportion of women and the growth rate. As p approaches $\frac{1}{2}$ then this positive feedback is reduced and when $p=1/2$ the proportion of females reaches equilibrium. This interpretation allows us to evaluate the strength of the positive feedback in recruitment of PhD students in the various subject areas.

For mathematics $r=0.054$ and for engineering and technology $r=0.068$, giving a slightly stronger feedback for the latter subject area. In contrast, the positive feedback has been almost four times as strong in agricultural sciences $r=0.206$ and almost twice as strong in the natural sciences $r=0.094$ than in mathematics. The strength of these positive feedbacks are important, because they show that Swedish mathematics departments' failure to increase the proportion of female participation is not simply due to the low initial levels. Natural sciences had a greater female participation in 1973 than mathematics, but participation also grew more rapidly over the next 40 years. Agricultural studies had a similar level of female participation as mathematics and grew much more rapidly. The rapid feedback experienced in agricultural studies is by no means an exception. The growth rates r are between 0.186 and 0.276 for other subjects (Table 1). Independent of the initial level of participation, most subject areas have seen a similar growth curve for female participation. The clear exceptions are mathematics and engineering and technology.

DISCUSSION

The aim of this paper was to investigate whether mathematics as a subject has followed the same trend as other subjects regarding women participation in PhD education, and if not, (1) in what way the growth rate differed, and (2) given the 60/40 gender policy in Sweden, how long it would take to reach this bench mark. Mathematics, together with engineering and technology stood out, showing old patterns of gender segregation (Bergström, 2007). As gender structures (Bjerrum Nielsen, 2003), they show, compared to the other subjects, slow dynamics and appears to be strong male gendered organisations (Acker, 2006; Budig, 2002; Connell, 2006). If mathematics departments are left to continue in the same way, it will take another 15 years before they pass the 40% level. This is nine years slower than engineering and technology. Just as Connell (2006) concluded, a result like this indicates that this is more than a question about glass ceilings, even though the glass ceiling seems to be exceptionally low in mathematics. Both Connell (2006) and Husu (2005) talk about power relations, including implicit and explicit power, and symbolism. Compared for instance to natural sciences, mathematics departments have not been as successful attracting and keeping

women despite decades with laws and decree of equity and equal opportunity promotions.

The logistic growth model we have fitted here assumes that female participation increases due to positive feedback. The model fits the overall pattern in the data, suggesting that the main difference between maths and other subjects is that the feedback between current participation and future growth is much weaker in maths. If current PhD students in mathematics in Sweden follow the conceptions indicated by female mathematicians that decided to leave partly because of hostility (Sumpter, 2014b), these conclusions gain further support. Considering the data presented here, at the aggregate level, along with survey results at the micro-level, the clear implication is that if mathematics departments want to create strong feedback between female participation and further recruitment then they need to improve their working environments.

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TRACES OF CLASSROOM DISCOURSE IN A POSTTEST¹

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Generally, we use two theoretical frameworks – Documenting Collective Activity (DCA) and Abstraction in Context (AiC) for investigating knowledge construction and knowledge shifts in classrooms. In this paper, we show that differences in the depth of teacher questioning during whole class discussions may leave traces in individual students' knowledge, which we were able to capture in students' explanations in a written post-test.

INTRODUCTION

In the course of the last few years we have been investigating knowledge construction and knowledge shifts among different settings in the classroom: the individual, the small group and the whole class community. In these investigations, we used two theoretical frameworks – Documenting Collective Activity (DCA) for investigating the whole class setting and Abstraction in Context (AiC) for investigating individuals and small groups (Hershkowitz, Tabach, Rasmussen & Dreyfus, 2014; Tabach, Hershkowitz, Rasmussen & Dreyfus, 2014). The goal of the current study is to investigate if and how the teaching-learning discourse in the whole class setting has some longitudinal effect on individual students' knowledge as expressed in a post-test. For this goal we analysed data from parallel whole class discussions and from the post-tests of two classes on the same topic.

THEORETICAL BACKGROUND

Argumentation, and learning from a socio-cultural perspective

A socio-cultural perspective helps us appreciate the reciprocal relationship between individual thinking and the collective intellectual activities of groups (Vygotsky, 1978). We use different forms of talk, and especially argumentative talk, to transform individual thought into collective thought and action, and conversely to make personal interpretations of shared experience. Generally, argumentative talk has a crucial role for school learning: (1) the process of generating arguments individually or collectively involves producing explanations/justifications and as such, encourages learning. (2) Argumentation is often initiated to refute a position, or a claim, and as such deepens

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understanding of the problem space (Hershkowitz & Schwarz, 1999). (3) The special structure of argumentative discourse that interweaves data, claims, warrants etc., improves knowledge organization (Krummheuer, 1995; Toulmin, 1958).

Research shows that quite often, argumentative talk is not part of classroom mathematical talk. Teachers have considerable difficulties in guiding classroom inquiry talk. The dominating genre of talk consists of recitation style discourse patterns such as Initiation-Response-Evaluate (IRE) (Cazden, 2001). Moreover, teacher interventions in teacher-led classroom discourse are often not tied to students' ideas. As Yackel (2002) claimed, to tie her interventions to the students' ideas the teacher must first identify the students' threads of thought, and then find a way to advance their reasoning. Some researchers have proposed that teachers provide *generic prompts* (e.g., prompts for encouraging argumentation, mostly prompts expressed as questions), that somehow break the IRE patterns and bring the classroom talk closer to argumentative forms of talk. Such generic prompts have been organized in what Mercer calls *ground rules* that not only encourage students to interact, but also to inter-think (Mercer, 2000).

Theories for studying classroom discussions

In recent years, researchers have come to realize that understanding learning and teaching in mathematics classrooms requires coordinated analysis of individual learning and collective activity in the classroom (Yackel & Cobb, 1996). Four types of processes are intertwining in the classroom: processes of knowledge construction by individuals (1) while working alone (these are frequently hidden); (2) while collaborating in a small group; (3) processes by which knowledge becomes part of the collective activity of the classroom community; and (4) processes of knowledge shifts among the different settings in the class. Researchers need to investigate all four types of processes in parallel, in order to reach a comprehensive understanding of how knowledge is constructed and becomes part of the collective activity of the classroom community, while focusing on the roles of the participants, and considering both cognitive and social processes. This requires a solid background of theoretical-methodological perspectives. One option for such a background is presented next.

Abstraction in Context

Abstraction in Context (AiC) is a theoretical framework for investigating processes of constructing and consolidating mathematical knowledge (Hershkowitz, Schwarz, & Dreyfus, 2001). Abstraction is defined as an activity of vertically reorganizing (Treffers & Goffree, 1985) previous mathematical constructs within mathematics and by mathematical means, interweaving them into a single process of mathematical thinking so as to lead to a construct that is new to the learner.

According to AiC, the genesis of an abstraction passes through a three-stage process, which includes (i) the need for a new construct, (ii) the emergence of the new construct, and (iii) the consolidation of that construct. A central component of AiC is a model, according to which the emergence of a new construct by an individual or a small group

of learners is described and analyzed by means of three observable epistemic actions: *Recognizing* (R), *Building-with* (B) and *Constructing* (C). Recognizing refers to the learner seeing the relevance of a specific previous knowledge construct to the problem at hand. Building-with comprises the combination of recognized constructs, in order to achieve a localized goal such as solving a problem. The model suggests constructing as the central epistemic action of mathematical abstraction. Constructing consists of assembling and integrating previous constructs by vertical mathematization to produce a new construct.

Documenting Collective Activity

Collective Activity is a sociological construct that addresses the constitution of ideas through patterns of interaction and is defined as the normative ways of reasoning which have developed in a classroom community. Such normative ways of reasoning emerge as learners solve problems, explain their thinking, represent their ideas, etc. A mathematical idea or a way of reasoning becomes normative when there is empirical evidence that it functions in the classroom *as if it were shared*. The phrase “function as if shared” is similar to “taken as shared” (Cobb & Bauersfeld, 1995) but is intended to make a stronger connection to the empirical approach which uses Toulmin’s (1958) model of argumentation to determine when ideas function in the classroom as if they are mathematical truths (Rasmussen & Stephan, 2008).

The concepts of knowledge agent & uploading and downloading of ideas

A *knowledge agent* is a member in the classroom community who initiates an idea, which subsequently is appropriated by another member of the classroom community (Hershkowitz, et al., 2014; Tabach, et al., 2014). Thus, when a student in the classroom is the first one to express an idea according to the researchers’ observations, and others later express this idea, then the first student is considered to be a knowledge agent. Such shifts of ideas may be observed from a group to the whole class (*uploading*), or within the whole class, or within a group, or from a group to a second group, or from the whole class to a group (*downloading*).

In the present study we focus on shifts from the whole class to individual students, by seeking traces of the collective activity of the whole class in the individual students’ knowledge, as it is expressed in post-test responses. We ask: Do the differences in students’ responses in a post-test between two classes reflect at least partially differences between whole class discussions that occurred in these classrooms in the course of the learning process? And if yes, how can this be explained?

METHODOLOGY

A 10-lesson learning unit in probability was implemented and video-recorded in several eighth grade classes. Two of these classes, those of teachers D and M, were selected for the present study, as the differences between both classes were prominent. The mathematical theme of the study is calculating probabilities in 2-dimensional sample space for cases, where each dimension has only two possible simple events

(binomial sample space), which are not necessarily equi-probable. We analysed the whole class discussion concerning this topic in both classes using AiC and DCA. The analyses focused on several variables: numbers of turns (total, teacher and students); identifying arguments in the classroom discourse in terms of claims, data, warrants, backings, qualifiers and rebuttals and the participants who raised them; the length of the argumentative chains (the number of utterances in the discussion on an idea); and characterizing the teachers' questions to conceptual vs. procedural, and also according to the epistemic action they aim to elicit – Recognizing, Building-with or Constructing. These variables were categorized and quantified.

In addition, students' responses on the corresponding question in the post-test were analysed (Azmon, 2010). The question presented a situation and two contradictory (correct and incorrect) replies of two virtual students about the probability of an event. Students were asked to choose what they think is the correct reply and to justify their choice. We analysed the students' responses regarding the correct reply and its correct justification. We further categorized the correct justifications into procedurally based justifications and conceptually based justifications, and continued to refine these analyses. Finally, we interpreted the differences between the post-test findings of the two classes on the basis of the findings from the whole class discussions.

FINDINGS

Findings from the whole class discussions

A first quantification on turns within the whole class discussion in both classes reveals quite similar results concerning the total number of turns, the number of teacher turns and of student turns, and the number of teacher questions (Table 1).

Table 1 – The number of turns of the different categories in the two classes

	Total no. of turns	Students turns	Teachers turns	Teacher turns with questions	Teacher prompts
Class D	65	39 (60%)	26 (40%)	21 (80%)*	10 (38%)**
Class M	67	32 (48%)	35 (52%)	29 (83%)*	20 (57%)**

* Percentage of questions out of all teacher turns; **Percentage of prompts out of all teacher turns

There was one exception: the difference in the number of teacher prompts. This difference is especially interesting, taking into account that the numbers of the two teachers' questions is quite similar. This may point to different patterns of interaction in the two whole class discussions. We further analysed the teachers' questions by three criteria: (1) whether the teacher draws the students' attention to procedural or conceptual mathematical issues; (2) what kind of epistemic action the teacher is trying to elicit – recognizing, building-with or constructing; and (3) whether the elicitation was for data, warrants or backings? Note: in this paper we didn't investigate criterion 3. Table 2 summarizes these findings. The distribution of questions by the epistemic actions they seem to elicit is again similar for both teachers. In both classes conceptual

questions were asked most. In M's class more than half of the questions were conceptual. In D's class about a quarter of the questions were rhetoric - that is they were answered by the teacher herself, hence not providing the students the opportunity to answer.

Table 2 – The teachers' question* types in the two classes

	Mathematical issue		Rhetorical	Intended epistemic action		
	Conceptual	Procedural		Recognizing	Building-with	Constructing
D's Class (N=30)	13 (44%)	10 (33%)	7 (23%)	12 (40%)	7 (23%)	11 (37%)
M's Class (N=29)	17 (59%)	12 (41%)	--	15 (52%)	4 (14%)	10 (34%)

* Teacher D had 8 turns with two questions each

To illustrate the difference between the discussions in the two classes, we bring next a short excerpt from each class discussion regarding one argument. We start with M's class. The situation under discussion is the Arrows problem: "Ora and Aya each shoot one arrow aimed at the target. The probability of Ora hitting the target is 0.3. The probability of Aya hitting the target is 0.5." Students were asked to draw a square model to represent the probabilities, if both Ora and Aya shoot one arrow each. After the class identified the events presented by each of the rectangles in the square model and calculated their probabilities, the following discussion, initiated by the teacher M took place:

- M62 M: ...Now, how can we check if we don't have any mistake?
M63 Yael: $15\% + 15\% + 35\% + 35\% = 100\%$
M64 M: Why does it have to be 100% when adding all these?
M65 Itamar: Because 100% is the whole
M66 M: Because this is the whole, and here we describe all four cases that can happen when two people each shot an arrow. Do you understand this task?

In 62 teacher M is prompting critical thinking, in order to check the correctness of the probability calculations. Yael (63) provides data (probability of each of the four events) and a claim (the sum of the probabilities is equal to 100%). In 64 M prompts again, asking for a warrant, and Itamar (65) provides a warrant. In this episode, Yael functions as knowledge agent and Itamar follows her by completing the argument. Note that these five turns constitute one argument of length five. The teacher's two questions (62, 64) are conceptual.

A similar question was raised by D in her class:

- D64 D: How can we check that it is correct what we wrote here?
D65 Yaad: You add and get 1
D66 D: Everybody added? You got 1?
D67 Students: Yes!

Like M, D initiates a discussion on the same issue. But, when Yaad (65) provides the answer, the teacher D, in contrast to teacher M, does not ask for data or warrant, and push for procedural action – calculate (66).

We counted the number of arguments and turns in the discussion in each class. In D's class we found eight arguments expressed by 22 turns, on average 2.75 turns per argument. In M's class we found six arguments and 31 turns, on average 5.17 turns per argument. That is, in M's class the discussion around each argument was more developed. We move next to analyse individual students' knowledge as reflected in one relevant post-test item.

Findings from the post-test

The post-test question was: "The 'Tel-Aviv' school offers a variety of extracurricular programs. The probability of encountering a student who is in the drama program is 0.9. The probability of encountering a student who is in the philosophy program is 0.2. Gal claims that the probability of encountering a student who is in both, the drama program and the philosophy program is $0.2+0.9$. Yam claims that the probability is 0.2×0.9 . Which of them do you think is correct? Explain."

Student responses were analysed and categorized with respect to correctness and explanations. More than 90% of the students in each class answered the question correctly and determined that Yam is correct. Also, 88% of students in each class provided correct explanations for their choice. Three categories emerged while analysing students' correct explanations (Table 3):

- A. *Explanations relying on the multiplication principle.* These belong to a few sub-categories relating to the characteristics of explanations:
- i. *Explanations relying only on the multiplication principle* and indicates that the student is aware of the principle that "in probability we multiply probabilities", but shows no evidence of the student's understanding why a multiplication is required. Explanations in this category are focusing only on a description of the solution procedure.
 - ii. *Explanations using the area model*, by providing a diagram with partition lines according to the given probabilities, and calculating the probabilities according to the relevant rectangle area. This strategy involves more complex processes than (i). Multiplication reflects calculating the area of the representing rectangle.
 - iii. Explanations using "*part of*" by calculating the required probability according to the portion of the second probability out of the first one (e.g., "Cause to find part of something we have to multiply").
- B. Explanation according to the "*probability can't be greater than 1*" principle. Many students chose to support the claim that "Yam is correct" by the claim that "Gal is wrong - $0.2+0.9$, the sum of probabilities, will lead to a probability that is greater than 1, an impossible situation", or "the square area cannot be more than 100%".

C. Explanation combining both principles. For example, "the result of multiplying 0.9 by 0.2 is 0.18, a probability that is smaller than 1".

Table 3 shows the frequencies in percentages of the explanations of the students in each class. We can see that there are differences between the two classes: While in M's class more students used explanations relating to the area model, in D's class more students justified their choices by multiplication only. That is, explanations provided in M's class might be used as evidence for an understanding while those provided by D's class showed mainly procedures. Also, we can consider explanations from categories Ai and B to be of superficial nature, as opposed to categories Aii and Aiii which are of a deeper nature. In this case, for D's class there are 73% superficial explanations vs. 11% deep ones, while in M's class there are 33% superficial explanations vs. 42% deep ones.

Table 3: categories of explanations in percentages

Class	A				B	C
	i	ii	iii	ii and iii		
D	43	3	5	3	30	8
M	12	38	4	0	21	13

DISCUSSION

On the surface, quantitative analysis showed similar patterns in the whole class discussions of the two classes. The only hints for possible differences were teacher prompts and questions. However, these differences point to a different nature of the two whole class discussions (Mercer, 2000). Indeed, our analysis of the two whole class discussions shows this clearly. Although we could only bring one episode from each, the difference in depth of argumentation between the two whole class discussions was consistent over all lessons. The whole class discussion in M's class throughout the learning unit included more developed arguments, in terms of students providing data and warrants to claims raised, and the teacher's prompts asked for explanations.

This depth of argumentation left traces in individual students' knowledge and beliefs about what constitutes an acceptable explanation, which we were able to capture in students' explanations in the post-test item. Can we point at a possible explanation for these differences? We were able to point to differences in the teachers' moves in terms of types of questions asked, and in providing prompts to elicit students thinking beyond the classic IRE. Hence, we think that the way of teaching explains, at least partially, the difference. More research in this direction is needed.

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IMAGES OF MATHEMATICS LEARNING REVEALED THROUGH STUDENTS' EXPERIENCES OF COLLABORATION

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This study focuses on students' images of mathematics learning and their relationships with mathematics. In this paper we consider how students described collaboration in mathematics classrooms, through the examination of students' autobiographical interviews and drawings. Our analysis revealed that many students considered mathematics learning mainly as an individualized and isolated process and did not perceive peer talk or collective exploration as meaningful. Our cross-analysis with students' feelings revealed that those who had positive feelings towards mathematics tended to find group work less helpful. Our findings illuminate a perceived gap between teachers' widespread use of group work as a teaching strategy and students' understanding and appreciation of the goals of such instruction.

PURPOSE OF THE STUDY AND LITERATURE REVIEW

The study from which the findings presented here are derived explores students' experiences of learning mathematics in Canadian schools and post-secondary institutions. This paper focuses specifically on how students perceive group work and collaboration in mathematics classrooms. Through students' descriptions of their experiences of collaboration in mathematics classrooms, we attempt to reveal their images of, and assumptions about, mathematics learning and how these relate to students' emotional relationships with mathematics.

Collaborative working has been implemented across disciplines as a tool for providing rich academic and social learning opportunities to students and group work is widely recommended as a teaching strategy in mathematics classrooms. For example, in its Principles and Standards for School Mathematics, the National Council of Teachers of Mathematics outlines the importance of group work for communicating, explaining, and justifying mathematical ideas among learners (National Council of Teachers of Mathematics, 2000). Collaboration, problem solving, and learning how to learn—essential components of the 21st century skills needed for navigating a rapidly changing society—can be developed through group work (Darling-Hammond et al., 2008; Trilling & Fadel, 2009). The kinds of learning that emerges from group work, however, cannot be taken for granted in mathematical classrooms. If the physical conditions and communication space for collaboration are not well prepared, learning by talking with peers cannot be guaranteed (Barron, 2003; Sfard & Kieran, 2001).

Collaboration and collective mathematical thinking are highly related to students' mathematical dispositions (Towers, Martin, & Heater, 2013). Over the past 30 years, researchers in the field of mathematics education and psychology have examined the interplay between the affective domain (beliefs, attitudes, and emotions) and teaching

and learning mathematics (Di Martino & Zan, 2011). Many of the studies investigating affect and mathematics in the field of cognitive psychology tend to focus on negative aspects, such as “math anxiety,” associated with mathematics (e.g., Ahmed, Minnaert, Kuyper, & van der Werf, 2012; Young, Wu, & Menon, 2012). Understanding a wider breadth of students’ emotional connections to mathematics is thus essential for designing mathematics instruction that enhances students’ dispositions for learning mathematics (Boaler, 2011). While various aspects of learning through group work has been researched in the mathematics education community (e.g., Barron, 2003; Esmonde, 2009; Ryve, Nilsson, & Pettersson, 2013; Webb, 1991; Yackel, Cobb, & Wood, 1991), little investigation has looked at the connection between students’ emotions, images of mathematics learning, and group work experiences. This research examines students’ emotional experiences and images of learning mathematics, in relation to the specific instructional context, group work.

THEORETICAL FRAMEWORK

This research is framed by enactivism, a theory of embodied cognition that emphasizes the interrelationship of cognition and emotion in learning (Maturana & Varela, 1992; Varela, Thompson, & Rosch, 1991). Enactivism recognizes human development and the surrounding environment as structurally coupled (Maturana & Varela, 1992) and therefore learning, in this frame, is seen as reciprocal activity. Students’ mathematical learning is not determined (solely) by the teacher or the learning environment, but is dependent on the kind of teaching experienced and the kind of mathematical milieu in which students are immersed. Enactivist thought reorients us to the significance of this mathematical milieu in shaping not only what students learn in school but also their emotional connections and relationships with the discipline. This enactivist frame, then, prompts us to seek to understand how students come to have particular relationships with mathematics, what being mathematical means to them, and the kinds of teaching and learning structures (such as group work) that are relevant as students develop particular dispositions for mathematics. Guided by enactivist thought, our investigation tries to understand how instructional contexts and the mathematical milieu in which students are immersed can influence students’ (emotional) relationships with mathematics learning.

RESEARCH DESIGN

The data on which we draw for this paper were gathered in the province of Alberta, which is located in Western Canada. The study’s participants are Kindergarten to Grade 12 students, post-secondary students, and members of the general public, but we focus here on data collected in the first phase of the study, which includes students from Kindergarten to Grade 9. Forms of data include semi-structured interviews, drawings (that represent participants’ ideas about what mathematics is, as well as their feelings when doing mathematics), and written and oral mathematics autobiographies (accounts of participants’ histories of learning mathematics).

To date, 94 interviews with Kindergarten to Grade 9 students (41 girls and 53 boys) have been conducted. We have also collected 95 mathematics autobiographies from post-secondary students and members of the general public through an online submission form.

All of the interviews were transcribed verbatim. In this paper, we mainly focus on elements of the transcripts that featured students' descriptions of group work and/or pair work in mathematics classrooms. In order to reveal students' images of mathematics learning, in relation to their experiences of group work, we also conducted thematic analyses of their drawings and associated descriptions of their feelings when doing mathematics.

FINDINGS: STUDENTS' IMAGES FOR GROUP WORK AND MATHEMATICAL LEARNING

Across grades, group work or pair work was frequently reported as a classroom learning structure, although the ways in which, and the extent to which, group work was used varied. Students reported that they often worked with their desk partners (those sitting next to them in class), their friends, and project members. Tasks that were used for group work also varied. In some classes, group work was used only for projects. In other classes, group work or pair work was used regularly for completing a worksheet. However, no students reported working on tasks specifically tailored towards group work [such as group-worthy tasks described in Cohen and Lotan (2014)].

Overall, students' preferences were split: 37.3% of the students preferred individual work to group work and/or pair work and 29.4% of the students preferred group work and/or pair work to individual work. For 31 % of the students, their preference was mixed: it depended on types of tasks and peers working together for group work. There was only one student who reported to have no preference. While slightly more students talked negatively about group work in elementary grades, the difference across grades was not outstanding.

Through the cross-analysis focusing on students' feelings about math and group work preference, it was revealed that both positive and negative feelings towards mathematics could influence students' preferences for group work. Our analysis suggests that the students who had positive feelings towards mathematics tended not to find group work very helpful. Among students who preferred individualized learning to working with peers, 57.8 % (11 out of 19) were good at mathematics and 10.5 % (2 out of 19) had negative relationships with mathematics. Among the students who preferred group work to individualized learning, 26.6 % (4 out of 15) had positive relationships with mathematics and 26.6 % (4 out of 15) had negative relationships with mathematics.

Students preferred individual work for various reasons. For those who are good at mathematics, they felt group work was unnecessary and could be distracting. They said, for example, "(I prefer individual work) because I know how to do it and those

things like math,” or “Sometimes I see people copying and making noise and I can’t focus on what I’m doing.” When explaining a preference for individualized learning, a Grade 2 student said it was, “Because you have your own space and people can’t copy you.” A Grade 3 student described how he felt about being asked for help from peers as follows:

Interviewer: So do people ever come to you then and ask for help?

Student: Sometimes.

Interviewer: Sometimes, yeah. Do you like helping them or do you find that a bother?

Student: I don’t know what the word is, but yeah it just disturbs me while I’m trying to work independent.

Similarly, a Grade 1 student said, she preferred individual work “Because when I’m working with a friend they’re talking and I’m trying to work and I say ‘Please will you be quiet?’ and they keep talking.” A Grade 2 student said he would not like group work “Because in groups, it’s not so quiet.” In fact, some students perceived “talk” in the classroom as noise and distraction. For example, a Grade 5 student compared learning environments at home and at school and said: “Well, my mother is kind of strict of, um, getting it. That’s why I always get it right. Because I make up strategies and then school with my teacher I kind of, you know, have a lot of noise and that’s why I get sometimes slow in writing.” Another Grade 5 student said, “Sometimes when I don’t have noise around me I can focus and I like it a bit more but sometimes when it’s noisy I can’t focus and I can’t do it. But I usually like math when it’s quiet.” Similarly, when describing group work, a Grade 1 student said, “I sit over with a friend but sometimes I see people copying and making noise and I can’t focus on what I’m doing.” These students’ comments depict mathematics classrooms where learning and thinking are essentially individualized and thus talking with others (and others’ copying their work) is considered to be a distraction and disturbance.

While group work and pair work were used regularly in our respondents’ mathematics classrooms, students’ autobiographical interviews and drawings did not communicate an image of collaboration and collectivity for mathematics learning. In their drawings, most of the students represented isolated and individualistic images of classroom mathematics learning—predominantly with the drawings of a student sitting at a desk working alone (see Figure 1). Only a few early grade students drew drawings of collaborating with others.

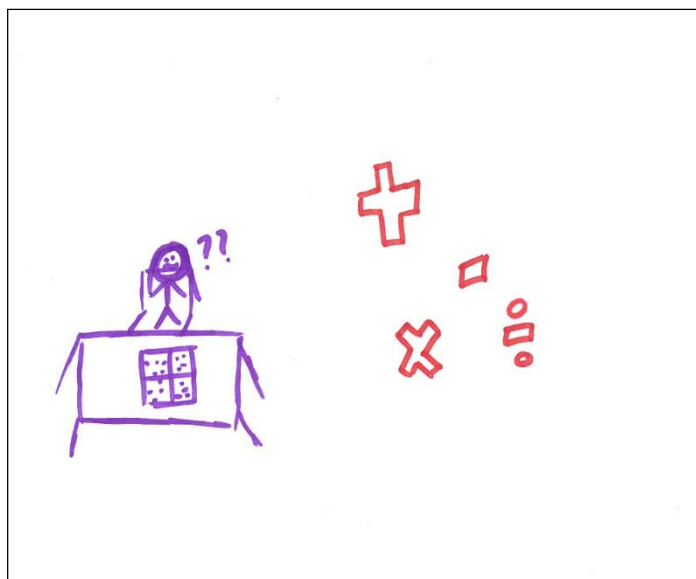


Figure 1: Typical student drawing of mathematics learning

For most of the students, when they got stuck on mathematics problems their strategy was to try to figure out the answer on their own, rather than collaborating with others. Many students said that they would sometimes seek help from a teacher or classmates but mostly they would try to work on their own. For example, a Grade 8 student said, “I usually like to work hard but in math it gets really hard. When it’s a hard stuff and I usually go up to the teacher several times, but he asks us to try and figure it out ourselves or ask friends and stuff.” Even when they were encouraged to ask their friends, many students across grades said they would still try to figure it out on their own. As students get older, they tend to rely more on themselves rather than seeking help from others, as represented by a quote from a Grade 8 student, “I developed the skill to always figure it out on my own until I could not.”

There was one exceptional but informative case wherein a Grade 5 student described how she liked to spend sufficient time to work on mathematics; and therefore she preferred working alone. This student enjoyed learning mathematics and working on problems. She said:

Normally I don’t like, really am a fan of working with someone else. When I work with other people they will want to do all the work and when I go up to the teacher answers will be wrong, and, but I take a lot of time. Once I took 25 minutes, um, to complete a math sheet that had 3 questions on it because I took my time.

In explaining why she likes to take time in mathematics, she said: “Well actually it’s quite fun, because the more actually slower you go the more better. Like in the hare and the turtle when they were racing the slower beat the faster.” This student’s description implies that, for her, collaboration and working with others are not compatible with spending time and exploring problems in depth.

In contrast to the above-introduced quotes, some students preferred group work because it helped them understand mathematics better by working with others. For example, a Grade 5 student said he preferred working with peers, “Just because if I don’t know something that they know then they can help me. Just, they don’t tell me the answer but they can tell me how to do it better.” A Grade 7 student said, “I prefer working in a group because it’s more fun and it just makes everything easier when there is more than one mind at work.” Similarly, a Grade 8 student said, “I can understand what they’re thinking and they can understand what I’m thinking and we can put that together and finish the question.” As these quotes indicate, these students recognized the benefits of group work and learning with peers. However, the number of students who recognized the benefits of working collectively with others was rather small (8.5%, 8 out of 94 students). Furthermore, most of the students perceived group work as a way of offering and/or receiving help for individualized tasks but not necessarily as an opportunity for creative collaboration. Our analysis shows that the majority of Kindergarten to Grade 9 students did not appreciate working with others and collaborating with others for deeper mathematics learning.

DISCUSSION AND EDUCATIONAL IMPLICATIONS

While collaborative learning and group work have been frequently used in mathematics classrooms in Canada and other countries, most of the students we interviewed still held images of mathematics learning that were mostly individualistic and isolated. Our analysis shows that merely experiencing group work does not convince all students of its usefulness. In our cross-analysis focusing on students’ feelings, it was revealed that those who had positive feelings towards mathematics tended not to find group work or pair work very helpful. The students who considered themselves to be adept at mathematics reported that group work and pair work were not beneficial because they mainly gave help to others but did not receive much in return. In fact, many students perceived the talk during group work as distracting and noisy.

Mathematics activities used during group work and pair work were characterized by the students as tasks in which finding a solution to the posed problems was the goal, rather than exploring multiple aspects of the problems and solutions. Because students considered mathematics mainly as an individualized and isolated process, many students did not perceive peer talk or collective exploration as meaningful, contrary to the perception of group work in other classrooms we have studied where the teacher deliberately structured mathematics learning through group activity (see, e.g., Towers et al., 2013).

As indicated in the interviews by some students, when they got stuck, trying to figure things out on their own was a commonly-observed solution. Even when they needed help and assistance in the very process of “figuring out,” they often did not have access to sufficient help or collaboration with others. Also, for those who think they are adept at mathematics, a lack of meanings for collaboration can deprive them of potential

opportunities to learn, because students can benefit from explaining and participating in discussions (Chizhik, 2001; Webb, 1985).

The picture of classroom mathematics learning we have described in this paper is problematic—especially given that some students who require help may hesitate to seek help in contexts where, despite the use of grouping in the classroom, value is placed more on individual competence and success. In our analysis focusing on immigrant students' mathematics learning experiences in Canadian schools, none of these students preferred group work over individualized work (Takeuchi & Towers, 2015). These students could not see the benefits of group work, even though newly-arrived immigrant students could have benefited from group work with peers who can draw out the expertise of immigrant students (Takeuchi, 2015).

Our research reminds us of the importance of creating a mathematics group work pedagogy that is deliberate, that embraces students' questions and dilemmas as a resource for meaningful mathematical learning, and that helps students to understand why they are being asked to work together and what they can learn from collaboration. Our findings suggest that there is a gap between teachers' use of group work in mathematics classrooms (which is widespread) and students' understanding of, and appreciation for, the potential benefits of this pedagogical approach. We see this as both a significant concern and a gap that is ripe for further study.

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WHEN IS A PROBLEM REALLY SOLVED? DIFFERENCES IN THE PURSUIT OF MATHEMATICAL AESTHETICS

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In the context of looking back, the fourth step of Pólya's problem-solving process, this study examined the question of when mathematics problems might be completely solved. In particular, it investigated the aesthetic principles that guided expert mathematicians in their professional experience as problem solvers and the aesthetic considerations that motivated mathematically gifted students in their problem solving experience. Our findings demonstrated that mathematical aesthetics might be a learned skill, instead of an innate characteristic of problem solvers.

INTRODUCTION

A number of pedagogical recommendations to improve and assess mathematics problem-solving experience have focused on the development of aesthetic appreciations, where learners are to recognize many different approaches and value those considered to be mathematically “beautiful” (Dreyfus & Eisenberg, 1986; Karp, 2008; Leikin & Lev, 2007). Indeed, affects and meta-affects connected with aesthetics in mathematics can serve as an indicative, and possibly predictive, measure of problem solvers' depth of mathematical comprehensions (Sinclair, 2004). Aesthetic aspects were particularly considered in many studies connected with mathematicians' preferences in problem-solving approaches (Hadamard, 1945; Krutetskii, 1976; Pointcare, 1946). Much less attention, however, was devoted to understanding whether aesthetic appreciation of mathematical “beauty” might be viewed by grade school level students. This study examined different aesthetic considerations that might motivate different groups of problem solvers.

THEORETICAL AND EMPIRICAL BACKGROUND

The seminal work of Pólya (1945) identified four steps in the process of solving mathematics problems: understanding the problem, devising a plan, carrying out the plan, and looking back. The fourth step, looking back, was proposed to imply that a solved problem did not mean the end of problem-solving process. It was necessary to examine the obtained result by checking the arguments along the way. Alternatively, it would be valuable to derive the obtained result by using a different approach. Given the many possible different approaches to solve the same problem, a decision to choose one approach over other approaches might be less than arbitrary (Leikin & Lev, 2007). Aesthetic aspects were particularly considered in many studies connected with preferences in problem-solving approaches (Krutetskii, 1976).

Silver and Metzger (1989) assessed the role of aesthetics in a study involving university professors in mathematics. They examined the aesthetic influence on mathematical

problem-solving experience in two assessments. In one assessment, they monitored the role of aesthetic value in the process of problem-solving as discussed by Poincare (1946) and Hadamard (1945). In another assessment, they analyzed the sense of aesthetics in the evaluation of the completed solutions as described by Kruteskii (1976) or the problems themselves. Silver and Metzger (1989) found that these expert problem solvers displayed signs of aesthetic emotion. On one occasion, a subject resisted the temptation to resort to the use of calculus in solving a geometry problem, acknowledging the possibility of a “messy equation” (p. 66). Only after some unsuccessful attempts to seek a geometric approach did the subject concede to solving the problem using calculus. Although successful, he felt that “calculus failed to satisfy his personal goal of understanding, as well as his aesthetic desire for ‘harmony’ between the elements of the problem and elegance of solution” (p. 66). On another occasion, having solved another geometry problem algebraically, the same subject appeared unsettled, recognizing that a geometric approach could be “more elegant” (p. 66).

Using a similar scope of analysis as Silver and Metzger (1989), Koichu and Berman (2005) examined how three members of the Israeli team participating in the International Mathematics Olympiad coped with conflict in their conceptions of effectiveness and elegance. An effective approach led directly to a final result in answering a mathematics problem with minimum memory retrieval of concepts and terms and procedural knowledge. An elegant approach was considered to have clarity, simplicity, parsimony, and ingenuity in solving a mathematics problem with minimum intellectual effort and few mathematical tools. In their study, Koichu and Berman (2005) observed that when solving geometry problems, these mathematically gifted students consistently directed greater aesthetic appreciations towards geometric approaches than algebraic or trigonometric approaches. However, when such a geometric approach was not readily accessible to them, they immediately resorted to algebraic or trigonometric approaches as long as the approaches effectively solved the problems. Only later on when students had built up their confidence could they develop the desired geometric approach to satisfy their need for aesthetic appreciations. This experience marked the point at which students successfully managed to balance the need for elegant approaches with the time constraint requiring effective approaches.

In the studies by Silver and Metzger (1989) and Koichu and Berman (2005), mathematics professors as well as International Mathematics Olympiad team members did not only find geometric explanations or approaches to problems to be more appealing than other explanations or approaches, but they also demonstrated persistence in finding approaches characterized by geometric reasoning or interpretations even after they had acquired non-geometric solutions to the problems. The study by Silver and Metzger (1989) demonstrated evidence that there appeared to be an agreement among mathematics professors with regard to their strong preference in geometry. Likewise, one might argue that because of the specific training that they received in preparation for the International Mathematics Olympiad, possibly as a direct influence of the heavy emphasis on simplicity in the scoring criteria of such

competitive mathematics pinnacle (Olson, 2004), the three mathematical Olympiads in the study by Koichu and Berman (2005) would gravitate towards personal preferences with such guidance, although they might not develop on their own a natural preference towards geometric approaches.

The present study examined the question of when mathematics problems might be completely solved. In particular, it investigated the differences in the pursuit of mathematical aesthetics as perceived by expert mathematicians and mathematically gifted students. It focused on an under-studied aspect of Pólya's fourth step because unlike the other three steps, the fourth step attracted much less consideration in the mathematics education research (Schoenfeld, 1985). This study might also be of value because little was known about the extent to which mathematical aesthetics might be viewed proportionately by different groups of people (Sinclair, 2004).

METHODOLOGY

Three expert mathematicians volunteered to participate in the study (Professors 1, 2, and 3). They were editors of a number of well-respected professional journals in pure and applied mathematics, and shared among themselves a total of 48 years of research experience. Nine mathematically gifted students also participated in the study to take the paper-and-pencil test consisting of the three non-standard mathematics problems which could be solved using 15 different approaches (see Table 1). The students were enrolled in one of the nine specialized high schools in New York City, where less than five percent of the approximately 30,000 applicants were admitted after passing an entrance examination (NYCDOE, 2011). These problems were carefully chosen to allow for many different approaches not immediately apparent to average students, yet readily accessible with typical high school mathematics knowledge and curriculum, which included arithmetic, algebra, and geometry (CCSSI, 2010). The first problem was an arithmetic-inequality problem (Problem 1) with four approaches (P1A1, P1A2, P1A3, and P1A4), the second problem was an algebra-of-two-variables problem (Problem 2) with eighth approaches (P2A1, P2A2, P2A3, P2A4, P2A5, P2A6, P2A7, and P2A8), and the third problem was a geometry-of-angle-measurement problem (Problem 3) with three approaches (P3A1, P3A2, and P3A3).

Problems	Descriptions
Problem 1	Fill in the blank with one of the symbols $<$, \leq , $=$, \geq , or $>$. $\sqrt{2009} + \sqrt{2011}$ _____ $2\sqrt{2010}$
Problem 2	Given $x^2 + y^2 = 1$, find maximum of $x + y$.
Problem 3	Given triangle ABC with median \overline{CD} and $CD = BD$, find measure angle ACB .

Table 1: Three non-standard mathematics problems

The researcher interviewed the three expert mathematicians individually. In each interview, the researcher presented each expert mathematician with the three problems and 15 approaches. The researcher first asked each expert mathematician to choose his or her most preferred approach for each of the three problems. The expert mathematicians were then to rank order the approaches for each problem from the most preferred to the least preferred, and to provide careful explanations for why they placed those approaches in such order. A collective choice was determined if at least two of the expert mathematicians ranked the approach the same, or in the case where each expert mathematician assessed different ranks for the approach, the mean rank of each approach was computed and the lowest mean was utilized. Furthermore, the three expert mathematicians' explanations as to why they preferred each of the 15 approaches more or less to the others were analyzed qualitatively. The researcher identified a couple of premises that were shared in these expert mathematicians' explanations, namely, simplicity and originality, consistent with how mathematical aesthetics were discussed in earlier studies. With respect to simplicity, the 15 approaches were coded as follows: very simple, somewhat simple, not quite simple, and not simple were coded as 1, 2, 3, and 4, respectively. With respect to originality, the 15 approaches were coded as follows: very original, somewhat original, not quite original, and not original were coded as 1, 2, 3, and 4, respectively. Explanations of the three expert mathematicians were then synthesized for each of the 15 approaches.

The students were explained that they were to creatively solve the three problems using as many different approaches as they could without calculator and without time limit. They were reminded several times that they could take as much time as they needed to think about and write down in their test as many different approaches as possible. After the test, the students' written responses were examined for correctness. The nine students were interviewed individually to elicit their explanations for the approaches they supplied in the test. In the interview, they were asked to explain how they came up with their approaches to the three problems.

Following the interview, the students were provided with the 15 approaches and were surveyed to examine the students' thoughts on the 15 approaches for the three problems, their most preferred approaches, and their overall reactions to the aesthetic view of expert mathematicians. Some questions included whether they understood each of the 15 approaches, whether they thought they had learned in their previous mathematics courses the mathematics content involved in each of the 15 approaches, which of the 15 approaches would they prefer the most, and whether any of the three approaches that mathematicians considered to be the most "beautiful" approaches (i.e., P1A1, P2A1, and P3A1 for Problems 1, 2, and 3, respectively) appealed to the students to any extent. In this sense, the students were explicitly informed that mathematicians' preferred approaches were considered by these mathematicians themselves to be the most "beautiful" approaches.

The findings of the paper-and-pencil test and the interviews with the students were analyzed to comprehend similarities in the justifications provided by the nine students

to supply particular approaches to the three problems. The responses to the survey items were tallied to determine the students' understanding of each of the 15 approaches, their acknowledgement of having learned in their previous mathematics courses the mathematics content involved in each of the 15 approaches, their most preferred approaches for each of the three problems, and their attraction to each of the three approaches most preferred collectively by the mathematicians.

FINDINGS

Collectively, P1A1, P2A1, and P3A1 were preferred the most, while P1A4, P1A8, and P3A3 were preferred the least, by the three expert mathematicians for Problems 1, 2, and 3, respectively (see Table 2). The approaches that were rated as most simple and original were characterized mainly by the surprising manner in which the information given in the problems were interpreted so unusually that the solutions to the problems revealed themselves naturally. For instance, P1A1 for Problem 1 was considered to be very simple since it did not treat 2010 as a single number, but rather as an average of two numbers, namely, 2009 and 2011. P1A1 was also very original because it was resolved by recalling the visual concavity of the square root function, allowing the proof to be comprehended effortlessly. The approaches that were rated as least simple and original were characterized mainly by the blunt manner in which the information given in the problems were processed without any refinement so that the solutions to the problems appeared strained. For example, P1A4 for Problem 1 was considered to be the least simple and original not only because it required tedious arithmetic calculations of four-digit multiplications, but also because it construed square roots in a most elementary concept as an arithmetic operator.

	P1A1	P1A2	P1A3	P1A4	P2A1	P2A2	P2A3	P2A4	P2A5	P2A6	P2A7	P2A8	P3A1	P3A2	P3A3
Prof 1	1	2	3	4	1	2	3	4	5	6	7	8	1	2	3
Prof 2	3	2	1	4	2	6	3	1	4	7	5	8	1	2	3
Prof 3	1	3	4	2	1	4	2	8	5	6	7	3	3	2	1
Collective	1	2	3	4	1	2	3	4	5	6	7	8	1	2	3

Table 2: Expert mathematicians' choices of mathematically "beautiful" approaches

The students who took the paper-and-pencil test were generally able to finish the test in less than one hour. One student successfully solved all three problems, one student successfully solved Problems 1 and 2, four students successfully solved Problems 1 and 3, one student successfully solved Problem 2, and two students successfully solved Problem 3. The one student who successfully solved all three problems supplied two approaches for Problem 3 (i.e., P3A3 and P3A2), but only one approach for Problems 1 and 2 (i.e., P1A2 and P2A6). The other eight students solved Problems 1, 2, and 3 using only one approach (i.e., P1A4, P2A8, and P3A3).

The three problems appeared to pose some challenge for the students to solve, and in addition, the instruction to supply as many approaches as possible might be something

that the students were not familiar with. Despite the unrestricted time to work on the test, the students appeared to be easily content with supplying only one workable, yet mechanistic approach as long as they obtained a correct answer to each problem. One might describe such problem-solving experience as lacking in reflective thinking, apart from flexibility and creativity.

It was clear that the role of aesthetics was limited in the students' considerations as they problem-solved. Much more evident in the students' written responses than elegance was impulsiveness. During the interviews, several students acknowledged in preferring P1A4 in Problem 1 that "when dealing with square roots ... what usually comes to me first was squaring both sides," and then "you just kind of hacked away at it [because] you do this big multiplication, and you finally get this large number is bigger than that large number." In resorting to P2A8 in Problem 2, the students confirmed that short-term memory recall of their most current mathematics course (i.e., AP Calculus) prompted their reflex to take derivative of an objective function for, as one student explained, "I'm learning calculus right now, so I figure why not use calculus, which is still fresh, more fresh." In using P3A3 in Problem 3, many students revealed their confidence and comfort in building up information step by systematic step until the solution appeared, as one student said, "I chose [P3A3] because of the whole logical following it." The students' choice of using the approaches in the paper-and-pencil test might therefore be viewed as an instinctive one with the sole intention to find, in the shortest amount of time and the least number of steps, the answers to the problems, albeit without any other meaningful aesthetic considerations.

Furthermore, there was no direct relationship between mathematicians' and students' views of "beauty" in mathematics. These views were grounded not only in how they perceived the three problems, but also in how they approached them. Although majority of the students indicated that they had no difficulty in understanding the mathematicians' most preferred approaches, only a few would prefer them to the rest of approaches for those three problems. Even those students who were in agreement with the mathematicians' choice of most preferred approaches were for the most part not able to provide adequate explanations for the aesthetic value of those approaches. They were only able to see the outward appearance of those "beautiful" approaches. For instance, P1A1 was considered to be "beautiful" because of the relatively shorter lines of argument, P2A1 because of the "helpful" presence of the graph accompanying the solution, and P3A1 because of the physical shape of the parallelogram that resembled "a diamond." Clearly, the mathematicians' preferred approaches did not appeal to the students as "beautiful" in the sense of the deeper structure of the mathematical arguments involved in those approaches.

CONCLUSION AND DISCUSSION

The present study sought to analyze to what extent mathematical aesthetics might be viewed different by different groups of people. It demonstrated differences in the motivations that guided different groups of problem solvers. While the three expert

mathematicians appeared to be more satisfied after a “beautiful” approach had been identified, the nine students in general stopped the problem-solving process once any workable solution was proved to have answered the problem. Unlike expert mathematicians who saw beauty in mathematics as an exhaustive consequence of simplicity and originality, students found mathematical elegance only in approaches that were efficient in terms of time and number of steps to solve the problems. While expert mathematicians considered students’ most preferred approaches to be the least “beautiful,” students, showing no enthusiasm, considered expert mathematicians’ most preferred approaches to be no more attractive than their own approaches.

The mismatch between the students’ most preferred approaches and those of the mathematicians did not appear to be a consequence of the students’ lack of mathematical proficiency, but rather, at least partially, the students’ lack of appreciations of mathematical “beauty.” This evidence suggested that the presence of such appreciations among mathematicians indicated to a certain extent that such competence might have been learned, cultivated, shared, and recognized within the community of professional mathematicians quite possibly beyond the high school level. To some extent, there appeared to be a profound lacuna in the understanding of mathematical aesthetics that might inadvertently subdivide the state of mathematics problem solvers into two groups: one group of professional research mathematicians and another group of those whose affects might be waiting to be nurtured.

Despite the rigorous selection process of students in the study, it became clear that mathematical “beauty” was not a consideration that young problem solvers grasped automatically, but also that they had not been exposed to such aesthetic appreciations, as defined by expert mathematicians, until much later when serious work of mathematics might be involved. Related to the findings by Silver and Metzger (1989) and Koichu and Berman (2005) was the three expert mathematicians’ constant reference to geometric reasoning in their explanations of their most preferred approaches for the three problems with respect to simplicity and originality. Nonetheless, such persistent pursuit of geometric interpretations did not appear greatly in the ways that the nine students explained their most preferred approaches. To some extent, therefore, aesthetic appreciations evolved partly around geometric interpretations, and more importantly, the search for such geometric interpretations, as part of aesthetic considerations, might be a learned skill, instead of an innate skill.

As the methodology employed in the study suggested, problem-solving experience using many different approaches, as well as the discussion that compared and contrasted their advantages or disadvantages, might be facilitated in a mathematics classroom setting. Given this frequent accumulation of different approaches either discovered by themselves or presented by their classmates or teachers, students might begin to grow their sense of mathematical aesthetic appreciations. To this end, mathematics curriculum might find the consideration of mathematical aesthetics, conceivably as a measure of flexibility and creativity, to be worthwhile if not exigent.

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TENSIONS IN STUDENTS' GROUP WORK ON MODELLING ACTIVITIES

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In this paper we study the modelling activity of secondary school students through the lens of Cultural Historical Activity Theory (CHAT) perspective. Our focus is on the tensions emerged throughout students group work. A particular mathematical modelling task was implemented in a lower and an upper secondary mathematics classroom (9th and 11th Grade respectively). The analysis of 24 episodes of tensions in the two classrooms revealed (a) task-based and group-based sources of tensions; (b) different resolution processes (bringing to the fore a given tool or providing a new mathematical or non-mathematical tool); and (c) who acted as facilitator in the above processes, namely the teacher or the group itself. Finally, commonalities and differences between the lower and the upper students' group work are also considered and discussed.

RATIONALE OF THE STUDY

Modelling activities refer to using mathematics to solve realistic situations and open problems. Among others, students' involvement in modelling activities provides opportunities for students to observe, communicate, explain, reflect, and thus build mathematical concepts based on meaning and inquiry (Maaß, 2006). The importance of modelling activities in today's world is highlighted by many researchers (Barbosa, 2006; Blum & Borromeo Ferri, 2009; Sriraman & Lesh, 2006; Wake 2015). In out of school practices problem solving involves mathematical processes as interpretation, description, explanation and argumentation more than computation or deduction (Shiraman & Lesh, 2006; Wake 2015). Also in school practices, as Christiansen (2001) argues, it is not only the content but the social organization of classroom activity as well that play a decisive role to the modelling activity outcome. In general, negotiation of meaning of a specific situation (in school or in out of school joint activities) might cause tensions and conflicts among the participants. The focus on tensions on classroom modelling activities is mainly related to tensions experienced by teachers when developing modelling-based lessons (de Oliveira & Barbosa, 2013) or to tensions experienced by students when they have to make necessary connections between abstract mathematical models and physical phenomena (Carrejo & Marshall, 2007).

In this study, we focus on tensions emerged during students' group discussions while they are working on solving the same modelling task in a lower and an upper secondary Greek class. In particular, the study was guided from the following research questions:

Q1: What are the main sources of tensions identified in student groups' modelling activity? How are these tensions resolved?

Q2: How the sources of tensions and the resolution process differentiate between lower and upper secondary groups' modelling activities?

THEORETICAL FRAMEWORK

Our main theoretical and methodological tool to explore tensions (dilemmas, conflicts) emerged among group members while participating in a modelling activity is the expanded model of Cultural Historical Activity Theory (CHAT), and in particular the work of Engeström (1999). The fundamental recognition of activity theorists is the fact that in an activity the relationship between the subject and the object of the activity is mediated by a series of situational factors, including the means of production (tools, materials), the subjects' local needs, and the community's traditions and rules (Engeström, 1999). The activity as a whole is characterized by inner contradictions (dilemmas, conflicts, disagreements) which are realized as tensions within the activity system. According to Engeström (ibid.) contradictions and tensions are important aspects of activity systems because they lead to change and development.

In this study, we analyze groups' tensions emerged during solving the 'Solar panels' task (object of the activity) in a lower (Grade 9) and an upper (Grade 11) secondary school classroom. We consider as *subjects* the student groups; as *tools* the means that mediate groups' discussions (e.g., contextual resources provided by the teacher, mathematical tools and processes); as *rules* the social conditions which control groups' actions and as *division of labour* the distribution of actions and operations in which students are engaged. We also consider the two classrooms as *two different learning communities*, due to the difference between them regarding students' experiences in mathematics. Finally, we consider as tensions the conflicts and disagreements emerged among the group members as well as dilemmas expressed by them throughout their modelling activity.

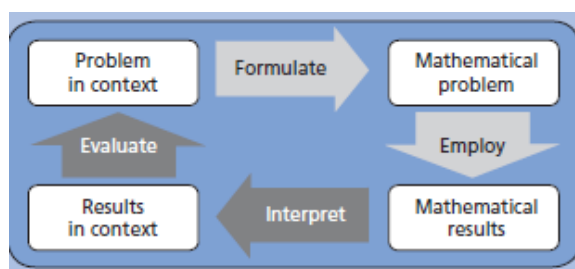


Fig. 1: The modelling circle (OECD, 2013, p.26)

In order to analyse students' modelling activity we adapt the model suggested by PISA 2012 diagram (Fig. 1). This model captures the cyclic nature of the activity and identifies four main processes that underlie a mathematical modelling route: In particular, the processes are: *Formulating* where the problem in context is transformed into a mathematical problem which is amenable to mathematical treatment; *employing* that involves mathematical reasoning that draws on a range of concepts, procedures,

facts and tools to provide the mathematical solution; *Interpreting and evaluating* that involves making sense, and considering the validity, of the mathematical results/solution obtained.

METHODOLOGY

The context

This paper refers to a study that took place in the context of a European project, Mascil (see: www.Mascil-project.eu). This program is intended to enhance the mathematical experience of students through fostering inquiry based learning by using modelling activities on authentic workplace settings. In this report we concentrate on two implementations of a particular Mascil task, the Solar panels problem, one in a 9th Grade and one in an 11th Grade mathematics classrooms. The two implementations took place during the school year 2014-15 and lasted two teaching hours each. In both cases, the students in the classroom were separated into groups of 4-5 students and all groups worked collaboratively for the solution of the problem.

The task

The *Solar panel* problem was about the installation of solar panels on a house roof top. The object of the activity was to calculate the maximum number of solar panels that could be placed on the roof of a house. Solving the problem required students to calculate the projected area of the panel on the roof by using trigonometric ratios. Students in both classrooms have been taught trigonometric ratios, but their familiarity with applications in geometric solids (e.g., projections) was rather limited. Moreover, modelling activities are not so common in the Greek Mathematics Curriculum.

In Figure 2 we present a brief description of the task and some of the representations that were included in the given worksheets.

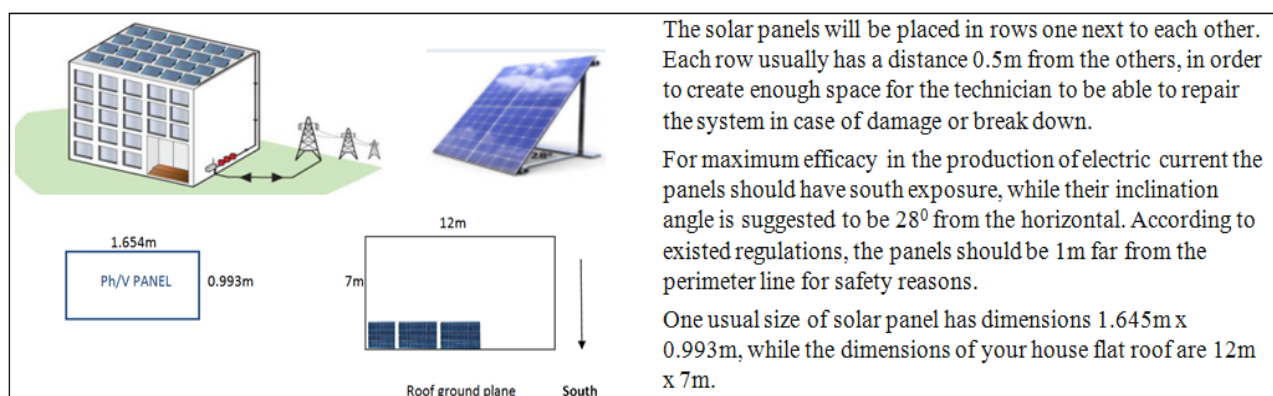


Fig. 2: Brief description of the task.

A detailed description of the solar panel problem is available in the mascil-project website (<http://www.mascil-project.eu/classroom-material>).

Participants and data

In this study, we focused on tensions that emerged while five groups of students were working on solving the *Solar panel* problem in two different classrooms, three groups

in a Grade 9 classroom (13 students) and two groups in a Grade 11 classroom (8 students). These five groups worked as a team and exhibited strong interactions among their members. The data consisted of the audio recordings from the five groups' discussions during the two - teaching hour implementation of the task while students' written work and video recording of the classroom activity were additional sources of data.

Data analysis

Qualitative content analysis has been employed for the analysis of groups' emerged tensions (Mayring, 2000). Initially, the data were transcribed so we could distinguish the main group's modelling actions, namely the central actions followed by the majority of groups in their attempt to solve the problem. A plurality of right or wrong hypotheses and ideas could be identified in each modelling action. Some of them were overlooked by the group, while others caused episodes of tensions (conflicts and dilemmas) within the group. Our analysis focused on the latter case. We distinguish episodes of tensions where one or more group members questioned their classmates' ideas or strategies or posed an alternative idea under discussion, and the group engaged in a process to respond or resolve the dispute. In each episode, we were particularly interested in: (a) what was (were) the source(s) of a tension? (b) who facilitated the resolution process? and (c) how was the tension finally resolved? We analyzed 24 episodes of tensions in groups' discussions in total. First we coded the various episodes in terms of what, who and how, as we described above, and then we classified the emerged codes into general categories. The produced scheme of the general categories was tested by the three researchers through the whole set of data. Finally, we traced the above categories as they appeared in the two learning communities.

FINDINGS

We distinguished three main modelling actions in groups' work that constituted parts of the students' routes: (action 1) calculate the useful roof area, the area of the panel and divide them. This action was faulty since students simplified the problem into a two - dimensional base by ignoring (consciously or not) that the panels were placed on the roof top with an inclination; (action 2) translating the problem in the three-dimensional space by utilizing the projections of the panels on the roof through the use of trigonometric ratios; and (action 3) examining alternative ways to place the panels on the roof. All groups engaged in the 1st modelling action but episodes of tensions among group members acted as catalyst in helping the groups reconsidering their strategy, re-formulating the problem and continuing successfully with the 2nd action. This was the case in all groups but one, the one of the 11th Grade groups, where the members failed to overcome the emerged tension.

In Figure 3, we present the scheme of categories that emerged from the data analysis as regards the sources of tensions, the facilitators of tensions treatment and the ways the tensions resolved. As far as the sources of the tensions are concerned, we identified two main categories: (a) task-based sources and (b) group-based sources. Among the

task-based sources we distinguished three sub-categories: (i) Neglect given information, namely when some students in the group overlooked the given contextual information regarding the panel's inclination while others had a more global view of the situation; (ii) Misleading interpretations of a contextual information, as for example, the group members interpreted the given restriction "the panels should be placed 1m far from the perimeter line" differently. Particularly, some students suggested subtracting one meter from each dimension, while others insisted on subtracting two meters from each dimension; and (iii) Limited understanding of the underlying mathematical notions (e.g., some students in the group had difficulties to employ trigonometric ratios in their mathematical solution). The group-based sources concerns the case that the modelling route was developed at different pace among the group members, something that also created tensions in group's activity.

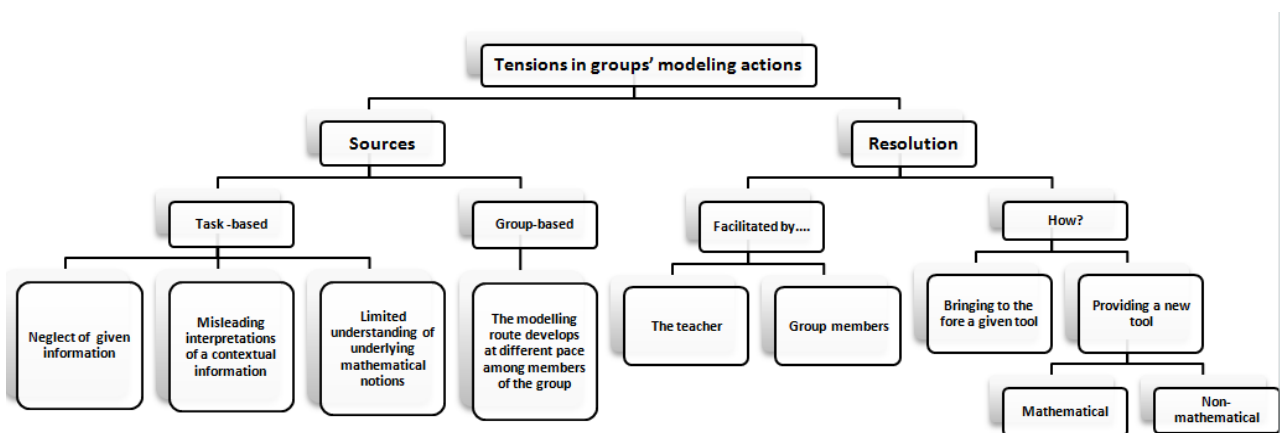


Fig. 3: The scheme of categories

A tension was resolved either internally by the support of some group members or externally by the support of the teacher. In both cases, two types of actions were employed by the facilitators: (i) bringing to the fore a given tool (i.e. contextual information, representation etc.) and (ii) providing a new tool. Among these new tools, we discerned the use of mathematical tools (e.g., arguments, notions, questions); and the use of non-mathematical tools (e.g., everyday objects, technologies, drawings, gestures) in facilitators' actions.

Differences identified between upper and lower secondary learning communities

As regards the sources of tension, the task-based sources were common in both communities while the group-based source was present only in the case of the upper secondary class. This could be explained by the fact that in Grade 11 some students seemed to have a strong mathematical profile and be ahead of the other members, while in Grade 9 all group members seemed to progress at the same rate. Regarding the facilitator, there was a clear difference between the two communities. The groups in Grade 11 resolved the observed tensions internally, either by exchanging ideas or by the assistance of a leader-student, usually the one with a strong mathematical profile. The students in this community asked rarely for the teacher's help and when they did so, the teacher preferred not to intervene. On the contrary, in Grade 9, in most cases

the tensions were resolved by the teacher's intervention. The teachers' different approaches in classroom management seemed to have an impact in groups' modelling work in the two communities. Finally, the ways the tensions were resolved were similar in both communities, but the employment of mathematical tools was more frequent in the case of the 11th Grade groups. Moreover, the employment of new tools in tension treatment, seemed to be rather important for changing the members' views and stance, and resulted to a development in the modelling activity. On the contrary, when a group tried to resolve a tension exclusively based on given information, the group missed the chance to open up to alternative directions/strategies. This is the case in the one 11th Grade group where the members didn't manage to complete the activity successfully.

Below, we present two characteristic examples from the two communities that illustrate some of the above findings.

Episode 1: Group B_Grade 9

stA: Are we interested in the area [of the solar panel];

stB: Yes we are, in order to find how many [panels] can be placed on the roof top.

stA: Yes, but are we going to lay them down [on the roof]?

stB: I believe that it is necessary. [The group after a lot of discussion decides to ask for the teacher's help]

stC: How could we find the number of panels we can place on the roof?

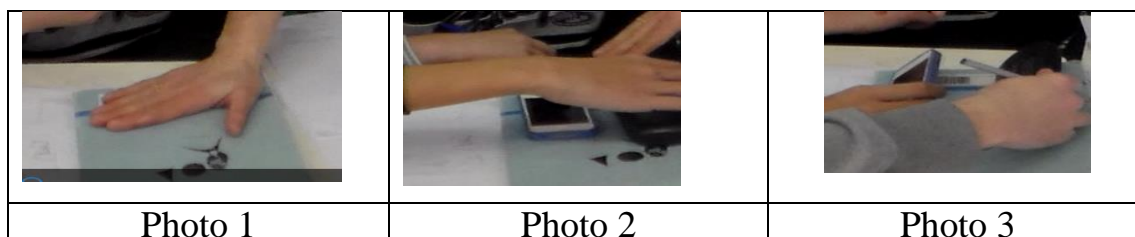


Table 1: Teacher's intervention and students' responses.

Teacher: This is the roof [the textbook], and these are the panels [two cell phones]. Can you show me how I could place them on the roof? (see photo 1).

StC: [places cell phones on the textbook without taking into consideration that the panels should be on an inclination, see photo 2].

Teacher: [refers to the rest of the group] Do you agree?

StA: No [he places them with inclination, see photo 3].

Episode 2: Group B_Grade 11

StX: Can I ask a question? In order to find the maximum number of the panels that fit up here, isn't it reasonable to find the exploitable area of the roof, then the area of the panel and then see how many can be placed here by dividing them?

StY: What you are suggesting is not valid since the panels will be placed under inclination, so we need to consider the projection area of the panel.

StX: Yes but isn't it the same?

StY: No it is not. Look at me, you are going to place the panel this way not that way [makes gestures to show that the panel will be in inclination] Look! Does the hypotenuse of a triangle have the same length with the one of the vertical sides?

SBX: Aaa, ok! I understand now.

Both episodes are based on a common task-based source of tension i.e. neglect of using given information, but the tension was resolved differently in each community. The 11th Grade group resolved the tension internally when StY brought to the fore the appropriate information and provided new tools (posed a mathematical questions, provided mathematical arguments and used gestures) in order to convince his classmate. On the other hand, in the 9th Grade group, StA brought to the fore the panels' inclination but he did not had the appropriate tools (mathematical arguments) to convince his classmates, so the group decided to ask for the teacher's help. The teacher treated the tension by posing inquiry questions and employing non-mathematical tools (e.g., the cell phones as panels).

CONCLUDING REMARKS

The lens of CHAT helped us to gain insight on the situational and social factors that influence student groups' modelling activity. We consider as situational factors the everyday objects, gestures and arguments employed by the community members as they were facilitating the emerged tensions. As social factors we consider the teachers' approaches on classroom management and the group members' interactions. Our analysis indicated that all student groups faced tensions during the process of formulating mathematically the real situation. These tensions played a decisive role to the modelling activity outcome. Moreover, the tensions emerged affected the linearity of the students' modelling activity, since as shown above, the groups returned in previous processes before completing the modelling cycle (Fig. 1). The non-linearity of the modelling cycle have been also discussed by other researchers (e.g., Blum & Borromeo Ferri, 2009). In addition, what seemed to be rather important for the resolution of a tension, was the enrichment of the activity with new tools (e.g., everyday objects, gestures and mathematical arguments) beyond those given in the task. Such tools proved to mediate effectively the tensions treatment since they were acting as a resource for the negotiation of new meanings. In this way, the group managed to overcome the tensions fruitfully and moved progressively from the one problem state to the next. The emerged categories of sources of tensions, revealed three dimensions regarding the complexity inherent in students' group work on modelling activities: the mathematical content (use appropriately mathematical tools); the real context (understand and simplify the real situation); and the social environment (collaborate fruitfully with peers). Group-based modelling activities can support the development of a harmonious interplay among these dimensions, and therefore it is important to strengthen their role in the mathematics classrooms.

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A NEW FRAMEWORK BASED ON THE METHODOLOGY OF SCIENTIFIC RESEARCH PROGRAMS FOR DESCRIBING THE QUALITY OF MATHEMATICAL ACTIVITIES

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This paper proposes a new framework for describing the quality of mathematical activities under radical constructivism. It is based on Lakatos' philosophy of science, instead of his philosophy of mathematics. We focus on a structural similarity between mathematical problem-solving activities and scientific research programs. While Lakatos' philosophy of mathematics is only a model of a progressive activity, the new framework can distinguish between progressive and degenerative activities. To show its usefulness, we provide a sample analysis. Based on the analysis, we hypothesize that the zig-zag process of solving a mathematical problem is driven by a hard core: A set of one's unrevised assumptions that one would like to continue to maintain. The necessity of further research with the proposed framework is suggested.

INTRODUCTION

Lakatos' (1976) logic of mathematical discovery (LMD), known as proofs and refutations, is one of the most cited philosophies in mathematics education research. It characterized mathematics as an informal repeated process of conjecturing, proving, and refuting. Based on the LMD, several scholars have advocated a fallibilistic nature of learning mathematics (e.g., Confrey, 1991; Ernest, 1998; Lampert, 1990). The application range of the LMD is wide: From problem-solving at the elementary school level (Lampert, 1990) to theorem reinvention at the undergraduate level (Larsen & Zandieh, 2007). However, as Sriraman and Mousoulides (2014) point out, “[t]he didactic possibilities of Lakatos' thought experiment abound but not much is present in the mathematics education literature in terms of teaching experiments that try to replicate the ‘ideal’ classroom conceptualized by Lakatos” (p. 513).

The rare replications of the LMD style in classrooms stem from the gap between naïve and sophisticated mathematical activities. Although the LMD suggests a fallibilistic nature of learning mathematics, disagreements about a conjecture do not always contribute to mathematical development in a classroom. Note that the LMD originates from sophisticated activities among professionals, not among novices. We need more empirical data on the relationship between naïve and sophisticated activities.

This paper proposes an alternative theoretical framework for describing mathematical activities. The proposed framework is based not on Lakatos' (1976) philosophy of mathematics, but his philosophy of science (1978): The methodology of scientific research programs (MSRP). The LMD is useful for describing relatively sophisticated

activities (e.g., Larsen & Zandieh, 2007), but the proposed framework based on the MSRP will be able to describe both naïve and sophisticated mathematical activities and will provide a descriptive framework for contrasting the two.

This paper consists of the following sections: (1) an overview of the MSRP, (2) an overview of radical constructivism (RC) proposed by von Glasersfeld (1995), (3) the proposal of a new theoretical framework, and (4) a sample analysis. Through the analysis, we will argue the usefulness of the proposed framework for describing mathematical activities.

LAKATOS' PHILOSOPHY OF SCIENCE

The scientific research program (SRP) is a series of activities with the same paradigm carried out by scientists. An SRP contains a hard core and protective belts. The hard core is a set of theoretical assumptions and the protective belts are auxiliary hypotheses, and any scientific claim in the SRP is based on both. If a counterexample of the claim is observed, either parts of the core or some of the belts are false. Thus, scientists, like pseudo-scientists, do not have to give up their own hard core and can protect it by revising some of the belts. This process is called a problem shift. In principle, the assumptions in the hard core can be arbitrarily selected. Lakatos (1978) abstracted this methodology from the history of science.

Although Yuxin (1990) pointed out the similarity between the LMD and the MSRP, there is a significant difference between them: The spirit of the LMD is “antidemarcationist,” while that of the MSRP is “demarcationist” (Ernest, 1998, p. 111). That is, Lakatos provided a distinction between good and bad scientific activities: Science must predict the next empirical evidence. If an SRP predicts the next empirical evidence, its problem shift is called progressive; if not, it is called degenerative. In principle, the LMD cannot require mathematicians to completely give up a mathematical research program because the LMD is related to informal mathematics and not pseudo-mathematics. On the other hand, MSRP requires scientists to completely give up an SRP if it cannot predict the next empirical evidence.

RADICAL CONSTRUCTIVISM

RC is a philosophy which begins from “the assumption that knowledge, no matter how it be defined, is in the heads of persons, and that the thinking subject has no alternative but to construct what he or she knows on the basis of his or her own experience” (von Glasersfeld, 1995, p. 1). This assumption leads to the possibility that even if an observed behavior looks irrational from the observer’s perspective, it is rational from the behavior’s own perspective. Therefore, any learner’s behavior should be interpreted as at least *locally rational* from his or her own perspective at that moment (Confrey, 1991; Uegatani & Koyama, 2015).

For our purpose, we introduce two key concepts in RC: viability and action scheme. The concept of *viability* is: Pieces of knowledge are viable “if they fit the purposive or descriptive contexts in which [learners] use them” (von Glasersfeld, 1995, p. 14).

Action schemes (AS) consist of the following three parts: “1 Recognition of a certain situation; 2 a specific activity associated with that situation; and 3 the expectation that the activity produces a certain previously experienced result” (von Glasersfeld, 1995, p. 65).

Let us consider the example of an AS and its viability. Suppose a learner needs to solve an equation $x^2 = 3179$ (1. Situation). His or her next activity will be to test the divisibility of 3179 by 2, 3, 5, and 7 (2. Activity) with an expectation that 3179 is divisible by a certain number (3. Expectation). Using the AS means testing the consistency between the expected and the actual results of the activity. If the results are consistent, they will become more viable. If not, they will become less viable or be revised.

An AS can be revised in certain ways. Importantly, when a learner senses inconsistency, he or she cannot uniquely determine what causes it. In the above example, since 3179 is divisible by neither 2, 3, 5, nor 7, the learner may sense inconsistency. Then, he or she can arbitrarily suspect at least either the suitability of divisibility testing in the situation or the sufficiency of testing integers from 2 to 7. If the learner chooses the former, he or she may solve the inconsistency by considering the activity not suitable for the situation. If he or she chooses the latter, he or she may solve the inconsistency by considering that the activity should test divisibility by 11. The AS can be arbitrarily revised as long as the inconsistency is solved (Uegatani & Koyama, 2015). “The viability of concepts [...] is not measured by their practical value, but by their non-contradictory fit into the largest possible conceptual network” (von Glasersfeld, 1995, p. 68).

A NEW THEORETICAL FRAMEWORK

There is a structural similarity between an AS and an SRP. The concept of viability corresponds to that of progressiveness. An AS has the following three features. (AS-a) If the AS predicts the next expected result, it remains viable and if not, becomes less viable. (AS-b) Even if the AS is viable at one moment, there may not be any consistency between the expected and the actual results in the next moment. (AS-c) When dealing with an inconsistency, the AS can be arbitrarily revised whether it becomes more or less viable. Similarly, an SRP has the following three features. (SRP-a) If the SRP predicts the next empirical data, it remains progressive, and if not, becomes degenerative. (SRP-b) Even if the SRP is progressive in one moment, there may not be any consistency between the predicted and the actual data in the next moment. (SRP-c) When dealing with an inconsistency, the SRP can be arbitrarily revised, whether it becomes progressive or degenerative (though a degenerative SRP is not qualified as science). Thus, in the analogy with an SRP, when we observe a revision of an AS, we will be able to identify the elements corresponding to “protective belts” and “a hard core.” In this context, *protective belts* can be defined as pieces of knowledge used by the learner to predict a result, but recognized as inappropriately used; a *hard core* can be defined as a set of unrevised assumptions the learner would

like to continue to maintain. We propose the MSRP based framework, which focuses especially on the hard core of a mathematical activity. The advantage of the new framework compared to the LMD based framework is that it enables us to describe the quality of mathematical activities as progressive or degenerative, for example, to understand the variation between progressive and degenerative activities.

SAMPLE ANALYSIS

To show the usefulness of the framework, we provide a sample analysis.

Background of a sample

The sample episode was videotaped in a part of the first author's mathematics lesson. This is a transcript of 11th grade students' group work. The group members (all names are pseudonyms) were Mr. Ham (leader), Ms. Uts (subleader), Mr. Ike (recorder), Mr. Tak (calculator), and Ms. Hor (presenter). Although each member was given his or her role to enhance the group discussion, the roles were often forgotten because of the heated discussion. The given task was identifying more digits of 2^{54} than other groups. The following episode is a vignette taken while performing the task.

Episode in a group work

Ike had already predicted the need for a logarithm before the task was presented:

6 Ike: Maybe, we are to refer to the table of common logarithms.

7 Ham: Really? ... Like enough.

The reason why they predicted the need for a logarithm seems to be that they had learned to use the table of common logarithms in the last class. After the task was presented, Ham immediately decided to use common logarithms.

8 Ham: OK, take the common logarithm. The common logarithm of 2.

17 Ike: OK, well, 0.3010 (Referred to the table of common logarithms).

18 Ham: Calculate 54 times 0.3010. (Said to the student with the calculator, Tak)

22 Tak: (using a pocket calculator) 16.254.

On the other hand, Hor, who observed the boys' approach in silence, suddenly started to calculate 2^{54} by paper and pencil with Uts, but independent of the boys:

25 Hor: [Inaudible] ... let's calculate 2^{54} . (Said to Uts, and started to calculate)

26 Uts: Oh....

27 Ham: So, is the value between 16^{th} and 17^{th} powers of 10?

28 Ike: Yes, yes.

Ham and Ike continued their approach without paying attention to the girls:

29 Ham: Ah ..., so, then ..., 16 digits ..., Uh

30 Ike: So, after that, so, taking the logarithm of it, 16. ..., 16.254. So, try to find a value as near as possible to 16.254 repeatedly. Maybe, we should take the antilogarithm of the value.

However, Tak alone started trying to directly calculate 2^{54} with a pocket calculator after observing the girls' approach. Hor noticed that, stopped calculating, and tried to communicate with the boys:

- 41 Hor: Hey, how many digits can the pocket calculator use? (Said to Tak)
 42 Tak: Um ..., 1, 2, 3, 4, 5, 6, 7, 8.
 43 Ham: So, how can we calculate $10^{16.3}$, for example? (Said to Ike)
 44 Hor: No way. (Said to Tak)
 45 Ike: We can do it if we can calculate $10^{0.3}$. (Said to Ham)
 47 Ham: Wow! Oh! That's true!

Immediately after hearing Ham's exclamation, Hor asked Ham:

- 48 Hor: What did you say? What of 10? (Said to Ham)
 49 Ham: So, so, decompose ..., in case of 16.3^{th} power ..., $10^{0.3}$..., and what is 10^{16} ?
 50 Ike: Ah, so, let's use the table of the common logarithms. If you find the value whose common logarithm is 0.2 in the table,

However, Ham and Ike were absorbed in their thinking and perhaps unintentionally neglected Hor. Then, Hor gave up her communication with them.

After that, Hor and Uts continued to calculate by paper and pencil together. Ike began to seek the next promising step alone, and Tak proposed his opinion:

- 58 Ham: Ike might solve alone
 60 Tak: Let's calculate 2^{54} in a step-by-step fashion!
 61 Ike: (Laugh) I don't recommend it.
 62 All: (Laugh)
 63 Uts: But, now she is calculating (Pointing to Hor)
 64 Hor: Without thinking difficult math, ah ..., simply 1024^5 times 16.
 65 Ike: Do you have enough courage to calculate it?
 66 Hor: Yes, let's calculate it.
 67 Uts: Now, she has already been calculating.
 68 Hor: Yes, now I am calculating.

Despite this communication, Ike and Ham ignored Hor's approach. Tak began trying to directly calculate 2^{54} independent of the other members of the team.

Although Ike had directed Ham in solving the task until that time, Ike's original plan started becoming unstable. Consequently, they began supporting each other.

- 71 Ike: The direct reference (to the table of the common logarithms) might be better. So, the target is $10^{16.254}$..., 0.254, 254, (searching the nearest value of 0.254 in the table of common logarithms) ... about 1.8?
 72 Ham: No, (the common logarithm of) 1.79 is nearer (to 0.254).
 73 Ike: 1.79 ..., so, oh, what can we do next?

- 74 Ham: Multiplying 10^{16} ... (Writing down “17900000000000000”). So, “nearly equal” is not clear. In this case, “equal to or more than” is suitable, isn’t it? “More than,” isn’t it?
- 75 Ike: But, any further inquiry is impossible because of (the precision of) our table of common logarithms.
- 76 Ham: Umm, in that case, is it better to calculate this (pointing to the common logarithm of 1.8 in the table). 1.8 means multiplying 2.54? No, it doesn’t.
- 77 Ike: It means (Writing down $1.79 \times 10^{16} < 2^{54} < 1.80 \times 10^{16}$). (Note: Their judgment was mathematically incorrect because their consideration to a margin of error is not proper.)

Finally, after identifying some digits of 2^{54} , their discussion became deadlocked:

- 89 Ike: Now, what can we do next? There is no cue (for raising the precision)
- 90 Ham: Improving is impossible by using our table of common logarithms, isn’t it?
- 91 Ike: Now what can we do?

Then, Ike noticed Hor and Uts’s progress:

- 93 Ike: ... Oh, you all have been really calculating by paper and pencil!
- 94 Hor: Really, we are still calculating.
- 95 Ike: Really?
- 96 Hor: If our calculation is finished (Hor and Uts had already finished calculating 2^{40} and 2^{14}), then we will finish all.
- 99 Ike: Oh, what can we do? What can we do? (Laughing and looking around)

Discussion

From the beginning of the episode, Ike and Ham seemed to share the same AS. Although they often found inconsistencies between the expected and actual results of their activities (e.g., #29, #49, #73, and #75), they immediately tried to change the interpretations of either their situations, or their activities in order to eliminate their sensed inconsistency (e.g., #30, #50, #74, and #76). Therefore, we can say that the pieces of knowledge that formed the rejected interpretations were the protective belts, while the unrevised assumptions that using logarithms is a better approach were the hard core, and that using logarithm seemed to be a policy rather than a conclusion. Although Yuxin (1990) argued that the term “hard core” in the MSRP corresponded to the term “main conclusion” in the LMD, this correspondence were not observed in the activity. In addition, when the inconsistency made Ike anxious that they could not find the next promising step, he tried to communicate with other members (#61, #65, and #93). Since Ike and Ham’s approach could not predict the next expected result, it became degenerative (e.g., #89 and #99).

On the other hand, Hor seemed to have a different AS than Ike and Ham. As the inconsistency made her anxious that her direct calculation might not end in time, she tried to communicate with other members twice. The first time she tried to use Tak’s pocket calculator (#41), and the second time she tried to get inspiration from Ike and Ham’s approach (#48). However, since she was not inspired, she finally continued to

calculate by paper and pencil. She seemed to adhere to her approach not only because she was in rivalry with Ike and Ham but also because she believed the rationality of her approach (#64). Thus, we can say the unrevised assumptions that direct calculation is a better approach were the hard core.

Although Uts and Tak calculated directly, they rarely contributed to the group activity. Because their cores were not hard enough, that is, because they were not confident enough of the validity of their cores, they seemed to follow Hor's lead.

In the above description with the MSRP based framework, we can observe the role of the hard cores in problem-solving activities. We hypothesize that the zig-zag process, the repeated process of confronting and eliminating inconsistencies, is driven by a hard core. Because of their hard cores, Ike, Ham, and Hor could take the initiative in problem-solving, at least temporarily. On the other hand, since Uts and Tak's cores were not hard enough, they could only follow Hor's lead. In addition, when the confidence in their cores was shaken, Ike and Hor tried to communicate with others in the group, and follow them. This suggests that taking the initiative in problem-solving in group work is related to the hardness of the core. In fact, while both Ike and Ham and Hor's ASs were progressive, they were incommensurable, and students did not need to communicate with the others in the group.

Implication for practice in mathematics education

Of course, a progressive series of the revised ASs offers no more guarantee of success in problem-solving than a progressive SRP offers of approaching the truth. However, if our hypothesis is valid, we can say that the existence of a single hard core is a necessary condition for a progressive mathematical activity. The participants can discuss and support each other if they share the same core like Ike and Ham, while they cannot effectively communicate with each other because of the incommensurability of their own hard cores like Ike and Hor. Thus, if the teacher intends to enhance his or her students' progressive mathematical activity, he or she must support them in constructing an appropriate shared hard core.

The origin of the hard core is not necessarily mathematical. For example, Ike and Ham created their core by predicting the contents of that day's lesson. On the other hand, Hor created her own core because she felt that Ike and Ham's approach was too complicated. Although the three students were in almost the same social or cultural settings, their hard cores were different. This means that RC cannot claim that the social or cultural settings themselves have an influence on one's core (cf. Lerman, 1996); it must state that depending on subjective interpretations of the social or cultural settings, different hard cores can be created even in the same settings.

Even if a core is created from the other participants' cores, it might be too weak to maintain like Uts and Tak. On the other hand, too strong hard cores will make the AS degenerative. If one wants to keep a progressive mathematical activity, then one may sometimes need to give up one's hard core and create a new core. Further empirical research is needed to explore what helps learners create their own core.

CONCLUSION

In this paper, we proposed the MSRP based framework as an alternative to the LMD based framework for describing mathematical activities. It enables us to describe the quality of mathematical activities as progressive or degenerative. We provided one sample analysis to show the usefulness of the framework. Based on the analysis, we hypothesized that the zig-zag process of solving a mathematical problem is driven by a hard core. For this reason, two persons with different hard cores are incommensurable. Although progressiveness does not always offer a guarantee of success in problem-solving, a hard core seems to be needed for progressive mathematical activities. The sample analysis empirically supports the validity of our hypothesis, even though that is not the purpose of this paper. We need further empirical research with the proposed framework to explore what helps learners create their own cores and have an appropriate shared core.

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THE ROLE OF LEARNERS' EXAMPLE SPACES IN EXAMPLE GENERATION AND DETERMINATION OF TWO PARALLEL AND PERPENDICULAR LINE SEGMENTS

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This study examines the role of middle school students' example spaces in generation and determination of two parallel and perpendicular line segments. Data was collected from 83 middle school students in grades 6 and 7 via two tasks having items on the example generation and determination of parallel and perpendicular two line segments in the grid paper. Data analysis indicated that many of students could not provide fully complete and correct responses when generating and determining parallelism and perpendicularity of two line segments because of limited example spaces under the influence of prototypicality and overgeneralization and undergeneralization errors. This study proposes a catalogue on common limitations in students' example spaces about parallelism and perpendicularity of line segments.

THEORETICAL BACKGROUND

Mathematics educators and mathematicians agree that the use of examples in teaching and learning as a communication tool between learners and teachers is very useful in helping students comprehend mathematical concepts (e.g. Bills et. al, 2006; Watson & Mason, 2005). In this sense, Zaslavsky and Zodik (2014) define *example space* as “the collection of examples one associates with a particular concept at a particular time or context” (p. 527). Example space has been used as a similar term with the concept image (Tall & Vinner, 1981). *Concept image* is the set of all the mental representations associated in the students' mind with the concept name. The image might be nonverbal and implicit. According to researchers, if students are encountered limited examples having common figural features of a geometric concept in school or other context, these examples lead to prototypes phenomenon. The *prototype examples* are usually the subset of examples that had the “longest” list of attributes all the critical attributes of the concept and those specific (noncritical) attributes that had strong visual characteristics” (Hershkowitz, 1990, p. 82). By the influence of prototypical examples and non-examples, learners begin to exhibit two types of common errors as *undergeneralization* and *overgeneralization* (Klausmeier & Allen, 1978). *Undergeneralization error* occurs when examples of a concept are encountered but are not identified as examples. For example, if a learner does not admit a rotated square as an example of square and he or she take this rotated square as an non-example in square set, which indicates he or she makes an undergeneralization error. On the other hand, *overgeneralization error* occurs when examples of other concepts treated as members of target concept (Klausmeier & Allen, 1978, p. 217). For example, if a learner treats

a regular hexagon as an example of parallelogram without considering the number of sides, he or she makes an overgeneralization error.

Researchers suggest that it is important to detect all details of the limitations in students' examples spaces in order to develop effective examples and tasks when teaching mathematical concepts. In this regard, asking students to generate examples of a specific concept and to determine examples of the concept among a set of examples that involves both examples and non-examples can get more details about students' comprehension about specific mathematical concepts. By this way, it can be possible to assert limitations in students' example spaces because example generation can be seen as an indicator of example space (e.g. Sağlam & Dost, 2015). Moreover, example generation and example determination activities help teachers and educators in order to understand less accessible and more accessible examples in students' mind (Zaslavsky & Zodik, 2014). Such activities allow entering learners' "*personal example spaces*" that constitute a collection of examples in learners' mind when facing a particular task (Watson & Mason, 2005). Thus, considering students' personal example spaces and their accessibility of the examples can give big chance to the teachers in terms of developing a didactic way when choosing of examples in their teaching activities in order to construct and enrich learners' examples spaces.

Many of mathematical concepts depend on lower order concepts (Skemp, 1971) or sub-concepts. In high school or universities, teachers assume that learners know and understand these lower order concepts and sub-concepts. However, among the learning domain of mathematics, students are generally exposed prototypical examples of the concepts in the instruction of geometrical concepts and textbooks rather than encountering non-prototypical examples or less-accessible examples. As a result, studies indicate that students have limited knowledge about the different forms of geometric concepts and their use of examples is limited (e.g. Moore, 1994). As basic geometric concepts, parallelism and perpendicularity of line segments have critical importance in terms of developing correctly and completely students' conceptions about the concepts of the altitude, perpendicular bisector, median, angle, slope and the subjects of quadrilaterals, coordinate system, and three-dimensional figures, as well as developing students' proficiencies in proof and argumentation. Many of research revealed both teachers and students have difficulties in some geometric concepts like altitude of triangle (e.g. Gutierrez & Jaime, 1999) and trapezoid (e.g. Ulusoy, 2015) because of inadequate knowledge about parallelism and perpendicularity of two line segments. As a reasonable argument, Zazkis and Leikin (2007) proposed that students' example spaces should be examined in terms of different perspectives such as accessibility, correctness, richness and generality. However, in the literature, there is limited study that directly focused on students' examples about parallelism and perpendicularity of two line segments. Considering the importance of parallel/perpendicular line segments in geometry and the influences of students' example spaces in comprehension of geometric concepts, I decided to investigate the role of middle school students' example spaces in example generation and

determination of two parallel/ perpendicular line segments in the current study. In line with this purpose, I tried to answer following research question: What is the role of middle school students' example spaces in example generation and determination of two parallel and perpendicular line segments? As a concluding remark, as stated in the literature, examining students' example spaces is crucial to present a catalogue of responses showing the characteristics of students' limitations in example spaces of two parallel/perpendicular line segments. For this reason, I mostly concentrated on students' partial and incorrect examples to prepare a catalogue that shows the role of students' limited example spaces in example generation and example determination of two parallel/perpendicular line segments.

METHODOLOGY

The school, in which participants were selected, was chosen in Ankara, Turkey with regard to easy accessibility to the researcher. The students were average-income families' children. In this school, 83 middle school students in Grade 6 and 7 (ages 11 to 13) were determined as the participants of the study. There were 40 students in Grade 6, 43 students in Grade 7. Studies dealing with concept formation highlight the role of carefully selected examples and non-examples in supporting the distinction between critical and non-critical features and the formation of rich concept images and example spaces (e.g. Watson & Mason 2005; Zodik & Zaslavsky, 2008). For this reason, I made a great effort in preparation of examples and non-examples in the tasks by focusing on the studies related to exemplification and basic geometric concepts. In this sense, I prepared two tasks as "example generation task" and "example determination task". The first task included 10 example generation items and the second task included 11 example determination items related to parallel and perpendicular two line segments. In the example generation task, there are two sections. In the first section, there are two items that ask students to generate two parallel/perpendicular line segments in the grid paper. These items were prepared to understand how students generate examples of perpendicular and parallel line segments. In the second part of example generation task, there are eight items to understand the role of prototypical and non-prototypical position of a line segment in a grid paper. These items requested students to generate a parallel or perpendicular line segment to the given another line segment in the grid paper (see fig. 1). For example, while "item3", "item4", "item7", and "item8" can give information about students' example spaces of parallel/perpendicular two line segments in terms of prototypicality, remaining items in fig. 1 can provide information about students' example space in terms of non-prototypicality.



Figure 1: Item3-6 for the construction of a perpendicular line segment and Item7-10 for the construction of a parallel line segment to the given another line segment

On the other hand, example determination task includes 11 items that asked students to determine whether given two line segments in the grid paper are perpendicular/parallel or not (see fig. 2). Items were arranged randomly in the task; however, I arranged and named them as in fig. 2 to provide clear explanation about their characteristics. While “item11” and “item16” are prototypical examples of parallel and perpendicular line segments, “item12”, “item14” and “item17” were added as the main non-prototypical examples. “Item13” was prepared to evaluate students’ example space in terms of verticality and perpendicularity. Furthermore, “item14 and “item21” were prepared to understand the role length of line segments on their example spaces about parallelism and perpendicularity of line segments. “Item15”, “item18”, and “item19” can give idea about students’ limited conceptions. Finally, “item20” was added to the task to understand students’ example spaces in terms of perpendicularity and perpendicular bisector. Before conducting data, the suitability of all items was asked two mathematics teachers and a mathematics educator who makes research on geometric concepts. Finally, I piloted all items in both tasks with sixteen seventh grade students in a different school by making semi-structured interviews.

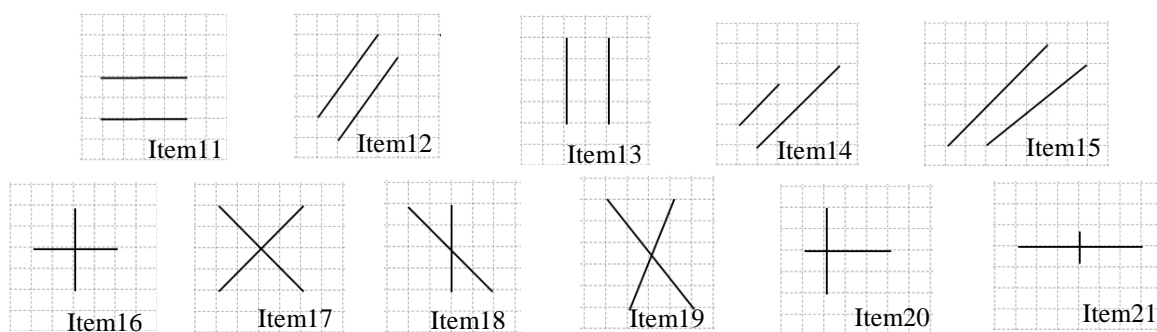


Figure 2: Items on determination of perpendicularity/parallelism of two line segments

Example generation task firstly implemented to the classrooms. After they finished responding to the task, I started to apply example determination task. The tasks took totally 40-45 minutes in each classroom. In both tasks, students asked to explain and justify the reason why they think these two line segments are parallel/ perpendicular or not. For the data analysis, all student-generated examples and written responses reflecting their decisions and justifications were analyzed in terms of correctness and completeness for each item. Then, common limitations in example generation and determination items were grouped in order to present a catalogue of responses that shows the characteristic of students’ example spaces involving partial or poor concept images on parallel and perpendicular line segments. Finally, I made themes for the common limitations in students’ example spaces.

RESULTS

The role of students’ example spaces in example generation task

Student-generated examples in the example generation task showed that most of the students generally provided prototypical and more-accessible examples of both parallel and perpendicular two line segments in first two items of the task. On the other hand,

while 20 students (24%) generated incorrect parallel line segment examples, 29 students (35%) made incorrect perpendicular line segment examples. In students' incorrect perpendicular line segments examples, it was observed the negative influence of mixing verticality and perpendicularity because they generated examples involving only a line segment or two vertical parallel line segments as an example of perpendicular two line segments. Besides, the most common two limitations in students' examples of parallel line segments were observed as generating only one inclined line segment or disregarding the properties of grid paper when generating inclined two parallel line segments. Furthermore, students' examples to "item3 to 10" supported the idea of most of students' examples spaces constitutes only prototypical examples of two parallel/perpendicular line segments.

The role of students' limited example spaces about parallelism of two line segments in example determination task

Limitation to see intersection of line segments by extension. A huge number of students (n=59) decided the example in "item15" as an example of two parallel line segments without considering the meaning of parallelism. They partially focused on the information that two lines on a plane which never meet. However, they made an overgeneralization error because they could not consider two line segments in "item15" can eventually cross over each other when extending both straight line segments. Their constructions in example generation task for especially "item9" and "item10" also supported the students' limitations to generate parallelism in two vertical parallel line segments because they generated two line segments that cross each other in case any extension.

Limitation to see parallelism in two vertical parallel line segments. Students (n=32) generally admitted the example in "item13" as a non-example of two parallel line segments although "item13" constitute an example of two parallel line segments. Instead, they treated this example as a member of perpendicular line segments. In written explanations, students made similar comments like in the following: "These line segments are not parallel. They are perpendicular because they are vertical to the base". In this regard, students' responses indicated their confusion between vertical line segments and perpendicularity of two line segments. These incorrect responses showed the presence of undergeneralization errors in students' example spaces. Moreover, such errors in learners' example spaces can be evaluated as an indicator students' inadequate knowledge about the meaning of parallelism.

Considering length of line segments as a critical factor. Some students (n=17) considered the length of line segments as a critical factor when determining the parallelism of two line segments in some examples like "item14". For example, these students made following explanations about the example: "These line segments cannot be parallel because they are not same length" or "One is short and another one is long, so they are not parallel". At this point, they could not establish a relation between a line and the concept of parallelism. Instead, they merely focused on the visual

appearance of line segments in the grid paper. Consequently, students' statements clearly showed the students' limited example spaces that were formed under the influence of undergeneralization errors. Thus, they treated an example of two parallel line segments as if they were non-examples because they could make a distinction between critical and non-critical features of an example.

The role of students' limited example spaces about perpendicularity of two line segments in example determination task

Seeing enough the presence of crossing two line segments at an angle close to 90° . A number of student's ($n=35$) example spaces involved a concept image about perpendicularity like in "item19". They made similar written explanations like in the following: "Because these line segments are crossing each other, they are crossing perpendicular". These students were aware of the requirement of crossing of two line segments for perpendicularity. However, they disregarded the crossing of two line segments at right angles, which indicated the possible influence of students' limited example space on the examples formed by overgeneralization errors.

Verticality vs. perpendicularity. Students' determination of perpendicularity for the example in "item13" and their written explanations revealed that a case of student ($n=27$) mixed concepts of vertical line segments and perpendicular line segments. This confusion can be the reason of limited concept image in students' mind. For this reason, they overgeneralized perpendicularity situation by admitting a non-example in "item13" as if it is an example of perpendicular two line segments. They did not consider perpendicularity of two line segments requires crossing at right angles to each other. As a result, they treated non-crossing vertical line segments as an example of perpendicular line segments. On the other hand, when I analyzed students' decisions for the example in "item18", I realized that a case of student found enough the intersection of a vertical line segment at any angle to another line segment. One of them made following explanation: "Perpendicularity of two line segments requires a vertical line segment and crossing of two line segments. In this example, there is a vertical line segment and another one cross it. So, they are perpendicular."

Considering length of line segments as a critical factor. Similar to the situation in parallelism of line segments, some students ($n=7$) considered the length of line segments as a deterministic factor for perpendicularity of two line segments. Although the example in "item21" is a member of the set of two perpendicular line segments, students did not admit the example in "item21" as perpendicular because of the non-equal length line segments. This situation showed that they treated an unnecessary condition as if it is a necessity for the perpendicularity under the influence of partial concept image, which case an undergeneralization error.

Perpendicularity vs. perpendicular bisector. A few students' ($n=5$) example spaces involved a pell-mell about the concepts of perpendicularity and perpendicular line segments. For this reason, they thought that perpendicular two line segments have to form perpendicular bisector. As a result, they treated the example in "item20" as a non-

example of perpendicular line segments. In such situations, the students do not have a wrong concept image of perpendicularity, but they made an unnecessary restriction due to the limitations in their example spaces.

CONCLUSION

This study aimed to examine the role of students' example spaces on their example generation and example determination of two parallel and perpendicular line segments. The results of this study revealed that there are different limitations in students' example spaces related to parallel and perpendicular line segments. For instance, student-generated examples generally showed the striking influence of students' limitations arising from partial concept images based on prototypical examples. Besides, students' responses in example determination task mostly allowed seeing students' limitations originating from overgeneralization and undergeneralization errors. For example, many of middle school students are unable to see parallelism in two vertical parallel line segments, and to see crossing of non-parallel line segments when making an extension. Another important result showed the role of students' limitations in example spaces of perpendicular line segments because they generated incorrect or partial correct examples related to the perpendicularity of two line segments due to the mixing of perpendicularity and verticality. They generally tended to treat examples of perpendicular line segments as that of non-examples due to the inadequate knowledge about the meaning perpendicularity, median, and perpendicular bisector. Some results resemble similarities with Gutierrez and Jaime's (1999) study in which they examined preservice primary teachers' understanding of the concept of altitude of a triangle. Students' limitations in example spaces can be related to their mathematics teachers' choices of examples in the instruction of the concepts. Since teacher choices of example either facilitate or impede students' example spaces, I recommend that future studies should concentrate on teachers' choices and treatment of examples related to perpendicularity and parallelism. Furthermore, the catalogue I prepared to show students' limitations in example spaces of parallelism and perpendicularity of two line segments can be utilized in prospective teacher education programs to show the boundaries of students' example spaces about parallelism and perpendicularity. Educators can give opportunities prospective teachers to analyse students' examples. This kind of an analysis can provide prospective teachers with insights when they become teachers with the responsibility to teach these concepts to their students. Thus, they can have a chance to expand and enrich their students' example spaces beyond the prototypical and more-accessible examples to more sophisticated examples (Zaslavsky & Zodik, 2014) by purifying students' overgeneralization and undergeneralization errors (Zodik & Zaslavsky, 2008). Additionally, further studies can examine learners' determination process of properties of quadrilaterals or slope of lines and in making proof and argumentation processes by selecting participants who have limited example spaces of perpendicularity/parallelism by referencing the catalogue. Finally, I suggest that it may be useful to ask students compare their examples in the classroom to enrich their example spaces.

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COGNITIVE AND AFFECTIVE CHARACTERISTICS OF YOUNG SOLVERS PARTICIPATING IN 'KIDUMATICA FOR YOUTH'

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The following research focuses on the characteristics of gifted math students from ages 9-10. This research is based upon a year-long documentation, which included observations of the problem solving activity of 19 gifted students who participated in a prestigious program called 'KIDUMATICA'. Qualitative analysis of the findings showed that the characteristics found in previous studies of gifted adolescents aged 11 and older were also present among younger gifted students. Moreover, it showed two additional characteristics, which were identifiable in this study precisely because they are particularly characteristic of younger students. Therefore, this research shows quite clearly the benefits younger solvers, and thus serves as an additional validation for the creation of programs aimed particularly at younger gifted students.

INTRODUCTION AND THEORETICAL BACKGROUND

Terman (1926), one of the pioneers of the research in this field, defined giftedness as "the top 1 percent level in general intellectual ability as measured by the Stanford-Binet Intelligence Scale or a comparable instrument" (p. 43). Over the years, additional studies of giftedness followed his, and the definition was expanded beyond the measure of intelligence to include additional factors. Thus, for instance, mathematical giftedness came to be defined by the aesthetics of the student's problem solving – their ability to provide a clear, simple, short and elegant solution (Krutetskii, 1976). Continued study also saw the rise of various models and theories, such as the "theory of multiple intelligences" (Gardner, 1985) and the "three rings" model, which, in addition to cognitive components, also takes the students' motivation into account (Renzulli, 1986). As the concept of giftedness expanded, the task of identifying gifted children became more complex and challenging, since students could be gifted in one field, but not necessarily gifted in others.

The literature on the subject now addresses both cognitive and affective characteristics in its attempt to identify gifted students and develop models for learning that are appropriate to their special needs (Hong & Aquino, 2004). Nevertheless, cognitive characteristics continue to feature more prominently, first because they are perceived to have more influence on giftedness, and second because they are methodologically easier to identify (DeBellis & Goldin, 2006).

Cognitive characteristics of giftedness include creativity, originality, fluency and flexibility (Amit, 2010; Leikin & Lev, 2013; Mann, 2006; Polya, 1957; Torrance, 1968), generalization and reflections (Amit & Neria, 2008; Sriraman, 2003), and argumentation (Tirri & Pehkonen, 2002). Studies have shown that there is a clear connection between giftedness and high cognitive abilities (Greenes, 1981), but that

these can also be improved and enhanced to some extent in *all* students. They suggest that students' cognitive ability is in part a function of the teaching methods and the learning environment to which they have been exposed, and even of the problems that have been selected to be solved in class (Amit & Gilat, 2012; Zohar & Nemet, 2002). Affective characteristics refer mainly to self-perception, motivation and determination. While gifted students have been found to have higher levels of these characteristics as well, here too studies have found that a supportive learning environment can significantly improve the affect system of all students (Debellis & Goldin, 2006; Hong & Aqui, 2004).

Most studies of gifted students today focus primarily on adolescents, and most of the special educational programs that are currently available are for students ages 12-18. While studies and programs that address younger students do exist, their numbers and their scope are still too small. This study therefore focuses on mathematically gifted students at the age of 9-10. Its goal is to identify the characteristics of these younger students and compare them to the characteristics that earlier studies have found in adolescents.

METHODOLOGY

Research questions

- 1) What are the characteristics of gifted students aged 9-10?
- 2) Which of these characteristics are unique to ages 9-10?

Research population

The study population was composed of 19 gifted 4th grade students aged 9-10, carefully chosen by their teachers from 10 schools in Southern Israel that were recommended by the regional coordinator. These students agreed to participate in a pilot program launched in 2013 as part of the 'Kidumatica' mathematics club.

Setting and teaching program

The 'Kidumatica' Mathematics club includes about 550 students aged 11–16. The project is aimed at addressing the special needs of students (most of whom come from underdeveloped or struggling areas), who possess mathematical ability and who are interested in learning more about mathematics. In 2013 the club expanded to include younger students, aged 9-10. The first year with the younger students served as a pilot program, in the sense that the lessons learned from it served as the basis for changes to the program in years to come.

Research approach (method)

This study was originally designed to employ a mixed methods approach, using an observation journal to document the students' lessons, as well as questionnaires with a variety of problems. Once we got to know the students, however, we decided not to use the questionnaires, since the lessons had already made clear that there was a gap between the students' possession of a correct and interesting mathematical idea and

their ability to express it in writing. The young students tended instead to explain their ideas orally, or by means of pictures, hand gestures and stories. We therefore decided to abandon the quantitative tools, which, despite their convenience, could not yet represent our students' understanding, and to focus on the observations as our primary source of data instead.

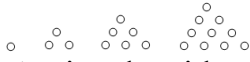
DATA AND ANALYSIS

Observations can be divided into various types of observation, based on the position taken by the researcher in the study. In this case, the observer was a source of interest to the students at the beginning, but in time the students became accustomed to her regular presence in class and she became like a "fly on the wall". The students were observed throughout the school year in a total of 25 meetings, which were then fully transcribed. Each meeting was 4 hours long and composed of 2 workshops, adding up to about 100 hours of observed learning. We analyzed the data by identifying categories according to "grounded theory" (Shkedi, 2003). The categorization process was conducted in the following stages:

- General orientation.
- Orientation through the lens of theory.
- Refinement and additions to theory.
- Building the category tree and verification.
- Recurrence of categories.

RESULTS

Our observations revealed 5 cognitive characteristics and 5 affective characteristics that appear in the research literature on gifted adolescents. In addition, we found 2 additional cognitive characteristics that seem to be particular to younger students. A detailed description, including an explanation and example of each characteristic, can be found in Table 1.

Category	Short explanation	Examples from the observations
Creativity-Originality	Non-routine approach to PS that leads to a solution	T: How can we divide the number 188 and get 200? S: 188
Creativity-Fluency	Ability to offer many solutions to a single problem	T: Fill in the blanks: $1_+3_+5_ = 111$ S: You need a total of 21 so there are lots of answers: $7,7,7$ or $6,7,8$ or $5,7,9$ or $3,9,9$ or $4,9,8...$
Creativity-Flexibility	Ability to move freely between different mathematical representations	T: What's the next member?  S1: A triangle with a base of 5... S2: The sum of the differences sequence and the first member: $1+2+3+4+5...$

Reflection	Ability to look at your PS both before and after in order to learn from what you have done	<p>T: Can 405 be a square number?</p> <p>S: That doesn't work with what we said before about</p> <p>20. Because 20 squared is 400, so 21 times 21 is more than 405...</p>
Generalization	Ability to transition from an individual case to a general rule	<p>T: how many ways are there to frost a cake divided into 2 pieces?</p> <p>S: 4.</p> <p>T: And when it's divided into 3?</p> <p>S: 8! Twice as much as the previous answer...like before but with the option of frosting or not frosting the fourth piece...</p>
Argumentation	Ability to formulate a claim and justify it with supporting evidence	<p>The students learned about Goldbach's hypothesis, according to which every even number over 2 can be presented as the sum of two prime numbers.</p> <p>S: I have a theory. Odd numbers can't be the sum of</p> <p>two prime numbers, because prime numbers are odd and the sum of two odd numbers is even."</p>
Connectivity	Ability to connect math PS to different – not necessarily mathematical topics	<p>T: What is a sequence?</p> <p>S: It's like a TV series; it's not a movie, it keeps going and going and appears on regular days. So a sequence in math also keeps going and going and has regular rules...</p>
Virtualization	Ability to address a problem as a story and imagine the situation without jumping straight to calculations	<p>T: A log is cut into 4 pieces in 12 seconds. How long would it take to cut it into 6 pieces?</p> <p>Half of the students answered 18 (a classic but wrong answer). Explanation: the ratio between 4 and 12 is equal to the ratio between 6 and 18. Half answered 20 (correct).</p> <p>S: If I take a stick and break it 3 times to get 4 pieces, each break takes 4 seconds, so 5 breaks will take 20 seconds..."</p>

Impronovation	Ability to improvise and find an innovative solution despite lacking the proper tools	T: The circumference of a rectangle is 144. Its length is 3 time as long as its width. Find the length and width of the rectangle. S: The length and width together are 72. 72 divided by 4 is 18. So one side is 18 and one is 72 minus 18.
Self confidence	The students' perception of themselves	The students expressed their opinions, even when in the minority. Most were unafraid to go up to the board, and even after making a mistake they overcame it easily and continued to participate in class.
Motivation	What drives the student to learn and succeed	Despite their young age and the difficult hours, the students came to the club regularly and happily. The students took an interest in the lessons and most signed up for another year at the club.
Determination	Ability to spend a long time on a PS and not give up until you solve it	The students often asked for more time to work on riddles, begging not to be told the solution. Sometimes they even refused to be given a hint.
Competitiveness	The strong desire to be first in any task	The children checked their scores often, and were very concerned with who won and who lost in any task or game.
Skepticism	Ability to doubt the words of the teacher	The students were unafraid to ask questions and challenge the teacher, or results that did not sit well with them.

(PS- Problem Solving, T- Teacher, S- Student/s)

Table 1: Categories table

Recurrence of categories

"Significant behavioral event" was defined according to when a student asked a question or made a comment that reveals one of the characteristics. Throughout the year, 267 significant behavioral events were observed - 152 in the cognitive context and 115 in the affective context. The findings were quantified according to the frequency of the events, as seen in the following tables:

Cognitive characteristics N=152						
Creativity	Reflection	Generalization	Argumentation	Connectivity	Virtualization	Impronovation
30%	15%	13%	12%	10%	10%	10%
(N=45)	(N=24)	(N=20)	(N=18)	(N=15)	(N=15)	(N=15)

Table 2: Recurrence of cognitive characteristics

Affective characteristics N=115				
Self confidence	Motivation	Determination	Competitiveness	Skepticism
35%	25%	15%	13%	12%
(N=40)	(N=29)	(N=17)	(N=15)	(N=14)

Table 3: Recurrence of affective characteristics

DISCUSSION

Cognitive characteristics

Creativity was the most common of the cognitive characteristics (see table 2). This corresponds to the findings of previous research, which has claimed creativity as a central characteristic that sets gifted students apart (Mann, 2006; Torrance, 1968). Studies of creativity's components have found that originality is the most influential of the three, and that it is also the only one of them that cannot be improved by educational means (Leikin & Lev, 2013). This means that the originality of their solutions can be used to identify gifted students even at young ages, since time and maturity do not play a central role. Other cognitive characteristics found in previous studies in adolescents (Amit & Neria, 2008; Greenes, 1981; Sriraman, 2003) appeared in the current study too, which provides additional justification for the claim that gifted children can be identified at an early age.

UNIQUE CHARACTERISTICS

Virtualization- Virtual Reality

Much has been said about visualization. "Virtualization," however, refers not only to a visual image, but to the creation of virtual reality. Virtualization is the students' ability to address a problem as a tangible, visual story. One of the authors of this paper, who teaches gifted high school students, gave them the "wooden log" problem (see table 1). When she did so only one student out of 26 gave the correct answer of 20. The older students' immediate reliance on familiar algorithms, which interfere with their ability to see the simple story underlying the problem, is most likely the product of the educational system. The younger students were able to see the problem as more than words on paper, while the older students, who had many years of experience with solving word problems, immediately began to look for a solution by calculating ratios. In this context it is important to note that students who are still in elementary school are more strongly exposed to teaching methods based on visual representation, which could also be a positive influence on the students' virtualization ability.

Imprnovation- Improvisation & Innovation

Imprnovation refers to the ability to improvise solutions to a problem when you lack the customary mathematical tools. This characteristic was revealed in our young students when they were given a problem for which the classical solution relied on mathematical tools that they had not yet been taught. Surprisingly, it was this "lack" that led them to find a successful solution of their own. In other words we can say that

sometimes "less is more". The "rectangle" problem (see table 1) is a good illustration of this, where instead of constructing two equations with two variables like older students would likely have done, they were able to improvise an innovative solution. Young students are unfamiliar with the various branches of mathematics, and as far as they are concerned mathematics are not divided into various compartments by topic. This lack of compartmentalization led them to make more spontaneous connections between different topics, which often helped them reach a solution more quickly.

Affective characteristics

Self-confidence and motivation were the most common of the affective characteristics (see table 3). This finding echoes those of other studies (DeBellis & Goldin, 2006; Hong & Aquil, 2004). Interestingly, this study did not reflect the BFLP phenomenon (Big Fish Little Pond) - a common negative effect noted in the literature, in which proximity to other stronger students minimizes the prominence of a student's qualifications, thereby causing a decline in self-confidence (Marsh & Aducci, 2003). In the present study BFLP was avoided, and the main reason for this is the supportive community framework offered by the club.

Conclusion

This study is consistent with other research on gifted students in that it found that despite their young age – its population showed characteristics commonly noted amongst older gifted children. The uniqueness of this study is that it found two additional characteristics, which were revealed due to students' young age. This study thus reinforces the need to apply special study programs at an early age and promotes the development of similar models in the future.

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THE NATURAL NUMBER BIAS AND ITS ROLE IN RATIONAL NUMBER UNDERSTANDING IN CHILDREN WITH DYSCALCULIA: DELAY OR DEFICIT?

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There exists already a large body of literature both on learners' struggle with understanding the rational number system and on the role of the natural number bias in this struggle. However, little is known about rational number understanding of learners with dyscalculia. In this study, we investigated the rational number understanding of learners with dyscalculia, with a specific focus on the role of the natural number bias in this understanding. The results suggest that in addition to a delay in their general mathematics achievement, learners with dyscalculia have an extra delay, but no deficit in their rational number understanding, compared to their peers.

INTRODUCTION

A good understanding of the rational number domain is of essential importance for learners' mathematics achievement (Siegler, Thompson, & Schneider, 2011). At the same time, rational numbers are known to form a stumbling block for many learners (Mazzocco & Devlin, 2008; Vamvakoussi, Van Dooren, & Verschaffel, 2012; Van Dooren, Van Hoof, Lijnen, & Verschaffel, 2012). Previous research indicates that a large part of learners' difficulty with rational numbers can be explained by the natural number bias, which is defined as the tendency to apply natural number properties in tasks with rational numbers, even when this is inappropriate (Ni & Zhou, 2005). Learners are found to make systematic and predictable errors in rational numbers tasks where the use of prior natural number knowledge leads to the incorrect answer (incongruent tasks), while they are much more accurate in rational number tasks where reliance on prior natural number knowledge leads to the correct answer (congruent tasks) (Vamvakoussi et al., 2012).

Previous research on the natural number bias mainly focused on three aspects in which rational numbers differ from natural numbers and where errors are known to occur: their dense structure, the way their numerical size can be determined, and the effect of the four basic operations (Vamvakoussi et al., 2012; Van Hoof, Vandewalle, & Van Dooren, 2013).

While there exists a large body of literature both on learners' struggle with understanding the rational number domain and on the role of the natural number bias in this struggle, little is known about the rational number understanding of learners with dyscalculia (further abbreviated with LWD). Nonetheless, a better insight of the understanding of rational numbers of LWD is important to provide adaptive (remedial)

instruction with the aim to increase LWD's understanding of the rational number system. One of the few studies that did investigate LWD's rational number understanding is the study of Mazzocco and Devlin (2008). They found that learners with mathematical learning difficulties compared to learners without mathematical learning difficulties struggle more to accurately order rational numbers, even more than learners with a low general mathematics achievement.

THE PRESENT STUDY

The goal of the present study is to extend the study of Mazzocco and Devlin (2008) in several ways. First, in the study of Mazzocco and Devlin (2008) learners were solely matched on their 'chronological age'. In this study we will also match learners on their mathematical ability level. This allows to investigate whether there is a 'deficit' or a 'delay' in LWD's rational number understanding (Torbeyns, Verschaffel, & Ghesquière, 2004). If LWD's rational number understanding is significantly lower than that of learners of the same age (= chronological age match), but not significantly different from learners without dyscalculia but with the same mathematics achievement level (= ability match), who are typically younger, this implies that LWD's rational number understanding reflects their mathematics achievement level and thus that the development of LWD's rational number understanding is only characterized by a *delay* rather than *deficit*. However, if LWD's rational number understanding is not only significantly lower than that of learners of the same age but also significantly lower than (younger) learners with the same mathematics achievement level, the development of LWD's rational number understanding does not represent their mathematics achievement level, thus their rational number understanding is characterized by a *deficit* (Torbeyns et al., 2004). This leads to the first research question (Research Question 1), namely whether LWD's rational number understanding is characterized by a 'delay' or a 'deficit' compared to learners without dyscalculia. Second, while Mazzocco and Devlin (2008) measured learners' rational number understanding in a rather general sense, we will pay particular attention to the differences between congruent and incongruent rational number tasks to map the natural number bias in the three groups of learners. This way, we aimed at answering our second research question (Research Question 2): Is the strength of the natural number bias in LWD comparable with that of normally developing children? Based on the available research literature, no specific prediction could be made for both research questions.

METHOD

Participants

Three different groups of learners were included. A first group consisted of sixth graders with an official clinical diagnosis of dyscalculia ($n = 16$). Next to these LWD, we included two control groups: a chronological age match and an ability match group. The first control group were sixth graders without dyscalculia ($n = 56$), further referred to as the sixth grade control group. The mean age (in months) of the LWD was 143.97

($SD = 10.39$), while the mean age of the sixth grade control group was 142.13 ($SD = 3.53$). An independent samples t-test showed that this difference in age was not significant, $t(70) = 1.13$, $p = .26$. The ability match control group were fourth graders ($n = 51$), further referred to as the fourth grade control group. We chose to include this age group because we needed younger learners with a mathematics achievement level that we could expect to be comparable to that of sixth graders with dyscalculia, but who would also be able to solve the rational number test. The mean mathematics achievement level of the LWD was 92.81 ($SD = 40.13$), while the mean mathematics achievement level of the fourth grade control group turned out to be much higher, i.e., 134.75 ($SD = 29.56$). An independent samples t-test showed that this difference was significant; $t(65) = -4.53$, $p < .001$. This result shows that we did not succeed in creating an ideal ability-matched group. However, we could not include an even younger group as an ability match control group, as younger students would hardly have any relevant rational number knowledge. In order to address this, we opted to take into account learners' mathematical ability as a control variable and correct for remaining differences between groups. This way, we were able to investigate whether there was still a difference between both groups' rational number understanding that could not be explained by a difference in mathematics achievement level, but by having dyscalculia.

Instruments

Rational number understanding

Learners completed a shortened version of the "Rational Number Sense Test" (RNST; Van Hoof, Verschaffel, & Van Dooren, 2015) as a measure of their rational number understanding. The shortened test consisted of 49 items. Both congruent ($n = 16$) and incongruent items ($n = 33$) from the three aspects of the natural number bias (density, size, and operations) were included. Examples can be found in Figure 1. As a measure of the strength of the natural number bias, we used learners' accuracy levels on the incongruent rational number tasks.

	Congruent	Incongruent
Density	Write a number between $\frac{1}{4}$ and $\frac{3}{4}$	Write a number between 3.49 and 3.50
Size	Choose the largest number: 4.4 or 4.50	Choose the largest number: $\frac{3}{2}$ or $\frac{9}{8}$
Operations	Is the outcome of $50 * \frac{3}{2}$ smaller or larger than 50?	Is the outcome of $40 * 0.99$ smaller or larger than 40?

Figure 1: Examples of congruent and incongruent items.

Mathematics achievement

Learners' mathematics achievement was measured by means of the Tempo Test Automatiseren (De Vos, 2010). This test measures the automated knowledge of the four basic operations.

Intelligence.

Two measures of intelligence were used. First, Raven's Progressive Matrices test (Raven, Court, & Raven, 1995) was used as a measure of learners' non-verbal intelligence. Second, the SiBO test measured learners' verbal intelligence (Hendrikx, Maes, Magez, Ghesquière, & Van Damme, 2007). Because high correlations were found between both IQ measures (Raven and SiBo) ($r = .41, p < .001$), we created one general intelligence score by first calculating z-scores for each measure separately and then taking the mean of these two scores for each learner. We standardised the intelligence score for the sixth graders and fourth graders separately (leading to a mean score of 0 in both groups), and then calculated the z-scores of the LWD group using the sixth graders as a reference group.

Reading achievement.

Because comorbidity was allowed, two measures of reading achievement were included as control variables to ensure that the results were not due to lower reading achievement. The één-minuut test (one minute test, further abbreviated with EMT) was used as a measure of learners' word recognition ability. The goal of the test is that learners correctly read out loud as many words as possible within one minute. Standardized scores were used based on existing norm tables (Brus & Voeten, 1972). The Klepel was used as a second measure of learners' reading ability. Contrary to the EMT, the words in this test are pseudowords. Standardized scores were used based on existing norm tables (van den Bos, Spelberg, Scheepstra, & de Vries, 1994). Also for the two reading achievement measures, high correlations were found ($r = .92, p < .001$). Therefore, we also calculated each learner's mean standardized score on the two measures as a general score of reading achievement.

RESULTS

Table 1 presents the descriptive statistics for the control variables. The results for the dependent variable (performance on congruent and incongruent rational number items) are shown in Figure 2.

		LWD	4 th graders	6 th graders
Age (months)	Mean	143.97	119.34	142.13
	SD	10.39	3.54	3.53
IQ (Raven + SiBo)	Mean	-0.99	0	0
	SD	1.12	0.79	0.83
Reading achievement (EMT + KLEPEL)	Mean	7.13	9.99	10.82
	SD	3.78	2.49	2.48
Mathematics achievement	Mean	92.81	134.75	171.27
	SD	40.13	29.56	22.30

Table 1: Descriptive statistics

Comparison between LWD and sixth grade control group

As can be seen in Figure 2, LWD's accuracy on congruent items was significantly lower than the accuracy of the sixth grade control group. Because our aim was to have a chronological age match design, in a next step we added learners' age (in months) as control variable in the comparison between both groups' accuracy on congruent rational number tasks. Moreover, because both groups differed in their general IQ and reading achievement, we also included these two as control variables. An ANCOVA indicated that, even after controlling for learners' age, IQ, and reading achievement, the sixth grade control group still significantly outperformed the group of LWD on congruent rational number tasks, $F(1,67) = 4.35$, $p = .04$, $\eta^2 = .06$, but the effect size was only small.

LWD's accuracy on incongruent items was also significantly lower than the accuracy of the sixth grade control group, see Figure 2. In a next step we again additionally controlled for learners' age, IQ, and reading achievement. An ANCOVA indicated that, even after controlling for these variables, the sixth grade control group still significantly outperformed LWD on incongruent rational number tasks, $F(1,67) = 102.16$, $p < .001$, $\eta^2 = .60$; the effect size was large.

The partial eta squared values reveal that the difference in accuracy between the group of LWD and the sixth grade control group is much higher in incongruent than in congruent rational number items.

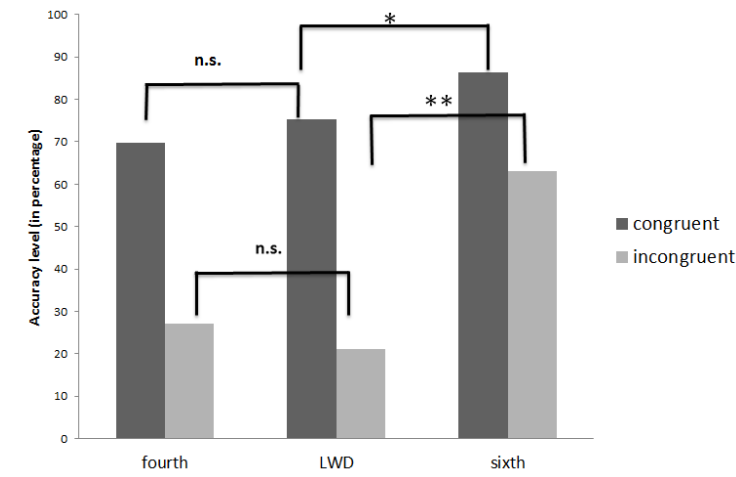


Figure 2: Learners' accuracy on congruent and incongruent rational number tasks per group

Note. * = $p < .01$, ** = $p < .001$

Comparison between LWD and fourth grade control group

As can be seen in Figure 2, LWD's accuracy on congruent items was not significantly different from the accuracy of the fourth grade control group. As stated above, we did not succeed in realizing an optimal ability match with fourth graders. Therefore, in a next step we added learners' mathematics achievement as a control variable in the comparison between both groups' accuracy on congruent rational number tasks. Moreover, because the LWD and the fourth grade control group also differed in their general IQ and reading achievement, we also included these two measures as control variables. An ANCOVA indicated that, after controlling for learners' mathematics achievement, IQ, and reading achievement, no significant difference was found between both groups' accuracy on congruent rational number tasks, $F(1,62) = 3.69$, $p = .08$, $\eta^2 = .03$.

LWD's accuracy on incongruent items was lower than the accuracy of the fourth grade control group, see Figure 2. This difference between the two groups was however not significant $F(1,65) = 2.06$, $p = .16$, $\eta^2 = .03$. For the same reasons as above, in a next step we additionally controlled for learners' mathematics achievement, IQ, and reading achievement. An ANCOVA indicated that, after controlling for these three measures, no significant difference was found between both groups' accuracy on incongruent rational number tasks, $F(1,62) = 1.75$, $p = .19$, $\eta^2 < .01$.

DISCUSSION

Concerning the first research question, results showed that LWD's rational number understanding is significantly lower than that of regular learners, but not significantly different from younger learners, even after statistically controlling for mathematics achievement, both in congruent and incongruent rational number tasks. These results suggest that the development of LWD's rational number understanding is characterized by a *delay* rather than a *deficit*. Concerning the second research question, there was a

big difference in accuracy level between LWD and their peers on incongruent rational number tasks, while no significant difference could be found with fourth graders' accuracy on the same incongruent rational number tasks. The difference in accuracy on incongruent rational number tasks revealed that the strength of the natural number bias is higher in LWD compared to normally developing learners of the same age, but is not different from younger learners. This finding confirms that LWD's rational number understanding is characterized by a delay rather than a deficit. We further found that LWD's rational number understanding is not significantly different from younger learners with a higher mathematics achievement level. This suggests that the role of dyscalculia is less strong for learners' accuracy levels on rational number tasks compared to learners' accuracy level on a mathematics achievement test measuring learners' automated knowledge of the four basic operations. Moreover, the finding that LWD's rational number understanding is not significantly different from younger learners with a higher mathematics achievement level, gives us reason to hypothesize that LWD have a lead in their rational number understanding compared to even younger learners with the same mathematics achievement. Our findings have implications for mathematics education. Although our results pointed out that LWD struggle even more with incongruent rational number tasks than their peers, they also indicated that this struggle is not characterized by a deficit but with a more general delay. This implies that it is possible for LWD to develop more insight in the rational number system and, therefore, more (remedial) instructional attention should aim at the enhancement of LWD's understanding of the rational number system. As stated above, LWD are more affected by the natural number bias compared to learners of the same age. Therefore, more instructional attention should go to the differences between the natural number system and the rational number system. While this should be implemented in all classrooms, it deserves especially attention when teaching LWD.

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THEORIZING THE MATHEMATICAL POINT OF BUILDING ON STUDENT MATHEMATICAL THINKING

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Despite the fact that the mathematics education field recognizes the critical role that student thinking plays in high quality instruction, little is known about productive use of the in-the-moment student thinking that emerges in the context of whole-class discussion. We draw on and extend the work of others to theorize the mathematical understanding an instance of such student thinking can be used to build towards—the mathematical point (MP). An MP is a mathematical statement of what could be gained from considering a particular instance of student thinking. Examples and non-examples are used to illustrate nuances in the MP construct. Articulating the MP for an instance of student thinking is requisite for determining whether there is instructional value in pursuing that thinking.

The field of mathematics education recognizes the critical role student mathematical thinking plays in planning and implementing quality mathematics instruction (e.g., National Council of Teachers of Mathematics [NCTM], 2014). Researchers have made progress on understanding how instruction can be improved by using tasks that are likely to engage students in meaningful mathematical activity and by working to maintain the cognitive demand of those tasks throughout instruction (e.g., Stein & Lane, 1996). We also know many of the benefits of teachers understanding common ways that students think about and develop mathematical ideas (e.g., Fennema et al., 1996). We know less, however, about productive ways of taking advantage of the student mathematical thinking that emerges during instruction. Recent work (e.g., Smith & Stein, 2011) has begun to help us understand how to effectively use written records of student work, but much less is known about how to effectively use the in-the-moment mathematical thinking that emerges during classroom mathematics discourse. We need to understand this important aspect of effective use of student thinking because whole-class discussion is fertile ground for the emergence of valuable student mathematical thinking (Van Zoest et al., 2015a, 2015b), yet many teachers, especially novices, fail to notice or to act on opportunities to use this valuable thinking to further mathematical understanding (Peterson & Leatham, 2009; Stockero, Van Zoest, & Taylor, 2010).

Our work investigating teachers' use of in-the-moment instances of high potential student thinking to further students' mathematical understanding during whole-class discussion has led us to conclude that an important reason for the slow pace of reform in this area is that what exactly can be learned from making a particular instance of

student thinking the object of discussion has been under theorized. Thus, the purpose of this paper is to theorize the mathematical point that an instance of student thinking can be used to build towards. Before beginning that theorizing, we first outline the theoretical framework that guides our thinking and then situate our thinking in the context of other related research.

THEORETICAL FRAMEWORK

The MOST research group (e.g., Leatham, Peterson, Stockero, & Van Zoest, 2015; Van Zoest, Leatham, Peterson, & Stockero, 2013) defined MOSTs—**M**athematically **S**ignificant **P**edagogical **O**pportunities to **B**uild on **S**tudent **T**hinking—as “instances of student thinking that have considerable potential at a given moment to become the object of rich discussion about important mathematical ideas” (Leatham et al., 2015, p. 90). They conceptualized MOSTs as occurring in the intersection of three critical characteristics of classroom instances: student mathematical thinking, significant mathematics, and pedagogical opportunities. For each characteristic, two criteria were provided to determine whether an instance of student thinking embodies that characteristic. Foundational to our work is the student mathematical thinking characteristic, for which the two criteria are student mathematics and mathematical point. To meet the student mathematics criterion, one must have sufficient evidence to make a reasonable inference about the mathematical thinking a student is expressing. The articulation of this mathematical thinking is called the *student mathematics* (SM) of the instance. To meet the mathematical point criterion, one must be able to “articulate a mathematical idea that is closely related to the student mathematics of the instance—what we call a *mathematical point*” (p. 92). It is this use of *mathematical point* (MP) that we theorize in this paper.

MOSTs are instances of student thinking worth *building* on—that is, “student thinking worth making the object of consideration by the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea” (Van Zoest et al., 2015b, p. 4). Such use encapsulates core ideas of current thinking about effective teaching and learning of mathematics, including social construction of knowledge and the importance of mathematical discourse (NCTM, 2014). Thus, building on MOSTs is a particularly productive way for teachers to engage students in meaningful mathematical learning. In this paper, we both draw on and extend the MOST framework by theorizing the MP—the mathematical understanding particular instances of student thinking can be used to build towards.

RELATED RESEARCH

Perhaps the work most closely related to this theorizing, at least on the surface, is Laurie Sleep’s 2009 dissertation, *Teaching to the Mathematical Point: Knowing and Using Mathematics in Teaching*. Sleep, however, defines mathematical point “to include the mathematical learning goals for an activity, as well as the connection between the activity and its goals” (p. 13). This is a broad definition that foregrounds the meaning of “point” as “something that is the focus of attention, consideration, or

purpose” and backgrounds the meaning of point as “a separate, or single item, article, or element in an extended whole” (Oxford English Dictionary [OED] cited in Sleep, 2009, p. 13). The first column of Figure 1 lists mathematical points articulated in Sleep’s dissertation. What is notable about these points is the lack of specifics they provide about the mathematical ideas related to them. Consider, for example, the first point in Figure 1. Although “reviewing and practicing strategies for adding multiple addends” (Sleep, 2009, p. 107) is important to be doing in a 2nd grade class, the statement does not say anything about the mathematics that makes up those strategies. That is, it fails to articulate the mathematical idea that is to be learned.

<i>Mathematical Points from Sleep (2009)</i>	<i>Mathematical Understandings from Charles (2005)</i>
“reviewing and practicing strategies for adding multiple addends” (p. 107)	“Numbers can be broken apart and grouped in different ways to make calculations simpler.” (p. 16)
“learning that halves are two equal parts” (p. 162)	“The bottom number in a fraction tells how many equal parts the whole or unit is divided into. The top number tells how many equal parts are indicated.” (p. 13)
“teaching the addition and subtraction algorithm [for fractions]” (p. 244)	“Fractions with unlike denominators are renamed as equivalent fractions with like denominators to add and subtract.” (p. 16)

Figure 1: Comparison of Sleep’s (2009) Mathematical Points and Charles’ (2005) Mathematical Understandings.

Although the importance of teachers having mathematical goals in mind for their teaching has been well established (e.g., Corey et al., 2010; NCTM 2014), most research, like Sleep’s (2009), has remained at the level of looking at whether teachers have mathematical goals and how those goals affect their instruction (e.g., Fernandez, Cannon, & Chokshi, 2003), rather than investigating the articulation of the goals and how that articulation affects teachers’ ability to support their students’ learning. Some work has been done around the articulation of intended instructional outcomes, however, in the context of courses on pedagogy. For example, the Brigham Young University Mathematics Educators (n.d., unpublished manuscript) developed a document for supporting preservice secondary school teachers in writing lesson goals focused on mathematical concepts that is now used by several universities. They emphasized that a *key concept* is not a topic or a step-by-step method for doing something; rather, it is “something that you want your students to *understand*. Concepts deal with *meaning*, *why* something works, *ways of imagining* or *seeing things*, and *connections*” (p. 1, italics in original).

Charles (2005), to “initiate a conversation about the notion of Big Ideas in mathematics” (p. 9), proposed a set of Big Ideas for elementary and middle school and their corresponding *mathematical understandings*. Charles described a mathematical

understanding as “an important idea students need to learn because it contributes to understanding the Big Idea” (p. 10). The second column of Figure 1 lists mathematical understandings from Charles (2005) that bear some relationship to Sleep’s (2009) mathematical points. Charles’ mathematical understandings do articulate the mathematical idea that is to be learned, and thus they are at a grainsize more appropriate for looking at the mathematical understanding particular instances of student thinking can be used to build towards.

DEFINING MATHEMATICAL POINT (MP)

Although Charles (2005) did not formally define mathematical understanding in his paper, drawing on his examples and explanations, we use the term *mathematical understanding* to refer to *a concise statement of a non-subjective truth about mathematics*. This definition specifies something that students can actually come to understand, as opposed to a topic for them to study or an outcome of their learning. Articulating mathematical understandings is useful for a number of teaching-related activities, such as determining goals of a lesson, analysing the mathematics students might learn from a task, and guiding the formulation of questions to ask in the midst of a lesson. Yet another reason for articulating mathematical understandings, and the one that is the focus of this paper, is to determine whether student thinking is worth building on in the moment in which it occurs.

Our focus is on instances of student mathematical thinking that emerge during whole-class discussion. We follow Leatham et al. (2015) in defining an *instance* as “an observable student action or small collection of connected actions (such as a verbal expression combined with a gesture)” (p. 92). In our ongoing research, we have found that for instances of student thinking for which *student mathematics (SM)* can be inferred, one can articulate related *mathematical understanding(s)*. Since student thinking is not always constrained by the teachers’ plan for the lesson, these mathematical understandings may or may not be within the confines of the planned lesson. Additionally, the mathematical understandings that are within the confines of the planned lesson may or may not be most closely related to the SM. Although we agree with Hintz and Kazemi (2014) that it is important that “the discussion goal acts as a compass as teachers navigate classroom talk” (p. 37), we also contend that a parallel goal is to honor student thinking. That is, making a decision about whether or not to pursue a particular instance of student mathematical thinking requires first identifying the SM of the thinking and then identifying the mathematical understanding most closely related to it. Otherwise, there is a risk of undermining a core principle of quality mathematics instruction—that of positioning students as legitimate mathematical thinkers (e.g., NCTM, 2014).

Thus, in the context of our work on productive use of student mathematical thinking during instruction, a *mathematical point (MP)* is the *mathematical understanding* that (1) students could gain from considering a particular instance of student thinking and (2) is most closely related to the SM of the thinking. That is, the MP is a mathematical

statement of what could be gained as a result of students making sense of the mathematics contained in the expression of the student thinking. Note that it is only when the MP is articulated that a clear decision can be made about whether the student thinking should be pursued. (Leatham et al., 2015, describe a tool for distinguishing instances of student thinking that provide opportunities to productively build on students' mathematical thinking from those that do not—the MOST Analytic Framework.)

We have identified four things to keep in mind when considering MPs. First, an MP exists only in relation to a specific instance of student thinking. That is, unlike a mathematical understanding, which can stand alone, an MP cannot. Specifically, one must talk about an MP in relation to what can be gained from considering a particular instance of student thinking. Second, in order to be an MP, the mathematical understanding must be gained from considering the student thinking itself. An instance of student thinking may often prompt teachers to ask a question, introduce an idea or pose a task that furthers student learning of a mathematical understanding related to the instance. Although these are important teaching tasks that *use* student thinking, we want to be clear that we do not consider them *building* on student thinking. In order for building to occur, the thinking itself must become the object of discussion. Third, not all instances of student thinking give rise to an MP. For example, suppose a student asked, “What is the formula for the volume of a cube?” This instance of mathematical thinking is related to the mathematical understanding: *The formula for the volume of a cube with side s is s^3 .* That mathematical understanding, however, is not something that students could gain from considering this particular instance of student thinking. They might be able to recall it, or they might be able to figure it out from a task that the teacher poses in response to the instance, but it would not result from considering the student thinking. Fourth, there are acceptable variations in the articulation of SMs, mathematical understandings, and MPs. What is presented here is the consensus of the authors, but other articulations may also be defensible.

To further illustrate our theorizing, Figure 2 contains instances of student mathematical thinking, the MP that would serve as the discussion goal of the conversation in which the instance of student thinking is the object of discussion, an example of a mathematical understanding that does not meet the “most closely related” criteria for that thinking and an example of a related statement that is not a mathematical understanding. Recall that MPs are a subset of mathematical understandings, thus both the second and third column contain examples of mathematical understandings.

Instances of Student Mathematical Thinking	Mathematical Point	Not the Most Closely Related Mathematical Understanding	Not a Mathematical Understanding
1. In an introductory lesson on adding fractions with like denominators, a student writes the following on the board: $2/5 + 1/5 = 3/10$.	Adding fractional pieces of the same size changes the number of pieces, but not the size of the pieces.	Adding two quantities means combining the amounts together.	How to get a common denominator when adding fractions.
2. During the second day of a lesson on solving simple linear equations, when the teacher solves the equation $x + 2 = 5$ and writes $x = 3$ on the board, a student remarks, "Hey, wait a minute, yesterday you said x equals two and today you're saying x equals three!"	A letter can be used to represent an unknown quantity in an equation and can represent different quantities for different equations.	"Letters are used in mathematics to represent generalized properties, unknowns in equations, and relationships between quantities." (Charles, 2005, p. 18)	The meaning of variable.
3. In a beginning algebra lesson on solving simple linear equations, a student says, "To get m alone on the left side of the equation $m - 12 = 5$, you can subtract 12."	Any term can be removed from one side of an equation by adding its additive inverse to both sides of the equation.	Adding a number and subtracting that same number are inverse operations.	Solving linear equations.

Figure 2: Examples and Non-examples of Mathematical Points for Instances of Student Mathematical Thinking

Since the MP is the most closely related mathematical understanding, we first look at ways in which statements may fall short of being a mathematical understanding (see Column 4 of Figure 2). "How to get a common denominator when adding fractions," for example, states a mathematical process without explicating that process. "The meaning of variable," refers to a concept without elaborating what it is, while "Solving linear equations," merely states a topic. Note that all of these mathematical statements fail to specify the *non-subjective truth about mathematics* that the statement encapsulates.

The mathematical understandings in Column 3 of Figure 2 are mathematical understandings for the corresponding instances of student thinking, but they are not as closely related as those in the *Mathematical Point* column. For example, although closely related on the surface—Instance 1 is certainly about addition of two quantities—the MP for this instance better captures the specific non-subjective truth about mathematics that students could gain by making this instance of thinking the object of discussion. The importance of the MP being the mathematical understanding most closely related to the SM of the instance is related to the idea of honoring student thinking. For example, if the teacher were to turn the student thinking in Instance 2 over to the class and navigate the discussion (Hintz & Franke, 2014) toward the goal of better understanding how letters are used to represent unknowns in equations, the student likely would not feel that their thinking was the object of the discussion. Again, that is not to say that making the student thinking the object of discussion is always the optimal teaching move; rather, it is to say that articulating the MP allows teachers to make an informed decision about how best to respond to the thinking. If there is an MP, the MOST Analytic Framework (Leatham et al., 2015) is a mechanism for determining whether to make the thinking the object of discussion for the class or to address it in some other way.

CONCLUSION

An important reason that instruction based on student thinking has not lived up to its potential may be that our target has been too broad. Focusing on teachers' goals for the lesson lends itself to the teacher using student thinking to make the point the teacher has in mind, rather than building on student thinking. Changing the grainsize to the MP for the SM in instances of student thinking may be a productive shift in how we think about using student thinking as part of instruction that will allow us to achieve the full potential of instruction based on student thinking.

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INCORPORATING MOBILE TECHNOLOGIES INTO THE PRE-CALCULUS CLASSROOM: A SHIFT FROM TI GRAPHIC CALCULATORS TO PERSONAL MOBILE DEVICES

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A case study in a high school in Mexico served as a scenario for investigating the process of incorporating two different kinds of mobile technology into the Mathematics classroom; the use of TI-Nspire calculators during the first year of the fieldwork and the use of personal mobile devices in the second one. This research project takes an instrumental genesis perspective in order to describe the process of mobile technology incorporation, considering both the individual and social aspects of the instrumental geneses developed in the classroom. The paper shows the different types of instrumental orchestrations performed along both stages of the fieldwork, and shows how the different types of instruments share similarities in their use but imply relevant differences in the kind of classroom interaction that can be promoted.

RATIONALE OF THE STUDY AND RESEARCH QUESTIONS

Research in the field of mobile technology in education has shown several of the constraints and possibilities of incorporating different kinds of mobile devices into Mathematics classroom contexts; from graphic calculators (Artigue, 2002; Drijvers, Kieran et al., 2010; Robutti, 2009) to the use of student-owned mobile devices (Kim, Hagashi, et al, 2010). However, as recently pointed out by the OREAL/UNESCO (United Nations Educational, Scientific and Cultural Organisation, 2013), part of the research in this field should be focused on how proposals of technology incorporation are developed by teachers for particular teaching and learning purposes.

This paper presents some of the results of a case study in a high school in Mexico where a Pre-Calculus teacher decided to explore and incorporate two different mobile devices into her Mathematics classroom: Texas Instruments graphic calculators (TI-Nspire calculators) together with a Navigator System along the first year of the study and personal mobile devices (PMD) a year later when TI-N calculators were not available any more.

The research questions that guide this project are stated as follows: 1) What are the different instrumental orchestrations performed along two periods of the fieldwork? 2) How are the mathematical meanings brought up and developed in relation to the performance of particular instrumental orchestrations? However, in this paper I will only show the results and a brief discussion around the first research question.

THEORETICAL FRAMEWORK AND LITERATURE

In order to give an answer to the first question of research, the study was carried out under the Theory of Instrumental Genesis, aiming at characterising instrumental practices (Drijvers, Godino et al., 2013; Guin and Trouche, 2002) in terms of the constraints and potentialities of the artefacts being used and paying particular attention to the way mobile devices were used to achieve particular mathematical and didactical tasks in the classroom.

As described by Drijvers, Godino, Font and Trouche, “an artefact is an -often but not necessarily a physical - object that is used to achieve a given task. It is a product of human activity, incorporating both cultural and social experience” (2013, p. 26). Artefacts in a classroom context can be defined in a wide range of ways from a pencil to a calculator, a dynamic geometry software or a networking system but its definition is fundamentally related to the task which is meant to be performed by the user (teacher or students). An artefact can be described as an instrument “if a meaningful relationship exists between the artefact and the user for a specific type of task” (Drijvers, Kieran, et al., 2010, p. 108).

The theory of Instrumental Genesis considers both the individual and the social aspects of the mediated activity in the classroom. The individual aspect focusses on the process of instrumental genesis by which an artefact becomes an instrument, defined by the bilateral influence between artefacts and users, where students’ knowledge influences the way in which the artefact is being shaped as an instrument (instrumentalisation), while at the same time the artefact’s affordances and constraints “influence the way the student carries out a task and the emergence of the corresponding conception” (instrumentation) (Drijvers, Godino et al., 2013, p. 4).

In relation to the social aspect of the theory of Instrumental Genesis, Guin and Trouche (2002) introduced the concept of instrumental orchestration, defined as “the intentional and systemic organization of the various artefacts available in a computerized learning environment by the teacher for a given mathematical situation, in order to guide student’s instrumental geneses” (Drijvers, Kieran et al., 2010, p. 112). In other words, the Instrumental Orchestration is the way in which the teacher decides to use the different artefacts available in the classrooms in order to attain the learning objectives of the lesson. The instrumental orchestration consists of three basic elements, namely the didactic configuration, its exploitation mode and the didactic performance (Trouche 2002, Drijvers 2010).

The Didactic Configuration refers to the arrangement of the learning environment, the artefacts to be used and the different tasks that students should accomplish along the lesson. The mode of exploitation of the didactical configuration refers to the decisions made by the teacher in order to specify how the artefact(s) should be used in order to accomplish the tasks and therefore the didactical intentions of the lesson. Relevant to the mode of exploitation is the way in which the teacher introduces the tasks and the way these tasks are meant to be worked on, considering the schemes and techniques

that should be developed by students and the roles that the different artefacts play in the process of instrumentation.

Finally, the Didactical Performance, the third element of the Instrumental Orchestrations defined by Drijvers (2010) refers to the enactment of the process of instrumentation in the classroom foreseen and planned in the didactic configuration and its mode of exploitation. Therefore, the didactic performance includes the ad hoc decisions taken by the teacher to guide the use of artefacts in the learning environment (2010).

METHODOLOGY

The fieldwork consisted of two stages carried out in a high school in Xalapa, Mexico, each of which lasted for four months. The first here referred to as study 1 was carried out at the beginning of 2013, and the teacher of a pre-Calculus class decided to use Texas Instruments graphic calculators together with a Navigator System that could allow her to monitor students' work on their handheld devices. Study 2 was carried out a year later with the same teacher and the same course but with a different group and different technological devices. As TIN calculators and NS were not available any more along study 2, the teacher decided to ask students to bring personal mobile devices to the classroom (smartphone, i-pod, i-pad or tablet) where they would have access to a graphing software such as Geogebra.

The methodological framework incorporates ethnographic strategies for data collection based on the observation and video recording of ten different lessons along each stage of the fieldwork, as well as semi-structured interviews with teacher and students. Lesson plans were used to obtain useful information concerning the didactical configuration and mode of exploitation of each instrumental orchestration developed along each lesson, such as the learning goal, a definition of the artefacts to be used, the teaching setting and the description of the different tasks students should accomplish. However, for the aims of this paper I will only refer to the video recordings of lessons, paying particular attention to the description of the techniques developed while using particular artefacts along the didactic performance and consequently, the interpretation of the corresponding schemes involved in the instrumental activity (Drijvers, Kieran et al., 2010, p. 108).

The analysis of the video recorded lessons was carried out under a multimodal approach (Jewitt, 2013), where teacher and students actions were categorised following Drijvers' global inventory of instrumental orchestrations (2010). This inventory was originally integrated by six different types of orchestrations, namely Technical-demo, Explain-the-screen, Discuss-the-screen, Link-screen-board, Spot-and-show, and Sherpa-at-work. The first five orchestrations are concerned with the use of a DME by the teacher and applets by the students, and as their names state, they are also related to the way these artefacts are being used in order to either provide a technical demonstration, explain or discuss what is happening on a main screen (which is usually an example of student's work), etc. Sherpa-at-work is a type of orchestration that

characterises the role of particular students played in the performance of the instrumental geneses and in concerned with the interaction between students. In recent research other types of orchestrations, which are not directly involved with the use of the technology under scope, were added to the inventory, such as Work-and-walk, Guide-and-explain and Link-screen-board (2014).

INTERPRETATION OF RESULTS

The following table shows the different types of orchestrations performed along both studies of the fieldwork, specifying the number of lessons where these orchestrations were found and the participants involved in each type of orchestration (Advisor student, Teacher Assistant, Teacher or Students).

Instrumental Orchestrations	Study 1		Study 2	
	Number of lessons	Participants	Number of lessons	Participants
Technical-demo*	2	Advisor	-	-
Technical-support*	4	TA, Advisor	-	-
Explore- TIN	2	Sts	-	-
Discuss-the-screen*	8	Teacher, Sts	-	-
Explain-the-screen*	9	Teacher, Sts	-	-
Link-screen-board*	5	Teacher, Sts	-	-
Question-NS	2	Teacher	-	-
Monitor-NS	7	Teacher	-	-
Graph-TIN / PMD	9	Sts	9	Sts
Link-screen- notebook	9	Sts	9	Sts
Discuss-the-board	4	Teacher, Sts	3	Teacher, Sts
Explain-the-board	7	Teacher, Sts	8	Teacher, Sts
Show-and-tell	-	-	6	Sts, other teachers
Advisor-at-work	8	Advisor	-	-
Question-Sts	7	Sts	-	-
Walk-and-work*	4	Teacher	5	Teacher

* Global Inventory of Instrumental orchestrations (Drijvers 2010 & 2014)

Table 1: Typology of Instrumental Orchestrations along Study 1 and 2

The types of instrumental orchestrations identified along study 1 of the fieldwork were consistent to the global typology described by Drijvers, as the Navigator System used by the teacher resembled the DME artefact and the TIN calculators to the handheld devices used in previous research (2010). However, it was found that the global inventory of instrumental orchestrations was not sufficient to characterise the different orchestrations found in the empirical data, considering that the kind of artefacts being used in both stages of the fieldwork were quite different to the ones that have been previously investigated. Therefore, new types of orchestrations were characterised. The following sections shows the types of instrumental orchestrations found along each of the two stages of the fieldwork.

According to the table, several types of instrumental orchestration were found to be performed simultaneously and distributed for short periods along the lesson, as in the case of Explain-the-screen, Discuss-the-screen and Link-screen-board. This situation made categorising instrumental orchestration a difficult part of the analytical procedure. and has been previously reported by Drijvers (2014, add reference).

Other types of orchestrations took most of the lesson time, and were very frequent along several lessons (8 out of 10 lessons in study 1), as in the case of Monitor-NS, Spot-and-show, Graph-TI and Link-screen-notebook. In this case, it was found that the teacher used the NS to monitor students work on their TI-N calculators (following a similar interaction as in the Walk-and-work), and spot some of the problems students faced while graphing using their handheld devices in order to show to the rest of the class through the main screen and provide the corresponding feedback. The tasks performed by students related to these orchestrations were related to the graphing of different types of functions where students had to analyse the graphs and provide a written explanation of their findings and conclusions on their notebooks, which in many cases was performed along with a Link-screen-notebook type of orchestration.

Two other types of orchestrations were characterised in relation to the use of TIN calculators and the Navigator System: Explore-TIN and Question-NS. Explore-TIN was found along the first two lessons of study 1 (group A and B), where the main aim of the lesson was for students to familiarise with their handheld devices. The didactical configuration for this orchestration considered the exploration of the artefact in terms of using the graphic function of the calculator in a free-style, so students were able to choose the kind of functions they wanted to graph. The exploration was not limited to algebraic functions, as data shows students were also interested in graphing trigonometric functions on a polar plane and 3D graphs. This orchestration was found to be performed simultaneously with the Technical-support type of orchestration where advisor students played a fundamental role. In the following lessons, Explore-TIN orchestration became less frequent, was only performed by advisor students and took place simultaneously with a Technical-demo orchestration, where the teacher asked advisor students to show on the main screen and through the NS what they have found new about the use of their calculators, so the rest of the students could follow and

replicate. In any case, the mode of exploitation of the explore-TIN orchestration attempted to get students familiarised to their handheld devices.

Question-NS in an orchestration where the teacher raises a question to the whole class and gets an answer from each one of the students. The didactical configuration requires the teacher to use the Navigator System to pose the question and to provide an account and statistics of all the students' answers after 30 seconds. Students use their TIN calculators to provide their answers. The mode of exploitation for this type of orchestration was different each time it was performed. The first time the teacher decided to use this type of orchestration was as a close-up activity at the end of the lesson where the teacher raised several yes/no questions in order to confirm the understanding of several mathematical concepts. Once all the answers were shown on the main screen, the teacher opened a short session to elicit the different answers from students and to discuss them. During students participation, the teacher never provided any feedback, but gave the right answers at the end of the discussion. The second time the Question-NS was performed was as part of the term exam. In this case, students had to answer to several yes/no questions as part of an evaluation, where feedback was not provided.

In study 2 of the fieldwork there is a significant variation of instrumental orchestrations performed along the lessons. As it was expected, some of the orchestrations that depended on the particular use of the NS and the TIN calculators are not present in study 2, but some others, as in the case of the Graph-TIN, it was found to be present in all lessons related to the use of personal mobile devices. This type of orchestration is characterised by the use of the graphing software as part of a task where students should analyse different functions graphically or just to get the graph in order to copy it either on their notebooks or to prepare material to present as in a poster. Therefore, the orchestration was performed along side with a Link-screen-notebook orchestration as in study 1.

A common type of instrumental orchestration found in study 2 which is not found along study 1 is the Show-and-tell orchestration where students should present different mathematical topics to the rest of the class or to external participants (other teachers or students in the school). The didactical configuration of this orchestration does not depend on the use of a technologically enriched environment as in the case of the orchestrations performed along study 1, and it is not strictly related to the use of a particular technological device. Instead, students are free to use their personal mobile devices in any way they find useful. As a mode of exploitation, students participate in a presentation as a way of review of mathematical concepts in order to help students get ready for the next exam. The didactical performance showed for example, that under a project called Mathematical labyrinth, students worked in groups of 5 to 8 students to prepare and present different types of functions and their graphic analysis to an external audience. Students prepared and presented their work mostly in open spaces, where personal mobile devices were always available.

In this case, it was clear how the use of personal mobile devices was not at the centre of the orchestrations as it was the case of other orchestrations in study 1. However, in all cases observed along study 2, students used their personal mobile device to graph different types of functions which were later on sketched either on posters or on their notebooks.

According to these results, it seems clear that the different types of instrument orchestrations can be grouped in three ways: 1) around the mobile technology used, as in the case of Explore-TIN, Monitor-NS, and Question-NS, 2) around typical classroom technology such as white/black board, notebook, posters such as Explain-the-board, Link-screen-board or notebook, and 3) instrumental orchestrations not specifically related to the use of particular technology but closely related to the role played by participants such as the teacher assistant, the technology coordinator or the so-called advisor students, as in the case of Sherpa-at-work or Advisor-at-work, Technical-support and Question-St.

GENERAL DISCUSSION AND CONCLUSIONS

The results at this first stage of the analysis show that the kind of instruments developed along two different stages of the fieldwork and their geneses share some properties, particularly concerned with the kind of tasks addressed mainly by students. However, the way these instrumented practices were organised in each case was significantly different as was their relation to how teacher orchestrated the activities and the broader learning tasks in which those instrumental tasks were embedded.

For example, relevant differences were found in terms of the roles played by teacher, assistant and advisor students in some of the instrumental orchestrations performed in the classroom. Technical-demo orchestrations, which are related to demonstrations of the technical issues regarding the use of TI calculators was found to be provided not by the teacher but by advisor students and in rare cases by teacher assistant. Nevertheless, in study 2, nor advisor students nor teacher assistant were available, and even though students were asked to used personal mobile devices as graphing instruments, technical demonstration and support rely mainly on other students in a kind of peer support. The role of advisor students became also significant when it consisted of providing not only technical but mathematical support to the rest of the students and the teacher. These conclusions raise awareness on the relevance of considering the role of participants involved in each type of orchestration and shows that further analysis on the type of interaction and the use of other psychological tools mediating the action in the classroom should be included in order to better explain the impact of particular types of orchestrations in the processes of teaching and learning.

The technological teaching setting, also proved to be relevant in order to promote some types of orchestrations with the specific participants. In study 1, all lessons were observed in a technologically-enriched classroom where the support of teacher assistant and technology coordinator are available. Besides, the teacher assigned specific roles to proficient students as advisor students which participation was not

limited to technical support but also provided mathematical guidance to other students. On the contrary, lessons in study 2 were observed in traditional classroom and outdoor spaces, where the use of personal mobile devices was always accessible.

Finally, a close attention to the types of tasks embedded in the orchestrations show that the interaction performed in each of them could allow a different kind of participation from students. As in the case of questions raised by teacher or by students. This however, requires further analysis in terms of the roles played by each participant.

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DEVELOPING ALGEBRAIC THINKING: THE CASE OF SOUTH AFRICAN GRADE 4 TEXTBOOKS

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This paper investigates the extent to which three Grade 4 South African mathematics textbooks attempt to develop learners' algebraic thinking. Although the current South African curriculum compares well with international practice in terms of algebraic thinking development, a need existed to determine the extent to which local textbooks reflect this. The textbooks reviewed reflect a good understanding of the expectations of the curriculum. The ways in which this understanding was used to develop teaching and learning material, however, vary considerably. While two textbooks provide learners with opportunities to develop conceptual understanding through investigative activities, one simply states what learners need to know, thus promoting rote learning. There is a clear development in sophistication of mathematical ideas; however, the sequencing of some of these raises questions.

BACKGROUND TO THE STUDY

Algebra is a fundamental part of mathematics, since “algebra is the language for investigating and communicating most of mathematics” (South Africa, 2011). To improve the quality of learning in algebra, it has been widely recommended that fundamental knowledge and skills be developed in learners' primary school years. This approach is known as early algebra or algebraic thinking. School curricula in many countries, including South Africa (SA), have been changed accordingly.

In an earlier study, certain requirements for an effective algebra curriculum were formulated (Vermeulen, 2007). That study concluded that the current South African curriculum to a large extent satisfies these requirements, is well-aligned with current international thinking regarding developing algebraic thinking, and should therefore succeed in developing younger learners' algebraic thinking.

One of the questions that remain to be answered, is the extent to which SA mathematics textbooks reflect this curriculum. This paper reports on the extent to which three Grade 4 South African textbooks achieve this.

EARLY ALGEBRA OR ALGEBRAIC THINKING

Early algebra is not an attempt to introduce symbol manipulation earlier to younger children, but rather an attempt to reform the teaching of arithmetic in a way that stresses its algebraic character. It requires understanding of how the arithmetic concepts and skills can be better aligned with the concepts and skills needed in algebra so that learning and instruction is more consistent with the kinds of knowledge needed in the learning of formal algebra (Carpenter et al., 2005). Kieran (2004) offers the following definition of algebraic thinking:

Algebraic thinking in the early grades involves the development of ways of thinking within activities for which letter-symbolic algebra can be used as a tool but which are not exclusive to algebra and which could be engaged in without using any letter-symbolic algebra at all, such as, analysing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modelling, justifying, proving, and predicting. (p. 149)

For the purpose of this paper, I will focus on the following two key areas of algebraic thinking, as presented by Blanton (2008:3):

- (1) using arithmetic to develop and express generalisations (algebra as generalised arithmetic), and
- (2) identifying numerical and geometric patterns to describe functional relationships (algebra as functional thinking).

Generalised arithmetic primarily refers to building generalisations about operations on and properties of numbers (Blanton, 2008). Thus, generalising arithmetic includes helping children notice, describe (conjecture) and justify patterns and regularities in operations on and properties of numbers (Blanton, 2008), thus becoming aware of structure (which includes equivalence).

Under algebra as functional thinking the following can be understood:

“Functional thinking is thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations for that relationship across instances.” (Smith, 2008:143). According to the South African curriculum documents (South Africa, 2011) this type of thinking should be introduced in Grades 4 to 6:

The study of numeric and geometric patterns develops the concepts of variable, relationship and function. The understanding of these relationships by learners will enable them to describe the rules generating the patterns. This phase has a particular focus on different, yet equivalent, representations to describe problems or relationships by means of flow diagrams, tables, number sentences or verbally (p. 9).

While the focus in Grades 1 to 3 is on recursive thinking (i.e. identifying and applying the rule within a single sequence of values), from Grade 4 the focus moves towards co-variational thinking (i.e. analysing how two quantities vary simultaneously), and describing this co-variation using flow diagrams, number sentences or in words (Smith, 2008).

RESEARCH QUESTIONS

In view of the aforementioned, the following research question was formulated: To what extent do SA Grade 4 mathematics textbooks succeed in developing learners' algebraic thinking? In particular, to what extent

(i) do they offer learners opportunities to develop generalised arithmetic skills; specifically, is structure addressed, primarily properties of operations and equivalence? (Sub-research question 1)

(ii) do they offer learners opportunities to develop functional thinking; specifically, are the notions of relationship and variable developed, and are relationships between variables demonstrated in several ways, i.e. in words, using flow diagrams and tables? (Sub-research question 2)

(iii) do they offer learners learning opportunities that promote investigation, conjecturing, verification and justification skills? (Sub-research question 3) and

(iv) is there a development in sophistication of mathematical ideas within the textbooks? (Sub-research question 4)

THE ROLE OF TEXTBOOKS IN THE TEACHING AND LEARNING OF MATHEMATICS

From literature about textbooks, it is clear that textbooks play an important role in the teaching and learning of subject matter. Studies on curriculum materials suggest that textbooks can impact both what and how teachers teach, as well as what and how learners learn (Herbel-Eisenmann, 2007). Lemmer et al. (2008) state that textbooks are expected to provide a framework for what is taught, how it may be taught and in what sequence it can be taught.

Rymartz and Engebretson (2005, in Newton et al., 2006) found that “a textbook made a big difference to the quality of teaching”. They point out that “Most teachers and particularly new teachers and those teaching outside their area of expertise found that they taught better, that they fostered better quality thinking, and assessed more purposefully”. “Teachers teaching outside their area of expertise” is of importance for the SA context. Most primary school teachers are generalists rather than specialist mathematics teachers, and can to a large extent be viewed as “teaching outside their area of expertise”.

A THEORETICAL FRAMEWORK FOR THE ANALYSIS OF TEXTBOOKS

Herbel-Eisenmann (2007) cites Otte (1983) stating that written materials can be examined as subjective scheme and as an objectively given structure. When examining textbook materials as subjective scheme, the focus is on the interaction between a reader and the material. When analysing textbooks as objectively given structures, the structure and discourse of the written unit is the focus. According to Herbel-Eisenmann (2007), this approach allows one to focus on the potential of the textbook material for supporting or undermining the ideological and epistemological goals of the curriculum on which it is based. As such, the present study analyses textbooks as objectively given structures.

LeBrun et al. (2002) emphasise the importance to conduct comparative analyses of curricula and textbooks, such as the present study attempts to do.

For the purpose of the present study, I will use the framework proposed by Tarr et al. (2006), which is arranged around three key dimensions, namely mathematics content emphasis, instructional focus and teacher support.

METHODOLOGY

The study took the form of a qualitative, descriptive case study. Three textbook series were selected on the basis that they are widely used in South African schools, and copies of learner's books and teacher's guides are readily available. Therefore, both purposive and convenient sampling methods were used.

In all cases, the learner's books were meticulously screened for incidences that reflect the four research sub-questions. These incidences were recorded and notes were made regarding the extent to which these incidences answer each research sub-question. The corresponding sections in the teacher's guide were subsequently consulted in an attempt to gain a deeper understanding of the authors' aims and suggested procedures for the teaching and learning of the observed incidences in the learner's book. These were also noted.

SELECTED FINDINGS

Summary of findings for Research sub-question 1

All textbooks in this study attempt to develop learners' *generalised arithmetic* skills, specifically as far as *structure* is concerned, primarily *properties of operations*. However none of them address *equivalence* explicitly. Textbooks vary considerably in their approach: Two provide opportunities for learners to investigate and to conjecture and reflect, while the third simply states the rule, without explanation or opportunity for developing conceptual understanding. Teacher guides also vary considerably in terms of teacher support. While two would provide detailed rationales and teaching guidelines, the third would simply provide solutions. Figures 1 to 4 show examples from textbooks:

<p>1. Complete these pairs of addition number sentences:</p> <p>a) $12 + 6 = \square$ $6 + 12 = \square$ b) $22 + 8 = \square$ $8 + 22 = \square$ c) $13 + 11 = \square$ $11 + 13 = \square$ d) $30 + 40 = \square$ $40 + 30 = \square$</p> <p>2. Write down what you notice in your own words.</p>	<p>1. Complete these pairs of addition number sentences. The brackets tell us that we must add the numbers in the brackets first.</p> <p>a) $6 + (4 + 13) = \square$ $(6 + 4) + 13 = \square$ b) $14 + (6 + 17) = \square$ $(14 + 6) + 17 = \square$</p> <p>2. Write down what you notice about each pair of sentences.</p> <p>3. Which was easier: the first set of number sentences or the second?</p> <p>4. What made it easier?</p>
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Figure 1: Commutative and associative properties for addition (Textbook 2)

Note

We can change the order of numbers for addition and multiplication and the answer stays the same.

$$3 + 5 = 8 \text{ and } 5 + 3 = 8 \quad \text{AND} \quad 3 \times 5 = 15 \text{ and } 5 \times 3 = 15$$

In subtraction and division the answer changes when we change the order.

Figure 2: Commutative property for addition and multiplication (Textbook 3)

2. Solve these calculations.

- $4 \times 8 = \square$ $8 \times 4 = \square$
- $3 \times 10 = \square$ $10 \times 3 = \square$
- $5 \times 7 = \square$ $\square \times 7 = 35$
- $2 \times 9 = \square$ $9 \times \square = 18$
- $21 = 3 \times \square$ $\square \times 3 = 21$
- $3 \times \square = 8 \times 3$ $6 \times 8 = \square \times 6$
- $4 \times \square = 10 \times 4$
- $\square \times 3 = 3 \times 9$

3. Solve these problems.

- $6 + 9 = \square$ $9 + 6 = \square$
- $15 + 7 = \square$ $7 + 15 = \square$
- $25 + 10 = \square$ $10 + 25 = \square$
- $120 + 80 = \square + 80$
- $\square + 50 = 50 + 350$

4. Do you get the same answers when you swap numbers in a subtraction calculation? Explain.

5. Do you get the same answers when you swap numbers in a division calculation? Explain.

Figure 3: Commutative property for addition and multiplication (Textbook 1)

Multiply and solve problems

There are different ways to split up numbers when multiplying.

Example
 $23 \times 6 = \square$
 Multiply by splitting up the bigger number. To make things easier we can split up the 23 into $20 + 3$ and make two arrays.

$20 \times 6 + 3 \times 6$

You can use brackets to show how you have grouped the numbers together.

$$(20 \times 6) + (3 \times 6)$$

$$= 120 + 18$$

$$= 138$$

So, $23 \times 6 = 138$.

Figure 4: Using an array to demonstrate the distributive property (Textbook 1)

Summary of findings for Research sub-question 2

All textbooks attempt to develop learners' *functional thinking*, using words, flow diagrams and tables to express rules between input and output values. However, the notions of *relationship* and *variable* are not explicitly mentioned. As before, teacher guides vary considerably in terms of teacher support.

The various textbooks follow more or less the same approach as for Sub-research question 1: whereas some would allow for investigation or provide opportunities for conceptual understanding, others would be more direct in their presentation of knowledge.

One textbook states for example: "Using special rules, you can make patterns with numbers. The special rule lets you know what numbers will follow in the pattern. We can use a flow diagram to show a rule. A flow diagram has a starting number, a rule and an answer." Elsewhere the textbook also explains the notions of input and output numbers. The first examples and exercises require learners to **express the rule in words**, and to use the rule and the input values to find the corresponding output values (see Figure 5).

EXERCISE 4.1

Look at the flow diagram below:
 Input → $+5$ → output

1. Write down the rule in words.
2. Write down the output number for each of these input numbers:
3, 5, 8, 11, 0

Figure 5: Using a flow diagram to write the rule in words, and to determine output values.

Another textbook would, for the first encounter, simply do the following (Fig. 6):

Find the value of the output number. Write a number sentence in your book.

a.	<div style="display: flex; justify-content: space-between;"> <div>Input</div> <div>Rule</div> <div>Output</div> </div> <div style="text-align: center;"> </div>	b.	<div style="display: flex; justify-content: space-between;"> <div>Input</div> <div>Rule</div> <div>Output</div> </div> <div style="text-align: center;"> </div>
	<div style="display: flex; justify-content: space-between;"> <div>17</div> <div></div> <div></div> </div> <div style="display: flex; justify-content: space-between;"> <div>24</div> <div></div> <div></div> </div> <div style="display: flex; justify-content: space-between;"> <div>52</div> <div></div> <div></div> </div> <div style="display: flex; justify-content: space-between;"> <div>39</div> <div></div> <div></div> </div>		<div style="display: flex; justify-content: space-between;"> <div>64</div> <div></div> <div></div> </div> <div style="display: flex; justify-content: space-between;"> <div>21</div> <div></div> <div></div> </div> <div style="display: flex; justify-content: space-between;"> <div>98</div> <div></div> <div></div> </div> <div style="display: flex; justify-content: space-between;"> <div>73</div> <div></div> <div></div> </div>
c.	<div style="display: flex; justify-content: space-between;"> <div>Input</div> <div>Rule</div> <div>Output</div> </div> <div style="text-align: center;"> </div>	d.	<div style="display: flex; justify-content: space-between;"> <div>Input</div> <div>Rule</div> <div>Output</div> </div> <div style="text-align: center;"> </div>
	<div style="display: flex; justify-content: space-between;"> <div>23</div> <div></div> <div></div> </div> <div style="display: flex; justify-content: space-between;"> <div>15</div> <div></div> <div></div> </div> <div style="display: flex; justify-content: space-between;"> <div>44</div> <div></div> <div></div> </div> <div style="display: flex; justify-content: space-between;"> <div>9</div> <div></div> <div></div> </div>		<div style="display: flex; justify-content: space-between;"> <div>9</div> <div></div> <div></div> </div> <div style="display: flex; justify-content: space-between;"> <div>71</div> <div></div> <div></div> </div> <div style="display: flex; justify-content: space-between;"> <div>34</div> <div></div> <div></div> </div> <div style="display: flex; justify-content: space-between;"> <div>67</div> <div></div> <div></div> </div>

Figure 6: Using flow diagrams to determine output values and to write the rule as a number sentence.

Some textbooks deal with single operation rules for quite a while, while others include double operation rules fairly early. All three textbooks develop the notion of functional relationships in the contexts of number patterns as well as geometric patterns.

While initially rules and input values are given, and learners have to calculate the output values, gradually activities also appear where input and output values are provided, and learners need to determine the rule. Rules mostly consist of a single operation, but a few cases of double operation rules also appear. As elsewhere, the level of guidance provided by the Teacher's Guide differs considerably.

Tables are introduced and used as a means to "record" the input and corresponding output values. Thus, relationships between input and output values are explicitly represented in various ways, as prescribed by the curriculum.

Summary of findings for Research sub-question 4

There are incidences where it can be questioned whether the sequencing promotes increasing sophistication, for example: In one textbook, patterns, flow diagrams and rules are introduced early in Term 1. These concepts are developed within the context of "Multiplication and division flow diagrams". Some activities contain one operation, while others contain two. The two operations invariably are multiplication (e.g. $\times 7$ and then $\times 2$), followed by an activity with a single operation ($\times 14$) (Refer to Figure 7). The idea here is for learners to realize that to multiply by the large number (14), one can break the large number into its factors, and multiply by the factors consecutively.

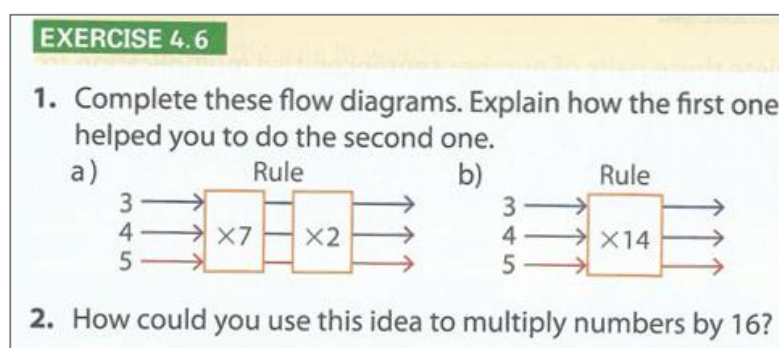


Figure 7: Multiplication by breaking up bigger numbers into smaller numbers.

However, in the subsequent *revision* activity flow diagrams with multiplication as well as addition are included, which seems out of place here, given the discussion above, as well as the fact that these types of activities are only really dealt with much later in the year.

In one textbook, the development of learners' ability to find a rule reaches its peak on p.137, where two operations, multiplication and addition, are involved. Learners are required to find co-variational rules, i.e. relating the input and output values, thus reinforcing the concept of a relationship between two variables. It is therefore strange to find that in the very next section number sequences are dealt with, where the relationship concept is absent, and learners need to continue a number pattern, using a

recursive strategy. It is also strange to find that later on rules contain only one operation (multiplication), rather than providing more two-operation rules.

Conclusion

The authors of the textbooks reviewed seem to have a good understanding of the expectations of the curriculum to develop learners' algebraic thinking. Two textbooks offer learners opportunities to develop conceptual understanding through investigative activities, thereby allowing for important abstractions and generalisations (Watson & Mason, 2006). The third one simply states what learners need to know, thus not encouraging discourse. This is of concern given the proven low pedagogical content knowledge of many primary school mathematics teachers in South Africa and the current research into teachers' mathematical discourse in instruction (Adler & Venkat, 2015). In all cases there is a development in sophistication of mathematical ideas; however the sequencing of some of these raises questions. This implies that there are not consistent, clear learning trajectories, showing a disregard of Rowland's (2008) argument that "choices of examples and their sequencing are neither trivial nor arbitrary" (p. 150).

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PRE-SERVICE TEACHERS' BELIEFS ABOUT MATHEMATICS EDUCATION FOR 3-6-YEAR-OLD CHILDREN

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The objective of this paper is to present the results of a questionnaire aimed at collecting pre-service teachers' beliefs about the role of mathematics teaching and learning in preschool. The questionnaire was mainly structured around four dimensions identified in the research literature as being important in determining whether mathematics instruction is implemented in early childhood classrooms. The results show that, at the beginning of their studies, pre-service teachers prioritise artistic and physical development over mathematics. By the end of their studies, pre-service teachers' beliefs have evolved significantly. In particular, they think that mathematics is an important goal for preschool, and that the teacher plays an important role in the development of mathematical competencies.

THEORETICAL BACKGROUND

The importance of early number competencies

Attention has increasingly been drawn to the importance of early mathematics education in recent years (Chen, McGray, Adams & Leow, 2014; Jordan, Kaplan, Ramineni & Locuniak, 2009; Platas, 2014). Many authors consider counting as the most fundamental tool offering access to arithmetical abilities during the first grades of primary school. However, the numerical competencies do not spontaneously develop, although there is an innate perceptual process, the so-called “number sense” (Dehaene, 2001). These competencies have to be learned and, in this context, the role of preschool education and the part played by families are of crucial importance (Cannon & Ginsburg, 2008).

From an educational perspective, it is generally acknowledged that the development of these competencies does not require formal learning, but can be developed through meaningful activities in everyday situations. However, according to Cannon & Ginsburg (2008), while everyday situations offer meaningful contexts, these are still not sufficient to develop the basic number competencies that are necessary for first-grade children. Consequently, adults need to plan specific goals for young children's mathematical learning processes and intentionally create opportunities to learn important mathematical competencies. However, preschool education seems traditionally to be primarily focused on the development of language and socio-emotional and motor development (Ginsburg, Kaplan & Cannon, 2006).

Teachers' beliefs about mathematics in preschool

Few studies have examined preschool teachers' beliefs about early childhood mathematics learning and teaching (Chen, McGray, Adams & Leow, 2014; Lee &

Ginsburg, 2007; Platas, 2014). According to Herron (2010), these beliefs need to be further investigated in order to achieve high-quality mathematics education in the preschool classroom. A major cause of failure of reforms has been the failure to take into account teachers' pedagogical knowledge and beliefs (Handal and Herrington, 2003). Philipp (2007) compares beliefs to lenses that shape the world we see. This research suggests that beliefs influence teachers' perceptions and judgments, thus shaping their actions in the classroom.

It is important, however, to be aware that promoting the teaching of mathematics in preschool may represent a significant change of practice for preschool teachers. Research literature on preschool teachers' beliefs about mathematics shows that in general, early childhood teachers experience fear and/or dislike of mathematics (Hachey, 2013; Lee & Ginsburg, 2007). This research has also shown that teachers do not attach much value to teaching mathematics or devote much time to this subject; instead, they tend to see preschool as an environment for encouraging socio-emotional and physical development rather than giving instruction in academic subjects (Lee, 2006). And on the academic side, teachers regard language as the most important subject. Such attitudes are obviously not without consequences for the teaching of mathematics in preschool.

The majority of studies of preschool teachers' beliefs are qualitative and are based on semi-directed interviews. Among these studies, those by Lee and Ginsburg, conducted on a large sample (around 70 teachers) are quite interesting. These authors identified nine very common misconceptions expressed by teachers in interviews, including, "Young children are not ready for mathematics education", "Language and literacy are more important than mathematics", "Teachers should provide an enriched physical environment, step back, and let the children play" (Lee and Ginsburg, 2009).

Incidentally, in overall terms, it is clear from all these qualitative studies that teachers' beliefs can be structured around four dimensions regarded as decisive for teaching practices (Lee & Ginsburg, 2009):

1. The primary goals of preschool instruction: This dimension relates to questionnaire items, which measure whether mathematics is regarded by teachers as a primary goal of preschool.
2. The age-appropriateness of mathematics instruction: This dimension relates to questionnaire items asking teachers whether they think that preschool pupils are mature enough to learn mathematics.
3. The classroom locus of generation of mathematical knowledge (teacher versus child): Some teachers think that mathematical knowledge is developed spontaneously in children through experiences and activities – in other words, that the locus of knowledge generation is situated in the child. Others think that this knowledge is developed through activities planned and managed by the teacher. They give an important role to the teacher and situate then the locus in the teacher.

4. Confidence in mathematics instruction: This concerns teachers' confidence about mathematics and their teaching of it.

Recently, Platas (2014), on the basis of the results of this qualitative work, designed and validated a quantitative questionnaire measuring these four dimensions. It is on the basis of this work that we designed our questionnaire for pre-service teachers.

The study presented in this paper is part of a larger project, whose main purpose is to develop early number competencies in preschool children in school and family contexts. In particular, the project aims to design, implement and support a play-based mathematics approach in preschool to the development of early number competencies by providing tools and a professional development programme as well as a specific model based on the involvement of parents.

METHOD

Our questionnaire, based on Platas' work (2014), was submitted to the 258 pre-service teachers of the "Bachelor en Sciences de l'Education" (BScE) programme of the University of Luxembourg during the academic year 2013-2014. This programme takes four years and prepares future primary, preschool and special education teachers. All students follow the same programme, regardless of the type of education that they will choose in their professional lives. There is no specialisation. Finally, the training programme offers few courses directly related to preschool. Three courses are taught, one on language, a second on science and a third on mathematics. In Luxembourg, preschool is for children aged three to six years and covers three school years. The first year is optional, whereas the last two are compulsory.

Our two research questions (RQ) were as follows:

- RQ1: How do pre-service teachers' beliefs regarding mathematics in preschool evolve from the first to the fourth year of their studies?
- RQ2: Do pre-service teachers' beliefs differ depending on the subject matter, i.e. mathematics, language, psychomotricity and the arts?

The paper-and-pencil questionnaire, based on that developed by Platas, consists of six-level Likert items. However, we had to make some changes to the initial questionnaire. We first translated the items into French, then adapted some of them to the Luxembourg context. Despite these modifications, each of the four dimensions presented a high degree of internal consistency for mathematics, with Cronbach's alphas all greater than 0.70 (see Table 1), the value regarded in the literature as the minimum acceptance threshold.

	N. of items	Cronbach's α
Primary goals of preschool instruction	6	.73
Age-appropriateness of mathematics instruction	5	.83
Classroom locus of generation of mathematical knowledge	12	.73
Confidence in mathematics instruction	6	.84

Table 1: Cronbach's α of each dimension of the questionnaire

To analyse the students' development over the course of their studies and answer the first research question (RQ1), we compared the views of first-year students ($N = 72$) with those of fourth-year students ($N = 75$).

To answer the second research question (RQ2), Platas' four dimensions were investigated with regard to mathematics in comparison with the other main areas of the preschool curriculum, i.e. language, psychomotricity and the arts. In order to validate the comparison, we had to change our way of constructing variables, as the alphas of the other areas for the first dimension, "Primary goal of preschool instruction", unlike those for mathematics, failed to reach the threshold level of 0.70 and thus could not be regarded as a one-dimensional scale. We therefore decided to merge the first two dimensions, calling this new dimension, "Relevance of mathematics in preschool". This decision was consistent with the research literature, as is clear in particular from the work of Koedinger and Nathan (2004), who observed that teachers organise their teaching according to their (sometimes erroneous) beliefs about the capabilities of their students. Table 2 below presents the Cronbach's alphas of this new dimension, all of which are now higher than 0.70 regardless of the type of activity (Mathematics, Language, Arts and Psychomotricity).

Dimension	Variables	Number of items	Cronbach's α
Relevance of mathematics in preschool	Mathematics	11	0.89
	Language	11	0.79
	Arts	11	0.75
	Psychomotricity	11	0.75

Table 2: Cronbach's α of the new dimension "Relevance of mathematics in preschool" for the four domains

As well as the dimensions of Platas, we also measured the allocation of time to the different preschool areas, classroom practices, and mathematical content. Finally we asked the students to what extent they agreed or disagreed with the nine misconceptions identified by Lee & Ginsburg (2009), which present a traditional view of mathematics in preschool.

In this article, we will analyse in particular the development of pre-service teachers' beliefs about the dimension "Relevance of mathematics in preschool" and about the "Classroom locus of generation of mathematical knowledge".

To identify the development of the results, we calculated the mean of students' positions on the different items of each scale, ranging from "completely disagree" (coded 0) to "completely agree" (coded 5). The mean scores calculated in this way could thus theoretically vary from 0 to 5. To compare the responses of the first- and fourth-year students, one-way ANOVAs were performed, treating the year of study as a fixed factor and the mean position as the dependent variable.

RESULTS

Relevance of mathematics in preschool

Table 3 below shows the results for the first-year and fourth-year students.

	Mean-1 st year	Mean-4 th year	Difference	
Mathematics	3.09	4.17	1.08	$p < 0.001$
Language	3.93	4.34	0.41	$p < 0.001$
Arts	4.11	4.32	0.21	$p < 0.03$
Psychomotricity	4.26	4.39	0.13	NS

Table 3: Mean of views of 1st and 4th year students on the various items of the dimension "Relevance of mathematics in preschool"

Table 3 first of all shows that the ranking of activities remains almost the same regardless of the moment in the programme (1st year or 4th year). Psychomotor activities are regarded as the most relevant to preschool (means of 4.26 and 4.39), followed by artistic activities (means of 4.11 and 4.32), then language activities (means of 3.93 and 4.34) and finally mathematics (means of 3.09 and 4.17). Note, however, that in the fourth year, the order changes slightly, with languages scoring slightly higher than the arts and thus gaining second place in the ranking of relevance.

A second notable point is that in both the first and the fourth year of the programme, languages are considered more relevant than mathematics. This confirms what research conducted in the field has shown, namely that language activities are considered more important than mathematics in preschool activities.

Finally, it is noticeable in Table 3 that there is a clearer difference between first- and fourth-year students' mean scoring of mathematics (a difference of 1.08) than of other activities. Fourth-year students are significantly and substantially more likely to regard mathematics as important for preschool than first-year students.

One hypothesis for this difference in students' views about mathematics is that first-year students regard mathematics as formal learning and as making little contribution to the social and emotional development that, according to the research literature

presented earlier, is regarded as the most fundamental area of learning by preschool teachers. Students' responses to the item worded as follows: "If a teacher spends time engaging in language / mathematics / arts / psychomotor activities, children's social and emotional development will be neglected" support this hypothesis. For language, arts and psychomotricity, a majority (over 90%) of both first-year and fourth-year students disagreed with this item. In other words, they did not think that the teaching of these subjects impairs social and emotional development. This does not hold true for mathematics, for which only 68% of first-year students disagreed with the statement. Thus 32% of them thought that the teaching of mathematics in preschool can hinder social and emotional development. By the end of the programme, 90% of students disagreed with this statement, in line with the views expressed by all students in the other areas. This (significant) difference in the position of students with regard to mathematics probably reflects a more integrated and more social vision of the learning of subjects, including mathematics.

The locus of the generation of mathematical knowledge

The results of the dimension "Locus of the generation of mathematical knowledge" are presented in Table 4. It will be recalled that these items measured where the students located the source of knowledge: with the teacher or with the child.

The continuum goes from the teacher (the minimum score of 0) to the child (the maximum score of 5). The closer the mean is to zero, the more students favour the idea that the teacher is the locus of knowledge generation. Conversely, the closer the mean is to 5, the more the child is favoured as the locus.

1 st year (%)	4 th year (%)	Difference	
2.44	2.02	-0.42	p < 0.001

Table 4: Mean of views of 1st and 4th year students on the various items of the dimension "Locus of the generation of mathematical knowledge"

We can see in Table 4 that in the 1st year, students present a mean position midway between the two extremes, with a mean of 2.44.

In the fourth year, students favour the teacher as locus more strongly, with a mean of 2.02, which is significantly different from the mean of the first-year students. This means that the fourth-year students assign a more important role to the teacher. This "teacher locus" does not mean that these students favour traditional classroom activities such as completing worksheets or doing exercises, however. We analysed the choices of activities by students depending on whether they identified a teacher locus (average ≤ 2) or a child locus (average > 2). The activities listed were of four types: 1) formal activities such as completing worksheets, 2) activities based on equipment such as logic blocks, 3) everyday activities such as following a cookery recipe, and 4) number games such as battleships. The following question was asked: "To what extent are these activities appropriate for developing pupils' number competencies?" The

results show that, regardless of the locus identified by the trainee teachers, the choice of activities was the same: overall, activities such as following a recipe or games were more favoured by students; by contrast, formal activities such as completing worksheets were relatively unpopular. Ultimately, then, the identification of a teacher or a child locus apparently will not be expressed in a particular choice of activity, but probably in the way these activities are managed: in the former case, the teacher will play an important role in the achievement of educational objectives, whereas in the latter case, children will be left more on their own, without the teacher intervening in the learning of number competencies, with the idea that these will develop spontaneously through the performance of activities.

Finally, on the last dimension, “Confidence in mathematics instruction”, we unsurprisingly find a similar pattern of development to that observed for the other dimensions. The fourth-year students said that they were significantly more confident of their ability to teach mathematics in preschool than those at the beginning of their training.

CONCLUSIONS

The results described above show that first-year students have a rather traditional view of the role of mathematics and the role of the preschool teacher in learning it. Overall, it is clear that trainee teachers at the start of their programme share the views of working teachers revealed in research conducted on the subject (Ginsburg, Kaplan, & Cannon, 2006; Ginsburg, Lee & Boyd, 2008; Hachey, 2013).

By the end of their programme, students’ beliefs about mathematics in preschool have been profoundly altered, across all dimensions. Although the mean results observed for mathematics do not fully coincide with those for other areas of the curriculum, they have definitely drawn closer to them. Future teachers now say that mathematics is an essential goal of preschool, almost to the same degree as other areas, and are more likely to take the view that the teacher should play an important role in learning mathematics. Their training appears to have played a major role in this. Platas (2014) also pointed to the importance of the training received by teachers in relation to the four dimensions measured, with those who had received training that included courses directly focusing on mathematics in preschool differing significantly from other teachers. The students of the “Bachelor en Sciences de l’Education” also took a course on the subject, although only a modest one. However, the change in viewpoints is probably the result of various factors in the programme, on both the theoretical and the practical side. As they progress, students have probably developed a more social and integrated vision of academic learning such as mathematics, reflecting a different and more appropriate notion of the abilities of young children, whom they now consider capable of learning mathematics.

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ENLISTING PHYSICS IN THE SERVICE OF MATHEMATICS: FOCUSSING ON HIGH SCHOOL TEACHERS

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There has always been a deep and close connection between mathematics and physics throughout history. Nevertheless, the linkage between the two sciences is almost neglected in mathematics education. In this study I describe an initial analysis of four types of tasks that connect the subjects and were presented to mathematics in-service teachers attending a course in mathematics education. Additionally, I show the types of the problems that were selected by the teachers to give a presentation on and examine the principles they employed in choosing appropriate tasks. Finally, I discuss the importance of reciprocal relationship between physics and mathematics, which can be used in mathematics teacher-education and in the high school mathematics classroom.

BACKGROUND

NCTM (2000) states that mathematical activities should include problems in context arising from areas outside mathematics. On the one hand, the literature indicates that enhancing mathematical understanding can promote one's perception of physical concepts (Bing & Redish, 2009). On the other hand, there are several studies that highlight the role of physical understanding on learning mathematical concepts. As an example, some of these studies propose the integration of calculus and kinematics (e.g., Planinic, Milin-Sipus, Katic, Susac & Ivanjek, 2012) in mathematics lessons. However, the linkage between mathematics and physics is almost neglected in mathematics education, in spite of the deep interconnection of mathematics and physics throughout history (e.g., Blum & Niss, 1991; Domínguez, de la Garza & Zavala, 2015).

Learning mathematical concepts through mathematical modelling and using examples from physics promote a better understanding of mathematical concepts (Blum & Niss, 1991). Using examples from physics constitutes a type of mathematical modelling, i.e., the process of translating the real world into mathematics and vice versa (Blum & Niss, 1991). For example, introducing concepts and arguments from physics into the teaching of geometry provides a better understanding of the theorems (Hanna & Jahnke, 2002). Moreover, introducing mathematical concepts with an emphasis on the interaction between mathematics and physics can provide a meaningful context for a better understanding of the creation of mathematical knowledge (Kjeldsen & Lützen, 2015).

Regardless of the proven importance of the integration of physics and mathematics, these subjects are taught separately (e.g., Planinic et al., 2012). The main reason for this separation is that their teaching applications are highly demanding and require

mathematical and extra-mathematical knowledge (Ferri & Blum, 2010). One way of promoting teachers' content knowledge is to offer specific university courses (Kaiser & Schwarz, 2006) with compulsory hands-on teaching experiences (Ferri & Blum, 2010). This can be done by offering a course in which mathematics and physics are taught simultaneously (Domínguez, de la Garza & Zavala, 2015).

This study presents different types of problems that were used in a mathematics course for mathematics teachers. These types of problems were given to the students at the first part of the course, and later selected by them independently. The selected problems, in their opinion, represent the desired connection between physics and mathematics.

METHOD

The present study is based on the Teacher Development Experiment (sf. Simon, 2000) conducted during a 56-hour course attended by 31 experienced in-service mathematics teachers. All teachers possess a BA in mathematics, a teaching certificate and went on to attain an MA degree in mathematics education. The teachers had basic knowledge of physics and did not teach physics in secondary school.

The setting included two types of sessions. Type A sessions were held during the first half of the course, while type B sessions were held during the second half,

A: Problem-solving sessions in which the teachers were exposed to problems of different types from the point of view of the linkage between mathematics and physics. The teachers were asked to solve the problems under an instructor's guidance, after which they presented their solutions to the whole group and discussed the solutions.

B: Problem-solving sessions guided by the teachers themselves. The teachers (in pairs) were asked to select problems from various scientific and educational resources that connect mathematics and physics and then to teach a session to the other teachers participating in the course.

All the sessions included use of either technological tools (GeoGebra and applets available on the internet) or "hands-on" physical experiments. All sessions were videotaped and all artifacts were collected for later analysis.

The goal of the study

The goal of the study was to analyze development of teachers' conceptions related to the problems that connect physics and mathematics, paying particular attention to their views of the mathematical and didactic power of the tasks (Jaworsky, 1992) as well as development of their success in solving and classifying the problems.

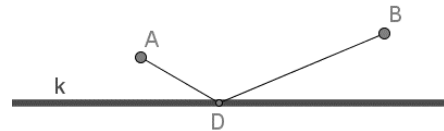
The goal of this paper is twofold: First, I present an initial analysis of the problems as they were devolved to the teachers. 4 problem types taken from in-service training courses that served as example problems are analyzed; second, I present the types of problems chosen by the teachers for the type B sessions.

ANALYSIS OF THE PROBLEMS IN SESSION A

The problems in session A were of 4 types: (I) A mathematical model of a physical phenomenon (e.g., Polya, 1954) (II) A mathematical problem with physics-based proof (e.g., Hanna & Jahnke, 2002) (III) A mathematical problem in the context of physics (IV) A physical problem and its mathematical context (Figure 1). The analysis of the tasks according to the connection between the two subjects (physics and mathematics) is presented in Table 1.

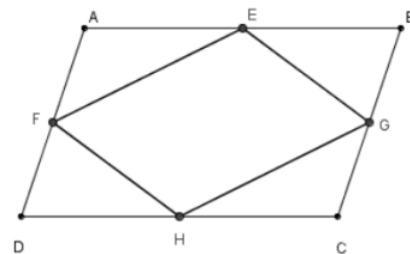
Example of Type I: Heron problem

Given two points A and B on one side of a straight line k , find point C on line k such that $|AC|+|CB|$ is as small as possible.



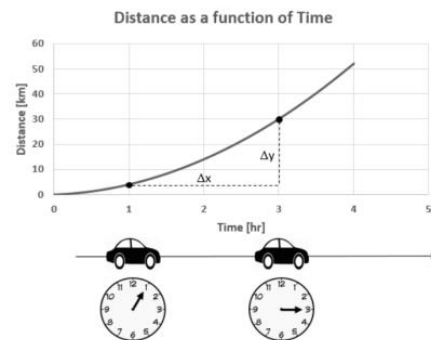
Example of Type II: Varignon theorem

Prove that the midpoints of successive sides of a quadrilateral form a parallelogram.



Example of Type III: Average and instantaneous velocity

A car is moving along a straight line whose distance from its origin after t hours is $s(t) = 3t^2 + t$ km. (a) What is the average velocity of the car in the time frame of 1h to 3h? (b) What is the instantaneous velocity at 1 h after beginning the movement?



Example of Type IV: Free diver

A free diver dived from the surface to a depth of 100 m while holding his breath. The volume of his lungs at sea level is 6 liters. What happens underwater to the volume of his lungs?

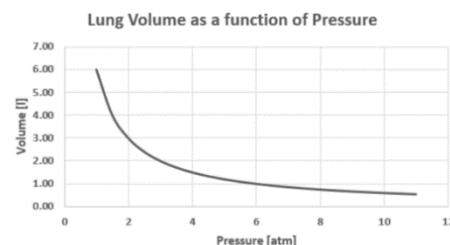


Figure 1: Examples of four types of the problems presented in session A

Type	Source	Problem exposition	Mathematical	Physical	Relationship
		(M) Mathematical (P) Physical	basis of solution	basis of solution	between mathematics and physics
I)	P	(M) Given two points P and Q and a straight line, all in the same plane, both points on the same side of the line. On a given straight line, find a point such that the sum of the distances from the two given points will be minimized. (P) A ray of light is travelling from point P, strikes a plane mirror and is reflected from the mirror to point Q. Find the point at which the light is striking the mirror.	Symmetry, Line reflection, the minimum of the continuous function	The propagation of a light, reflection from a plane mirror, Two laws of reflection	A minimum principle in describing the physical phenomena, the optimization problem
II)	M	(M) Prove that the midpoints of successive sides of a quadrilateral form a parallelogram. (P) Prove that the midpoints of successive sides of a quadrilateral form a parallelogram by means of mechanics principles.	Parallelogram, The line segment joining the midpoints of two sides of a triangle	Lever principle, center of gravity of a system of masses	Arguments from physics in a mathematical proof

Table 1: Analysis of the problems according to the connection between mathematics and physics

Type	Source	Problem exposition	Mathematical basis of solution	Physical basis of solution	Relationship between mathematics and physics
		(M) Mathematical (F) Physical			
III)	M or P	<p>(M) For the function $f(x) = 3x^2 + x$</p> <p>(a) What is the average rate of the change between $x=1$ and $x=3$? (b) What is the derivative of the function at $x=1$?</p> <p>(P) The car is moving along a straight line and its distance from its origin after t hours is $s(t) = 3t^2 + t$ km.</p> <p>(a) What is the average velocity of the car in the time frame of 1h to 3h? (b) What is the instantaneous velocity at 1 h after beginning the movement?</p>	The slope a function at a point, the derivative of a function at a point, the rate of change	The average velocity, the instantaneous velocity	Context problem contributes to the development of meaningful mathematical concepts
IV)	P	<p>(P) A free diver dived from the surface to a depth of 100 m while holding his breath. The volume of his lungs at sea level was 6 liters. What happens underwater to the volume of his lungs?</p> <p>(M) What are the properties of the function $f(x) = 6/x$?</p>	The relationship between the pressure and the volume of an enclosed gas when temperature remains constant: $PV=k$ (Boyle's law)	The properties of the function $f(x)=k/x$	Physical problem and related mathematical concepts

Table 1: (cont.) Analysis of the problems according to the connection between mathematics and physics

TYPES OF PROBLEMS SELECTED BY THE TEACHERS

The teachers were instructed to present problems in session B, after the four types of problems were taught to them in session A. The majority of teachers provided problems of one of the four types presented. However some of them had chosen problems of a different type (type V) which describes physical and mathematical problems with the same keyword in the concept (e.g. a circle). In these cases the teachers were unable to connect the physical and mathematical facets of the task, so the two problems were presented separately.

Example of Type V: (M) Given a circular arc AB is rotated through a given angle into a position AB'. Prove that the straight lines through the pairs of points corresponding under the rotation all pass through a fixed point. (P) A particle of dust travelling in a clockwise direction moves uniformly at a speed of 3.5 m/s on a disk in a circle with a diameter of 10 cm. Calculate its centripetal acceleration.

The mathematical bases of the solution of M-problem are a circle, rotation around a point, symmetry, reflection in a line and a fixed point. The physical bases of the solution of P-problem are circular motion and centripetal acceleration.

Interviews with the teachers revealed several principles they employed when choosing a task for the final work and presentation. These were: (1) The personal interest in the mathematical facet of the task and its accessible physical connection (2) The knowledge of mathematical and physical concepts involved in the problem (3) The ability to demonstrate the physical facet of the problem by an experiment or by means of technological tool.

The most frequent choice of problem was represented by type I (N=10). The mathematics educators' second choice was type IV (N=8). The number of teachers preferring to present problems of either type II or type V was similar (N=3 or N=4, respectively). Six teachers selected type III.

DISCUSSION

Kaiser & Schwartz (2006) claim that it “is insufficient to simply impart competencies for applying mathematics only within the framework of school curriculum”. Students should deal with tasks that stress the relevance of mathematics for the other sciences (in our case physics) and should acquire competencies that enable them to solve real mathematics problems.

In the framework of the proposed course, the teachers were introduced to four types of problems that connect mathematics and physics (session A) and were required to present a problem drawing connections between the two subjects (session B). All the problems were based on the topics that are learned in secondary schools. The choice of problems by the teachers can be subdivided into five types. Type I is a mathematical problem with a “hidden” physical analogy. This type of problem stresses the physical aspect of the mathematics. Type II employs the physical proof for mathematical problems. This type of problem “may reveal the essential features of a complex

mathematical structure or point out more clearly the relevance of a theorem to other areas of mathematics or to other scientific disciplines” (Hanna & Janke, 2002, p. 40). Type III is a mathematical problem in a physical problems “suite”. This type of problem requires transference from a physics-based content to a mathematical one. It proposes a didactic tool for teaching mathematical concepts in a more illustrative way. Type IV is a physical problem that points to the mathematical concepts involved in solving it. It is related to the mathematical knowledge that is essential for success in its solution. Placing more emphasis on the meaning of these mathematical concepts will lead to a better understanding of physical concepts (Johansson, 2015). Type V as proposed by the students focuses on the differences and similarities of mathematical and physical concepts that contain the same keyword but are not actually conceptually connected.

For all the mathematics teachers in this study, dealing with the physical aspects of problems in their mathematics lessons was a totally unfamiliar experience. On the whole, they perceived physics as a difficult subject and therefore avoided using it. It can be assumed that their acquaintance and experiences with physical contents as well as their beliefs about mathematics guided their choice of the problem types for their final presentations. For example, teachers that are familiar with the physical concepts started their presentations with an experiment or a demonstration and selected problems of type II to IV. In contrast, teachers who lacked sufficient knowledge of physics concentrated on topics of type V. As a result of the course, teachers’ predisposition to using physics in solving mathematical problems was enhanced. For example, when the teachers were presented with two different solutions to a type II problem (physical proof to a mathematical problem) all of them agreed that the solution based on physical intuition was more “elegant.” Nevertheless, most of the teachers considered the solution using mathematical apparatus to be more reliable.

It is my view that integration of physical phenomena in the mathematics curriculum can foster a more meaningful view of mathematics among teachers and students alike. However, this cannot be done, in my estimation, without mathematics educators attaining a satisfactory understanding of the linkage between physics and mathematics. In addition, the choice of appropriate problems is essential to developing an understanding of the connections between physics and mathematics.

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REFLECTIVE PRACTICE AND TEACHER IDENTITY: A PSYCHOANALYTIC VIEW

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This article explores issues that are central to changed mathematics pedagogy. It engages general debates about teaching reflexivity and within that, more specific debates in relation to identity. It uses theoretical concepts derived from Lacanian psychoanalysis as a way of understanding what structures a teacher's narrative about his practice. Thus the article is both a study of one teacher's reflections on a sequence of algebra lessons at the secondary school level, and an exploration into a range of theoretical issues about identity construction, about knowing, and about effective practice.

INTRODUCTION

A major focus in mathematics education today is the enhancement of pedagogical effectiveness. The focus is based on the realisation that the teacher is a key resource for enhancing student achievement (Copur-Gencturk, 2015; Drageset, 2015; Oonk, Verloop, & Gravemeiger, 2015) and is a critical feature in the promotion of equitable classrooms (see Anthony & Walshaw, 2007; Norton & McCloskey, 2008, Owens, 2015). A contemporary interest, centred on the teacher as reflective practitioner (for example, Muir & Beswick, 2007), adds a compelling layer to our understanding of effective teaching. Teacher reflection, it has been proposed, provides a way of authoring the teacher's self into an account of pedagogy and, hence, is a way of promoting change. The practitioner's reflective analysis is a reaction "against a view of practitioners as technicians who merely carry out what others, outside of the sphere of practice, want them to do" (Zeichner, 1993, p. 204). Specifically, teacher reflection is presented as a counter to the effects of researcher power, privilege, and perspective, and as a catalyst for an empowering dialogue focused on pedagogical change.

With its roots in the critical social science of the Frankfurt School, the notion of the reflective practitioner has been instructive in debates surrounding pedagogical questions. Personal narratives as experienced and told by teachers about their practice with a view towards development are propelled by assumptions to the effect that 'experience' is self-evident and that pedagogical change is specifiable. However, in the view of Brown (2008), the reflections and changes proposed merely provide "a mask for the supposed life behind it, a life with attendant drives that will always evade or resist full description" (p. 1). They fail to engage in a critical examination of the way in which change, and hence reflections, are actually produced. In particular, they overshadow the "relationships and forms of reciprocity and obligation that are embedded within them for understanding the identities and practices in which [teachers and researchers] engage" (Thomson, Henderson, & Holland, 2003, p. 44). As a

transformative strategy that claims emancipation, transcendence and freedom from ineffectivity, reflective practice fails to theorise how processes of change are lived out ‘experientially’, performatively, at the level of the individual.

In this article, the objective is to provide a vocabulary and a lens for explaining and analysing shifts in mathematics pedagogical practice, consequent on a practitioner’s reflective practice. In offering empirical and theoretical insights on what counts as pedagogical change, I argue for the strategic use of concepts drawn from Lacanian psychoanalysis (for example, Lacan, 1977) to bring about transformation in the context of the mathematics classroom. Specifically, Lacan’s arguments about narratives of the self, and Žižek’s (1989, 1998) related examination of how subjectivities are constructed across sites and time, are applied to a research project focused on shifts in pedagogical practice. Thus, this is both a study of one teacher and his reflections on a sequence of lessons, and an exploration into a range of theoretical issues about identity construction and change processes in mathematics teaching and research. In theorising the connection between narrations of the self and wider processes and events, the analysis provides a counterpoint to current thinking about researcher reflexivity.

CONTEXTUALISING THE EXPLORATION

Data for the project were collected through classroom video records, interviews with and classroom researcher observations of the teacher (Dave) who had been identified by the local mathematics teaching community as an effective secondary school practitioner. In his fourth year of teaching, Dave taught in a large co-educational school, catering for students from, in the main, the middle socio-economic sector. Students in his class of 30 formed one of two extension classes at the Year 9 (aged 13 years) level. This was a class that included “some very top students who conceptually pick things up very quickly” (Interview, post research). The classroom research component focused on 10 consecutive lessons that represented a unit on algebra—specifically, formulating linear equations, substitution, and solving linear equations.

In the analysis of Dave’s data, the intent is to unpack the ways in which his identity as a teacher is mobilised, reconceptualised and reformed through his participation in the research project. The analysis involves uncovering and exposing the mechanisms through which he comes to an understanding of his classroom practice (Brown & England, 2004). The work of Lacan and Žižek allows me to engage critically with the ideological frameworks through which Dave, as teacher-as-reflective practitioner, produces a narrative of his classroom work. Methodologically, in taking the reflective self to task, the psychoanalytic interest in how Dave produces his narrative, acknowledges the interdependencies and the realities that shape not only classroom life, but also the research process itself. It will involve looking at the intersection of the teachers’ subjectivity, the researcher’s subjectivity and intersubjective negotiations and the place of emotions between both.

WORKING WITH IDENTITY AND REFLECTIVE PRACTICE

Understanding the self-in-conflict

In interview following teaching of the ten algebra lessons Dave explained that his teaching goals for the unit were twofold: (i) that students will learn to use and understand equations to solve problems and (ii) that they will develop an understanding of the meaning of equality ($=$). My classroom observations recorded the content of the ten lessons as follows: Lesson 1 revised understanding of basic understanding of algebraic terms and fundamental algebra manipulation. Lesson 2 developed a strategy for writing simple linear equations and for solving them using a ‘1-step’ approach (for example, $x + 17 = 29$). Lesson 3 proceeded to a 2-step approach to the solution of simple linear equations (for example, $3x + 5 = 41$). The understanding and solution process was further developed in lesson 4 (for example, $2x + 5 = 19$). In lessons 5 and 6 real-world applications of solving simple linear equations were explored. Lessons 7 and 8 investigated the equals sign further and strategies were extended in Lesson 7 in order to solve equations with x -terms on both sides (for example, $3x + 4 = 2x + 9$) and, in Lesson 8, negative values (for example, $3x - 3 = -2x + 7$) were incorporated. Lesson 9 introduced fraction and decimal solutions (for example, $3x - 1.5 = 12.3$). The sequence of lesson culminated in lesson 10 in which real-world applications of solving equations with non-integer solutions were explored.

In developing students’ understanding of the equals sign, in lesson 2 Dave drew a number of balanced scales, weighing icons that represent the four suits of a pack of cards. For example, in one diagram, the left hand side of the balance scales held five clubs and the right hand side—a diamond as well as five spades. The task was to determine the value assigned to a spade and to a diamond. Dave pointed out to the class: “The puzzle is saying if we have a set of perfectly balanced scales then the left hand side and the right hand side must be the same.” For this and other similarly rich open-ended problems in lesson 2, Dave anticipated a range of possible solutions and accepted a ‘guess and check’ method to find values for a spade and a diamond. He then proceeded to more difficult problems in which writing an equation was a prerequisite for a solution.

As a researcher observing his classroom practice, I formed an impression of Dave’s teaching as immensely effective. I observed the quiet undivided attention he gave to his students and witnessed the kinds of intellectual exchanges and sophisticated mathematical argumentation developed within the classroom. Particularly uplifting was the way he enabled individual students to appreciate for themselves that the values they had found for a spade and for a diamond were (or were not) mathematically sound. Because of this, I wanted to observe his teaching, and the positive influence his teaching had on student outcomes. I wanted to hear about his lesson objectives and witness their attainment. He said in an interview after lesson 2:

With the balancing of scales, I am trying to sow the seed for later on in terms of manipulating each side...They got the idea that there were scales that needed to be

balanced and by manipulating what goes on the sides of the scales was really what it was all about.

I asked him: “So the balance idea, each side must be different?” He replied:

Yes, because later on they are going to need to understand that the equal sign doesn’t just mean...and up until now most of them think the equal sign means ‘works out to be’, or ‘I get this’, whereas later on I am going to have to adjust their view of what that equal sign means and think in terms of scales. And so later on when I talk about scales, they will have a reference point for it.

In lesson 3, Dave introduced the ‘magic box’ (sometimes known as the ‘function box’). He took a step-by-step approach to solving $3x + 5 = 41$, taking x first, multiplying it by 3, posting the $3x$ card into a box, then posting a ‘+5’ card into the box, and exiting the number 41. He explained to the students about reversing the order of operations, and proceeded to carry out the reversal process in order to find the unknown variable, and hence to solve the equation. Dave then repeated the ‘magic box’ trick with ‘ $2x$ ’, ‘-6’. He asked students to write the first part of the equation, in the same way that he had shown them to do during the first magic box episode. Once the right hand side number had been provided, students then substituted the value obtained for x in the equation to verify the result. In lesson 4 Dave again used the magic box trick, illustrating the process of solving equations by reversal using two different equations that students had already worked on and solved, one of them being $2x + 5 = 19$.

In lesson 7, Dave discussed with his students the meaning of equality and the importance of developing an understanding of ‘equals’ appropriate for the task at hand. He then used a data projector to show an animation of balance scales for a different equation: $3x + 2 = 2x + 3$. Again, using a step-by-step approach, he placed ‘3 x ’s’ and ‘2’ on the left hand side of the balance scales. Students immediately noticed that the scales became unbalanced. Dave then placed ‘2 x ’s’ and ‘3’ on the right hand side, to achieve equilibrium. He worked through a solution of the equation, using the procedure of ‘doing the same to both sides’. The visual display illustrated that ‘doing the same to both sides’ guaranteed to produce balance in the scales.

After lesson 7, I remarked to Dave in interview: “In a couple of earlier lessons you used a model of the box where you put in something and a process happened and then got you back to the original. You had to reverse or undo or go backwards. So that is a different way of thinking about equations.” Dave replied:

When they arrive they tend to have this idea that that’s what an equal sign does. It’s a command that gives you an answer after you have done certain things. So I was keeping the traditional view of what an equation is about. You do something to ‘ x ’, then maybe you subtract a number from it and then you get an answer. You push the equal button and out comes this answer. And I was trying to process that if we reverse that idea we can undo what has happened and get back to ‘ x ’.

In lesson 8, Dave used a different piece of software on the data projector to show an animation of solving equations. The representation was of a set of balanced scales, as before, but in this case weights corresponded to the addition of an entity.

T: ...How do you think we could represent for example 'minus 3'. How do I get $3x$ minus 3 on the left hand side of my scale? Plus 3 is a weight blocks pulling down. What do you reckon minus 3 might be James?

S: Lifting it up.

T: Lifting it up. So what kind of symbol do you think we could use to represent lifting the side up?

S: A helium balloon.

T: A helium balloon. All right let's try it.

The class then watched an illustration on the data projector and the use of weights and balloons for solving $3x - 3 = -2x + 7$. More discussion on the process developed and then the class set to work on examples from their textbook. In our discussion immediately after the lesson, while watching the video clip of the lesson, Dave pointed out:

...they could picture if you had two balloons pulling one side up and you take them away, the impact is going to be the same as if you put something on it to weigh it down....The idea of having a balanced scale, having them visually see what is essentially working; visually step by step is really helpful. To be able to say right 'we are taking away three from this side and then go to the software and take three away' and see it is not balanced and you need to keep it balanced so what do we do? Step by step process, going from the working to the visual really works very well.

Just before all these observations were made, I had pointed out that it was not entirely clear to me what the balloons and the blocks represented. I was also unclear about the use of multiple representations, namely, the 'magic box', the balance scales, in addition to the balloons and blocks. In response to the balloons and blocks question, he explained: "If I wanted $3x$, I had to have three little blocks built up, and if I wanted negative $2x$ each of the two balloons represented a negative x so I needed two of them to represent the negative." Reflecting on the lesson he drew attention to "lots of learning. It was a really packed lesson, the coming together of ideas and putting them in place."

In the final interview—the interview requested by him and which took place a few weeks after the classroom data gathering had concluded—Dave reflected on his teaching:

The one idea that I haven't one hundred percent really settled on is again that 'equals sign'. My approach was 'what do they know, and what knowledge have they brought into the classroom?' and predominantly it was that 'equal' sign...it's 'give me the answer', strike the calculator and give me the answer and write it down on your paper after the work and see what the answer is. And that is what they brought into the classroom, so I used that initially to get them thinking about how to solve the equations, 5 times x plus 3 equals

something and then we will reverse that process to figure out what the original number for 'x' was. And then later I introduced the idea of 'same' the two sides of an equation being the same and you can swap the order around there is no direction from left to right it's just a set of scales that are balanced and that is when I brought in more complicated equations with variables on both sides. And they responded well to that but I have never really been sure whether I should have brought that idea of 'same' straight away and I am still not sure.

At an overt level the research data foreground the construction of a coherent classroom identity that developed in response to a set of themes to do with pedagogical skills, knowledge and agency. At a more covert level, Dave's talk evoked traces of other events and other interpersonal relations, as well as defences, that created a rationale and a sense of cohesion to his interview. Together these two levels opened up important aspects of his subjectification in relation to being a mathematics teacher. It was not simply the present that factored into the construction of teacher identity: past as well as anticipated experiences, in a wider range of sites, also played their part in how Dave lived his subjectivity as a teacher. As Žižek (1989) has claimed: "identification is always identification on behalf of a certain gaze in the Other" (p. 106).

Dave, like any other effective teacher, was constantly trying to close the gap between how he sees himself and how he thinks others see him, always attempting to reconcile what he is with what he might become. It is not an especially obvious procedure, but nevertheless, in its subtlety, it was extremely powerful in establishing the parameters along which his identity as a mathematics teacher will be constituted. It is in this sense that we can understand how the terms that enter into the production of a mathematics teaching identity are "outside oneself, beyond oneself in a sociality that has no single author" (Butler, 2004, p. 1). What Dave was looking for is an instance, a moment, or what Lacan calls a 'quilting point', that will provide him with a marker, a strategic place from where he could make his choices about how to close the gap between his own and others' views of him as a teacher.

In Dave's case, in the instance of the final interview, a 'quilting point' was, among other things, the researcher's element of doubt over the representation of balloons and weights during the data show in Lesson 7. Although immediately after the lesson he had assessed the lesson as productive, in his reflections on his teaching during the final interview, his 'true' sense of self at that moment was betrayed. Fictions and fantasies of practice competed for Dave's attention, operating beyond his comprehension, provided a censoring device as a defence against a set of fears and concerns. They shaped his lived experience, defending against his anxieties, and informing the kinds of interpretations he made about his teaching in the future. It is in this sense that we can understand the psychoanalytic claim that the 'core' inner self is not 'core' at all; rather, a sense of self is constructed through language and intersubjective images projected onto us by others (teachers, students, parents, principals, researchers, and so forth) of how they would 'see' us within a set of given social relations.

CONCLUSION

Research on teachers' reflections of practice offers a productive site for exploring questions of identity and change. Contemporary theories of meaning making and subject formation remind us of the inadequacy of language to capture lived experience. In claiming that the narrative of lived experience can never coincide completely with experience itself, these approaches have been an important resource in this article. I have taken particular inspiration from psychoanalytic writing as a means of probing the difficulties of narrating the experience of teaching mathematics, in any straightforward way, and as a way of problematising the use of experience to initiate change. In acknowledging the complexity and complicity operating when teachers engage in reflective moments of their practice, the approach foregrounds the insufficiency of knowledge, the constitutive interplay of subjectivity, obligation and reciprocity and the psychical dynamics at play in narrating oneself. In doing so, the psychoanalytic approach closes the affective-cognitive separation that characterises conventional notions of reflective practice.

There are significant differences between the conventional approach to reflective practice and that developed through psychoanalytic theory. For Lacan and Žižek identity claims can never achieve final or full determination; the past is always implicated in the present. Since memories of practice are constructed from past investments and conflicts, always with a gaze towards the Other, "narratives are not the culmination of experience but constructions made from both conscious and unconscious dynamics" (Pitt & Britzman, 2003, p. 759). Those constructions are inevitably destined to miss the mark, continually subverted within a kind of metaphorical space between people, never fully understood and never fully captured by language.

Narratives of pedagogical practice will never reveal a fidelity to truth. There can never be a 'truthful' account of the mathematics teacher's reflections because "the fictions of subject positions are not linked by rational connections, but by fantasies, by defences which prevent one position from spilling into another" (Walkerdine, Lucey, & Melody, 2003, p. 180). However, that realisation does not in any way prevent us from working at understanding how intersections of fictions and fantasies of practice are lived by teachers. To the contrary, exploring how the subjectivity of the teacher is produced at the interpersonal level is more pressing than ever in any discussion of teacher change. It is pressing in that it alerts us to the fact that teachers' reflections are more than instruments of change; they are also instruments of social reproduction. Paradoxically, then, reflective practice is as regulatory as it is emancipatory. For the politically motivated researcher, the goal will be to make transparent the epistemic constructions that compete for attention about what will count as mathematics teaching in schools. It is in that sense that a psychoanalytic approach operates as a test-bed for innovation, and a catalyst for pushing ideas about teacher change forward.

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WHAT TEACHERS SHOULD DO TO PROMOTE AFFECTIVE ENGAGEMENT WITH MATHEMATICS—FROM THE PERSPECTIVE OF ELEMENTARY STUDENTS

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This study surveyed a nationwide sample of elementary school students in Taiwan to explore students' perspectives on what teaching behaviors promoted their affective engagement in learning mathematics. Factors contributing to the teaching behaviors were identified by conducting exploratory and confirmatory factor analyses on lists of teaching behaviors obtained from empirical studies. This study identified a three-factor structure with factors of cognition, extrinsic motivation, and activity. The results also showed that when considering enhancing affective engagement, Taiwanese students prefer teacher help on their cognition and teacher management of teacher–student interaction and relationships compared with working on various hands-on or explorative activities.

INTRODUCTION

Engagement with mathematics influences students' development of mathematical literacy (Attard, 2012). Studies have shown that engaging students is more difficult in mathematics classes compared with classes for other subjects (Plenty & Heubeck, 2011). Therefore, methods of increasing student engagement in mathematics warrant research. Engagement is usually considered a multidimensional construct that encompasses behavioral, affective, and cognitive components (Fredricks, Blumenfeld, & Paris, 2004). Affective engagement, relating to willingness to learn and enjoyment in learning, is a crucial consideration in the literature on engagement in mathematics. However, motivating students to learn in mathematics classes is not easy (Maehr & Midgley, 1991). Teachers' instruction has been considered a powerful contributor to student engagement (Mark, 2000). Students' low engagement is at least partly due to teachers' inability to engage them and maintain their engagement (Bodovski & Farkas, 2007). Therefore, information specific to what teachers should do to increase students' affective engagement in mathematics is influential and valuable. This was the focus of the present study.

The significance of this study was to provide national representative lists and latent factors of teaching behaviors for promoting students' mathematical engagement from students' perspectives. The use of students' perspectives is an endeavor to adopt a student-centered view (Murray, 2011), which was advocated by Taiwan's mathematics curriculum reform in order to make students "insiders" rather than "guests" in their mathematics classes (Hsieh, 1997). With the use of a nationwide sample in Taiwan, the present study addressed the following research questions:

- (1) What factors contribute to the teaching behaviors that promote students' affective engagement from students' perspectives?
- (2) Do students consider the factors of teaching behaviors obtained in (1) to be equally influential in promoting students' affective engagements?
- (3) Do students' perspectives differ with gender, interest in mathematics, and mathematics achievements?

RESEARCH METHOD

Conceptual framework

Affective engagement

Affective engagement plays a major role in activating and maintaining cognitive engagement (Sancho-Vinuesa, Escudero-Viladoms, & Masià, 2013), and the term has been used interchangeably with motivation in numerous studies (Fredricks et al., 2004). Most engagement studies have focused on students' emotional reactions to the school, the academic schoolwork, and the people at the school when examining factors such as interest, enjoyment, preferences, happiness, and curiosity (e.g., Bodovski & Farkas, 2007). Some research has considered students' willingness and persistence in learning as major aspects of affective engagement (Steinberg, Brown, & Dornbush, 1996). In addition, some research has related affective engagement to students' appreciation and the value of specific subjects (Fredricks et al., 2004).

Teacher instruction effects on student affective engagement

The literature shows that students' affective engagement is enhanced by teachers' instructional management such as by using clear, concise, and meaningful explanations (Cavanagh, 2011); real-life examples (Attard, 2012); timely feedback (Sancho-Vinuesa et al., 2013); challenging or interesting tasks (Attard, 2012); and hands-on activities (Blumenfeld and Meece, 1988); and by cooperating with peers in small-group work or discussion (Bodovski & Farkas, 2007; Cavanagh, 2011).

Studies have suggested that teachers instruct students with various methods to cater to students with different backgrounds and needs (Attard, 2012; Cavanagh, 2011). Insufficient empirical research has examined what types and aspects of teacher performance most effectively promote engagement in students with various demographic, achievement, and affective backgrounds (Fredricks et al., 2004).

Design and Instrument

This study was conducted in two stages. In the first stage, a qualitative study employing open-ended questions was conducted on 238 high school students to obtain their opinions regarding what a great mathematics teacher would do when teaching. A content analysis of the students' responses and a literature review were performed to obtain dimensions and items related to mathematical teaching competence from university mathematics educators and researchers, school-based supervisors of prospective mathematics teachers, and expert school mathematics teachers. The

dimensions and items obtained in this stage were used to develop the instruments for the second stage of the study.

In the second stage, two questionnaires with dichotomous items were developed. One questionnaire was for the secondary school study, and the other was for the elementary school study. The items in the two questionnaires were identical. In the questionnaires, students were asked to state whether a great mathematics teacher should perform the described teaching behaviors in a variety of teaching contexts. The affective engagement items obtained from the first stage were prompted by “In order to raise our learning motivation, when teaching mathematics, a great elementary school teacher should....”

Participants

The sample comprised 1,039 elementary school students from 78 classes in 26 schools in 25 cities in Taiwan. The sampled schools were randomly selected, and in each, one Grade 4, one Grade 5, and one Grade 6 class were chosen randomly. The students in the fourth, fifth, and sixth grades constituted 33.3%, 33.4%, and 33.3% of the sample, respectively. Table 1 shows some critical characteristics of the sample. Regarding the demographic, affective, and achievement backgrounds, this study asked students their gender, interest in mathematics, and usual mathematics grade, respectively.

Data Analysis

The data analyses included exploratory factor analysis (EFA) and confirmatory factor analysis (CFA). Because our sample was large, we randomly separated it into two halves, with one half for EFA and the other half for CFA as suggested by the literature (Reis & Judd, 2000). For the first research question, this study performed EFA with oblique rotation to determine the factor structures of students' perceptions of what teachers should do to promote their affective engagement. EFA, as its title indicates, is exploratory and data-driven. It was suitable for this study because the hypothesized structures were absent. For the second and third research questions, clean factor loadings were required to calculate descriptive information. This study conducted CFA by using the structures identified through EFA to obtain clean factor loadings.

In this study, EFA and CFA were conducted with M-plus 6.12 by using a robust weighted least squares estimator that is typically considered robust to nonnormal data. The model fit for EFA and CFA was evaluated using a comparative fit index (CFI), Tucker–Lewis Index (TLI), and root mean square error of approximation (RMSEA). The estimates of $CFI \geq 0.90$, $TLI \geq 0.90$, and $RMSEA \leq 0.08$ indicate a good fit (Kline, 2011). The number of eigenvalues larger than 1 was also examined according to the Kaiser–Guttman rule to determine the number of latent factors extracted using EFA.

The weighted average percentage of checking (POC) for each latent factor was also computed. The factor loadings estimated through CFA were employed as the weights for the indicators when calculating the weighted average POC for each factor (DiStefano, Zhu, & Mîndrilă, 2009). To examine whether students considered the

factors equally influential, a paired t test, an independent samples t test, and an analysis of variance combined with post hoc analysis were conducted. In addition to statistical significance, Cohen's d as a measure of effect size was also reported. Values exceeding 0.2, 0.5, and 0.8 indicate a small, medium, and large effect size, respectively (Cohen, 1992).

Characteristics	Percentage		
	Female		Male
Gender	47.9%		52.1%
Interests in mathematics	Like mathematics		Dislike mathematics
	59.2%		40.8%
Usual mathematics grades	High achieving	Middle achieving	Low achieving
	42.6%	33.4%	24.0%

Note. The high-achieving, middle-achieving, and low-achieving students were those whose usual grades were 90 points and above, 80–90 points, and below 80 points, respectively.

Table 1: Sample characteristics

RESEARCH FINDINGS

Factor Structure

Thirteen teaching-behavior items regarding what teachers should do to promote students' affective engagement were obtained in the first stage, as listed in Table 2. The POC of every item was higher than 70%, except for M201 (52%) and M208 (68%). Five items even received endorsements of more than 90% from the students.

The EFA of the 13 items yielded three factors, as shown in Table 2. The factors explained 64% of the total variance. The model fit was good (CFI = 0.993, TLI = 0.987, RMSEA = 0.049). The first factor, *cognition*, included a group of teaching behaviors that considered students' learning regarding understanding, meaning, challenging, and prompt feedback from teachers. The second factor, *activity*, included a group of teaching behaviors related to arranging mathematics activities for students. The third factor, *extrinsic motivation*, consisted of teaching behaviors that provide extrinsic motivation by giving rewards, developing a favorable classroom climate and teacher–student relations, and applying extra aids or media.

According to Deci, Vallerand, Pelletier, and Ryan (1991), people have three basic psychological needs: competence, autonomy, and relatedness. The degree to which students' perceived classroom context meets their needs affects their engagement (Fredricks et al., 2004). The three factors identified in the present study reflected these three basic needs. The approaches of the *cognitive* factor tended to meet students' need for competence involving attaining internal outcomes. The approaches of the *activity* factor tended to meet students' needs for autonomy and relatedness with peers through classroom discussion or various learning activities such as hands-on explorations and games. The approaches of the *extrinsic motivation* factor included facilitating positive

teacher–student relationships or meeting students’ need for competence involving external outcomes. In addition, the three factors could be placed along two continuums: one relating to the cognition vs. affection teaching objectives (Krathwohl, 2002) and the other relating to interaction with objects (content bound; Piaget, 1936) vs. interaction with people (social bound; Vygotsky, 1978, see Figure 1).

Teaching-behavior Item		CFA Loading	POC
<i>Cognition</i>			
M212	Take into account how well we understand in order to keep us willing to learn	0.786	0.95
M213	Provide us immediate feedback, encouragement, or suggestions to our test results	0.651	0.93
M207	Tell us why we need to learn a new math idea/concept to facilitate our learning willingness	0.662	0.88
M204	Give out challenge questions during class to raise our learning interests	0.417	0.74
<i>Activity</i>			
M205	Leave time for us to discuss to help us like learning in class	0.624	0.75
M206	Arrange appropriate activities during class for us to learn (ex., hands-on, games, groups, and exploration)	0.560	0.70
M201	Make the handouts pretty and organized to help us learn in a good mood	0.609	0.52
<i>Extrinsic motivation</i>			
M211	Be energetic and spirited during class to keep us from feeling bored	0.543	0.94
M210	Provide appropriate encouragement when we have good performance	0.716	0.93
M203	Use his/her enthusiasm to spark our interests and keep us from giving up learning	0.789	0.92
M202	Share his/her academic and life experiences during class constantly	0.547	0.81
M209	Use various teaching aids or media to arouse our curiosity	0.593	0.79
M208	Tell stories of mathematical history to raise our learning willingness	0.626	0.68

Note. The CFA loadings were standardized coefficients.

Table 2: CFA loadings and POC

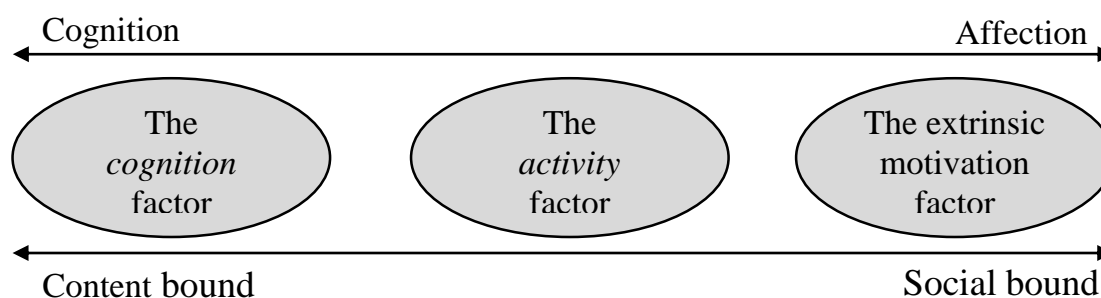


Figure 1: Continuums for the factors of teachers' approaches to enhancing student affective engagement

Comparison of POCs among Factors and Students with Different Backgrounds

The factor loadings of affective engagement items on the hypothesized factors in CFA are shown in Table 2. The model fit was good (CFI = 0.991, TLI = 0.988, RMSEA = 0.015). All the factor loadings were adequate (≥ 0.3).

The weighted average POCs of the *cognition*, *activity*, and *extrinsic motivation* factors were 0.90, 0.72, and 0.87, respectively. These high weighted average POCs showed that the factors were all influential in promoting students' affective engagement. The weighted average POCs of *cognition* and *extrinsic motivation* were significantly higher than that of *activity* ($p < .01$ and $p < .01$, respectively) with a medium effect size ($d = 0.63$ and $d = 0.53$, respectively). The difference between the weighted average POCs of *cognition* and *extrinsic motivation* was also significant ($p < .01$) and almost reached a small effect size ($d = 0.193$). The contrastive end, *cognition*, was the most influential factor in enhancing affective engagement. Providing autonomy has been reported to be effective in enhancing elementary school students' affective engagement (Ryan & Connell, 1989); this was reflected in our findings with the high weighted average POC of the *activity* factor. However, compared with *activity*, *extrinsic motivation* registered a higher endorsement, which means that Taiwanese elementary school students cared more about their teachers' management of teacher–student interaction and relationships than about their autonomy.

Table 3 shows that the weighted average POCs of each factor between students with different demographic, affective, and achievement backgrounds were not significantly different except for students with different interests in mathematics in the factor of *cognition* ($p < .01$; $d = 0.30$). These findings are not consistent with those of other studies, which have claimed gender as a factor in the degree of student affective engagement (e.g., Plenty & Heubeck, 2011).

CONCLUSION

What teachers should do to enhance student affective engagement in mathematics learning is a practical and crucial issue. The present study identified a three-factor structure by using a Taiwanese national representative sample of elementary school students. Our results indicated that the *cognition* and *external motivation* factors are more effective than the *activity* factor in promoting affective engagement. This

phenomenon indicates that, when considering enhancing affective engagement, Taiwanese students prefer their teachers' help with their cognition and management of teacher–student interaction and relationships compared with working on various hands-on or explorative activities. Practically, rather than developing time-consuming learning activities, a Taiwanese teacher may first focus on teaching approaches embedded in the *cognition* and *external motivation* factors to promote students' affective engagement. However, further research is required to determine whether this principle applies to teachers in other counties.

Factor	Gender		Interest		Achievement		
	Female	Male	Like	Dislike	High	Middle	Low
Cognition	0.91	0.90	0.92	0.87	0.91	0.90	0.89
Activity	0.73	0.71	0.73	0.70	0.73	0.72	0.70
Extrinsic Motivation	0.88	0.86	0.88	0.85	0.88	0.85	0.86

Note. Like = like mathematics. Dislike = dislike mathematics. The shaded pair of values is significantly different.

Table 3: Weighted average POCs of students with different backgrounds

Another crucial result of the present study is that, for students with different demographic, affective, and achievement backgrounds, the efficiencies of the factors are not different except for students with different interests in mathematics in the factor of *cognition*. The teaching behaviors involving more mathematics content (the *cognition* factor) work equally well for students with different achievement levels, but work differently for students with different interests in mathematics. It is possible that students with high achievement understand most of the mathematics content taught by teachers, and that an increase in understanding would not change their willingness to participate in class; by contrast, students with a high interest in mathematics benefit from understanding, which may be a prior barrier to their participation. However, further research is required to make any additional conclusions.

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NONLOCAL MATHEMATICAL KNOWLEDGE FOR TEACHING

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The notion of practice-based models for mathematical knowledge for teaching has played a pivotal role in the conception of teacher knowledge. In this work, teachers' knowledge of mathematics that is outside the scope of what is being taught is considered more explicitly. Drawing on a cognitive model for the development of mathematical knowledge for teaching, this paper explores the implications for the underlying theory being applied to (nonlocal) knowledge beyond what is being taught as being influential for the teaching of (local) mathematics.

INTRODUCTION

Teacher's mathematical knowledge, and the role that it plays in classroom practice, has been a central question in mathematics education for a long time. Some scholars have developed frameworks that describe various domains of that knowledge (e.g., Ball, Thames, & Phelps (2008); others, have focused on its' development (e.g., Silverman & Thompson, 2008). Much of this work has focused on how a teacher should understand the content that they teach; yet, when it comes to knowing content that is *beyond* what one teaches, there is little consensus as to its importance or its implications on classroom practice. In this paper, we draw on Silverman and Thompson's (2008) cognitive model for the development of mathematical knowledge for teaching as a means to adapt and explore the theoretical ramifications when one considers knowledge of content that is *beyond* what is being taught. This work has been informed by five years of research studies and projects with teachers (e.g., Wasserman, 2015a; Wasserman, 2015b).

MATHEMATICAL KNOWLEDGE FOR TEACHING

In one of the more broadly-adopted frameworks, Ball, Thames, and Phelps (2008) described their conception of Mathematical Knowledge for Teaching (MKT), which built on Shulman's (1986) work and proposed three sub-domains of subject-matter knowledge (SMK) and three sub-domains of pedagogical content knowledge (PCK). In the realm of SMK, the third category, horizon content knowledge (HCK) – which is the most associated with knowing mathematics beyond what one teaches – was only provisionally included. Although others have worked to further conceptualize HCK (e.g., Wasserman, Mamolo, Ribeiro, & Jakobsen, 2014), it has remained underdeveloped because of difficulties conceptualizing it in relation to classroom practice and as distinct from the other sub-domains. Ultimately, we propose a different division for considering teachers' mathematical knowledge; but first, we briefly discuss existing ideas about its' development.

Silverman and Thompson (2008) outlined a two-step cognitive model for the development of mathematical knowledge for teaching. Their model posited that powerful mathematical understandings – related to Simon’s (2006) notion of *key developmental understandings* (KDUs) – were the first step toward the development of mathematical knowledge for teaching. Simon (2006) described KDUs as a “conceptual advance... a change in [one’s] ability to think about and/or perceive particular mathematical relationships” (p. 362). In other words, KDUs are mathematically powerful understandings that change perceptions about content, effect ontological shifts in understanding, and influence mathematical connections. According to Silverman and Thompson (2008), however, while such understandings are mathematically powerful, they are not intrinsically pedagogically powerful. A second step, of transforming such understandings into having pedagogical power – which then affect classroom practice – was necessary for developing mathematical knowledge for teaching. Ultimately, a teachers’ understanding of the content they teach is one of the primary mediators for the way that they teach that content.

A MATHEMATICAL LANDSCAPE

This paper proposes a different approach for considering mathematical knowledge for teaching. In particular, instead of partitioning knowledge into SMK and PCK, we propose a different division, based on the relative location of mathematical ideas within a broader mathematical landscape. In particular, such a division more directly tackles the notion of mathematical knowledge *beyond* what ones teaches. Indeed, since the act of teaching deeply involves teachers in the mathematics of what they teach, we regard such a distinction as incredibly practical. And since there is continued debate around such knowledge, this work also contributes to the broader conversation about teachers’ content knowledge in mathematics education.

Although knowing ideas beyond what one teaches may be interesting to discuss in every subject area, in the teaching of mathematics, it takes on an even more important role. Compared to many other disciplines, mathematics is fairly linear in its developmental trajectory – new ideas and concepts are progressively *built on* and *refined from* older ones throughout the course (often, over a decade) of one’s mathematical study. This means that in mathematics what one teaches now is often revisited at a later point, and thus more directly linked to ideas that are beyond the current scope – which also has implications in the reverse direction as well.

With this in mind, we define the *local mathematical neighbourhood* as those mathematical ideas that are relatively close to the content being taught. “Close” in this sense entails both the degree to which mathematical ideas are closely connected, but also temporally close in relation to when mathematical ideas are typically developed (Wasserman, 2015a; Wasserman, 2015b). In other words, we are using a topological description about the landscape of mathematical ideas, defining two regions: the *local mathematical neighbourhood* of the mathematics being taught, and the *nonlocal mathematical neighbourhood*, which consists of ideas that are farther away. This idea

is connected to and was influenced by the notion of a “mathematical horizon.” Indeed, the image of a(n) (epsilon) neighbourhood allows for the inclusion of mathematical ideas that are “behind” as well as “beyond” the content being taught – not just a forward-looking horizon but also one in the rearview mirror. From Shulman’s (1986) notion of *vertical* curricular knowledge, we also might consider the inclusion of a curricular mathematical neighbourhood, which could separate the nonlocal mathematical neighbourhood for K-12 teachers into those ideas within the scope of school mathematics and those in more advanced mathematics (Figure 1).

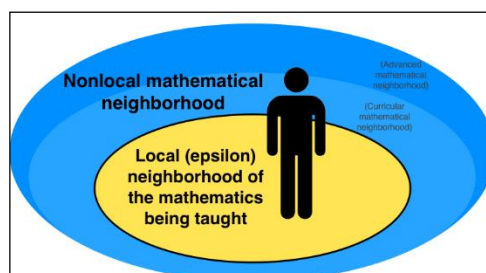


Figure 3. Mathematical Landscape

Briefly, we elaborate on two aspects of this partitioning. Firstly, as a discipline, mathematics has been a forerunner in defining both *content* and *process* standards as important educational aims (e.g., NCTM 2000). That is, the local neighbourhood of mathematics necessarily includes specific content, but it also includes more general ways of doing and engaging with mathematics. Such processes – e.g., problem solving, reasoning and proof, etc. – are reminiscent of Shulman’s (1986) portrayal that teachers should know their subject’s organizing structures, principles of inquiry, core values, etc. Indeed, both the local and nonlocal neighbourhoods have this dichotomy. Secondly, describing the set of mathematical ideas within the local neighbourhood can be difficult: there is no “distance metric” between mathematical ideas by which one might determine precise boundary regions. However, for many mathematical ideas, even without explicit definition, which neighbourhood they fall into is clear (e.g., groups in abstract algebra are outside the local mathematical neighbourhood for a K-12 mathematics teacher). Yet there are also advantages in the generality of the definitions; in particular, it allows for interpretations of various grain sizes. For example, one might consider not only the neighbourhood of one year of mathematics (e.g., 6th grade mathematics), but a much smaller neighbourhood of ideas being taught. This allows for a broader interpretation about how knowledge outside the scope of what is being taught – even if it is within the scope of content the teacher is going to teach – may influence teachers’ practices in the classroom.

Finally, two comments more specific to this paper: i) most examples stem from secondary mathematics education, as this is where the majority of the work has been. However, the intent is that the model is broad enough to incorporate other levels of teaching (e.g., elementary, university); and ii) most of the discussion leverages more advanced mathematics encountered at the university level – e.g., abstract algebra, real

analysis – as the means to address knowledge beyond what one teaches. But, again, a more flexible interpretation that would make sense in other contexts is also intended.

NONLOCAL MATHEMATICS AND PEDAGOGICAL POTENTIAL

In this section, we adapt the two-step cognitive model for developing mathematical knowledge for teaching in consideration for how knowledge of nonlocal mathematics interacts with the teaching of local mathematics. But first, we make more explicit one of the inherent difficulties with considering knowledge outside the scope of what one teaches: teachers *should not* end up teaching this content to their students. That is, we are discussing content that should, theoretically, not arise, explicitly, in instruction – secondary teachers should be teaching algebra, not abstract algebra; however, it should simultaneously be influential for their teaching. Therein lies the tension.

Key Developmental Understandings

Essentially, in accord with Silverman and Thompson (2008), one of the primary mechanisms by which we view connections to teaching has to do with teachers' own mathematical understandings. More specifically, with regard to mathematics outside what one teaches – the nonlocal mathematical neighbourhood – we adapt the first step in their cognitive model in a specific sense: teachers' understanding about nonlocal mathematical ideas must serve as a KDU for the (local) content they teach – which includes both mathematical content and disciplinary processes. This is to say that knowledge of *nonlocal* mathematics becomes *potentially* productive for teaching at the moment that such knowledge alters teachers' perceptions of or understandings about the *local* content they teach. We see this adaptation as aligned with the development of mathematical knowledge, and as a natural extension of the cognitive model, but also as very different from other perspectives. We contrast this (third) view with two other common perspectives about more advanced mathematics.

Advanced (Nonlocal) Mathematics as being for Mathematics' sake

First, some would argue that teachers should learn mathematics beyond what they are going to teach because they should. Mathematics, regardless of whether it relates to future teaching, is important. The most compelling arguments for this have something to do with the development of “mathematical confidence.” That is, the essential role of learning more advanced mathematics – i.e., mathematics beyond what they will be teaching – is to build a degree of confidence in their knowledge of the subject. Such confidence, we note, does have potential teaching benefits (Brown & Borko, 1992). However, particularly after the broad acceptance of PCK, such arguments, which, for the most part, are completely disconnected from the work that teachers do in the classroom, are met with a healthy degree of scepticism. It certainly becomes more difficult to justify that a secondary teacher needs to know that $Q(i):Q$ is a finite field extension (Heinze, et al., 2015). We depict this perspective – which contends that, regardless of connection, more advanced mathematics is important to study – by the two mathematical neighbourhoods being disjoint (Figure 2a).

Advanced (Nonlocal) Mathematics as related to Local Mathematics

Next, we consider those that view more advanced mathematics as important for teaching when it is related to the local mathematics. Perhaps the first to popularize this idea was Felix Klein, who wrote *Elementary mathematics from an advanced standpoint* (1932). The Conference Board of Mathematical Sciences' *Mathematical Education of Teachers II* (CBMS, 2012) has a similar position of applying more advanced mathematics to the content that the teacher will be teaching: for example, it would be “quite useful for prospective [secondary] teachers to see how C can be ‘built’ as a quotient of $R[x]$... [and] Cardano’s method, and the algorithm for solving quartics by radicals can all be developed... as a preview to Galois theory” (p. 59). Cuoco (2001) summarizes a principle for redesigning the undergraduate experience of prospective teachers this way: “Make connections to school mathematics” (p. 170). At the heart of this perspective is a desire to make more advanced mathematical study related to what a teacher is going to teach. Yet we regard the more general argument, that by the simple merit of some advanced topic – e.g., Galois Theory – being related to the content of school mathematics that such knowledge is important for teachers, as tenuous. We do not presume such a “trickle down” effect to teaching. We depict this perspective – which contends that more advanced mathematics is important when it is connected to school mathematics – by the two mathematical neighbourhoods interconnected at several places (Figure 2b).

Advanced (Nonlocal) Mathematics as related to *Teaching* Local Mathematics

Although these two perspectives about advanced mathematics both have potential value, if one adopts Silverman and Thompson’s (2008) model, the powerful understandings gained from nonlocal mathematics must serve as KDUs not (only) for their knowledge of nonlocal mathematics, but for the teachers’ understanding of the *local* mathematics they teach. Essentially, the first mechanism for bringing about connections to teaching is by tying the nonlocal mathematical knowledge as not only connected to but as fundamentally important for their own mathematical understanding of the local content they teach. We depict this perspective – which contends that more advanced mathematics becomes potentially important for teachers when it serves as a KDU for the local content they teach – by the two mathematical neighbourhoods overlapping, where the overlapping local region has been fundamentally altered (i.e., a new colour) by this connection (Figure 2c).

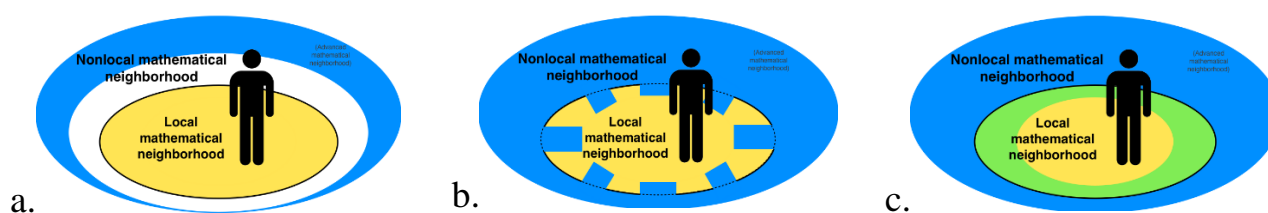


Figure 2. Depicting three perspectives on advanced mathematics

Having an idea in more advanced mathematics, such as the algebraic structure of a group, serve as a KDU for local content sets a high bar. That is, a teacher’s

understandings about and perceptions of, say arithmetic properties, must be fundamentally different because of the advanced mathematics. We note that this perspective of advanced mathematics, based on being a KDU for the local mathematics one teaches, is *very* different from others. It is different from simply having nonlocal knowledge serve to orient oneself in the mathematical landscape (e.g., Ball, 2009), or from applying more advanced mathematical techniques to school mathematics (e.g., Klein, 1932). Essentially, in accord with the cognitive model, we argue that very little that could be productive for teaching will transpire unless the nonlocal knowledge serves as a KDU for the local content.

PEDAGOGICAL POWER

This third perspective has been studied less but has the most potential for connection to classroom teaching due to its assimilation into a cognitive model for developing mathematical knowledge for teaching. Yet these understandings about local content, still, only provide a sense of pedagogical potential. We briefly describe three areas where such understandings might become pedagogically powerful.

On specific local mathematics content areas

One of the ways that knowledge of nonlocal mathematics might influence the teaching of local mathematics is in *specific* content areas. For instance, knowing the Calculus concept of derivative can influence how a teacher teaches about linear functions, slopes, and rates of change – three *specific* content areas. Accordingly, Wasserman (2015b) argued that understanding abstract algebraic structures might influence instruction in four specific content areas: arithmetic properties, inverses, structure of sets, and solving equations. Such instructional changes about specific content areas stem first from teachers' local understandings having been transformed by their nonlocal knowledge.

On specific pedagogical actions in teaching mathematics

Another way that knowledge of nonlocal mathematics might influence the teaching of local mathematics is in some *specific* pedagogical ways. For example, Wasserman (2015a) clarified a few specific actions in mathematics teaching by making a distinction between the local versus nonlocal mathematical neighbourhood. Two of the classroom actions – *foreshadowing* and *abridging* – were specifically in response to the teacher being aware of nonlocal mathematical complexities. Both of these classroom practices are examples of the kinds of pedagogical actions that transcend particular content areas, yet stem from teachers' nonlocal mathematical knowledge.

On general mathematics processes

Lastly, although all of mathematics can be a place to learn important mathematical *processes*, more advanced mathematics is potentially uniquely helpful for further refining and grasping some of these disciplinary ideals. Teachers at all levels need to help students understand what doing mathematics is all about. As an example, Real Analysis is a proof-based course that attends to a rigorous development of real numbers

and real-valued functions, and sets the foundation for important ideas in Calculus. Since the content of real analysis – indeed more so than many other mathematics courses – is extremely explicit with both definitions and assumptions, and producing rigorous deductive arguments, interaction with this nonlocal mathematics can serve as a place to strengthen these disciplinary practices.

NONLOCAL MATHEMATICS INFLUENCING TEACHING PRACTICE

To summarize, knowledge of nonlocal mathematics can influence both teachers' understanding of and teaching of local content. The primary mechanism is having such knowledge serve as a KDU for the content they teach – which can be specific content as well as general processes. These KDUs then can influence instructional practice across three different aspects: specific content areas, specific pedagogical actions, and general mathematics processes. Recently, Stockton and Wasserman (under review) posited five forms of knowing advanced mathematics that might be particularly applicable for teaching: *peripheral* knowledge, *evolutionary* knowledge, *axiomatic* knowledge, *logical* knowledge, and *inferential* knowledge. These represent some particular understandings about more advanced content that might help foster development as KDUs for local content that also have pedagogical power. Figure 3 summarizes the theoretical considerations for content outside the content being taught as potentially influential on the teaching of local content.

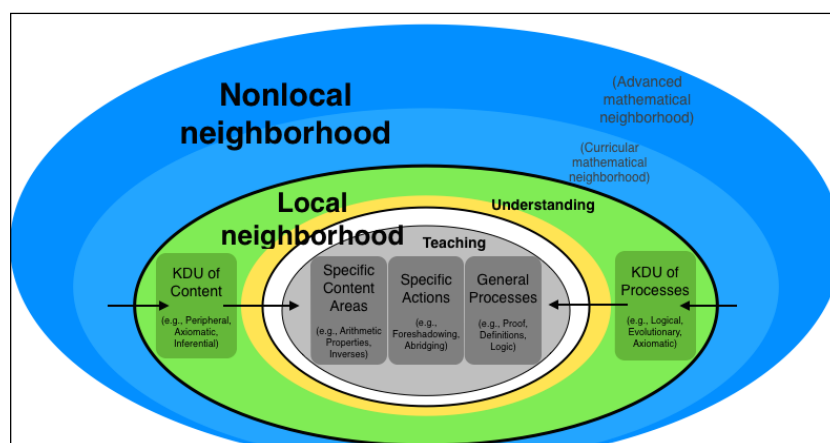


Figure 3. Nonlocal mathematical knowledge interacting with local teaching

IMPLICATIONS AND CONCLUSIONS

In conclusion, we mention one of the primary implications from this paper: that of considering how the teaching of more advanced mathematics might take place as a part of teacher preparation. We do not advocate that teachers need fewer advanced mathematics courses, but rather that the teaching of these ideas be more informed by and related to their future professional needs. To that end, these ideas suggest and support a model for teaching more advanced mathematics that explicitly has course content “build up from” and “step back down to” teaching practice. In other words, instead of hoping for a trickle-down effect, one ramification of our adaptation of the cognitive model is that the teachers’ development of and understandings about

nonlocal mathematics must not only relate to the *content of school mathematics*, but to the *teaching of school mathematics content*. The field of teacher education as a whole must better identify and use desired pedagogical changes – in specific content areas, pedagogical actions, or processes – to help build and develop teachers’ key understandings about nonlocal content in ways that can be pedagogically powerful.

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EXPLORING MIDDLE SCHOOL GIRLS' AND BOYS' ASPIRATIONS FOR THEIR MATHEMATICS LEARNING

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This study sought insights into the aspirations of over 3500 middle school girls and boys for their mathematics learning, with the intent of not only informing teachers of the nature of students' hopes more broadly but also to offer teachers a tool they can use with their own students and against which their own students' responses can be compared. The students responded to a free-format prompt and generated a wide range of aspirations related to their goals for learning, and their affect, interest and effort, along with specific insights into the features of tasks, working arrangements, and interactions with teachers. This paper discusses a comparative analysis of boys' and girls' responses in terms of the nature and frequency of their expressions of aspirations.

The considerable disengagement of middle school students in mathematics in recent years (e.g., Middleton, 2013) highlights the importance of finding ways to plan, teach, and assess mathematics that better align pedagogies with students' own aspirations. Middle school students have been described as showing less interest, less self-efficacy, and poorer achievement over time (Gottfried, Marcoulides, Gottfried, Oliver, & Guerin, 2007). This study sought to explore this issue from the perspectives of middle school students themselves (9 to 13 years old) – their own views on what matters to them, what goals they might hold, and what they perceive as desirable for their learning. It intended to consider and perhaps challenge assumptions and preconceptions about middle school students' engagement and learning. It was believed that finding out more about aspirations from the students themselves might allow teachers to respond in productive ways.

Researchers have investigated differences between boys' and girls' attitudes, engagement, and motivation in learning mathematics and have drawn diverse conclusions. Yet there is consensus that both boys and girls experience decreased motivation in the middle years. This study provided an opportunity to compare and explore the self-generated aspirations of boys and girls as another way to investigate this issue. It was assumed that awareness of students' own aspirations can help teachers to reflect on and respond to students' voice, and broaden their repertoire for teaching mathematics in ways that positively influence boys' and girls' motivation and achievement. This paper addresses the following research questions: *What aspirations do middle school boys and girls express related to their mathematics learning? What evidence of mastery or performance goals do boys and girls spontaneously generate?*

BACKGROUND

It was anticipated that the students' responses might relate to their own goals for learning mathematics. This paper focuses on the nature of boys' and girls' self-generated aspirations, and evidence of different types of goal orientations. One theoretical perspective relates to four different types of goals an individual might hold for their learning in a particular domain: *mastery* and *performance* goals (e.g., Ames, 1992) intersecting with another dichotomy of *approach* and *avoidance* goals. *Mastery goals* focus on improving one's own learning and making progress in task- or skill-based outcomes whereas *performance goals* focus more on *comparing* oneself with others, such as through test results or competitive situations. Approach and avoidance goals describe how competence is valenced: how a situation or experience involves inherent attraction leading to approach, or aversion leading to avoidance. The resulting two-by-two goals framework is presented in Figure 1. It has been substantially supported by empirical research for three out of the four types and more recently also with mastery-avoidance (e.g., Elliot & Muryama, 2008).

Mastery-approach goal <ul style="list-style-type: none"> • <i>Interest and curiosity</i>: learning something interesting • <i>Task</i>: mastering a task • <i>Challenge</i>: mastering a challenge • <i>Improvement or attainment</i>: Learning as much as possible; improving my knowledge; understanding the content as thoroughly as possible; acquiring new skills 	Performance-approach goal <ul style="list-style-type: none"> • <i>Appearance</i>: demonstrating competence / ability • <i>Normative</i>: performing better than other students • <i>Evaluative</i>: Demonstrating my ability relative to others in the class (as judged by authority figure such as a teacher)
Mastery-avoidance goal <ul style="list-style-type: none"> • <i>Task</i>: Avoiding forgetting what I have already learnt • <i>Improvement or attainment</i>: Avoiding losing my skills / abilities / knowledge; avoiding stagnation or lack of development 	Performance-avoidance goal <ul style="list-style-type: none"> • <i>Appearance</i>: avoiding looking incompetent / 'dumb' • <i>Normative</i>: Avoiding performing poorly in the class • <i>Evaluative</i>: Avoiding demonstration of lack of ability relative to others (as judged by authority figure)

Figure 1: Conceptualising four types of student goals using mastery-performance and approach-avoidance dichotomies (Ames, 1992; Hulleman et al., 2010)

Brophy (2005) raised the issue that most research using goal theory has involved measurement with experimental induction procedures or Likert-scale questionnaires, which do not allow investigation into the degree to which students *spontaneously* generate different goal orientations. He suggested that there is very limited evidence to indicate that students actually do generate performance goals that relate to "looking good in comparison with their classmates" (p. 171).

Boys' and girls' goal orientations in mathematics learning

In recent years, studies on goals and motivation have examined differences between year levels and gender. Although some have drawn differing conclusions, it is generally agreed that both boys and girls can experience a decrease of motivation in the middle years. A review of studies found that overall, boys tend to report a higher interest in learning mathematics than girls (Meece, Glienke, & Burg, 2006). Chouinard, Karsenti, and Roy (2007) found that more girls reported mastery goals and higher effort than boys. A study of 1244 German secondary students found that nearly half reported

believing that boys achieve more, one fifth reported that girls achieve more, and the rest indicated no gender difference. Of those students who reported that they believe *girls* achieve more, the three most frequently cited reasons were effort, concentration, and ambition – not ability (Kaiser, Hoffstall, & Orschulik, 2012).

In an Australian context, Watt (2004) found that boys maintained a higher interest in and liking for mathematics and a higher perception of competence (ability rather than effort) than girls throughout adolescence. In contrast to these findings, Leder and Forgasz (2002) studied over 800 lower secondary students and found that the majority viewed mathematics as a gender-neutral domain in terms of ability or achievement, and reported believing that *girls* are more interested in mathematics and *enjoy* it more, whereas the boys are more likely to find it difficult and boring – that they need more help to learn it than girls. Another Australian study of 1801 secondary students found that the middle school girls demonstrated more mastery goals and more effort than boys (Green, Martin, & Marsh, 2005). An across-country longitudinal study of secondary students found that Australian girls had significantly lower intrinsic value for mathematics than the boys, unlike those in Canada and the US. Yet they did not show lower perception of ability than the boys, as did the girls in Canada and the US (Watt et al., 2012).

This study provided the opportunity to consider what boys and girls themselves choose to focus on when expressing their aspirations for their mathematics learning, evidence of their spontaneously generated goal orientations and how these findings might give teachers insight into teaching mathematics at middle school levels.

RESEARCH DESIGN

There seems to be an increasing understanding of the value of consulting learners about issues that affect them, for making teaching and learning more effective (e.g., Flutter & Rudduck, 2004). Much of the research literature describes structured surveys and Likert scales, typically used in large-scale studies, to examine students' goal orientations from a normative view (di Martino & Zan, 2010). As a complementary yet alternative methodological approach, this study invited students to express their aspirations in their own words. It used an open-response survey to enable inductive and interpretive analysis for investigating different facets of goals and motivation from students' perspectives, and with no use of a priori constructs to influence their responses. The study's purpose was not to infer causal relationships but to understand more about what students choose to focus on when articulating their aspirations and how these might relate to different goal orientations, motivational issues, or experiences in mathematics learning. The students were asked: *If you had one wish for your mathematics learning, what would it be?* Although seeking qualitative data, the survey generated responses from 3562 middle school students (93% response rate for this item within a larger survey as part of the Encouraging Persistence Maintaining Challenge (EPMC) project funded by the Australian Research Council). Responses ranged from a few words to long paragraphs.

The study employed inductive, researcher-driven (Corbin & Strauss, 2008) line-by-line coding and interpretive analysis, rather than automated software procedures, despite the large data set. The use of NVivo 10 supported this process and enabled cyclical comparisons of coding frequencies and adjustments to categories throughout the process to improve intra- and inter-coding reliability (Miles & Huberman, 1994). The program documented the process by forming an audit trail of the coding undertaken (author and research assistant).

DISCUSSION AND IMPLICATIONS

The following discussion focuses on the nature of the aspirations that girls and boys described spontaneously, and also on evidence of different goal orientations in their use of language. Table 1 presents the coded categories of the students' responses, the percentage frequencies, and comparative ratios.

CATEGORISATION OF RESPONSE	Percentage of students (n=3562)	Percentage of boys (n=1610)	Percentage of girls (n=1952)	Ratio boys/girls	Ratio girls/boys
About learning or achievement	60.22	55.53	64.09	0.87	1.15
About understanding: being able to understand, knowing or having knowledge	11.99	10.62	13.11	0.81	1.23
About performing: marks / grades / standard, good / smart at maths	10.44	11.61	9.48	1.23	0.82
About improving: becoming smarter or better (in general)	7.61	7.20	7.94	0.91	1.10
About fluency: being able to learn or answer quicker or more easily or efficiently	7.19	5.28	8.76	0.60	1.66
About more challenge: being challenged, doing harder work, being in higher class	5.98	6.58	5.48	1.20	0.83
About appropriate learning: at my level, more or new or useful things, choice	4.46	3.91	4.92	0.80	1.26
About wanting to learn multiple strategies, ways to solve, how others solve	4.24	2.61	5.58	0.47	2.14
About retaining: not forgetting, remembering, memorising, off by heart, revising	4.18	2.61	5.48	0.48	2.10
About thinking: using mental faculty well, improving ability to think	2.27	2.30	2.25	1.02	0.98
In comparison with others: being better than, the best	1.18	2.24	0.31	7.27	0.14
About explaining my understanding, showing what I know	0.67	0.56	0.77	0.73	1.37
Type of task	43.46	40.37	45.95	0.88	1.14
Better at a specific topic or concept	(37.00)	(33.29)	(40.01)	0.83	1.20
Fractions / Decimals / Percentage	14.15	12.98	15.11	0.86	1.16
Times tables	8.25	6.71	9.53	0.70	1.42
Division	5.70	5.90	5.53	1.07	0.94
Algebra or equations	2.86	2.55	3.07	1.21	0.83
Other (operation, measurement, geometry etc.)	6.04	5.16	6.76	0.76	1.31
More creative / visual / hands-on tasks, games	2.47	2.67	2.31	1.16	0.86
More technologies	1.85	2.48	1.33	1.87	0.54
About set exercises / homework	1.46	1.61	1.33	1.21	0.82
About problem solving tasks	0.42	0.25	0.56	0.44	2.27
Less technologies	0.25	0.06	0.41	0.15	6.60
About affect or motivation	12.61	10.81	14.09	0.77	1.30
Emotional response: feelings, engagement, interest, confidence	4.35	2.42	5.94	0.41	2.45
Wanting to make more effort – working / studying harder, not giving up	4.18	3.42	4.82	0.71	1.41
More enjoyment, fun	2.50	2.73	2.31	1.19	0.84
Wanting to make less effort – <i>not</i> working harder, doing easier work	0.81	1.37	0.36	3.81	0.26
Not study mathematics at all	0.28	0.43	0.15	2.83	0.35
Working arrangement	9.74	7.95	11.22	0.71	1.41
More work with others, pairs	6.43	5.47	7.22	0.76	1.32
More individual work	1.43	0.68	2.05	0.33	3.00
With others on same level	0.79	0.81	0.77	1.05	0.95
Need quiet or spacious environment	0.56	0.56	0.56	0.99	1.01
Needs to be at my own pace	0.53	0.43	0.61	0.71	1.41
About being taught	8.34	5.71	10.50	0.54	1.84
Teachers explaining more or in depth or clearly	2.27	1.49	2.92	0.51	1.96
About being helped in general	1.46	0.75	2.05	0.36	2.75
Teachers using particular or different approach or teaching strategy	1.15	1.06	1.23	0.86	1.16
Being given more time in class, on tests, for revision, more lessons	0.67	0.50	0.82	0.61	1.65
About receiving one-on-one or small-group help	0.65	0.12	1.08	0.12	8.66
Teachers giving harder or more work or more strategies	0.51	0.31	0.67	0.47	2.14
Teachers giving encouragement, empathy	0.34	0.31	0.36	0.87	1.15
About receiving topic-specific help	0.31	0.06	0.51	0.12	8.25
Teachers giving easier work	0.20	0.25	0.15	1.62	0.62
Teachers making maths fun or interesting	0.14	0.25	0.05	4.85	0.21
Other	0.65	0.62	0.67	0.93	1.07
Utility for the future: higher year level, career, scholarship	1.38	1.61	1.18	1.37	0.73
Nothing to wish for – everything is fine	0.31	0.19	0.41	0.45	2.20
Unclear / irrelevant response	1.49	2.24	0.87	2.57	0.39

Table 1: Categories with percentage frequencies for students overall, boys, and girls, and ratios

The students' responses demonstrate a wide range of aspirations, and there are responses from boys and girls coded in every category. Most are about learning or achievement (60% overall – 56% of the boys and 64% of the girls). The next most frequent type of aspiration is about features of tasks, with most referring to a specific mathematics topic or concept. *Fractions, decimals, and percentage*, and *times tables* were both key areas. The next three most frequent types of aspirations are about affect or motivation, working arrangements in lessons, and being taught (explicit reference). Across all of these five categories, a slightly higher proportion of girls made responses, suggesting that they were more likely to have made survey responses that required coding in more than one category.

Figures 2 and 3 present the five most frequent categories for boys and girls alongside comparative percentages for the other gender. It can be seen that two categories relate to specific topics and the other three to learning and/or achievement.

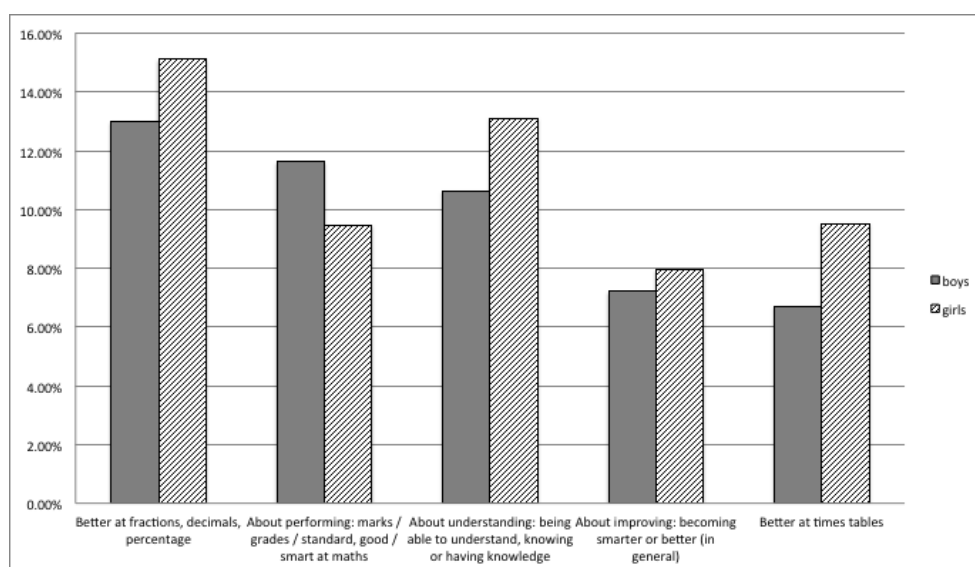


Figure 2: Five most frequent categories for boys

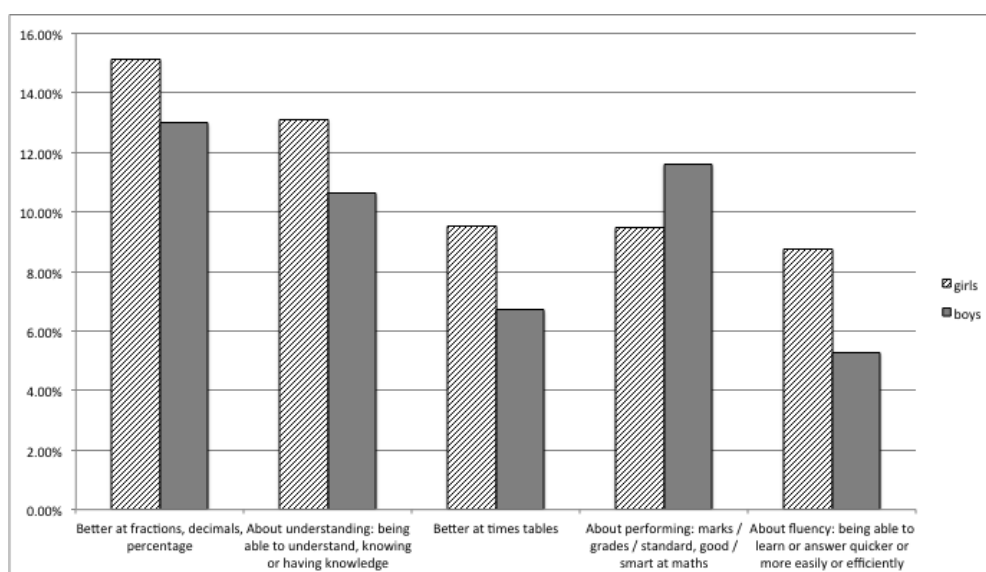


Figure 3: Five most frequent categories for girls

Four out of the five categories are common to both genders yet are in a different order. The girls' fifth category – *about fluency* – does not appear in the boys' list. The boys' fourth category – *about improving (in general)* – does not appear in the girls' list, even though relatively more girls were coded in this category. The two previously mentioned topic-specific categories appear in both lists, suggesting that these areas of mathematics are of concern to both boys and girls.

The framework of different types of goal orientations (Figure 1) was used to look for evidence of these in the five most frequent categories. Wanting to improve in a particular concept or skill was the focus of two categories for both genders and is suggestive of mastery-approach goals. For the girls, the most frequent non-topic-specific category was *about understanding: being able to understand, knowing, or having knowledge*. This is also suggestive of a mastery-approach goal because of the focus on “understanding the content as thoroughly as possible.” The second most frequent (non-topic-specific) category for the girls was *about performing: marks or grades or standard, good or smart at maths*. This could relate to mastery or performance goals, depending on whether or not an individual seeks validation through performing well that they have mastered a task and acquired new skills (mastery) or that they have *demonstrated* competence and *appear* to have ability (performance). Hulleman et al. (2010) emphasised the need to distinguish between these reasons for wanting to perform well; even though on the surface such language about grades and results might look like performance goals, it is important to look explicitly for the desire to be *compared* favourably against *other people*. The girls' third most frequent (non-topic-specific) category was *about fluency: being able to learn or answer quicker or more easily or efficiently*. It is unclear as to whether they wanted fluency for improved learning (mastery-approach), or for not wanting to appear incompetent or dumb to others by being slow to understand or answer (performance-avoidance).

In the boys' list of five most frequent categories, *about performing* was more frequent than *about understanding* – the reverse of the girls' list. Their third most frequent learning category was *about improving: becoming smarter or better (in general)*. It does not appear on the girls' list. It was unclear from the students' responses whether or not the reason for wanting to improve related to a comparison with other people and therefore cannot be used as evidence for one particular goal type.

Those categories where students' spontaneously generated language evidenced a clearer link to a particular type of goal are presented in Table 5 along with the percentage frequencies of boys and girls. It can be seen that many more categories evidenced mastery-approach goals than any other type. Within the mastery goal types, it appears that interest and curiosity, and challenge were key foci for the boys' descriptions of their aspirations, whereas learning different approaches and retaining knowledge were key foci for the girls. Yet both genders spontaneously generated responses that were coded in every category.

<p>Mastery-approach goal (% boys, % girls)</p> <p><i>Interest and curiosity:</i></p> <ul style="list-style-type: none"> • About appropriate learning: at my level, more or new or useful things, choice (3.91, 4.92) • More creative / visual / hands-on tasks, games (2.61, 2.31) • More enjoyment, fun (2.73, 2.31) • Teachers making maths fun or interesting (0.25, 0.05) <p><i>Task / Challenge:</i></p> <ul style="list-style-type: none"> • About wanting to learn multiple strategies, ways to solve, how others solve (2.61, 5.58) • About more challenge: being challenged, doing harder work, being in higher class (6.58, 5.48) • Teachers giving harder or more work or more strategies (0.31, 0.67) <p><i>Improvement or attainment:</i></p> <ul style="list-style-type: none"> • About understanding: being able to understand, knowing or having knowledge (10.62, 13.11) • About thinking: using mental faculty well, improving ability to think (2.30, 2.25) • Better at a specific topic or concept (33.29, 40.01) 	<p>Performance-approach goal</p> <ul style="list-style-type: none"> • <i>Normative:</i> In comparison with others: being better than, the best (2.24, 0.31) • <i>Appearance:</i> About explaining my understanding, showing what I know (0.56, 0.77)
<p>Mastery-avoidance goal</p> <ul style="list-style-type: none"> • <i>Improvement or attainment:</i> About retaining: not forgetting, remembering, memorising, off by heart, revising (2.61, 5.48) 	<p>Performance-avoidance goal</p> <ul style="list-style-type: none"> • <i>Appearance or evaluative:</i> Emotional response – explicit reference to feeling embarrassed, left out, left behind, less smart than others (0.19, 0.72)

Table 2: Codes evidencing particular goals with % boys and % girls

The category that provided evidence of mastery-avoidance goals was *about retaining: not forgetting, remembering, memorising, off by heart, revising*. Just over 4% of students made such a response; the girls' responses were more than twice as frequent as the boys'. Although this type of orientation has only recently been empirically validated (e.g., Elliot & Muryama, 2008) this study provides some evidence that more girls than boys may hold this type of goal.

There is more to be analysed, and other frameworks for analysing the large data set of middle school students' own responses to being asked their wish for mathematics learning. Perhaps a conclusion that can be drawn from the work to date is that boys and girls both express a wide range of aspirations, which are overwhelmingly positive and often focussed on mastery-approach goals. Their language is often quite specific, suggesting that these students do know what they desire for their learning. Rather than generalising about what matters to middle school students, teachers might do well to view their classes as comprised of *individuals*, to seek information about their specific aspirations, and find ways to incorporate students' own suggestions for promoting their engagement, learning, and achievement.

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PROSPECTIVE ELEMENTARY TEACHERS' TALK DURING COLLABORATIVE PROBLEM SOLVING

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This paper examines prospective elementary teachers' qualities of talk during collaborative problem solving in mathematics. Data were collected throughout one semester. The 16 participants who attended a problem solving class, worked in four groups of four members each, with non-routine mathematical problems which could be solved by alternative approaches. Their discussions during collaboration were audio-recorded and later transcribed. A discourse analysis revealed ten tentative qualities of talk, common to all four groups. Some ideas for further analyses and future work built on our tentative framework are presented at the end of the paper.

INTRODUCTION

In recent years, extensive research interest has been expressed on collaborative problem-solving (CPS) in mathematics (Hurme and Järvelä, 2005; Greiff, 2012; Mercer and Sams, 2006). It is, therefore, not surprising that the OECD (2013) has set CPS as a high priority for PISA 2015 by proposing an analytical framework for assessing pupils' skills and competence in collaborative environments. Furthermore, colleagues working in the field of mathematics teacher education have looked at student teachers' heuristic strategies (Bjuland, 2007), social and socio-mathematical norms (Tatsis and Koleza, 2008), and beliefs (Xenofontos, 2014, 2015) related to CPS. While introducing prospective teachers to environments that support the idea that quality teaching talk is essential for learning and communication (Kyriakou, 2016), little is known about prospective teachers' qualities of talk in such environments. Along these lines, this paper presents and discusses some preliminary findings from an ongoing project that investigates potential strategies that might enhance talk during CPS. In particular, we explore both students' and teachers' qualities of talk while working collaboratively on solving non-routine mathematical problems.

THEORETICAL CONSIDERATIONS

Research has established that the verbalization of mathematical ideas and thinking improves mathematical understanding (Bills and Grey, 2001; Carpenter et al. 2003; Pirie and Schwarzenberger, 1988; Smith, 2010). As educators, it is crucial to provide learners with opportunities to talk about mathematics during classes, across all levels of education. However, while raising quality classroom talk has been, for many decades, a target for many educational systems around the world, there is no consistent evidence indicating it has been succeeded and, if so, how that might be so (Kyriakou, 2016).

This study is built on the Vygotskian premises of social constructivism. According to this perspective, knowledge is constructed through social interaction, while higher mental functions are developed through interactions either with adults or more capable peers (Vygotsky, 1978). For Vygotsky, the use of language as externalized thought acts both at the social (intermental) and self-directing (intramental) level, eventually remaining within the mind as inner speech. The view of language as externalized thought underlines the link between thinking and talking, which mutually act upon learning (Smith, 2010). Improving understanding through managing classroom talk can provide more insight into thinking in the classroom.

Based on cross-cultural data, Alexander (2008, p. 30) identifies five types of classroom talk:

- **Rote** (teacher-class): the drilling of facts, ideas and routines through constant repetition
- **Recitation** (teacher-class or teacher-group): the accumulation of knowledge and understanding through questions to stimulate recall or to cue pupils to work out the answer from clues provided in the question
- **Instruction/ exposition** (teacher-class, teacher-group or teacher-individual): telling the pupil what to do, and/or imparting information and/or explaining facts, principles or procedures
- **Discussion** (teacher-class, teacher-group or pupil-pupil): the exchange of ideas in view of sharing information and solving problems
- **Dialogue** (teacher-class, teacher-group, teacher-individual, or pupil-pupil): achieving common understanding through structured, cumulative questioning and discussion which guide and prompt, reduce choices, minimize risk and error, and expedite the ‘handover’ of concepts and principles

According to Alexander, discussion and dialogue are met less frequently within primary classrooms while the first three types constitute the basic oral teaching repertoire. More recently, in their systematic review of studies from 1972 onwards, Howe and Abedin (2013) conclude that the situation remains static for over 40 years, as classroom talk has not yet refrained from traditional patterns of talking where the teacher is the one making the questions with a focus on short and predictable answers by a single pupil. Of course, no lesson can be characterized by a single type of talk, as the boundaries among types of talk are permeable (Teo, 2013). Discussion and dialogue have their merit within a larger oral repertoire that might as well include rote, recitation and exposition (Alexander, 2008). Yet, research needs to find ways of bringing these two types of talk to the fore.

PARTICIPANTS, DATA COLLECTION AND ANALYSIS

The participants of this study were 16 undergraduate students (11 female, 5 male), reading for a degree in primary education with a qualified teacher status. Eight of the students were Greek-Cypriots, six were from Greece, while two of them were half Greek and Greek-Cypriot. The language of instruction in the Republic of Cyprus (at public schools, and of the undergraduate programme the students were attending) is

Standard Modern Greek (SMG), and in sociolinguistic terms, Greek-Cypriots can be labelled as bidialectal (Yiakoumetti and Esch, 2010), since they speak two variations of the same language (SMG and the Greek-Cypriot dialect).

All participants attended a class on Problem Solving in Primary Mathematics, taught by the first author. The class lasted 12 weeks and the students and instructor met once a week for three hours. All lessons included practical workshop elements during which students worked on solving mathematical problems as learners. For about half of the classes the lessons were designed to include studying issues from the mathematical problem-solving literature (i.e. heuristic strategies, affective factors and problem solving, problem solving and mathematics teaching), while the rest were entirely practical. During the latter part, students spent the whole class time working in small groups of four, solving non-routine mathematical problems in order to promote Alexander's (2008) last two types of talk, discussion and dialogue. Each student was randomly assigned to a group at the first meeting. The groups did not change throughout the semester, while the instructor's input was kept at a minimal. At various points, the instructor visited each group to observe its progress, ask questions to clarify ideas and provide guidance where necessary. At other times, during and at the end of each class, students were invited to a whole-class discussion, so that each group shared some of their ideas and approaches with their peers. Each group's talk during these practical classes was audio recorded and later transcribed.

Below are presented three of the problems given to the groups. Problem 1 was taken from www.nrichmaths.org, while problems 2 and 3 were given to the first author by Prof. Paul Andrews (Stockholm University, Sweden) and are presented in Xenofontos (2015). In fact, problem 3 is a slightly adapted version of a problem from TIMSS video study. The problems were carefully chosen so that they could be solved by several alternative strategies (Borasi, 1986), in order to enable students to engage in discussion and dialogue.

Problem 1 – The cards problem

I have fifteen cards numbered 1– 15. I put down seven of them in a row on the table.



The numbers on the first two cards add to 15.

The numbers on the second and third cards add to 20.

The numbers on the third and fourth cards add to 23.

The numbers on the fourth and fifth cards add to 16.

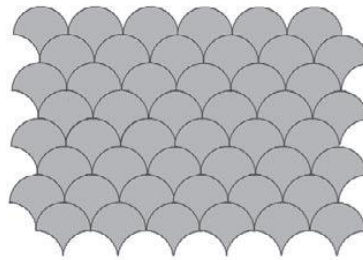
The numbers on the fifth and sixth cards add to 18.

The numbers on the sixth and seventh cards add to 21.

What are my cards? Can you find any other solutions? How do you know you've found *all* the different solutions?

Problem 2 – The fish scales problem

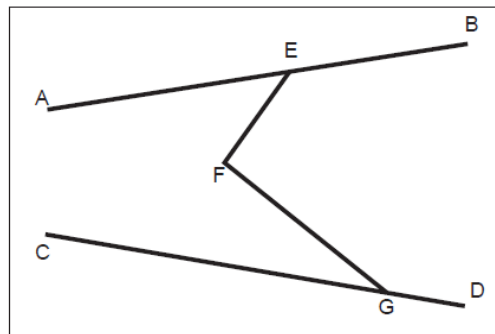
The figure below shows some of the scales of a fish found in the North Atlantic Ocean. Each scale comprises three arcs of a radius of 2mm.



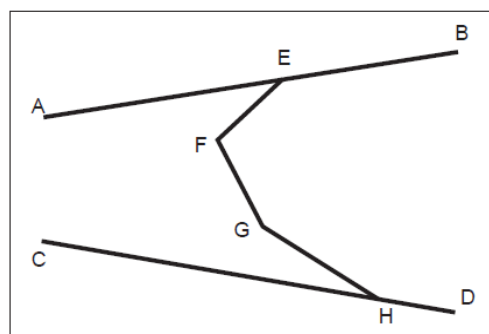
- 1 Calculate the area of one of the scales.
- 2 Repeat the calculation for a radius of 3mm.
- 3 Repeat the calculation for a radius of 4mm.
- 4 What do you notice about the results?
- 5 Find a solution process that explains the above.

Problem 3 - The farmers' field boundary problem

The illustration below shows the boundary, EFG, between two fields. Each field is owned by each of the two farmers and both agree that their lives would have been easier if the boundary were straight. Where might we draw a straight boundary, in order to preserve the areas of both fields?



What if the boundary was as shown below? Where would the straight boundary be now?



What general geometrical principle is outlined in this problem?

The data were approached from the perspective of discourse analysis, that is, an investigation of the purposes for which language is used (Brown and Yule, 1983). By exploiting the constant comparison process outlined by Strauss and Corbin (1998), as well as the ideas of coding and categorization (Miles & Huberman, 1994), the two authors worked individually, trying to identify different qualities of students' talk. Later, the authors brought their individual works together, had discussions on identified qualities of talk, and reached an agreement on ten tentative themes. Readers should be reminded that this is a work in progress and that these qualities are subject to revision and alteration.

RESULTS

Table 1 presents the ten identified qualities of talk of our tentative framework. It should be noted that these qualities are not mutually exclusive, as two or more qualities could be linked to a particular part of students' talk.

Quality of talk	Description
<i>Brainstorming</i>	Students throw in ideas on how to begin or proceed with a solution process
<i>Explanations</i>	Students engage with peers by elaborating on concepts when they are not clear to other group members, and/or provide explanations on the method followed
<i>Realization of errors</i>	Students spot mistakes in calculations or realize that a particular method they followed does not work for them
<i>"Aha!" moments</i>	Students have insightful moments and bring in an idea that proves to be effective
<i>Complementarity</i>	Students build on and complement peers' knowledge and ideas
<i>Decision taking</i>	Students reach an agreement on how to proceed
<i>Confusion – blackout</i>	Some or all group members experience confusion/blackout and express it explicitly
<i>Pauses – moments of silence</i>	Some group members remain silent for a variety of reasons (i.e. to think, make mental or written calculations, or experience confusion)
<i>Disagreements – contradictory ideas</i>	Students explicitly express disagreements with peers and provide contradictory ideas
<i>External input</i>	Students engage in input coming from outside the group, i.e. the instructor (through direct guidance or questioning) or members of other groups, during whole-class discussion.

Table 1: The ten tentative qualities of talk

Below are examples of these qualities of talk along with supporting extracts from the transcripts. These may come from different groups, while the student number indicates that a different person is speaking each time. In this short paper, examples of all qualities cannot be presented.

Brainstorming and decision taking

- Student 1: Hmm, do you think 16, 18, 21 have something to do with this?
- Student 2: You mean if it's a product or something?
- Student 1: Or maybe with the same numbers.
- Student 3: I have a thought, let's, erm, let's say the first card is a, the second is b, then...
- Student 1: Give them a name.
- All together: Give them a name.
- Student 4: And then, maybe, find relations between them?
- Student 3: Yes, say $a+b = 15$
- Student 4: Yeah, I see what you mean.
- Student 3: Then, $b+c = 20$
- Student 4: Yeah, ok
- Student 2: Let's find the equations then, and see their relations.
- Student 4: Yeap. Let's do that!

Disagreement

- Student 1: But if we don't try it, how will we know that it doesn't work?
- Student 2: We don't have much information, it doesn't advise, "If you put this with this you get this". Are you saying we should try some random values?
- Student 1: Are we going to spend more time disagreeing on this? If we don't try it, we won't know. [note: the student starts writing something]
- Student 2: What are you writing there?
- Student 1: I'm trying this idea. I'll get back to the previous one later.
- Student 2: Pff, you'll get back to nothing. Absolutely nothing.

CONCLUDING REMARKS

As already indicated, this paper is work in progress and the ten qualities of talk are tentative. In later analyses, we will be interested in examining a number of issues. For example, we'd like to see whether there are specific patterns in the ways each quality is associated with others, especially within discussions and dialogues. Also, considering that Greek-Cypriot students are bidialectal while Greek students are not, we'd like to examine whether there are significant differences in how they engage in

mathematical talk. Another interesting idea is to use the emerged framework to analyse mathematical talks in other age groups, like, primary school pupils, or to investigate whether these same qualities appear in mathematical talks across countries. Fostering classroom dialogue is important at all levels of education, and our work intends to contribute to the identification of strategies that lead towards dialogic teaching and learning.

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TEACHERS' BELIEFS TOWARDS THE VARIOUS REPRESENTATIONS IN MATHEMATICS INSTRUCTION

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The aim of this paper is to investigate in-service teachers' beliefs towards the use of manipulative models, realistic pictures, abstract pictures, word problem tasks and the symbolic language in mathematics instruction. The focus is on the formation of new concepts in algebra. The results indicated that neither the cognitive development of students, nor the abstractness of the content are sufficiently recognized as an important criterion when choosing representations.

Mathematics teaching theories are based on the consensus that mathematical ideas are communicated through different representations (manipulatives, pictures, diagrams, narratives, symbols), which are interiorized in the learning process (Ainsworth, 2006; Dreyfus 1991; Goldin, 2014; Presmeg, 1997; Terwel et al., 2009). The importance of multiple representations and the method of their use are emphasized by Dreyfus (1991). Dreyfus defined the phases of learning (considering the use of representations) as the use of one representation in the first phase and a flexible use of multiple representations in the last. In that process, the hierarchical relationship and gradual nature of representation development from concrete to abstract have an important role (Goldstone & Son, 2005; Hiebert & Carpenter, 1992; Sfard, 2000; Smith, 2006). Previous research (Brizuela & Schliemann, 2004; Carraher et al., 2007; Kieran, 1996; Radford, 2000; Stephens, 2003) showed a tendency toward the use of the symbolic language as dominant and often the only representation, especially in algebra. In this paper, we are dealing with the character of pedagogical representations and their development in the first cycle of mathematics education in Serbia, considering different topics of school mathematics, with the emphasis on algebraic representations.

THEORETICAL FRAMEWORK

As a starting point, we will take the theoretical view in which the use of symbolic language is considered as an abstract representation, and the use of physical objects (e.g. manipulative models) or pictures (e.g. diagrams) and/or the conceptualization of abstract ideas in real situations (for example through word problems) is considered as a concrete representation. Recent studies (e.g. Cai, 2004; Koedinger et al., 2008) showed that abstract representations are more efficient than the concrete ones in the process of solving complex problems. "Expressive" and communicative representations assume pointing to what is important, and they are a predecessor to more abstract representations (Terwel et al. 2009). On the other hand, Goldstone and Son (2005) emphasized that learning of simple mathematical principles in an abstract context could be inefficient, because that way pupils could obtain only ready-made knowledge. The answer to the question of the best level of representation abstractness

in the formation of a new concept could be found in the usage of multiple representations (Mason et al., 2007).

There is no doubt that teachers' beliefs of what mathematics is are affecting their choice of representation when introducing mathematical concepts (Huang & Cai, 2007; Philipp, 2007). The choice of an appropriate representation is an important decision for which the teacher should consider at least two perspectives: 1) the nature of the mathematical content which should be learned and 2) the developmental characteristics of students, i.e. the mind of the students who learn the content (Ball, 1993). A pedagogical representation should emphasize the important properties of the mathematical matter that teacher wants to teach and to provide a known and accessible context for students in which they could expand and develop their capacities for reasoning and understanding the ideas (Huang & Cai, 2007).

In this paper we will highlight the algebraic representations because numerous authors consider the use of various representations as an important component of algebraic thinking (eg. Kieran, 1996). We will consider linear equations and inequalities as the representatives of algebraic ideas in the first cycle of schooling. Voluminous research deals with the understanding of the structure of equations (Macgregor & Stacey, 1997; Stephens, 2003). Panasuk (2011) considers that an important indicator of the conceptual understanding of a linear equation with one unknown is the pupils' ability to identify the same relations presented in different representations and to flexibly transform one representation into another. Because of the abstractness and difficulties in understanding the symbolical forms of equations, a number of authors (e. g. Radford, 2000) proposed introducing some sort of a "transitional language" before the standard algebraic notation, e.g. $32 + \square + \square = 54$. Similarly, Kieran (2004) emphasizes that the current approach to teaching inequalities does not consider the development of meaning. The results of research that Verikios and Farmaki (2008) conducted showed that the use of different representations (graphs, tables, word problems, symbols) when introducing inequalities helped students to assign meaning to symbols and understand the procedure of solving inequalities.

The overall goal of our research was to identify types of representations that teachers use in different topics of mathematics, especially in early algebra. Hence we focused on several research questions: 1) Does the level of abstraction of representations that teachers use differ at the beginning and at the end of the first cycle of schooling (1st and 4th grade)? 2) Are there any preferred topics (Arithmetic, Algebra, Geometry, Measurement) when teachers use a particular representation? 3) Are teachers' beliefs regarding the preferred representation implemented in examples that teachers use when introducing new algebraic concepts?

METHODOLOGY

In-service teachers voluntarily answered a questionnaire during the Teachers' Gathering in Belgrade, Jun, 13th 2015. A hundred and three in-service teachers participated, 55 of them were teachers from urban and 48 from rural schools in Serbia.

The questionnaire consisted of ten tasks. Eight tasks were sets of five Likert items, while the 9th and 10th tasks had an open ended form. All the tasks referred to representations used in the first or fourth grade when teaching new concepts in one of the four major topics in primary mathematics. Hence, we are representing three groups of tasks: the first one refers to the first grade (tasks 1, 3, 5 and 7), the second one refers to the fourth grade (tasks 2, 4, 6 and 8), and the third group consists of the open ended tasks (tasks 9 and 10). The first group of tasks is presented in Table 2, the second group is analogue to the first one. Each of the Likert items presented in Table 2 consists of 5 points ranging from 1- strongly disagree to 5- strongly agree. For example, Task No.3 reads: “When introducing a new concept in algebra in the first grade, I use: A. Manipulative models, B. Realistic pictures, C. Abstract pictures (e.g. diagrams, schemes), D. Word problem tasks, E. Symbolic language” (Table 2). Task No.9 reads: “Write a typical example for introducing equations with an unknown addend in the first grade”, and task No.10: “Write a typical example for introducing inequalities with an unknown addend in the fourth grade”.

Task No.	Topics	Representations
1	Arithmetic	Manipulative models
3	Algebra	Realistic pictures
5	Geometry	Abstract pictures (e.g. diagrams, schemes)
7	Measurement	Word problem tasks Symbolic language

Table 2: Summarized Likert items in the questionnaire referring to the first grade.

RESULTS AND DISCUSSION

The obtained data are analyzed with Cronbach’s alpha for internal consistency, Median and Interquartile Range are used as measures of central tendency, and Wilcoxon signed rank test was used to evaluate the differences in teachers’ opinions. For addressing the first research question, we have summarized the data regarding the use of various representations in the first and fourth grades (see Table 3). The results showed that teachers expressed their attitude towards the use of diverse representations at the beginning and at the end of the first cycle of schooling (see Med and IQR values in Table 3). Hence, teachers consider that conceptual understanding is achieved through the use of multiple representations as proposed by other authors (Goldstone & Son, 2005; Mason et al., 2007). Their opinion about abstract pictures in the first grade is polarized (Mdn=3, IQR=2, Table 3).

Teachers seem to agree that symbolic language should be used more in the fourth than in the first grade (Z5, p5, Table 3). We assume that teachers see the use of diagrams and word tasks as more abstract representations, and so they expressed their belief that they should rather use them in the fourth than in the first grade (Z4, p4; Z3, p3, Table 3). It is interesting that teachers don’t make distinctions between using manipulative models and realistic pictures at the beginning and at the end of the first cycle (Z1, p1;

Z2, p2, Table 3). Hence, we can't conclude that the teachers' beliefs completely go along with the attitude of various authors that the increase of representation abstractness should accompany the increment in the difficulty of mathematical problems (Cai, 2004; Koedinger et al., 2008) and developmental characteristics (Ball, 1993).

Representation	First grade			Fourth grade			Wilc. sign rank	
	Mdn	IQR	C. Al.	Mdn	IQR	C. Al.	Z	p
Man. models	4.0	1.5	0.78	4.0	1.5	0.80	$Z_1=-0.68^a$	$p_1=0.493$
Real. pictures	4.0	0.5	0.70	4.0	1.0	0.75	$Z_2=-0.47^a$	$p_2=0.637$
Diagrams	3.0	2.0	0.84	4.0	1.0	0.80	$Z_3=-6.81^b$	$p_3=0.000$
Word problem	4.0	0.5	0.71	5.0	1.0	0.74	$Z_4=-4.82^b$	$p_4=0.000$
Symbolic	4.0	1.0	0.76	4.5	1.0	0.78	$Z_5=-5.51^b$	$p_5=0.000$

Table 3: Cronbach's alpha, Median and Inter Quartile Range values for each representation. Wilcoxon sign rank test (a-based on positive, b-based on negative ranks) performed on data obtained in 1st and 4th grade (4th -1st)

To answer the question which representations teachers preferably use when teaching different topics of mathematics we used Friedman test and Wilcoxon signed rank test for the post hoc analysis (with Bonferroni correction, $p < 0.008$). The Friedman test showed that there is a preferred topic when using all but symbolical language and abstract pictures in the 4th grade (see Sig. values less than 0.05 in Table 4). As the beliefs about the symbolical language do not vary through the topics, we can say that teachers' beliefs go along with the previous findings (Cai, 2004; Koedinger et al., 2008) that abstract representations are significant for the development of mathematical ideas in all topics. This is not surprising since the use of symbolical notation is present to a significant extent in the Serbian mathematics curriculum. Variable as the unknown is introduced in the first grade, and by the end of the fourth grade, the structure of the natural number system is introduced including the generalization and symbolical notation of arithmetic rules.

	Man. models		Real. pictures		Ab. pictures		Word context		Symbolic	
Grade, n	1 st ,94	4 th ,95	1 st ,98	4 th ,94	1 st ,95	4 th ,97	1 st ,96	4 th ,94	1 st ,99	4 th ,97
$\chi^2(3,n)$	46.80	51.79	15.04	19.78	17.54	5.38	13.12	11.71	4.95	6.59
Sig.	.000	.000	.002	.000	.001	.146	.004	.008	.175	.086

Table 4: Results of Friedman test for each representation

We will report here only on the most interesting results of the post hoc analysis. Teachers expressed that they prefer to use manipulative models when introducing concepts in geometry and measurement rather than in arithmetic (see val. Z1, p1; Z2, p2; Z4, p4, Z5, p5, Table 5). This is not surprising since the most natural means of learning lie in initial geometry and measurement models and their pictures. But, they

did not express a significant difference in the opinion towards using manipulative models (Z_3 , p_3 ; Z_6 , p_6 , Table 5) and realistic pictures ($Z_{rp1}=-1.10$, $prp1=.271$; $Z_{rp4}=-2.14$, $prp4=.032$) in arithmetic and in algebra, both in the 1st and 4th grades. This means that they do not consider that the increment of the level of abstractness in the transition from arithmetic to algebra should cause a reduction in manipulative models and realistic pictures use.

Topic	First grade			Fourth grade		
	Geo/Ar	Meas/Ar	Alg/Ar	Geo/Ar	Meas/Ar	Alg/Ar
Man. models	$Z_1=-3.37$, $p_1=.001$	$Z_2=-4.94$ $p_2=.000$	$Z_3=-0.07$ $p_3=.945$	$Z_4=-3.18$ $p_4=.001$	$Z_5=-4.77$ $p_5=.000$	$Z_6=-1.51$ $p_6=.130$

Table 5: Results of post-hoc signed rank Wilcoxon test for use of manipulative models

We have especially analyzed the teachers' beliefs about representations in early algebra. Results showed that in the first grade teachers prefer to use realistic pictures (RP), word problems (W) and symbols (S) rather than manipulative models (M) (see RP-M, W-M, S-M values, the 1st and the 2nd row, Table 6), while all representations are more preferable than abstract pictures (AP) (AP-M, AP-RP, W-AP and S-AP values, the 1st and the 2nd row, Table 6). In the fourth grade, word problem tasks and the symbolic language are preferred (see the values in the 3rd and the 4th row in Table 6). Hence, symbolic representations and word tasks are the most preferred representations in initial algebra. This implies that there is a mismatch between the practice in Serbia and the previous research (Cai, 2004; Radford, 2000; Verikios & Farmaki, 2008) which showed that abstract representations are justified in complex problem solving, but concrete representations should be preferred when introducing concepts.

Rep.	RP-M	AP-M	W-M	S-M	AP-RP	W-RP	S-RP	W-AP	S-AP	S-W
1 st Z	-4.11 ^b	-3.54 ^a	-4.20 ^b	-3.34 ^b	-5.66 ^a	-1.52 ^b	-.45 ^b	-6.40 ^b	-6.05 ^b	-1.66 ^a
p	.000	.000	.000	.001	.000	.128	.656	.000	.000	.097
4 th Z	-4.11 ^b	-3.66 ^b	-6.57 ^b	-6.59 ^b	-.37 ^b	-5.15 ^b	-5.24 ^b	-6.19 ^b	-6.30 ^b	-.16 ^a
p	.000	.000	.000	.000	.709	.000	.000	.000	.000	.874

Table 6: Wilcoxon signed rank for analyzing the use of different representations in algebra. a – based on positive and b – based on negative ranks

Through tasks No. 9 and 10, we have analyzed how teachers' beliefs are implemented in teaching algebraic topics. Sixty two teachers (60%) provided the example for introducing an unknown addend in the 1st grade (task No.9). There was no example with manipulative models, which goes along with their beliefs described in the previous section. From the 62 reported examples, 2 teachers (3%) used pictures of realistic objects, 16 (26%) word problem tasks, 12 (19%) algebraic language and 17

(27%) transitional language. Multiple representations were used by 15 teachers (24%), from which 3 used more than two representations. Most of the multiple representations (5 of 15 i.e. 33%) were the use of word tasks and symbolical (algebraic and transitional) language. It is interesting that teachers expressed that they prefer realistic pictures in the first grade, while only 2 of them made such an example, while 5 more used this kind of pictures with symbols. A classic example in which teachers use realistic pictures is shown in Fig. 1 A. Only one teacher gave an example with the use of scheme (Fig.1B). The presented scheme is an appropriate mental image of the structure of equations that could be suitable for different word tasks, one of which is shown in the picture.

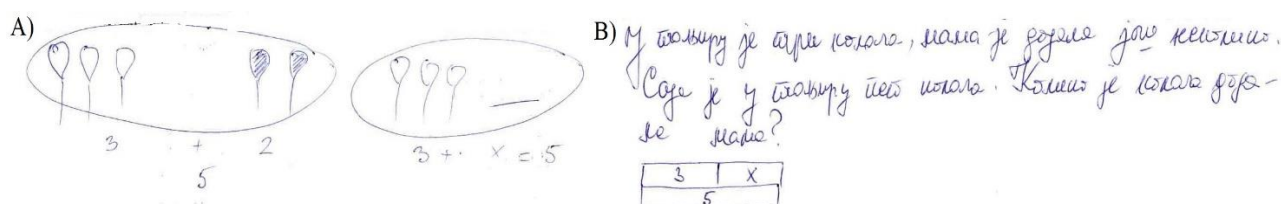


Fig. 1: Teachers' example of A) realistic pictures and B) abstract pictures

Still, in the largest number of examples, symbolic language is used – in 29 as the only representation and in 9 together with the word problem task. Teachers mostly try to cross the semantic complexity of algebraic forms of equations by introducing transitional language (number) (in the sense of Kieran, 1991; Radford, 2000).

Regarding the use of representations in the 4th grade, teachers expressed beliefs toward using symbolical language and word problem tasks in algebra, and in their example for introducing inequality (task No.10) they used exclusively these representations. No one gave an example of manipulative models, realistic pictures or schemes. The example is provided by 50 teachers (48.5%) of whom 14 (28%) used the word problem task, algebraic language 26 (52%), transitional language 3 (6%), while multiple representations was used by 7 teachers (14%), all of them word problem tasks with algebraic notation.

CONCLUSION

The results related to our first research question indicated that the grade in which teachers teach is not a criterion when choosing representations. There is not enough difference in their answers regarding the 1st and 4th grades. On the other hand, the second research question indicates that the abstractness of mathematical content is also not a significant criterion for choosing a representation. Algebraic concepts are the most abstract in the curriculum (they are introduced in the 1st grade in Serbia) and for their introduction teachers choose abstract representations without the attempt of reducing the level of their abstractness with the use of more concrete representations. Based on the examples that teachers created, it seems that teachers in Serbia still primarily use symbolic representations and word problem tasks (Stephens, 2003), while abstract pictures as schemes, diagrams, and the line segment model (Panasuk, 2011) are neglected. Teachers showed through their beliefs that they do not recognize

the importance of using abstract pictures as representations. Future research should answer the question whether teachers use some of these representations, without recognizing their designation and classification, or they do not recognize the importance and effect of their use as representations. In the former case, insufficient knowledge about the types of representations blocks communication and the exchange of ideas with colleagues and educators. In the latter case, if all systems of representations are not included in teaching, the result could be the formation of formal, semantically empty knowledge.

The use and creation of different systems of representations and their importance in forming of the mathematical knowledge should be an important part of teachers' education curriculum in Serbia, and their path of professional development. As researchers, teacher educators and professional developers we are generally not interested solely in the measuring of teachers' beliefs but also in changing them.

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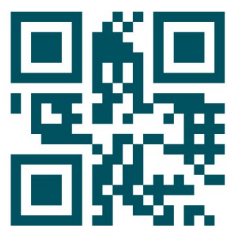
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