



MATHEMATICS  
EDUCATION  
*How to solve it?*



# PROCEEDINGS OF THE 40<sup>TH</sup> CONFERENCE OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

EDITORS: CSABA CSÍKOS • ATTILA RAUSCH • JUDIT SZITÁNYI

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Mathematics Education:  
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# ELEMENTARY CHILDREN WITH LEARNING DISABILITIES AND DIFFICULTIES: INITIAL FRACTION UNDERSTANDINGS

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*Little to no information exists explaining the nature of initial conceptions of fractions held by students with learning disabilities. The current study extends existing literature by presenting key indicators of understandings of fractional quantity of 44 children with learning disabilities and difficulties as evidenced through their problem solving strategy, observable operations, and language across six tasks based in the measure and partitive interpretations. Constant comparison analysis of the children's work across the tasks documents indicators reflective of a framework. Pending future research, the framework may be a useful tool to practitioners wishing to document students' initial conceptions of unit fractions.*

## THEORETICAL FRAMEWORK

Elementary children labelled as having learning disabilities or difficulties (MD) begin their study of fractions with similar yet more diminished conceptual understandings of fractions as quantities than what is documented among their peers (Hecht, Vagi, & Torgeson, 2007). An incomplete understanding of fraction concepts impacts children's ability to operate or apply computational procedures with fractions in higher-level mathematics (National Mathematics Advisory Panel, 2008). These findings suggest a continued instructional focus on the development of conceptual understanding of fractions as quantities for these children is critical.

A review of literature on fraction understandings in special education revealed deficit depictions of children with MD evident in assessment and instructional procedures (e.g., needs for explicit demonstration of efficient strategies within instruction). Such a depiction may stem from views of children with MD as possessing innate cognitive factors that render them unable to hold understanding of concepts (Davis, Cannistraci, Rogers, Gatenby, Fuchs, Anderson, et al., 2009) or perhaps hold qualitatively different understandings (Mazzocco & Devlin, 2008).

The deficit view of children with MD seems to yield a conceptualization of "understanding" for these children as their responsiveness to static, teacher-led instruction (e.g., Butler, Miller, Crehan, Babbitt, & Pierce, 2003; Test & Ellis, 2005). Yet, understanding cannot be imposed onto children (Baroody, Cibulskis, Lai, & Li, 2004). Instead, supporting children's conceptual growth involves *children* "[solving] problems within [their] reach [while] grappling with key mathematical ideas that are comprehensible but not yet well formed" (Hiebert & Grouws, 2007, pp. 387). If teachers wish to support children's growing understandings in instruction, then a characterization of the understandings children with MDs do hold of fraction concepts provides an indispensable foundation (Daro, Mosher, & Corcoran, 2011). This study

extends current literature by presenting key understandings of unit fractions evidenced by 44 children with MD in semi-structured clinical interviews. The research question was, “What initial understandings of fractions do children with MD evidence through employed problem solving strategies and observable operations?”

## METHODS

We completed 44 interviews with children in the second, third, fourth, and fifth grades who assented and whose parents provided informed consent listed in Table 1. Twenty-one of the 44 children have documented, cognitively defined learning disability. Twenty-three of the children were labelled "Tier 2", indicating documented, pervasive performance issues in mathematics not necessarily cognitive in origin.

	LD (%) N= 21	Tier 2 (%) N= 23
Grade		
2	5%	13%
3	24%	13%
4	48%	31%
5	23%	43%
Gender		
Male	76%	57%
Female	24%	43%
Ethnicity		
Caucasian	10%	9%
Black	14%	30%
Hispanic	71%	61%
Burmese	5%	0%
Learning Disability*		
Working Memory	29%	0%
Processing	10%	0%
LTM	5%	0%
Fluid Reasoning	5%	0%
Comorbid	51%	0%

Table 1: Characteristics of the children

## Data Sources

Data sources included a set of six main problem tasks for use in the study based on pilot work and a synthesis of prior research (Charles & Nason, 2000; Empson, Junk, Dominguez, & Turner, 2005; Pothier & Sawanda, 1983; Steffe & Olive, 2010; Streefland, 1993; Tzur, 1999, 2007). Problem tasks were based either the measure or partitive interpretation of fractions; all problems were situated in the context of equal sharing situations. The number of sharers ranged from two to eight and the number of objects shared ranged from one to nine.

Semi-structured clinical interviews (Ginsburg, 1997) were done with each student individually and were audio and video recorded. Children were encouraged to solve each task in a way that made sense to them – they could use the manipulative materials, paper and pencil, or no materials to aid them in reaching a solution. Tasks were administered to children until it was evident that (a) no new insights into how each child conceived of unit and non-unit fractions emerged or (b) the child could no longer provide a solution to the problems on his/her own or with prompting.

### **Data Analysis**

First, constant comparison method was used to delineate elements of children's conceptions of unit fractions (Strauss & Corbin, 1997; Leech & Onwuegbuzie, 2007). Three researchers reviewed the first four videotaped interviews as a team, inspecting each task. Researchers examined (a) the way in which the child solved the problem (nuances within evidenced strategies along with the nature of associated conceptions led us to ultimately refer to ways of solving problems by levels) and observable operations he or she employed. We then gave each element of children's thinking an initial code. Researchers also informally noted indicators of fraction understanding that began to emerge in the data (discussed further below). As more tasks and interviews were coded, we carefully compared each new chunk of data (i.e., each problem solution) with data coded previously and searched for confirming and disconfirming evidence to ensure consistency and validity (Leech & Onwuegbuzie, 2007). The iterative process of coding, comparing, and refining continued through three additional rounds of independent coding until all tasks in all interviews were coded.

Next, emergent coding (Grbich, 2012) was used to uncover indicators of understanding for each interview across the tasks as they were evident in children's problem solving and operations. Researchers identified major categories of indicators evidenced *across the tasks* within each interview by each child. The indicators were considered in terms of development reflected in how they emerged in the data. Indicators were then placed in a framework, which was used by two researchers to code all remaining interviews to establish each child's overall understandings (i.e., level).

Finally, data visualization techniques (Ward, Grinsteind, & Keim, 2010) were used to visually examine trends in reference to the indicators across all children interviewed. A heat map of coded indicators (lowest level –orange, highest level- yellow) for each child. The maps were analysed to examine which indicators seemed to lead development at various levels and further refine the framework.

### **RESULTS**

Results of the data analysis are summarized below. Our presentation will provide in depth examples of each level of problem solving and operations uncovered along with discussion of the indicators depicted in Figure 1.

Trajectory Level	Divisibility of the Whole	Partitioning Plan	Coordination of Equal Units Within the Whole
<b>No Fractions (Pre-Participatory)</b>	<b>No Fractions</b> Will only share/deal out wholes. Whole not yet conceived as divisible. Does not act on the whole or create fractions.		
<b>Emergent Sharer (Participatory, Level 1)</b>	<b>Developing</b> Seems to reluctantly cut into pieces.	<b>Early</b> Trial and Error based in whole number in activity. <ul style="list-style-type: none"> <li>Partitioning across wholes and/or leftovers is difficult.</li> <li>May begin to use "half" in activity, but it is not meaningful to them as a quantity.</li> </ul>	<b>Early</b> Child's attention is on making a number of parts. <ul style="list-style-type: none"> <li>Parts created are not equal in size, and the child is not bothered.</li> </ul>
<b>Halves (Participatory, Level 2)</b>	<b>Solidified</b> Readily divides whole without hesitation.	<b>Developing</b> Plan becomes evident in dealing with the leftover in activity. <ul style="list-style-type: none"> <li>"Half" represents a meaningful quantity that children may use to create pieces.</li> <li>May link number of pieces to number of sharers.</li> </ul>	<b>Developing</b> Begins to coordinate equal parts in the whole "after the fact" when dealing with leftover. <ul style="list-style-type: none"> <li>Child pays close attention to creating equal size pieces.</li> </ul>
<b>Relational Sharer 1 (Participatory, Level 3)</b>		<b>Developing</b> Plans to create number of parts equal to number of sharers prior to activity. <ul style="list-style-type: none"> <li>If partitive task, parts may be planned across wholes.</li> <li>May use knowledge of multiplication/division.</li> </ul>	<b>Developing</b> Creates equal parts while exhausting the whole. <ul style="list-style-type: none"> <li>May explain and justify the value of a created part as the same as all other parts needed to recreate one whole.</li> </ul>

Figure 1: Framework

Qualitative analysis resulted in four levels of problem solving (i.e., *No fractions*, *Emergent sharing*, *Half*, and *Emergent relations/coordination*). Five percent of children with MD and 0% of children in Tier 2 evidenced *No Fractions* while 30% of children with MD and 35% of children in Tier 2 evidenced *Half* as their dominant problem solving level. Thirty-five percent of children with MD and 30% of children in Tier 2 evidenced *Emergent Sharing* as their dominant problem solving level, while of children with MD and 35% of children in Tier 2 evidenced *Emergent Relation or Coordination*.

Three levels of partitioning operations (i.e., No regard to equal parts, Equal "halves", Equal parts) and Iteration (present or absent) were uncovered in the analysis. Ten percent of children with MD and 0% of children in Tier 2 evidenced No regard to equality as their dominant operation while 20% percent of children with MD and 0% of children in Tier 2 evidenced equal halves. Seventy percent of children with MD and 100% of children in Tier 2 evidenced equal parts as their dominant operation. Ten percent of children with MD and 55% of Tier 2 children used iteration at varying points across the tasks.

Five overall indicators of children's understanding of fractions emerged across the interviews. Indicators aligned with Piaget and colleagues (1960) account of children's fraction understandings and included (a) *whole as divisible*, (b) *partitioning plan*, (c) *relation between partitions and parts*, (d) *exhaustion of the whole*, and (e) *creation of*

*equal parts*. The indicators evidenced a framework in terms of children's fractional knowledge (see Figure 1). Forty percent of children with MD were coded as a Level One; 20% of Tier 2 children were coded as Level One. Fifty percent of children with MD were coded as a Level Two; 45% of Tier 2 children were coded as Level Two. Ten percent of children with MD were coded as a Level Three; 35% of Tier 2 children were coded as a Level Three.

Figure 2 illustrates the trend analysis in a heat map. As illustrated in the heat map, seven out of 15 children who were holistically coded as Level One fully conceived of the whole as divisible (42%), while the remaining eight children's conception of a divisible whole was developing. An understanding of the need for equality of the parts was coded as developing in all but one of the children. Eleven children in Level 1 showed an early understanding of exhausting the whole when sharing. An unprompted plan for partitioning going into modelled activity was coded as early or developing in all children. Thirteen children coded at Level 1 seemed to confuse partition lines with parts in their activity. Independent iteration was not used. From the evident trends, it appears KDUs leading development at Level One include the child's developing (a) notion that the whole is divisible and (b) recognition of the need for equal parts/shares; these indicators showed as "1" or "0.5" in most children.

All 18 children holistically coded at Level 2 conceived of the whole as divisible without prompts. A notion that parts need to be equal becomes solid at this level of development, with all of the children evidencing recognition of the need for equal parts. The need to make equal parts, however, did not always reconcile with exhausting the whole, as children's propensity to exhaust the whole in their activity was coded as solidified in only six children. A partitioning plan ahead of modelled activity is coded as developing in all but one child. Parts created are now related to partitioning. Iteration also emerges among eight of the children. From the evident trends, it appears the KDUs leading development at this level is not only the child's evolving (a) coordination of created equal parts with exhausting the whole and (b) an a-priori plan for creating the parts. Such a plan becomes somehow linked to sharers as an afterthought in the child's activity.

For the eight children coded as Level Three, most KDUs were coded as solidified *in activity*. The whole is divisible for 100% of the children without prompts; parts are related to cuts for 100% of the children without prompts. A partitioning plan and exhaustion of the whole were solidified. Moreover, Iteration occurred unprompted. From the evident trends, what seems to lead development in Level Three is the testing of implicit or explicitly equal parts against the whole, seemingly in an effort to quantify the share in terms of the whole and/or to rectify creation of fractions in activity (i.e., "partitioned fractions") toward becoming mentally abstract quantity fractions "run".



Child #	Divisible Whole	Partitioning Plan	Exhaust Whole	Equality of Parts	Part/Cut Relation	Iterating	Trajectory Level
1	0.5	0	0	0.5	0	0	1
4	0.5	0	0	0.5	0	0	1
5	0.5	0	0	0.5	0	0	1
6	0.5	0.5	0	0.5	0	0	1
7	0.5	0	0	0.5	0	0	1
8	0.5	0	0	0.5	0	0	1
10	0.5	0.5	0.5	0.5	0	0	1
15	1	0	0	0.5	0	0	1
21	0.5	0	0	0.5	0	0	1
22	1	0.5	0	0.5	0	0	1
23	1	0.5	0	0.5	0	0	1
24	1	0.5	0.5	0.5	0	0	1
25	1	0	0	0.5	1	0	1
29	1	0.5	0.5	1	1	0	1
30	1	0.5	0.5	0.5	0	0	1
2	1	0.5	0.5	1	1	0	2
3	1	0.5	0.5	1	1	0	2
9	1	0.5	1	1	1	0	2
11	1	0.5	0.5	1	1	0	2
12	1	0.5	0.5	1	1	0	2
13	1	0.5	0.5	1	1	0	2
18	1	0.5	0.5	1	1	1	2
19	1	0.5	0.5	1	1	0	2
20	1	1	1	1	1	0	2
32	1	0.5	0.5	1	1	1	2
33	1	0.5	0.5	1	1	0	2
34	1	0.5	0.5	1	1	1	2
35	1	0.5	1	1	1	1	2
37	1	0.5	1	1	1	1	2
38	1	0.5	0.5	1	1	1	2
40	1	0.5	0.5	1	1	1	2
41	1	0.5	1	1	1	1	2
16	1	1	1	1	1	0	2
14	1	1	1	1	1	1	3
17	1	1	1	1	1	1	3
26	1	1	1	1	1	1	3
27	1	1	1	1	1	1	3
28	1	1	1	1	1	1	3
31	1	1	1	1	1	1	3
36	1	1	1	1	1	1	3
39	1	1	1	1	1	1	3

Figure 2: Heat Map

## Discussion

The results of the study support the notion that not only do children with MD have conceptions of fractions but that their conceptions evidence across a continuum not unlike that documented in prior research with children without MD (Empson et al., 2005; Tzur, 2007; Steffe & Olive, 2010). This work focuses on children labelled MD as *capable*; we worked to demarcate what is possible mathematically through their own reasoning and informal knowledge. Yet, the higher-level understandings that leads to abstracted notions of fractions as numbers evident in prior research with children without MD were not noted in our analysis. Thus, a continued instructional

focus on the development of conceptual understanding of fractions as quantities for these children is critical.

The “early” conceptions of fractional quantity documented in the current study are not necessarily limited to MD children or indicative of intrinsic mathematics difficulties. Initial results (not reported here) investigating significant differences in problem solving or operations between children with disabilities and children in Tier 2 educational settings yield no significant differences. It is possible that the nature of the performance gap in children’ conceptual understanding of fractions is related to malleable factors (Vukovik, 2012) such as underdeveloped number composites or multiplicative unit coordination.

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# MATHEMATICS TEACHER KNOWLEDGE-IN-INTERACTION: A DISCURSIVE PSYCHOLOGY APPROACH

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*Mathematics teacher knowledge has been researched from a wide variety of perspectives and using a variety of methods. In this paper the focus is on knowledge-in-interaction and how teacher knowledge can be examined through teachers' interactions with students, in the moment. Using a discursive psychology approach transcripts of whole class interactions are examined to reveal how we can consider teacher knowledge through looking at how the teacher designs their turns in response to what their students say. This enables not only an insight to the teacher's knowledge-in-interaction but also how this knowledge is used in guiding the students' mathematical activity.*

## **Mathematical knowledge for teaching.**

There have been different conceptualisations of mathematics teacher knowledge alongside different approaches to analysing or examining these conceptualisations (Rowland & Ruthven, 2011). This paper considers knowledge-in-interaction as distinguished from knowledge-in-action by Rowland, Huckstep, and Thwaites (2005). They describe knowledge-in-interaction as "revealed by the ability of the teacher to 'think on her feet' and respond appropriately to the contributions made by her students" (page 266). This is similar to the notion of in-the-moment pedagogy linked to mathematical awareness described by Mason and Davis (2013) that influences "teachers' capacities to engage flexibly and productively with their students" (p.184). Rowland et al.'s development of the notion of knowledge-in-interaction has focused on what they call contingency, where teachers face unexpected or unplanned for events. They make a distinction between what can be *anticipated* or predicted by the teacher yet cannot be *known* in advance. In interaction what students say, when they say it, and how they say it can be anticipated but not known, so all turns at talk in classroom interaction are contingent. In particular, a teacher's response to a student is contingent upon what that student said or did.

In this paper the approach to the analysis of the classroom interaction builds on the initial work by Barwell (2013). Barwell uses discursive psychology to examine how the use of knowledge claims by teachers in interaction with students treat the teachers as knowing but can also at the same time treat the students as not knowing. This treatment of students as not knowing can be pre-emptive where the teacher has anticipated the students not knowing something without any indication from students themselves. Discursive psychology is used in this paper to illustrate two different aspects of classroom interaction. Firstly the co-construction of teachers' knowing-in-

interaction and secondly how knowledge claims by teachers can also treat students as knowing.

This perspective treats teacher knowledge as both dynamic and active (Rowland, Thwaites, & Jared, 2015) as well as situated (Hodgen, 2011) but takes these conceptions further in that teacher knowledge is constituted by the interactional context, not just influenced or created by it (Lerman, 2002).

### **Discursive Psychology**

Discursive psychology is an approach to studying cognitive phenomena such as knowing, thinking or understanding developed by Edwards and Potter (1992). This approach draws on the methodological principles of ethnomethodology (Garfinkel, 1967) and conversation analysis (Sacks, 1992). Discursive psychology studies knowing as an interactional discursive object and practice, rather than as a cognitive state or process (Koole & Elbers, 2014). That is, rather than attempting to identify what a teacher does or does not know, discursive psychologists argue that we can examine how knowing, and in particular teacher knowing, is dealt with by the teacher and his students themselves in interaction. This includes how teachers and students manage knowing in the classroom, what social actions they are performing when they make knowledge claims and how any knowledge claims are designed for the particular recipients (for more detail about the concept of recipient design see Sacks, Schegloff & Jefferson, 1974).

The unit of analysis is thus the interactions between teachers and students and the focus of any analysis is on how the participants in the interaction display or use aspects of knowing through the organisation and design of their turns (Edwards, 1997). The questions driving the analysis offered in this paper are then:

- How does the teacher make knowing observable in his interactions with students?
- What social and pedagogic actions are these displays of knowing doing?
- How is this display contingent upon what the students have made observable?

### **T-Totals**

The analysis presented in this paper focuses on the T-Totals task. This task is well known by many teachers in England as it was a popular coursework task that students completed as a part of their GCSE qualifications before coursework was abolished. The teacher is an experienced mathematics teacher working with a class of 12-13 year old students. The teacher was video-recorded for 6 lessons during the final term of the academic year and the whole class interactions were transcribed using Jefferson transcript notation (2004). A discursive psychology approach entails examining naturally occurring classroom interactions and no instructions were given as to what or how to teach. Thus the teacher chose to work on this task with his students. The task occurred over two consecutive lessons. The original coursework task begins with an investigation of the total of the numbers inside a T-shape that is placed on a hundred



squares. As part of the investigation students are encouraged to extend this task by varying different features, such as the shape, different transformations of the shapes, and different grids but are not directed to these variations as to how to extend the task specifically.

The teacher is clearly familiar with the task and the analysis in this paper focuses on how knowing the task directs students' attention to focus on different aspects at different stages on the features that lead towards the goal of the task, which is itself discursively constructed in the interaction. The 'goal' this paper focuses on is that the algebraic expression  $5n + 30$ , which the students identify as the formula for the T-total, counts as a mathematical proof that the total "is always a multiple of 5". I also consider how this 'knowing' of the task serves to open up some choices within the task whilst restricting others. There are other aspects of teacher knowledge-in-interaction evident in the transcripts such as ways of being mathematical (Mason & Davis 2013), the nature of mathematical proof, and the role of specific and generic examples, but due to limitations of space these aspects will be discussed elsewhere.

### Offering Freedoms and Constraints

In this section three scenarios are examined which focus on the way different choices are treated by the teacher; opening them up as possibilities, placing constraints, or doing neither. These contrasting ways of treating these choices reveal aspects of the teachers' knowledge of the task.

Initially, as the task is introduced to the students, a hundred square is projected onto the interactive whiteboard at the front of the classroom. The teacher's first introduction to the task is in Extract 1 below.

#### Extract 1: Lesson 1 - introducing the task

- 1 T: ...it's all to do with this idea of a hundred square which we've used before. and I thought it would interesting to think about putting (1.2) to think about putting a letter on to the number square. I'm going to start with the letter t. maybe that can change later on. I'm going to put a t on to the square there and um how would you (.) how would you describe where that t (.) is. (4.1) there could be more than one way of doing it. B

In this introduction the teacher treats the students as knowing the hundred square, this is 'taken-as-shared'. The introduction of the T is treated as a choice. This choice is from a range of possible letters (as opposed to shapes etc.) but the freedom to choose other letters is not what they will be doing now, but can occur later. After two student descriptions of "where the T is" a third student offers another description:

#### Extract 2: Lesson 1 - describing the T

- 8 S: um (.) it's (.) facing with the long bit at the top and it starts with thirty five and then ends (with fifty).
- 9 T: thank you very much. when you said it's (face) was the long bit along the top what were you (.) drawing attention to
- 10 S: the (.) head of the t

- 11 T: this part here  
12 S: yeah  
13 T: okay were you envisioning that it might be (.) rotated or something. yes okay so that's a possibility for later isn't it. um what if we agreed that it was going to be this shape. that it's going to have three squares and then (.) two underneath. and what if we agreed it was going to be that way round. how might we describe where it is if we already know that that was the same shape and size as the t we just wanted to say where it was

In turn 13 the teacher responds to the student's description by focusing on the possibility of rotating the T. The student's description does not mention rotation but the teacher interprets the student as suggesting this through their description, which includes an aspect of orientation. This interpretation is accepted by the student by nodding to the start of the teacher's turn. The teacher accepts this as a possibility, and treats it as a shared awareness of possibility ('isn't it'), but locates this choice as something to make 'later', again treating this as a shared decision. In the rest of the turn, the teacher restricts the choices further, repeating the need to keep the shape the same, but also that it will be the same size (i.e. enlargement is not a possibility). The possibility of rotating occurs twice more in the interactions. On both occasions the students introduce the idea and the teacher accepts it as a possibility but also indicates that it is something to think about later.

In the next extract again possibilities arise as to how the students might investigate the task:

Extract 3: Lesson 1 - If we move the T lower

- 27 T: it would change. can you say more about that change (0.6) R?  
28 R: um if you move the t (.) lower down the t-board then the total will be higher  
29 T: if we move the t lower down the total will be higher. who agrees with that. a few people. maybe. um these are the sorts of things I thought we could investigate in other words what happens to the the total of the t (.) if the t's somewhere else. maybe for now we could keep it the same way round but you might like later on (.) to change that. you might have to think what happens to the t-total if we move it one place along. what happens to it if we move it down, what happens if we reflect it maybe? ...

Turn 29 finishes the introduction of the task to the class and after this turn the students work in pairs on the task. In this turn the teacher again asks the students not to rotate the T "for now" but continues to offer this as an option later. In addition the teacher states some options they can explore now, moving it one place along, moving it down and reflecting it. This is the first and only mention of reflection in the two lessons and the teacher treats it as something that could be explored now.

Following this turn the students work in pairs on the task whilst the teacher circulates the room. The last 10 minutes of this first lesson is then spent with pairs being invited up to the board to explain what they found out. The first pair report that the total changes by 50 if you move the T up one square and explain that this is because each of the numbers has changed by minus 10. The second pair report a formula for the T-

total as  $5x + 30$ . The lesson ends during this second pairs report but their results are displayed on the board at the beginning of the second lesson on the task. This second lesson begins with a discussion on what the students understand by the word proof before returning to pairs presenting their findings but this time from their seats. Extract 4 is taken from the second pair of pupils sharing their findings in this lesson. The previous pair have reported that they have found that the total is always a multiple of 5. In this extract a student offers a transformation of the T that the teacher neither opens this up for further exploration nor places a constraint on not exploring this particular transformation of the T, or delaying this exploration until later:

Extract 4: Lesson 2 – with six it will always be a multiple of six

- 98 S: um (.) we found out that with five (.) it will always be a multiple of five and six would always um they'd always be a multiple of six
- 99 T: when you say six what, six what
- 100 S: if there's like um three on the top an:d (0.7) yeah three on the bottom. it say just add one [at the bottom]
- 101 T: [you mean] like that ((drawing a T on the board))
- 102 S: yeah. um (0.3) if you add all the numbers up and divide it all, it will always be a multiple of six
- 103 T: excellent right so so the t total for a t with a bit sticking out the bottom or a bigger t, a longer t, is always a multiple of six that's interesting. um. and I think I didn't know that anyone else had worked on that, that was an interesting idea so to change the t shape to make it look a little bit longer. thank you. um sorry I stole your thunder about (.) multiples of five. I forgot it was you that did that. do you want to say something else about what you did? th- tha- tells excellent. um yes

In turn 103 the teacher summarises what the student has said, evaluating both the student's turns ("excellent") and the idea expressed in them ("interesting idea"). He then indicates that this result was unexpected, in that he "didn't know that anyone else had worked on that". He then partially repeats the student's idea again focusing just on how the T has changed, with no mention of the result that this is always a multiple of 6. After this partial repeat the teacher turns to the pair who had spoken previously and returns to the idea of multiples of five.

The teacher ends the lesson by reporting that several groups told him that the T-total was always a multiple of 5 but that he wasn't convinced by just being shown lots of examples. This leads to a discussion resulting in  $5x+30$  being identified (in a joint construction by the students and teacher) as being always a multiple of 5 and therefore representing a proof of this (stated by the teacher).

## Discussion

In the last 3 extracts, the teacher has not made any explicit claims about knowing the task and which choices will support the students in noticing that the total is always a multiple of 5, and which will distract from this or end in different results. Yet in each extract he has treated the choices differently. Those transformations of the T, such as

rotation and enlargement, which generate different formula, are those that the teacher treats as constraints whereas the transformations of the T, such as translation and reflection, which result in a total that is a multiple of 5 are allowable choices that can be explored immediately. In extract 4 the transformation is neither opened up as something to explore nor constrained as something to do later. So whilst the teacher has not stated that he knows the impact of these freedoms and constraints on the task, and has not told us in any interviews (the usual approach to identifying this type of knowing), it is evident from the different ways in which he has treated the choices in interaction with his students that different choices affect the task in a different way. With the final extract, we cannot say whether the teacher is familiar with this transformation of the T or the consequences of this transformation (which only works for the two scenarios given by the students and does not generalise further) but he does not offer it as a choice for exploration, as he did with translating or reflecting the shape, and does not discourage students from exploring it, as he did with rotation and enlargement.

This analysis using discursive psychology not only enables us to examine teacher knowledge-in-interaction but also how this knowledge-in-interaction affects the activity both in the interaction, and also in the lesson as a whole. In particular, how the teacher uses knowledge claims to guide the students to a goal of the task, whilst enabling the students to construct the mathematics for themselves. This is what Edwards and Mercer (1987) refer to as “the teacher’s dilemma”. Each of the choices discussed in this paper are used to guide students towards noticing that the T-total is always a multiple of 5 and eventually the proof of this through the creation of an algebraic formula. These goals are not explicitly stated, either in the initial presentation of the task or as the students are working on the task. Instead they are co-constructed by the teacher and his students in interaction through the knowledge claims that are made and the way these claims are used by the other participants in their turns. This approach addresses some of the issues raised by studies that focus on teachers’ own accounts of their teaching where what teachers emphasise retrospectively does not always align with what they were attending to during teaching (Speer, 2005; Mason & Davis, 2013).

Finally, the way the teacher displays his knowledge-in-interaction also treats his students as knowledgeable. In extract one the introduction of the hundred square is treated-as-shared, as is the possibility of rotation but leaving it to later in extract 2. The students are the first to introduce the idea of rotation, how the T-total changes as the T is translated, the pattern that the total is always a multiple of 5, and the pattern that adding a square at the bottom results in the total always being a multiple of 6. In each case the teacher accepts these claims in a variety of ways, thanking the students, evaluating the claims, building on the claims or simply acknowledging them as a possibility. This is in contrast to Barwell’s analysis (2013) where it is the teacher largely making the knowledge claims and many of these claims treat the students as not knowing.

## CONCLUSION

In this paper I have shown how a discursive psychology approach can enable us to not only directly examine teacher knowledge-in-interaction but also how this knowledge-in-interaction affects the mathematical activity in the lesson. We can examine how both teachers and students make aspects of their knowledge observable, the social actions these displays are performing and how this influences the subsequent interactions. We can also examine how teachers use students' knowledge claims and how students use teachers' knowledge claims. Finally, these knowledge claims can also be used to position the other participants as knowing or not knowing. This approach therefore offers an additional analytical tool to examine how teacher knowledge influences the learning opportunities of students both in terms of gaining new mathematical concepts but also in ways of behaving mathematically.

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# NARRATIVES AND THE DEVELOPMENT OF THE SKILL OF NOTICING

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*Previous studies have focused on examining contexts and paths for the development of the skill of noticing children's mathematical thinking. However, the development of this skill is not an easy task and deserves further research. We analyze how the task of writing narratives could allow pre-service teachers to develop this skill. During their practices at schools, 22 pre-service teachers were asked to write a narrative identifying and describing a noteworthy situation related to the students' mathematics learning. Results show that writing narratives helped pre-service teachers to identify important mathematical elements of the situation, interpret students' mathematical thinking and decide instructional decisions based on students' mathematical thinking.*

## INTRODUCTION

The new educational perspectives on mathematics teaching in the XXI century have proposed a change in the way of dealing with interactions in the classroom. From this perspective, teaching involves observing students, listening attentively to their ideas and explanations, planning objectives and using the information to make instructional decisions (NCTM, 2000). This perspective calls for a greater flexibility of teachers in recognizing students understanding while they are teaching (van Es, & Sherin, 2002) and suggests the development of the skill related to being aware of what happens in their classrooms and how to manage it. In this sense, the skill of noticing has been identified as a way to give effective responses in the classroom, and to take appropriate educational decisions.

The skill of noticing has been conceptualized from different perspectives (Mason, 2002; Sherin, Jacobs, & Philipp, 2011) but all of them emphasize the importance of identifying the relevant aspects in teaching-learning situations and interpreting them to support instructional decisions. Over the last decade, studies have reported different contexts to help pre-service and in-service teachers to develop this skill: watching video clips or interviews (Coles, 2012; van Es, & Sherin, 2002; 2008), and participating in online debates (Fernández, Llinares, & Valls, 2012).

On the other hand, previous research has used narratives as a means of capturing and studying practice (Chapman, 2008). The connection between narratives and teacher learning is supported by the fact that narratives are a form of expressing teachers' practical understanding of mathematics teaching (Chapman, 2008; Connelly & Clandinin, 1990; Ponte, Segurado, & Oliveira, 2003; Smith, 2003). Our study links these lines of research, using narratives as a tool that can help pre-service teachers to develop the skill of noticing children's mathematical thinking.

## THEORETICAL BACKGROUND AND OBJECTIVE

### The skill of noticing children's mathematical thinking

We used the conceptualization of the skill of noticing children's mathematical thinking given by Jacobs, Lamb, and Philipp (2010) as three interrelated skills: (i) attending to students' strategies; (ii) interpreting students' mathematical thinking and (iii) deciding how to respond on the basis of students' mathematical thinking. These authors provided growth indicators to identify shifts in prospective and in-service teachers' professional noticing of students' mathematical thinking: a shift from general strategy descriptions to descriptions that include the mathematically important details; a shift from general comments about teaching and learning to comments specifically addressing the children's understanding; a shift from overgeneralizing children's understandings to carefully linking interpretations to specific details of the situation; a shift from considering children only as a group to considering individual children, both in terms of their understandings and what follow-up problems will extend those understandings; a shift from reasoning about next steps in the abstract to reasoning that includes consideration of children's existing understandings and anticipation of their future strategies; a shift from providing suggestions for next problems that are general to specific problems with careful attention to number selection (p. 196). So, the noticing skill could be developed by moving from a focus on teachers' actions to students' conceptualizations and by moving from evaluative comments to interpretative comments based on evidence (Bartell, Webel, Bowen, & Dyson, 2013).

Fernández, Llinares, and Valls (2012) has shown that online debates have been an appropriate instrument to promote the development of this skill in the specific domain of proportional reasoning since results showed a pre-service teachers' movement from describing general strategies to identify evidence of how students were developing proportional reasoning. Coles, Fernández, and Brown (2013) postulated that meetings between in-service primary teachers, sharing the work they have been done at school, helped them to develop their noticing of children's mathematical thinking. In the area of early numeracy, Schack et al. (2013), showed that after the participation in an intervention (using video) that progressively nests the three interrelated skills of professional noticing (attending, interpreting and deciding), prospective elementary school teachers gained expertise in the three component skills.

Although these studies have focused on examining contexts and paths for the development of the skill of noticing children's mathematical thinking, the development of this skill is not an easy task and deserves further research. Therefore, we wonder whether the task of *writing narratives* could help pre-service teachers to develop the skill of noticing students' mathematical thinking during their practices at schools.

### The narratives and objective

As narratives are a form of expressing teachers' practical understanding of mathematics teaching (Chapman, 2008; Ponte et al., 2003), they can allow us to understand the way in which teachers organize their work and act in professional

contexts. A narrative is a story that tells a sequence of events that are significant for the author and has an internal logic that makes sense to him/her. Pre-service teachers can be considered as storytellers in the context of teacher education programs. So, narratives are a tool that can help pre-service teachers to make sense of their experience during their teaching practice.

Consequently, the narratives of pre-service teachers, describing a mathematics teaching-learning situation in which they think that primary school students are developing some aspects of the mathematical competence, could focus their attention on students' mathematical thinking, instead of general observations and teaching management.

The objective is to analyze how the task of writing narratives allows pre-service teachers to focus their attention on specific aspects of students' mathematical thinking.

## METHOD

### Participants and context

Participants were 22 pre-service primary school teachers enrolled in the last year of their degree to become primary teachers. They were in the period of practices at primary schools (practicum). The first part of their practices was a *period of observation* where pre-service teachers had to interpret noteworthy interactions of the classroom and students' mathematical thinking.

### Instrument: The narratives

During the observation period (two out of eight weeks of the period of practices), pre-service teachers had to write a narrative identifying and describing a noteworthy situation related to the students' mathematics learning (these narratives are the data of this study). They could use the theoretical knowledge from didactics of mathematics from previously mathematics education courses (one related to numerical sense and other related to geometrical sense) to identify and interpret what they considered a noteworthy situation and provide evidence of students' mathematical understanding. To perform this task, we provided pre-service teachers with specific guidelines based on Jacobs et al. (2010)'s conceptualization of the skill of noticing students' mathematical thinking:

- *Describe in detail the mathematics teaching-learning situation:* The task (curricula contents, materials, resources...). What did the primary school students do? For example, you can indicate some students' answers to the task, difficulties... What did the teacher do? For instance, you can describe the methodology and some aspects of the interactions.
- *Interpret the situation.* Indicate some evidence of students' answers about the way in which they have achieved the objectives (students' understanding of the mathematical content) and the difficulties they had.

- *Complete the situation.* Modify or propose a new situation in order to help students to develop other aspects of the mathematical competence identified (that is, supporting students in their conceptual development).

## Analysis

Three researchers analyzed, individually, the narratives written by pre-service teachers looking for evidence of how they were noticing students' mathematical thinking. The agreements and disagreements were discussed to reach a consensus on these issues.

We briefly explain what we consider evidence of how pre-service teachers noticed students' mathematical thinking: i) if in the descriptions of student's responses, pre-service teachers included mathematically important details; ii) if in the interpretations of students' responses, pre-service teachers addressed students' mathematical thinking and linked students understanding with specific details (mathematical elements) of the situation; iii) if in the decisions about the next instructional steps, pre-service teachers provided specific problems based on students' mathematical thinking.

## RESULTS

In this section, we show some characteristics of how pre-service teachers notice students' mathematical thinking. 12 out of 22 pre-service teachers identified important mathematical elements in students' responses and used these elements to interpret students' mathematical thinking. 7 out of these 12 pre-service teachers provided, as an instructional decision, specific problems based on students' mathematical thinking, and 5 out of these 12 pre-service teachers provided only general comments that were not based on students' mathematical thinking.

Next, we present an excerpt of a narrative written by a pre-service teacher. We report on this narrative as there is clear evidence of how the pre-service teacher notices students' mathematical thinking (identifying important mathematical elements and using them to interpret students' mathematical thinking). The context of this narrative is a classroom of 1st grade and the mathematical content *the numerical sequence*. This pre-service teacher described the situation writing in her narrative:

The teacher gave each child an abacus, and a red and blue pen. The teacher asked each child to represent a number at the abacus and to write the number in a sheet representing tens in red and units in blue. The numbers used were 8, 10, 14, 16, and 19. Students' answers were:

- (1) Some students represented the number 16 as 13 and the number 14 as 15.
- (2) Several students did not represent the number 10.
- (3) Some students wrote in a wrong way the numbers 5 and 7 (reversed).

Firstly, this pre-service teacher described the mathematical task observed in class underlying the type of numbers used. Next, she identified some students' answers that provided him with important mathematical elements of the situation, particularly, those related with the numerical sequence, the position of figures in the number writing and

the idea of grouping. This allowed him to interpret students' understanding of Decimal Numbering System in relation to the idea of grouping and place value. She interpreted the three students' answers described before pointed out:

- (1) These students made mistakes in counting units (i.e., when reciting the numerical sequence counting units).
- (2) These students had difficulties grouping 10 units to have 1 ten "and there are not units"
- (3) These students had difficulties with the number writing.

Finally, this pre-service teacher made a summary of her interpretation:

This task is related to the Decimal Numbering System: place value (units and tens). Difficulties of students are connected with number writing (some of them are written reversed), with reciting the number sequence (difficulties in counting) and with the idea of grouping 10 units in 1 ten.

Although this pre-service teacher showed evidence of students' mathematical thinking and highlighted some of their difficulties, she was not able to propose an instructional decision based on students' mathematical thinking.

Following, we report on a pre-service teacher narrative that shows evidence of how the pre-service teacher interprets students' mathematical thinking (identifying important mathematical elements and using them to interpret students' mathematical thinking) and proposes specific instructional decisions taking into account the students' mathematical thinking.

The context of this narrative is a classroom of 4th grade. This pre-service teacher described the situation and the students' difficulties:

*The graph represents the spending made by the council to maintain the natural pool (during the last year)*

1395 €	1530 €	1643 €	1519 €
May	June	July	August

1) How much money have they spent in total in the four months? 2) What is the daily cost in May? What about June? In which month the daily cost is lower? 3) The council thought the possibility of charging people for a bath if the daily cost exceeds 60 €. In which months had the council charged people for a bath last year?

With the first question there were no problems, [...] all students knew that the total amount was the sum of the spending of the four months. Regarding the second question, most of the students had problems because they believed they had to multiply the sum of the amounts by the four months.

In the third issue, they also had difficulties. Some students answered that they had to divide the cost of each month between 60 and others that they had to divide 60 between the days of each month.

This pre-service teacher identified important elements of the situation: difficulties with the understanding of the word problem in relation to data that are not explicit or are

not needed to solve the problem. The pre-service teacher wrote as an interpretation of these answers:

In the second question, students did not consider the days of the month (30 or 31) as data problem because maybe it does not appear explicitly in the word problem.

In the third issue, students thought that when a number appears in the word problem, necessarily, they must operate with it.

Writing the narrative allows this pre-service teacher to identify some relevant mathematical elements and interpret the students' difficulties in relation to these elements (for example, that children do not identify the use of the division in a real context and seem that had difficulties in managing the meaning of division as quotative or partitive). Thus, in the final part of his text, the pre-service teacher proposed a specific instructional decision in relation to his interpretation of students' understanding:

Insisting on the meaning of the division as: distribute into equal groups. They do not identify the use of a division in a problem. For example, let them think about: If the expending is not the same every day, could we solve the problem with the operation that we have done?

With this specific task, this pre-service teacher defines a specific learning objective, focused on the different meanings of the division derived from his interpretation of students' mathematical thinking.

## **DISCUSSION AND CONCLUSIONS**

In this study, we analyze how the task of writing narratives helps pre-service teachers to develop the skill of noticing students' mathematical thinking during their practices at schools. From the results, we, firstly, underline that narratives seem to be a useful instrument that helps pre-service teachers to focus on specific mathematical elements and on students' mathematical thinking. In spite of being the first time they worked with narratives, 12 out of the 22 pre-service teachers who participated in this experience identified important mathematical elements of the teaching-learning situation and interpreted students' understanding providing evidence. Even though our data are positive since more than half of pre-service teachers addressed students' mathematical thinking, some of them only provided descriptions of classroom events without identifying evidence of students' understanding. These data confirm the fact that the development of this skill is not an easy task (Coles et al., 2013; Fernández et al., 2012; Schack et al., 2013).

Secondly, only seven pre-service teachers who had interpreted the students' understanding proposed specific instructional decisions focused on students' conceptual development (in other words, focused on students' mathematical thinking). So our results support that the skill of proposing instructional decisions (Jacobs et al., 2010) is the most difficult one.

Thirdly, the fact that ten pre-service teachers provided only general comments invites us to reflect on why students seem to ignore the attention directed toward a particular aspect of the teaching-learning situations (Mason, 1998). We wonder about if these pre-service teachers that failed in attending and interpreting students' mathematical thinking in their narrative would be able to show a progression in successive narratives receiving some help, such as getting feedback from the tutor of practices, or using on-line discussions (Fernández, Llinares, & Valls, 2012).

From this evidence, we think that writing narratives allowed pre-service teachers to begin to theorize practice in practical contexts during the observation period (practicum) (Smith, 2003) when attending and interpreting students' mathematical thinking. In this sense, we can see the act of writing as a mediator in pre-service teachers' learning (Wells, 1999). For further studies, it could be interesting that pre-service teachers write a narrative during the second part of their practices when they are embodied in their own teaching practice.

Finally, we want to highlight that we are aware that it cannot be expected from this type of experiences (in this case, writing narratives) that they cause immediate changes in pre-service teachers but is, certainly, a step in the achievement of their professional development. Although the skill of noticing students' mathematical thinking is developed and sustained over long periods of time, effort and experience (Litle, 1993; van Es & Sherin, 2008), narratives seem to appear as an effective tool on its development.

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# MATHEMATICAL PROBLEM SOLVING WITH TECHNOLOGY BEYOND THE CLASSROOM: THE USE OF UNCONVENTIONAL TOOLS AND METHODS

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*This paper addresses mathematical problem solving with technologies in a beyond school web-based competition. We aim to disclose the ways mathematical and technological knowledge are used and combined for solving the given problems. A specific conceptual framework for accounting both these components was developed. By means of the Mathematical Problem Solving with Technology model (MPST) we report the case of Marco, aged 13, solving and expressing a geometrical problem. His ability in perceiving affordances in the tools that he chose is in line with the efficient use he made of them in the development of mathematical understanding that was crucial for finding and expressing the solution. Results suggest that digital thinking and experience have to be seen as relevant as the mathematical cognitive resources.*

## INTRODUCTION

Mathematical Problem Solving has acquired the status of a research field within Mathematics Education over the last decades of the 20<sup>th</sup> century, after an intense research activity following the influential work of George Polya later developed by the seminal work of Alan Schoenfeld. The turn of the century brought new research objects and impetus, which have diverted the interest of the research community regarding this topic and, particularly, regarding the problem solving activity that occurs beyond the classroom (English & Sriraman, 2010). Recently, the constant availability and usage of sophisticated digital tools in out-of-school and beyond-school contexts are requiring new thinking about the sort of skills that may become especially important in the technological, global and interconnected society of the 21<sup>st</sup> century (Hoyles, Noss, Kent & Bakker, 2010). Thus, problem solving with new methods and new tools holds up as a central competence to meet the challenges of active life, as technological tools are altering ways of thinking and acting (Lesh, 2000).

This paper reports on a study that addresses students' mathematical problem solving with technology in a beyond-school context comprising of a web-based mathematics competition, named SUB14<sup>®</sup>. The competition is aimed at middle graders (12-14 years-old) of every school of the south of Portugal. Its Qualifying stage consists of solving a non-routine problem proposed every two weeks, either through e-mail or an online text editor available on the competition website. Participants may solve the problems using their preferred methods and tools but are explicitly required to report on their solving process and must offer a complete explanation of their reasoning.

Our main research goal is to uncover the generally unknown ways in which students

use and combine their mathematical and technological knowledge outside the classroom, particularly when they are allowed to pick any digital tool of their choice and use it to achieve mathematical purposes. In this paper we hope to provide evidence for the claim that problem solving with digital tools can only be partially described by previous frameworks that took mathematical thinking and experience as the primary cognitive resources. What one must realize is that digital thinking and experience have to be seen as equally central and fundamental cognitive resources. Here we draw on the case of a student solving a geometrical problem, which shows much of his mathematical thinking going on when he handles similarity and ratios among circles inscribed in triangles, but also reveals specific actions and processes related to his use of digital tools for the analysis of the geometric figure presented in the problem. Thus, we seek to address this need to redesign and expand well-known earlier theoretical models and suggest more efficient ways to describe the connection between mathematical knowledge and the affordances of digital tools that solvers bring into the problem space.

## **MATHEMATICAL PROBLEM SOLVING WITH TECHNOLOGY**

The prevailing theoretical models of the problem solving activity appear inadequate as tools for interpreting the role of technology and to explain the interaction between individuals' technological and mathematical competences in their problem solving activity (Santos-Trigo & Camacho-Machín, 2013). This has the development of a new specific conceptual framework that might account for both components of the problem solving process.

Solving a non-routine mathematical problem is here understood as the development of a productive way of thinking about a challenging situation (Lesh & Zawojewski, 2007) where the solver must adopt a mathematical point of view in order to carry out mathematization processes. Problem solving is also conceived as a synchronous process of mathematization and expression of mathematical thinking (Carreira, Jones, Amado, Jacinto & Nobre, in press). This means that solving a problem encapsulates both the required answer and the creation of an explanation for that answer.

In terms of student's interaction with digital media in performing complex tasks, such as non-routine problems, we draw on the concept of perception of affordances in the tools (Gibson, 1977). "Perceiving affordances is placing features, seeing that the situation allows a certain activity" (Chemero, 2003, p. 187). This suggests that the solver's effective use of a tool is grounded on the recognition of its particular features that will be useful for developing an approach to the problem. Affordances emerge from the relationship between the capabilities of the solver and the properties of the tool (Norman, 2013), insofar as one is not "specifiable in the absence of specifying the other" (Greeno, 1994, p. 338), which leads us to consider the impossibility of separating the solver's mathematical and technological competences.

The DigEuLit Project proposed a model that sets a list of processes performed while solving a task or problem that requires the use of a digital resource, comprising:

*statement* – clearly state the problem and the actions likely to be required; *identification* – identify the digital resources required to achieve the solution; *accession* – locate and obtain those digital resources; *evaluation* – assess the accuracy and reliability and relevance of the digital resources; *interpretation* – understand the meaning they convey; *organization* – organize them in ways that may enable the solution; *integration* – bring these resources together in relevant combinations; *analysis* – examine them using concepts and models that will enable the solution; *synthesis* – recombine them in new ways to achieve the solution; *creation* – create new knowledge objects, units of information or digital outputs that contribute to achieve the solution; *communication* – interact with others while solving the problem; *dissemination* – present the solution to others; *reflection* – consider the success of the task performed (Martin & Grudziecki, 2006, p. 257).

Although several actions in this list resemble well known problem solving models, a mathematical lens is needed. Alan Schoenfeld’s (1985) model for describing students’ mathematical problem solving performance seemed useful to this task. His five stage model comprises: *read* – time spent “ingesting the problems conditions”; *analysis* – attempt to fully understand the problem “sticking rather closely to the conditions or goals” that may include a selection of ways of approaching the solution; *exploration* – a “search for relevant information” that moves away from the context of the problem; *planning and implementation* – defining a sequence of actions and carrying them out orderly; *verification* – the solver reviews and assesses the solution (pp. 297-298).

Mathematical problem solving with technology (MPST)	
Grasp	Appropriation of the situation and the conditions in the problem, and early ideas on what it involves.
Notice	Initial attempt to comprehend what is at stake, namely the mathematics that may be relevant and the digital tools that may be necessary.
Interpret	Placing affordances in the technological resources in pondering mathematical ways of approaching the solution.
Integrate	Combining technological and mathematical resources within an exploratory approach.
Explore	Using technological and mathematical resources to explore conceptual models that may enable the solution.
Plan	Outlining an approach to achieve the solution based on the analysis of the conjectures explored.
Create	Carrying out the outlined approach, recombining resources in new ways to create new objects that convey both mathematical and technological understanding of the situation, which will contribute to solve-and-express the problem.
Verify	Engaging in activities to explain or justify the solution achieved based on the mathematical and technological resources.
Disseminate	Present the solutions or outputs to relevant others and consider the success of the problem-solving process.

Communicate – Interact with relevant others whilst dealing with the problem or task.

Table 1: Processes underlying mathematical problem solving with technology

By comparing and relating the processes proposed by Martin and Grudziecki and the stages identified by Schoenfeld, we reached a proposal of merging these two frameworks by means of fusing some of the processes of digital tool usage and

segmenting some of the stages of mathematical problem solving (Jacinto & Carreira, to appear). Table 1 presents a summary of the processes involved in solving a mathematical problem with technology.

## RESEARCH METHOD

The larger study from which we extract the data covered here focuses on students' use of freely chosen technological tools for solving and expressing the mathematical problems posed by SUB14. The explorative nature of the study demanded an interpretative approach that involved qualitative techniques for data collection and analysis (Quivy & Campenhoudt, 2008).

Data collection is based on two different sources: the solutions submitted by the participants throughout the Qualifying phase, and two clinical interviews that took place at the participants' home with the permission of their parents. The second interview, video-recorded, included the observation of the student while solving a problem posted at the competition's website and thinking aloud.

This paper reports on the case of Marco (pseudonym) who usually resorts to a variety of technological tools to solve the problems and present his explanations. The data refer to the observation of Marco while working on a problem. We used NVivo for organizing the data, transcribing the interviews, segmenting and coding. Marco was asked to choose one out of three problems posted for this purpose on the SUB14 website, and to solve it by performing as closely as possible to his usual problem solving process in the competition. Marco chose to solve the problem "Decorative Drawing" (Figure 1) and resorted to several technological tools during the process. Marco's processes of problem solving-and-expressing with technology will be considered and interpreted through the lens of the MPST analytical framework. The following section presents a summary of our findings.

The picture shows a decorative drawing that will be used in the construction of a stained glass window. The equilateral triangle has a height of 12 cm. The circles are all tangent to the triangle and also each small circle is tangent to the large circle. Which is the radius of the smaller circle?

**Don't forget to explain your problem solving process!**

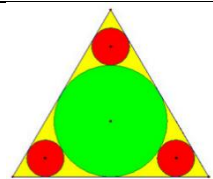


Figure 1: Statement of the problem 'Decorative Drawing' chosen by Marco

## DATA ANALYSIS

Marco is a 13 years-old student, very enthusiastic about digital tools. At school, his math teacher usually uses the whiteboard and sometimes takes the class to a computers room with specific tasks to perform. Marco is quite familiar with a diversity of digital tools; at school, in particular, he learned to use GeoGebra, while studying geometric transformations. Usually, he submits his answers to SUB14 in a spreadsheet file.

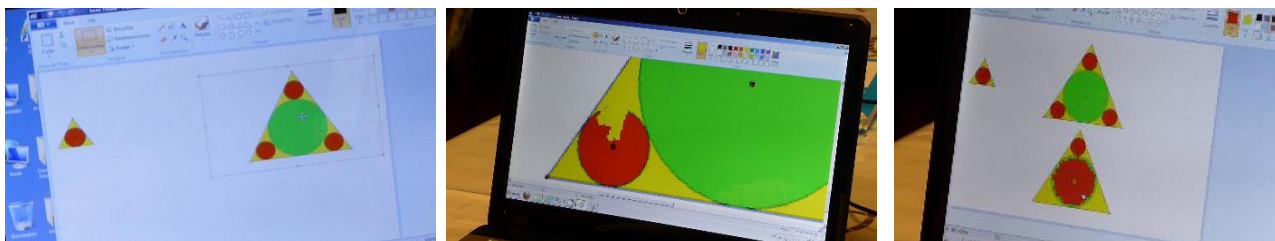
### Marco solving the problem "A Decorative Drawing"

After carefully analysing the three problems on the SUB14 website, Marco chose to solve his favourite. When asked about his reasons, he explains: "It has to do with

triangles and stuff like that, besides in the 7<sup>th</sup> grade I got 100 [%] in both tests (...) I studied congruent triangles and such..." His choice seems grounded on the immediate recognition of the mathematical notions that may be necessary to solve the problem and, simultaneously, on how familiar and self-confident he feels (*grasp*).

As Marco starts to interact with the figure shown in the problem and displayed at the competition's website; he develops several arguments that lead to a conjecture about the solution. Initially, this includes attempts to understand what the problem involves (*notice*), and in each argument he makes considerations about mathematical ways of approaching the solution (*interpret*). The sequence of arguments was as follows: i) "Since the triangle is equilateral, if I reach the circle in the centre, I might get to the others"; ii) "It's like it divides in half. Dividing from each vertex to the midpoint of the opposite side; iii) "It has 12 cm. At the middle of the triangle it is not 12, for sure. But it could be 4. Dividing these parts... Because they are tangent... I can tell they are the same length". While Marco is thinking aloud, he 'interacts' with the figure on the screen: he points, estimates distances, hides areas with his hands. Developing a visual approach to the problem, Marco considers the possibility of decomposing the figure mentally simulating different transformations – cutting, reorganizing, and recolouring.

He finally conjectures: "If we draw a triangle here (...) this is an enlargement of the other triangle. If it is 12, 12 divided by 3, [is] 4... Maybe the radius of the smaller circle is 2". By this, Marco is considering the construction of a small equilateral triangle at the top of the given one. This triangle is obtained using the Snipping Tool: Marco sets up a region at the top of the original triangle and saves the new image as a separate file. He uses the same process to obtain an image similar to the original triangle (*integrate*).



A) Pastes the two images cropped.

B) Covers the red circles with yellow.

C) Paints the central circle in red.

Figure 2. Three steps of the image processing with MS Paint

He then pastes both these triangles in MS Paint (Figure 1A) and attempts to overlap the two of them. As the images have a white background, the overlapping is not satisfactory for him, so he decides to edit the original triangle to make it similar to the smaller one by removing the red circles (Figure 2B) and recoloring the central circle, initially green, as red (Figure 2C). Marco is developing and exploring a conceptual model for explaining the similarity of these two triangles (*explore*) that will guide him in the construction of the solution. At a certain point, he opens a blank spreadsheet. Then, never 'leaving' the computer screen and without resorting to any other exterior tool – neither a notepad nor a pencil – Marco keeps moving between the website, which displays the problem, the image processing tools, and the spreadsheet, where the

solution will be expressed. (*plan*). The original image and the two manipulated figures (Figure 4C) become a mathematical argument that he resorts to while assembling his answer in the spreadsheet. These images support his understanding of the problem specifically the way he envisions the similarity between the two triangles. By integrating mathematical ideas, related to similarity of triangles, and the deconstruction of the triangle by means of the editing tools, he reaches a conceptual model of the situation (*create*). Actually, the spreadsheet contains the three images and a verbal text where he reports the whole process.

...that smaller triangle is a reduction of the larger triangle; since it is a reduction all I have to do is 12:3 (which is twice the radius of the green circle plus the height of the smaller triangle) and I got 4, which is the radius of the green circle; as the smaller triangle is a reduction of the larger one and its height is 4, to obtain the radius of the red circle one must divide 4:3 which is  $\frac{4}{3}$ . (Excerpt of Marco's written part of the solution).

As he engages in writing an explanation and analysing the images processed (*verify*) Marco reaches the solution to the problem – the radius of the smaller circle is  $\frac{4}{3}$  – which is actually different from the conjecture that he formulated at the beginning and that guided his approach. Throughout the process, Marco occasionally interacts with the researcher for clarification of wording (*communicate*) and, when finished, he submitted his answer to the competition using the editor embedded in the SUB14's webpage (*disseminate*).

Marco's initial activity seems to have a recurrent nature, where each argument is formulated as he tries to make sense of the mathematics that may be relevant (notices) and considers mathematical ways of approaching the solution (interprets) while he interacts with the figure on the screen. This cyclic activity leads Marco to a final conjecture – “the radius of the smaller circle is 2” – which is his first answer to the problem and will trigger the subsequent exploration activity. Marco's success in achieving the solution to the problem seems to be related to his ability in recognizing the affordances of the selected tools, which empower his thinking process, and, ultimately, influence the expression of his reasoning. Starting with exploring the first conjecture, Marco's elaboration of images in the graphic environment leads him to find the correct ratio of similarity. The move to the spreadsheet environment supports the combination of objects because it affords an easy organization of images and textual inscriptions (the images move freely, formatting is easy as well as cell merging).

## DISCUSSION AND CONCLUDING REMARKS

The analysis presented above shows that unconventional tools, such as Paint or the Snipping Tool, can be used efficiently to develop mathematical understanding that becomes crucial for finding and expressing a solution to a problem. Cropping, reconstructing or recolouring images lead to the creation of new objects that convey mathematical and technological understanding of the situation. These new objects not only contribute decisively to finding the answer, but they also become a crucial part of the solution as they allow to establish a roadmap to the approach developed (Lesh &

Doerr, 2003).

Moreover, the effectiveness of these technological tools as ‘problem solving tools’ seems mainly arising from the digital representations they afford, which allow manipulating images and for this reason can foster a geometrical interpretation of the situation that, in turn, enhances the development of a conceptual model. Marco’s elaboration of images played a paramount role in every phase of the processes of mathematization and expression of thinking. In fact, the development of his mathematical thinking seems to take advantage of the affordances of the tools that he found helpful in finding the solution to the problem. This is in line with the theory of affordances, namely the most recent developments that contribute to explain human-computer interaction (Norman, 2013).

Additionally, the youngster’s constructions and explanations are crucial elements that assume a double role: they simultaneously support the finding of the solution and the reporting of the answer. Thus, this case highlights the artificiality of the boundaries between solving the problem (i.e., the processes followed in obtaining the solution) and constructing the answer (i.e., the solution in the file to be submitted), since the mathematical thinking is developed in a continuum and is refined whilst the explanation is being produced. This strengthens the idea that solving and expressing are simultaneous mathematizing activities. Hence, solving-and-expressing is a way of accounting for the youngsters’ mathematization processes, particularly when technological and mathematical knowledge come into play in the development of an approach to the problems (Carreira et al., in press).

While there are powerful models that account either for the processes of mathematical problem solving, or for the processes taking place with digital tools in general tasks, the MPST model provides the means for describing problem solving with technology, letting the combination between mathematical and technological knowledge and skills to emerge throughout the whole process. The levels of description achieved within this model, grounded on the more general conceptual framework, allow to acknowledge the role of technological tools in mathematical problem solving, even when such tools appear deprived of mathematical affordances. Today’s real world problem solving activity, highly impregnated with digital tools, requires such a framework with a broader scope, capable of supporting the specificities of the digital tools considering their affordances in terms of the mathematical thinking needed for achieving an elegant solution and communicating it effectively.

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# DISCUSSING SECONDARY PROSPECTIVE TEACHERS' INTERPRETATIVE KNOWLEDGE: A CASE STUDY

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*This paper focuses on the mathematical knowledge mobilized by student teachers when interpreting and giving feedback to students' productions in a problem concerning powers of ten. The special kind of knowledge involved here was elsewhere called by us the interpretative knowledge. In this study we want to deeply investigate how the interpretative knowledge depends from and is intertwined with CCK and SCK (Ball et al., 2008). To this end, our sample of student teachers is taken here from a class of students of a Master's degree in Mathematics.*

## INTRODUCTION

Problem solving is largely recognized all around the world as one of the core mathematical activities/competencies that students must develop in their school curricula (e.g., National Council of Teachers of Mathematics, 2000). Much of this emphasis has its origin in the pioneering work of Pólya (1945) on different strategies and skills for solving problems.

Teachers and different aspects of their knowledge play an essential role developing and enhancing students' ability in problem solving (Ball, Thames & Phelps, 2008; Carrillo, Climent, Contreras & Muñoz-Catalán, 2013). First of all, teachers are themselves required to be able to solve problems. But this ability should be complemented with a particular kind of knowledge and awareness that would enable them to have a broader and deeper view of mathematics and its teaching. This particular and specialized content knowledge, linked with problem solving, also involves what we have called *interpretative knowledge* (e.g., Ribeiro, Mellone & Jakobsen, 2013). It corresponds to the knowledge required to interpret, make sense of, and explore the productions and comments of students—in particular the non-standard products and those that are based on errors.

The nature of and focus on using tasks in teacher education are an important contribution to the process of improving teachers' knowledge and abilities (see, e.g., Llinares & Krainer, 2006). In order to develop this interpretative knowledge in teachers, especially in prospective teachers, there is a need to design and implement specific tasks. In this paper we present a particular kind of task already used with prospective primary teachers (see, e.g., Jakobsen, Mellone & Ribeiro, 2014). The task is organized in the following sequence: First we ask prospective teachers to solve a problem. Then we ask them to make sense of students' solutions to the same problem and finally to propose fruitful feedback to students. The problem used here derives

from a “critical” item regarding operations with powers of ten, taken from the Italian national examination for grade 10 (Mellone, Romano & Tortora, 2013).

We proposed this task to math students attending a course in Mathematics Education, i.e., student teachers, in order to deepen our understanding of the nature of interpretative knowledge and of the mathematical aspects involved in it. In particular we want to address the following research question:

What knowledge is revealed by fourth-year university mathematics students when they are confronted with an interpretation task of students’ productions on a problem concerning powers of ten, and what can we learn from this in order to improve (prospective) teachers’ training?

## **THEORETICAL FRAMEWORK**

Since Shulman’s (1986) work, teachers’ knowledge with its specificities has received much attention in mathematics education communities. When studying such specificities, several approaches can be taken. One approach is to consider the domain of pedagogical content knowledge as a core part (e.g., Krauss, Baumert & Blum, 2008).

Another approach is to focus more on the specificities of the mathematical aspects of teachers’ knowledge (e.g., Ball et al., 2008; Carrillo et al., 2013), recognizing in particular that teachers need a kind of mathematical knowledge different from that needed in other professions. In this study we refer to this last research trend. In particular, combining the work of Ball and colleagues (2008) on the conceptualization of the Mathematical knowledge for teaching (MKT) with the approach to errors and non-standard reasoning as learning opportunities (Borasi, 1996), the notion of interpretative knowledge was developed (e.g., Ribeiro et al., 2013; Jakobsen et al., 2014). This refers to the need for teachers to possess a rich and ample knowledge of the possible examples, strategies, representations, and errors for problem solving that allow them to make sense not only of solutions similar to their own, but also of students’ solving processes different from their own. This implies a complex, deeper, and more ample mathematical knowledge than knowing for oneself; rather, it requires a mathematical knowledge that shapes teachers’ ability to support students in building their mathematical knowledge from their productions, as well as from the not standard or incorrect one. Indeed, according to Borasi (1996), we perceive ambiguity, anomalies, and contradictions as elements that would be highlighted and capitalized on as a motivating force by teachers. But this obviously requires very special mathematical sensitiveness and awareness by teachers. In a previous research study we recognized the interpretative knowledge linked to both Common and Specialized Content Knowledge (CCK and SCK) sub-domains of MKT model (e.g., Ribeiro et al., 2013; Jakobsen et al., 2014). In this study we want to deepen knowledge of which roles are played by CCK and SCK in student teachers interpretation knowledge/capacity, by focusing on student teachers with a strong CCK.

Powers of ten and the operations involving them is a crucial mathematical topic with transversal interest. Indeed, we can recognize, for example, its connections with

number sense, decimal system of number representation and multiplicative structure, but at the same time, we can also trace its powerful use in the scientific notation of magnitudes. Many studies reported students' difficulties on powers, and in particular on powers of ten (e.g., Mellone et al., 2013). The students' difficulties, teacher's role in students' learning and the specificities of teachers' knowledge led us to place our inquiry in this specific mathematical topic.

## CONTEXT AND METHOD

In previous studies (e.g., Ribeiro et al., 2013; Jakobsen et al., 2014) we explored the nature of the interpretative knowledge of prospective primary teachers during their professional training for becoming teachers. In this study we decided to change our sample; in particular, we chose to examine interpretative knowledge of mathematical students who had no specific training as teachers at the time of the study. Indeed, our aim was to deepen our understanding of the nature of interpretative knowledge and, in particular, of the mathematical aspects involved in it, and the sample population for this study is supposed to have strong knowledge in mathematics.

The expression  $10^{37} + 10^{38}$  is also equal to:

A.  $20^{75}$

B.  $10^7$

C.  $11 \cdot 10^{37}$

Figure 1: Item taken from the annual Italian national assessment for grade 10.

For this study we designed a task using an item (Figure 1) from the annual Italian national assessment for grade 10 by INVALSI (Istituto Nazionale per la VALutazione del Sistema educativo di Istruzione e di formazione) as a starting point. This item was from the year 2010-2011 and it was one of those where the Italian students encountered major difficulties. In a previous research work (Mellone et al., 2013) the same item was administered to students in grade 10, using a slightly different approach. In particular, students had more time to answer the item and were asked to support their given answer. Some of these collected students' productions have been used in the task for the study we are presenting here.

The task was administered to 32 mathematics students during the Mathematics Education course, placed in the fourth year of the Master's Degree in Mathematics (in Italy). Most of the students attending this course want eventually to become secondary teachers, and for this reason we can call them student teachers (ST). Following the kind of task we designed in a previous research work (e.g., Jakobsen et al., 2014), we first asked the ST to solve the item by themselves. Afterwards seven different students' productions to the same item were given. Some of those productions contained errors

while others were mathematically valid but followed non-standard procedure. The ST were asked to reflect and comment on the mathematical correctness of these productions, and afterwards to propose possible feedback that could be given to the students in order to support their mathematical learning.

Due to space constraints, here we include only two of the students' productions presented in the task and the ST's reactions to those two. Emanuela's production (Figure 2a), although providing a right answer, contains three different mistakes:  $10^{37} + 10^{38} = 10^{37} + 100^{37} = 110^{37} = 11 \cdot 10^{37}$ . The first error is  $10^{38} = 100^{37}$  in which we can recognize a wrong use of parentheses:  $10^{38} = 10 \cdot 10^{37} \neq (10 \cdot 10)^{37} = 100^{37}$ ; the second one is  $10^{37} + 100^{37} = 110^{37}$ , a wrong application of linearity  $(a + c)^b \neq a^b + c^b$  (the same typical error which occurs in the square of a binomial  $a^2 + b^2 \neq (a + b)^2$ ); the third one is  $110^{37} = 11 \cdot 10^{37}$ , again a wrong use of parentheses:  $110^{37} = (11 \cdot 10)^{37} \neq 11 \cdot 10^{37}$ .

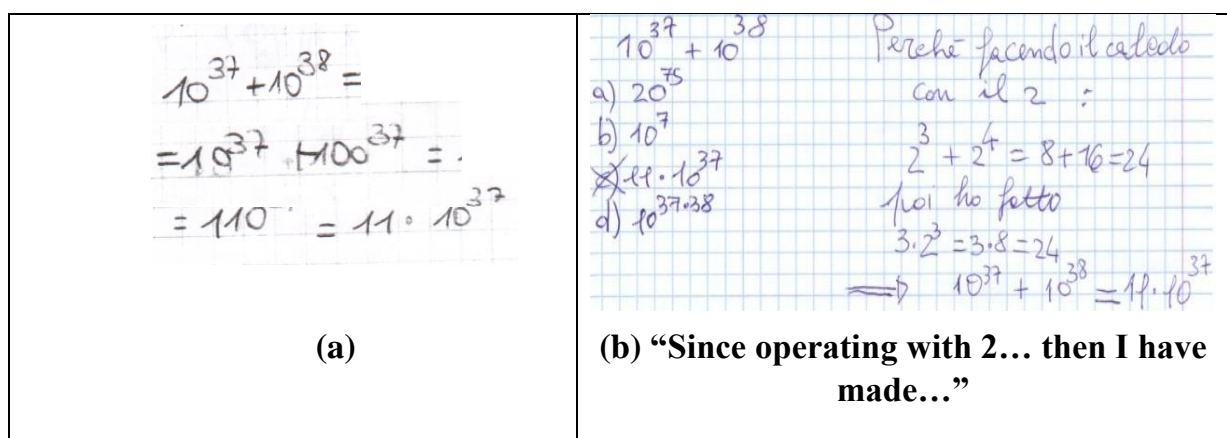


Figure 2: Emanuela's production (a) and Giuseppe's production (b).

In Giuseppe's production (Figure 2b) the right answer is reached using a procedure that has some algebraic features, but stays only in the arithmetical domain. It seems that the sum  $10^{37} + 10^{38}$  appears to Giuseppe as a particular case of an expression containing two variables, that is,  $a^n + a^{n+1}$ . Denoting formally  $P(a, n)$  the equality  $a^n + a^{n+1} = (1 + a)a^n$ , then  $P(10, 37)$  corresponds to  $10^{37} + 10^{38} = 11 \cdot 10^{37}$ , which is the right answer for the item. In his production Giuseppe checks if  $P(2, 3)$ , meaning  $2^3 + 2^4 = 3 \cdot 2^3$ , is a valid equality and then, using the implication  $P(2, 3) \Rightarrow P(10, 37)$ , he provides the right answer. This implication is not valid, unless we regard  $P(2, 3)$  as a generic example (Balacheff, 2004). Of course, we don't know how aware Giuseppe was of this, but, as a matter of fact, the presence of the right answer in a multiple choice task can be a good reason for employing different cases as generic, as Giuseppe does.

Emanuela's and Giuseppe's productions are perceived as an excellent prompt to discuss, for example, the perception of the powers rules and their meaning, as issues connected to the meaning of example, argumentation and proof.

Three categories emerged from the ST' answers to the second part of the task:

- (i) *absence of interpretation of students' solving processes*: when no comment is made for students' productions or when ST explicitly mention they cannot interpret it;
- (ii) *interpretation of students' solving processes as incorrect*: when words like "incorrect", "not right", "wrong", "error", "mistake", or "inadequate" appear in the comments concerning all or part of the solving process presented;
- (iii) *interpretation of students' solving processes as correct*: when words like "correct", "right", or "adequate" appear in the comments concerning all or part of the solving process presented.

For the request of the second part of the task in which ST were asked to provide feedback to support solvers' (students') mathematical learning (in seeing their errors and reaching a correct answer or the possibility of using a "generic" example), we offer an overview and a qualitative analysis of ST' answers.

This task was used both as a tool to access and deepen ST' interpretative knowledge mobilized by this prompt, as well as a tool to support ST in developing their MKT. Indeed, after applying the task, a lesson of three hours (audio recorded) was dedicated to discuss and reflect upon the given students' productions. Here we focus only on ST' written answers.

## RESULTS

For the first part of the task (solve the item for themselves), foreseeable since our sample was comprised of mathematics students, all of them provided the correct answer to the item. Concerning the second part of the task, we will separately present the ST' answers to the two productions discussed in this paper.

For Emanuela's production, two ST provided no interpretation, 26 considered her answer incorrect and four considered her answer correct. Moreover, it is important to note that only two ST detected the three errors contained in her production, referring to them explicitly. Some of the other ST considered some of the wrong manipulation as correct, or referred only to some of the errors, for example:

- |     |  |
|-----|--|
| ST1 | The result is correct but the process is not adequate. She decomposed $10^{38}$ in $100^{37}$ and it is right but the next step is not using correctly the property of powers because $10^{37} + 100^{37}$ is not equivalent to $110^{37}$ , the error is in adding the bases.                         |
| ST2 | Emanuela did several conceptual mistakes, maybe also by writing $10^{38} = 100^{37}$ ? If we write $100^{37}$ as $(10^2)^{37} = 10^{74} \neq 10^{38}$ . Moreover, in the next step she has $110^{37}$ , which she wrote as $(11 \cdot 10)^{37}$ raising to the power of 37 only the 10 and not the 11. |

These explanations, together with the four ST who considered Emanuela's solving process as correct, reveal some gaps in these ST' CCK. Regarding the request of the task concerning providing feedback to solvers, only eight ST answered. Moreover, eight ST answers considered providing feedback as showing some examples (counter-

examples) with different numbers, both in terms of the bases and exponent used, or to brush up the rules of operating with powers:

ST3 Calculate  $10^3$  and  $100^2$  and compare if they are the same thing?

ST4 I think that for all the errors made are due to a mix up of the powers. She is confused— she cannot distinguish what to add and what to leave equal — she cannot distinguish the exponent and the base.

For Giuseppe's production, eight ST didn't give any interpretation, 12 considered his answer incorrect and 12 considered his answer correct. Among the ST who provided no interpretation for Giuseppe's production, some of them explicitly mentioned they did not understand the process followed and assumed he copied, as can be seen in the following answer:

ST5 I cannot follow his reasoning. It is completely different from what I would do. I think that the steps in his reasoning completely lack connections. For this reason, I think that he copied without not even understand what he has written.

Some of the ST who considered Giuseppe's answers incorrect presented justification for such judgment, mentioning the lack of rigor in his argument:

ST6 The answer is correct, but the reasoning is not general and rigorous enough, so for me it is not mathematically correct.

Those ST considering Giuseppe's answer correct provided some interesting attempts to translate the presented reasoning in an algebraic language. Some examples are:

ST7  $a^n + a^{n+1} = a^n + a^n a = a^n(a + 1)$  is a correct reasoning but I would make him work more with the powers of 10.

ST8 He tried to see what happens with smaller base and then he generalizes he, saw that  $x^y + x^z = (x + 1)^{\min(y,z)}$ .

Regarding the request for providing feedback, only four ST answered. Although these ST mentioned explicitly they did not understand the reasoning, they provided some generic indications, like the following:

ST9 I don't know what to ask Giuseppe, since I don't understand his reasoning, and in the case he copied I advise him to not do it again because in real life this would not lead him anywhere.

As we can see in this last advice, as in the previous comment of ST who could not interpret Giuseppe's production, refers to the possibility of copying from a classmate. Similar comments were found in eight other ST' answers.

## DISCUSSION AND FINAL COMMENTS

Although all the ST in our sample correctly answered the given item, they had problems when mobilizing their mathematical knowledge for interpreting students' productions. Indeed, they were able to "see" some of the mathematical aspects

involved in these students' productions, but a large amount of other aspects remained hidden to them.

As we tried to show previously, Emanuela's and Giuseppe's productions are rich of mathematical and psychological issues like perception and meaning of arithmetical rules, argumentation and proof. We found very few references to these issues in ST' answers. In particular in giving sense to Emanuela's production the ST stayed only at a descriptive level by illustrating some or all of her errors. Moreover, the ST's feedback to her production are basically built on the proposal of showing examples that prove the incorrectness of her computation. None of the ST discussed, for example, the rationale and the meaning of the power rules, or suggested the opportunity to explore them with the student to a deeper extent.

For Giuseppe's production, even if our sample is quite small, it is interesting to notice that almost a third of the ST can't follow his reasoning. For the ST who were able to give sense to his production, we found that some of them referred to Giuseppe's particular use of the example with powers of two, but none of the ST proposed a way that could rely on this to support the student in developing his mathematical knowledge. Moreover, among the ST who had problems to interpret his production, many did referred to the possibility of copying from a classmate. This could reveal a sort of fear of moving toward a path they do not (immediately) control.

The ST's difficulties in providing feedback to students' productions are an indication of the need to improve mathematics teachers training in this direction. Moreover, the observed difficulties reinforce the importance of such kind of professional learning task (Smith, 2001) and of the space of reflection and work opened by it (Ribeiro et al., 2013; Mellone, Jakobsen & Ribeiro, 2015). Indeed, this task was also used in the following classes of the Mathematics Education course to further work with the ST on the mathematical aspects involved in these students' productions and on their MKT.

Our analysis shows that, although in their schooling the ST had explored advanced mathematical topics, most of them were not able to transpose and to exploit such advanced knowledge when interpreting others' solutions. This supports the idea that the interpretative knowledge is more linked to the SCK than to the CCK. Moreover, these results open a new space of inquiry, that is to identify which components of the mathematical knowledge better support teachers in two crucial abilities of their work: namely to interpret students' solutions, and to develop students' mathematical knowledge by starting from their productions.

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# **PREDICTING EARLY DROPOUT FROM UNIVERSITY MATHEMATICS: A MEASURE OF MATHEMATICS-SPECIFIC ACADEMIC BUOYANCY**

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*At University, mathematics freshmen often drop their studies as a consequence of the excessive demands that they felt not able to cope with. Academic buoyancy describes students' ability to effectively handle academic challenges and setbacks and may therefore be an important factor to be considered when examining freshmen dropout. As a consequence, we conceptualized academic buoyancy in the context of college mathematics. This study focuses on the development and initial empirical evaluation of an 11-item questionnaire assessing mathematics-specific academic buoyancy. Analyses on internal consistency and on structural, content, predictive, convergent and discriminant validity are reported. Overall, our findings suggest the instrument to provide reliable and valid measures of mathematics-specific academic buoyancy.*

## **INTRODUCTION**

For mathematics studies, universities have to face high dropout rates (Heublein, Hutzsch, Schreiber, Sommer, & Besuch, 2010; Chen, 2009). In Germany, for instance, about 38% of college students leave their studies during the first year (Dieter, 2012). Students often explain their decision to quit with the excessive demands they encountered and not felt able to cope with (Heublein et al., 2010). Therefore, persisting the study of mathematics not only seems to be due to cognitive abilities, but also due to the individual's ability to handle challenges encountered during the studies. With respect to education in school, an individual's ability to cope with everyday setbacks, challenges, and pressures in a learning context has been defined as academic buoyancy (Martin & Marsh, 2008). Having said this, the idea of academic buoyancy might be helpful to describe mathematics freshmen's ability to handle challenging circumstances, as well.

## **THEORETICAL BACKGROUND**

In the transition from school to university mathematics students often recognize major changes in the way of learning mathematics; and, not rarely, these changes lead to difficulties students have with the studies of mathematics (Hoyles, Newman, & Noss, 2001). The goal of college mathematics, for example, is to promote the character of mathematics as a scientific discipline, whereas the goal of school mathematics is to promote general education; likewise, college students need proving skills and deductive logic as tools of the trade, whereas school students at school mostly focus on performing calculation schemes (Hoyles et al., 2001; Rach & Heinze, 2011). Among first semester students, however, proving skills are often only poorly developed and as a consequence, difficulties in writing proofs occur (Brandell, Hemmi, &

Thunberg, 2008). This situation is aggravated by the fact that students are often obligated to work on weekly homework assignments (Rach & Heinze, 2011). With respect to the scientific nature of college mathematics, those assignments usually require proving skills and can therefore be very challenging. Overall, mathematics freshmen seem to encounter major challenges concerning the character of mathematics and in particular the character of mathematical exercise assignments.

In the context of school education, the construct of academic buoyancy has been introduced to describe “students’ ability to successfully deal with academic setbacks and challenges that are typical of the ordinary course of school life” that is “students’ everyday academic resilience” (Martin & Marsh, 2008, p. 53). Academic buoyancy therefore refers to everyday adversities, setbacks and pressures experienced by students in an educational context such as poor grades, difficult schoolwork, competing deadlines or exam pressure (Martin & Marsh, 2008). Up to now, however, research on academic buoyancy has only focused on school students and has only been operationalized from a general perspective. What is missing, are conceptualizations of academic buoyancy in the context of college education, and specified for particular disciplines. Given the fact that for many mathematics freshmen at university everyday adversities seem to affect their decision to quit the studies, and given the lack of research on academic buoyancy for this target sample, the present study reframes the concept of academic buoyancy for the context of college mathematics by introducing the construct of mathematics-specific academic buoyancy.

## **MEASURING MATHEMATICS-SPECIFIC ACADEMIC BUOYANCY**

Assuming that most of the mathematics freshmen experience setbacks and frustration due to the compulsory homework assignments, we conceptualized mathematics-specific academic buoyancy in the context of these assignments. In particular, we conceptualized five situations, in which we think mathematics-specific academic buoyancy comes into play: 1) When students keep working on an exercise persistently even when there is no perceptible progress in learning or solving the exercise, 2) when students start working on exercises again and again even if they failed on these exercises before, 3) when students work persistently on exercises (as in 1 and 2) even if they are not interested in the content of the exercise or 4) the learning goal is unclear, and 5) when students keep studying mathematics despite of difficulties in solving the given assignments. Based on this conceptualization, we developed a self-evaluation questionnaire, the “Measure of Academic Buoyancy – Mathematics (MAB-M)”. The MAB-M consists of eleven items addressing the five above-noted aspects (see Table 1). Each item contains a statement and a 7-point rating-scale (1 = “strongly disagree”, 7 = “strongly agree”).

Given the fact that academic buoyancy has not been investigated in the context of university education so far, and, thus, no instrument assessing mathematics-specific academic buoyancy is known yet, the quality of the newly developed instrument above needs to be investigated in-depth. In particular, there is no evidence available on

whether there is something like mathematics-specific academic buoyancy or how it can be measured from an empirical perspective. Hence, the present study focussed on an initial empirical evaluation of the MAB-M and approached the following research question: To what extent does the MAB-M provide reliable and valid measures of mathematics freshmen's mathematics-specific academic buoyancy?

Item No.	Statement
1	Math problems which need hours just for the basic idea how to solve them are not for me.
2	I don't mind spending a whole afternoon or longer on a complicated math problem.
3	I don't like to start extremely difficult assignments that even with a team require several sessions to solve.
4	Even if I don't know how to solve a difficult math problem after several tries, I keep trying to solve it.
5	If I don't see any progress in solving a math problem even after three attempts, I give up.
6	I am quick to drop a less interesting math problem if I don't know how to approach it.
7	Even if the problem is from a less fascinating mathematical topic, I won't drop an assignment after several failed attempts.
8	I persistently work on a math assignments even if I don't deem them useful.
9	If I don't see a learning objective of a difficult math assignment, I'll just drop it after 1 or 2 attempts.
10	If after some weeks' time I am still unable to solve advanced problems as well, I'll give up on studying mathematics.
11	Even if I fail difficult assignments again and again, that won't stop me from studying mathematics.

Table 1: Items of the MAB-M

## METHODS

### Design and Sample

In order to gain insight in the empirical quality of the newly developed instrument, we gathered data of  $N = 661$  mathematics freshmen (57% females, 42% males, 1% missing; mean age 20.4 years,  $SD = 3.6$  years). The overall sample is made up by two sub-samples, which will be described in more detail.

Sample 1 contained mathematics freshmen, who started their studies in 2014 at Kiel University (Germany). Here, data were collected in scope of a mathematics preparatory course prior to the first semester (measurement T1,  $N = 85$ ) and at the beginning of the second semester (measurement T2,  $N = 91$ , 48 of these participated in measurement T1 and T2). At T2, data collection was performed in scope of the calculus tutorials, which are obligatory for every student who intends to complete the first year. The data of  $N = 48$  students who participated in data collection of both T1 and T2 were used for longitudinal analyses.

Sample 2 included mathematics freshmen from Munich University ( $N = 292$ ), the Royal Institute of Technology in Stockholm ( $N = 50$ ), and from Kiel University ( $N = 234$ ), who started their studies in autumn 2015. Data were collected in the first weeks of the semester. A second assessment for longitudinal analyses will be performed in spring 2016 (i.e. by the end of the first semester, or the beginning of the second semester respectively).

### **Instruments**

Additional to the MAB-M, we employed five other instruments in order to investigate different validity aspects. To investigate convergent and discriminant validity, we administered four self-evaluation instruments, (1) a well-established questionnaire assessing general resilience (see Wagnild & Young, 1993;  $\alpha = .83$ ), (2) Big Five personality scales (see John, Donahue, & Kentle, 1991;  $\alpha$  from .68 to .81), (3) a measure on mathematics-specific self-concept (Kauper et al., 2012; i.e. “I am good at mathematics”;  $\alpha = .8$ ), and (4) a measure on interest in mathematics (adapted from Köller, Baumert, & Schnabel, 2001; i.e. “I enjoy working on mathematical problems”;  $\alpha = .83$ ). Instruments (1) and (2) were employed in both samples, instruments (3) and (4) in a subsample of sample 2 only (that is within  $N = 526$  students from Kiel and Munich University). All instruments were employed at the first measurement, that is, at the beginning of the first semester 2014 or 2015 respectively. Each item of the employed instruments consisted of a statement and a Likert-scale (instruments 1 and 2: 7-point scale; instruments 3 and 4: 4-point scale).

To investigate content validity, we designed a special questionnaire (5) for the second measurement (i.e. at the end of the first semester resp. the beginning of the second): Based on a list of 8 key challenges, which mathematics freshmen have to face in their mathematics studies (e.g. preparing for the examinations, visiting the lectures, working on homework assignments), the participants were asked to indicate the extent to which those activities stress them on a 10-point rating scale. Besides, at measurement T2, we also assessed students' grades in the final examinations of the first semester to investigate predictive validity. Since the calculus tutorials – where we performed the second study – are compulsory for second semester students, we additionally used the presence in these tutorials as an indicator for dropout. Note, that data on the rating of challenges, the grades and the presence in tutorials is, by now, only available for sample 1, but will be collected for sample 2 in spring 2016 as well.

## RESULTS

In order to address the research question, we analysed the gathered data with respect to structural, convergent, discriminant, content and predictive validity and reliability.

### Structural validity

To investigate the structure of the newly developed instrument, we performed a principal component analysis (PCA). Prior to that, we checked and found the sample adequate (Kaiser-Mayer-Olkin test,  $KMO = .9$ ) and the correlation matrix significantly differing from the identity (Bartlett test,  $\chi^2(55) = 2674.5$ ,  $p < .001$ ). The actual PCA then indicated a one-factor structure of the instrument. The factor loadings of the items were  $> .52$  and the identified component explained 44% of the variance.

### Convergent and discriminant validity

To investigate convergent and discriminant validity, we correlated students' mathematics-specific buoyancy measures with their measures of general resilience and the Big Five personality traits (Table 2). We found the MAB-M to correlate with general resilience ( $r = .37$ ,  $p < .001$ ) and conscientiousness ( $r = .34$ ,  $p < .001$ ). These findings indicate that the respective constructs are related but clearly distinct from each other. Evidence for discriminant validity was provided by the fact that there were no more than weak correlations of the MAB-M with extraversion, openness, neuroticism and agreeableness. Furthermore, we found only moderate correlations of MAB-M with mathematics specific self-concept ( $r = .33$ ,  $p < .001$ ) and students' interest in mathematics ( $r = .59$ ,  $p < .001$ ) in a subsample of sample 2 ( $N = 526$  students from Kiel and Munich University).

	(1)	(2)	(3)	(4)	(5)	(6)
(1) MAB-M						
(2) General Resilience	.37***					
(3) Conscientiousness	.34***	.41***				
(4) Extraversion	-.01	.27***	.08*			
(5) Openness	.19***	.34***	.12***	.15***		
(6) Neuroticism	-.13***	-.22***	.01	-.11**	.00	
(7) Agreeableness	.10**	.21***	.16***	.07	.10**	-.06

Table 2: Pearson correlations for MAB-M, General Resilience and personality scales;  
Key: \*  $p < .05$ ; \*\*  $p < .01$ ; \*\*\*  $p < .001$ .

### Content validity

In order to investigate to which degree the weekly exercises in fact are viewed as a main factor of pressure in the first semester, we asked  $N = 91$  second semester students of sample 1 to indicate to which extent they were challenged by different key aspects of the mathematics studies. The results suggest that, from the view of mathematics students, the most challenging activities in the first semester are preparing for

examinations ( $M = 8.5$ ) and working on homework assignments ( $M = 8.3$ ). None of the other aspects were rated comparably high ( $M < 5.5$ ). Since it can be assumed that the preparation for examinations may be a daily activity but not a daily pressure, this finding underpins the hypothesis that the weekly exercises are the most pressing daily activity of the first semester.

### **Predictive validity**

In order to investigate predictive validity, we analysed the data of  $N = 85$  students of sample 1 from the first measurement T1 and the second measurement T2 regarding dropout in the first semester. We identified 48 students in both, T1 and T2, thus indicating that these students persisted in the studying of mathematics. The remaining 37 did not participate in a compulsory course of the second semester at the time of measurement. Logistic regression revealed that students' probability of participating in the second semester is significantly higher with higher academic buoyancy measures at the beginning of the first semester ( $\exp(B) = 1.66$ ,  $SD = 1.28$ ,  $p < .05$ ). This corresponds to a significant biserial correlation of  $r_b = .3$  ( $p < .05$ ) between the MAB-M measures and the persisting in mathematics studies. These findings should be considered only tentatively indicating predictive validity, given the only small number of data at hand.

### **Reliability**

The MAB-M showed a good internal consistency (Cronbach's  $\alpha = .87$ ). The internal consistency did not increase if one of the items was excluded from the measurement. We therefore kept the instrument as is.

## **DISCUSSION AND CONCLUSIONS**

The main aim of the present study was to adapt the construct of academic buoyancy to the context of mathematics freshmen, and to provide an instrument assessing this construct. Applying the newly developed instrument, a principal component analysis showed that the MAB-M can be considered as unidimensional. Moderate correlations of the MAB-M with a valid and reliable measure of general resilience indicated evidence for convergent validity. The fact that we did not find a higher correlation may be explained by the theory that resilience concepts (e.g. academic buoyancy) depend on the context of the challenges (e.g. Weber, Glück, Sassenrath, & Heiss, 2003). Hence, a student may behave resilient in one situation and vulnerable in another. Additionally, the MAB-M correlated moderately with the Big Five factor conscientiousness. Given the fact that conscientiousness includes facets like self-discipline, dutifulness and achievement striving (Costa, McCrae, & Dye, 1991), the found correlation provides evidence for convergent validity as well. The other Big Five personality scales were found to correlate with the MAB-M only weakly, if at all; this finding provides evidence for discriminant validity of the MAB-M. Furthermore, it was possible to separate the MAB-M measures from students' interest in mathematics and mathematics-specific self-concept. A survey of second semester students revealed that homework assignments actually are the most pressing daily activity in the first

semester, indicating content validity of the instrument. First evidence for predictive validity was found since the MAB-M proved to be able to predict students' dropout in the first semester. A good internal consistency revealed evidence for the reliability of the questionnaire. Overall, our findings indicate that mathematics-specific buoyancy, in fact, seems to be a latent ability to successfully deal with everyday academic setbacks and challenges that are typical of mathematics studies. Moreover, the respective newly developed MAB-M seems to provide valid and reliable measures of this new construct.

Despite these promising findings, our study faces some limitations. The most pressing limitation lies within the only small sample we consulted for the search for predictive and content validity. These evidences should therefore be considered as only tentative with the need for corroboration in larger samples as well. Such corroboration may be found in the data still to be collected for sample 2 at the second measurement.

Nevertheless, our study provides first evidence that the MAB-M could be helpful to identify mathematics-specific academic buoyancy as an important factor of mathematics students' success in the first semester. In particular, our study suggests that academic buoyancy could turn out an important factor in modelling college dropout. Hence, future research on academic buoyancy could be useful to counteract the problem of dropout from mathematics studies. As such, our study contributes implications for further research. For example, little is known about whether mathematics-specific academic buoyancy can be trained. To this end, an experimental study could employ a treatment fostering freshmen's mathematics-specific academic buoyancy compared to a control group and investigate whether the treatment in fact results in higher buoyancy measures and smaller dropout rates. Likewise, future research should investigate in even more detail which learning conditions and demands exactly are challenging for mathematics freshmen. Both these ideas also imply practical contributions. For example, if the treatment on mathematics-specific buoyancy is in fact effective, it could be included as an element of bridge courses or tutorials in the first year of study. Knowing more about the challenging learning conditions might influence how college teaching of mathematics could be improved to decrease the need for mathematics-specific academic buoyancy. Hence, with better support and adequate demands put on them, even students with low academic buoyancy could be able to master this crucial phase of the mathematics studies and exploit their whole potential as future mathematicians and mathematics teachers.

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# AN ALTERNATIVE APPROACH TO ASSESSING ACHIEVEMENT

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*Traditional exams typically assess general achievement by testing procedural knowledge across a sample of mathematical domains. Here we explore whether achievement can be assessed by testing conceptual understanding across domains. This follows previous work in which we showed that comparative judgement, based on pairwise expert judgements of students' work rather than rubrics and scoring, can be used to measure understanding of a specific concept (e.g. fractions). In the present study, school students ( $N = 197$ ) sat open-ended tests sampling a range of concepts, and their responses were comparatively judged. Analysis supported the validity of the approach for assessing general achievement. We conclude that comparative judgement could help improve the assessment of mathematics.*

## INTRODUCTION

Summative mathematics tests, such as examinations at the end of secondary schooling, typically comprise short questions sampling across mathematical domains. The question scores are then summed to produce a measure of overall mathematical achievement. However, questions tend to test the recall and application of facts and algorithms, thereby privileging procedural knowledge (e.g. Noyes, Wake, Drake & Murphy, 2011). Few questions test understanding of concepts and their interconnections, and as such conceptual understanding is underrepresented in assessments of overall mathematical achievement (Burkhardt, 2009).

One reason for this is that conceptual understanding is difficult to assess reliably. Scientific instruments, such as the Mathematical Equivalence Assessment (Matthews, Rittle-Johnson, McEldoon & Taylor, 2012), require painstaking work to develop and validate, and such a process is not practical for routine test production. In previous work we proposed a novel and efficient approach to measuring understanding of a given concept (Jones, Inglis, Gilmore & Hodgen, 2013). Students were administered tests that contained a short prompt followed by a blank page for a response (an example prompt and student response from the present study is shown in Figure 1). We observed that students produced a wide variety of response types, making use of symbols, diagrams and natural language to express their understanding.

Such open-ended tests can be assessed validly and reliably using a novel comparative

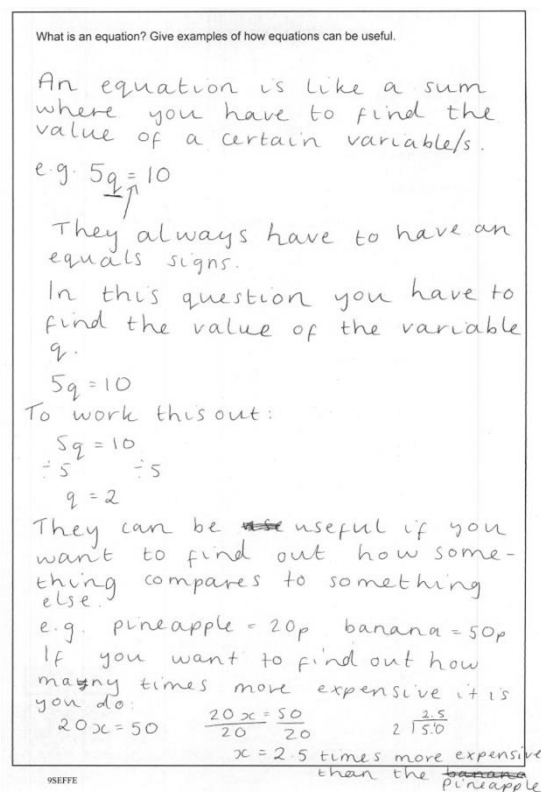


Figure 1: Example test question and student response.

judgement technique (Jones & Alcock, 2014; Jones et al., 2013). Mathematicians are presented with pairs of student responses and asked to decide which is “better” in terms of a global construct such as “conceptual understanding”. The decision data is fitted to a logistic model (Pollitt, 2012) to produce a parameter estimate of the “quality” of each response as collectively perceived by the judges. The parameter estimates can then be used for routine assessment procedures such as evaluating validity and reliability (Jones et al., 2013), and assigning grades (Jones & Alcock, 2014). Comparative judgement requires no scoring rubrics, and instead validity is grounded in the collective expertise of the judges making direct comparisons of students’ work. The approach is supported by a corpus of psychophysical research demonstrating that human beings are more consistent when judging one object against another than when judging a single object in isolation (see Laming, 1984).

Whereas in previous work we focussed on measuring understanding of single concepts (Jones & Alcock, 2014; Jones et al., 2013), here we explored aggregating parameter estimates across several tests to produce an assessment of overall mathematical achievement. The method described below was designed for research purposes and is not proposed as directly usable for routine summative assessment, nor were the open-ended tests designed as wholesale replacements for procedural questions. Rather, we evaluated whether a comparative judgement approach can yield a meaningful assessment of overall mathematical achievement, and so offer a complement to traditional testing procedures.

## METHOD

*Materials.* Twenty-five open-ended test questions were written by a researcher, covering the topics listed in Figure 2. An example is shown in Figure 1.

*Participants and test administration.* The tests were administered to students ( $N = 197$ ) aged 12 to 14 years in a large secondary school with a culturally and socioeconomically diverse intake. Students sat the tests in a single lesson under examination conditions and the supervision of their mathematics teachers. Teachers selected which tests to administer to their classes, and were advised to allow at least ten minutes per test. The outcome was 686 completed tests, and each student completed between 1 and 5 tests (mode = 4) over the course of 50 minutes. In addition, national test scores for mathematics and reading were obtained for 163 of the participants.

*Comparative judgement.* The 686 student responses were anonymised, scanned and uploaded to an online comparative judgement engine ([www.nomoremarking.com](http://www.nomoremarking.com)). Eleven mathematics PhD students were recruited, all of whom had undertaken judging work for previous studies, and allocated 600 pairwise judgements each. For each pair they were asked to decide, based on the evidence in front of them, which student was “the better mathematician”. Most pairings presented to the judges were student responses to two different test questions.

The judgement decision data were fitted to the Bradley-Terry model (Pollitt, 2012) to produce a final parameter estimate (mean = 0, SD = 2.1) for each test response. The internal consistency of the parameter estimates was calculated using the Scale Separation Reliability (analogous to Cronbach’s  $\alpha$ ) and found to be acceptably high, SSR = .87. Inter-rater reliability was estimated using a split-halves technique (iterations = 100) whereby the judges were randomly split into two groups, the decision data refitted to the Bradley-Terry model, and the resulting two groups of parameter estimates correlated. Inter-rater reliability was found to be acceptably high,  $r = .75$ . Note that this is an underestimate because the split-halves technique involves effectively discarding half the data in order to calculate a correlation coefficient.

*Response analysis.* To help evaluate the validity of the outcomes we investigated how judges made their decisions. In previous work we have done this through post-judging surveys and interviews (Jones & Alcock, 2014; Jones & Inglis, 2015). However, these methods were unable to access possible subconscious processes, and findings could not be linked directly to the actual judgement decisions used to produce student scores. In the present study we sought to overcome these limitations by classifying the student responses using an adapted coding scheme (Hunsader et al., 2014). For brevity the coding scheme is only summarised here and readers are directed to Hunsader et al. (2014) for further details.

### Coding scheme.

*Real world.* Response uses a context outside of mathematics (no coded 0, yes coded 1). The mean score for real world across all responses was 0.30.

*Connections.* Response introduces a relevant concept not explicitly prompted in the test question (no coded 0, yes coded 1). For example, if a student connected percentages and decimals in the question about percentages, this would be coded 1. The mean score for connections was 0.07.

*Graphics (graphs, pictures, tables).* No use of graphics (coded 0). Use of graphics that are superfluous to the mathematics (coded 1), explicitly illustrate the mathematics (coded 2), are required to interpret the mathematics (coded 3). The mean score for graphics was 0.71.

*Numbers.* Numbers may be present but not as part of an expression of equation (coded 0). Numbers used in an expression (e.g.  $3 + 2$ , coded 1), or an equation (e.g.  $3 + 2 = 5$ , coded 2). The mean score for numbers was 0.69.

*Letters.* Letters may be present but not as part of an expression or equation (coded 0). Letters used in an expression (e.g.  $x + 1$ , coded 1), or an equation (e.g.  $y = x + 1$ , coded 2). The mean score for letters was 0.20.

For illustration, the response shown in Figure 1 was coded “1” for the categories *real world* and *connections* because there is a non-mathematical context and interconnections between the concepts equation and ratio respectively. It was coded “0” for the category *graphics* because there are no graphs, tables or pictures. It was coded “2” for the categories *numbers* and *letters* because the response contains equations such as  $5q = 10$ . The codes were intended to be hierarchical, such that a score of (say) 2 for letters reflects a “more mathematically sophisticated” response than a score of 0 or 1. This enabled a total score (min. = 0, max. = 9) to be calculated for each response by summing across the codes (for example, the response in Figure 1 scored 5). The Spearman rank-order correlation between comparative judgement parameter estimates and coding scheme total scores was moderate,  $\rho = .34$ ,  $p < .001$ . This provides support for a meaningful relationship between the two measures.

## ANALYSIS AND DISCUSSION

The outcomes were evaluated in terms of criterion validity, the basis of judges’ decisions, and the performance of individual test questions.

*Criterion validity.* We explored the extent to which the parameter estimates reflected mathematical achievement rather than a mathematics-irrelevant construct such as written communication skills. To do this an achievement score was assigned to each student by calculating the mean parameter estimate across tests. Students who completed fewer than three tests ( $N = 23$ ) were removed from this analysis because our interest was in whether sampling across topics can be used to assess general achievement. Mathematics and reading scores based on national tests at the end of primary schooling were available for 148 of the students who sat three or more tests. We hypothesised that mathematics scores, but not reading scores, would be a significant predictor of mean parameter estimates. Multiple linear regression explained 22% of the variance,  $R^2 = .22$ ,  $F(2, 145) = 20.04$ ,  $p < .001$ . Mathematics scores

significantly predicted parameter estimates,  $\beta = .40$ ,  $t(145) = 4.94$ ,  $p < .001$ , but reading scores were not a significant predictor,  $\beta = .10$ ,  $t(145) = 1.02$ ,  $p = .310$ . This result lends support to the criterion validity of the assessment.

However, most of the variance in the data (78%) was not explained by this analysis. Three limitations of the available achievement data may have contributed to this. First, the national tests were taken two or three years prior to the open-ended tests. We repeated the multiple linear regression for the younger ( $N = 61$ ) and older children separately, and found these analyses explained 34% and 13% of the variance respectively. This suggests that, unsurprisingly, the national test data becomes less informative as children progress through secondary school. Second, the reading scores acted as a proxy for written communication skills and their suitability for this purpose could not be verified. Third, the mathematics scores were based on largely procedural tests whereas the parameter estimates were based on conceptual tests, and so the two sets of scores were intended to measure different, albeit related, constructs.

*Response analysis.* A multiple linear regression analysis was conducted to investigate the extent to which the classification of responses (coding scheme) predicted parameter estimates (comparative judgement). We included two further predictors that were evident to the judges when making their decisions. The first was test question, which can be expected to impact on assessors' perceptions of the mathematical quality of a response (Good & Cresswell, 1988). The second was file size (of the scanned responses), which acted as a rough proxy for the quantity written for each test response. The analysis explained 35% of the variance,  $R^2 = .35$ ,  $F(7, 678) = 52.39$ ,  $p < .001$ , and the results are summarised in Table 1.

Variable	$\beta$	95% CI
File size	0.50***	[0.01, 0.02]
Numbers	0.15***	[0.18, 0.50]
Graphics	0.12***	[0.12, 0.41]
Connections	0.11	[-0.04, 0.99]
Letters	0.05	[-0.04, 0.43]
Test question	-0.04	[-0.02, 0.00]
Real world	0.03	[-0.17, 0.45]

Table 1: Multiple linear regression results for features of student responses as predictors of parameter estimates. \*\*\* $p < .001$ .

File size was the strongest predictor. This is unsurprising given that the more that is written the more mathematics is visible to judges to influence their decisions. Student responses with the smallest file sizes typically contained little or no mathematics, such as “I don’t know”. However caution must be exercised. File size is only a proxy for “quantity communicated”. Moreover, variance in the length and style of question prompts will have contributed to variance in file size.

Connections was not a significant predictor, (although it was marginal at  $p = .70$ ). This has relevance for the type of conceptual understanding being assessed using these tests, which mainly focus on individual concepts rather than interconnections between concepts. The test questions used here perhaps should be enhanced by the use of unstructured or semi-structured problem-solving tasks that require students to draw on multiple mathematical domains (see Jones & Inglis, 2015). Another option is to develop more open-ended test questions similar to those in Figure 1, but which require the explicit connection of two or more concepts. In the present study, seven of the 25 questions were of the type “What are the differences between multiplication and division? Give as many examples as you can as to how and why they are different.” (Note that responses to these tests were only coded “1” for connections if students introduced a concept not prompted by the question).

As expected, the use of numbers was a significant predictor, as was the use of graphics. However, the use of letters was not a significant predictor, which is perhaps surprising given the high status of symbolic algebra in school mathematics. This may have been due to the relatively rare use of letters in the responses. Two test questions focussed on equations (as in Figure 1), and we might expect letters to be more prominent in the corresponding responses than for other questions. Indeed, the mean score for letters for the two equations tests ( $N = 73$ ) was 1.00, and the mean for the non-equations tests ( $N = 613$ ) was 0.11. The Spearman rank-order correlations between parameter estimates and letter scores for the responses to equations and non-equations tests were significantly different,  $\rho = .34$  and  $\rho = .08$  respectively,  $z = 2.17$ ,  $p = .03$ . Therefore we expect the use of letters to be more influential on judges’ decisions for test questions that explicitly focus on symbolic algebra.

Test question was not a significant predictor, suggesting a given student’s parameter estimate was not strongly dependent on which questions he or she sat. Figure 2 shows parameter estimates by question and reveals some variation. A one-way ANOVA revealed a significant difference across mean question parameter estimates,  $F(24, 661) = 5.07$ ,  $p < .001$ ,  $\eta^2 = 0.155$ , and post-hoc tests (Bonferroni corrected) suggested this difference was mainly driven by four questions close to the extremes of Figure 2 (namely “translate/rotate”, “equation 1”, “Pythagoras” and “enlarge”). This reinforces the importance of aggregating across several questions when estimating overall achievement. While “extreme” questions should be avoided, some variance in question performance is to be expected, and can be monitored.

Real-world applications have an important role in many mathematics curricula around the world, but we found that student use of real-world context was not a significant predictor of parameter estimates. This might be due to only a few of the test questions explicitly requesting real-world examples (three out of 25), or the research mathematicians who undertook the judging valuing pure over applied mathematics. These two hypotheses will be investigated in further work.

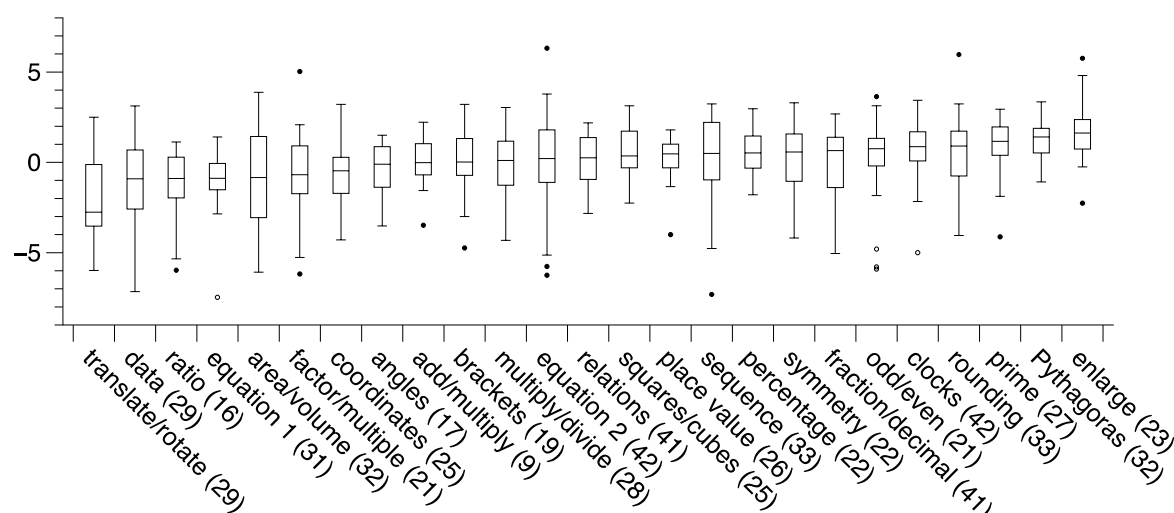


Figure 2: Boxplot of parameter estimates by test question. The number of student responses obtained for each question are shown in brackets.

We were unable to evaluate the possible influence of other features, including mathematically-irrelevant constructs such as neatness, or sought-after learning outcomes such as mathematical creativity. Such features will be investigated in future studies using adaptations of the methods reported here, along with standardised instruments from the literature (for example the Creative Behaviours in Mathematics Questionnaire, see Leu & Chiu, 2015).

## CONCLUSION

We investigated whether students' general mathematical achievement can be assessed using open-ended conceptual test questions and a comparative judgement technique. Analysis supported the validity of the approach in terms of students' national test scores and researcher classification of the test responses. Further work is required to better understand the features of responses that are most valued by expert judges. This will enable the design of test questions and judging procedures that maximise the validity of the approach, and therefore the confidence of stake-holders that outcomes are legitimate measures of overall mathematical achievement.

Assessment processes should match the objectives of curricula. There is a current focus in many countries on improving students' conceptual understanding of mathematics, and as such assessments should capture conceptual understanding. An approach based on open-ended test questions evaluated using comparative judgment has been described here, and our findings offer promise for complementing and enhancing the common practice of aggregating students' scores on procedural questions. This could lead to richer, more valid examination practices for which outcomes are based on both procedural knowledge and conceptual understanding. This in turn might lead to mathematics teaching and learning that better reflects what is valued by teachers, policy-makers and other stake-holders of schooling systems.



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# TEACHER OFFLOADING, ADAPTING AND IMPROVISING WITH THE TEXTBOOK – A CASE STUDY

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*The study presented in this paper investigates how two mathematics teachers utilize textbook in a traditional curricular environment. Using Design Capacity for Enactment framework, we analysed four lessons, observed in each teacher classrooms. The results of the study showed that teachers used the textbook extensively in their teaching practice. Most of the time teachers offloaded lessons from the textbook for various activities: teaching a new content, practice exercises or homework. The teachers offloaded lessons on the textbook because the textbook supported their current teaching goals and was aligned with their beliefs for particular lessons. Other types of textbook utilization such as adapting and improvising were present but to a lesser extent. They were used when textbook did not satisfy teacher's goals and beliefs.*

## INTRODUCTION

Curriculum materials such as textbooks and teacher guides can be regarded as mediators between the intended and enacted curriculum (Valverde et al., 2002). Moreover, those materials can be seen as the potentially enacted curriculum because they include mathematical problems and exercises for teachers to use while teaching, and often suggestions for mathematical activities. Despite the availability of various materials, the mathematics textbook still remains the major resource for teaching and learning mathematics in many countries (Pepin & Haggarty, 2001; Fan, 2013). But it turns out that teacher is the one who decides which textbook or textbooks to use, and where and when to use it in the classroom (Pepin & Haggarty, 2001).

However, the teacher's role in education was recently reconsidered – the focus moved from the role of the teacher as just the mediator between curriculum and students to a designer of curriculum instructions (Brown, 2009; Remillard, 2005). According to this perspective, teachers and curriculum materials participate in a dynamic and collaborative relationship. Teachers work with curriculum materials to develop planed curriculum and to construct enacted curriculum (Beyer & Davis, 2009). This perspective is not fully researched yet; there is still the lack of understanding how teachers use curriculum materials to craft instruction and how these materials can constrain and afford teacher practice. Some authors such as Lloyd, Remillard & Herbel-Eisenmann (2009) tried to provide feedback how new and experienced teachers used reform-oriented textbooks. Such materials are connected to curricular reforms and designed to emphasize mathematical thinking, conceptual understanding, and problem-solving in realistic contexts. Since Croatia still has traditional character of mathematics education, we found challenging to understand better the relationship between traditional textbooks and experienced teachers. In this paper, we tried to investigate the

dynamic of teacher-textbook relationship, particularly whether the teachers in the classroom step out the traditional requirements and why. These questions could help us better understand enacted curriculum as a dynamic process and provide quality guidelines for the reform that Croatian educational system is facing.

## THEORETICAL BACKGROUND

### Framework for the teacher-curriculum relationship

To investigate and explain the relationship between teachers and curriculum resources, Brown developed a theoretical framework called Design Capacity for Enactment framework (Brown & Edelson, 2003). The Design Capacity for Enactment framework (DCE) represents the idea that the curriculum as well as teacher's personal resources influence designing and enacting the instruction. Curriculum resources encompass physical objects, representations, and procedures of the domain, while teachers' personal resources mean subject matter knowledge (SMK), pedagogical content knowledge (PCK), beliefs and goals. The DCE framework "provides a starting point for identifying and situating the factors that can influence how a teacher adapts, offloads, or improvises with curriculum resources" (Brown, 2009, p. 27). *Offloading* denotes relying mostly on the curriculum material for the delivery of the lesson; *adapting* indicates an equally-shared responsibility for the delivery of the lesson between teacher and curriculum materials; and *improvising* means teacher relies mostly on external and own resources for delivering the lesson. Teachers either omit components of a lesson, or replace one component with another, or completely create new components during the adaptation process (Sherin & Drake, 2009).

Brown introduced the term *pedagogical design capacity* (PDC) to describe the capacity of teachers to perceive and mobilize existing resources in creating deliberate and productive instructional episodes:

PDC is not simply an indicator of whether a teacher will be likely to design something for the classroom; it is an indicator of whether the teacher's designs are pedagogically beneficial. (Brown, 2009, p.31)

According to Brown, the PDC is a vital and important part for the actions involved in resource mobilization. It characterizes a process where resources such as SMK and PCK are mobilized, and it can be viewed only during classroom activities (Brown, 2009). Textbooks as curriculum materials provide opportunities to learn school subjects and have their characteristic impact on classroom activities (Valverde et al., 2002, p.10). They are very important as means for teachers' lesson preparation, teaching new content, as well as sources of worked examples, practice exercises and homework (e.g. Pepin & Haggarty, 2001). Hence, from the perspective of textbook utilization, teacher's development of PDC is the vital part of teacher's interactions with the textbook.

## **Previous research**

The study presented in this paper is a follow-up of the large-scale study reported in Glasnović Gracin (2011). That large-scale study investigated nearly one thousand mathematics teachers on the utilization of mathematics textbooks in lower secondary education in Croatia (grades five to eight), using a questionnaire with multiple choice items. The results showed that teachers used the textbook to a great extent for various activities: lesson preparation, teaching a new topic, exercising and assigning homework. Also, the results showed that teachers used textbook more than other curriculum materials.

Glasnović Gracin (2011) also analysed the content and requirements of Croatian mathematics textbooks. The results showed the dominance of operation activities on the reproductive or simple-connections level with intra-mathematical content. The further analysis showed that the requirements of intended curriculum match the ones from textbooks, thus Croatian mathematics textbooks can be perceived as a 'conveyor of the curriculum' (Fan et al., 2013, p. 635). These results show that mathematics curriculum in Croatia has traditional character; it places more emphasis on algorithms and the view of mathematics as a tool than as a medium of communication (Heymann, 1996).

## **Research question**

Despite the fact that the previous research surely gave useful and interesting information on the textbook use and its requirements, it did not provide the answer about the nature of the relationship between the teacher and the textbook. It seems important to find the reasons what teachers take from the textbooks and what they change in the classroom practice and why. Therefore, we formed following research question: How and why does a teacher adapt, offload, or improvise with textbook content concerning the traditional curricular environment?

## **METHODOLOGY**

The study reported in this paper is a case study based on two female mathematics teachers from lower secondary education in Croatia (grades five to eight). Throughout the paper, we will call them Mrs. G and Mrs. A. Mrs. G and Mrs. A have a different educational background, where Mrs. G obtained her teaching degree from mathematics department and Mrs. A comes from Teacher College. Both teachers are experienced teachers; Mrs. G has 17 years of teaching experience and Mrs. A has 39. The main reason we had chosen these teachers for this study is connected with their extensive textbook utilization in the classroom. At the time of the study, Mrs. G and Mrs. A have been teaching grades five and seven. They use the textbook series from the same textbook publisher, and they work in the same school.

In this study, we used qualitative methods in the form of observations and interviews.

We observed four lessons in each teacher's classrooms; two lessons in grade five and two lessons in grade seven. Before each observation took place, we asked the teachers about mathematical content to be taught in the lesson, and we examined its structure and pedagogical approach in the textbook: motivation, definitions, types of worked examples and tasks, and pedagogical instructions written by the textbook author.

After the classroom observations, we conducted a semi-structured interview with the teachers to detect teacher's personal resources: goals, beliefs, parts of PCK and SMK, and questions related with their lesson preparation, teaching, textbook use, the influence of the textbook on the instruction. This helped us in making a coherent conclusion about teacher–textbook relationship.

## RESULTS

We coded episodes in observed lessons according to three types of textbook utilization: offloading, adapting and improvising (Table 1). Mrs. G taught two lessons in the grade five and two lessons in the grade seven. Mrs. A had a similar situation. Mrs. G and Mrs. A taught three lessons of the same type and with the same topic.

Grade	Topic (lesson type)	Teacher	Type of textbook utilization in the lesson duration (in minutes)		
5	Distributive property (acquisition)	Mrs. G	offloading	45 min	
		Mrs. A.	offloading/inserted adaptations	45 min	
7	Proportionality (acquisition)	Mrs. G	offloading 10 min	adapting 35 min	
5	Order of operations (acquisition)	Mrs. G	offloading/inserted adaptations 10 min	adapting 35 min	
		Mrs. A	improvising 20 min	offloading 10 min	improvising 15 min
7	Proportionality (practicing)	Mrs. G	offloading 10 min	adapting 35 min	
		Mrs. A	offloading/inserted adaptations 45 min		
7	Inverse proportionality (acquisition)	Mrs. A	offloading 10 min	improvising 10 min	offloading 25 min

Table 1: Utilization of textbook

## **Offloading**

Mrs. G offloaded the textbook content in 60% of the observed time. She used offloading for explanatory talk, worked examples and practice exercises in the first and third lesson while in the second and fourth lesson offloading was connected with checking homework at the beginning of the lesson.

Mrs. G provided a reason she leans strongly on the textbook in her lessons. The teacher believes that this particular textbook has a "good pedagogical approach for the students." Her decision to use the textbook is connected with students' self-regulated learning at home.

Always at the beginning of the year I tell them that a textbook is written for them, not for me, [...] and that, of course, they should use the textbook in repetition phase, for exercising and learning at home.

In order to direct students toward the use of the textbook, the teacher writes the lesson title on the blackboard always literally the same as the title in the textbook. That way the students can find at home easily what they did in school.

I like that the lesson title that I write on the board is the same as the one in the textbook... The definitions don't have to be the same...

Also, she offloads in practice exercises because the textbook provides routine, but also "non-routine, and very interesting tasks".

Mrs. A followed the textbook in 75% of the observed time. She used offloading for explanatory talk and worked examples in the first lesson, and for practice exercises and homework in all her lessons. Mrs. A elaborated the direct use of textbook for acquisition phase in teaching because she likes "the beginning, key concepts, and motivation" as presented in the textbook lessons. When asked why she offloaded practicing on the textbook, she explained that the task is the central term of mathematics education and that she appreciates the variety of tasks given in the textbook. She does not prefer quantity of textbook tasks, but rather their quality:

I find tasks [in the textbook] very important... It is not important to provide too many tasks, but good types of tasks... because they [students] can not solve all tasks from the textbook... so types of tasks, various types of tasks are important.

## **Adapting**

Mrs. G used adaptation in two her lessons, what makes 40% of her teaching time observed. Here the teacher used technology for a presentation of worked examples and practice exercises that were similar to ones in the textbook, but not the same. For example, she used authentic context that was close to students' experience and environment, what added adapting component to the material.

In the interview, Mrs. G. mentioned that she combines the textbooks and other materials in the lesson preparation because she wanted her students to experience the variety of tasks.

I use other sources like internet; materials from other colleagues...I like to check other [textbooks]... not just this [official] textbook, so that children cover all areas...

### **Inserted adaptations**

Besides longer episodes of adapting, we noticed short moments of adapting within longer offloading phases. We denoted these moments as *inserted adaptations* where the teacher inserted her content and broke episodes of pure offloading, but the textbook stayed dominant. Such situation was noticed when Mrs. G created problem situation as the motivation for the order of the operations in fifth grade.

In the interview, Mrs. G explained that she likes to break offloading by adding some short elements, like “historical stories or humor connected with mathematics.”

Mrs. A used inserted adaptations for the distributive property when she combined tasks from the textbook with tasks she invented at the moment of teaching. She also used authentic real life situations to connect them with proportionality and inserted mathematical symbol for proportionality that was not present in the textbook.

Mrs. A explained she finds mathematics in all aspects of everyday life, and accordingly she likes to insert examples of real life situations where mathematical content has its application:

Mathematics is life; I tell this often to my students. Everything around us is mathematics, wherever we look, in what area we look, it's math. I especially like to insert, add an application of mathematics in everyday life because this is very important to me.

### **Improvising**

Mrs. A improvised in two lessons, what makes 25% of her teaching time. In one lesson she used worksheets with her tasks for practicing. In another lesson, she had an episode where she invented problems involving the inverse proportionality without using the textbook.

Mrs. A stated in the interview that she does not always follow the textbook. For some lessons she has her ideas or she does not find the textbook approach appropriate.

However, I do have my idea, my combination... Sometimes I do not like what is offered [in the textbook]. Sometimes I try to combine. So...I find some other way to teach.

She also elaborated why she improvised and created problem situation when she was teaching inverse proportionality:

I always use problem situations; I tend to use some problem from real life, from everyday life, to bring math closer to a student. I want them to see why we learn mathematics.

### **Pedagogical design capacity**

From the observations and interviews, we conjecture that both teachers have a reasonably high level of PDC for the taught mathematical content. That rises from their adaptations of the curriculum material, inserted adaptations and improvisations. The adaptations and improvisations afforded meaningful opportunities for students to

engage with mathematical concepts, to move from the purely procedural approach to the more conceptual understanding of taught mathematical concepts.

Mrs. G stated that she constantly participates in the professional development because she believes she has to learn things she finds lacking in her practice. Mrs. A said that she was well educated for her job as a teacher but several times she emphasized her constant professional development:

There was professional development through years... I like to improve myself all the time, no matter how much of teaching practice I have.

## **DISCUSSION AND CONCLUSION**

Both teachers used the textbook in the observed lessons intensively. Offloading on the textbook was frequent when teachers taught new concepts and procedures. Here teachers relied on the pedagogical approach from the official textbook. The textbook was used also for worked examples, the practice exercises and homework. Adapting was visible only in Mrs. G's classroom. It was less used than offloading, and it was present during the acquisition phase of new mathematical content and for practice exercises. Improvising as the type of textbook utilization was present only in Mrs. A's classroom. However Mrs. A's episodes with improvising were shorter than to the adapting episodes of Mrs. G. Improvising was used for the acquisition of new mathematical content. The study showed that during offloading episodes teachers used inserted adaptations i.e. they inserted own content and broke episodes of pure offloading, but the textbook stayed dominant. Those inserted adaptation were various: motivation, new mathematical symbols, authentic real life component of mathematics.

Although these two teachers offloaded lessons on the textbook, the reason was not their inexperience, but because the textbook supported their teaching goals and was aligned with their beliefs for particular lessons. Similarly, Brown (2009) provided a case where teacher extensively offloaded lesson on the textbook because he recognized a pedagogical benefit of such teaching and this offloading enabled him to achieve the desired goal.

Both teachers used the same textbooks and had the same conditions for teaching because they work in the same school. But they taught the same content differently. Both teachers showed high level of PDC for the taught mathematical content, although they had different educational background. This shows that educational background does not have to be only or dominant factor in PDC development, but it seems the professional development can have greater influence on its development. Both teachers stepped out from traditional curricular guidelines in the textbook utilization under the influence of professional development. Here the textbook was still a dominant curriculum material but interrupted with inserted adaptations, improvising and using the textbook only partly. In this way teachers transformed curriculum material for their teaching practice. Brown (2009) states that teachers with high PDC can deconstruct curriculum materials, identify important elements, and reconstruct them in a way that suits their needs.



The results of this study are important for the development of new textbooks and other curriculum materials, especially for Croatian curriculum reform that is in its beginning ([www.kurikulum.hr](http://www.kurikulum.hr)). However, we wonder if the teachers would be making those inserted adaptations if they worked with the reform-oriented textbook.

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# CONCEPTUALISING FUNCTION AS COVARIATION THROUGH THE USE OF A DIGITAL SYSTEM INTEGRATING CAS AND DYNAMIC GEOMETRY

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*This paper reports ongoing classroom research focusing on 16 year-olds' construction of meanings for function as covariation. The students worked on modelling tasks related to optimization problems using a digital environment that connects CAS and dynamic geometry. We analysed students' activities on functional dependencies in different settings including physical device, dynamic geometry, magnitudes and mathematical functions. The results indicate students' transition from covarying quantities to mathematical functions as an abstraction process mediated by the use of the available tools.*

## THEORETICAL FRAMEWORK

This paper reports classroom research aiming to explore 16 year-olds' construction of meanings for function as covariation while exploring and solving contextual problems through the use of the digital environment Casyopée (Lagrange, 2010), which links Computer Algebra Systems (CAS) and dynamic geometry. The students worked in groups of two to solve problems involving modelling geometrical dependencies. Casyopée offers opportunities for students to experience variations and covariations of quantities and to decide, through appropriate feedback, if couples of corresponding covarying magnitudes can define functions. In this case, students can create a function and make sense of it using multiple integrated representations.

The notion of function occupies a central position both in school mathematics curricula and research in mathematics education. Existing research confirms the complexity of issues involved in students' conceptualisation of function (Thompson, 2011). A number of researchers described students' transition from a focus on actions and processes to a gradual focus on structure and vice-versa in terms of the distinction between the process view and the object view of functions (e.g., Sfard, 1991). A gradual consideration of this distinction as a dynamic interplay led, from the middle nineties, a number of approaches to emphasize the covariation aspect of function (Carlson et al., 2002; Thompson, 2011; Lagrange, 2010; Psycharis, in press). The essence of a covariation view is related to the understanding of the manner in which dependent and independent variables change as well as the coordination between these changes. According to Carlson et al. (2002), *covariational reasoning* consists of “the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other” (ibid., p. 357).

Existing research on students' covariational reasoning indicated that coordination of two covarying quantities is embedded within an evolving/developing process in which

students initially construct an image of the coordination of two quantities that gradually moves in the direction of the continuous coordination of both quantities (Saldanha & Thompson, 1998). Carlson et al. (2002) studied college-level students' ability to reason about two covarying quantities when interpreting and representing dynamic situations (e.g., a bottle filling with water) by constructing their graphs. The outcome of their research was a covariation framework with five levels characterizing students' engagement in making sense and representing functional relationships through corresponding mental actions: (a) coordination (coordinating the change of one variable with changes in the other variable); (b) direction (coordinating the direction of change of one variable with changes in the other variable); (c) quantitative coordination (coordinating the amount of change of one variable with changes in the other variable); (d) average rate (coordinating the average rate of change of the function with uniform changes in the independent variable); (e) instantaneous rate (coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function).

A critical point in students' conceptualisation of function as covariation is related to their ability to pass from quantitative reasoning - "based on quantities themselves and images of them that entail their values varying" (Thompson, 2011, p. 48) - to algebraic reasoning that involves describing the relation between two covarying quantities formally. This is a step far from trivial for the students. Lagrange (2014) indicated modelling dynamic situations with the use of specially designed digital tools (e.g., integrating geometrical and algebraic representations) as a context favoring students' transition from relations among quantities to mathematical functions. Based on the work done for the development of Casyopée, he described students' work with dependencies through a *modelling cycle* (ibid.) involving four settings: (a) a physical device allowing dependencies of items to be experienced by humans; (b) a dynamic figure modelling these (physical) dependencies into a digital tool (e.g., dynamic geometry); (c) magnitudes standing for measures of quantities independently of the unit in which they are measured; and (d) algebraic functions. In this approach, students' transition from experiencing dependencies in a physical system to the world of functions is expected to be mediated by their work with covarying magnitudes and the use of multiple representations such as formulas, graphs and tables. In the present study, we adopted this approach and we considered students' passage from physical dependencies and covarying quantities to mathematical functions as an abstraction process of meaning generation (described by the idea of *situated abstraction*, Noss & Hoyles, 1996) evident in students' identification and expression of relationships through the use of the available tools.

The general aim of this study is to shed light on students' conceptualisation of function as covariation, as they are engaged in solving modelling tasks involving geometrical dependencies with the use of concrete materials (e.g., manipulatives) and Casyopée. Our focus is on how the students used the available representations in order to attribute meaning to two covarying quantities from the level of physical dependencies to the

level of magnitudes and mathematical functions. We were also interested in exploring the role of the available tools (digital and non-digital) in shaping students' activity towards more abstract conceptions of the relation between two covarying quantities.

## THE DIGITAL ENVIRONMENT

Casyopée (Lagrange, 2010) is a digital environment that combines CAS (i.e. a symbolic window with registers: numeric, graphic and symbolic, Fig. 4) and dynamic geometry allowing students to treat functions using interconnected representations.

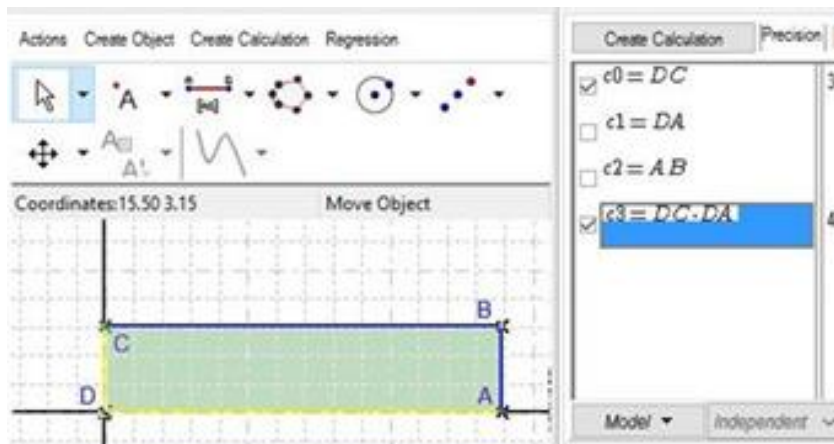


Fig. 1 Windows of dynamic geometry and geometric calculations in Casyopée

Students can observe covariation of quantities in the dynamic geometry window and define independent magnitudes (related to free points) and dependent ones (using the “geometric calculations” window, Fig. 1) involving distances (e.g., lengths),  $x$ -coordinates,  $y$ -coordinates, areas, etc. In Casyopée, magnitudes are labelled as

$c0$ ,  $c1$ ,  $c2$ ,... in the geometric calculations window. The movement of a free point changes the measures of all magnitudes depending on it. Casyopée provides the opportunity to check if two covarying magnitudes (i.e. chosen one as independent and the other one as dependent) are in functional dependency or not through the “automatic modelling” functionality. In the former case, the system automatically exports the formula of the corresponding function to the symbolic window, while in the latter case an error message indicates that functional dependence is not possible. The new function can be treated by students using its formula, a table and a graph (Fig. 4).

## METHODOLOGY

The research reported in this paper is the first part of an ongoing classroom-based design research (Cobb et al., 2003) aiming to study meaning generation for function as covariation by 16 year-old students, who work in groups with concrete materials (e.g. manipulatives) and Casyopée to model a series of dynamic real life situations. The experiment took place in a secondary school with one class of twenty 11<sup>th</sup> grade students (10 groups of two), one researcher who acted as teacher (called teacher in the paper) and another one who had the role of participant observer in the classroom. The class had totally 14 teaching sessions (45 min each one) over 3 months (one teaching session per week). At the time of the study, the students had been taught about function as correspondence (according to the curriculum), monotonicity and extreme points.

The activity sequence was divided in two phases and for each one of them we designed a series of tasks related to optimization problems. In the sequence of tasks, covariation appeared from simple to more complex situations. In the first phase (2 teaching sessions), after an introduction to main features of Casyopée (e.g., dependencies emerging by moving free points, definition of geometric calculations, automatic modelling) the students were asked to find the minimum distance of a point  $M$  on a parabola from a given point  $A$ . They had to create a function by defining the  $x$ -coordinate of point  $M$  and the length of  $AM$  as covarying magnitudes and use that function to solve the problem. In the second phase (12 teaching sessions), the students were engaged in modelling three realistic situations through the tasks: *Gutter*, *Front of a Store*, *Oil Tank*. In this paper, we analyze the first task and its implementation (4 teaching sessions) as well as the work of three groups of students (groups 1, 2, 3).

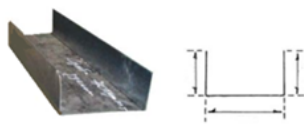


Fig. 2 Gutter design

In the *Gutter* task, the students were engaged in exploring the construction of a gutter model (Fig. 2) that allows the biggest amount of water passing over it. The task was divided in subtasks corresponding to students' activities on functional dependencies in the different settings of the modelling cycle outlined previously: (1) experiment with folding a piece of paper (10cm X 20cm) to explore the

construction, notice if some quantities change together and express their relation formally with the use of one variable; (2) design a dynamic figure with one free point that models the situation in Casyopée and explore; (3) use the software tools to propose a function modelling the problem (e.g., create appropriate geometric calculations of magnitudes and test if functional dependency can be defined); (4) solve the problem using the available tools.

The data, which consisted of video and audio recordings, outcomes of students' work and screen captures software files, were analyzed under a data grounded approach (Strauss & Corbin, 1998). Through the analysis, episodes were selected: (a) to have a particular and characteristic bearing on the students' interaction with the available materials/tools; (b) to represent construction of meanings for function as covariation highlighting students' transition towards more abstract conceptions of the relation between two covarying quantities. Then, the selected episodes were categorized in themes indicating different aspects of the students' work with dependencies as well as their progressive passage from covarying quantities to mathematical functions.

## MEANING GENERATION FOR FUNCTION AS COVARIATION

In this section, we provide an account of the emerging themes of episodes characterising students' activity throughout the implementation of the *Gutter* task.

### Identifying dependencies in the paper model

In the first teaching session, the students were given a piece of paper (10cm X 20cm) and were asked to explore the creation of a gutter model (Fig. 2) that maximizes the amount of water passing over it. They experimented by folding the paper in different ways. For instance, they folded it in different points along its length (Fig. 3) and width and through this they recognized that the amount of water depends on the dimensions of the rectangular cross section of the gutter. Until the end of this teaching session the students conceived the interdependence of the dimensions of this rectangle and its area perceptually.



Fig. 3 Experimentation with the paper model

In the second teaching session, all groups were engaged in using algebraic notation to represent the length and width of the rectangle (cross section). Most of them used two variables  $x$  and  $y$  without linking them. Only students of group 1 and 2 in the class used one variable to express the dimensions of the cross section. Group 2 students recognized that folding the paper along its length (in three segments symbolized as  $x$ ,  $20-2x$ ,  $x$ ) gives a better solution than folding it along its width (symbolized as  $x$ ,  $10-2x$ ,  $x$ ) by comparing  $20-2x$  and  $10-2x$ , without providing a solution to the problem. Group 1 students noticed the interdependence of the covarying quantities (i.e. width of the cross section, cross-sectional area) and recognized that the optimization of the area leads to the solution of the problem.

To sum up, in the first two teaching sessions students' interaction with the given piece of paper allowed them to experience the problem sensually and through this to conceive the dependencies of the covarying quantities perceptually. Only two groups of students took a step further in identifying the interdependence of the covarying quantities, whose covariation leads to the function that models the problem.

### Modelling dependencies in dynamic geometry

In the third teaching session, the students were asked to construct a dynamic figure in Casyopée modelling the problem (i.e. a rectangle with one dimension depending on the other) with one free point. Then, they were asked to define appropriate magnitudes as geometric calculations (e.g., the area of  $ABCD$ ) so as explore further their covariation for solving the problem. Four groups of students in the class (including groups 1, 2, 3) completed the construction successfully in the dynamic geometry window. First of all, the students of these groups renamed the origin as point  $D$ , then, they constructed a free point  $C$  on the  $y$ -axis and represented the height of the gutter through the segment  $CD$  (Fig. 1). In some cases (e.g., group 1), the students did not take into account that point  $A$  was dependent on point  $C$  at the beginning of their work and constructed it as a fixed point. The teacher had to intervene indicating that the coordinates of the defined points (e.g.,  $x_C$ ,  $y_C$  in Casyopée) could be used for defining new ones. Finally, the students defined successfully all points and expressed the



dependence of points  $A$ ,  $B$  and  $D$  to the free point  $C$  through expressions such as  $A(20-2*CD,0)$  or  $A(20-2*yC,0)$ ,  $B(xA, yC)$ . Then, according to the task, the students were engaged in exploring further the given problem by creating magnitudes as geometric calculations in the corresponding window so as to observe how changes in the dynamic figure changed their numeric values. For instance, group 1 students created four magnitudes (i.e.  $c0=DC$ ,  $c1=DA$ ,  $c2=AB$  and  $c3=DC*DA$  in Fig. 1). Next, we provide an episode showing how students' conceptualisation of covariation was influenced when they engaged in manipulating the dynamic figure. The group 1 students started to observe how dragging of  $C$  changes the numeric values of  $c0$  (i.e.  $DC$ ),  $c1$  (i.e.  $DA$ ) and  $c3$  (i.e. area of  $ABCD$ ) in the geometric calculation window. In the next excerpt, students were moving point  $C$  continuously and the teacher asked them to describe their observations.

- 415 R: [To S1] What do you observe by dragging the point  $C$ ?
- 416 S1: I look at the cross sectional area [i.e.  $c3$ ] to see when it [i.e. *the gutter*] has maximum capacity. I see that this seems to happen when the height [i.e.  $DC$ ] is nearly half of the base [i.e.  $DA$ ].
- 417 R: How did you find it?
- 418 S1: Here, we see that if we increase  $DC$  more than 5, the capacity of gutter decreases more and more. If we decrease it less than 5, the capacity decreases again. When it is equal to 5 or 4.9, the capacity seems to take its maximum value.

By continually moving the point  $C$ , S1 focused on the direction of change of  $DC$  (more or less than 5) and the corresponding changing of the area  $ABCD$ . Through coordinating changes in the figure with numeric changes of the corresponding magnitudes, she approaches dynamically the value of  $DC$  that seems to give the maximum value of the cross-sectional area. S1 appears not to be completely sure that the measure of the magnitude that maximizes the area is 5 or 4.9 since her dragging on point  $C$  is not stable. At the level of covariation, this episode indicates how the available tools supported students to link covariation of measures (in the dynamic geometry) and covariation of magnitudes (in the geometrical calculation window).

### Conceptualising magnitudes as dependent and independent variables

After designing dynamic models of gutters in the dynamic geometry window and defining the covarying magnitudes as geometrical calculations in Casyopée, the students decided to explore the problem through the definition of a new function in the “automatic modelling” window (fourth teaching session). One challenge that emerged at this point concerned the selection of a dependent and an independent variable. Students' selection of variables was facilitated by their interaction with the dynamic figure (e.g., as in the episode presented above). For instance, group 3 students came to select  $c0$  ( $=yC$ ) as an independent variable and  $c2$  ( $=\text{area of } ABCD$ ) as a dependent one by referring to their preceding dragging of the free point  $C$ . Explaining this choice, one student (S6) said: “By moving point  $C$ , we concluded that what is [i.e. independent] variable is  $C$  and what it changes is the area [i.e. of  $ABCD$ ]”. Based on her dragging on

point  $C$ , S6 seems to conceptualise the dynamically changing magnitudes  $c0$  and  $c3$  as a pair of independent-dependent variables. This is expressed by the students through a situated abstraction indicating their transition to conceptualizing covariation at the level of variables.

### Conceptualising function as covariation by connecting different representations

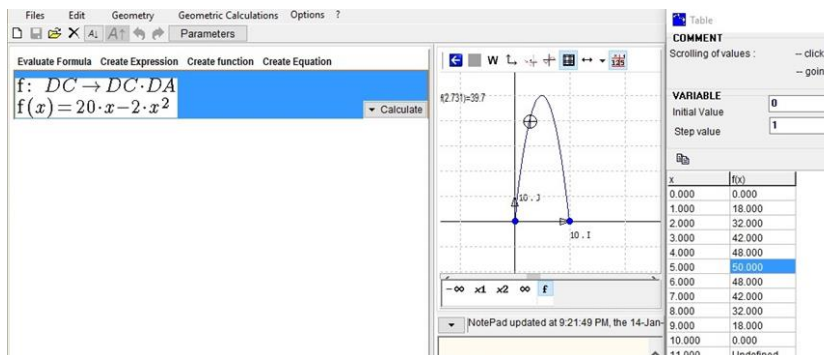


Fig. 4 Windows of Algebra, Graphs and Table

In the fourth teaching session, two groups of students (groups 1, 2) obtained a formula for a function modelling the situation ( $f(x)=20x-x^2$ , see Fig. 4) through “automatic modelling”. This indicated their transition from working with covariation of measures and magnitudes to

working with mathematical functions. The students’ exploration of the problem at that time was characterized by the progressive coordination of different representations of function. In the next episode, group 2 students were able to interpret their answer to the problem by linking three representations of function in Casyopée: formula (symbolic window), table and graph (Fig. 4).

- 620 S3: We see in the table that the area is maximized when the coordinate of the free point  $C$  [i.e.  $y_C$ ] is 5. That is, we have the maximum area when one side [i.e. of  $ABCD$ ] is half of the other.
- 621 S4: In the graph of the function,  $x$  is between 0 and 10. When  $x$  is equal to 5, the area is maximized.
- 622 S3: In the upper point of the graph, we have the greatest area ... when we are in the maximum of the function.

Here, meaning generation is progressive. The students conceptualise function as covariation by observing the graph and the values of the two variables in the table and, at the same time, they refer to: (a) the  $y$ -coordinate of  $C$  (as independent variable); (b) the relation between the two sides of the dynamic rectangle modelling the problem; (c) the domain of the function; and (d) how the extreme point in the graph is related to the solution of the problem.

### CONCLUSION

By analyzing students’ activity, we could trace their conceptualisation of function as covariation throughout their engagement in working with a contextual task. The analysis was structured around four themes of episodes indicating students’ progressive passage towards more abstract conceptions of the relation between two covarying quantities. Meaning generation in these themes of episodes involves: making sense of the interdependence of two covarying quantities sensually by modelling the



problem with a piece of paper; making sense of the dependency between the free point (i.e.  $C$ ) and the area (i.e.  $ABCD$ ) influenced by its move; conceptualising the creation of relevant magnitudes as geometric calculations; linking covariation of measures (in the dynamic geometry) to covariation of magnitudes (in the geometrical calculation window); conceptualising two dynamically changing magnitudes as a pair of independent-dependent variables; using this pair to define a function through “automatic modelling”; and conceptualising function as covariation by connecting different representations. Besides, the analysis indicates the critical role of “geometric calculations” and “automatic modelling” in facilitating students’ passage from the level of quantities and magnitudes to mathematical functions.

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# MAKING CONNECTIONS BETWEEN ANALYTIC AND VISUAL APPROACHES: DIFFERENTIAL EQUATION

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*This study has explored one student's struggle in making coordination between analytic and visual approaches to solve a first order autonomous differential equation. The potential difficulties related to confirmation of coordination were uncovered. The zigzag to make confirmation of coordination between analytic and visual approaches was suggested by the data, which in turn raises questions about the usefulness of the confirmation of coordination in mathematical thinking related to solve a first order autonomous differential equation.*

## INTRODUCTION

In the past three decades, there has been an effort to revitalize the DEs curriculum by connecting analytic and visual representations (Artigue, 1992). Prior to these reform efforts, introductory level Differential Equations (DEs) course were changing (Rasmussen, 2001). Concepts once treated only analytically were now explored visually (Kadunz & Yerushalmy, 2015) and this change required students to interpret and connect analytic and visual approaches simultaneously. The goal of these changes was to provide students with a rich understanding of the underlying concepts to provide insights into the various approaches. In order to realize these goals, more research is needed on students' experiences in making connections between analytic and visual approaches to solve first order autonomous DEs.

There was little research in this area, except the preliminary work done by Artigue (1992), who reported on the cognitive difficulties of one coordination task. The task asked students to find and justify the correct match between seven different DEs and corresponding graphs of solution curves. This was the first step to research about making connections between analytic and visual approaches in DEs education. After several years, much like the previous research, Zandieh and McDonald (1999) asked students to draw representative solutions to make connections between analytic and visual approaches. In another study that followed suit, Rasmussen (2001) investigated the mathematical development of six students in a reform-oriented DEs course at a large mid-Atlantic state university. One of the tasks required students to determine, with reasons, the appropriate match between eight DEs and four slope fields (four of the DEs did not have a matching slope field). This research tried to investigate students' understanding and difficulties while they tried to coordinate an analytic representation of DEs with an appropriate visual one. Students' confirmation of coordination suggests that they demonstrate a certain level of rich understanding to coordinate visual and analytical approaches. However, no previous research has examined the exact students'

mathematical thinking and confirmation of coordination between analytic and visual approaches simultaneously.

Knowledge of students' thinking, when struggling to make confirmation of coordination between approaches, could lead to rich understanding, new insight into instructional visualization practices, and revised and improved curriculum.

## **FRAMEWORK**

Constructivism, as used here to get new insights into mathematical thinking, was the belief based on which knowledge was constructed by the individual. In other words, "mathematical knowledge was not a thing that you have, but an activity that you (might) engage in" (Dubinsky, 1994, p. 224). In this paper, for example, Shayan engaged in harmonic and dynamic thinking to construct confirmation of coordination. Constructivism, as used here, provided the overall perspective on how a student zigzags between obtaining analytical and visual approaches simultaneously and analysing coordination between approaches to construct rich understanding. Constructivism deals essentially with the construction of a rich understanding as Hiebert and Lefevre (1986) characterize it as "a knowledge that was rich in relationships. It can be thought of as a connected web of knowledge" (p. 3). Regarding to this study, DEs' rich understanding was the knowledge that was rich in relationships between analytical and visual approaches. It could be a mathematically connected knowledge in which the analytical and visual approaches were not as the discrete method.

The analytical and visual coordination is not a passive process. Zazkis et al. (1996) found the coordination between analytical and visual approaches to be fairly complex. However, their contention is that it is important to understand how students switch between and combine them as "both visual and analytical thinking may need to be present and integrated in order to construct rich understandings" (p. 438). To account for analytical and visual connection as a rich understanding, Zazkis et al. (1996) have developed a model called the Visualization/Analysis (V/A) model. In this model a spiral going back and forth can be described as a rich understanding while the process continues until the learner regards visual and analytical approaches as intertwined and is able to use either of them and readily can change between them to make confirmation of coordination. Our work is also framed by the research of Krutetskii (1976) and Presmeg (2006) which provides an insightful framework to describe shayan's cognitive processes and mental images while he regards visual and analytical approaches as intertwined and is able to confirm a coordination between them. On the one hand Krutetskii (1976) identified, based on student's preferences, harmonic thinking as a cognitive process to describe a student who relies equally on verbal-logical and visual-pictorial processes. On the other hand Presmeg (2006) identified, based on student's preferences, a dynamic imagery as a kind of mathematical imagery to describe a student who combines visual and analytic imagery which produces high levels of mathematical rich understanding.

## METHODOLOGY

A suitable technique for exploring students' understanding was an in-depth task based interview. In order to provide in-depth information on shayan's rich understanding, one semi-structured interview was conducted. The interview was audio recorded and transcribed by the researchers. This provided an opportunity for the researchers to gain some insight into Shayan's understanding and it assisted the researchers in interpreting his mathematical thinking. Shayan was asked to "think aloud" during the interview and was asked about his thinking and reasoning.

The study reported here was carried out in the Fall, 2015 semester in one section of an introductory DEs course at the university of IAU, North West IRAN. The class used the ninth edition of the textbook Elementary DEs by Boyce. The class met for one 90-minute held every week in a classroom with no computer resources. One student from a class of 32 was asked to participate in this study. The participant was a 22 year old undergraduate student named Shayan (pseudonym). The first author has asked Shayan to participate in this exploratory study since, as he had been his teacher for calculus, he knew Shayan had a sound mathematical understanding and would be able to articulate his thinking. Our goal was to maximize information, not to facilitate generalization (Lincoln & Guba, 1985), and thus we selected Shayan as a critical case in order to be able to conduct an in-depth investigation and develop information-rich case.

## RESULTS

The task of the interview was the  $dy/dt = -1 + y$ , an autonomous DE, which should be solved in both analytical and visual approaches simultaneously. This type of activity did not appear anywhere in the textbook and required no new knowledge and therefore provided an opportunity to illuminate Shayan's mathematical thinking and his understanding of the relationship between analytic and visual approaches. In Figure 1, Shayan's analytic and visual approaches and their limitations are shown.

Shayan started to solve the task analytically. He computed its integral, and then used  $y(0) = y_0$  to estimate  $c = y_0 - 1$  to get his first analytic approach A(1):  $y = (y_0 - 1)e^t$  in Figure 1. Shayan demonstrated a strong preference for Krutetskii's (1976) analytic thinking as he started analytically. Then he solved the task visually. He reported that he "examined  $dy/dt = -1 + y$  sign at  $y < 1$  and  $y > 1$  simply" so he quickly drew the sample solution curves in his first visual approach V(1) in Figure 1. From Shayan's quick response to sketch the sample solution curves, his thinking might be representative of Krutetskii's (1976) visual type. In order to verify this early assertion, he was asked to explain how he had determined the sample solution curves.

Shayan: I got one, that the critical point [the equilibrium solution  $y = 1$ ] where it is held constant. The negative  $dy/dt$  is to be decreasing [sample solution curve] and positive  $dy/dt$  is to be increasing [sample solution curve]. We know that the direction of arrows on the phase line is downward when  $dy/dt$  has to be negative and arrows go upward when  $dy/dt$  has to be positive.

Shayan made no mention of symbolic interpretation of the sign and instead, quickly transformed the sign of  $dy/dt$  into the solution space according to the direction of arrows which changing on the phase line. This is an act of Presmeg's (2006) dynamic imagery, as our inferences from his work supported our belief that Shayan demonstrated a strong preference for visual thinking, and his imagery appeared to be dynamic.

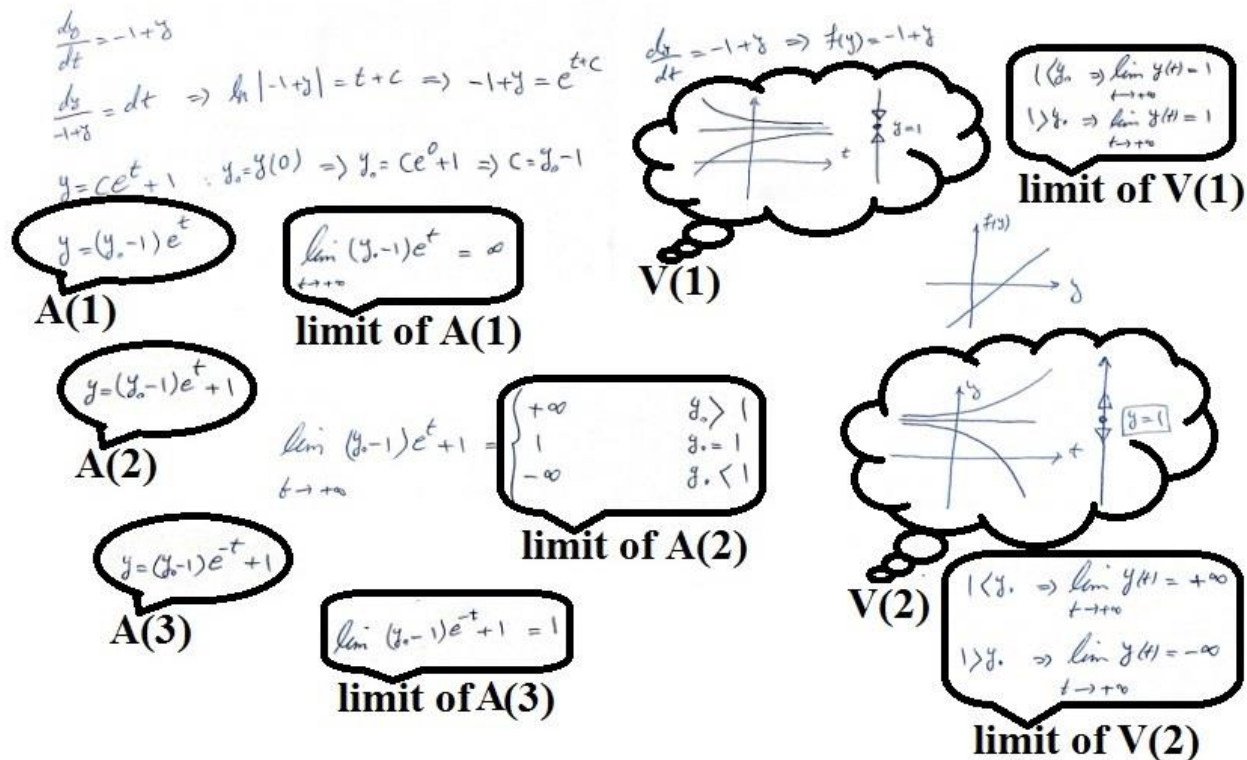


Figure (1): Shayan's visual: V(1) and V(2), and analytical: A(1), A(2) and A(3) approaches and the limits of them

After the first V(1)/A(1) approaches obtained, he was asked "what do you think to check out the coordination between the approaches?" As Stylianou's refinement V/A model, the first step was inferring additional consequences. Shayan had an idea how to check out the coordination by "the limits of both approaches [considered] should be the same." So he was in search for information and cues other than what was immediately sought as approaches.

For the limit of V(1) in Figure 1, Shayan translated from the visual representation to the symbolic representation; that is, he decided that the solution space was a piece-wise and estimated the intervals;  $y_0 > 1$  and  $y_0 < 1$ . When interviewer asked why he wrote two limits, he said, "Because there is a critical point here. So I split it up into two limits because they have two different intervals so I thought it would be suitable to write them separately." Apparently, for Shayan, the limits represented mainly as symbolic, and this emphasis suggests again that Shayan's thinking is representative of Krutetskii's (1976) analytic type.

In the line into the Stylianou's refinement V/A model, he investigated further by comparing the limit of the A(1) with the limit of the V(1). As a "verbal-logical" reasoning he said "the limit of this one [A(1)] is infinity while the limit of this one [V(1)] is one" to specify that there was no coordination between V(1) and A(1). This lack of coordination led him to set a new goal. In the Stylianou's refinement V/A model, Shayan shifted the direction of the problem solving process to find out the reason of the non-coordination as: "The question is why there is no coordination between approaches. Maybe something is going wrong. Let's check the approaches."

Shayan decided to review and monitor approaches in the second cycle of the spiral going back and forth of the V/A model to obtain a new V/A model. He started to control the analytic approach not visual one. Our interpretation was that this means he was analytical, but then we realized he "did not doubt" that his V(1) approach was "absolutely true that did not need to be checked" so our interpretation were changed and that this means that he was self-confident by visual approach.

Shayan found that, instead of  $y = (y_0 - 1)e^t + 1$  he had written  $y = (y_0 - 1)e^t$  by dropping "+1". So his second analytic approach A(2) was  $y = (y_0 - 1)e^t + 1$  and his second V/A was V(1)/A(2). He specified that there was no coordination, again, as he verbal-logically compared the limit of V(1) with the limit of A(2) as follows.

Shayan: Still, something has gone wrong. The limit of this [A(2)] still is infinite while this limit [V(1)] was one. The limit of this [A(2)] should be one. The limit of  $(y_0 - 1)e^t$  should be zero. The  $y = (y_0 - 1)e^{-t} + 1$  is correct, not  $y = (y_0 - 1)e^t + 1$ .

So he guessed that it should have been written  $e^{-t}$  instead of  $e^t$  on A(2). So his third analytical approach A(3) was  $y = (y_0 - 1)e^{-t} + 1$ . This time Shayan specified that there was a coordination between the limit of V(1) and A(3) as he said "both of the limits of the this [V(1)] and this [A(3)] are one." But the interviewer knew that A(3) could not be correct, so he set up questions to lead him to check out the A(3). Then Shayan specified that A(3) did not satisfy the  $dy/dt = -1 + y$  while A(2) satisfied it.

So Shayan came to conclusion that the A(2) was the correct approach while the limit of the V(1) was not coordinated to the limit of the A(2), which was the correct approach. So Shayan came to the conclusion that the V(1) was incorrect. Then he started to solve the task visually. His second visual approach, V(2) is shown in Figure 1. Then the spiral going back and forth of the V/A model continued until Shayan showed that the limits of both V(2) and A(2) were coordinated, as seen in Figure 1.

## DISCUSSION

In this paper, a task based interview, provided the overall perspective on how a student zigzags between "obtaining a V/A model" and "analyzing coordination between analytical and visual approaches" to solve a first order autonomous DE. Here, constructivism deals with the construction of rich knowledge as a connected web of analytical and visual thinking. As seen in the previous section, a potential difficulty was uncovered as Shayan appeared to check out the coordination between visual and

analytical approaches that he obtained. The difficulty, trying to check out the coordination between approaches, was an opportunity to move flexibly between analytic and visual approaches. It required Shayan to interpret and connect approaches. The goal of making connections between analytic and visual approaches was to provide him with a deeper understanding of the underlying concepts. This involves understanding how the behaviour of  $y(t)$  as  $t \rightarrow \infty$  depends on the initial value of  $y$  at  $t = 0$ . Calculating the limits of  $y(t)$  as  $t \rightarrow \infty$  for both approaches gave Shayan an opportunity to see that the limit of  $y(t)$  depends on the initial value of  $y(t)$ . The knowledge required by making connection between analytic and visual approaches was not previously taught. The task had not been discussed in the course. It provided Shayan with the opportunity to move flexibly between analytic and visual approaches. While resolving such a task, Shayan formed associations with his existing analytic and visual knowledge and demonstrated the ability to shift between these representations, and apply his existing knowledge in a different situation.

The insight presented in this paper can be adapted and used to create diagrams, such as the zigzag process. The image of a “zigzag” helps us to understand this procedure, as shown in Figure 2. As illustrated in this figure, making connection involves obtaining V/A approaches, analysing them for coordination between limits of them, and then looking for clues about what caused the non-coordination. These clues may include a misunderstanding in the sequence of solving a DE (e.g. the limit of the A(3) coordinated to the limit of the V(1) while they did not satisfied the DE), missing information in the sequence of solving the DE (e.g. instead of  $y = (y_0 - 1)e^t + 1$  he had written  $y = (y_0 - 1)e^t$  by dropping “+1”), or making a guess (e.g. “ $e^{-t}$ ” instead of “ $e^t$ ”) which can provide insights into some aspects of the sequence of the solving DEs.

In this zigzag process, the approaches are refined, developed and clarified. This process weaves back and forth between obtaining V/A approaches and analysis coordination, and it continues until the zigzag reaches a confirmation of coordination. Confirmation in making a connection between approaches required that the student makes a subjective determination that analytical and visual representations of approaches of a DE are the same mathematically.

When Shayan was explaining his mathematical work, he revealed the visual and analytic components of his thinking. His preference was to start the processes verbal-logically, and then he continued the processes visual-pictorially and switched between and combined them as a cognitive process which describes harmonic thinking that relies equally on verbal-logical and visual-pictorial processes. On the one hand Shayan’s approaches involved visual and analytic elements, and thus his strategies suggest that his thinking is representative of Krutetskii’s (1976) harmonic types. On the other hand this is an act of Presmeg’s (2006) dynamic imagery, as our inferences from his work supported our belief that Shayan demonstrated a strong preference for visual thinking, and his imagery about solving a first order autonomous differential equation appeared to be dynamic. By a spiral going back and forth in V/A approaches



he made confirmation of coordination into construction of rich understanding that analytical and visual approaches were not as the discrete approaches.

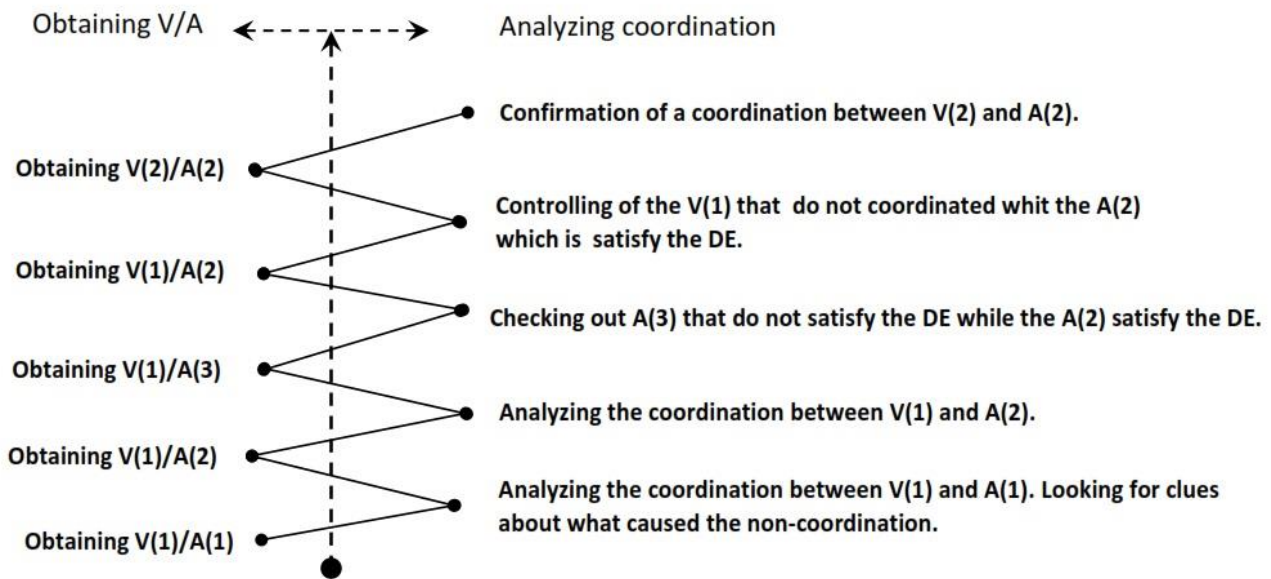


Figure 2: Shayan's zigzag between obtaining approachs and analysing coordination

Besides uncovering some mathematical thinking, this study also raised a further question: What role does confirmation of coordination play in the development of a students' mathematical thinking to move flexibly between analytic and visual approaches?

Regarding previous research, we know that there has been an effort to revitalise the DEs curriculum by connecting analytic and visual representations. Seeking students' challenges to correlate analytic and visual representations was a central goal in (Artigue, 1992; Zandieh & McDonald, 1999; Rasmussen, 2001)'s work. However, no previous research has examined the students' challenges to solve a first order autonomous DE analytically and visually simultaneously. We find that Shayan went through a rather complex back and forth movement between analytical and visual approaches to reach a confirmation of coordination. Meanwhile in the previous studies, students simply went through a correct association of analytic and visual representation. By the core of Shayan's mathematical thinking, we mean that his main challenge was confirmation of coordination. Regardless of whether he faced with some difficulties to solve a first order autonomous DE analytically and visually, our goal is to extract information to elaborate on how the confirmation of coordination could be opportunistically considered as an effort to revitalise the DEs education.

Hence, to some extent the results of this paper support previous research claim that the connection between analytic and visual representations was a dilemma. However, what this paper adds is that it seems not to be the student's disability that prevents him from succeeding. Rather, it seems to be a motive force of mathematical thinking.



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# PARADOX OR PROCESS? A DESCRIPTION OF NOVICE TEACHERS' POWERFUL MATHEMATICAL AFFECT AND THEIR INSTRUCTIONAL PRACTICES IN TEACHLIVE™ REHEARSALS

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*This paper proposes to investigate novice teachers' (NTs) powerful mathematical affect as they engage in "rehearsing" a standards-based mathematics lesson using TeachLivE™, a mixed-reality simulated classroom. To explore NTs' affect, Goldin et al.'s (2011) 'engagement structure' (ES) archetypes are applied, unpacking evidence of each structure with a description of the "in-the-moment" affective states that may influence NTs' traits. Findings suggest that rehearsal of teaching in TeachLivE™, as part of an investigation into novice teachers' praxes, proved to be a turning point for both their affective states and traits (as measured by self-efficacy). This study potentially connects NTs' affect with their behaviours while rehearsing, suggesting that the use of engagement structures may prove powerful for novice teacher praxes.*

## INTRODUCTION

Mathematics student achievement is a particular concern in the United States where students lag behind other developed nations (PISA, 2012; TIMMS, 2011); a lag compounded by within nation learning and opportunity gaps that fall across racial and socioeconomic lines (NAEP, 2015). One way to improve K-12 students' achievement with mathematics is to better prepare teachers to engage students with the subject (Hiebert et al., 2007), as a myriad of research has supported how teacher quality influences student achievement (Ingersoll & May, 2011). Researchers posit that improved quality in instructional practices has been linked to more efficacious teachers, which translates into stronger student engagement and performance (Boykin & Noguera, 2011). In mathematics classes, however, negative affect often dominates both student (Ma, 1999) and teacher (Phillips, 2007; Rayner et al., 2009) experiences. It follows then that attending to teachers' mathematical affect is an important educational lever (McLeod, 1994; Goldin, 2014), as evidenced by recent reform- and standards-based documents that speak to the importance of cognitive, social, and emotional teaching and learning (e.g. see NCTM, 2000, 2007; CCSSI, 2010).

One place to develop teachers' powerful mathematical affect is through teacher preparation programs, specifically in clinical experiences where learning to teach is situated in the practice of "doing." It is imperative that novice teachers (NTs) begin learning their practice in environments that will encourage positive affect (Ronfeldt, 2012). While there are numerous quantitative studies that document the relationship of teachers' affective *traits* (e.g. self-efficacy studies) and instructional performance (e.g. Blanchard, Southerland, Osborne, Sampson, Annetta, and Granger, 2010; Ottmar,

Rimm-Kaufman, Berry, & Larsen, 2013) few studies examine them qualitatively (Goldin, 2014). Fewer still examine “in-the-moment” affective *states*, and the possible links between affective states, traits, and teaching praxes. In fact, to date the authors know of only one study that describes teachers’ mathematical affective states (Lake & Nardi, 2014). Fenton-O’Creevy, Soane, Nicholson, & Willman (2011) pointed out that the dearth of research is due to logistical issues of “applied empirical work in field settings [on emotion in the workplace]” (p. 1045). Thus, one option that addresses the need for practicum settings, and that also lends itself to applied research, is a mixed reality classroom environment like TeachLivE™ (Dieker et al., 2008). Therefore, the purpose of this present study is to examine the extent to which NTs’ “rehearsals” of a reform-based math lesson (Lampert et al., 2013), may strengthen their powerful mathematical affect (Goldin, 2014).

## CONCEPTUAL FRAMEWORK

This inquiry is grounded in two bodies of theory and research: the work on the affective domain in mathematics (Goldin, 2014; McLeod, 1994), and rehearsals as clinical practice (Lampert et al., 2013). First, this paper makes use of Goldin’s (2014) characterization of powerful mathematical affect, as both “*traits* (characterizing different individuals’ most typical emotional responses in mathematical situations), and *states* (emotions as they occur in-the-moment when doing mathematics)” (p. 392). According to Goldin, Epstein, Schorr, and Warner’s (2011) theory, the architecture of traits and states can help link students’ “in-the-moment” emotions with their behaviours in learning math; the authors coined this “behavioural/affective/social constellation...[as]...engagement structures” (p.549). While other scholars have since connected engagement structures to teaching math (Khalil & Johnson, 2016; Lake & Nardi, 2014), this paper builds on these prior works, in order to further explore how engagement structures may relate NTs’ self-efficacy, in-the-moment affect, and teaching behaviours. To apply ‘engagement structure’ (ES) archetypes (e.g. *get the job done, look how smart I am, check this out, I’m really into this, don’t disrespect me, stay out of trouble, it’s not fair, and let me teach you*), a description of each structure’s “in-the-moment” states is unpacked via seven strands (e.g. *goal or motivating desire, patterns of behavior, affective pathways, expression of affect, meanings encoded by emotions, meta-affect, self-talk or inner-speech*) (p. 549).

Second, this study describes NTs’ powerful mathematical affect whilst rehearsing a reform-based math lesson in a simulated clinical environment (Dieker et al., 2014; Ronfeldt, 2012). Lampert and colleagues describe rehearsals as a cycle of enactment and investigation where NTs alternate between enacting teaching and investigating it (Lampert et al., 2013). In this study, the investigation took the form of Japanese Lesson Study, which involves “collaborating with fellow teachers to plan, observe, and reflect on lessons” (Takahashi & Yoshida, 2004, p. 439). While such investigations have proven to be favourable for NTs, field placements are in short supply around teacher preparation programs centred in urban areas, and “easier” working conditions during fieldwork are even less prominent (Ronfeldt (2012). Thus TeachLivE™ rehearsals

seem to be a viable alternative for early field work, as it allows NTs to rehearse in a controlled environment (TeachLivE™ has 5 levels for classroom behaviour), and teaching can be rehearsed free of assessment. Both conditions have been proven to be more conducive to improved overall performance (Dieker et. al, 2014).

## METHODS

The primary question driving this study is, in what ways can reform-based rehearsals in TeachLivE™ be used as a tool to further explore NTS' powerful mathematical affect? We hypothesized that NTS' participation in a modified lesson study intervention that included a rehearsal of a reform-based lesson allowed NTS to experience a myriad of affective states (engagement structures) that then improved their affective traits (self-efficacy). The lesson study and rehearsals were an integral part of the four-credit course, which emphasized a cycle of planning, practicing, coaching, reflecting, re-planning, and further practice. The sample for this study were ten NTS (all African American females) who were in their 3<sup>rd</sup> year completing the requirements of a Bachelor's, were enrolled in an undergraduate elementary mathematics methods and practicum course at a mid-sized Historically Black University in the eastern United States. In the fall of 2014, NTs took the course twice a week for two hours. All participants were asked to design a 90-minute lesson based on one CCSSM standard on fractions, as prior research on virtual rehearsals has shown that assigning the topic allows NTs to focus on how to teach as opposed to what to teach (Deiker et. al, 2014). To orient NTs to TeachLivE™ rehearsals, the instructors arranged for a 1-hour visit to TeachLivE™ where NTs interacted with student avatars. All NTs (i) received feedback from two teacher educators on their lesson plans that allowed them to further revise it, (2) were asked to prepare only 15-min of their 90 min lesson to rehearse in TeachLivE™ while being video-taped, (3) received further feedback/coaching after the first rehearsal, (4) reflected upon their experience of lesson planning and teaching, and (5) revised the same portion of their lesson plan to reteach in class or TeachLivE™ rehearsal (30 minutes of rehearsal teaching; two times of feedback from instructor). All NTs completed the *Mathematics Teaching Efficacy Beliefs Instrument* (MTEBI; Enochs, Smith, & Huinker, 2000) prior to any rehearsal experiences, and again after the second TeachLivE™ rehearsal. The MTEBI has 21 items on a 1-5 Likert scale (strongly agree to strongly disagree). The alpha reliability coefficient for the MTEBI was .88. The Personal Mathematics Teaching Efficacy (PMTE) subscale consisted of 13 items with an alpha reliability coefficient of .89 and .92 at Time 1 and Time 2, respectively. The Mathematics Teaching Outcome Expectancy (MTOE) subscale consisted of 8 items with an alpha reliability coefficient of .89 and .91 at Time 1 and Time 2, respectively. The two subscales measure the two-dimensional aspect of teacher efficacy; the PMTE subscale addresses NTs beliefs in their individual capabilities to be effective mathematics teachers, while the MTOE subscale measures NTs' beliefs that effective teaching can bring about student learning of mathematics. Additionally, the Reformed Teacher Observation Protocol (RTOP; Piburn et al., 2000) was used to observe rehearsal videos and assign scores that

represent NTs' instructional practices. RTOP provides a standardized means for detecting the degree to which classroom instruction is standards-based/learner-centred or engaged versus teacher-centred. Finally, all video-recorded rehearsals and reflections were coded for instances of the 9 engagement structures from Goldin et al. (2011) study. These engagement structures illuminated how NTs "in-the-moment" affect and emotions were tied to their teaching behaviours. To explore NTs' affect in relation to rehearsals, data analysis involved both contextual analysis and cross-teacher analysis of NTs' survey responses, video observations, and journal reflections after rehearsing in TeachLivE™ to compare evidence of NTs affective domain.

### POWERFUL MATHEMATICAL AFFECT: EVIDENCE FROM REHEARSALS

First, descriptive statistics and correlations were calculated to determine means and variability for each variable and relationships between each construct. Table 1 provides the descriptive statistics for PMTE and MTOE at Time 1 and Time 2.

Subscale	N	Minimum	Maximum	Mean	Std. Deviation
PMTE Time 1	10	41	62	50.20	7.61
PMTE Time 2	6	43	65	57.17	8.66
MTOE Time 1	10	19	40	31.00	6.23
MTOE Time 2	6	28	40	35.50	5.28
RTOP Time 1	10	23	67	41.50	12.47
RTOP Time 2	9	22	68	47.56	13.95

Table 1. Descriptive Statistics

Of the 10 students who completed the TeachLivE™ Mathematics Rehearsal, only 9 had both RTOP 1 and RTOP 2 scores. One participant only had a RTOP 1 score. As the study sought to explore the effects of a rehearsing a reform-based lesson in TeachLivE™ on NTS' mathematical affect, a bivariate correlation analysis was conducted to examine the relationship between PMTE, MTOE, and RTOP scores. There was a medium positive correlation between PMTE2 and RTOP2 ( $r = .37$ ), however, it was not statistically significant. Given the small sample size, it is possible that there was not enough power to detect a difference (note: 2-tailed test \*correlation is significant at the 0.05 level, \*\*correlation is significant at the 0.01 level).

Measure	PMTE1	PMTE2	MOE1	MOE2	RTOP1	RTOP2
PMTE1	1					
PMTE2	0.95**	1				
MTOE1	0.70*	0.73	1			
MTOE2	0.78	0.64	0.86*	1		
RTOP1	-0.16	-0.17	0.12	-0.18	1	
RTOP2	0.09	0.37	0.2	0.23	0.88**	1

Table 2. Bivariate Correlations

Next, a paired-samples t-test was conducted to compare *Personal Mathematics Teaching Efficacy* (PMTE) before and after TeachLivE™ Mathematics Instruction. There was a significant difference in the PMTE scores before TeachLivE™ ( $M=52.67$ ,  $SD=8.50$ ) and after TeachLivE™ ( $M=57.17$ ,  $SD=8.66$ );  $t(5)=-3.922$ ,  $p = .011$ . These results suggest that rehearsing a reform-based lesson in TeachLivE™ does influence *Personal Mathematics Teaching Efficacy*. Specifically, the results suggest that NTs beliefs in their individual capabilities to be effective mathematics teachers increased due to their rehearsals. A paired-samples t-test was conducted to compare *Mathematics Teaching Outcome Expectancy* (MTOE) before and after TeachLivE™ Mathematics Instruction. There was a not a significant difference in the MTOE scores before TeachLivE™ ( $M=34.17$ ,  $SD=4.83$ ) and after TeachLivE™ ( $M=35.50$ ,  $SD=5.28$ );  $t(5) = -1.195$ ,  $p = .286$ . These results suggest that perhaps a clear limitation of rehearsals in TeachLivE™; since avatars are not ‘real’ students, they cannot influence NTs beliefs that effective teaching can bring about student learning.

		Mean Difference	t	df	Sig. (2-tailed)
Pair1	PMTE1 - PMTE2	-4.50	-3.922	5	0.011
Pair 2	MTOE1 - MTOE2	-1.33	-1.195	5	0.286

Table 3. Paired-Samples T-Test Results

Finally, further perusal of NTs “in-the-moment” affect during rehearsed teaching suggested the experiential learning experience proved to be a turning point for many NTs. As one novice teacher explains, “rehearsal boosts my readiness and confidence in what I am teaching.” Evidence of NTs “in-the-moment” affect includes most of the engagement structures posed by Goldin et al (2011). Evidence of the strands that make up the engagement structures were NTs belief systems (i.e. self-efficacy and self-identity) as noted in their retrospective journal reflections, while others were characterized by their behaviours oriented toward fulfilling an emotion. NTs’ meta-affect and affective pathways were also strands that helped unpack how the constellation of emotions and behaviours relate in each engagement structure, particularly with regards to understanding “the sequence of emotional states interact[ing] with heuristics during [lesson planning]” (Goldin, 2000). These heuristics included completing the lesson, pacing, scaffolding questions, and were most apparent in NTs first rehearsals. They manifested themselves through the engagement structures i) *Get the Job Done*, where the “motivating desire for task completion [was to] evoke more procedural, time-efficient strategies” that lent themselves to “comple[ing] a task procedurally” (Goldin et al, 2011; 552), and ii) *Let me teach you*, where NTs wanted to demonstrate their mathematical ability to themselves, the avatar students, and their instructors. As one NT states: “I felt more comfortable in my knowledge of the information and was able to teach without the visual aid I prepared. I think I was effective in meeting my objectives. The students were able to answer all of the verbal

summative and formative assessments during and after my lesson. The students were able to recall information, including definitions and steps of the [butterfly] method.”

In addition to engagement structures that identified emotions related to the heuristics of planning and teaching math, several NTs clearly exhibited engagement structures that mirrored their motivation for an intrinsic reward. NTs displayed increased enthusiasm and excitement. These positive affects were apparent in iii) *Check this out*, where several NTs assessed and assisted CJ when he struggled through their lesson, and in iv) *I’m really into this*, where NTs exhibited deep interest or “flow” (Csikszentmihalyi, 1990 as cited in Goldin et al, 2011) in explaining operations with fractions to CJ links how one engagement structure may have branched into another. For a few NTs, the transition to engagement structures that had more “intrinsic payoff” (Goldin et al, 2011) was due to the decreased heurism in planning a lesson. As one NT explains: “The delivery was more genuine and I believe the students took well to my lesson, because I didn’t have to focus too much on classroom management as I did the first time and the students responded better overall.” She goes on to explain “This is one lesson that after taking the critique I received the first time and using that to improve, I felt extremely comfortable in my delivery and I believe the students learned what I expected them to at the end of it all.” Thus NTs’ increased attention to areas they could improve upon from their initial feedback allowed them to become more positively enthusiastic about their delivery of their lesson plans a second time. The constructive feedback they obtained from rehearsals provided NTs a higher positive self-regard for their abilities to perform teaching fractions, and was observed through the engagement structure v) *Look how smart I am*, in addition to *Check this out* and *I’m really into this*.

NTs more positive affect also seemed to influence avatar students’ engagement, thereby displaying the interconnectedness of a NTs modelling of affect and impacting student affect. An example of this is the engagement structure vi) *Don’t disrespect me*, which is illustrated in the way NTs commanded respect with quiet pride by stating policies and instructional directions in a no-nonsense tone. This engagement structure was markedly evident even within the very short duration of rehearsals, and was characterized by NTs showing respect for everyone’s dignity, requiring it back, and providing a “safe” space for students to ask math questions without worrying about belittlement. Perhaps the overwhelming evidence of this engagement structure was because it was “socioculturally dependent,” (Goldin et al, 2011), where the avatar students were set behaving like “urban” students and the NTs recognized this easily and put an end to it—however stereotypical it may have been. One NT reflects in a “self-talk” tone, “Although I was able to shut all these [student distractions] down within the first few minutes, I heard that they were worse with some of my classmates. I see how important it is to incorporate tasks that really engage the students so you can have their undivided attention for the time that you have.” This speaks to how engagement is “bidirectional” (Goldin et al, 2011), a transaction between teachers and

students (Boykin & Nogurea, 2011), and perhaps even between one's own learning and teaching.

While many of the affective states displayed in the aforementioned engagement structures were positive, several NTs did display behaviours that can be associated with negative affect. For example, for some NTs, *Get The Job Done* was perhaps a result of “disinvesting in further conceptual learning” (Goldin et. al, 2016, p. X). However, by far the most often observed engagement structure in the data related to less positive affect is vi) *Stay out of Trouble*, where NTs avoided interactions that may lead to distress and vii) *Its Not Fair*. One example of *Stay out of Trouble* is when one NT avoided feeling vulnerable in her perceived inability to answer a question (performance-avoidance goal), perhaps due to her initial self-concept that she has low capability. NTs showed evidence of the engagement structure *Its Not Fair* when voicing their unhappiness with rehearsing with avatars, as opposed to K-12 students in the field, and several describe their frustration with TeachLivE™ limited use of space and proximity with students in their journals. Both these engagement structures often seemed connected to lower self-efficacy as it pertains to the context (modelling operations with fractions). The desire to avoid an unfavourable situation was also displayed by NTs in viii) *Pseudo Engagement*, which was observed among NTs who exhibited apathetic behaviour during teaching and relief as teaching ends. However, it is to be noted, that despite evidence of less positive affective states and traits, NTs reform-based teaching showed marked improvement for most of the NTs across the two rehearsals, with an average RTOP score of 39.6 in their first teaching rehearsal in TeachLivE™ to an average score of 46.7 during the second rehearsal in TeachLivE™ [traditional lecturers score between 0-22, active lecturers score between 23-38, and student-centred instructors score above a 38]. This paradox, of low affect but high teaching performance, is intriguing and provides evidence for a theory that low self-efficacy does not necessarily mean poor teaching practices. This may be due to the various affective *states* a novice teacher experiences while teaching, and suggests further study of teaching praxes is needed to unpack the connection between teachers' affective states and traits.

## DISCUSSION AND IMPLICATION

The evidence provided above suggests rehearse teaching in TeachLivE™, as part of an investigation into their praxes, proved to be a turning point for many NTs' powerful mathematical affect. The significant increase in NTs self-efficacy seems to be connected to NTs' Lesson Study, and more specifically their rehearse teaching. Their first TeachLivE™ rehearsals seemed to boost several NTs belief in their instructional capabilities, and this seems to be related to their positive “in-the-moment” affect. The evidence from this study suggests that rehearsals in simulated environments can not only help bridge teaching and learning and translate learning to practice in effective teacher preparation programs, but that such experiences can also have an impact on NTs' affective traits (like self-efficacy) and states. As described in an another paper, engagement structures, first theorized for students, has transcendental value in that it



can connect teachers' prior 'in-the-moment' behaviour as math learners with their 'in-the-moment' behaviour as math teachers (Khalil & Johnson, 2016). *Transcendental engagement structures (TES)* can serve as both an analytical and practical tool by linking teachers' overall affect to their praxes, and intervening when it is apparent that affect is influencing teaching and learning behaviours.

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# IN THEIR OWN VOICES: FACTORS THAT AFFECT THE MATHEMATICS LEARNING OF AFRICAN AMERICAN UNDERGRADUATE STUDENTS

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*The purpose of this study is to identify the factors that influence the mathematics learning and achievement of African American undergraduate freshmen who attended a historically Black college/university (HBCU) by examining self-reported narratives which focus on their high school mathematical experiences. The data include narratives from a cohort in 2006, four years after the authorization of No Child Left Behind, and another in 2016, six years after the Common Core State Standards Initiative. The researchers identified common themes that emerged from students' voices. Findings describe the extent to which mathematics learning related directly to in-school factors versus others such as high stakes testing. All student voices revealed their mathematical learning, and testing, experiences were racialized. This study contributes to the growing body of research that conceptualizes mathematics as part of the unjust race relations in the U.S.A.*

## INTRODUCTION

Who should define the necessary mathematical experiences of students of color in the United States? Currently, policymakers, politicians, and philanthropists influence national, state, and private initiatives designed to improve teaching and learning outcomes in public schools. Though at one time conversations of mathematics education were localized to university programs and public schools, more recently mathematics education has become the focus of multiple stakeholders including federal and state agencies, nongovernmental lobbyists, and departments of education (Ravitch, 2014). This focus has resulted in high stakes accountability policies, such as *No Child Left Behind* (NCLB), that then promoted a need to assess the effectiveness of standardized-based curricula (Sadovnik, 2008).

High stakes accountability policies, however, rely upon “technicist” metrics, such as progress monitoring and assessment, and limit context-specific decisions based upon the unique dynamics of the classroom (Zeichner, 2010). Emphasis upon such metrics inhibit learners from equitable opportunities to learn. Although recent studies have indicated that the gap in achievement among ethnic groups has narrowed over the years, the fact still remains that African Americans currently score lower than any other racial group, even when controlling for class (NCES, 2015). For this paper’s researchers, understanding the learning experiences and achievement of African American students in relationship to other ethnic groups is valuable for at least two reasons. First, it provides in an in depth lens into understanding how learners may perceive or explain their “achievement gap” as a manifestation of systemic inequities

of educational opportunity. Second, it refocuses the question back to the needs and learning of the individual student and systems that claim reforms.

## PERSPECTIVE

Most inferences about student achievement are made from data collected in quantitative studies. Often times, these are large-scale, longitudinal studies which involve tens of thousands of students. Walberg's (1984) seminal work, "Improving the productivity of America's schools," (1984) discusses a model of educational productivity that examines the interaction of factors affecting success in school learning. This model, based on an economic theory of productivity, has been continuously built on to examine relationships among inputs and outputs of "effective" educational systems. The model examines nine factors, which fall into three categories: student aptitude, instruction, and psychological environment. Student aptitude is operationalized as (a) ability or prior achievement; (b) development; and (c) motivation or self-concept. Instruction includes (d) quantity of time students engage in learning; and (e) quality of the instructional experiences. The psychological environment encompasses (f) the home; (g) the classroom peer group; (h) the peer group external to school; and (i) use of out-of-school time.

Using the education productivity model on the National Education Longitudinal Study of 1988, Thomas (2000) reports that 81% of the variance in mathematics achievement can be accounted for by the productivity factors. But even this percentage, noted Thomas (2000), "could, in fact, be an underestimate of the actual measure of the relationship of the productivity factors to [mathematics] achievement..." For African Americans, this variance can be explained by their experiences inside and outside the classroom, in spite of all the claims about reform, as well as efforts by professional organizations such as the National Council of Teachers of Mathematics (NCTM). To point, the first principle of the *Principles and Standards for School Mathematics* (NCTM, 2000) is the equity principle, aimed to address inequitable opportunities to learning mathematics experienced by underserved communities in the U.S.

Critical race scholars have noted possible explanations for African American achievement—as opposed to "achievement gap." Williams (2012) noted that "what is lacking is research that focuses on the characteristics of African American students who are successful in science, technology, engineering, and mathematics" (p. 56). Interestingly, he found that the majority of experiences of successful African American in STEM took place within their communities and was mediated by family members and other community members. McGee (2014) had similar findings, where African American pre-service teachers in her study "cited Black male fathers and close male relatives as their first mathematics teachers." Thus, the early experiences of *successful* African American scientists, engineers, and mathematicians were not attributed mainly to experiences in the formal school environment. To this end, the present study was designed to explore the factors that influence the mathematics learning and achievement of two groups of African American students by examining self-reported

narratives which focus on their high school mathematical experiences. The researchers identified common themes and factors that emerge from these students' voices and explore how these factors relate to school experiences (e.g., interactions with teachers), systemic experiences (e.g. high stakes national assessments), and other structural factors that may have influenced their mathematics learning in the United States.

## **METHODS**

The study uses phenomenology to guide the analysis to these two questions: (1) What factors do African American undergraduate freshman write about as relevant to their mathematics learning experiences and achievement? (2) Are those factors consistent across diverse groups of African American students from different times? Phenomenology is "grounded" by data (Strauss & Corbin, 1990) collected from students' recollections and perceptions of prior experiences (Smith & Osborn, 2008). The strength of this method is that data comes directly from learners and it has both theoretical and practical implications for instructional improvement (Ramsden, 1992).

### **Context**

This study took place at a private historically Black university (i.e., HBCU) in the mid-Atlantic region of the United States. The university is a culturally diverse, comprehensive, research intensive institution of higher learning which provides, per its mission, "an educational experience of exceptional quality at the undergraduate, graduate, and professional levels to students of high academic standing and potential, with particular emphasis upon educational opportunities for Black students." For the current study, the mathematics instructors for the two cohorts of students taught at the same HBCU. Cohort 1 students were "rising" freshmen enrolled in a 3-week developmental mathematics class in the summer of 2006. These students had been conditionally admitted to the university. Their performance in this course and on other measures would determine whether they were invited to return to the university for the fall semester. The class met for 2 hours each day for two weeks. The math instructor and the diverse group of students bonded quickly. During that time, the she got to know some of the students; but, clearly two weeks was not enough time for a teacher to get to know *all* of her students well enough to provide the kind of instruction that was necessary to guarantee a favourable outcome in the high-stakes decisions that were going to be made at the of end of the third week. Therefore, she asked them to write a letter to her at the end of the first week in which they were to describe their mathematical experiences during their high school years. She wanted to hear "in their own voices" about their experiences in their high school mathematics classrooms and with family or friends. This request resulted in 28 "Dear Professor" letters. At the time of writing the letters, students had just finished high school under the reauthorization of the Elementary and Secondary Education Act, better known as No Child Left Behind (2002). Cohort 2 students were enrolled in a semester long freshman class for aspiring educators in the spring of 2016. For class, students were required to conduct a comparative study to learn more about school systems here and abroad. Among the

many topics that arose was the discrepancy between the qualities of schools within the U.S. The course instructor sought to understand students' perceptions of their education's quality by asking students to write their "mathematics autobiography." She wanted to hear "in their own voices" about their experiences with mathematics and the factors they believed helped or hindered them along the way. This request resulted in 12 autobiographies written in the first person. This autobiography also served to help the instructor prepare to teach students taking her mathematics methods course in the near future. At the time of writing the letters, students had just finished high school under the Common Core State Standards Initiative (CCSSI, 2010).

### **Participants and Analysis**

Cohort 1 included 28 African American freshmen (16 females; 12 males) who were conditionally admitted to the business school at the university. The students were conditionally admitted based on their *Scholastic Assessment Test* (SAT) score and their grades mathematics and English courses while in high school. Cohort 2 included 12 African American freshman (all females) who were conditionally admitted to the Elementary Education program. The students were conditionally admitted until they passed their national teacher certification test, *Praxis I*, which is designed to measure "academic skills in reading, writing and mathematics." Students in both cohorts came from diverse geographic regions across the United States, including 19 different states.

Students' narratives are the primary source of data for this paper. To elicit the rich details that explore students' perceptions of their K-12 mathematics experiences, the instructors shared their own narratives prior to eliciting students' responses. To understand the patterns and themes of each student's experience with math, data analysis involved both contextual analysis of each student and cross-student analysis across the two cohorts. The researchers first coded inductively by using the various emerging patterns and themes that were first identified; and then deductively by comparing the interactions, motivations, attitudes, opportunities, and challenges that shaped the mathematical experiences of African American students while they were enrolled in high school, and contemplating how these experiences might influence their higher educational experiences – and beyond (Creswell, 2009).

### **FINDINGS AND DISCUSSION**

Findings include students' reflections of their high school environments and achievement and a description of these factors' effect on their engagement, motivation, and feelings about mathematics. Descriptions of their environment included students' recollection of interpersonal relationships and how these relationships affected their racialized mathematics experiences. For both Cohort 1 and Cohort 2, Walberg's model of educational productivity, which includes nine factors, was used as a basis for categorizing the themes that emerged from students' voices as they wrote about their high school mathematical experiences.

For Cohort 1, the two factors mentioned most often were (1) the quality of instructional experiences, especially their experiences with teachers; and (2) their own motivation

or self-concept about mathematics. These two factors may not be mutually exclusive because for most students, and African American students in particular, feelings of self-concept or motivation to do mathematics are shaped by their instructional experiences (Bryant, 2014; Ladson-Billings, 1997; Williams, 2012).

Quality of instructional experiences accounted for 91% (n=10) of the student responses under the category “Instruction” (N=11) which also included quantity of time students engaged in learning (9%) (n=1). Students wrote about negative interactions with their teachers at a rate four times that of positive interactions with their teachers. One student wrote about “experiences of struggling to learn from unkind, impatient, middle and high school teachers with little compassion for students.” Another student wrote about a “desire for better high school math teachers...because they were young and inexperienced and disconnected from their students.” Several students described their mathematics teachers as “older and checked out” or as “moving too quickly” (through the material). Still others viewed their classroom interactions as “racialized experiences of isolation or disconnection from [their] teachers and classmates.” Not all reported experiences were perceived as negative. One student reported successfully learning mathematics from “helpful middle school teachers”, and another student had “high praise” for her mathematics instructor in the special developmental program for “her ability to explain mathematics in a way that students could understand.”

Students’ levels of motivation and self-concept about their ability to do mathematics were evident in statements about a “fear of looking or sounding dumb or unintelligent” during instructional time. These feelings of low self-efficacy in doing mathematics prevented students from asking clarifying questions during mathematics instruction. Some students also recognized that they, too, have a role in shaping their mathematical experiences. For example, one student wrote about a “commitment to perseverance and hard work moving forward;” and another student observed that “dedication to taking personal responsibility and owning [her] role in being successful” would come by studying and focusing on her school work.

Other relationships that could have influenced students’ feelings regarding mathematics included both family and community members. However, unlike the experiences of *successful* African American scientists, engineers, and mathematicians mentioned above, students enrolled in the developmental math course reported far fewer influences from the family or community environment.

The 12 African American students in Cohort 2 reported many of the same racialized experiences in mathematics reported by the African American students in the development mathematics course (i.e., Cohort 1), but also reported experiences similar to the early experiences of the successful African American scientists, engineers and mathematicians (Williams, 2012). The students in Cohort 2 were preparing to be elementary school teachers whose professional responsibilities would most likely include the teaching of mathematics.

Using Walberg's model as a basis for analysing their responses about influences on their mathematical experiences, we found that the factors cited most often were the same as for Cohort 1: (1) the quality of instructional experiences and (2) their motivation and self-concept about mathematics. What is different in their results, however, is the relationship between their perceived positive and negative interactions with teachers during their high school years. For these students, 83% (n=12) of the responses under "Instruction" (N=10) were attributed to interactions with teachers and 17% (n=2) were attributed to quantity of time students engaged in learning. Furthermore, the number of reported negative and positive interactions with teachers were almost equal.

Motivation/self-concept and ability also played a role in the overall experiences of the students in Cohort 2. These two factors fell under the category of "Student Aptitude". Unlike the students in Cohort 1, many of these students had taken four years of mathematics while in high school. Several students had taken Honors or AP Calculus, most had taken and performed well in Algebra I, Algebra II, Geometry, and Pre-calculus. Enrollment in advanced-level mathematics courses recorded by these students does not mean that they all had positive instructional experiences. Most of these students did well because they exhibited relatively high self-confidence, persistence and perseverance in doing mathematics and being successful in it. For example, one student described how her experiences with her mathematics teachers and her own approach to learning changed as his progressed through high school: "I now entered Algebra I, ... I grasped some concepts and struggled with others...My teacher did a poor job of thoroughly explaining some concepts...I received a C my freshman year. After freshman year, I vowed to myself to always pay attention in math class and always ask questions ...as I needed to in order to understand the content...From sophomore to senior year, I completed math with a B...Overall, I believe my level of engagement and teacher-student discourse plays a huge role in my understanding of mathematical concepts."

Another student wrote: "Freshman year, I took Algebra one and I absolutely loved algebra. Being one of the few people in my class to actually understand and do well always made me feel good. Me and my teacher were always really close because of that and that really had a positive effect on my math experience for the year....Junior year I took algebra 2 and I loved this class as well..., but the teacher and I did not really did not clique...This took away from my math experience, but I still received an A."

Some students had negative experiences early in high school, but still managed to stay connected with mathematics. One student reported: "My freshman year of high school I remember I had Geometry with my teacher...and he wasn't the most clear and understanding teacher...I had Algebra 2 with my favourite teacher...an amazing teacher because he was so knowledgeable in math, and could explain even the most difficult concepts well. He was thorough and made math a very smooth class that year. My Pre-calculus teacher was always telling jokes, barely taught, and would get very irritated when you had a question...After I escaped from that class, I took AP

Statistics...I know I am good at math, I just need the right teachers to explain concepts to me well enough so I can understand and master them.” Another student wrote: “When I arrived in high school the content got more difficult and I was not encouraged to learn and understand the content...my teachers did not know how to engage their students and I get distracted easily, so I did not try harder.”

The students in Cohort 2 were also mindful of the fact that in order to teach in some school districts in the U.S., they had to be prepared to take and pass standardized tests in order to be licensed and certified. Several students juxtaposed traumatic K-12 mathematical experiences caused by high-stakes standardized testing with positive ones in what a couple of students described as a “love/hate” relationship with mathematics. The trauma, or “storm” as one student described it, was more due to the high-stakes preparation including timed testing. For students who tested poorly, enduring such “trauma” caused them to develop anxiety *with respect to standardized test-taking* and to doubt their ability to perform well in assessments. Alternatively, for students who performed well on assessments, test preparation was not the focus of their narrative; rather, the narratives’ foci were on the factors that influenced their ability in and enjoyment of performing well in mathematics.

## IMPLICATIONS

Aside from racialized experiences, positive and negative interactions with teachers, and participant affect with respect to math in general and standardized testing in particular, further research and analysis is needed. To generate more understanding, particularly as it relates to the high stakes national policies that influence mathematical learning in the United States, a larger mixed-method study could shed light on more theory and provide data of their generalizability. Findings from this pilot study will be used to guide the design of mathematics courses at this HBCU and other Minority Serving Institutions. This study will offer important implications for how student reflections may be used to foster better student-teacher interactions. Establishing discussions on identifying a paradigm for understanding how context matters, and how it connects to students’ affective domain and mathematical learning, is pivotal in creating opportunities to learn mathematics and close the “achievement” and opportunity gap (Boykin & Noguera, 2011).

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# EXPLORING REAL AND COMPLEX IMAGES OF THE SQUARE ROOT CONCEPT AMONG UNIVERSITY STUDENTS

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*The goal of the study was to characterize the images of the square root concept of university students with a focus on its different meanings in the fields of real and complex numbers. The characterization was made in terms of the set approach (two opposite numbers that when squared equal to the radicand) and the principal root approach (a singular number either from the right or the upper half of the complex plane). The data was collected with an online questionnaire that was answered by 115 students enrolled in an Israeli technological university. Statistical data analysis revealed that in the field of real numbers students distinguish between the notions of a square root and the radical sign. In the field of complex numbers students' concept images were incoherent and depended on the stimulus.*

## FOCUS OF THE STUDY

In their mathematical journey from primary school to high school, students expand the number system in which they work. Some scholars notice that the journey from natural through integer and rational to complex numbers bares resemblance to the historical development of mathematics (see Jankvist, 2009 for an elaborated review). A common justification for each extension is that in the new number system we can solve problems that we could not solve before. For example, the extension from natural numbers to integers allows subtracting a bigger number from a smaller number, the extension from real to complex numbers provides two solutions to the equation  $x^2 + 1 = 0$ .

In this expansional approach the emphasis is naturally put on concepts and problems that emerge in a new number system and could not have been addressed in the previous one. However, in some cases shifting to a new system of numbers entails a modification of concept definitions, meanings of symbols and problem-solving approaches. For example, for determining  $\sqrt[3]{8}$  in the field of real numbers, we could employ an exponential law and get 2. Accordingly, the symbol  $\sqrt[3]{8}$  represents a singular number that when cubed equals to 8. In the field of complex numbers we can represent the number 8 in polar coordinates, determine its third roots with an extension of the de Moivre's formula and get three results. Consequently, in this field the exercise  $\sqrt[3]{8} - \sqrt[3]{8}$  is either meaningless or requires a redefinition of notations (see Zazkis, 1998 for additional examples of concepts that are sensitive to the number system).

The study reported in this paper is a part of a larger international research on students' and teachers' understanding of mathematical concepts. The study was concerned with the images of the square root concept of university students, with a focus on its different meanings in the fields of real and complex numbers.

## MOTIVATION FOR FOCUSING ON THE ROOT CONCEPT

Roots or radicals can be considered a fundamental concept due to its multiple connections and appearances in various mathematical contexts. For example, in the Israeli curriculum the square root is introduced in the 7<sup>th</sup> grade in the context of exponents of integers. In the 8<sup>th</sup> grade students learn to determine square roots of rational numbers and use them in geometrical context in the Pythagorean theorem and for calculating the length of a side of a square with a given area. In the 9<sup>th</sup> grade square roots are recalled before solving quadratic equations. In high-school, a square root is extended to an  $n$ -th root, and a focus is put on its connections with exponential and logarithmic functions. At the end of the 12<sup>th</sup> grade, students are introduced to complex numbers, where roots appear again.

In undergraduate mathematics the square root function is frequently used in calculus for exemplifying continuous and inverse functions. In the course of complex functions a root is a common example of a multivariable function. In abstract algebra the concept is generalized for a group structure. Based on the above, it can be argued that developing a coherent image of the root concept is crucial at multiple stages of mathematics learning. In this paper I focus on a square root as the most basic but still interesting case of the overarching concept of a root.

## THEORETICAL BACKGROUND

### Textbook perspectives on the square root

Although the square root concept could be considered in the larger context of rational exponents, the picture is complicated enough when focusing solely on it. In Israel there is no consensus on the definition of a square root of a real number. In the curriculum for the 7<sup>th</sup> grade, a number  $y$  is a square root of a number  $x$  (a *radicand*) if  $y^2 = x$  (e.g., Yaquel, 2004; Goren, n.d. a,b). Accordingly, a square root of a positive radicand is a *set* of two opposite number, each of which is a square root of  $x$ . However, according to definitions of other textbooks (e.g., Zaslavsky et al., 2012)  $y$  is a non-negative number. Thus, under this definition a square root is a singular number.

While textbook authors agree that the radical sign ( $\sqrt{\phantom{x}}$ ) is preserved for the positive root of a number, the set approach entails that determining roots of a positive  $x$  is not equivalent to determining  $\sqrt{x}$ . However, in the exercises the authors use such formulations as “*Provide both roots of 9*” (e.g., Goren, n.d. a) and conceal the difference between roots and a radical sign.

In the field of complex numbers the authors of Israeli high-school textbooks seem to agree that every number has two square roots, which is in accordance with the set approach. However, disagreements exist in regard to the radical sign: For Yaquel (2004),  $\sqrt{z}$  is a set of two numbers for any non-zero  $z$  (*set approach*). Goren (n.d. b), on the other hand, uses the sign only once for defining  $i = \sqrt{-1}$ .

Providing a singular result as a root of a complex number can be justified with the *principal root approach* for complex functions (e.g., Starkov & Stepenko, 2007). Indeed, for a complex variable a square root is a multivalued function with two single-valued branches. Accordingly, in this approach the radical sign of a number  $z = re^{i\theta}$  is  $\sqrt{z} = \sqrt{r}e^{i\frac{\theta}{2}}$ , which is a singular number either from the right half of the complex plane (for  $-\pi < \theta \leq \pi$ ) or the upper half of the plane (for  $0 \leq \theta < 2\pi$ ). Note that in this approach the meaning of the square root sign for real non-negative radicands is the same in the fields of real and complex numbers.

In Israel the set approach seem to dominate in high-school textbooks in the context of the radical sign of complex numbers (e.g., Yaquel, 2004). It is also notable that in many textbooks the only real number the roots of which are readdressed in the field of complex numbers is 1 (see Focus of the Study again for the idea of extending the number system). Accordingly, the repertoire of learners' opportunities to appreciate the difference between working in the field of real numbers (e.g.,  $\sqrt{9} = 3$ ) and complex numbers (e.g.,  $\sqrt{9} = \pm 3$ ) are limited. Thus, it is interesting to find out whether students are aware to the differences between the meanings of square roots in these fields.

### Concept image of a square root

*Concept image* has been introduced by Tall and Vinner almost 35 years ago and it remains one of the most used constructs in mathematics education literature (Bingolbali & Monaghan, 2008). The concept refers to the total cognitive structure that a learner associates with the concept, which includes all the mental pictures, properties and processes (Tall & Vinner, 1981). Tall and Vinner suggested that it is unlikely that a learner operates with the whole concept image at once and assumed that various stimuli evoke partial concept images. Accordingly, mathematics education research has been often concerned with exploring tensions among the evoked concept images (e.g., Bingolbali & Monaghan, 2008; Hiebert, 2014; Tall & Vinner, 1981).

While experienced teachers are well-aware to the complexity of teaching and learning roots, empirical literature on students' images of this concept is scarce. Pitta-Pantazi, Christou and Zachariades (2007) indicated that understanding of roots is a generalized level of exponent understanding. In her study on perfect squares, Hiebert (2014) found that high-school students confuse the notions of square number and square root. The researcher probed students' images of square numbers concept with different stimuli and revealed that they are sensitive to number representation. In my study I was also interested in exploring the robustness/sensitivity of students' concept images.

### RESEARCH GOALS AND QUESTIONS

The overarching goal of the study was to characterize the images of the square root concept of university students. Specifically, I was interested in exploring their sensitivity and robustness to variations of the fields, radicands and formulations of the stimuli. Accordingly, the questions of the study were:

1. What approaches to the square root concept prevail in students' images in the fields of real and complex numbers?
2. Do students change their approaches when considering square roots of positive radicands in the fields of real and complex numbers?
3. Do students change their approaches as a response to variations of the stimuli?

## METHOD

### The questionnaire

The data was collected with a questionnaire consisting of fifteen items divided between two main parts. Cronbach's alpha of the questionnaire was 0.92, which indicates a high level of internal consistency with this specific sample.

The first part of the questionnaire was targeted at research questions 1 and 2, and it comprised multiple-choice items that asked participants to determine square roots of positive, negative and non-real numbers. Half of the items were formulated verbally (e.g., Indicate root(s) of 9 ...) and in the other half a radical sign was used (e.g.,  $\sqrt{9} =$ ). Each item was accompanied by two possible answers: opposite numbers that when squared equaled to the radicand. Participants could choose either one of the answers, both of them or provide answers of their own. The second part of the questionnaire was targeted at the third research question, and it contained items that asked participants to provide numeric answers to basic calculation exercises with radical signs (e.g.,  $\sqrt{25} - 3i =$ ,  $\sqrt{-1} \times \sqrt{-9} =$ ). The participants were encouraged to explain their answers, however due to space limitations of this paper their explanations are not addressed.

### Participants

The questionnaire was spread in a closed Facebook group for students who are enrolled in one of the most prestigious technological universities in Israel. One hundred fifteen students responded to the questionnaire. At the time of data collection the average respondent was 24 years old ( $SD=3.54$ ) and 96% were in the first year of their undergraduate studies in engineering. All the participants had already taken at least one university course in calculus or linear algebra, and thus were familiar with complex numbers from high-school and tertiary mathematics.

### Data analysis

The data analysis focused on students' answers that corresponded either with the set or the principal root approaches. Accordingly, the approach was a dichotomous dependent variable that was explored in the study. For determining the prevailing approach (research question 1) the Chi-Square-Goodness-of-Fit-Test was applied (Guilford & Fruchter, 1987). The null hypothesis of the test was that the proportion of students who responded in accordance with the set approach is equal to the proportion of those who responded in accordance with the principal root approach.

The McNemar's test was used for exploring whether students' change of approach between two items is statistically significant (research questions 2 and 3). This test is applied for comparing two corresponding proportions and it can be considered as a modification of the paired-samples t-test for a dichotomous variable (Guilford & Fruchter, 1987). The null hypothesis of the test was that the proportion of students who answer in accordance with the principal root approach in the first item is equal to the proportion of those who respond in accordance with this approach in the second item. The data was analyzed with the IBM SPSS STATISTICS 23 software.

## FINDINGS AND DISCUSSION

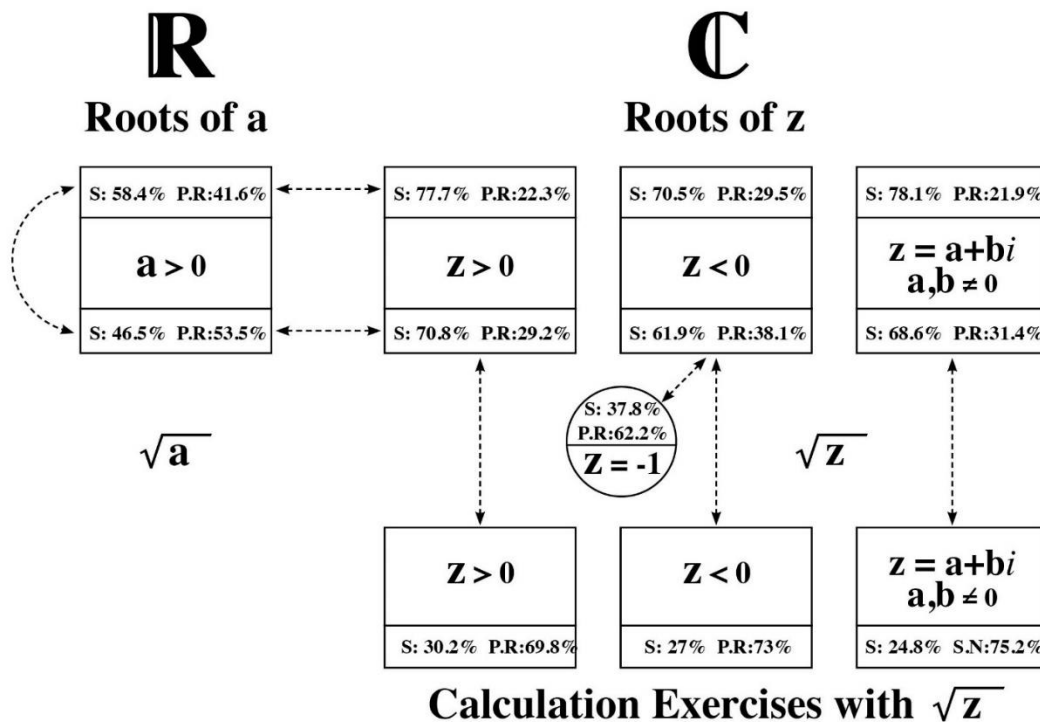
The findings of the study are schematically summarized in Figure 1. The dashed arrows represent a statistically significant change in students' approaches. Next, the findings are explained in more details.

The data analysis showed that in the field of real numbers there was no statistically significant difference between the proportion of students who responded in accordance with the set approach and the proportion of those who answered in line with the principal root approach. This finding was robust to item formulation. In regard to the verbally formulated items, the finding is in line with Israeli textbooks, in which some authors consider a square root of a number as a set (e.g., Yaquel, 2004; Goren, n.d. a, b) and others as the principal root (e.g., Zaslavsky et al., 2012). However, it was unexpected that for 47% of the students a square root sign will evoke the concept image of two numbers. To recall, the participants of the study have three years of high-school experience and they studied at least one course in calculus, where a radical sign was used for denoting a function that produces a singular output. In addition, the McNemar's test revealed a statistically significant change in students' approaches between items with verbal and symbolic formulations ( $p = 0.002$ ). Thus, it can be concluded that in the field of real numbers, students distinguish between a square root and the principal root of a number denoted with  $\sqrt{\phantom{x}}$ , which concurs with the Israeli curriculum. However, students do not agree on the meanings of both of them.

The McNemar's test revealed a significant change in students' approaches to square roots and radical signs of positive radicands between the fields of real and complex numbers ( $p = 0.0$ ). Indeed, in the field of complex numbers the proportion of students who responded in accordance with the set approach was significantly different and was larger than the proportion of those who responded in line with the principal root approach for positive and non-real radicands ( $\chi^2(1) = 19.549, p = 0.0$  and  $\chi^2(1) = 14.486, p = 0.0$ ). Thus, it can be concluded that students' image of the square root concept is sensitive to the field.

When negative radicands were under consideration, the proportion of responses in accordance with the set approach prevailed ( $\chi^2(1) = 6.45, p = 0.01$ ) as long as the radicand was not  $-1$ . Indeed, when determining  $\sqrt{-1}$ , the proportion of students who answered  $i$  was significantly different and larger than the proportion of those who responded  $\pm i$  ( $\chi^2(1) = 7.00, p = 0.0$ ). In line with this finding a statistically

significant change was identified in students' approaches to this item compared with positive, negative and non-real radicands ( $\chi^2(1) = 33.23, p = 0.0$ ;  $\chi^2(1) = 19.18, p = 0.0$  and  $\chi^2(1) = 22.13, p = 0.0$ ). Accordingly, it can be concluded that in students' concept image  $\sqrt{-1}$  has a special status compared to other complex radicands. I suggest that this is related to a number  $i$ , which is often defined as the square root of -1 (Yaquel, 2004; Goren, n.d. b).



S – Set approach; P.R – Principal Root approach; S.N – Singular Number approach.

Figure 1: Findings of the study

A statistically significant change was indicated between students' answers to the first and the second parts of the questionnaire ( $\chi^2(1) = 18.23, p = 0.0$ ,  $\chi^2(1) = 22.67, p = 0.0$ ,  $\chi^2(1) = 19.37, p = 0.0$ ). In the items where a square root sign was a part of a calculation exercise, about seventy percent of the students responded with a singular number. This approach was significantly different and prevailed over the set approach ( $\chi^2(1) = 17.95, p = 0.0$ ,  $\chi^2(1) = 15.68, p = 0.0$ ,  $\chi^2(1) = 15.97, p = 0.0$ ). When a square root sign was applied to real radicands, students' answers were in accordance with the principal root approach, i.e. belonged to the positive half of  $x$  axis for a positive radicand and to the positive half of  $y$  axis for a negative radicand. However, in the exercises such as  $\sqrt{(-3 - 4i)^2 - 1}$  the prevailing answer was  $-4 - 4i$ .

To summarise, in the field of complex numbers students' concept image seem to be sensitive to the stimuli: If a stimulus is concerned with determining roots of a number that is not -1, the set approach is employed. If a stimulus contains a calculation with a real radicand, the majority of the students respond with a singular number that

corresponds with the principal root approach. In the case of non-real radicands it seems that students follow the intuitive rule of “*roots cancel exponents*”.

## CONCLUDING REMARKS

The literature contains many examples of mathematical concepts with incompatible definitions (e.g., Tirosh & Even, 1997). Research has also accumulated a considerable amount of evidence for students holding incoherent and malformed concept images (e.g., Hiebert, 2014; Tall & Vinner, 1981). In this study I considered both phenomena suggesting that the lack of consensus among high-school textbooks on the definition of a concept (e.g., Yaquel, 2004; Goren, n.d. a, b; Zaslavsky et al., 2012) does not enhance students’ development of a coherent concept image. Indeed, the study showed that in the field of complex numbers for a critical mass of university students a concept image of the square root is sensitive to the stimulus. Thus, relearning the concept from high-school at the tertiary level does not seem to contribute to students’ concept image.

The situations in which mathematics textbooks do not speak with one voice increase the responsibility of the teacher. Indeed, s/he should be aware to the existence of competing definitions and her or his choice of either one of them should be informed by its mathematical consequences. Tirosh and Even (1997) suggest that considering alternative definitions may become a fascinating experience for the teachers, in which they are exposed to the nature of mathematics. I concur and suggest that such experiences can contribute to students as well and shape their views of mathematics. Moreover, for developing a coherent concept image, I call upon teachers to inform their students of the existence of alternative definitions (e.g., a set vs. the principal root) and provide them with stimuli that highlight the differences between the alternatives.

In this study students’ images of the square root concept were unorthodoxly explored with statistical methods. On the one hand, such analysis allows making cautious suggestions regarding large populations. On the other hand, it does not take into account statistically insignificant groups of learners the (mis)conceptions of whom deserve attention. For instance, 55 students responded that in the field of real numbers  $\sqrt{9} = \pm 3$ ; in the field of complex numbers 14 students provided a different number of answers when determining  $\sqrt{169}$  and  $\sqrt{529 + 0i}$ . Moreover, the analysis only suggested whether students’ concept images correspond either with the principal root or the set approach. Accordingly, in the larger study the images of the square root concept will be further explored.

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# STATIC AND DYNAMIC: NEGOTIATING INCOMPATIBLE PERSPECTIVES ON ANGLE

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*This study is concerned with tensions between the two different perspectives on the concept of angle: angle as a static shape and angle as a dynamic turn. The goal of the study was to explore how teachers cope with these tensions. We analyse scripts of 16 in-service secondary mathematics teachers, which feature a virtual dialogue between a teacher and students around the following statement: “The sum of the exterior angles of a polygon is  $360^\circ$ ”. The findings show that while addressing a variety of intellectual needs of their imagined students, in many cases the teachers compromise the mathematical rigour.*

## INTRODUCTION

The concept of angle is fundamental in elementary geometry. Research shows that this concept is challenging for various populations of students (e.g., Hadas, HersHKovitz & Schwarz, 2000). The difficulties students experience have been attributed to the existence of various perspectives on the concept (e.g., Freudenthal, 1973), and research has focused on developing the pedagogy for bridging among them (e.g., Mitchelmore, 1998).

However, Freudenthal (1973) argued that some perspectives stem from different axiomatic systems and consequently, they are not always compatible. In this paper we place particular emphasis on tensions between the different perspectives on the concept of angle and explore how teachers plan to address these tensions in an instructional interaction. To evoke the tensions we asked in-service secondary mathematics teachers to write a script for a virtual dialogue in which teacher- and student-characters explore *The Conjecture*: “The sum of the exterior angles of a polygon is  $360^\circ$ ”.

## THEORETICAL BACKGROUND

In the first part of this section we provide a mathematical perspective on the concept of angle and the Simple-Closed-Path Theorem, a variation of which we presented to our teachers. The second part consists of the framework of intellectual needs that we used for formulating research questions and data analysis.

### Mathematical perspective

Freudenthal (1973) distinguishes among four perspectives on the concept of angle, two of which are relevant to our study: an angle as a static geometric shape and an angle as a dynamic turn. In the *angle-shape* perspective, an angle is a part of a plane captured between an unordered pair of half lines (rays) with a common origin. In this way two angles are created, but the conventional angle is the smaller one between the two. The angles are measured with a semicircle protractor and their magnitude is positive and

less than  $180^\circ$ . In the *angle-turn* perspective, an angle is created between an ordered pair of half lines with a common origin. According to this perspective angles are measured with a circular protractor, when conventionally a positive measure is obtained counter-clockwise, and a negative measure is obtained clockwise. Thus, the angles are directed and calculated modulus  $360^\circ$ .

For the purposes of this study we are interested in the sum of the exterior angles in a polygon. An exterior angle is formed between a side of a polygon and an extended adjacent side. According to the angle-shape perspective two exterior and congruent angles are located at each vertex, when either one (and only one) of them is considered for the sum. Note that in the case of a concave polygon, at least one of the exterior angles is located in the interior of a polygon. We will refer to its vertex as *the concave vertex*. When measuring exterior angles with the angle-turn perspective, the reader should imagine herself/himself walking along the perimeter of a polygon. A measure of an exterior angle is the directed amount of turn needed to pass from one side of a polygon to the adjacent side at each polygon's vertex. At each vertex two such turns can be made: counter-clockwise and clockwise. The conventional turn is the smaller one among the two.

In the case of simple (i.e., non-intersecting) polygons, The Conjecture can be approached with the Simple-Closed-Path theorem, which states that, "A total turning in a simple closed path is  $360^\circ$ " (Abelson and diSessa, 1986, p. 24). Note that the theorem is concerned with turning, not angles, but "angle-turn" perspective is implied. In the case of convex polygons The Conjecture can be also approached with the angle-shape perspective. It can be proven by mathematical induction, or by relying on the formula for the sum of interior angles, or by other means. However, the sum of measures of the exterior angles in a concave polygon, according to the angle-shape perspective – where the measure of each angle is positive and less than  $180^\circ$  – results in more than  $360^\circ$ . Thus, The Conjecture is invalid in the case of concave polygons when one takes this static view on the angle concept.

### Intellectual needs

In the classical view of Piaget (1985), learning occurs when one tries to resolve a mental disequilibrium in an attempt to achieve a temporary equilibrium through assimilation and accommodation processes. Harel (2008) associates a mental disequilibrium with situations, in which the current state of knowledge is insufficient or incompatible and additional piece of knowledge should be acquired.

According to Harel (2008), the new piece of knowledge is instigated by five types of *intellectual needs* that a learner can experience. We elaborate on four of them, as these are relevant for our purposes: (1) *The need for certainty* emerges when a learner has doubts about the correctness of a particular conjecture; (2) *The need for causality* is a desire to identify the reasons for the occurrence or non-occurrence of a particular phenomenon; (3) *The need for communication* is the need to establish common definitions, notations and conventions to persuade others that a statement is true; and

(4) *The need for connection and structure* consists of the desire to organize obtained knowledge into an easy approachable structure. Harel's (2008) list can be extended with Koichu's (2008) principle of intellectual parsimony. The principle states, that when solving a problem a person tends not to make more intellectual effort than the minimum needed for obtaining a solution. We associate this principle with *the need for efficiency*.

## RESEARCH GOALS AND QUESTIONS

The goal of the study was to characterise the approaches that teachers employed in scripted instructional interactions for addressing potential tensions between the angle-shape and angle-turn perspectives. The research questions that guided our investigation were: What perspectives on angle were applied for various polygons considered in teachers' scripts? How are the tensions between angle-shape and angle-turn perspectives on angle addressed? What intellectual needs of imagined students were fulfilled with the chosen approaches?

## METHOD

Sixteen secondary mathematics teachers participated in this study. Their mathematics background and experience varied significantly, but all held degrees in Mathematics or Science and had at least three years of teaching experience. The teachers responded to a script-writing task, in which they were asked to create a virtual dialogue between teacher- and student-characters, who explore The Conjecture. The participants were explicitly requested to associate themselves with teacher-characters and address any 'problematic' issues that they anticipate with of The Conjecture. Teachers' scripts comprise the data corpus of our study.

We chose script writing as a data collecting tool in light of its benefits in advancing and investigating teacher knowledge (e.g., Koichu & Zazkis, 2013). To script-writers, writing a virtual dialogue provides an opportunity to develop and exhibit their professional competency. To researchers, the created scripts provide a window into writers' mathematical understanding of concepts and claims, their anticipation of student difficulties and their pedagogical sensitivity in helping students. Our underlying assumption was that scripts reflect the actions that the participants would carry out in compatible classroom interactions.

The collected scripts were analysed according to the principles of content analysis (Krippendorff, 1980). At the first stage, we focused on the similarities among the scripts. Each script was fragmented into self-contained sections, each of which focused on a particular tension between the (often implicitly) perceived conceptualization of angle and/or of The Conjecture. At the second stage, we examined the differences between data excerpts related to compatible topics. As a result, we identified various approaches to exploring The Conjecture. We organize our findings according to these approaches.

## FINDINGS

Only in one script (out of 16) The Conjecture was explored for convex and concave polygons with the angle-turn perspective exclusively. Consequently, no tensions between the perspectives on angle were identified. In what follows we present three approaches to The Conjecture with a focus on the tensions between the perspectives on angle that were identified in the remaining scripts. In some of the scripts more than one approach were employed. We explain the choice of a particular approach by noting what intellectual needs it may address.

### **The Conjecture is explored for regular polygons only**

Two script-writers employed angle-shape perspective and focused their virtual dialogues on regular polygons exclusively. This focus was motivated by teacher-characters who claimed that *“Regular polygons, it is a wonderful thing to investigate”*. Then the discussions turned to developing a well-known formula for calculating the sum of angle-shapes in regular polygons. Afterwards, the formula was extended to the sum of exterior angles in regular polygons. Interestingly, one of these scripts ended with the Teacher-Character saying: *“We have run out of time for today. For the next lesson please consider our discussion and review our conjecture for other polygons...”* This quote demonstrates that the script-writer is aware of the existence of other polygons, but decides not to address this issue.

The decision to limit virtual discussions to regular polygons can be explained by *the need for efficiency*. Indeed, excluding other polygons from discussions prevented the need for shifting to angle-turn perspective and enabled script-writers to direct discussions to less complicated issues. Moreover, the angle-shape perspective that script-writers used in this approach is more common in a local school setting. Thus, it is more familiar and consequently less effort-demanding for the participants and their imagined students. In a way, employing this approach also satisfied *the need for certainty* in the correctness of The Conjecture and the need for keeping the structure simple by limiting the example space.

### **Attributes of angle-shape and angle-turn perspectives are conflated**

In six scripts the characters explored The Conjecture with a particular perspective on angle, but integrated attributes from another perspective. For instance, in one of the scripts the characters agreed:

- 1     S1:     Look at the exterior angles of a simple ordinary square. It has eight exterior right angles.
- 2     S2:     No, I think you're only supposed to measure one exterior angle per vertex. I'm trying to remember why that is, but I think it's explicitly part of the problem.
- 3     S1:     Well, it [The Conjecture] works as long as you always measure the angles which point in the same direction around the polygon: clockwise or counter-clockwise. You can't measure some that go clockwise and others that go counter-clockwise.

- 4 T: OK, then we'll rephrase our conjecture: "The sum of the exterior angles of a polygon, taken one per vertex and in the same direction, sum to  $360^\circ$ ".

Indication of two equal angles at each vertex ([1]-[2]) is consistent with the angle-shape perspective. However, the mention of an angle direction ([3]-[4]) corresponds to the angle-turn perspective. As a result, the formulation in [4] contains an unnecessary condition.

The words of the Teacher-Character from one of the scripts suggest an explanation for the appearance of such conflating formulations. She said: "*When the polygons become more sophisticated, we have to be fussier about what we mean. We want to make sure that anyone else that reads our proof is clear on what we mean and can follow our ideas.*" Consequently, mixing the attributes of angle perspectives can be governed by *the intellectual need for communication*.

### **Angle-shape perspective is extended with a negative measure**

In the beginning of ten virtual dialogues a teacher-character presented The Conjecture and student-characters verified it for convex polygons with the angle-shape perspective. At some point a concave polygon was introduced as a counter-example to The Conjecture, however teacher-characters engaged student-characters in further discussion. As a result, the idea of a negative measure was introduced to resolve the 'problematic' exterior angle in a concave polygon.

The spectrum of student-characters' reactions to the described extension of the angle-shape perspective was rather broad. For example, in one of the scripts the extension was rejected and a student-character said: "*Teacher, it seems like you are just changing the signs of the two exterior angles of the concave sections of the polygon to fit your conjecture. You have no good reasons to back up your random decision except for the need of wanting to be right!*" In this case, the rejection can be explained with two intellectual needs: The extension was not supported mathematically for concave polygons, and thus *the need for certainty* remained unfulfilled. Moreover, limiting The Conjecture to convex polygons addresses *the need for structure*, as the structure is kept simple.

In three scripts an extension of the concept of angle as a shape with a negative measure was immediately adopted by virtual characters. Thus, the script-writers overlooked a contradictory situation in which an angle with a negative measure existed between unordered half-lines. Some student-characters were disturbed by the extension and said: "*I don't understand. How can an angle in a polygon be negative?*", "*How does it make sense that an exterior angle is on the interior of the polygon?!*". However, the extension was forced by other student-characters, who argued: "*Don't you want the conjecture to work? Let us call it 'not-always-exterior angles' and the problem is solved*". The fact that the extension "made the conjecture work" for concave polygons fulfilled *the need for certainty*. Introducing the phrase "not-always-exterior angles" appeals to *the need for communication* as the tension between the name of the angles and their possible location appeared resolved.

In one of the scripts *the need for causality* was addressed by presenting an original justification for the extension: “Originally when we extended the sides of polygons to make exterior angles they went outside, but at concave vertices the extended side went inside the polygon - that’s opposite. When we extended sides outside the polygon, we added angle. So instead of adding, the angles that go **inside**, we have to **subtract** them”. Inspired by Stavy and Tirosh’s (2000) research on intuitive rules, we refer to this justification as “Inverse A–Inverse B”. It contains an invented rule that creates a correspondence between two binary variables: The first variable represents the location of an exterior angle, either outside (A) or inside (inverse A) of a polygon. The second variable corresponds to the sign of measure of an angle under discussion, which can be positive (B) or negative (inverse B). Although the invented explanation was attributed to a student-character, it was accepted and praised by the teacher-character. As such, we conclude that this explanation was acceptable for the script-writer.

## SUMMARY AND DISCUSSION

In this study we explored the approaches that in-service secondary mathematics teachers used for addressing possible tensions between the perspectives on angle as a static geometric shape and as a dynamic turn. The study contributes to the existing research on the concept of angle, which in the case of students has been mainly concerned with bridging between multiple perspectives on the concept (e.g., Mitchelmore 1998); and in the case of teachers, has been focused on the measurement perspective (e.g., Akkoc, 2008).

For capturing teachers’ approaches, we invited them to compose a script for a dialogue between a teacher-character and student-characters around The Conjecture “The sum of the exterior angles of a polygon is  $360^\circ$ ”. In their scripts, the teachers (in the role of teacher-characters) were passionate, motivating and sensitive to the variety of intellectual needs of their imagined students. However, in many cases the teachers did not coordinate their sensitivity to students’ needs with mathematical rigour.

The research tool of script-writing is aimed at capturing the amalgam of teachers’ mathematical knowledge and pedagogical considerations (see Zazkis & Kontorovich, 2016 for a possible way to separate between the two). Accordingly, the collected data does not provide a direct evidence for inferring that the employed approaches are reflections of teachers’ personal conceptualizations of angle concept. However, we suggest two indirect arguments for this inference: *First*, the research on conceptualizations of angles provides multiple evidence for students’ difficulties in coordinating various perspectives on the concept. For instance, Krainer (1991) showed that a thirteen year old student conceptualized an angle as a figure consisting of two rays (angle-shape perspective), but at the same time insisted on using arrows that pointed “upwards” for indicating angle’s measures and distinguished between ‘right’ ( $\perp$ ) and ‘left’ ( $\lrcorner$ )  $90^\circ$  angles (angle-turn perspective). Mitchelmore (1998) found that even when possessing the ideas of rotations (the angle-turn perspective) and an idea of an angle as two rays originating from the same point (the angle-shape perspective),

young students did not connect the ideas into a coherent whole. In the scripts of our teachers were also not coordinated. Accordingly, we surmise that the scripts reflect the lack of coordination that teachers have been carrying since school.

*Second*, Koichu and Zazkis (2013) invited mathematics education students to interpret a proof of a particular theorem in a form of a scripted dialogue. The researchers noted that the students who were unsuccessful in sense-making of the core ideas of the proof, devoted an extended attention in their scripts to unsophisticated details. Koichu and Zazkis referred to this approach as a “pedagogical shield”, that is, a choice of pedagogical approach that protects the script-writers from exposing personal difficulties with the ‘real’ problematics of the proof. In the case of our study, focusing only on regular polygons can be considered as such pedagogical shield.

Exploration of the origins of the identified approaches is an evident avenue for further research. However, based on the findings, we propose enriching teachers’ engagements in the multifaceted nature of the concept of angle with the cases, in which the facets are incompatible. The concept of angle is fundamental in mathematics, and these engagements can prepare teachers for exposing its full complexity to their students.

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# HOW DO STUDENTS SOLVE COMBINATORIAL PROBLEMS? SOME RESULTS OF A RESEARCH ABOUT DIFFICULTIES AND STRATEGIES OF HUNGARIAN STUDENTS

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*Our research is the first step of an extensive investigation of mathematics instruction in Hungary. Combinatorics had been chosen because teaching and learning combinatorics is a problematic and relatively new domain without tradition in our country. In the study we analyze how students, grade 6-7 and grade 10-11 solve combinatorial problems. We examine the effects of three main aspects of the problem: the cardinal number of the set, the number of conditions and whether students have to separate the problem into different cases. We analyze students' enumerative techniques and their representations. According to our main hypothesis we found that older students use mainly ready-made formulas and they fail when these formulas are not applicable.*

## INTRODUCTION

Teaching and learning combinatorics can encourage development of systematic thinking, text comprehension and problem solving ability in students. Combinatorial problems give students opportunities to engage in the mathematical processes of representation, reasoning, abstraction, and generalization (Sriraman, & English 2004). It follows that teaching combinatorics should be an important part of school mathematics and the investigation of its teaching methodology can help teachers. In Hungary, combinatorics is included in primary school mathematics since 1978 (Halmos, & Varga, 1978), but even in our days students have only a few combinatorics lessons per year (NAT, 2010). This persists even though there are more and more combinatorial problems in the final exam and in the entrance exam to secondary school as well. Teachers have learnt combinatorics mainly at university; they have not had a lot of experience about students' combinatorial problem solving skills in primary school. We suppose, teachers teach mainly the ready-made solving formulas for different types of combinatorial problems such as permutation, variation, combination with repetition or not, instead of useful strategies, processes of representations, and systematical thinking.

Based on Piaget and Inhelder the stages of combinatorial thinking are the following: students at the preoperational level use random listing procedures, at the concrete operational level use trial-and-error strategies and only from about the age of 12, at the formal operation level, they are able to think systematically (Piaget & Inhelder, 1951). English (1991) found that children can use systematic procedures for solving two- and three-dimensional combinatorial problems even before they reach the formal operation level. They were able to translate the problem into a graph representation and solve it by fixing variables. Students can develop their own strategies for solving problems

without realizing the common structure of problems in different contexts (Szitányi, & Csíkos, 2015). Some researches examined the difficulties of combinatorial problems. Hadar and Haddas (1981) listed the main pitfalls jeopardizing the solution of combinatorial problems. Eizenberg and Zaslavsky (2004) found that one of the major difficulty in combinatorial problem solving is verification, so they identified some types of verification strategies. In our research we focus on the features of the combinatorial problems that cause difficulties.

School problems are mainly enumerative problems, usually requiring to count the number of the elements of a finite set. In these problems, the main questions are the following: Have we counted different elements; Have we counted all of the elements (Varga & Dumont, 1973). The levels of combinatorial problem solving strategies are the following (Pintér, 2013):

- Distinction of the elements
- Random listing of the elements
- Regular listing (strategy: fixing; representations: tabular arrangement, graph)
- Applying formal methods (strategies: multiplication, addition, translating the problem into an equivalent problem, recursion, etc.; representations: e.g. box method)
- Recognition of the structure

The strategies of a level can be taught based on the previous levels' strategies. For example, students can create their own systematic procedures based on former experiments. Varga and his colleague therefore collected activities (Varga, & Dumont, 1973) for students from grade 1 to 8 based on manipulations with real objects. Some problems for grade 8 have the same context as in grade 1 with greater number of elements and conditions. The development of combinative operations were investigated by Csapó and Nagy (Csapó, 1988, Nagy, 2007), they have worked out the assessment of the combinatorial reasoning.

Our research concentrates on school mathematics, namely what are the difficulties of teaching and learning combinatorics. We focus on the features of the problems – the number of elements, the number of conditions, and the necessity of case separation – instead of the type of problem. We chose permutation problems because these are among the most difficult combinatorial problems but they become easier after instruction (Fischbein & Gazit, 1988). We are interested in the strategies used by students. Our hypothesis is that students have learnt mainly ready-made formulas and they cannot use them for solving problems of higher complexity.

### **Research questions**

- Is there a relation between solution rate and the features of the problems (number of elements, number of conditions, necessity of case separation)?

- Do students' strategies depend on features of the problems (the number of elements, the number of conditions, and the necessity of case separation)?
- Do students use visual representations (graphs, tables)?
- Is the choice of strategies related to age?
- When do students use ready-made formulas and what do they do if they cannot use them?

## RESEARCH METHODS

In order to address the research questions, we conducted a large-scale cross-sectional study. We have chosen our sample group of students as well as our test and the aspects of evaluation based on the targets of our research.

### Sample group of the research

As we were partly interested in whether the students would use formulas to determine the number of possible orders, we tested two age groups:

1. Students from primary school 6-7<sup>th</sup> grade (12-13 yrs.) who had not studied the necessary formulas
2. Students from highschool 10-11<sup>th</sup> grade (16-17 yrs.) who studied the necessary formulas

Schools from the capital and other parts of the country were equally represented in the sample. The test was filled by 429 students, 230 from group 1, 199 from group 2.

### Test structure

The test covered enumerative problems. They were constructed according to the following rationale:

- The number of elements of the set of countable possibilities. We established two categories: low and high number of elements. (The meaning of "low": at most 40 elements of the set of countable possibilities. In other words the "low" means that not too difficult to list the elements of the set of countable possibilities.)
- The number of conditions considered while counting. We also have two categories here: less and more conditions. (The meaning of "more": at least two conditions.)
- Is it necessary to separate cases or not.

In the test we presented 16 questions based on 6 different contexts. Table 1 shows the distribution of the 16 questions.

	Low number of elements	High number of elements
Less conditions	3+1(necessary to separate cases)	3+1(necessary to separate cases)
More conditions	3+1(necessary to separate cases)	3+1(necessary to separate cases)

Table 1: Distribution of the 16 questions of the test

Let us present 2 problems (including 7 questions) out of the 6 that illustrate our test (problems number 1 and 4 in the test).

1.

- a) Robert, John, Kate and Elizabeth are sitting on a bench next to each other. How many sitting arrangements are possible? (“**low number of elements** and **less conditions**”)
- b) Another boy, called Michael joined this team. How many different ways can they be seated on the bench if a girl can sit next to a boy and a boy can sit next to a girl only (if girls and boys alternate)? (“**low number of elements** and **more conditions**”)
- c) Robert, Michael, John, Kate, Maria, Elizabeth and Susanne line up (in a row) on the schoolyard. How many ways is it possible? (“**high number of elements** and **less conditions**”)
- d) How many different ways could this be done if boys and girls alternate? (A girl can stand next to a boy and a boy can stand next to a girl only.) (“**high number of elements** and **more conditions**”)

4.

- a) How many five digit numbers can be formed by using all these cards if each is used exactly once? (“**high number of elements** and **less conditions**”)

1 2 3 4 5

- b) How many five-digit numbers can be formed by using all these cards if each is used exactly once? (“**low number of elements** and **less conditions** and **necessary to separate cases**”)

0 0 1 4 5

- c) How many seven digit numbers can be formed by using all these cards if each is used exactly once? (“**high number of elements** and **less conditions** and **necessary to separate cases**”)

0 0 1 2 3 4 5

## Aspects of evaluation

We were pursuing two main aspects when evaluating our tests:

1. Is the answer correct? We compared the two age groups for each question and we further evaluated their success rates in those question types, where one question characteristic was the same (e.g. “**low number of elements** and **less conditions**” – “**high number of elements** and **less conditions**”; “**low number of elements** and **more conditions**” – “**high number of elements** and **more conditions**”). Finally, we measured the overall success rate of both age groups.
2. We analyzed the solving strategy applied (as far as it can be inferred by the problem solution). Wherever we could find a certain system, clear line of thought, we classified them in the following categories:
  - visualization of cases with a graph (tree)
  - different countable possibilities sorted in a tabular arrangement

- usage of “boxes”, “plates” and other figures to assist counting
- operations with concrete numbers
- application of a general formula

We analyzed the rate of usage of the strategies listed above within both age groups and compared the results.

## RESULTS

In Table 1 it can be seen that in both age groups the success of the solution was the highest in the case of problem with low number of elements and conditions. With the growth of number of elements and conditions the success of the solution diminished in both age groups. The differences between the age groups are significant in each problem category, while the difference in the distribution of the overall success rate is not significant ( $\chi^2(3)=0.43$ ;  $p>0.05$ ).

Solution rates	6-7 <sup>th</sup> grade	10-11 <sup>th</sup> grade	
Low number, less conditions	54%	83%	$\chi^2(1)=40,82$ ; $p<0,001$
High number, less conditions	25%	55%	$\chi^2(1)=39,20$ ; $p<0,001$
Low number, more conditions	20%	36%	$\chi^2(1)=14,81$ ; $p<0,001$
High number, more conditions	7%	17%	$\chi^2(1)=10,63$ ; $p=0,001$

Table 2: The proportion of successful students in four different task conditions

In Table 2 it can be seen that 6-7<sup>th</sup> grade students in the case of problems of small number of elements preferred listing and counting cases. In the case of problems of high number of elements they preferred systematic thinking. The difference of the ratio in using the two strategies are significant in every problem type except “**low number of elements and more conditions**”.

	Low number, less conditions	Low number of elements and more conditions	High number, less conditions	High number, more conditions	High number, less conditions and necessary to separate cases	High number, more conditions and necessary to separate cases
Listing and counting cases	35%	33%	18%	14%	18%	13%
Systematic thinking	24%	26%	27%	28%	25%	26%
	$\chi^2(1)=7.06$ $p=0.005$	$\chi^2(1)=2.67$ $p=0.06$	$\chi^2(1)=5.52$ $p=0.01$	$\chi^2(1)=13.48$ $p<0.001$	$\chi^2(1)=3.72$ $p=0.03$	$\chi^2(1)=13.39$ $p<0.001$

Table 3: The proportion of strategies of 6-7<sup>th</sup> grade students in different problem types

10-11<sup>th</sup> grade students solved the problems mainly by application of formula. It is notable however, in the case of problems of high number of conditions they tried to list and count cases. The difference of the ratio in using the two strategies are significant in every problem type.

	Low number, more conditions	Low number, more conditions and necessary to separate cases	Low number, less conditions	High number less conditions	Low number, less conditions and necessary to separate cases	High number, more conditions and necessary to separate cases
Application of formula	17%	29%	53%	63%	52%	63%
Listing and counting cases	30%	41%	17%	7%	7%	3%
	$\chi^2(1)=10.89$ p=0.001	$\chi^2(1)=6.97$ p=0.005	$\chi^2(1)=50.38$ p<0.001	$\chi^2(1)=105.01$ p<0.001	$\chi^2(1)=112.91$ p<0.001	$\chi^2(1)=187.12$ p<0.001

Table 4: The proportion of strategies of 10-11<sup>th</sup> grade students in different problem types

In both age groups one can notice the small number of occurrences of visual representations (Figures 1 and 2).

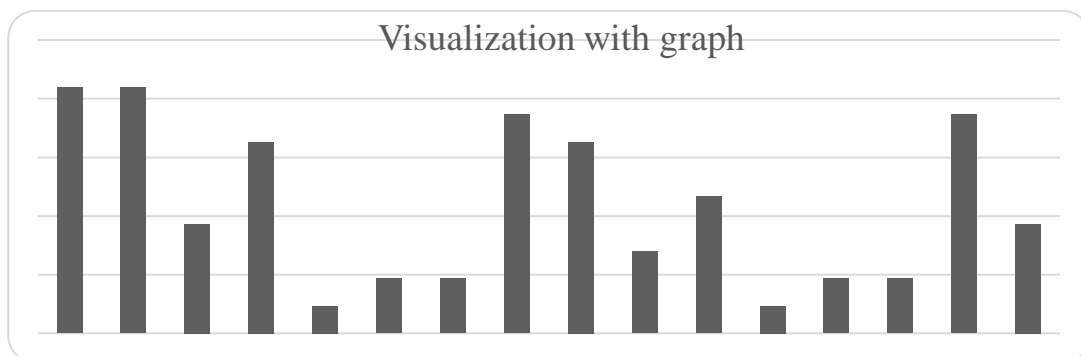


Figure 1: Occurrences of graphs (accumulated across age groups).

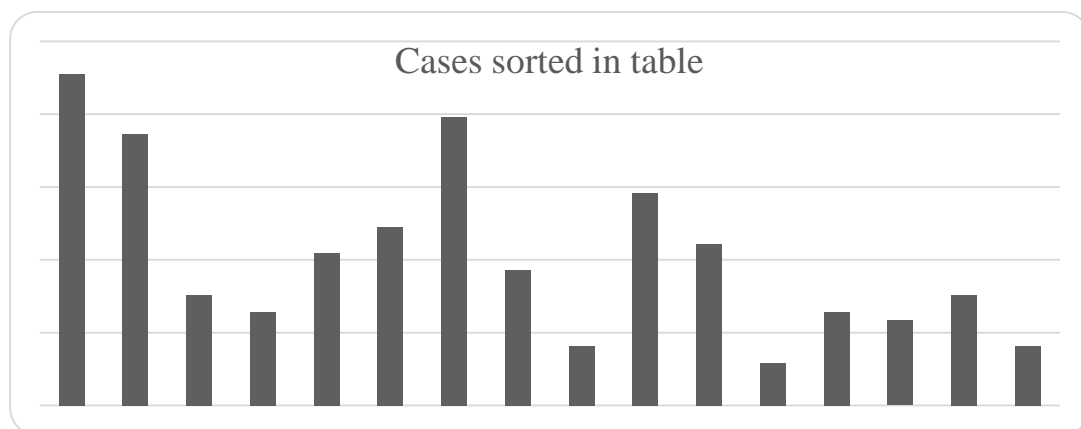


Figure 2: Occurrences of tables (accumulated across age groups).

In the case separation problems, it is noticeable that although 6-7<sup>th</sup> grade students solved the problems less successful than the 10-11<sup>th</sup> grade students but their solution strategies were better (however, they made a lot of counting mistakes). The differences between the age groups are significant, except the differences of the success rates for “**high number of elements and less conditions and necessary to separate cases**” problems.

Group	Low number, less conditions and necessary to separate cases		High number, less conditions and necessary to separate cases	
	6-7 <sup>th</sup> gr.	10-11 <sup>th</sup> gr.	6-7 <sup>th</sup> gr.	10-11 <sup>th</sup> gr.
Solution rate	5%	12%	3%	4%
	$\chi^2(1)=7.01$ ; $p=0.006$		$\chi^2(1)=0.26$ ; $p=0.40$	
Percentage of students with wrong results but good reasoning	20%	12%	14%	8%
	$\chi^2(1)=5.22$ ; $p=0.015$		$\chi^2(1)=4.40$ ; $p=0.025$	

Table 5: Reasoning and solution rates between age groups

Our statistical investigations were carried out by cross table methods. The homogeneity of the samples was tested by Pearson’s chi-square test, with 5% significance level. Due to the measurement level of the data, no parametric probes were used.

## DISCUSSION

The results show that increasing the number of elements, the number of conditions, and the necessity of case separation causes difficulties for students in both age groups.

The 6-7<sup>th</sup> grade students cannot list all of the elements when their number becomes higher. The 10-11<sup>th</sup> grade students cannot apply formulas in problems with another structure than the typical one, so students’ success is decreasing.

Unfortunately, visual representations were not very frequently used to help students’ systematic thinking. It could be an aim of instruction to teach applying graphs and tables to list the possibilities systematically.

The success of younger students in the atypical problems has convinced us that it is worth to teach the techniques of systematic thinking instead of ready-made formulas. Our result is in line with Melusova and Vidermanova (2015): They also found that students need experiences and the opportunities to list all the events systematically before generalization. The usage of graphs and tables should be taught in higher grades as well because formulas often do not provide enough help for combinatorial problem solving.

In the test students were engaged to write down their thoughts and the explanation of the solutions. But they didn’t write it, so we could draw conclusions only from the



written operations and numbers. In our next project we are going to use further methods, for example face-to-face interviews, for more detailed analysis of student's combinatorial reasoning.

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# LEARNERS' GESTURES WHEN LEARNING ALONE

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*At university level, mathematics learners are often left to learn independently, getting prepared without peers or teachers. In the present study, we direct attention to the gestures made by students when learning on their own. We considered undergraduate students accessing worked examples on the multiplication of complex numbers. Adopting an embodied cognition approach, we show how gestures help to trace the students' thoughts, but also how they may influence the learner's thinking.*

## INTRODUCTION

A common form of learning 'in isolation', that is, without peers, is in studying worked examples (Renkl, 2014). The investigation of an individual's learning in isolation provides a challenge from a methodological point of view since ideas and approaches are not accessible to the researcher by tracing the learners' exchange. Individual learning processes are often investigated by requesting the learners to 'think aloud' and thereby, to make their approaches explicit in speech. But taking note only of the verbal expression ignores the possibility of a more comprehensive picture of the learner's strategies and approaches. Furthermore, "researchers need to be aware that even thinking aloud, which makes inner speech external, cannot reveal deeper thought processes in their true complexity because they have to be simplified into words before anyone, even the thinkers themselves, can really know them" (Charters, 2003). Gesture analysis in particular may enrich think-aloud protocols by admitting additional modes of expression that go beyond consciously uttered thoughts (Salle, 2015). Furthermore, gestures may play an active part in individual learning when thinking aloud, considering that "gestures, together with language, help constitute thought" (McNeill 1992, p. 245). Both aspects, the possible benefit for the researcher as well as for the learner are kept in mind for this study when examining gestures' role when learning alone.

## THEORETICAL FRAMEWORK

Rather than defining gestures as "idiosyncratic spontaneous movement[s] of the hands and arms *accompanying speech*" (McNeill, 1992, p. 37, italics added) in thinking-aloud protocols, we consider gestures more generally as *accompanying thought* that becomes (partly) externalized in the verbal expression. Taking into account that "gesture and the spoken utterances [are] different sides of a single underlying mental process" (McNeill 1992, p. 1), the verbal expression in turn provides the frame to interpret the gesture and with this, make hypotheses about the underlying thought. Being interested in mathematical thinking processes, we restrict ourselves to those gestures that are interpreted as semantically affiliated to mathematical objects.

Furthermore, the co-timing of gesture and speech reveals and induces an affiliation of referential meaning in both modalities.

This link between thinking and gesturing is considered within the theory of embodied cognition (Lakoff & Núñez, 2000, Núñez, Edwards & Matos, 1999). Following this, being in the world and the embodied activity shapes out mathematical thinking, and, vice versa, our thinking becomes expressed as embodied in gestures.

A degree of conceptualization of a mathematical object to which a gesture refers can be described by the gesture's referential level (Krause, 2016). On *level 1*, gestures refer *concretely* to an entity already fixed within an existing inscription and have no intrinsic meaning. On the *second level*, gestures can ephemerally embed entities within an inscription that may be potentially 'thought into it'. Performed in a second layer in front of an existing inscription, they present *possibilities*. While gestures on level 2 already show some detachment from the concrete, they still need the context of the inscription to be interpreted. On *level 3*, gestures can refer to mathematical ideas in a *free*, almost *decontextualized* manner 'in the air'. The gestural representation on level 3 is reduced to core aspects considered important within the current train of thought.

Krause (2016) developed the referential levels above in the context of social learning processes. However, we found these levels also in individual learning processes and wondered about their role when interacting with one's self while accessing worked examples. Therefore, we aim to answer the following questions:

- (1) What does the use of gestures on the three referential levels reveal about the students' accessing of the worked example?
- (2) How may the students' use of gestures aid in accessing the worked examples?

## METHODS

33 undergraduate students were asked to consider the following worked example on the multiplication of complex numbers.

*"Transform  $s = 2+2i$  and  $t = 0.5 + 3i$  into polar coordinates, multiply the complex numbers and plot  $s$ ,  $t$  and  $s \cdot t$  into a coordinate system."*

The solutions provided involved: (1) The transformation of the Cartesian coordinates into the trigonometric form of polar coordinates, (2) the calculation of the product of the two complex number  $s$  and  $t$  represented in the trigonometric form, and (3) the geometrical representation of the calculated product (see also (Salle, 2015))).

The students learned individually with three worked examples printed on paper and without time limit. They were placed at a desk with a computer and were filmed by a webcam attached to the computer. During the intervention phase they were asked to think aloud. The use of a 'formula sheet' with definitions and formulas, a triangle ruler, and a calculator application on the computer were permitted. No guidelines on taking notes or gesturing were given.

In a first step of data analysis, we identified scenes relevant for answering the research questions. The search for these relevant cases was gesture-driven. In the second step, a qualitative analysis was performed, conducting first a speech-based analysis before integrating the students' gesture use. Therefore, the respective scenes have been transcribed in two versions, one leaving aside the non-verbal activities and gestures and the other one including them. For denoting the co-timing of gesture and speech we use a simplified transcription code in which squared brackets indicate end and beginning of movements. The gesture interpretation is always carried out within the contextual frame given by speech and with respect to a potential reference to inscription. Considering the latter, the gestures are grouped with respect to their referential level (Krause, 2016).

The following scenes provide examples for all three referential levels to show up various ways in which the gesture analysis may enrich the analysis of the learning-processes and how their use may aid accessing worked examples.

## CASE ANALYSES

### Levels 1 and 2: Julian's investigation of the graphic representation

In the following scene, Julian starts to address the third part of the worked example. Until now, he examined the multiplication of the complex numbers as polar coordinates, presented symbolically in the second part of the worked example. In that context, he already mentioned that he would need a direction and an angle to represent a vector within a coordinate system. While he does not consider this approach to be of any further importance for investigating the *second* part of the example, he comes back to this now.

- 1 well anyway, third part geometrical representation
- 2 (5s) ok thus [the (points at the sector of the angle (a) and to " $\gamma = 125.5^\circ$ " (b) as inscribed right there, see Fig. 2) [angle
- 3 One hundred twenty][five point five degrees]
- 4 Well, [ (places the edge of the hand on the coordinate system and turns his hand anti-clockwise as presented in Fig.1) [crosses in a way over the] y-axis

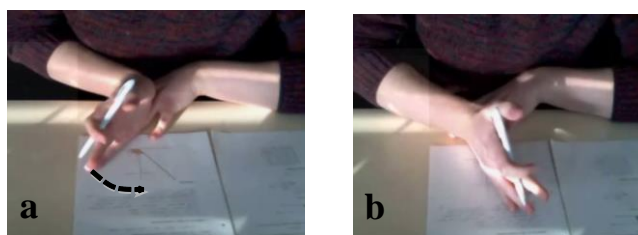


Figure 1: Julian's gesture performed on level 2 in line 04

- 5 (3s, coughs) ] And [then (points close to the origin of the coordinate system, Fig 3)
- 6 The length] is [about (points at the number "8.602" as inscribed at the left side of the last line in the determination of the vector  $s \cdot t$  in its polar coordinates, see Fig. 3)
- 7 Eight] point (moves the tip of the pen along the arrow representing the vector  $s \cdot t$ , beginning at the origin of the coordinate system, see Fig. 2 for the representation) [six ]
- 8 Yes . ok

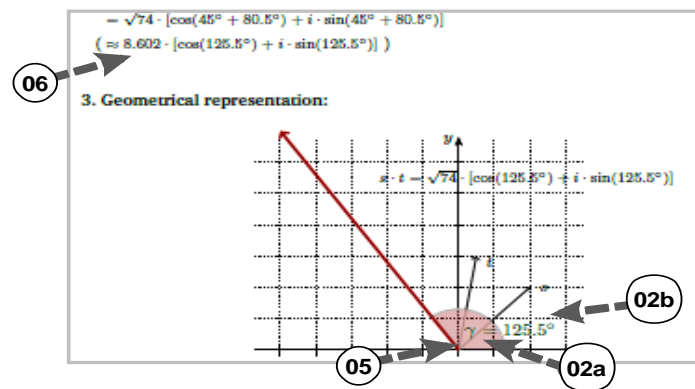


Figure 2: Location pointed at in lines 02, 05 and 06

Julian verbally refers to some angle of one hundred and twenty five degrees (02/03) and adds a dynamic component of the angle ‘crossing over the y-axis’ (04). This dynamic perspective on an angle may be related to the construction of the angle corresponding to the vector  $s \cdot t$ , namely  $\gamma = 125.5^\circ$ , as given in the coordinate system in the geometrical representation. After a pause of 3 seconds, he mentions some length of “eight point six” (06/07), probably the one of the vector  $s \cdot t$ . As described above, he noted earlier that for constructing a vector, one needs an angle and a length before examining the geometrical representation. While he identifies both for the vector  $s \cdot t$ , he appears to be satisfied with his examination of the graphical representation. However, he seems to focus on the product of the multiplication rather than on the process of multiplying the two vectors as is suggested by the task.

Julian’s gestures support that he is concerned with the components ‘angle from the x-axis’ and ‘length’ that constitute the vector. His first reference to the angle in lines 02 and 03 is accompanied by pointing alternately at the circular sector and at its label, “ $\gamma = 125.5^\circ$ ” (see Fig. 2). Following this pointing gesture on level 1, he puts the edge of his straight right hand on the paper close to the origin, directed with an angle of about  $50^\circ$  to the x-axis (04, Fig. 1a). While saying “[crosses in a way over the] y-axis” he turns the hand around his wrist in an anti-clockwise movement (Fig. 1). Performed on level 2, this gesture may provide an embodied visual approach, representing the emergence of the angle  $\gamma = 125.5$  corresponding to the vector  $s \cdot t$ . While this reference reveals a dynamic perspective on constructing an angle, it matches the dynamic approach explicated verbally when referring to ‘crossing over the y-axis’ (04). Furthermore, his hand may embody the vector, showing its direction. Julian now turns his attention to its length. While he identifies the length verbally, his gesture furthermore reveals that he connects two different representations of the length of the vector. First, he points at the origin while saying “then the length”, then at “8.602” (06) as inscribed on the left side of the last line of the equations in part 2 (see Fig. 2), continued to saying “about eight”. Third, he traces the vector  $s \cdot t$ , starting at the origin, while ending by naming the numerical value (“six”, 07). These three gestures are not regarded as separate but as linked in one movement. They follow each other very closely and are tied to only one sentence, specifying subject and measure. While in

relation to speech, his first pointing at the origin has no clear affiliate, it may reveal the reference to the length of the vector as length of the arrow in a *Growth Point* (McNeill 1992). The Growth Point is described as “the theoretical starting point, in a microgenetic sense, of a speech-gesture combination—‘growth’ in the sense that it is the seed out of which speech and gesture grows” (McNeill cited in Montredon et al. 2008, p. 173). This Growth Point can be recognized in retrospect: after explicating the length as “about eight point six”, Julian moves his finger back to where he pointed at first, now starting to trace the vector from that point upwards.

Julian’s gestural indication of the inscription “8.602” in line 06 could reveal a kind of anchor point within the material, being the fixation of the concrete length that he remembers from dealing with the calculation before. With subsequent gestural tracing of the arrow representing the vector, starting at the origin, Julian ties the length 8.6 to a measure of a geometrical object. Together with his pointing to the algebraic representation of the length, he reveals his thoughts of linking the geometrical and the algebraic representation of the length of the vector.

In this complex excerpt, both gestures on level 1 and 2 can be identified. The gestures on level 1 trace Julian’s orientation within the worked example. While the verbal references may appear disjointed, the gestures give a more comprehensive picture of what he considers important as given in the example. Furthermore, the concrete indications of inscriptions on level 1 (lines 02/03, 06, 07/08) provide a suggestion to where Julian turns his attention within the worked example and which inscriptions may be linked for him (07/08). The level 1-indications may also help Julian to direct his attention within the example and to keep his focus. They help to separate the important and related inscriptions within the entire example.

The gestures performed on level 2 reveal how Julian approaches the two aspects that define the vector; the angle giving its direction (04), and its length (08). Considering the gestures we can reconstruct that Julian considers the emergence of the vector dynamically and that he links the lengths of the inscribed arrow to the algebraic-numerical representation of the length as calculated in the second part of the example. The emergence of the angle, as well as the lengths of the vector, are *re-enacted* within the diagram and by this, singled out from inscription and *brought to life* in the third dimension. For Julian, this embodied approach to the components may help to approach the vector as object with its defining features in a physical way.

### **Level 3: Complementing the explanation in a general way**

Cornelius examines a second worked example, similar to the one given above but with different vectors and representations. In this example, the angle corresponding to the vector  $t$  is  $\beta = 225^\circ$ . While trying to understand how this angle is determined, Cornelius calculates  $\tan(\beta)$  ( $=1$ ) and compares this to  $\tan(45) = 1$  as given on an additional formula sheet. This approach is directed in the opposite direction as presented in the example where the angle is determined from the given values of the tangent-function. Cornelius may be referring to the periodicity of pi in the following:



- 1 Why is that equal .
- 2 In both cases one.
- 3 How do I come now, (incomprehensible) all the time
- 4 .. am I stupid now?
- 5 (4s, *uses the calculator*) all nonsense
- 6 In both cases one.
- 7 [ (*performs the gesture as visible in Fig. 3*) Because it probably always takes this shape ]
- 8 Ok . I just go on



Figure 3: Cornelius' gesture performed in his gestures space on level 3 in line 07

Cornelius seeks to explain to himself the phenomenon he observed (01-03). Not finding a sudden explanation for getting the value 1 for both different angles, he doubts his competence (04/05). Returning to the fact that the tangent function both times has the same value (06),  $\tan(225^\circ) = \tan(45^\circ) = 1$ , he refers to the graphical form of the function for a renewed explanation (07). This can be surmised from his verbal reference to a “shape” but is left unspecified without considering his gesture: Co-timed with saying “because it probably always takes this shape”, Cornelius shapes a wave-like form from his left to his right in the air on level 3 in front of him. The form of the gesture is reminiscent of a sine or cosine function rather than a tangent function. This does not seem to bother him and the explanation seems to be sufficiently satisfying for him to continue with further examining the solution (08).

The wave-like gesture on level 3 complements his verbal explanation with a visual component. It specifies his verbal utterance by clarifying the imprecise speech, making more precise what may be referred to by “it” by making visible what “this shape” may look like. Within the context of understanding the worked example, he may refer to some calculation or algorithmic pattern that is shaped equally in both cases, or to a graphical form, for example, the one of the tangent function. It is his gesture that clarifies his reference as being graphical in nature. However, Cornelius' gesture reveals that it is not the shape of a tangent-function he is thinking of but that the general feature satisfying his explanation may be the periodicity of the graph rather than its concrete form. While in social interaction, this mistaken form of the function may be caught and discussed further, there is no further activity on the concrete shape of the curve in the individual learning process.

It is up to speculation whether this gesturing is only performed because Cornelius uses (outer) speech to express his thoughts. In that case, it may become necessary to make his utterance, and also his explanation, semantically complete. The observation he seeks to explain is not directly linked to the worked example but rather a consequence of his individual exploration of it. Therefore, the gesture on level 3 may not help to access the worked example but to approach a more general hypotheses inferred from

its examination. Without an obligation to figure out details, the gesture can sketch an idea such that approaches not yet elaborated are formed and presented holistically and reflexively to himself, which may lead to new ideas.

## SUMMARIZING DISCUSSION

This paper presented first insights on the role gestures may play in individual learning. In a micro-grained analysis of thinking-aloud protocols, we investigated the gestures used by students when learning with worked examples. We showed how this gesture use discloses aspects of the students' approaches not identified from the analysis of speech and therefore helps the researcher to get a better understanding of the working process. Furthermore, we also claim that the gestures help the students while learning and presented ways in which that may happen.

On the one hand, gestures *complement verbal utterances* and give *additional information about how the students navigate within the worked examples*. For example, gestures may reveal visual approaches prior to their verbal articulation. Such a visual approach may concern *links made between different aspects* within the inscribed example, but also the *ephemeral shaping of one's own approaches* that are not directly related to the inscriptions given.

On the other hand, we have hypothesized how the use of gestures may provide a benefit for the students when approaching the multiplication of complex numbers through worked examples. Gestures can help to *keep a focus* and *realize visual links* that may aid the students by reducing cognitive load (Mayer & Fiorella, 2014). Furthermore, gestures may *embody* (representations of) mathematical objects or *re-enact* the process of their emergence such that the mathematics *comes to life* as a physical experience in which thinking becomes embodied (Núñez et al, 1999)

However, the potential influence of the gestures on the isolated working process may also be interpreted within a semiotic approach, considering the gestures as signs in a Peircean sense as “something which stands to somebody for something in some respect or capacity” (Peirce, CP, 2.228). Brought to the eyes as an external sign, the gesture becomes interpreted and may lead to further insights by fulfilling an epistemic function; see (Krause 2016, p. 25), and also (Hoffmann, 2005). Furthermore, gestures on levels 1 and 2 may also unfold the epistemic function of inscriptive signs, enabling the students to see the mathematics “through the sign” (Hoffmann 2005, p. 37).

The results here only concern how the gestures may help the students to access the worked example, but they are not necessarily linked to the actual learning process. To also trace the individual's learning with the worked example, self-explaining as “a constructive activity that engages students in active learning and insures that learners attend to the material in a meaningful way” (Roy & Chi, 2005, p. 272) can be taken into account. The activity of self-explaining is constituted by self-explanations. So far, self-explanations are considered *verbal* “units of utterance” (Chi 2000, p. 165) and the analyses of self-explaining has mainly been restricted to the students' speech. Salle (2015) emphasized that a large amount of self-explanations only become apparent by



analyzing the students' gestures in addition to their verbal utterances. Therefore, a good question to ask is, in what ways do gestures contribute to self-explaining as constructive activity.

Given the results presented here, we suggest gestures are an integral part of learning, even when learning alone. We claim that further investigations may lead to a better understanding of the ways gestures can enable students to help themselves.

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# EFFECTS OF SHORT-TERM PRACTICING ON REALISTIC RESPONSES TO MISSING DATA PROBLEMS

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*In an experimental study with fifth-graders ( $N=108$ ), we carried out a short-term intervention aimed at improving students' realistic responses to missing data problems. A control group received nonspecific instructions about solving problems with missing data. An experimental group additionally solved a sample problem and discussed possible solutions. At posttest, students from both groups solved missing data problems. In line with our expectations, the experimental group gave more realistic responses to problems with missing data in which the problem statement did not contain any numbers. However, problems that contained data that, at first glance, appeared sufficient to conduct the required calculations were still answered unrealistically by both groups.*

## INTRODUCTION

Problems encountered in everyday life often do not contain all of the relevant data needed to solve them effectively. Estimations, assumptions, and realistic considerations are necessary to produce solutions to such problems. Therefore, the ability to solve problems with missing data is an important part of mathematical education. Despite the relevance of such problems for everyday life, students seem to separate their real-world knowledge from their mathematical knowledge and tend to fail to consider the context of the problem statement (Verschaffel, Greer, & de Corte, 2000). Moreover, students have been found to have trouble identifying and making assumptions while solving problems that involve missing data (Galbraith & Stillman, 2001). Although different studies have investigated students' responses to realistic word problems (Greer, 1993; Verschaffel, De Corte, & Lasure, 1994; Verschaffel et al., 2000; Yoshida, Verschaffel, & De Corte, 1997), a dearth of research has focused on students' tendency to give unrealistic responses to problems involving missing data. The aim of our study was to investigate the effects of short-term practicing in the solving of problems with missing data on students' realistic responses to these problems in comparison with a control group that received only general instructions about missing data problems. Theoretical background and research question

## Problems with missing data

Word problems with missing data (also called problems with missing information or vague conditions) are problems in which the problem statement does not provide all of the data necessary to solve the problem. These problems are addressed as a part of "ill-defined" problems because they comprise problem elements that are unknown and they possess multiple solutions (for characteristics of ill-defined problems, also called ill-

structured problems, see Wood (1983), or Jonassen (2000)). Being able to solve such problems may have an impact on students' actual and later lives because most problems encountered in everyday life and professional practice are ill-defined (Jonassen, 2000). The importance of ill-defined problems and hence problems with missing data stands in contrast to their absence in mathematics teaching and in assessments via large-scale studies such as PISA (van den Heuvel-Panhuizen & Becker, 2003).

The effects of prompting students to solve problems with missing data have been investigated in research frameworks for open-ended problems (Silver, 1995; Stacey, 1995) and modelling problems (Galbraith & Stillman, 2001; Schukajlow & Krug, 2014; Schukajlow, Krug, & Rakoczy, 2015). The role of assumptions is thereby mentioned as an underestimated part of successful modelling because it influences the entire modelling process, including the mathematizing of real-world problems (Galbraith & Stillman, 2001). Case studies have shown that students have trouble making assumptions while solving problems with missing data and have suggested that students should practice these problems more often. An experimental study that showed the benefits of problems with missing data was conducted by Schukajlow and Krug (2014). In this study, students in the experimental group were prompted to construct multiple solutions for modelling problems involving missing data. Students in the control group solved modelling problems without missing data. The authors found a positive influence of prompting students to construct multiple solutions for problems with missing data on students' interest in mathematics. In a subsequent study (Schukajlow et al., 2015), an effect on student performance was found via the number of solutions students developed and their experience of competence. Thus, when students practiced solving problems with missing data, their abilities to solve these problems (including the number of realistic responses they gave) improved.

### **Characteristics of solving problems with missing data**

The process of solving problems with missing data includes three central activities. First, students have to recognize that data are missing. Second, students must identify which quantities have to be estimated. Finally, they need to make assumptions about the missing quantities.

Recognizing that data are missing from the task can be demanding for students, and its difficulty depends on the type of problem at hand. If problems with missing data do not contain numbers at all, the need to make assumptions cannot be overlooked. An example of such a problem is: "How many centimeters of toothpaste are used in one month?" In the following, we will call this problem type "problems with no numerical information" (NNI-problems). In other problems, at first glance, there appears to be sufficient data to do the required calculations. However, simply using the standard operations would ignore the realistic nature of the context. An example is: "Mr. Meier wants to have a rope long enough to stretch between two poles that are spaced 12 m apart, but he has only pieces of rope that are 2 m long. How many of these pieces would he need to tie together to stretch between the poles?" (adapted from Greer (1993)).

Realistic considerations are required in order to recognize that data are missing, such as the length of rope that is needed to tie two pieces of rope together and to tie the ropes around the poles. Research on such problems—called problematic problems (P-problems) in the literature—is addressed in the next paragraph.

### **Problematic problems**

#### **Research question**

Problems are considered P-problems if the modelling assumptions are problematic, at least if the real-world aspects of the context are taken seriously (Verschaffel et al., 1994). The important issue of P-problems is the strong tendency of students' to neglect their real-world knowledge while solving such problems. This finding was demonstrated by Verschaffel et al. (1994) and Greer (1993) and was replicated in several other studies (Dewolf, Van Dooren, Cimen, & Verschaffel, 2013; Verschaffel et al., 2000; Yoshida et al., 1997). The main reason for the large number of unrealistic responses to P-problems can be understood to come from the inappropriate frequent use of “standard word problems” and how they are applied in teaching-learning situations (Verschaffel et al., 2000). Such so-called standard word problems can be solved through a superficial straightforward application of one or more arithmetic operations to the given numbers, meaning that it is not necessary to make realistic considerations. This leads to restricted conceptions and beliefs about word problems, namely, that every word problem can be solved with a single numerical answer, that all of the relevant data is given, and that each given number is relevant to the solution (Reusser & Stebler, 1997; Verschaffel et al., 2000). Different kinds of arrangements were provided in several studies to challenge these beliefs and to improve the realistic aspects of the responses. The findings of these studies demonstrate that many short-term interventions for P-problems (such as providing a warning or illustrations) do not work (Dewolf et al., 2013; Greer, 1993; Yoshida et al., 1997). However, some studies did find positive effects of increasing the authenticity of the setting or of carrying out long-term interventions with a focus on mathematical modelling (Reusser & Stebler, 1997; Verschaffel & De Corte, 1997).

These considerations led us to pose the following research question:

Will students respond in a more realistic manner if they have been able to practice solving a problem with missing data and are taught to discuss different solutions in a short-term intervention? More precisely: Would a short-term intervention work differently for problems that can obviously be identified as missing data problems because they do not contain any numbers at all (NNI-problems) in contrast to problems for which realistic considerations are necessary to recognize the need to make assumptions (P-problems)?

Taking into consideration the analysis of the processes applied to solve problems with missing data from the previous section, it can be expected that the short-term practicing and discussing of solutions would have different effects on different types of problems. We expected that after practicing and discussing the solutions to a sample problem

with missing data, students would give more realistic responses to NNI-problems. For P-problems, however, we expected that greater efforts would be needed to change students' tendency to give unrealistic answers.

## METHOD

### Sample and design

Our sample involved 108 fifth graders (52 % females, mean age=10.8 years) from four high-track classes (German gymnasium) from two different schools. The classes did not differ in their level of experience in solving realistic word problems. One class from each school was randomly assigned to the experimental condition and another class to the control condition. In order to control for students' mathematical achievement, all students were administered a standardized mathematical computation test (we used the arithmetic section of DEMAT 5+) at the beginning of the study (see Figure 1). After that, both groups received brief general instructions about missing data problems:

“You will see that the following problems differ from the problems you are usually given because of missing data. These problems are nevertheless solvable. To solve these problems, you have to make assumptions for the missing data.”

In addition, the experimental group (EG) engaged in short-term practicing, which included solving a sample problem and discussing different solutions to this problem in the classroom. The sample problem was: “Some friends meet to play. They have 20 small bags of jelly babies. How many bags gets each of them?” Finally, a posttest with four missing data problems was administered to both groups.

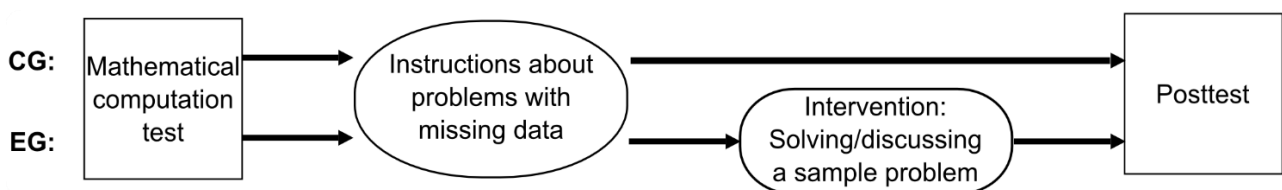


Figure 1: Overview of the study

### Missing data problems

In this study, we addressed the two types of missing data problems mentioned above: In NNI-problems, the problem statement does not contain numerical information at all, and thus, making assumptions cannot be overlooked, whereas P-problems provide the opportunity to apply standard operations and therefore require the problem solver to recognize the necessity to make assumptions (see Table 1).

The answers were scored as “realistic” if the solution included realistic assumptions.

NNI-problems	Toothpaste	How many centimeters of toothpaste are used in one month?
	Birthday	Max celebrates his birthday. He wants to eat chocolate marshmallows with his guests. How many packs does he have to buy with his mother?
P-problems	Rope	Mr. Meier wants to have a rope long enough to stretch between two poles that are spaced 12 m apart, but he has only pieces of rope that are 2 m long. How many of these pieces would he need to tie together to stretch between the poles?
	Present	Sina gift-wraps a book. The book measures 5 x 15 x 20 cm. Afterwards, she wants to tie the present with a ribbon. How much ribbon does Sina need?

Table 1: The NNI-problems and P-problems given on the test

These assumptions did not have to be explicitly stated. This means that for the P-problems, that students had to mention that additional rope was required to tie the pieces together and to tie the ropes to the poles (“Rope”) and that more ribbon was needed to tie a bow (“Present”). Arithmetical errors were tolerated. In addition, for the “Birthday” item, responses were still coded as realistic even if they did not state that Max also wanted to eat chocolate marshmallows (as long as they mentioned having enough for the guests). This decision was made to eliminate the “problematic” part of the item and ensure that it could be seen as an NNI-problem for which making assumptions was obvious.

## RESULTS

As a preliminary result, there were no significant differences between the experimental and control groups on the standardized mathematical computation test (EG:  $M = 6.13$ ,  $SD = 3.087$ ; CG:  $M = 6.36$ ,  $SD = 3.006$ ;  $t(106) = -.379$ ,  $p = .71$ ). The two groups could be seen as having equal levels of mathematical ability.

Descriptive results revealed that there were only a few realistic responses to the P-problems, a finding that is consistent with previous research on P-problems. The NNI-problems were noticeably more often answered in a realistic manner than the P-problems (Table 2).

To answer the research question, we computed a logistic regression with the realistic responses as the dependent variable and the short-term practicing as the independent variable (Table 3). Thereby, the single items from both problem types were considered.

Problem type	Item	Experimental group	Control group
NNI-problems	Tooth	65.38 % (34)	35.71 % (20)
	Birthday	59.62 % (31)	14.29 % (8)
P-problems	Rope	3.85 % (2)	8.93 % (5)
	Present	3.85 % (2)	1.79 % (1)

Table 1: Percentages of realistic responses and total numbers in parentheses

The results revealed a significant effect of practicing for the two NNI-problems: The odds of giving a realistic response were about 3 and 8 times greater for students in the experimental group than for students in the control group for the “Toothpaste” and “Birthday” items, respectively. In contrast to this, we found no effects of practicing on the realistic responses to the P-problems. Thus, as expected, the practicing was successful for the NNI-problems but not for the P-problems.

Problem type	Item	$\beta$	$SE$	Wald	$p$	$e^{\beta}$	$R^2$ (Nagelkerke)
NNI-problems	Toothpaste	1.17	0.40	8.56	.003*	3.22	.11
	Birthday	2.14	0.47	20.39	<.001*	8.46	.28
P-problems	Rope	-0.92	0.86	1.14	.287	0.40	.03
	Present	0.77	1.24	0.38	.535	2.16	.02

\* Significant at the 5 % level (Bonferroni-adjusted alpha level of .0125 per test (.05/4))

Table 3: Summary of the logistic regression analyses

## CONCLUSIONS AND DISCUSSION

The presented study investigated the impact of short-term practicing on students’ realistic responses to missing data problems. The effects were analyzed for two kinds of missing data problems. NNI-problems cannot be solved using standard operations because no numbers are presented in the problem statement, and thus, the necessity of making assumptions cannot be overlooked. P-problems, however, provide the opportunity to apply standard operations to the given numbers, and realistic considerations are necessary to recognize the need to make assumptions.

Descriptive findings showed only small numbers of realistic responses to the P-problems. This finding is in line with the results of other studies investigating P-problems, which, for example, have reported 0 % up to 8 % realistic responses to a slightly different version of the “Rope” item (Greer, 1993; Verschaffel et al., 1994; Yoshida et al., 1997). A distinctly higher number of realistic responses were given to the NNI-problems. It can therefore be assumed that recognizing the need to make

assumptions in order to provide a numerical answer is a key difficulty in solving P-problems because this feature marks the central difference between P-problems and NNI-problems. It can be expected that the opportunity to apply standard operations confirms students' restricted conceptions and beliefs about word problems and leads to unrealistic responses. This is consistent with the findings of other studies that consider students' conceptions and beliefs as a main reason for their neglect of real-world issues (Reusser & Stebler, 1997; Verschaffel et al., 2000).

The present study addressed these restricted conceptions and beliefs: Students in the control group received general instructions about problems with missing data. In the experimental group, the students additionally solved a sample problem and discussed its solutions in the classroom. The rationale behind the experimental manipulation came from the results of previous studies that showed the necessity of practicing realistic word problems in the classroom (Galbraith & Stillman, 2001). Indeed, constructing and discussing solutions in the classroom had positive effects on students' realistic responses. However, these effects were restricted to the NNI-problems, which do not provide the opportunity to apply standard arithmetic operations at all. No differences between the control and experimental groups were found for the P-problems. These results are in line with previous findings on P-problems, which showed no effects of illustrations, warnings, and other short-term interventions on students' responses (Dewolf et al., 2013; Verschaffel & De Corte, 1997; Yoshida et al., 1997). The open question is whether interventions, which have been shown to have an impact on other problems (e.g., the request to construct two solutions to a problem), can help to stimulate the construction of a situation model and thus increase the number of realistic responses. Schukajlow and Krug (2013) provided some evidence for the impact of prompting students to construct multiple solutions on planning and monitoring. Therefore, additional research is required to investigate the question of whether it is possible to change students' strong tendency to neglect the real world while solving P-problems by improving their monitoring strategies. Further, it seems to be a promising approach to investigate how working with missing data problems can be used to change students' strong tendency to give unrealistic responses to P-problems.

We acknowledge that our study used a limited number of items and item types. Future studies should increase both of these aspects in order to improve the reliability and generalizability of our results.

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# QUALITY OF CRITICAL ANALYSIS AS PREDICTOR OF TEACHERS' VIEWS ON COGNITIVE ACTIVATION IN VIDEOTAPED CLASSROOM SITUATIONS

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*To which extent does teachers' analysis of videotaped classroom situations make a difference for their subjective views on these classroom situations? Quantitative empirical evidence about interdependencies between teachers' situation-specific views and aspects of their competency of analysing classroom interaction is still scarce. Consequently, this study aims at providing insight: A reanalysis of data from more than 40 German in-service teachers uses a new top-down coding of their answers to open format questions. The results reveal that teachers with a higher quality of critical analysis had different views on classroom situations in comparison with teachers who did not connect specific observations with conclusions, suggestions or situation interpretations.*

## INTRODUCTION

In an article in *Educational Studies in Mathematics* (Kuntze, 2012) and earlier papers (e.g. Kuntze, 2006), results from video-based in-service teacher surveys were published which showed that there can be interdependencies between teachers' views on classroom situations on the one side and more general, non-situation-specific convictions on the other. However in these studies, teachers' answers to open format questions have not yet been analysed under the scope of the quality of the teachers' knowledge-based critical analysis of the classroom situations.

In the meantime in further research projects, our research focus has been extended to include also aspects of criteria-based noticing and teachers' competencies related to knowledge-based analysis of classroom situations (e.g. Dreher & Kuntze, 2015a, b; Kuntze, Dreher & Friesen, in press). This broader perspective raises the question to what extent the teachers' classroom situation-related views interdepend with their knowledge-based analysis of classroom situations. Taking the opportunity to reanalyse the teachers' answers to the open format questions from the Kuntze (2012) study with a new focus on the teachers' analysis of the classrooms, this study can extend existing research and advance our understanding of teachers' perceptions of mathematics classrooms.

In the following, we will introduce into the theoretical background of this new, more comprehensive approach and link it to existing research. After presenting the research questions, we will inform about design and methods, report results and discuss implications for theory development and practice.

## THEORETICAL BACKGROUND

The theoretical framework of this study is connected with (a) the background of the studies of Kuntze (2012) and Kuntze (2006) and (b) with the approach related to teachers' knowledge-based noticing and analysis as reflected in Dreher and Kuntze (2015a, b) and Kuntze, Dreher & Friesen (in press). As this paper links the two perspectives, we will in the following sum up key elements of the two approaches.

### **(a) Teachers' views of cognitive activation, intensity of argumentation and learning from mistakes in videotaped classrooms**

Teachers' instruction-related views such as for instance their epistemological beliefs and instruction-related orientations (e.g. Staub & Stern, 2002, cf. Törner, 2002) have shown to be significant impact factors on the quality of mathematics classrooms (e.g. Kunter et al., 2013; Staub & Stern, 2002) and thus can be considered as aspects of teacher expertise. As teachers' professional knowledge including beliefs and views (cf. Pajares, 1992) is organized episodically (Leinhardt & Greeno, 1986), content-related and situation-specific views (e.g. Lerman, 1990; Kuntze, 2012) are likely to play a key role as well. It hence makes sense to include them in a multi-layer model of professional knowledge (Kuntze, 2012) which uses Shulman's (1986) categories.

In Kuntze (2006), situation-related views were used to describe teachers' growth during a video-based professional development project: According to the findings from a cluster analysis (see Fig. 1; Kuntze, 2006) the participants' views developed during a professional development project. The teachers had been asked about their views on two contrasting videotaped classrooms on geometrical proof (video A and video B). The situations had been selected on the base of results from a prior video study (Kuntze & Reiss, 2004): According to these results, video A was marked by more discourse and more student-centred argumentation as well as by discourse among students around mistakes. In contrast, video B rather showed a teacher-centred small-step question-answer interaction which can be seen as typical for the dominant teaching script in Germany. More detailed information about these videotaped classrooms and even transcripts have been published in Kuntze (2008). Details about the quantitative methodology can also be found in Kuntze (2006) and Kuntze (2012).

The findings in Kuntze (2006, see Fig. 1) show in particular progress towards a more convergent view of all teachers and – most importantly – progress of a group of teachers labelled “traditionally oriented teachers” towards a more positive view of video A (which was marked more by discourse and less by small-step interaction). This was interpreted as an increase in expertise, as many of the teachers had rather negative views about cognitive activation, intensity of argumentation and opportunities of learning from mistakes in video A at the beginning of the professional development project.

According to the scope of Kuntze (2006, cf. also 2012) the teachers' views on video A thus play a key role. As during the professional development program the participants were encouraged to reflect on (different) videotaped classrooms and as cooperative

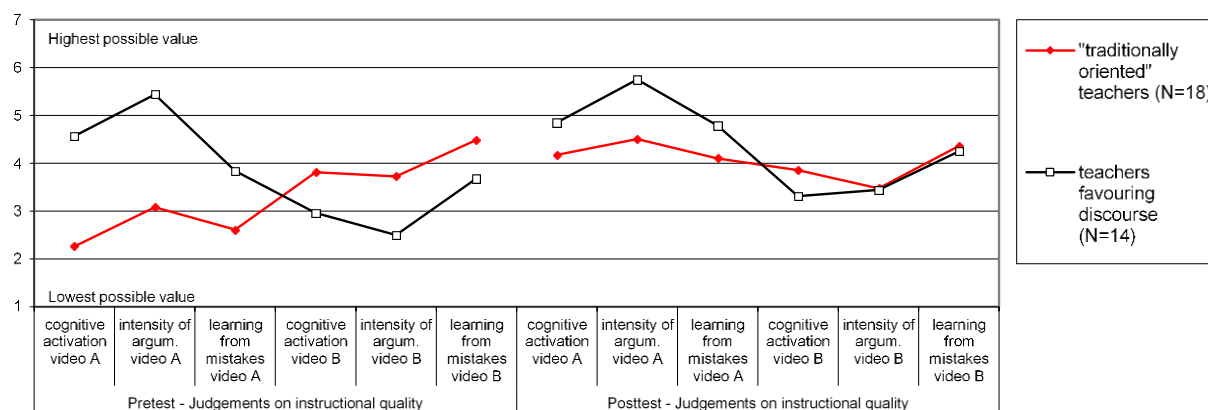


Fig. 1: Views on instructional quality (groups according to a cluster analysis on the base of pretest and posttest, Kuntze, 2006, p. 417)

analysis of classroom situations was in the foreground, the question whether the teachers' analysis of the classroom situations makes a difference for their views is key – and it has so far not been answered, as no corresponding coding of the teachers' open answers had been available. This is where more recent research approaches come in.

### (b) Teachers' analysis of classroom situations

There is a growing consensus that teachers' *noticing* or *professional vision* is a substantial indicator of teacher expertise (Sherin et al., 2011; Berliner, 1991). Especially when mathematics teachers are being confronted e.g. with videotaped classroom situations, the element of *knowledge-based reasoning* should be taken into account (cf. Sherin et al., 2011), as professional knowledge (such as e.g. pedagogical content knowledge) is often necessary to analyse classroom situations. With a content-specific emphasis, Kuntze, Dreher & Friesen (in press) use the notion of mathematics teachers' *analysing* of classroom situations. By *analysing* we understand an *awareness-driven, knowledge-based process which connects the subject of analysis with relevant criterion knowledge and is marked by criteria-based explanation and argumentation* (Kuntze et al., in press). Classroom situations can be the subject of analysis, in which teachers have to connect aspects of their pedagogical content knowledge with relevant situation observations.

Recent findings suggest that pre-service and even in-service secondary teachers encounter difficulties when they are asked to analyse the use of mathematical representations in classroom situations (Dreher & Kuntze, 2015a, b; Dreher, Novinska, & Kuntze, 2013). In that study, an indicator coding was used on teachers' answers about four text vignettes in which critical statements or conclusions had to coincide with the teachers' commenting on specific observations related to the change between representations in the vignettes. Consequently, for a first approach towards a more general indicator coding for teachers' knowledge-based critical analysis, it appears as a central aspect that teachers connect specific observations in the classroom situations on the one side with conclusions, suggestions or individual situation interpretations on the other.

## RESEARCH QUESTIONS

Corresponding to the research interest outlined above, a reanalysis of the qualitative open format question data of the Kuntze (2012) study could give insight whether situation-specific views of teachers interdepend with the quality of their critical knowledge-based analysis in the case of video A and B introduced above. In particular, the reanalysis affords answering the following research questions:

- (1) *When commenting on video A and B in open format questions, to which extent do teachers connect specific observations with their conclusions, suggestions or situation interpretations?*
- (2) *Which aspects of the classroom interaction do they analyse?*
- (3) *Do teachers who connect their conclusions with specific observations in their analysis of classrooms (as reflected in their open format answers) differ in their views about the observed classrooms (with respect of cognitive activation, intensity of argumentation and learning from mistakes)?*

## DESIGN AND METHODS

The data already analysed in Kuntze (2012, 2006) is mainly quantitative: For both video A and B, the teachers were asked to respond to standardised items which were coded on a 7-point Likert scale with respect of the criteria of cognitive activation (sample item: *“The students were encouraged to learn intensively”*), intensity of argumentation (sample item: *“The classroom interaction was characterised by an argumentative interchange between the students and the teacher”*), and learning from mistakes (sample item: *“The manner in which mistakes were treated in the classroom encouraged the students to construct meaningful knowledge that is relevant for tasks and problem solving”*). The reliability values of the corresponding scales were satisfactory ( $.77 < \alpha < .90$ , for more details see Kuntze, 2012). The sample consists of 42 German secondary teachers (14 female and 28 male, 19 aged 35 years or below; 8 aged 36–45 years; 13 aged 46–55 years; 2 aged 56 years or above) who had been teaching mathematics for on average 11.3 years ( $SD = 9.8$ ) at academic-track secondary schools. After having been shown video A and B, the teachers were (among other) first asked in open format whether they had suggestions for improvement of the situations – in general and concerning content as well as concerning classroom interaction in particular. Further, they were asked to compare the situation with their own teaching and they were given the opportunity of making further comments. The criteria-based standardised questions mentioned above were the last part of this questionnaire, again with the opportunity of making further comments.

The new analysis of the open format data was carried out according to a top-down coding scheme. It was coded whether the teachers connected any specific observations with any conclusions, suggestions or situation interpretations in their answers, regardless of the specific suggestion, conclusion or situation interpretation they made. The codes were: 0 – *no answer or no relevant answer*, 1 – *conclusion and observation*

*disconnected or only conclusion resp. observation given, 2 – conclusion and observation connected with respect of at least one issue raised in the comment.* For the comparative analysis of two groups, codes 0 and 1 were taken together under the label “conclusion and observation disconnected”. For control, it was further coded whether the comments were rather negative with respect of the video in question, or whether they were positive, neutral or both negative and positive. These codes were assigned to the teachers’ open answers for video A and B separately. Moreover in a bottom-up content analysis, the aspects raised in the teachers’ comments were coded in order to have a frequency overview of the teachers’ perceptions (cf. second research question). In the following, we will concentrate on the comments about the interaction in the classroom only, as these are most relevant for the criteria-based views about cognitive activation, intensity of argumentation, and learning from mistakes in videos A and B. As introduced in the theoretical background, especially the analysis related to video A was in the centre of interest, as views related to this situation were considered as indicator of expertise growth in Kuntze (2006).

## RESULTS

Proceeding in the order of the research questions, we would first like to illustrate the coding procedure by two examples. For example, Bernhard comments on the classroom interaction of video B in the following way (open format questions):

“The key answers were given by the teacher himself!!! False utterances of students were never corrected by other students. [...] I would not be satisfied if I would give the key answers myself! The major part of the class appeared to me quite bored. There was no real encouragement to verbalise facts beyond one-word-answers.

This comment gives interpretations/conclusions (e.g. “*The key answers were given by the teacher himself*”) which were seen negatively (“*I would not be satisfied*”) and which were supported by observations (e.g. “*...never corrected by other students*”, “*one-word-answers*”). For this reason, code “2” was assigned to this comment.

Andreas comments on the classroom interaction in video A (and B):

“[video A:] Perhaps rather hand over the evaluation of answers and of the responses to questions even more to the students (increase activity of passive students through questioning) [...] every now and then control of understanding of students who have not yet participated (make them repeat or evaluate argumentation) [...] - the class in example A obviously had by far better pre-conditions and motivation than the class in example B – I am not sure whether the qualitatively more elaborated contents in example A would have been understood by weaker students and whether they would have had the chance to work on them and deepen them at home [...] In A, mistakes were often remarked and corrected by other students, in B, the level was mostly too low for that”

Andreas gives several suggestions for improvement (e.g. “*every now and then control of understanding*”) and interpretations of the situations (e.g. “*the level was mostly too low for that*”). The connection with specific observations mostly appears as rather indirect, however he mentions, for instance, the observation that in video A, “*mistakes*

were often remarked...” which can be seen as connected with prior situation interpretations, so that still for the whole comment on video A, code “2” was assigned.

In contrast, comments by other teachers such as “*more structure, emphasise more the chain of thought*” were not coded as a suggestion or situation interpretation which is connected to any specific observation. Such comments were hence assigned the code “1” if only suggestions or situation interpretations of this type were given.

In total, 17 of the 42 teachers reached code 2 for classroom interaction in video A, 14 for video B. 94.2% of the teachers gave negative comments for video A, 4.8% of the teachers gave negative and positive comments. 81.0% of the teachers commented on video B negatively, 7.1% positively, 4.8% positively and negatively, and 7.1% gave neither a positive nor a negative comment.

For an explorative overview of the aspects the teachers mentioned in their analyses according to the second research question, the comments were grouped in a bottom-up content analysis. For video A, 43% commented on the teacher’s feedback, 36% on the representation on the blackboard, 26% on the speed of the interaction, 17% on teacher dominance, 17% on the low number of students participating in the interaction, 14% on the content structure, and 10% on the reciprocal understanding of students. For video B, 33% commented on the small-step interaction and the high content contribution of the teacher, 24% on teacher dominance, 14% on the low number of students participating in the interaction, and 5% on the teacher’s echoing of students’ answers.

For answering the third research question, the analysis codes for video A were used for distinguishing two groups of teachers, for which the mean views according to cognitive activation, intensity of argumentation and learning from mistakes could be compared. Figure 2 shows the profile of views of the two groups of teachers.

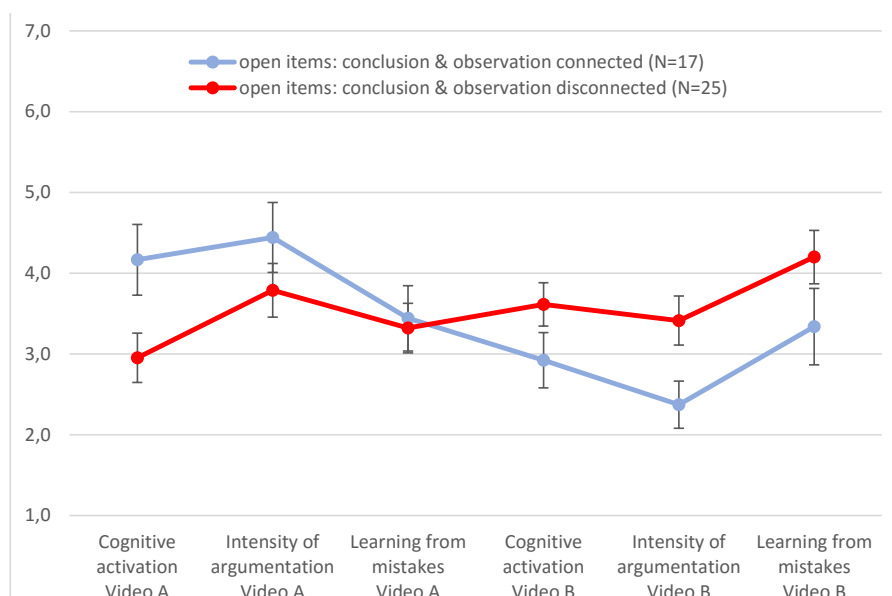


Fig. 2: Views of classroom situations for groups of teachers according to the coding of their analysis of video A (mean values and their standard errors)

The data in Figure 2 indicates that teachers who connected their conclusions/suggestions/situation interpretations with specific observations differed in their views from teachers who had not made such connections in their analysis. The views are significantly different for cognitive activation of video A ( $T=2.35$ ;  $df=40$ ;  $p=.024$ ;  $d=.73$ ) and for the perceived intensity of argumentation of video B ( $T=2.36$ ;  $df=40$ ;  $p=.023$ ;  $d=.76$ ). The  $d$ -values indicate medium resp. large effect sizes. Given the relatively small sample size of the subgroups, the other variables do not differ significantly despite two other  $d$ -values around .5.

## DISCUSSION AND CONCLUSIONS

The results indicate that less than half of the in-service teachers connected their conclusions or situations interpretations with specific observations in the classrooms both for video A and B. It would certainly be an over-interpretation of the evidence to conclude that more than 50% of the teachers were unable to connect with observations, as some of the teachers might have made such connections implicitly without mentioning them in their comments. However, the codes used for describing the quality of analysis appear to make a difference for the teachers' views of the classroom situations: There are tendencies of different views as far as cognitive activation, intensity of argumentation and learning from mistakes are concerned, and even given the low  $N$ 's some of the views differ significantly with considerable effect sizes. As most of the teachers' comments in the open answers were negative, video A was seen more positively by the group of teachers who connected observations and conclusions despite the fact that they raised critical points about this video in their comments and justified them with specific observations. This suggests that the codes related to the teachers' analysis really reflect a quality aspect of analysis and not only a sort of positive or negative general view about the classroom situations. It appears that a more careful analysis of the classroom interaction in video A corresponds to a more positive view of the elements of discourse in video A and a more critical view in particular of the argumentation quality of video B. As video B corresponded more to the dominant German small-step interaction teaching script and video A rather can be seen as a potential alternative classroom interaction, the results suggest that a high-quality analysis of teachers might enable them to develop a critical stance towards teacher-centred forms of small-step interaction and to see positive elements in more classroom discourse. The results call for follow-up research about the potential impact of professional development in teachers' analysis on their instruction-related views, also in the case of other content domains. Conversely, the role of teachers' situation-related views for teachers' analysis should also be examined more closely, as there might be a complex bi-directional interplay of these variables.

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# MODELING CHILDREN'S METACOGNITION DURING MATHEMATICAL PROBLEM SOLVING

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*Empirical results show that metacognition is a predictor for student mathematical learning and for mathematical problem solving. As such is the promotion of metacognitive activity in both domains advocated. Such advocacy can only be effective if the advocated process is being understood from early grades on. In this paper I have three goals in mind: to present Wilson and Clarke's (2004) framework and an adaptation of their multi-method interview technique to examine young children's mathematical metacognition, and to report on their validity and usefulness in the context of grade 2 and 4 mathematics. On this basis I offer concrete suggestions for how to further develop the method for the study of young children's metacognition.*

## INTRODUCTION

Over the years, metacognition has been linked to improved student outcome (Hattie, 2009). Especially in the field of mathematics, metacognition has been advocated as an important factor for student learning and problem solving (e.g., Brown, 1978; Lester, Garofalo, & Kroll, 1989; Schoenfeld, 1992; Veenman, Van Hout-Wolters, & Afflerbach, 2006). Metacognitive activities should, therefore, constitute a core intellectual and desired behavior during problem solving activity, and that from early grades on. In educational psychology, metacognition of children is described as still being incomplete; it starts developing early, at the age of 5 to 7, reaching its full development at the age of 12. It continues to do so during life span, parallel to the development of one's intellectual ability (Alexander, Carr, & Schwanenflugel, 1995), becoming more powerful and effective as a result of years of accumulated experience in making thought the object of thinking (Brown, 1978). However, the researchers cannot agree on the level of children's awareness of their thinking, and metacognitive skills, especially in domain specific areas. Moreover, what children actually do metacognitively when problem solving needs further research. Here, coherent and viable models of metacognition, and accompanying valid and reliable methods for analyzing children's metacognition are needed.

In this article, I present a model of metacognition of Wilson and Clarke (2004), and an adaptation of their method, so called multi-method interview technique (MMI), developed to study student mathematical metacognition. Their model was empirically validated, and they have also shown the effectiveness of MMI in the context of grade 6 problem solving. The usefulness of their framework and the general utility of MMI technique for the analysis of primary grade students' problem solving remains open, which were the focus of the study reported here. Their model and the adaptation of MMI were employed to study metacognition of grade 2 and grade 4 students during

mathematical problem solving. The results of the empirical study highlight the extent of usefulness of the model and the utility of MMI technique for the analysis of children's mathematical metacognition, and key aspects of children's metacognition.

## THEORETICAL BACKGROUND

### Modeling metacognition

Wilson and Clarke (2004) conceptualize metacognition as having three components: awareness of thought processes, individual evaluation and regulation of these thought processes. *Metacognitive awareness* refers to “individuals’ awareness of where they are in the learning process or in the process of solving a problem, of their content-specific knowledge, and of their knowledge about their personal learning or problem solving strategies” (p. 27). Moreover, it entails knowledge of what has been done, what needs to be done, and what might be done in order to attain a specific goal related to learning or a problem-solving situation. *Metacognitive evaluation* refers to judgments made with respect to one’s thinking processes, capacities, and limitations. For instance, individuals could be evaluating the effectiveness of their thinking. *Metacognitive regulation* occurs when individuals make use of their metacognitive skills to direct their knowledge and thinking. It draws on individuals’ knowledge (about self, possessed strategies and their use) and uses executive skills (self-correcting, setting goals) to optimize the use of their own cognitive resources.

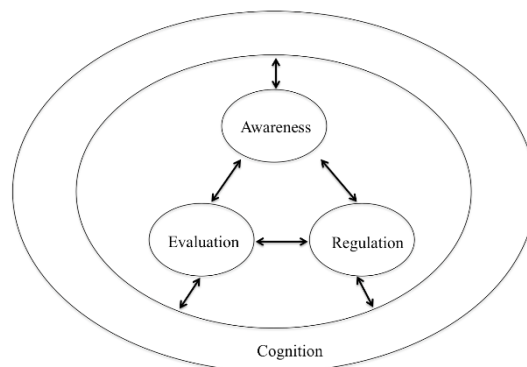


Figure 1: Wilson and Clarke's structure of a model of metacognition.

In mathematical problem solving, for the successful solution of any complex problem interplay of both cognition and metacognition is fundamental (Schoenfeld, 1992), as illustrated in Figure 1. The objects on which metacognition acts are cognitive objects. It is via cognitive behaviors that we interact purposefully with the world, and the overt actions are the results of cognitive activity, that itself is influenced by metacognitive activity (Wilson & Clarke, 2004).

### Monitoring metacognition

In developmental psychology no uniform, comparable measurement methods of metacognition exist, but rather a variety of methods is available (Ericsson & Simon, 1980). Wilson and Clarke (2004) criticize the widely known and used method for measuring metacognition during problem solving, namely think-aloud method. They

bring into question „the accessibility, veridicality, and completeness of verbal reports“ (Wilson & Clarke, 2004, p. 28). Others (e.g., Ericsson & Simon, 1980) also criticize the ability of the problem solvers to verbally report their thinking processes parallel to the problem solving process. Nowadays, a combination of online and offline methods got established as an appropriate method to analyze metacognition awarding it the highest degree of validity (Veenaman et al., 2006).

A multi-method interview technique (MMI) combines both these methods using several different instruments: a problem-based clinical interview (self-reporting using action cards), audio and video recording for stimulated-recall, and observation. The main feature of the clinical interview is a card-sorting procedure by which the children reconstruct their thought processes after the problem solving process based on the three functions of metacognition. For it, 14 action cards including individually listed metacognitive statements on cards, each associated with one of the three metacognitive functions, were developed. Problem specific cards listing cognitive behaviors are also included. In addition, MMI technique includes using videos to stimulate students' reflections on the constructed card sequence, which serves as an essential validating strategy for the card sequences. In their study, Wilson and Clarke (2004) demonstrated the effectiveness of the MMI for the study of student metacognition in the context of grade 6 mathematical problem solving.

## **METHODOLOGY**

### **Participants**

The participants of the study were grade 2 and 4 students from two urban schools. Teachers volunteered their classes as possible subjects for the study. In both classes the children had experiences with problem solving. All of the children were invited to participate in the study. From those children who volunteered, a group was identified that balanced gender and mathematical ability. The latter was assessed by the teachers. In total 36 children (18 students per grade level) participated in the study.

### **Research design**

The research data was collected during three problem solving sessions, and consisted of: (1) audio and video-data, (2) retrospective reconstruction using action cards, (3) document review, and (4) researcher's field notes. (1) Two video cameras recorded the audio and video data, which were pointed on the child, and on the child's work. Video data comprised of participants' not prompted verbal reports during, and prompted verbal reports after the problem solving process. Video data was not used for stimulated-recall in order not to overburden the child; (2) Retrospective reconstruction of the problem solving session was supported by action cards, as shown in Table 1. Children were given 12 metacognitive cards (4 cards per metacognitive function), 3-5 general and problem-specific cognitive cards (varied upon task), and empty cards, if their thought processes did not fit the provided action cards. I adapted the MMI technique of Wilson and Clarke (2004) by using fewer cards, shortening the statements and using the language appropriate for primary grade students; (3) During the problem

solving sessions, the children solved three problems (numerical, geometrical, and combinatorial), that were chosen so to provide them with opportunities to engage in metacognitive activities. They documented their problem solving process on an empty piece of paper. The problems given to grade 4 children varied only quantitatively from the problems given grade 2 students; (4) Researcher's field notes contained protocol of children's problem solving actions, and helped validate child's reconstruction of the metacognitive processes.

<b>Awareness</b>	<b>Evaluation</b>
I thought about math I know.	I thought about whether what I was doing was working.
I knew what math could help me.	I checked my last step.
I had an idea what I can do.	I checked my solution.
I thought about similar problems.	I wasn't sure if I can do it.
<b>Regulation</b>	<b>Cognitive cards</b>
I made a plan in my head.	I draw a figure. (general card)
I thought about another way.	I made a table. (general card)
I thought about my next step.	I calculated how many meters a snail makes in a day. (problem-specific card)
I decided to do something else.	

Table 1: Action cards with cognitive and metacognitive statements.

## Procedure

Data was collected in one-to-one setting between the child and the author. By solving an exemplarily problem, they got introduced to action cards. During each session a child solved one problem, which lasted about 15-20 minutes. The session started with a child receiving a problem and an empty piece of paper. After they were done solving the task, they were given the actions cards, and they reconstructed chronologically their problem solving process. If needed, they took an empty card and wrote down additional actions. Afterwards, we discussed the card sequence taking into consideration my field notes, so to validate or question the reconstructed sequence by the child. The same procedure was used for each following session.

## RESULTS

### Observed functions of metacognition

Regardless of the grade, all functions of metacognition were observed (see Table 2). Both groups engaged the most in metacognitive behavior of evaluation. Two action cards were used by both groups in more than 85% of cases, "I thought about whether what I was doing was working" and "I checked my solution". Both teachers were not surprised with these results as they often model and emphasize this behavior when problem solving. Both groups differed with respect to frequency of regulatory and

awareness actions. Grade 2 students engaged in more awareness actions than regulatory ones. Predominantly they chose the action card “I had an idea what I can do” (45.2%). Regulation was reported the least (14.7%). This behavior may be cognitively too demanding for your children or they may not be used to regulating their actions. In 70.6% of cases the children used the card “I thought about my next step”. Interestingly, they never used the card “I decided to do something else” and the card “I thought about another way” only once. The planning card was used in 29.4% of cases only. Hence, behaviors of grade 2 students reveal all functions of metacognition, but differ in their frequency and in the spectrum of different actions.

	Awareness	Evaluation	Regulation	Cognition
Grade 2 students	93 (26.7%)	102 (29.3%)	51 (14.7%)	102 (29.3%)
Grade 4 students	69 (17%)	117 (28.9%)	75 (18.5%)	144 (35.6%)

Table 2: Absolute and relative frequencies of reported metacognitive behaviors during three problem solving sessions.

Metacognitive portrait of grade 4 students was different. Regulatory actions (18.5%) preceded the awareness actions (17%), but only minimally. Predominantly they chose the action cards “I thought about my next step” (40%) and “I made a plan in my head” (44%). Thus, some children had a more gradual approach to regulating their problem solving process, whereas others were able to do it more globally. Taking into account the results of grade 2 students, it is likely that regulatory behaviors already develop during early-school years at a basic level, but become more sophisticated with time and in a specific context. With respect to awareness, grade 4 students most often chose “I thought of similar problems” (43.5%) card, followed by the card “I had an idea what I could do” (34.8%). They most often tried to remember a similar problem they once solved in order to solve the new problem. Both groups reported a minimal use of the card “I thought about math that I know”. It seems that both age groups approach problems “impulsively” and try to find a sort of support system (similar situations or problems), rather than focusing on the mathematical constructs behind the problem itself. It may be that both groups developed or adopted this strategy to use when confronted with problematic situations.

### Metacognitive structure sequences of primary grade students

Similar to results of Wilson and Clarke (2004), AER und ARE models of metacognition were evident in the empirical data. A simple AER sequence would be:

Action card 1: I had an idea what I can do [awareness]

Action card 2: I draw a figure [cognition]

Action card 3: I thought about whether what I was doing was working [evaluation]

Action card 4: I thought about my next step [regulation]

These two models were most often reported (61.1%), were embedded in longer sequences, and contained repeated similar actions, such as A, E, C, C, R, C, E (see

Figure 2), and A, R, C, C, C, E, E (see Figure 3). The positions of the three metacognitive functions are the same as in Figure 1.

Action card 1: I thought about similar problems [awareness]

Action card 2: I thought about whether what I was doing was working [evaluation]

Action card 3: I added two things und subtracted from 30 [cognition]

Action card 4: I checked my calculations [cognitive]

Action card 5: I thought about my next step [regulation]

Action card 6: I looked for two objects, which in sum would give 24 [cognition]

Action card 7: I checked my solution [evaluation]

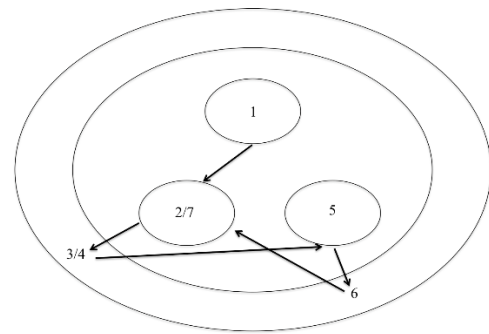


Figure 2: Exemplarily visualization of AER sequence (Jana, 2<sup>nd</sup> grade).

Besides AER and ARE, their longer models were also observed, such as AERER, ARERE, and ARAE. The reported sequences did not always include all functions of metacognition, but only one or two, such as A, AR, AE, E, RE, EAE. Short sequences were more common for grade 2 students. Majority of children (63.9%) started with an awareness card, which is a logical action. In a problematic situation a problem solver needs to call to mind what they know and have done other times or in similar situations. In 61.1% of cases evaluation ended the problem solving process.

Action card 1: I had an idea what I could do [awareness]

Action card 2: I made a plan in my head [regulation]

Action card 3: I calculated until I reached 9. I subtracted three [cognition]

Action card 4: I calculated +3, and the -1 [cognition]

Action card 5: I counted till I reached 9 [cognition]

Action card 6: I thought about whether what I was doing was working [evaluation]

Action card 7: I checked my solution [evaluation]

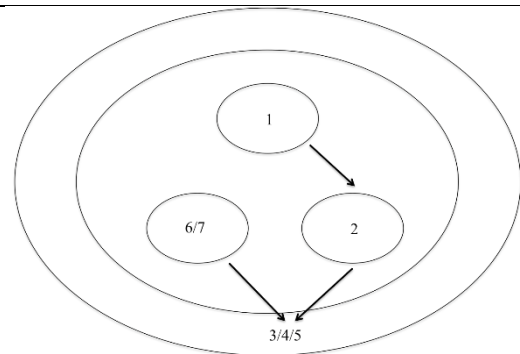


Figure 3: Exemplarily visualization of ARE sequence (Luis, 4<sup>th</sup> grade).

Some of the reported sequences started with either evaluation (5.6%) or regulation (2.8%), sometimes also preceded by a cognitive activity (30.6%). Such sequences are rather peculiar, because a child has to be evaluating or regulating something. When a metacognitive behavior was preceded by a cognitive action (e.g., “I draw a table” followed by “I thought about whether what I was doing was working”), such sequence makes sense. Some started with an action card “I wasn’t sure if I can do it”. The use of this card suggests that they had to consider their cognitive capacities and found them inadequate in order to make this statement. Three children started with a regulatory action preceded by a cognitive action. However, regulation implies a sort of awareness behavior. This implies that the behaviors typical for awareness were not registered and, therefore, not reported by some children. In 25% of cases the children reported ending with a cognitive action. Retrospective interviews revealed children’s last thoughts being of evaluative nature. Sequences starting with a cognitive action were also common (30.6%). It may be that awareness behaviors did not get reported or recognized, because a problematic situation was familiar or already automatized.

## DISCUSSION AND CONCLUSIONS

The framework by Wilson and Clarke (2004) and an adaptation of MMI have shown to be useful for the analysis of children’s problem solving, yielding insights into children’s metacognition. The results of the study showed that primary grade students’ problem solving process was a non-linear, dynamic interplay between cognitive and metacognitive actions (see Figure 4).

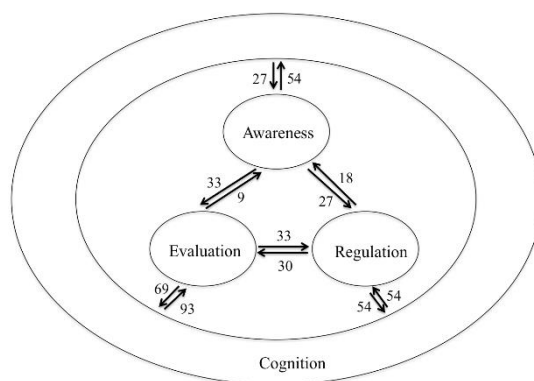


Figure 4: Primary grade students’ model of metacognition.

Regarding the expected metacognitive processes, it was difficult to determine in advance which are to be expected. For this purpose, only an ideal course could be represented – A, E, R and A, R, E – with cognitive activities between the individual components. The empirical results with primary grade students have confirmed this hypothesis. These sequences were, however, embedded in longer sequences. The reported metacognitive actions support the thesis that even young children are able to engage in different metacognitive actions. The study showed that these occur at a basic level at very early age, and may become more sophisticated and its range even wider with time as a result of a context or maybe even classroom influence. The model also illustrates that all possible transitions were employed, and that between two different



metacognitive behaviors, and between metacognition and cognition (illustrated by incoming and outgoing arrows). The highest reported transfer was from cognition to evaluation (n=93) and vice versa (n=69). The transfer from regulation to cognition was also high (n=54). High frequencies for these transfers have been expected. A higher frequency from awareness to cognition rather than in other direction can be explained by awareness actions being more crucial in the beginning phase of the problem solving process. When observing the children, behaviors of an affective nature were noticed. In the next step, extending the model with an affective domain would allow for a deeper insight into the phenomenon.

The utility of the multi-method interview technique has shown to be effective when working with primary grade students, but should be further developed, and examined. Reconstruction of metacognition was rather difficult for grade 2 students. They were not able to work with so many cards, and sometimes did not understand what the statement on the card meant, though they practiced this at the beginning. A possible development of the instrument could include supporting the text with an appropriate visual. The analysis showed that some processes were not reported by the children. For instance, cognitive actions did not get reported after each metacognitive action, and got less reported than metacognitive actions overall, which seems rather unreasonable. Development of a validating protocol for the interviewer may help overcome this. Another possibility would include just watching few video sequences to stimulate students' reflections on the card sequence instead of the entire video.

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# STUDENTS' ASSESSMENT FOR LEARNING PERCEPTIONS AND THEIR MATHEMATICS ACHIEVEMENT IN TANZANIAN SECONDARY SCHOOLS

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*The impact of Formative Assessment (FA) and Assessment for Learning (AfL) practices depends on whether students perceive and utilize the guidance provided by their teacher to improve their learning strategies. This study investigates the relationship between students' perceptions of their mathematics teacher's FA and AfL practices and their mathematics performance among 48 Tanzanian secondary schools. Data was collected in 54 Form three (grade 11) mathematics classes with 54 mathematics teachers and 2779 students. We used a correlational survey design and multilevel modelling for data analysis. Our results show that students' perceptions of their mathematics teacher's FA and AfL practice are significantly related to their mathematics performance. Practical and theoretical conclusions are drawn.*

## INTRODUCTION

Formative Assessment (FA) and Assessment for Learning (AfL) received increased attention over the past two decades (Black & Wiliam, 1998). FA and AfL stress the diagnostic purpose of assessment (i.e., improve student learning). AfL additionally highlights active involvement by students in the assessment process (Black & Wiliam, 2009), and students who conceive of assessment as an opportunity to direct their own learning have been shown to increase their learning outcomes (Brown & Hirschfeld, 2008). The impact of FA and AfL assessment practices depends on how they are perceived by students and teachers. AfL is a two-way process in which not only students adapt their learning in response to the information provided by assessments, but teachers adapt their teaching as well based on assessment results (Pat-El, Tillema, Segers, & Vedder, 2014). Hence, FA and AfL demand that both students and teachers adjust their actions based on assessment information. The success of FA and AfL hinges on student's willingness to engage in appropriate actions to close the gap between the target performance and their actual performance (Sadler, 1989) and whether students perceive and utilize the guidance provided by their teacher to improve their learning strategies (Pat-El et al., 2014). For example, Wiliam, Lee, Harrison and Black (2004) showed that improving formative assessment produces tangible benefits in terms of student performance in externally mandated assessments. FA literature provides extensive evidence that, if well implemented and well perceived by students, FA and AfL have the potential to improve student learning (Black & William, 1998, 2009; Wiliam, 2011), and especially for struggling learners (Black & Wiliam, 1998). More precisely, FA and AfL serve two functions: 'monitoring' to track student

progress and ‘scaffolding’ to help students improve their learning (Pat-El, Tillema, Segers, & Vedder, 2013; Pat-El et al., 2014; Stiggins, 2005).

Studies on the description and improvement of teacher practices have often focused on monitoring activities in the past (e.g., Reinhold, 2014; Veldhuis & van den Heuvel-Panhuizen, 2014). Nevertheless, tracking student progress is only a first step towards supporting learning in a formative way. Even though both practices are considered essential for formative assessment, it is not trivial to decide in a specific context if both have to be developed equally or if one of them is more important.

### **Formative Assessment for Mathematics Learning in Tanzania**

From a mathematics education perspective, some authors argue that formative assessment occurs naturally in context of good classroom instruction (Ginsburg, 2009; Veldhuis & Van den Heuvel-Panhuizen, 2014). Nevertheless, this is not easy to achieve. For example, in a study on learning from errors, which is closely connected to FA, Rach, Ufer and Heinze (2012) showed that even though students valued the way in which their teachers’ dealt with errors in the classroom, and even though they reported low fear of making errors, many of them did not use errors as a learning opportunity. The same study showed that it is far from trivial to support teachers in terms of classroom instruction that provides students with cognitive strategies to deal with errors, so that feedback can be used for learning. Also meta-analyses indicate that positive effects of FA in terms of student achievement are not easily achieved (Bennet, 2011; Veldhuis & Van den Heuvel-Panhuizen, 2014).

In this contribution, we study an existing programme that aims at Formative Assessment on a large scale: in 1976 Tanzania introduced Continuous Assessment (CA) in secondary schools, which was envisioned to serve as a formative practice. However, despite having CA in schools students consistently underperform in mathematics national examinations. For example, Basic Education Statistics in Tanzania (BEST, 2004-2013) indicate that for ten consecutive years (2004-2013) almost 79% of secondary schools students failed their mathematics national examinations. Students’ poor performance raises doubts about the formative effects of the mathematics assessment practices in Tanzanian secondary schools. Moreover, schools in Tanzania differ strongly in terms of their students’ mathematics performance. According to national examinations, schools are usually categorized into low performing, middle performing and high performing schools (MoEVT, 2013). Even though differences between these schools can have multiple reasons, they often go along with differences in the financial and personal resources of the schools. Thus, this school categorization is usually considered an important potential confounding factor for many issues regarding educational effectiveness.

Regarding the effects of the Continuous Assessment programme in Tanzania, student perceptions as indicators of their mathematics teachers’ formative assessment practices might explain inter-individual differences in their mathematics performance (Ginsburg, 2009). If the implementation and use of this assessment programme by

individual teachers or in individual schools is a central issue, these relations between students' perception and student performance occur on the classroom level, and are less pronounced on the individual level. In particular, the present study seeks to answer the following question: "To what extent are students' perceptions of their mathematics teacher's FA and AfL practices, in terms of perceived monitoring and perceived scaffolding, related to their mathematics performance?"

## METHOD

Data was collected in 48 secondary schools in Tanzania: 25 in the mostly urban Dar es Salaam region and 23 in the mostly rural Kilimanjaro region. Three criteria were used to achieve a representative sample: (a) mathematics performance (high, medium, low), (b) class-size ( $< 40$ ,  $\geq 40$ ), and (c) school-type (private, government). The combined mean GPA for the schools in the two sampled regions ( $M = 4.63$ ,  $SD = 0.69$ ) is representative for the country mean GPA ( $M = 4.85$ ,  $SD = 0.70$ ). A total of 2779 Form three (grade 11) students from schools varying in mathematics performance ( $N_{\text{high}} = 426$ ,  $N_{\text{middle}} = 999$ , and  $N_{\text{low}} = 1354$ ) and 54 Form three experienced mathematics teachers from these schools ( $N_{\text{high}} = 8$ ,  $N_{\text{middle}} = 19$ ,  $N_{\text{low}} = 27$ ) participated. The sample comprised classrooms from 30 private and 18 government secondary schools. Students had an overall mean age of 16.50 ( $SD = 1.12$ ) and girls ( $M = 16.31$ ,  $SD = 1.04$ ) were slightly younger than boys ( $M = 16.73$ ,  $SD = 1.16$ ),  $t(2743) = -9.94$ ,  $p < .001$ ,  $d = 0.38$ . Form three (grade 11) was suited for this study because it contains more teacher based assessment practices, which count into the final (grade 12) secondary education mathematics grade.

We applied a correlational survey design. To measure student perceptions of their teacher's FA and AfL practice, we used the 28 item Student Assessment for Learning Questionnaire (SAFL-Q; Pat-El et al., 2013) measuring 'perceived monitoring' (16 items, original Cronbach's  $\alpha = .89$ , sample item: 'My mathematics teacher inquires what went well and what went badly in my work') and 'perceived scaffolding' (12 items, original Cronbach's  $\alpha = .83$ , sample item 'When I do not understand a topic, my mathematics teacher tries to explain it in a different way'). Each scale was adapted to the mathematics context for the present study and students could chose to read the questions in English or Swahili. A 4-point Likert scale was used for all items: fully disagree (1), somewhat disagree (2), somewhat agree (3), and fully agree (4). In our sample the perceived monitoring scale had a Cronbach's  $\alpha$  of .93 and the perceived scaffolding scale a Cronbach's  $\alpha$  of .87. Students self-reported their mathematics performance in Form three (grade 11) terminal examination which is a teacher made examination. To analyse the data, a two-level (teachers and students) multilevel model was used to account for the hierarchical nature of the data: students are nested in teachers and leads to dependency which may otherwise bias the estimation of standard errors. The study considers only two levels because mostly one mathematics teacher was sampled from each school; hence, the impact of the school as a third level could not be included.

## FINDINGS

Table 1 shows the descriptive statistics and correlations for the perceived monitoring and perceived scaffolding scales, as well as for students' mathematics performance.

	Correlations							
	<i>N</i>	<i>M</i>	<i>SD</i>	Min.	Max.	1	2	3
1 Perceived monitoring	2779	3.16	0.72	1	4	-		
2 Perceived scaffolding	2779	3.39	0.56	1	4	.78**	-	
3 Mathematics performance	2767	43.13	18.58	0	99.90	.12**	.16**	-

Table 1: Descriptive statistics and correlations for perceived monitoring, perceived scaffolding and mathematics performance (\*\*  $p < .001$ ).

Table 1 indicates that students agreed to questions on scaffolding ( $M = 3.39$ ) more than those on monitoring ( $M = 3.16$ ), yet ratings are quite positive for both scales. Students' performance in Form three mathematics terminal examination was below average ( $M = 43.13$ ) with a large standard deviation ( $SD = 18.58$ ). Such a large variation in student performance indicates that this variability is most likely attributable to our sampling of students from clusters of high, middle and low performing schools. The intraclass correlations for 'perceived monitoring' ( $ICC = .20$ ) and 'perceived scaffolding' ( $ICC = .16$ ) indicated that students' perceptions varied systematically between classrooms to a large extent. This provides evidence for the validity of students' judgements as indicators of the implementation of FA in the different classrooms.

To analyse the relationship between students' perceptions of their teacher's monitoring and scaffolding practices and their mathematics performance, students' 'perceived monitoring' and 'perceived scaffolding' were first separately entered into the multilevel regression models to investigate their relationship to students' mathematics performance, and subsequently their combined impact was investigated. Finally, the school category was included in the analysis as a control. Table 2 shows the parameter estimates for each model. The low performing school category was used as a baseline group (reference category) against which the other school categories were compared.

Prior to the estimation of models, the empty model (model 1) was estimated which indicates that 25.02% of the variance in students' mathematics performance is due to differences between classrooms, which includes teachers' characteristics. Entering each of the two variables separately into the regression model shows that both, more intense perceived monitoring (model 2) and more intense perceived scaffolding (model 3) go along with higher mathematics performance. Nevertheless, only perceived scaffolding contributes notably to the explanation of differences between classrooms ( $R^2_{\text{Classroom}} = 3.79\%$ ). Even though this is a rather small contribution, it is larger than the contribution of perceived monitoring.

The change in Akaike's Information Criterion (AIC) was considered a good measure of model fit as it corrects for the number of estimated parameters. It indicates that

including both predictors in a joint model (model 4) resulted in best model fit among the first four models. In this joint model, perceived monitoring ( $B = 1.69$ ,  $\beta = .13$ ) shows a weaker relation to mathematics performance than perceived scaffolding ( $B = 3.21$ ,  $\beta = .31$ ) and both relations are weaker than in model 2 and 3. This indicates that the correlation between perceived monitoring and performance can partly be explained by differences in student perceptions of teacher scaffolding.

Since the students' self-reports of mathematics performance related to teacher-made tests, we checked if an introduction of random slopes would increase model fit. If the relations between students' perceptions and performance would differ between classrooms due to the different exams made by each teacher, this would be visible in a substantial improvement of model fit when introducing random slopes. However, AIC model fit indicators did not indicate an improved model fit with random slopes.

Parameter	Model 1	Model 2	Model 3	Model 4	Model 5
<i>Regression coefficients</i>					
Intercept	42.79** (1.31)	31.25** (2.00)	26.42** (2.39)	26.46** (2.39)	22.77** (2.58)
Perceived monitoring		3.63** (0.47)		1.69* (0.72)	1.72* (0.72)
Perceived scaffolding			4.80** (0.59)	3.21** (0.89)	3.15** (0.89)
High performing					10.10* (3.37)
Middle performing					6.77* (2.60)
<i>Variance explanation</i>					
Student Level $R^2$	---	2.13%	2.31%	2.55%	2.54%
Classroom Level $R^2$	---	--- <sup>a</sup>	3.79%	1.99%	21.67%
AIC	23389.95	23333.76	23326.48	23322.92	23315.95

Table 2: Regression coefficients (standard errors) between perceived monitoring and scaffolding practices and mathematics performance (\*\*  $p < .001$ , \*  $p < .05$ ).

<sup>a</sup>Variance estimate was negative, but only marginally different from zero.

Finally, to control for potential differences between high, middle and low performing schools, the school category was added (model 5). The relations between student perceptions and student achievement remained widely unchanged when doing so, indicating that these relations are not due to systematic differences between low, middle and high achieving schools. No substantial changes were also observed when adding school category as predictor to models 2 and 3.

## CONCLUSIONS

The main goal of this study was to identify relations between students' perceptions of their teacher's formative assessment practices and students' mathematics performance.

In this vein, we conducted a large cross-sectional survey study in a carefully selected sample of classrooms in Tanzania. We can draw several conclusions from our analyses. There are huge differences in mathematics performance between different classrooms. Reasons for this do not necessarily lie within the quality of teaching in general, and of formative assessment practices in particular, but also entrance selection for the different schools. Nevertheless, this variation warrants closer scrutiny of possible explanations for these systematic differences. The implementation of a large scale formative assessment programme might offer such explanations (Black & Wiliam, 1998).

Our analysis of intraclass correlations shows that students' perceptions indeed differ systematically between classrooms, which indicate that they carry substantial information about the actual assessment practices they experience in their schools. Furthermore, students' perceptions of monitoring and scaffolding practices are strongly related with each other. Finally, Tanzanian students do value their teachers' assessment practices, which replicates results from other countries (e.g., Rach et al., 2012). This holds for perceived monitoring as well as for perceived scaffolding.

Even though this is a good first sign, it has been shown that a positive perception of teacher practices does not necessarily go along with related learning processes and learning gain (Rach et al., 2012). Thus, we studied the relation between students' perceptions and their mathematics performance. The regression analyses indicated that more positive student perceptions of their teachers' scaffolding practices went along with higher mathematics performance, and this relation occurred systematically within classrooms. Even though this is in line with previous studies, which support that students' perceptions of teacher FA and AfL practice impact their performances (Pat El, Tillema, & Koppen, 2012), this relation was comparably weak. There are two possible explanations. First, the students' ratings of their teacher's assessment practices are only rough indicators of these practices. Second, the chain of effects from teacher actions and student learning is quite long and therefore strong relations are hard to find. More research is needed to examine these relations.

What is maybe more important is that the analogue relation for perceived monitoring was weaker, partly due to the correlation with perceived scaffolding, and did not occur in a similar strength on the classroom level. On the one hand, this may seem plausible because monitoring of student performance alone can hardly stimulate learning processes. On the other hand, monitoring is a necessary prerequisite for further formative (supporting) teaching practices. A potential systematic relation between the implementation of monitoring practices and student performance would have indicated that teachers do not sufficiently search for information about student achievement. In our study, we did not find evidence for this. Nevertheless, even though there might not be systematic differences in student perception of monitoring between teachers, a deeper qualitative analysis of typical monitoring practices, beyond the tests from the Continuous Assessment programme, might still offer starting points for improvement (e.g., Veldhuis & Van den Heuvel-Panhuizen, 2014).

Summarizing, our results indicate that in particular teachers' scaffolding practices go along with differences in student performance. Even though these results must be substantiated by further research to make definitive claims, it seems that in particular a development of teachers' scaffolding practices can be very promising to support the development of formative teaching in our sample (Pat-El et al., 2014; Wiliam et al., 2004). Stimulating such a development can build on and extend the monitoring through regular tests in the Continuous Assessment programme and teachers' own monitoring activities.

### **Restrictions and implications**

We selected a sample that should reflect teaching practices in Tanzania carefully, but our results have to be taken with a grain of salt. Of course, a cross-sectional survey design makes it impossible to draw strong causal conclusions. Nevertheless, we believe that we can gather substantial information with such a design that can guide and inform further research, for example using interventions that can substantiate or extend our results. Moreover, using student ratings of teacher practices might be criticized, however, our data provides some evidence of their validity. Nevertheless, even though teacher self-reports and document analysis have their restrictions, we plan to triangulate our results with such data in future studies.

Finally, our results might be specific to the Tanzanian context to a certain extent. In particular the fact that there is a large assessment programme in place might explain why we do not find strong connections of monitoring practices with student achievement. This was contrary to expectations from the formative assessment literature (Wiliam et al., 2004). Thus, our results might only be transferable to educational systems, which apply similar assessment programmes. For further research on the improvement of formative assessment, this pinpoints the necessity to take the specific context into account when conceptualizing teacher professional development on FA. Gathering data on the relative effects of different assessment practices in a specific context seems to be an important step before conceptualizing new interventions or adapting programmes from other contexts.

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# OBSERVING ROBOT A.L.E.X. TEACHING MATHEMATICS

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*The paper accounts on the main experiences gained from a two-year study that incorporated A.L.E.X., a puzzle game available on tablet devices, within the primary school geometry. The study took place in a public primary school in Cyprus. A group of fifteen (n=15) Grade 5 pupils (8 boys and 7 girls; aged 10-11), was purposefully selected to comprise the sample. The same group was visited twice within a period of two years and a teaching intervention was organized. In both interventions, the application A.L.E.X. accompanied by a student worksheet, constituted the main means of instruction. Results concur with those of previously conducted studies, suggesting that game apps hold a lot of promise as a tool for reforming mathematics education.*

## INTRODUCTION

The exponential rate of adoption of tablets and other smart mobile devices witnessed worldwide in recent years, has dramatically increased children's accessibility and usage of computing devices. The relatively lower cost of mobile devices compared to laptop/desktop computers has been leading to a rapid decline of the digital divide in technology access along ethnic and socioeconomic lines (Kabali et al., 2015). Today, the vast majority of children of all ages in the developed world, regardless of ethnic or socioeconomic background, have access to mobile devices and, although a large gap still exists between rich and poor nations, the sharp rise in access to mobile devices in developing countries has been helping to narrow this gap.

As research indicates, the main activity in which children engage when using mobile devices is to play games (Common Sense Media, 2013). Responding to this trend, there has been an explosive growth in the number of educational games apps targeting children available on the market (Chau, 2014). The increased popularity and proliferation of digital games, has led to a widespread interest in their integration into the mathematics curriculum. Several mathematics educators (e.g. Ke, 2008; Meletiou-Mavrotheris, 2013) have been experimenting with digital games, investigating how they can be brought into the mathematics classroom in order to capture students' interest and facilitate their learning of mathematical concepts. The existing literature strongly indicates the educational value of games (e.g., Ke, 2008; Kolovou, van den Heuvel-Panhuizen, & Köller, 2013) and their potential to serve as a powerful perspective for reforming mathematics pedagogy at the school level. There are strong indications in the literature that appropriately designed and constructively used games support experimentation in authentic contexts, and can be used as the machinery for engaging students in problem solving activities that can help raise their intrinsic interest in mathematics and promote the attainment of important competencies essential in modern society (see Lowrie & Jorgensen, 2015).

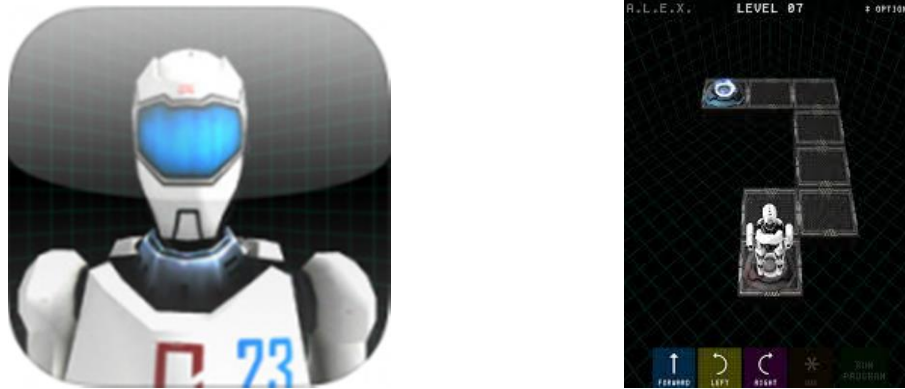
While digital educational games provide a range of potential benefits for mathematics teaching and learning, not all the available game apps are designed to promote optimal development among children. Many of the available educational games apps often include mediocre or even inappropriate content. They tend to be drill and-practice and to focus on basic academic skills, rather than on content creation or high-level thinking (Peluso, 2012). Nonetheless, although high quality, developmentally meaningful mobile education game apps are less common than hoped, some exceptional exemplars exist that can help create constructive, meaningful, and valuable learning experiences for children (Chau, 2014).

One promising type of game apps is coding apps, which teach children the concepts behind programming in a playful context. In many different countries across the world, some innovative, educationally sound game-based apps, supporting the development of computer programming skills from a young age, have begun to appear. Several educational apps are currently available for helping children, with no coding background or expertise, grasp the basics of programming through the exploration and/or creation of interactive games and other applications (e.g. Scratch Jr, HopScotch, Bee-Bot, etc.). Often, coding game apps enable children to share their games with others, and to play or edit games programmed by others.

The current article reports on the main experiences gained from a case study which incorporated the educational puzzle game app A.L.E.X. (Guida, 2014) within the primary school mathematics curriculum. A.L.E.X. (named after the developer's nephew) is an app available on tablet devices that uses programming logic in a game setting. It belongs to the constantly growing list of educational apps that aim at developing young children's rudimentary programming concepts through games. In this study, we explored ways of using this coding game app as a tool for engaging a group of students from a low socioeconomic background in authentic problem solving activities that can help raise their interest in mathematics, and promote the attainment of important competencies essential in modern society.

## **BACKGROUND TO THE STUDY: A.L.E.X. GAME APP FEATURES**

A.L.E.X. is entertaining programming puzzle game that lets players control a robot along a path. It is a free educational app suitable for downloading on iPad or Android tablets. The game is all-ages friendly. The lower levels of the game are suitable for children as young as six, while the higher levels might be challenging even for high school students or adults. A.L.E.X. has the potential to tacitly promote a number of concepts and procedures embedded in the mathematics curriculum. This becomes feasible by offering the user the opportunity to think and plan logically as he or she programs A.L.E.X. (see Figure 1a) with a sequence of commands, in order to complete each level.



**Figure 1:** Robot A.L.E.X. (Figure 1a) and screen's display of commands (Figure 1b)

The game has two modes, Play and Create. In the *Play mode*, players complete standard puzzles using the pieces provided to them. They begin at a start-point and have to "pre-plan" the robot's path. Once they plan the path by building a sequence of instructions, they execute these instructions, and watch the robot "walkout" their plan. If they have given the right instructions, the robot will reach its destination otherwise it will fall into oblivion. The levels start off fairly easy and increase in difficulty as the player advances. The free version includes 25 progressively demanding levels. There is also an upgrade available, which incurs a small cost, and provides 35 additional levels. At each level, players are evaluated on whether they successfully complete the level, how quickly they do so, and whether they take the shortest path. The *Create Mode* includes features for players to create their own puzzle. In this mode, players can devise their own levels by structuring the pathways they would like A.L.E.X. to follow, and play through their own levels.

The directions A.L.E.X. could follow are simple and symbolically expressed. For instance, the commands "turn left", "turn right" or "go forward" could be given when one touches the game's screen on the particular arrow pointing to the respective direction (see Figure 1b). The app is currently available only in English, however the instructions are simple to follow for non-native English speakers.

Despite being very simple to initially use, A.L.E.X. is a powerful educational game app that can help children improve their skills in directional language and programming through sequences of forwards, backwards, left and right 90 degree turns. At the same time, it can help improve understanding of mathematical ideas related to motion, direction, and geometry. Unlike the vast majority of geometry apps currently available, which are very limited in their ability to assist students in developing geometrical conceptual understanding (Larkin, 2015), A.L.E.X. provides a user-friendly venue for children to experiment with geometrical ideas – to make and test hypotheses, and to implement corrections based on feedback received.

## **METHODOLOGY**

### **Context and Participants**

The exploratory study described in the current paper took place in a public primary school, located in a rural area of Cyprus. The majority of its students come from low socioeconomic status families. High dropout rates before high school graduation constitute a usual phenomenon among the area population and this stance is often mirrored in parents' limited interest in their children's educational attainment. The researchers knowingly selected such a context to orchestrate a teaching intervention. Their goal was to explore the potential of iPad technologies for providing students with knowledge, skills and confidence in doing mathematics. Among the school community, a fifth-grade class consisting of fifteen pupils (8 boys and 7 girls), aged 10-11 years old, was selected to comprise the sample. Most of the students in this class had a low academic performance, and tended to suffer from negative self-efficacy and attitudes towards mathematics.

Although the teaching intervention was not designed as a two-stage comparative study, due to practical reasons it spanned two school years. Phase 1 was conducted towards the end of the 2013-2014 school year (students' fifth grade year), and there was insufficient time to complete the teaching intervention. For this reason, we continued its implementation in the Fall of 2014, using the same student cohort we had worked with in Phase 1 (there was no student attrition between the two phases).

At both phases, the first author was the class teacher and organized the class into three groups of five. Each group was given an iPad through which participants could have access to the application A.L.E.X. In both years, the teaching intervention was the only instance in which tablets were used by this class. Children did have some prior exposure to other educational technologies (e.g. mathematical applets), but not to coding applications.

### **Instruments, Data Collection and Analysis Procedure**

At the outset of Phase 1, participating students were asked to individually complete a questionnaire prior to their engagement with the A.L.E.X. app, in order for us to get a better understanding of the experiences and beliefs they held with respect to the use of tablets. The questionnaire, consisting of both open-ended and close-ended questions, focused on exploring students' familiarity with tablets, and also their perceptions and acceptance of mobile technology.

The application A.L.E.X. accompanied with a worksheet distinct for each year, constituted the main means of instruction. The worksheet was designed to integrate technology with core mathematical ideas. Symmetry was selected for year five, and measurement of the area and perimeter of a rectangle for year six. The latter concepts were purposefully selected among the geometrical concepts included in fifth and sixth grade's curriculum, respectively. Geometry, closely related to the constructionist approach of the A.L.E.X. game app, was chosen to serve as a mathematical basis for

revealing children's connected, but temporarily hidden mathematical concepts and thinking skills.

At the beginning of Phase I of the teaching intervention (Year 5), prior to students' engagement with the instructional activities, the teacher informed the class that the goal was for them to become familiar with a robot named A.L.E.X. No words relevant to the subject of mathematics were articulated, nor were any further explanations provided, because the principal idea of the intervention was for students to develop novel knowledge and skills and for the researchers to gather information regarding their attitudes towards technology-enhanced mathematics learning. During Phase 2 (Year 6), the teacher reminded students of the experience they had had during the previous year with A.L.E.X., and then the same procedure was followed. At both times, while the participants were focused on completing the tasks in their provided worksheets, the teacher approached different groups and deliberately posed clarifying questions that would, in turn, shed light on the process of data analysis.

In order to include children's authentic voice, and to strengthen the reliability of the collected data, a tape recorder was placed on each group's desk. At each phase, data collection spanned four consecutive 40-minute teaching sessions. Extensive field notes were also taken by the researchers during and immediately after each session. This helped to supplement the recording of the students' voices and clarified their actions and interactions with the app.

For the purpose of analysis, we did not use an analytical framework with predetermined categories due to the lack of well-established frameworks and methodological insights for studying mobile mathematics teaching and learning in the context of primary school classrooms. What we instead did was, through careful reading of the transcripts and field notes, and examining the various interactions for similarities and differences, to identify recurring themes or patterns in the data.

## **RESULTS**

The questionnaire completed prior to students' engagement with A.L.E.X., revealed that despite the low socioeconomic background of the sample, the majority owned their own iPad or other tablet. They had extensive prior experience and knowledge of playing games on tablets for their own leisure, but had very limited explicit understanding of the educational potential of game-based apps. This finding is likely indicative of the pervasiveness of mobile technologies in the daily life of almost all Cypriot youth, regardless of age or socioeconomic status.

### **Main insights from the Teaching Intervention**

Symmetry was central in Phase 1 of the teaching intervention. Findings suggest that not only did students make the link between the robot's movements and angles, but they also moved a step further to give an alternative name to the created angles.

- 1 S1: When he turned two times right or two times left it was 180 degrees, whereas when A.L.E.X. turned one time he created a right angle, that is, 90 degrees.

- 2 T: How about 180 degrees? What do they represent?
- 3 All: One straight line.
- 4 T: Perfect! What if A.L.E.X. turned four times right or four times left? What do you think would have been constructed?
- 5 S1: 360 degrees.
- 6 T: Could you tell us why?
- 7 S1: I did four times nine equals to thirty six. And then I put another zero and I got 360.
- 8 T: Bravo. Could you say where A.L.E.X. would have been after making this movement?
- 9 S1: At the same point.

While the participants were working on the activities, the class teacher observed that they were moving around the classroom, doing the movements themselves. Their body served in essence as a model of the robot, in respect to both its mental and physical actions.

- 10 S4: I first noticed the way A.L.E.X. was standing and then I took this position myself. I pretended that I was walking those moments.
- 11 T: Couldn't you just memorize the steps? Was it necessary for you to do the actual movements?
- 12 S4: Yes. I was surer this way.
- 13 T: So, what can we say about this game?
- 14 S5: You have to put yourself in the position of A.L.E.X. and to follow the guidelines as if the player was you.

The excerpt above is a characteristic example of how the game app's intuitive interface, accompanied with the mobility offered by the tablet device, provided children with independence and stimulated exploratory behaviour and engagement. Although clever use of a desktop computer and/or basic group work might have achieved similar behaviour and learning outcomes, the user-friendly nature and mobility of the touch-enabled app made it quicker and easier for the children to experiment with mathematical ideas. Supporting evidence for the latter claim appears in Phase 2 of the teaching intervention, as well. Students seem to have formulated a mental picture of the rectangular area and perimeter.

- 15 S3: A.L.E.X. helped us with the rectangles...
- 16 S4: It's difficult sometimes to understand the perimeter only with the formula, length plus width, times two...I mean by only doing calculations.
- 17 T: Okay. And how did A.L.E.X. help you understand the concept?
- 18 S2: By watching A.L.E.X. following the steps we told him, it was like walking on the perimeter myself. For me, perimeter means what is around the rectangle...I mean the surrounding line.
- 19 S3: The steps forward, left, right all together made the perimeter of the rectangle. And we saw this in action.

- 20 S5: And the small squares inside were for us the area of the shape.
- 21 S1: A.L.E.X. helps us to concentrate much more to mathematics and it entertains us, at the same time.

## DISCUSSION

The presented results are only suggestive. The full spectrum of findings is listed in Kyriakides, Meletiou-Mavrotheris and Prodromou (2015). Indications exist, albeit tentative given the nature of the study, that the integration of the coding game app had a positive impact not only on the level of engagement, but also on the learning of mathematics for the participants involved. In comparing the data from the two phases of the teaching intervention, interesting observations emerged. The first observation is that the students experienced their interaction with the A.L.E.X. game in different ways as a result of the app's affordances and/or limitations. For example, the app acted as a source that promotes the construction of mathematical concepts. The mobility of the A.L.E.X. robot, in combination with the mobility of iPads, seems to have promoted the construction and measurement of angles of 90, 180, and 360 degrees during the first phase of the teaching intervention.

Another observation to emerge from the investigation of the two phases of the teaching intervention is that the mobility of iPad resulted in altering the physical structure of the classroom, as well as promoting a more conducive to learning mathematics environment. The use of iPads altered the positioning of the teacher by allowing him to sit together with his students instead of being located at the front of the classroom. The altered structure of the classroom, and the use of iPads, promoted higher levels of student control over their learning of mathematics (Attard & Orlando, 2014). Nevertheless, achieving this transfer of control from teacher to students is highly dependent upon the teacher's level of experience and use of quality pedagogical approaches that can enhance learning and teaching of mathematics.

In this study, the class teacher's knowledge and pedagogy contributed towards the creation of an engaging environment in which students were encouraged to experiment with mathematical ideas, using the mobile learning environment as a tool. He consistently made connections to students' prior experiences and adapted instruction in response to feedback from learners. It is for this reason we claim that the integration of mobile technologies such as the iPad, and their potential use in the classrooms, necessitates careful professional development that highlights the importance of engaging teachers in regular reflection and evaluation of their current practices in relation to the successful implementation of transformative technologies into their teaching practices.

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# **‘LET ME TEACH YOU’ ADJUSTING ENGAGEMENT STRUCTURES FOR EXPERIENCED MATHEMATICS TEACHERS**

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*As part of a PhD exploring the role of positive emotions in the classroom, I have identified dominant archetypal structures for the ways in which experienced mathematics teachers engage with teaching. To this aim I have modified Goldin et al.'s (2011) Engagement Structures (ES) initially devised to describe student engagement with mathematical learning. ‘Let me teach you’ (LMTY) is a common ES for participants, and appears in various forms and in conjunction with different dominant ES for each participant. I draw from data from interview and observations, and other works on affect, in order to elaborate LMTY in the form of **affiliation**, **rejection**, **succorance** and **play** in addition to **nurturance**. I discuss implications of a teacher adopting LMTY in these forms for supporting engagement in mathematics.*

## **ENGAGEMENT STRUCTURES IN THE MATHEMATICS CLASSROOM**

Within a growing body of affect research within mathematics education, one of the current issues is how to examine fluid, complex emotions in the context of a mathematics classroom. This issue is particularly the case for teacher emotions, where there is less research as teachers engage in teaching mathematics (Frade et al., 2010). In this paper, the term affect is used to represent more stable affective traits such as beliefs, whilst emotions are used to indicate fluid states, experienced ‘in-the-moment’ (Hannula, 2012). Teachers, within an unequal power relationship, can condone and model selected practices, through language and use of emotions. ES offer potential to examine teaching relationships, as facilitating or restricting student engagement.

This paper reports from a study of UK secondary mathematics teachers. I use interview and observation data to establish dominant ES, and to explore the implications of ES in regards to engagement in learning mathematics. My research attends to expressed positive emotions, assuming that positive emotions are supportive of engagement and learning. For example through expressing enthusiasm (Kunter et al., 2008) where emotions serve to arouse, support readiness and coordinate effort.

Goldin et al. (2011) present Engagement Structures as meta-level structures, more complex than experienced emotions. A person can rapidly alter positions suggested by these structures, sometimes showing characteristics of more than one structure, although at any one moment there will be a dominant structure. Each ES has a different dynamic, although all have affective, cognitive and behavioural elements. For example, ‘Let Me Teach You’ and ‘Look How Smart I Am’ (LHSIA), are two ES that, if activated, impact on how mathematics is communicated. In action, two ES that evoke different responses yet both are formed from a position of mathematical knowledge. If

we consider a student who has a solution to a problem, evoking LMTY indicates a desire to share the solution, whilst LHSIA is to share that the person has a solution which implies that others should recognise the person as mathematically able.. According to Positioning Theory (Harré & Langenhove, 1999), the interlocutors could accept, decline or negotiate the proffered expected response, altering their own positions as a result. For LMTY, the ES I focus on in this paper, this response might mean accepting a passive position as an ignorant recipient, declining the proffered teaching or responding by attempting a repositioning of self or other in a different way. Examining such interaction, including non-verbal communications, such as establishing LMTY through leaning forward, as approach behaviour, reveals the participants motivating desires, whilst knowing some underlying motivations or beliefs through interview can predict which ES might be activated. One purpose of this research is to evaluate ES initially created for use with students for use with mathematics teachers (Lake & Nardi, 2014). LMTY is a common factor within Engagement Structures as teachers expect to, and are obliged to, engage in LMTY as part of a teaching role. LMTY is however not necessarily the dominant ES for teachers, whilst the nature may change as activated in conjunction with other ES.

## LMTY

I begin by clarifying student LMTY as appearing in the work of Goldin et al. (2011) and extend the use of **nurturance** as one form *affection need* (Murray, 1938), an extension that may account for LMTY differences between participants. Return to this original source provides some alternative models for LMTY in context. I illustrate these alternatives in relation to participant data, concluding with implications of LMTY as potentially conducive to or oppositional to engagement in teaching and learning.

LMTY specifically for students whilst learning mathematics is summarised in Table 1. If LMTY is considered as a whole, Goldin et al. (2011) give an example of a student setting out to ‘*Get The Job Done*’, summarised as to just follow instructions. But the student might realise they understand a task better than their peers, so they move to ‘*Let Me Teach You*’. This ES might prove a stronger motivator for the student than ‘*Get The Job Done*’. Yet the ES might be declined by the response of peers, who may not value the student’s mathematical knowledge. The student might then move into a performance governed ES such as ‘*Look How Smart I Am*’ in order to impress.

However, a student attempting to assist a peer is not necessarily the same as for a teacher trying to help by explaining or demonstrating. The evocative social situation of awareness that another person does not understand lies at the core of teaching as the teacher is a ‘more knowledgeable other’. In relation to a teacher, ‘*Get The Job Done*’ might appear as achieving the lesson objectives, and a teacher is more likely to adopt ‘*Let Me Teach You*’ because of their mathematical expert role and because their role is to notice where help is needed.

Let Me Teach You <i>Strands</i>	Description	Belief
Desire	To help another understand or solve the mathematical problem.	Math has understandable internal logic, and the person has high self-efficacy beliefs.
Affection Need	<b>Nurturance</b>	
Evoked by	Evoked where a person who has insight or mathematical knowledge to share, notices someone who does not understand.	Understanding and helping others are both valued.
Behaviour	Behaviour includes trying to help by explain or demonstrate	
Emotional satisfaction	Satisfaction is derived from the other person or people learning mathematics and/or appreciating help.	

Table 1: Summary of the ES LMTY for students (Based on Goldin et al. 2011)

LMTY as an Engagement Structure, is complex within the identities of a mathematics teacher, one that would be difficult not to adopt as the designated one who helps. This designation has implications if students decline LMTY, or try to renegotiate. Declining in this case has implications since, unlike peers, and as Hagenauer & Volet (2013) suggest, “empathic emotions are more likely to occur if someone feels responsible for others, just as teachers frequently do, at least to some extent, for their students” (p.255), whilst simultaneously, school teachers hold responsibility for student success. If a teacher holds strong beliefs within this ES, and values helping others highly, or if the teacher self-efficacy is not robust, then declining of the LMTY ES by students threatens a teacher’s self-efficacy, and declining the teachers’ logical explanation of the mathematics can be negative over time and repetition.

Murray described a need as a "potentiality or readiness to respond in a certain way under certain given circumstances" (p.83). The underlying human gratification of reward from appreciation for helping resolve confusion, or providing a successful explanation is universal, but particularly applicable within the context of mathematics teaching. Alongside a **nurturant** attitude, as used in student LMTY above (Table 1), “To nourish, aid or protect a helpless O [Other]. To express sympathy. To 'mother' a child.” Murray (1938) adds four more affection needs, “to do with affection between people; seeking it, exchanging it, giving it, or withholding it”. The original definitions are as follows;

**“Affiliation** (Affiliative attitude). To form friendships and associations. To greet, join, and live with others. To co-operate and converse sociably with others. To love. To join groups.

**Rejection** (Rejective attitude). To snub, ignore or exclude an O. To remain aloof and indifferent. To be discriminating.

**Succorance** (Succorant attitude). To seek aid, protection or sympathy. To cry for help. To plead for mercy. To adhere to an affectionate, nurturant parent. To be dependent.

To these may be added with some hesitation: **Play** (Playful attitude). To relax, amuse oneself, seek diversion and entertainment. To 'have fun,' to play games. To laugh, joke and be merry. To avoid serious tension.” (p.83)

All apply by degree, but some may be more important to a mathematics teacher than others, and so guide LMTY in action. In the next section, I draw from analysis of three teachers, Gus, Bertha and Carol to illustrate permutations of the five needs as one specific component of LMTY.

## **DRAWING ON THE DATA AS ILLUSTRATIONS**

As part of my PhD research exploring the role of positive emotions in mathematics classrooms, I interviewed, observed and discussed with each teacher episodes from their lessons. In this paper, drawing from this data, I discuss whether LMTY is applicable to mathematics teachers. I retain LMTY as described for mathematics students, but extend how the underlying needs differ in the new context. The participants discussed here, Gus, Bertha and Carol, are experienced UK secondary mathematics teachers. Gus talks of teaching higher achieving students, whilst Carol discusses her mathematically promising year 10 students. Bertha works predominately with younger students who find mathematics difficult, and may lack motivation.

### **The case of Gus: LMTY as affiliation and play**

To illustrate how LMTY might appear in different forms, I first consider the case of Gus. LMTY is not the primary ES for GUS, the most experienced participant. LMTY for Gus is likely to be activated along with LHSIA, modelling engaging successfully in mathematics. One observed episode shows Gus explaining the importance of communication via correct mathematical language. The episode evolves into fast-paced banter about sport and politics. In interview, Gus emphasises the importance of talking mathematics, and his desire for social appreciation appears to be deeply rooted. He talks of emotions predominantly from a social perspective.

“You set things up and then they happen whether or not you are in the room. So you can sit there and relax, and tell jokes and what I do is set up situations where they can enjoy and relax, and not necessarily really learn from me but learn from themselves and process.”

Additionally, I observed self-enjoyment through performance, yet he actively sought student contributions. Gus was visibly emotionally expressive through smiling and body language and the students responded positively. My interpretation is that this behaviour structures LMTY as ‘Let me entertain you and you will learn as well’. The distinguisher is the level of communication, as equals rather than say as a parent to a

child (Berne, 1964). This form of LMTY positions students as able. Gus tells of enjoyment as increasing learning directly providing an example of where the **affiliative** affection needs dominate because of an overt social and enjoyment focus. Later he adds, “Because if the stress levels get high for the class, then reptilian brain kicks in and you cannot learn.” which mirrors Murray (1938) for **play**, defined as “To ‘have fun’, to play games. To laugh, joke and be merry. To avoid serious tension.”

### The case of Bertha: LMTY as succorance, nurturance and rejection

Bertha provides an example of potential adjustments to LMTY. Her LMTY is less dominant than individually orientated ‘*Don’t Disrespect Me*’, ‘*Get the Job Done*’ and ‘*Stay Out Of Trouble*’. Murray’s (1938) affection need **succorance** summarises a need to seek aid, protection or sympathy, to cry for help, plead for mercy, and for a teacher to be dependent on others, for example, for solutions to problems. When talking about problems with class behaviour, Bertha allocated her own concerns to others by talking of another teacher having problems with the same class. She does this sotto voce, presenting as depreciating and assigning guilt or shame through dropping to a whisper. In interview, Bertha said,

“I am very emotional, that’s something that probably you should need to know before you do it... very, very emotional. I wear my emotions very much on my sleeve and there has been more than one time when I’ve cried in the classroom because kids says things that affect me very much and I’ll take a child outside and they’ll start telling me their life story and I’m in floods of tears and they’re going are you all right miss and things like...”

Murray (1938) defines LMTY in connection with a **rejective** attitude. The intensity of emotions as revealed to the students in the observed lesson was predominantly negative, such as frowns, sharpness and impatience (Fig.1). An episode where she is controlling one student’s data contributions in a lesson on collecting statistical data illustrates this negativity. The bold type in the extract indicates shouting.

- Sam: The gender is er... male...
- Teacher: Yes, I know. [Sarcastic. Some nasty laughter from other boys in the group]
- Sam: ...arm length...
- Teacher: No. **Foot length**. [Shouted. Eye contact briefly as shouting]
- Sam: [quieter] Sorry, ah, 25...
- Teacher: Hand span? [Imperative]
- Sam: 24, ah yeah.
- Teacher: ...rapid reaction? [Imperative]
- Sam: 1.306
- Teacher: 1.306. You need one of those [hands out sheet without looking at paper or student]. Um. Kevin. Your information [makes brief eye contact with Kevin]
- Kevin: For my height I’ve got 5 foot 1

- Teacher: How much is that, how much is that in centimetres?
- Kevin: I don't know... er... [Teacher waits]
- Teacher: **Go away, find it out and come back again** [shouting, waves arm in dismissive manner, averts eyes and turns away directly to next student].

This extract comes from a longer episode where I interpreted the dominant emotion within the episode. Tracing dominant emotions might be considered as an affective pathway as shown below (Fig.1).



Figure 1: Affective pathway, Bertha.

Murray assigns a **nurturant** attitude as a protective sympathetic form for LMTY. Bertha refers in interview to her students as ‘a bit of a naughty’ or ‘bless him’ a parent to child form of discourse, also used by Bertha in the classroom, “Well just, Terry, just say it, have more confidence in yourself sweetheart...” She says in interview that she wants students to be friends with her and values when this desired connection happens. But at the same time she refers to students as “little boys and girls” and that “they love to be talked to.” The same manner applies to mathematics learning, a deficit model that Hardy (2000) assigns as ‘I expect you to have difficulties’.

### The case of Carol: LMTY as nurturance and affiliation

Carol illustrates that **Nurturance** within LMTY can appear as positive. In interview, she talks of a strong desire to help others understand and to solve problems, so the beliefs associated with LMTY in peer-to-peer student form align. She talks directly to students of resolving confusion, “So although I wanted you to find some of the patterns yourself, I don’t want you to be completely confused.” Seeing need and meeting it, including that mathematics has an understandable internal logic, she places a strong emphasis on explaining and demonstrating. Her primary reward is recognition or appreciation of help, or that students learn. When asked about a particular student she responds from a **nurturant** and **affiliative** teaching position,

“Andrea. Yeah. She feels she’s quite weak in the group. She needs a lot of reassurance, but she’s actually better than she thinks. Very thorough with what she does. She’ll often ask questions, often write things down fully. I think I’ve got quite a good rapport with her.”

Carol does not show positive emotions overtly in the class, yet indicates a strong emotional attachment to students in interview, and in addition to teaching mathematics, she has a nurturing pastoral role. In Murray (1938) terms, indicative of positioning as a teacher as being discriminating, indicative of a need to remain aloof and not become emotionally attached and draws attention to the role of emotional regulation in the classroom. Her LMTY appears in conjunction with positive potentially engaging ES of ‘I’m Really Into This’, a positive modelling ES, but also with ‘Check This Out’

CTO. The data for Carol indicates **affiliation**. This social emphasis becomes clearer as she says, “I’ll get excited about it [mathematics], I’ll keep talking, so when I am doing that, it’s me... don’t stop... stop myself.” Later she adds,

“...my view of myself is quite difficult, I think I am a teacher first, then a maths teacher and I don’t know that everyone would say that, I think a lot of people I meet in maths specifically really love the subject, are really keen to impart that information.”

This suggests Carol’s beliefs prioritise the social over the mathematics, in contrast with Gus, who says, “Maths was just like breathing.” The social dimension for LMTY is also evident in Carol’s mathematics teaching discourse. In this extract she points out a graphical method used in a textbook to a group of students,

“This is what we are looking for. [Shows graph in book to the students] Right, here is a graph. They have put frequency up the side, they’ve gone for decimals, with the number of trials along the bottom. Ok. So they have marked it every hundred, really, ok, although you have not got time to do that. So I was hoping you would go up at perhaps, every 16, 32... [Nods head to show pattern continues] and then every time you have done a chunk then stop and see what you have got.”

Her language seems to **affiliate** Carol with the students, whilst the textbook, representing mathematics becomes the other. This interpretation is confirmed after this lesson as she talks of two groups in relation to perceived ability, which additionally highlights her motivating affective need to **nurture**,

“I was probably more comfortable talking to that group than I was when we were looking at the one earlier, (right) for a variety of reasons. Just because I think, I suppose cos they are weaker, so I feel I can help them more. They engage better.”

## DISCUSSION AND IMPLICATIONS

I have reconsidered LMTY, an Engagement Structure applicable to teachers, but to varying degrees and centred on differing underlying affective needs of **nurturant**, **affiliative**, **rejective**, **succorant** and **playful** attitudes. I have introduced LMTY as initially conceived for students, and suggested how LMTY applies to teachers. I have suggested differing patterns related to the dominance of LMTY, how it appears for teachers in relation to other ES that are likely to be evoked in action, and supported the discussion with data from Gus, Bertha and Carol.

There are indications from the above that LMTY has a great variation for teachers. The example of Carol supports **nurturance** in a positive form, but Bertha shows it might also act to block positive engagement. Gus for example, as **affiliative** and **playful**, and for Carol, **affiliative** attitude appears in conjunction with **nurturance**, both indicative of a more socially orientated need. Therefore, the important characteristic may be a social orientation to LMTY in order to support engagement. Especially as **affiliative** is not characteristic of LMTY for Bertha, whilst LMTY with succorance may act as a barrier to student engagement. I would suggest that proposed model modifications are worth exploring further, seeking classroom interaction patterns to explore in more depth for use with mathematics teaching. In particular, exploring positive emotions as



supportive of engagement for teachers who meet their needs for **affiliative** and **playful** attitudes whilst teaching mathematics, such as Gus as described above. Such an approach may be useful within teacher training as the model is practically appealing as a lens. Examining engagement in this way addresses complexity and fluidity and adds an affective dimension to existing models of classroom interaction.

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# IMPACT OF A PROFESSIONAL DEVELOPMENT PROGRAM ON MATHEMATICAL QUALITY OF ELEMENTARY TEACHERS' INSTRUCTION

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*To investigate the effect of a professional development program on teachers' instructional practices, a study was conducted with four 4<sup>th</sup> grade teachers. This study illuminates the change in quality of teachers' instruction using the dimensions of the Mathematical Quality of Instruction (MQI) as a lens.*

## INTRODUCTION

NCTM's Principles to Action (2014) provides six guiding principles for school mathematics. The Teaching and Learning strand promotes an approach to teaching that emphasizes meaning-making and strengthens students' abilities to make sense of mathematical ideas and reason mathematically. To achieve these learning goals, high quality teaching is essential. Every proposal to reform schools emphasizes teacher professional development (PD) as an important vehicle in efforts to bring about change (e.g. Guskey, 2002; Sowder, 2007). Thus, efforts have been made to measure teaching quality, and many studies have focused on developing effective PD programs to increase the quality of mathematics teaching. As an example, Hill and her colleagues (2008) developed a tool called mathematical quality of instruction (MQI). The MQI is a research-based instrument developed by Learning Mathematics for Teaching Project and designed to assess the quality of mathematics instruction by evaluating several dimensions of instruction.

In this study we used the dimensions of the MQI instrument as a lens to investigate the effects of PD program on elementary teachers' instructional practices in the case of four 4th grade elementary mathematics teachers. Specifically, we focused on answering the following research questions: (1) what changes do teachers demonstrate in terms of their teaching practice before and after the PD? and (2) how does the PD influence teachers' quality of mathematical teaching?

## THEORETICAL CONSIDERATIONS

### Professional Development Programs for Mathematics Teachers

Education reform movements, which began in the 1980's in the United States, advocated for effective mathematics instructional practices. Calls for these changes led to state standards developed in the 1990's to facilitate instruction focused on developing students conceptual understanding and procedural fluency. As these changes specifically impacted the work of teachers, policy makers and educational reformers demanded PD as opportunities to improve classroom practices. Thus, the implications of effective PD remain particularly crucial. Prior studies (e.g. Boyle

Lamprianou, & Boyle, 2005; Hofman & Dijkstra, 2010) suggest that PD programs that provide teachers with opportunities to collaborate on teaching and examine their own instructional practices have a positive impact on teaching. Additionally, Gravani (2008) reported that school and university partnerships is an effective format of PD which enables educators and researchers to collaborate and determine appropriate ways to address teaching practices and student learning based on school context, culture and needs. As a result, the PD program in this study was designed to emphasize teacher collaboration through co-teaching and coaching. Both of which integrated an instructional feedback component that allowed teachers to analyse classroom practices. In addition, the PD was based on a school university partnership by implementing workshops that focused on research-based teaching practices.

### **Assessments to Evaluate Quality of Teaching Practice**

Prior research (e.g., James & McCormick, 2009; Ross & Bruce, 2007) demonstrates that PD programs that have a systematic assessment of teaching integrated within them have a positive impact on teaching. This is because such programs allow teachers to use empirical data to unpack teaching practices (Hiebert & Grouws, 2007). The Mathematical Quality of Instruction (MQI) is one such model that has been developed to specifically analyse mathematics teaching and learning (Hill et al, 2008). The MQI instrument that we used (2010 version) consists of five dimensions: (1) *Instructional Format* which captures the proportion of lesson time spent on mathematics in classroom; (2) *Richness of the Mathematics* which captures the depth of the mathematics provided to students; (3) *Working with Students and Mathematics* which captures whether teachers can understand students' reasoning and respond to them appropriately; (4) *Errors and Imprecision* which captures teachers' errors or imprecision in spoken language and written notation; and (5) *Student Participation in Mean-making and Reasoning* which captures evidence of students' involvement in cognitively activating classroom work.

In this study, we have adapted MQI framework as a lens to investigate the effects of our PD on teachers' quality of instructions since it has the capacity to impact teaching and learning by providing a medium to evaluate mathematics practices with key characteristics, which is outlined to be components of each of the dimensions. According to prior research, teachers' scores on some dimensions of the MQI instrument are related to their students' mathematics achievement (e.g., Hill, Kapitula, & Umland, 2011).

## **RESEARCH METHOD**

### **Participants and Context of the Study**

As part of larger study, four 4th grade elementary teachers from two school districts in the U.S (Midwestern region) were invited for this present study. The teachers have participated in 3-year long PD program and this paper focused on the data from the first year. In the first year PD, to improve teachers' mathematical knowledge and instructional practices, teachers met with the professional developers for 40 hours in

the summer and eight 8-hour sessions throughout the year. A week-long summer workshop consisted of four sections such as (1) Morning math focused on mathematical knowledge and problem solving; (2) Getting familiar with Common Core State Standards for 4<sup>th</sup> grade Mathematics; (3) Student-centered instruction based on learning trajectory; (4) Five practices for orchestrating productive mathematics discussion (Smith & Stein, 2011). Monthly workshops were focused on three things: (1) Mathematics content four teaching 4<sup>th</sup> grade students such as place value, fraction concept, fraction computation, transformations, measurement of length and area. (2) Effective instructional practices and students' thinking through video-based discussion; (3) Collaboration through coaching and co-teaching experience. Coaching sessions consist of three phase. Before coaching, each teacher met with an assigned mathematics educator to get support about his/her mathematics topic to teach. During the teaching, the mathematics educator observed the teacher's lesson. After their teaching, each teacher met with the mathematics educator again to get feedback on the teaching. Regarding co-teaching, a pair of teachers who teach different grade levels planed a lesson together and implemented the lesson for their mixed grade level students. After teaching, the pair of teachers met to debrief their teaching while watching the recorded lesson. Through all of these activities teachers were provided several opportunities to not only improve their math content knowledge, but they also were provided opportunities to improve and receive feedback regarding their teaching practice.

### **Data Collection**

In this study, we used pre-and post- teaching videos as the main data sources along with materials used in a week-long summer workshop and monthly workshops. In the spring of 2013, at the beginning of PD, all four teachers were asked to video-record their mathematics lessons. These videos were used as the pre-teaching video. In the spring of 2014, after a year-long PD, we asked teachers to plan a lesson and teach the lesson in their classroom, which happened almost the end of the PD. Teachers collaborated with a researcher to plan a math lesson and implement the lesson through the end of the PD. We used these videos as our post-teaching videos.

### **Data Analysis**

Two pairs of researchers who had completed a 16 hour-long training to use the MQI instrument, analysed each teacher's 40 minute-teaching video after choosing an 8-minute video clip that is the densest with mathematical activity. First, each researcher coded teachers' videos using the three words of "High/Mid/Low" for each aspect of four dimensions in the MQI instrument. But, for the Instructional Format dimension, we coded "Yes or No" following the directions from the MQI instrument. Then we compared our coding as a pair and reconciled our discrepancies. Finally, we assigned scores to the three levels – 1 point for low, 2 points for mid, and 3 points for high – and Yes/No coding – 1 point for Yes and 0 point for No. Only for the dimension of Errors and Imprecision, we assigned points in a reverse way because teachers' low

error indicates high quality of instruction. There were five major dimensions and each major dimension involved several sub-dimensions (see Table 1) and thus, the perfect MQI scores were 58 points. After the initial analysis, the researchers reconciled their results, and then created a detailed summary about each teacher's teaching practice. Based on the summary of each teacher, we examined overall tendency of each major dimension from four teachers' pre-and post-teaching videos. Then we identified which teacher demonstrated the most noteworthy improvement in each major dimension and extracted some examples from the teacher's pre-and post-teaching videos to show his/her change of teaching practice.

Major Dimensions	Sub-dimensions
Classroom work is connected to Mathematics	Linking and Connections
Richness of the Mathematics	Explanations
	Multiple Procedures or Solution Methods
	Developing Mathematical Generalizations
	Mathematical Language
Working with students and Mathematics	Remediation of Student Errors and Difficulties
	Responding to Students Mathematical Productions in Instruction
Errors and Imprecision	Major Mathematical Errors
	Imprecise Language or Notation (Symbols)
	Lack of Clarity in Presentation of Mathematical Content
Student Participation in Meaning-making and Reasoning	Students Provide Explanations
	Student Math Questioning and Reasoning
	Enacted Task Cognitive Activation

Table 1: Major dimensions and sub-dimensions of MQI instrument.

## RESULTS

### Result of Quantitative Analysis

In this section, we share the changes in quality of teachers' mathematical instruction around the four dimensions of MQI. In the pre-teaching videos, the average of the four teachers' MQI scores was 36.25 but in the post-teaching videos, the average increased to 47.5. Every teacher showed the increase of MQI scores from 5 to 19. Teacher A demonstrated the most significant change and Teacher B showed a small amount of change (see Table 2).

Name	Pre-MQI scores	Post-MQI scores	Change
Teacher A	27	46	+19
Teacher B	45	50	+5
Teacher C	28	38	+10
Teacher D	45	56	+11
Average	36.25	47.5	+11.25

Table 2: Major dimensions and sub-dimensions of MQI instrument.

In terms of the five major dimensions, teachers demonstrated the biggest improvement in the aspect of “richness of mathematics” and followed by “working with students and mathematics” (see Table 3). Also, teachers showed pretty small improvement in the aspect of “student participation in meaning-making and reasoning.” Moreover, teachers demonstrated the smallest change in the aspect of “errors and imprecision” because they tended to show high scores in both pre-and post-teaching videos.

Major Dimensions	Teacher A		B		C		D		Total	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Classroom work is connected to Mathematics	1	1	1	1	1	1	1	1	4	4 (+0)
Richness of the Mathematics	7	11 (+4)	12	15 (+3)	6	11 (+5)	12	18 (+6)	37	55 (+18)
Working with students and Mathematics	3	9 (+6)	8	9 (+1)	4	6 (+2)	4	9 (+5)	19	33 (+14)
Errors and Imprecision	10	12 (+2)	12	12 (+0)	10	10 (+0)	12	12 (+0)	44	46 (+2)
Student Participation in Meaning-making and Reasoning	4	8 (+4)	7	8 (+1)	5	8 (+3)	9	10 (+1)	25	34 (+9)

Table 3: PSTs’ Changes of MKT scores in each dimension.

Thus, for qualitative analysis, we focused only on three areas: richness of the mathematics; working with students and mathematics; and student participation in meaning-making and reasoning. Because of space limit, we only present a teacher’s case who showed the biggest or the second biggest change in one of the three areas of MQI scores. More specifically, in the category of students’ participation in meaning-making and reasoning, we selected Teacher A who showed the biggest change. In the category of working with students and mathematics, we selected Teacher D’s case – although Teacher A showed one point more increase than Teacher D – because Teacher A’s case was already selected to show the category of the students’ participation in meaning-making and reasoning. Similarly, in the category of richness of the mathematics, the case of Teacher C was selected although Teacher D showed the highest change in this category.

## RESULT OF QUALITATIVE ANALYSIS

### Change in Richness of the Mathematics

This dimension includes two main elements: (1) attention to the meaning of mathematical facts and procedures by linking between representations and

explanations and (2) engagement with mathematical practices by focusing on multiple strategies/solutions and by generalizing patterns. When analysed the pre-teaching video of Teacher C, there was no indication of linking and connection in teaching how to make the largest or the smallest numbers with given digits. The class simply focused on making large or small numbers with the digits they drew from a deck of number cards. Also in terms of multiple strategies, even though different examples were shared during instruction by using different numbers, Teacher C used only one strategy (e.g., putting the lowest numbers at the front and larger numbers at the back to make the smallest number with given digits). However, in the post-video, the instruction was set to connect a multiplication problem to an area model. The class focused on representing a multiplication problem (e.g., number of combinations of a drink with 7 types of fruits and 16 beverages with 10 square-free and 6 non-sugar free options) by using an area model. To do so, students were given dot papers to work on, which allowed them to develop multiple strategies in finding the product of the multiplication such as counting single dots that represents the drinks, using repeated addition (e.g.,  $16+16+\dots+16$ ), and using distributive property (e.g.,  $16 \times 7 = (10+6) \times 7 = 10 \times 7 + 6 \times 7$ ).

### **Change in Working with Students and Mathematics**

This dimension includes two sub-dimensions such as (1) remediation of student errors and difficulties and (2) responding to student mathematical productions in instruction. In the pre-video of teacher D, she noticed a student's error during class, but did not address the error. For example, when a student was asked to find the perimeter of a square with 9 inch as a side, the student wrote  $9+9=18$ ,  $18+18=36$ ,  $36 \div 4 = 9$  and  $36 \div 9 = 4$  on white board. The teacher asked the student to tell more about her division equations. However, when the student did not answer anything, the teacher moved onto the next problem without addressing why division does not work in the context of the given problem. Also, in the pre-video, teacher D tried to understand students' thinking but did not seem to clearly understand students' thinking when considering that the teacher did not revoice students' thinking in a way that was similar to the student's method. However, in the post-video in which to find area of rectangular shape was taught though covering the area with colour tiles, the teacher captured students' main conceptual error (e.g., meaning of multiplication:  $A \times B = A$  groups of  $B$ ) and gave the students a chance to compare the visualization of two different operations (e.g.,  $9+12$  and  $9 \times 12$ ). That is, the teacher helped students see their thinking through visual representation and allowed them to ponder the meaning of multiplication as repeated addition by comparing it with adding two digits. In addition, in responding to student mathematical productions during class, the teacher demonstrated clear understanding of students' thinking by correctly revoicing it and also brought up the student's idea during whole class discussion for further learning.

### **Change in Students' Participation in Meaning-Making and Reasoning**

This dimension includes three sub-dimensions such as (1) students' provision of explanation, (2) students' engagement in math questioning and reasoning and (3)

students' participation in cognitively demanding activities. In the pre-teaching video, teacher A tried to encourage students' participation in mathematical content but most of that was on a procedural level. For example, in teaching coordinate system on a plane, Teacher A asked students to show the given points on the coordinate plane but she did not ask how they found them. This prevented students from providing explanations; as a result, their engagement with the content remained at a low cognitive level, such as using procedures or known facts. Also, students did not appear to engage in any math questioning and reasoning during class. However, in the post-teaching video, Teacher A gave more voice to students and tried to engage them in meaning-making while they were working on problems. More specifically, in a division problem to distribute 256 students to 8 groups, students were encouraged to show their work and explain their models by using Base-10 blocks. However, even though students were actively explaining their solutions in the post-video, the teacher could have encouraged students to elaborate more on their reasoning by explaining why their solutions actually worked.

## **DISCUSSION**

In this study, we investigated the impact of a PD program on four 4th grade elementary mathematics teachers' teaching practices by using the dimensions of the MQI instrument. The improvement in the dimensions of "richness of mathematics" and "working with students and mathematics" were evident in the data. However, the improvement in the dimension of "student participation in meaning-making and reasoning" was minimal. These results indicate that our PD was helpful for teachers to improve their teaching practice by allowing them to explicitly connect the different domains of knowledge embedded in the MQI. In particular, regarding the dimension of richness of mathematics, morning math and video discussion sessions provided during the monthly or summer workshops seemed to be useful for teachers to have deeper understanding of mathematical concepts and to share multiple solutions, and developing mathematical generalizations. Also, in the workshop the emphasis of five teaching practice such as anticipating, monitoring, selecting, sequencing, and connecting students' solutions (Smith & Stein, 2011) seemed to help teachers respond to students' mathematical productions in instruction more actively, which is related to the aspect of "working with students and mathematics." Also, coaching sessions seemed to impact teaching because it provided a medium for teachers to collaboratively plan their lessons with a math educator and reflect on their instructional videos after the implementation of the lesson. However, PD did not seem to impact teachers practices regarding how they engaged their students in meaning-making and reasoning due to the low scores obtained on the MQI. This fact indicates that beyond teaching practices, the classroom climate needs to change in ways that encourage students to participate in meaning-making and reasoning activities. To do so effectively, requires establishing socio-mathematical norms which include mathematical discourse, unpacking mathematical problems and evaluating different solution strategies. In this regard, this study contributes to the current literature on teaching practices by



elaborating on strategies that can be used to improve the quality of PD programs. Specifically, this study sheds light on the importance of PD programs integrating strategies that can be used to promote students' participation in meaning-making and reasoning.

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# REFUTATIONAL TEXT AND MULTIPLE EXTERNAL REPRESENTATIONS CAN IMPROVE THE INTERPRETATION OF BOX PLOTS

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*In order to remediate the area misinterpretation of box plots we tested two teaching methods. First, we used refutational text to explicitly state and invalidate the area misinterpretation of box plots. Second, we used multiple external representations, using histograms as an overlay on box plots in order to give students a better insight in the way box plots represent data distributions. Additionally, we combined refutational text and multiple external representations. We found that refutational text was and that multiple representations were not successful in improving students' interpretation of box plots. The addition of multiple external representations also did not increase the effect of refutational text.*

## INTRODUCTION

The area misinterpretation of box plots occurs frequently in university students and has recently also been investigated more systematically (Lem, Onghena, Verschaffel, & Van Dooren, 2013). The goal of this study was to remediate this misinterpretation using refutational text and multiple external representations.

### The area misinterpretation

Box plots are used to present the distribution of one variable in a visual way, by dividing the data into four parts, each representing approximately one quarter of observations. In Figure 1, the minimum value is 3 and the value of Q1 is 5, meaning that about 25% of the observed data is situated between 3 and 5. The median (Q2) is 9, while Q3 is 18. Despite the interval between 9 and 18 being much wider than the interval between 3 and 5, both intervals contain approximately the same number of observations. Research has shown that this discrepancy between the size of an interval and the number of data points it represents is difficult to understand for students, and even for expert users of box plots (Lem et al., 2013, 2014).

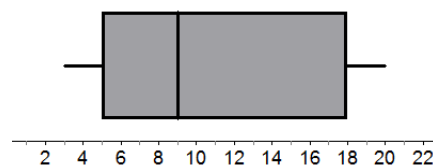


Figure 1: Example of a box plot.

## **Refutational text**

In a refutational text, a common but incorrect conception is explicitly stated, after which it is refuted and replaced by an alternative (correct) theory (Hynd, 2001). This type of text has been used frequently in the conceptual change research tradition (Sinatra & Broughton, 2011), especially for the remediation of misconceptions in science education (Tippett, 2010). Studies on the effect of refutational text have been conducted in different areas, like physics (McCrudden and Kendeou, 2012) and epistemological beliefs (Gill, Ashton, and Algina, 2004). The present study wants to test the effect of refutational text on the area misinterpretation of box plots. Lem, Kempen, Ceulemans, Onghena, Verschaffel, and Van Dooren (2015) have already successfully used refutational text in the remediation of the area misinterpretation, but we will be doing this in a more systematic way.

## **Multiple External Representations**

When using multiple external representations (MERs), a representation is presented simultaneously with a second, more familiar representation. This way, the interpretation of the first representation is “constrained” by the interpretation of the second presentation (Ainsworth, 1999). It has already been shown in various domains that MERs can indeed improve students’ learning, for instance in geometry (Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2004), chemistry (Corradi et al., 2012), and genetics (Tsui & Treagust, 2003). One underlying mechanism is that by using the simultaneous presentation of a second, more familiar, representation the interpretation of the first representation can be “constrained” by the interpretation of the second presentation (Ainsworth, 1999). Also for the use of MERs in teaching box plots, some empirical evidence exists. Bakker (2004) used interactive applets in which box plots were built on top of dot plots in order to teach secondary school students about box plots. Although there were no pre- and post-tests and no control group, he concluded that the area misinterpretation did not occur frequently in his participants, suggesting that the use of this combination of representations helped the students. Lem et al. (2013) based their intervention on the study of Bakker (2004) by also building box plots on top of dot plots. This intervention of Bakker seemed to be successful in most participants, although some participants’ reasoning did not improve at all. Moreover, reaction time patterns revealed that even those who did improve their interpretation were still affected by the area misinterpretation. Also Lem et al. (2015) and Pierce, Chick, and Wander (2015) have successfully used this MERs approach to teach box plots to, respectively, students in an introductory statistics course and teachers who had to learn to interpret school assessment data.

## **METHOD**

### **Participants**

Participants were 199 undergraduate students enrolled in the introductory statistics course taught by the third author. The average age was 18.14 years. Participation was

voluntary, but it was stressed that all material taught during the intervention was part of the regular curriculum.

### **Pre- and post-test**

The same test was used before and after the intervention, with only differences in the contexts and numbers in the tasks. Both tests consisted of two parts: four box plot fact knowledge questions, and eight items in which we specifically aimed at revealing the area misinterpretation. In five of the area items (box plot only items) one or more box plots had to be interpreted and in the other three area items (matching items) participants had to match a histogram to a box plot (or vice versa).

### **Intervention**

The intervention was done during the practical sessions of an introductory statistics course, about one month after the pre-test. Before the intervention, students were already taught about descriptive statistics and histograms, but not yet about box plots. A 2x2 design was used, varying whether refutational text and/or MERs were used in the intervention or not. This led to the following four conditions: control condition ( $n=46$ ), refutational text condition ( $n=58$ ), MERs condition ( $n=49$ ), and the combination condition ( $n=46$ ).

The intervention consisted of a short lecture of about 20 minutes by the second author. The three experimental lectures were based on the lecture in the *control condition*, in which histograms and box plots were explained, without making a connection between both representations and without explicitly saying anything about possible misinterpretations. In the *refutation* condition, this connection between histograms and box plots was not made, but the area misinterpretation was explicitly stated and refuted, and the correct interpretation was subsequently offered. This is an example of a refutational text as used in the refutation condition:

When you look at a box plot you might think that a larger part of the box represents more results than a smaller part of the box. This is incorrect! In each of the four parts of a box plot approximately the same number of observations is represented. The size of the parts of the box does hence not give an indication of the number of represented observations. A larger part of the box means that the observations here are more spread out. In a smaller part of the box plot the observations are hence less spread out. In each of the four parts of a box plot approximately 25% of the observations is represented.

In the *MERs* condition, histograms were used as a basis to explain box plots. This was done by adding lines to a histogram, in such a way that a box plot was formed (see Figure 2 for some example images). In the *combination condition*, finally, MERs and refutational text were combined. This means that box plots were explained by using histograms as a basis, but also that the refutational text used histograms to show how box plots should be interpreted. Directly after the intervention, the post-test was administered.

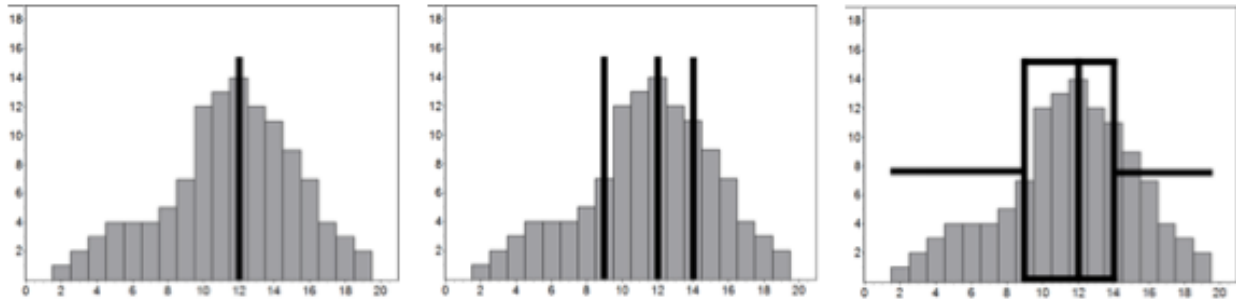


Figure 2: Example of how a box plot was built on top of a histogram in the MERs and the combination condition.

## RESULTS

### Box plot fact knowledge

Using a generalized linear mixed model (see Table 1) with test moment, refutational text, MERs, and their interactions as independent variables, and accuracy as the dependent variable, we found that accuracy was higher in the post-test (73.49%) than in the pre-test (39.45%), and that condition had no effect on accuracy. This suggests that box plot knowledge was the same in all conditions, not only before the intervention, but also afterwards, implying that all conditions were equally successful in improving participants' box plot fact knowledge.

	Num DF	Den DF	<i>F</i>	<i>p</i>
Test moment	1	1389	178.51	< .001*
Ref	1	1389	0.18	.667
MERs	1	1389	0.35	.553
Test moment * Ref	1	1389	0.76	.383
Test moment * MERs	1	1389	0.02	.880
Ref * MERs	1	1389	0.88	.347
Test moment * Ref * MERs	1	1389	0.78	.379

Table 1: Results of the generalized linear mixed analysis with accuracy on the box plot fact knowledge items as dependent variable. Ref = refutational text,

MERs = multiple external representations,

\* indicates effects that are significant at the .05-level.

### Accuracy area misinterpretation items

The accuracy rates for the area misinterpretation items are presented in Table 2. A generalized linear mixed model with test moment, refutational text, MERs, and their interactions as independent variables, and accuracy on the area misinterpretation items

as the dependent variable was fitted. The results of this analysis are presented in Table 3. First, we found a main effect of test moment. This effect shows that accuracy was higher in the post-test (71.19%) than in the pre-test (45.23%). Second, we found a main effect of refutational text, showing that participants exposed to refutational text solved more items correctly (75.38%) than participants who were not exposed to refutational text (52.95%). Finally, we found an interaction effect of test moment and refutational text. This effect shows that participants exposed to refutational text showed a larger improvement (from 47.48% to 77.16%,  $OR = 3.73$ ) than participants who were not exposed to refutational text (from 42.67% to 64.64 %,  $OR = 2.55$ ).

We did not find a significant effect of the exposure to MERs, and, moreover, the lack of a significant interaction effect of refutational text and MERs shows that the addition of MERs to refutational text did not significantly add to the positive effect of refutational text.

	Pre-test			Post-test		
	No Ref	Ref		No Ref	Ref	
No MERs	40.76	46.12	43.75	65.67	76.94	71.96
MERs	44.64	49.18	46.84	63.68	77.45	70.36
	42.76	47.48	45.23	64.64	77.16	71.19

Table 2: Accuracy rates (in %) for area misinterpretation items in the pre- and post-test, per condition.

	Num DF	Den DF	$F$	$p$
Test moment	1	2979	228.46	< .001*
Ref	1	2979	10.40	< .001*
MERs	1	2979	0.27	.600
Test moment * Ref	1	2979	7.98	.005*
Test moment * MERs	1	2979	1.08	.300
Ref * MERs	1	2979	0.02	.893
Test moment * Ref * MERs	1	2979	0.26	.609

Table 3: Results of the generalized linear mixed analysis with accuracy on the area misinterpretation items as dependent variable.

\* indicates effects that are significant at the .05-level.

We also analysed whether there was a different effect of the interventions for the two item types: box plot only items and matching items. An overview of the percentages of correct responses per item type is provided in Table 5. For the box plot only items,

we found a main effect of refutational text,  $F(1,795) = 26.65$ ,  $p < .001$ ,  $OR = 2.74$ , but not of MERs,  $F(1,795) = 0.02$ ,  $p = .875$ , and also no interaction effect between both teaching techniques,  $F(1,795) = 0.87$ ,  $p = .352$ . For the matching items, on the other hand, accuracy rates were much more similar between conditions, resulting in no main effect of refutational text,  $F(1,397) = 1.71$ ,  $p = .191$ , nor of MERs,  $F(1,397) = 0.33$ ,  $p = .567$ , and also no interaction effect between both,  $F(1,397) = 1.26$ ,  $p = .263$ . These results suggest that while refutational text had a positive effect on the solving of box plot only items, none of the tested teaching techniques had an effect on the solving of the matching items.

## DISCUSSION AND CONCLUSION

Our results show that the use of refutational text improves the accuracy on items on the area misinterpretation significantly more than the use of MERs. We did not find a similar effect of MERs, and as we also did not find an added effect of the use of MERs next to refutational text, we could conclude that the use of MERs did not improve participants' interpretation of box plots. When looking at the two different item types, box plot only items and matching items, all conditions scored equally well on the matching items, in which both histograms and box plots had to be interpreted simultaneously, while refutational text clearly led to higher accuracy in the items in which only box plots had to be interpreted. This is slightly different from the results of Lem et al. (2015), where matching tasks tended to be solved better when MERs were used as an intervention. A possible interpretation of this difference in results is that MERs do not help students in *learning* about box plots as such, but that they are more effective in constraining the incorrect interpretation of box plots during *problem solving*.

Although these results are promising, we must take some limitations into account. First, the design of our study did not allow collecting extensive qualitative data: By only using multiple-choice questions, it is difficult to get a full understanding of the way participants interpret box plots. Second, because of curricular restrictions, we could not perform a *delayed* post-test. Because of this, we do not know whether the effects of the intervention lasted after the post-test. It is hence possible that the interventions only led to short-term effects or that for example the addition of MERs to refutational text led to longer lasting effects than refutational text alone.

The results of this study have several implications for theory. First, we have united two teaching techniques from different fields, namely the conceptual change literature and the instructional design literature. Both have fundamentally different points of departure. While refutational text specifically focuses on what students interpret incorrectly, MERs depart from a representation that is already interpreted correctly by students to teach another representation. This difference makes that combining them could make a strong combination of teaching techniques. Although we could not find that this combination was more successful than the use of refutational text alone, Lem et al. (2015) did find a trend in this direction, warranting further research. A possible

explanation for the absence of this effect in the current study is that the instruction time might have been too short to fully process the combination of refutational text and MERs. In the study of Lem et al. the processing time was much longer, given that the intervention was presented as a home study text and participants had one week to process the text. Second, our results suggest that, in the interpretation of box plots, MERs were more successful in improving students' *problem solving* than in improving their *learning*. More specifically, while participants who were exposed to refutational text had a clear advantage in solving items in which only box plots had to be interpreted, all participants scored equally well when box plots and histograms had to be matched to each other. This suggests that the constraining function of MERs (Ainsworth, 1999) had its effect for all conditions during the solving of the matching tasks, while this function was less successful in teaching participants about box plots.

The results of this study are also important for educational practice. They show that refutational text helps students in understanding box plots and overcoming their misinterpretation. As this teaching technique is relatively easy to implement and does not require much extra teaching time, it can be recommended for teachers to implement it for the teaching of box plots, but also to use refutational text in the teaching of other topics.

Further research could focus on improving the interventions we used and testing their effect, both for the interpretation of box plots and for other topics in which misconceptions occur. Furthermore, it would be interesting to collect and study process data in order to get a deeper insight in the way students interpret box plots after having been exposed to refutational text and/or MERs. Finally, it could be studied whether refutational text, MERs, and their combination can benefit learning and problem solving in other content fields.

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# ASSESSING MATHEMATICAL CREATIVITY OF PRE-SERVICE TAIWANESE AND U.S. TEACHERS

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*This paper presents the preliminary findings of a small-scale international comparative study on assessing pre-service teachers' mathematical creativity. Participants include 38 pre-service elementary teachers from Taiwan and 26 of their counterparts from the U.S. Their performance on three cognitive dimensions of divergent production (fluency, flexibility, originality), as well as the overall mathematical creativity is described and compared. No significant difference of performance between Taiwan and U.S. groups were captured for three cognitive dimensions and mathematical creativity for all but one test item.*

## INTRODUCTION

Mathematical creativity has long been an interest of many researchers in the context of school mathematics because of its connections to mathematical ability, mathematical achievement, problem solving and problem posing (Haylock, 1997; Silver, 1997). It is important that the pre-service teachers should be provided with such tasks to develop their own mathematical creativity (Vale, Primentel, Cabrita, Barbosa, & Fonseca, 2012). In Taiwan, despite of high mathematics achievement across all school levels, the ministry of Education discourages an over emphasis on rapid response and convergent reasoning that potentially hurt the development of creativity and innovation (Ministry of Education, 2003). In the U.S., the words “creative” and “creativity” were documented in the Common Core State Standards of Mathematics (Common Core State Standards Initiatives, 2010) as needed for mathematical modelling, part of the high school mathematics standards.

Despite of this consensus, however, only a handful of studies have examined mathematical creativity in the context of teacher education. Leung and Silver (1997) examined the relationship among U.S. pre-service elementary teachers' (PSTs) ability to pose arithmetic problems, their mathematics knowledge, and their mathematical creativity as measured by the verbal subtest of the Torrance Test of Creative Thinking (TTCT-V). They found that pre-service elementary teachers in their study were able to pose plausible arithmetic problems. Recent study by Bolden, Harries, and Newton (2009) found that PSTs in the U.K. hold a narrow view about mathematical creativity and had difficulty developing activities and assessments that can be used to support students' development of mathematical creativity in their future classrooms. To our knowledge, there is no study that compares the performance of mathematical creativity between pre-service teachers from different countries. Research studies comparing achievement for typical school mathematics topics indicate that Taiwanese PSTs

outperformed their U.S. counterpart (Center for Research in Mathematics and Science Education, 2010; Authors, 2011). However, Cai and Hwang (2002) found that while the Chinese sixth grade students outperformed U.S. students on all the problem solving tasks, their performance was identical to their U.S. counterparts on the problem posing tasks which is an indicator for mathematical creativity that is less emphasized in mathematics curricula than problem solving. Will the same pattern exist in performance on a mathematical creativity assessment between Taiwanese and U.S. PSTs? In this paper, we report the results of an international comparison study that addresses this gap.

## **THEORETICAL FRAMEWORK**

When surveying psychological literature, Silver (1997) identified two distinct views of creativity. The first one, the genius view of creativity, treats creativity as a rare mental trait and not likely to be influenced by instruction. The second view suggests that creativity is closely related to “deep, flexible knowledge in content domains” (p.75) and is subject to instructional influence. This latter view encourages the development and implementation of creativity-enriched instructional activities that may benefit a broader range of students. Mathematics education researchers commonly used three key dimensions of creativity: fluency, flexibility and originality (referred as novelty by some researchers, e.g. Silver, 1997 and Leikin & Lev, 2007) to assess mathematical creativity. Roy (2011) explained fluency as the “number of correct responses,” flexibility as “the number of categories of the responses,” and originality as “the statistical infrequency of the responses” (p.72).

Haylock (1997) identifies three different types of tasks that have been used to assess this construct: problem solving, problem posing and redefinition. Redefinition requires students to “re-define the elements of a situation in terms of their mathematical attributes” (p.72). An example of such a task given by Haylock is to find all the things that are the same for the numbers 16 and 36.

Previous studies on mathematical creativity found that the nature of the task (conventional vs. non-conventional) had an impact on student performance. Leikin and Lev (2007) found that gifted students performed similarly to their non-gifted counterparts on a conventional task but outperformed them on a non-conventional task. Van Harpen & Sriraman (2013) also noted that the geometrical task used in their study was less likely to create cultural biases.

## **METHODOLOGY**

Situated in the theoretical framework outlined above, this international comparative study seeks to answer the following research question:

What are the similarities and differences between groups of Taiwanese and U.S. pre-service teachers' performance on mathematical creativity assessment in terms of fluency, flexibility, originality, and overall creativity?

Participants were PSTs from two elementary teacher preparation programs. Based on objective information such as university ranks and college standardized entrance test scores, those participants' content knowledge were in the top 25% among the entire PSTs' population in their own nation. In Taiwan, the elementary teacher preparation program is designed for fostering future first to sixth grade teachers, while the U.S. program was for future kindergarten to eighth grade teachers. There are 38 Taiwanese PSTs enrolled in a college mathematics course for future teachers took the test of mathematical creativity as part of their in-class activity. And there are 26 U.S. participants took the same assessment in the required math methods course. Most of the U.S. participants were juniors or seniors, while their Taiwanese counterparts were mostly sophomores.

## Instrument

Based on the theoretical framework and literature reviews, a five-item mathematical creativity instrument was developed and implemented. The duration of the assessment is 50 minutes. Item #1 and #5 were problem-posing tasks. In Item #1, PSTs were asked to pose multiple word problems by using only one operation: division. In Item #5, PSTs were asked to pose question about the diagram in Figure 1a). For Item #2 and #4, they were problem solving tasks. Item #2 was adapted from Vale, et al. (2012) that asked PSTs to develop multiple ways to find the total number of apples in the Figure 1b). In Item #4, PSTs were asked to draw different quadrilaterals. Item #3 was the re-definition task developed by Haylock (1997) that asked PSTs to identify common attributes between 16 and 36.

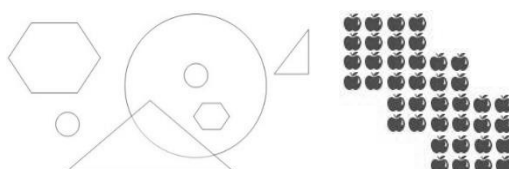


Figure 1a

Figure 1b

The validity of the instrument was established using the six criteria suggested by Haylock (1997) for an effective task in revealing and differentiating levels of creativity. That is, all these five items allow the participants to: (1) show a range of different responses, (2) provide the possible number of such responses larger than 20, (3) have consistent interpretations; (4) contain several obvious solutions, (5) afford unique solutions, and (6) demonstrate mathematical importance and creativity in those unique solutions.

## DATA ANALYSIS

The calculation of the mathematical creativity score was based on the assumption that all three cognitive dimensions made equal contribution. This score is established through a multi-step process described below.

*Calculation of the raw fluency score for each item.* Each participant response, if judged to be valid, receives 1 point. The total number of valid responses is the raw score for that item.

*Calculation of the raw flexibility score for each item.* First, identify the number of different categories of response each participant generate (flex). Second, the raw flexibility score for each item is the fraction of flex/Maxflex, where Maxflex is the total number of different categories of response generated by all participants.

*Calculation of the raw originality score for each item.* First, for each category, calculate the percentage of all participants who gave that variety. These percentages are  $a_1, a_2, \dots, a_{\max flex}$ . Second, for each participant who has given  $n$  categories, he or she has a corresponding set of percentage  $b_1, b_2, \dots, b_n$ , which is a subset of the  $\{ a_1, a_2, \dots, a_{\max flex} \}$ . The raw originality score for this participant on this item is calculated with the following formula:

$$\frac{(1-b_1)+(1-b_2)+\dots+(1-b_n)}{(1-a_1)+(1-a_2)+\dots+(1-a_{\max flex})}$$

*Calculation of the standardized T scores of fluency, flexibility, originality, and overall mathematical creativity for each participant.* First, let A/B/C be the participant's raw fluency/flexibility/originality score. Use mean= 50 and standard deviation 10 to obtain a T-score to get the standardized individual scores. Second, each participant's mathematical creativity score for each item is the mean score of those three individual scores. Third, each participant's final fluency, flexibility and originality scores are the means of their respective T-scores on all items. Lastly, each participant's overall mathematical creativity score is the mean of his or her final fluency, flexibility and originality scores.

*Descriptive statistics and independent-samples t test.* Descriptive statistics including mean values and standard deviations were applied to summarize Taiwanese and U.S. PSTs' performance of T scores on individual items and all test items by nationality regarding the dimensions of fluency, flexibility, originality and mathematical creativity. Independent-samples t test was used to compare whether difference of T scores exists between Taiwanese and U.S. PSTs' performances on individual item and all test items regarding to those dimensions.

## RESULTS

The analysis of individual items shows that no significant difference exists between Taiwanese and U.S. PSTs on T scores for any dimension at a significant level of .05 ( $\alpha = .05$ ) on items #1, #3, #4 and #5 (See Tables below)

Dimension	Nation	N	Mean	Std. Deviation	t-test for Equality of Means		
					t	df	Sig. (2-tailed)
Fluency	Taiwan	38	46.46	10.74	-1.57	62	.12
	U.S.	26	50.52	9.24			
Flexibility	Taiwan	38	42.78	6.21	.93	62	.36
	U.S.	26	41.42	4.93			
Originality	Taiwan	38	42.71	5.73	1.38	61.88	.17
	U.S.	26	41.02	4.07			
Creativity	Taiwan	38	43.98	5.53	-.25	62	.81
	U.S.	26	44.32	5.03			

Table 1: Results for Performance of T Scores on Item #1

Dimension	Nation	N	Mean	Std. Deviation	t-test for Equality of Means		
					t	df	Sig. (2-tailed)
Fluency	Taiwan	38	47.81	9.64	-1.10	62	.28
	U.S.	26	50.52	9.70			
Flexibility	Taiwan	38	50.32	7.76	-1.18	62	.24
	U.S.	26	52.66	7.81			
Originality	Taiwan	38	53.08	8.43	-1.26	45.13	.21
	U.S.	26	56.24	10.73			
Creativity	Taiwan	38	50.40	7.80	-1.32	62	.19
	U.S.	26	53.14	8.59			

Table 2: Results for Performance of T Scores on Item #3

Dimension	Nation	N	Mean	Std. Deviation	t-test for Equality of Means		
					t	df	Sig. (2-tailed)
Fluency	Taiwan	38	55.11	4.91	.35	62	.73
	U.S.	26	54.63	5.96			
Flexibility	Taiwan	38	62.94	7.23	1.50	62	.14
	U.S.	26	60.16	6.74			
Originality	Taiwan	38	61.53	8.92	1.16	62	.25
	U.S.	26	58.94	8.49			
Creativity	Taiwan	38	59.83	5.98	1.27	62	.21
	U.S.	26	57.91	5.90			

Table 3: Results for Performance of T Scores on Item #4

Dimension	Nation	N	Mean	Std. Deviation	t-test for Equality of Means		
					t	df	Sig.(2-tailed)
Fluency	Taiwan	38	45.73	13.16	-.30	62	.77
	U.S.	26	46.71	12.74			
Flexibility	Taiwan	38	46.85	10.03	-.80	62	.43
	U.S.	26	48.92	10.26			
Originality	Taiwan	38	48.45	6.84	-.98	62	.33
	U.S.	26	50.19	7.18			
Creativity	Taiwan	38	47.01	9.61	-.65	62	.52
	U.S.	26	48.60	9.63			

Table 4: Results for Performance of T Scores on Item #5

For Item #2 (finding the number of apples), Table 5 below reveals that a significant difference exists between Taiwanese and U.S. PSTs on T scores for the dimensions of flexibility and mathematical creativity, but not for the dimensions of fluency and originality at a significant level of .05 ( $\alpha = .05$ ).

Dimension	Nation	N	Mean	Std. Deviation	t-test for Equality of Means		
					t	df	Sig.(2-tailed)
Fluency	Taiwan	38	53.34	6.72	1.62	42.37	.11
	U.S.	26	49.91	9.30			
Flexibility	Taiwan	38	48.96	6.51	2.98	62	<.01
	U.S.	26	44.29	5.62			
Originality	Taiwan	38	44.90	5.53	1.62	62	.11
	U.S.	26	42.63	5.49			
Creativity	Taiwan	38	49.07	4.82	2.73	62	<.01
	U.S.	26	45.61	5.18			

Table 5: Results for Performance of T Scores on Item #2

For the performance of all test items, Table 6 below reveals that no significant difference exists between Taiwanese and U.S. PSTs on T scores for any dimension at a significant level of .05 ( $\alpha = .05$ ).

Dimension	Nation	N	Mean	Std. Deviation	t-test for Equality of Means		
					t	df	Sig.(2-tailed)
Fluency	Taiwan	38	49.69	6.99	-.47	62	.64
	U.S.	26	50.45	5.44			
Flexibility	Taiwan	38	50.35	43.5	.81	62	.42
	U.S.	26	49.49	3.87			
Originality	Taiwan	38	50.13	4.18	.31	62	.76
	U.S.	26	49.80	4.17			
Creativity	Taiwan	38	50.06	4.45	.13	62	.90
	U.S.	26	49.92	4.10			

Table 6: Results for Performance of T Scores on All Test Items

## DICUSSION

This paper describes findings of performance between Taiwanese and U.S. PSTs on mathematical creativity assessment. The results of this study add to the literatures of PSTs' mathematical creativity. The tasks used in this study can serve as a springboard for potential ideas that can be used for students (and teachers alike) at all levels to develop their mathematical creativity.

The results indicate that there is no significant difference for the performance of all test items on any dimension of creativity for all but one item. In addition, the results show that positive and negative t values are mixed in different dimensions. Thus, regardless whether significant statistics difference exists, Taiwanese PSTs are not always higher or lower than their U.S counterparts numerically in T scores. This finding does not support the findings of prior international comparison studies such as TEDS-M that identify Taiwanese PSTs outperformed their U.S. counterparts in the mathematics content knowledge (Center for Research in Mathematics and Science Education, 2011). In sum, in terms of mathematical creativity, no significant gap between Taiwan and U.S. was captured like the gap between them in the results of international comparative studies of mathematics achievement at all levels.

Further, except performance for Item #2 (finding the number of apples) on the dimensions of flexibility and mathematical creativity, the results suggest that no significant difference exists for each individual item on any dimension. For Item #2, Taiwanese PSTs outperformed their U.S. counterparts in the dimensions of flexibility and mathematical creativity. The difference of performance on mathematical creativity was contributed from the difference of performance on flexibility rather than fluency and originality. The degree of flexibility is determined by the number of response categories. This finding suggests that Taiwanese PSTs used more response categories than their U.S. counterparts for this item. Item #2 contains a stronger nature of problem solving than the other items. Smith (2004) identified a similar disparity in the flexibility on problem solving. In a comparative study of U.S. and Japanese lessons, she found



that the Japanese teacher presented more than one solution method while her U.S. counterpart only presented one solution method.

The findings of this study are limited because of the small size of the participants. Nevertheless, the results provide insights into the assessment of mathematical creativity. Those insights can be used to design a large-scale study, and to generate responses to the enduring challenge of including mathematical creativity in school mathematics. They can be used to serve as basis for improving the instrument to be used to test samples of different populations such as gifted and non-gifted school students in different countries.

### Acknowledgement

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# FLOW: A FRAMEWORK FOR DISCUSSING TEACHING

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*In this paper I explore Mihály Csíkszentmihályi's notion of flow as a lens to analyse the teaching practices of two very different teachers. Results indicate that flow is not only a good theoretical framework for drawing attention to the differences in teaching style, but also for describing these differences in ways that is grounded in what we know about good learning. The possibility of shifting flow from a descriptive framework to a prescriptive one is also explored.*

## FLOW AND THE OPTIMAL EXPERIENCE

In the early 1970's Mihály Csíkszentmihályi became interested in studying, what he referred to as, the *optimal experience* (1998, 1996, 1990),

“a state in which people are so involved in an activity that nothing else seems to matter; the experience is so enjoyable that people will continue to do it even at great cost, for the sheer sake of doing it.” (Csíkszentmihályi, 1990, p. 4.)

The optimal experience is something we are all familiar with. It is that moment where we are so focused and so absorbed in an activity that we lose all track of time, we are un-distractable, and we are consumed by the enjoyment of the activity. As educators we have glimpses of this in our teaching and value it when we see it.

Csíkszentmihályi, in his pursuit to understand the optimal experience, studied this phenomenon across a wide and diverse set of contexts (1998, 1996, 1990). In particular, he looked at the phenomenon among musicians, artists, mathematicians, scientists, and athletes. Out of this research emerged a set of elements common to every such experience (Csíkszentmihályi, 1990):

1. There are clear goals every step of the way.
2. There is immediate feedback to one's actions.
3. There is a balance between challenges and skills.
4. Action and awareness are merged.
5. Distractions are excluded from consciousness.
6. There is no worry of failure.
7. Self-consciousness disappears.
8. The sense of time becomes distorted.
9. The activity becomes an end in itself.

The last six elements on this list are characteristics of the internal experience of the doer. That is, in describing an optimal experience a doer would claim that their sense of time had become distorted, that they were not easily distracted, and that they were

not worried about failure. They would also describe a state in which their awareness of their actions faded from their attention and, as such, they were not self-conscious about what they were doing. Finally, they would say that the value in the process was in the doing – that the activity becomes an end in itself.

In contrast, the first three elements on this list can be seen as characteristics external to the doer, existing in the environment of the activity, and crucial to occasioning of the optimal experience. The doer must be in an environment wherein there are clear goals, immediate feedback, and there is a balance between the challenge of the activity and the abilities of the doer.

This balance between challenge and ability is central to Csíkszentmihályi's (1998, 1996, 1990) analysis of the optimal experience and comes into sharp focus when we consider the consequences of having an imbalance in this system. For example, if the challenge of the activity far exceeds a person's ability they are likely to experience a feeling of anxiety or frustration. Conversely, if their ability far exceeds the challenge offered by the activity they are apt to become bored. When there is a balance in this system a state of, what Csíkszentmihályi refers to as, *flow* is created (see fig. 1). Flow is, in brief, the term Csíkszentmihályi used to encapsulate the essence of optimal experience and the nine aforementioned elements into a single emotional-cognitive construct.

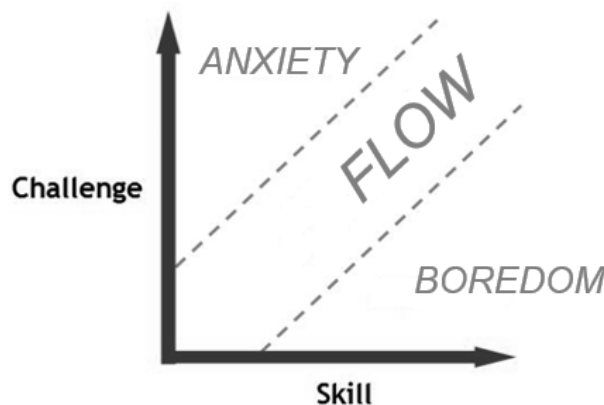


Figure 1: Graphical representation of the balance between challenge and skill

## FLOW IN MATHEMATICS EDUCATION

Flow is one of the only ways for us, as mathematics education researchers, to talk productively about the phenomenon of engagement. The nine aforementioned elements of flow gives us not only a vocabulary for talking about aspects of the subjective personal experience of engagement, but it also gives us a way to think about the potential environments that occasion engagement in our classrooms.

Williams (2001) used Csíkszentmihályi's idea of flow and applied it to a specific instance of problem solving that she refers to as discovered complexity. Discovered complexity is a state that occurs when a problem solver, or a group of problem solvers, encounter complexities that were not evident at the onset of the task and are within

their zone of proximal development (Vygotsky, 1978). This occurs when the solver(s) "spontaneously formulate a question (intellectual challenge) that is resolved as they work with unfamiliar mathematical ideas" (p. 378). Such an encounter will capture, and hold, the engagement of the problem solver(s) in a way that satisfies the conditions of flow. What Williams' framework describes is the deep engagement that is sometimes observed in students working on a problem solving task during a single problem solving session.

Extending this work, I used the notion of flow to look at situations of engagement extended over several days or weeks wherein students return to the same task, again and again, until a problem is solved (Liljedahl, 2006). The results of this work showed that although flow was present in each of the discrete problem solving encounters, what allowed the engagement to sustain itself across multiple encounters was a series of minor discoveries in each session linking together to form what I referred to as a *chain of discovery*.

### FLOW AS A FRAMEWORK FOR DESCRIBING TEACHING

Thinking about flow as existing in that balance between skill and challenge, as represented in figure 1, obfuscates the fact that this is not a static relationship. Flow is not the range of fixed ability-challenge pairings wherein the difference between skill and challenge are within some tolerance. Flow is, in fact, a dynamic process. As a doer engages in an activity their skills will, invariably, improve. In order for the doer to stay in flow the challenge of the task must similarly increase (see fig. 2).

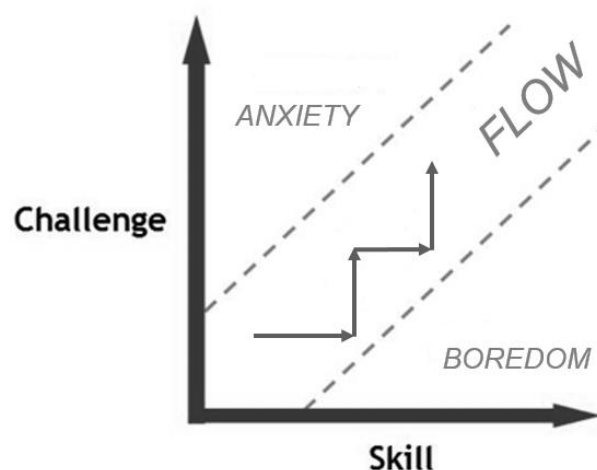


Figure 2: Graphical representation of the balance between challenge and skill as a dynamic process

If the doer is then a student in a learning setting, such as a mathematics classroom, it is up to the teacher to manage the increases in challenge as the student's skill increases. There is a chance then, however, that the student's skill will increase either too quickly or too covertly for the teacher to notice resulting in a student previously in flow slipping into a state of boredom (see fig. 3). Likewise, there is also chance that when the teacher

does increase the challenge that increase will be too great for the student to stay in flow, causing them to enter into a state of anxiety (see fig. 4).

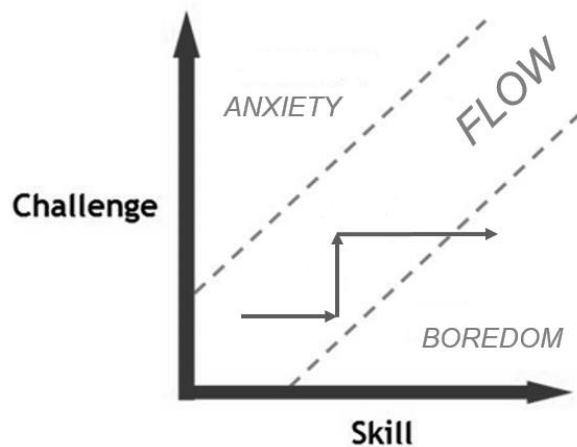


Figure 3: Too fast an increase in skill

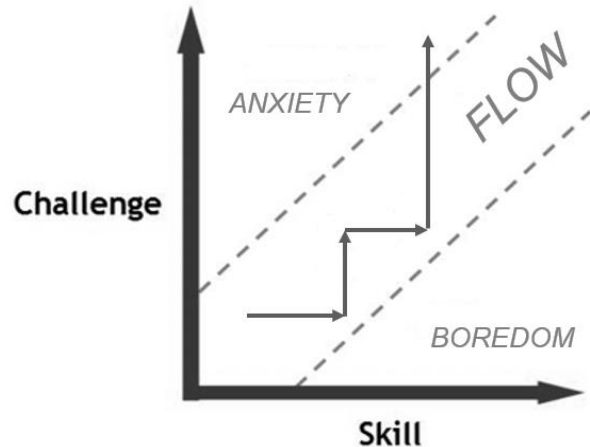


Figure 4: Too great an increase in challenge

In this study, I look at the practices of two teachers through the lens of flow in general and their ability to set clear goals, provide instant feedback, and maintain a balance between challenge and skill in particular.

## METHODOLOGY

The participants in this study were selected through purposive sampling. I was looking to compare two very different teaching styles and thus asked a number of school administrators to recommend to me high school mathematics teachers they believed to be effective mathematics teachers and they believed to be confident in their practice. Through this process a pool of 15 teachers were recommended to me. Of these 15 teachers, six agreed to have one of their 'typical' mathematics lessons filmed. Each of these videos was produced using a camera which followed the teacher during the lesson.

For the research presented here I have selected for analysis two of these videos – one of a teacher named Claire<sup>1</sup> and one of a teacher named Connor. Claire has been teaching high school mathematics for 15 years. She teaches primarily the senior (grades 11 and 12) academic courses. Connor, on the other hand, has been teaching for only 8 years, teaches all levels of high school mathematics – both academic and non-academic – and also teaches some junior (grades 8 -10) science.

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<sup>1</sup> These are pseudonyms.

## RESULTS

In what follows I give a brief synopsis of Claire's and Connor's lesson marked with a time stamp as to when key moments of each lesson occurs. For brevity purposes these synopses focus only on the actions of each of the teachers.

### Claire's Lesson (grade 11)

- 00:00 Claire begins with a brief review of the previous lesson.
- 06:30 Claire delivers a 'lesson' on calculating the angle  $\theta$  ( $0^\circ < \theta \leq 360^\circ$ ) given a trigonometric ratio  $r$ . This lesson involves her giving several examples of how to solve such tasks.
- 22:00 Claire asks the class to solve for  $\theta$ :  $\sin \theta = 0.8$ ,  $\cos \theta = 0.32$ , and  $\tan \theta = 1.2$ . During this activity Claire circulates and checks on how students are doing. When a student puts up their hand she quickly moves to them and answers their question. The first two questions asked by students pertained to the fact that the ratio for the third question ( $\tan \theta = 1.2$ ) is greater than 1.
- 26:15 Claire stops the activity to re-explain the limitations on the ratios for each trigonometric relationship.
- 31:00 Claire calls the class to attention and goes over the solutions to each of the three questions.
- 36:30 Claire gives the next question for the students to solve (solve for  $\theta$ :  $3\sin \theta + 1 = 2.8$ ;  $0^\circ < \theta \leq 360^\circ$ ). Almost immediately many students put up their hands. Claire helps two students to understand the task and begin to solve it.
- 40:00 Claire calls the class to attention and reviews how to solve the equation  $3x + 1 = 2.8$ .
- 42:30 Claire refocuses the students on the original task:  $3\sin \theta + 1 = 2.8$ .
- 50:00 Claire calls the class to attention and goes over the solutions to the question.
- 55:30 Claire assigns homework.

### Connor's Lesson (grade 11)

- 00:00 Connor reviews how to multiply two first degree binomials on the board:  $(x + 2)(x + 3) = x^2 + 5x + 6$ . He then asks the question, "what do the binomials have to be if the answer is  $x^2 + 7x + 6$ ?"
- 01:30 Connor places the students into random groups and asks them to work on vertical whiteboards to find the answer. He then begins to circulate amongst the groups as they begin to work on the task.
- 05:00 Connor stops to speak with a group who is having trouble understanding the task. He re-writes the example as follows:

$$\begin{aligned}(x + 2)(x + 3) &= x^2 + 5x + 6 \\ (?) (?) &= x^2 + 7x + 6\end{aligned}$$

He then points to the question marks and asks, "what has to go in here so that the product of the two binomials is this (pointing at the quadratic expression)? I'll give you a hint – look at the last number."

- 06:00 Connor repeats the above process with another group.

07:00	Connor asks a group who has an answer to check their solution by multiplying the binomials. Once the solution is confirmed he gives the group a new task: $x^2 + 6x + 8$ .
07:30	Connor helps another group in the same fashion as above.
08:30	Connor gives a new task to two groups asking them first to check their answer.
09:00	For the next 32 minutes Connor moves around the room giving new tasks to groups that have finished and checked their solution, and helping groups that are stuck. Sometimes he works with two or three groups at the same time. Eventually Connor projects a list of 20 progressively challenging tasks onto a wall. These range from the initial task of $x^2 + 7x + 6$ to tasks as complex as $6x^2 + 10x - 4$ . The groups start to move through these tasks one by one solving each and checking their answers.
41:00	Connor gathers the students around one whiteboard and asks them to walk him through how to solve the question $x^2 + 5x - 24$ . Connor forces the students to articulate their thinking at each step.
47:00	Connor suggests that the students sit down and write down some notes for themselves on how to solve tasks of the type seen during class.
52:00	Connor projects five more tasks on the wall and asks the students to solve them on their own.

## ANALYSIS

As mentioned, these videos were analysed using the framework of flow (Csíkszentmihályi, 1998, 1996, 1990). In particular, I looked for instances of the teacher providing clear goals, providing feedback, and maintaining balance between challenge and skills in each of the videos. In what follows I discuss the results of this analysis organized according to the aforementioned aspects of flow.

### Providing Clear Goals

Claire's approach to providing clear goals for her activity involves demonstrating how to solve several similar tasks. Still, she is faced with students that are not clear about one of the tasks. Claire chose to deal with this by addressing the class as a whole about the confusion. When Claire poses the next task ( $3\sin \theta + 1 = 2.8$ ) she is again faced with students who are not clear what the goal of the activity is. She again pulls the whole class together to further explain the task *and* she offers an analogue for how to solve it.

Connor, on the other hand, begins his lesson by reviewing how to multiply binomials and then asks the students to find the two binomials if given the resultant quadratic. Regardless of his differing start to the lesson, Connor is still faced with groups who do not understand the goal of the activity. He chooses to deal with these situations one by one. In all, he needs to work with four groups (two of them at the same time) who are struggling to understand the goal of the activity. After this initial push to help groups get started he has no further issues with groups not understanding what the goal of the activity is.

## **Providing Feedback**

Claire's primary method for providing feedback is whole class recitation. She does this at the end of every activity. During the activity she spends her time helping individual students who are stuck or confused. During this time she will occasionally acknowledge correct solutions and when she encounters a student with an incorrect solution she works with them one-on-one to solve the task correctly.

At the beginning of the activity, Connor is providing feedback on student solutions in much the same way as Claire did – acknowledging correct solutions and working with groups who are struggling. However, Connor very quickly shows the students how they can check their own solutions by multiplying the binomials that they obtained. In so doing, he has established a way for the groups to get instant feedback on their actions.

## **Regulating the Challenge of an Activity**

For the most part, Claire regulates the challenge of an activity with the class as a whole. In the first activity the challenge of the tasks is reduced for the whole class through the explicit lesson on, and examples of, how to solve these types of tasks. She then lowers the challenge even further in the middle of the activity by reviewing the range of ratios possible for the various trigonometric relations. Regardless of the skill level of each individual student coming out of the first activity, everyone was given the second more challenging activity. The challenge of this activity is then reduced for the whole class when Claire shows how to solve an analogous algebraic task.

Connor, on the other hand regulates the challenge of each task according to the ability of each individual group. Groups who need help receive individualized help. Groups whose ability allows them to solve a task are given a more challenging task to work on. Eventually, the students are shown how to check their own solutions and from then on they are able to increase the challenge of the activity as their ability increases.

## **DISCUSSION AND CONCLUSIONS**

The research method used does not allow me to determine if any of the students in either class are in flow (Csíkszentmihályi, 1990, 1996, 1998). More detailed video data, along with interviews and potentially written evidence, is needed to make such a conclusion. From the results presented above I can conclude, though, that it would be very unlikely that a student in Claire's class would be in flow. Few of the necessary conditions are present for this to happen. Although it could be argued that Claire was very careful in ensuring that the goals of the first activity were clear, these goals did not transfer to the second activity. Feedback in her class was primarily provided during her recitations on the solutions of each of the tasks. These were both infrequent and impersonal – providing feedback at times often inappropriate for the diverse stages of solving each of her students were in. Finally, her management of the level of challenge was synchronized with her teaching schedule and not the dynamic abilities of her students.



Connor, on the other hand, created an environment rich in the necessary conditions for occasioning flow. The goals of the activity were established early on and did not waiver as the lesson went on. Feedback was timely and individualized to the groups' varying progress – eventually becoming instantaneous when they learned how to check their own answers. Finally, the challenge of the activity increased in step with each groups evolving ability, first under the individualized attention of Connor and then through each groups own self-regulation.

It is clear from brief synopses of each teachers' lesson (above) that Claire is a traditional teacher while Connor is a progressive teacher. But aside from being able to state the obvious differences in their teaching style there are not many ways to describe these differences in ways that are linked to what we know about good learning. Analysing these episodes through a lens of Csíkszentmihályi's flow (1990, 1996, 1998) is one such ways. Doing so allowed us to see what aspects of Claire's teaching were antithetical to student engagement and what aspects of Connor's teaching helped foster an environment conducive to engagement.

Going forward, it would be interesting to explore the possibility of shifting the descriptive characteristics of an environment conducive to flow – providing clear goals, providing instant feedback, and regulating the challenge of an activity so as to keep it in balance with students' skills – into a prescription for how to construct such an environment.

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# THE QUALITY OF STUDENTS' ARGUMENTATION USED IN A FOURTH-GRADE CLASSROOM

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*The study was to identify the quality of students' argumentation in a fourth grade classroom where the teacher participated in a professional program. The aim of the program was to assist teachers incorporating conjecturing into mathematics that allows students' argumentation to take place. 24 students participated in the study. Data consisted of: conjecturing tasks, transcripts of audio- and video-taped, and students' worksheets. The results indicate that the improvement of students' argumentation was a result of conjecturing but also related to the mathematics contents. The quality of argumentation was improved from Level 1 advanced into Level 2. The improvement includes three aspects: accuracy of warrants, types of rebuttals, and completeness of the elements of argumentation.*

## INTRODUCTION

The reform recommends that students should have early opportunity of conjecturing, explaining, and justifying in classrooms (NCTM, 2000). Numerous studies on the effects of students' argumentation suggest that when opportunities are made available to students to participate in argumentation, the quality of their own explanations and justification are enhanced (e.g., Stylianides, 2007). Argumentation may help advance conceptual understanding, such as, refuting common misconceptions. This is resulted from students engaging in argumentation involving the skills of persuasive arguments for themselves, not just providing an audience for the teachers' reasoning. Here, argumentation refers to the process of persuading or convincing peers, refuting a disagreed claim, giving warrants for a claim. However, the use of valid argument does not come naturally and is acquired only through practice (Kuhn, 1991), which is why students in early or middle grade with weak mathematical knowledge and less experience of argumentation, need argumentation as a form of discourse which needs to be taught explicitly through suitable activity.

However, strategy to launch argumentation in mathematics classrooms becomes a pedagogical issue. For instance, a student doing mathematics often makes a conjecture by generalizing from a pattern of observations made in particular cases, and then tests the conjecture by constructing either a logical verification or a counterexample (NCTM, 1989). A conjecture might be true or false. When a plausible conjecture is justified or proved before it is supported, then an argumentation immediately follows. Conjecturing is a pivot to launch mathematical argumentation. Thus, this study was designed to support in-service teachers in designing the conjecturing tasks and enacting the tasks in elementary classrooms that allow argumentation take place. Under the context of conjecturing, what is the quality of argumentation improved when students engage in conjecturing in a fourth-grade classroom?

## **CONCEPTUAL FRAMEWORK**

Social psychologists view learning as a process that emerges during interaction (Blumer, 1969). Argumentation is seen as a social resource (Schwarz, 2009). The relationship between learning and argumentation includes learning to argue and arguing to learn. Learning to argue involves the acquisition of general skills such as justifying, challenging, refuting, or conceding. Arguing to learn refers to achieving a specific goal through argumentation. Learning to argue about mathematical ideas is fundamental to understanding mathematics. Palincsar and Brown (1984) point out that understanding is more likely to occur when one is required to explain, elaborate, or defend one's position to others; the burden of explanation push one to evaluate, integrate and elaborate knowledge in new ways. Schwarz (2009) suggests that learning to argue and arguing to learn are dependent; they are intertwined and often seem inseparable when they observe discussion in classrooms. Argumentation involves in arguing to learn and learn to argue. It involves the process of assembling claims, data, warrants, and backings that contribute to the content of an argument (Toulmin 1958). Thus, students are in the argumentation not only developing conceptual understanding but also promoting high level of mathematical thinking.

Cobb and his colleagues (2001) adopted the social interaction stance, successfully describe how collective argumentation led to autonomy and shared understanding. Following the social interaction perspective, our interest is mathematical argumentation as social constructs. Argumentation is viewed as an interactive process of knowing how and when to participate in the exchange. Given this, learning argumentation should be considered in classroom contexts.

The thinking students express in elementary classrooms probably includes many illogical elements but they are essential to the development of their thinking. Thus, the arguments that occur in elementary classrooms cannot be analysed using formal logic (Knipping, 2008). The increasing studies on analysing argumentation in classrooms are adopted from Toulmin's scheme (Toulmin, 1958). According to Toulmin, the scheme consists of data, warrant, backing, and conclusion. The support given for a claim is the data. A warrant explains the legitimacy of the data. Backing provides further support for the warrant. The conclusion is a statement that is made as though it is certain. If a datum requires support, a new argument in which it is the conclusion can be developed. Kinpping (2008) elaborates Toulmin's scheme on clarifying how individual student's explanations in the classroom collective are interactively constituted. This study adapted Kinpping's analytical method as an approach of analyzing mathematical argumentation structures in classroom.

## **METHOD**

The study was to identify how students' argumentation was improved during the terms of participating in a professional program. Only one out of six classrooms is selected for this paper based on two reasons: (1) to explore how argumentation can be evolved via conjecturing, thus, looking at what might be considered a successful rather than

typical teaching practice; (2) the teacher played the role of the pilot, generating new conjecturing tasks and its implementation. Jing (teacher) had limited experience in designing tasks for conjecturing. 24 students were in grade 4 participated in the study. They had been engaged in conjecturing and argumentation for one year in grade 3.

Three tasks involve in the volume of a rectangular solid (2014.10), recognizing triangles (2014.11), and quadrilaterals (2015.04) were incorporated conjecturing into regular instruction without taking extra hours. Each task was designed with ADDIER model: analysing the textbook, developing the potential tasks, designing the draft tasks, implementing the tasks, evaluating, and revising the task for the following day or another classroom. In addition, the design of each task followed five stages of conjecturing, adapted from Cañadas & Castro's (2005). The five stages include: constructing cases, formulating conjectures, validating the conjectures, generalizing, and justifying the generalization. Each stage was an attempt to be explored from individual, group, to whole class.

The topic of recognizing the properties of triangles is presented as the example throughout the paper. Two bases Jing relies on to incorporate conjecturing into the content of the textbook include: (1) a risk of students accepting the mathematical properties by one or two examples without generalizing and justifying; (2) students might be deprived of the opportunity to explore more properties of triangles when directly told by teachers. The task which were briefly described as follows.

Task: Giving each student 30 sticks (5 colors, 6 sticks each) in each group.

- (1) Each of you makes 2 triangles in a group. (*Construct cases*)
- (2) How do you categorize the triangles? Why? Comparing the similarities and differences among the 8 triangles. Why? (*Organize the cases students construed*)
- (3) For each group, you are given 7 triangles cards with all angles identified. Sorting the 7 triangles into categories. (*Organize the cases*)
- (4) Observe the angles in each category. What did you discover? Write it down. (*Formulate conjectures*)
- (5) Share your conjectures and check with a group. (*Check the conjectures*)
- (6) Are the conjectures you made still true in a new case? (*Validate the conjectures*)
- (7) Are the conjecture you made still true in all triangles? (*Generalize the conjectures*)
- (8) How do you convince your conjectures to your friends? (*Justify the conjectures*)

The subtasks (1) to (3) were designed to recognize various triangles, while the subtasks (4) to (9) were to recognize the properties of triangles, e.g. “*The sum of three angles is  $180^{\circ}$* ”; “*Each angle of an equilateral triangle is  $60^{\circ}$* ”; “*Isosceles triangle has two equal angles*”; “*An isosceles triangle could be acute, right, obtuse triangle*”, and etc...

### **Data Collection and Data Analysis**

The data for the study derived from a large database of a three-year project included: transcripts of 3 videos and 3 lessons of students' individual and group work. The quality of students' argumentation was analysed in two stages. This first stage was to

draw the argumentation structures by systematically analysing elements of arguments. The analysis of argumentation was adapted from Knipping's (2008) method. The first step was to divide the classroom conversation into several segments and to reconstruct the sequencing and meaning of conversation. The second step was to identify the elements of argumentation including data, warrant, backing, rebuttals, and conclusions. The symbols of circle, diamond, and rectangle represented data, warrant and refutation, and conclusion, respectively. The sequence of arguments with different elements formed an argumentation "stream". The third step was to make comparison between local argumentations structures and global argumentation structures. The argumentation stream AS3 as an example, part of the argumentation structure is excerpted from the episode of triangles. The initial conjecture formulated by a student was "adding two small angles of a right triangle is 90" (S15) ended up with the conclusion "the sum of two small angles in a right triangle is 180" (S16).

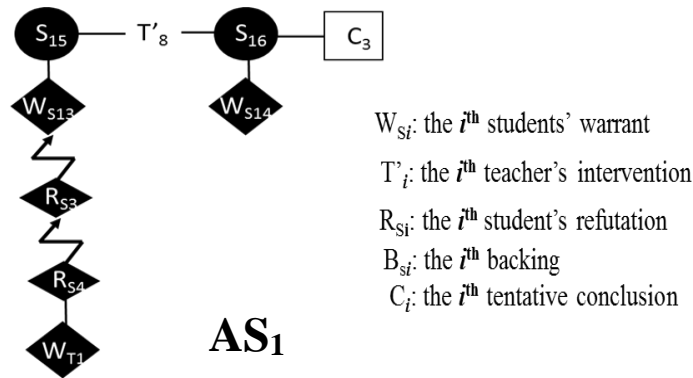


Figure 1: Argumentation stream of "the sum of a two small angles in a right triangle is 90"

- 65 S5: adding two small angles of a right triangle is 90. [ $S_{15}$ ]  
 66 T: why?  
 67 S5: a right angle is 90.  $180-90=180$ . [ $W_{S13}$ ]  
 68 S9: no, all are in the table [ $R_{S3}$ ]  
 69 S11: t is true for all triangles even they are not in the table. [ $R_{S4}$ ]  
 70 T: It must be true for all cases on the table. [ $W_{T1}$ ]  
 71 T: is it still true in any right triangle? [ $T'_8$ ]  
 72 All: yes, the sum of two small angles in a right triangle is 180. [ $S_{16}$ ]  
 73 S5: because the sum of three angles is 180. [ $W_{S14}$ ]

# 5 student proposed the conjecture (S15) (line 65) with a warrant ( $W_{S13}$ ) (line 67), since "the sum of three tangles is 180". #9 student disagreed her with a refutation ( $R_{S3}$ ) (line 68), because it is only based on several cases in the table. The invalid refutation from #9 student was immediately refuted by #11 student ( $R_{S4}$ ) (line 69). The teacher joined in this discussion for generalizing the conjecture S15 to all right triangles ( $W_{T1}$ ) by asking a question "is it still true for any right triangle? ( $T'_8$ )" (line 71). It leads to the conclusion ( $S_{16}$ ) (line 72) with the warrant ( $W_{S14}$ ) (line 73).

The second stage was to determine what progress students made based on the argumentation structures drawn in the first stage. Osborne, Erduran and Simon's analytical framework for assessing the quality of argumentation (2004) was revised for the study. The quality of argumentation was coded with four aspects: accuracy of the warrants, types of refutation, completeness of the elements, and the validity of the initial conjectures leading to conclusions. Each sub-construct was ranked from 0 to 3. The higher rank stands for higher quality of argumentation, seen in Table 1.

	Level 0	Level 1	Level 2	Level 3
Ay occuracf warrants	no warrant	irrelevant	plausible	valid
Types of rebuttals	no rebuttal	against for one warrant	against for more than one warrants	against an invalid rebuttal
Completeness of the elements	claims without warrant	claims with a warrant	claims with warrants & a backing or rebuttal	claims with warrants & more than one backings or rebuttals
Validity of initial claims	irrelevant math.	relevant math properties without generalizing	no precise but relevant math property & generalizing	precise with relevant math properties & generalizing

Table 1: Rubrics of the quality of argumentation

Based on the argumentation structure, the accuracy of the warrants given by the students was counted. There were 8 warrants to be calculated; the scores seen in Table 2. The accuracy of the warrants given by students was scored by  $(1 \times 8 + 2 \times 9 + 3 \times 3) \div 20 = 1.75$ , it stands for the quality of the warrants students provided at the level 1.75, between level 1 and 2. The rest of the aspects of the quality of argumentation is calculated in the same formula.

Level	Level 0	Level 1	Level 2	Level 3
Scores	0	1	2	3
Rubric	no warrant	irrelevant	plausible	valid
Warrants		WS5,WS6,WS8,WS9, WS12,WS13,WS14,WS16	WS1,WS2,WS4,WS10,WS11, WS17,WS18,WS19,WS20	WS3,WS7, WS15
Frequencies	0	8	9	3

Table 2: Quality of the accuracy of warrants students engaging in the topic of triangles

To increase the reliability of analysis, two graduate students coded each argumentation structure. All episodes were read independently by two coders, who then compared their codes and resolved differences through identifying episodes from classroom talk. Afterwards, the researcher joined their analysis on the comparison of local and global argumentation structures.

## RESULTS

Figure 2 shows that the students in grade 4 gradually improved the quality of argumentation as engagement in conjecturing increases. There are improvement on three aspects: accuracy of warrants, types of rebuttals, and completeness of the elements of argumentation.

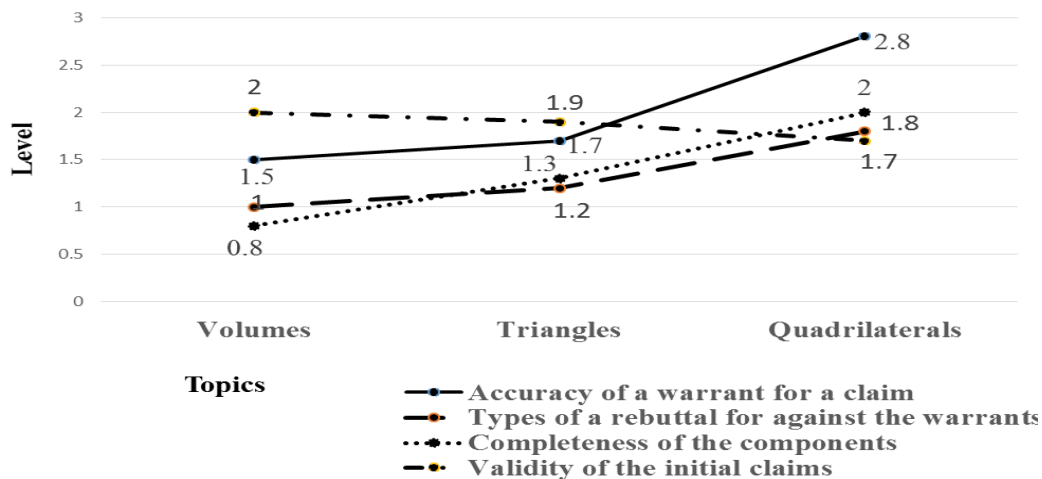


Figure 2: The improvement of the quality of students' argumentation in a year

### Students' improvement on the accuracy of warrants

Note that the conjectures students formulated were not always accompanied by warrants, but the warranted conjectures were improved in the three topics from 72.7%, 77.3%, up to 90.9%. In the warranted conjectures, the percentages of the warrants given by students without teacher's intervention were 37.5%, 80.0%, and 91.7% for the three topics. The data suggests that with increasing experience of engaging conjecturing, students were more able to make the conjectures with warrants. Moreover, the warrants students gave could be different quality, such as some warrants are irrelevant to the conjectures, plausible, or valid. For instance, the valid warrant for the conjecture "the sum of three angles is  $180^\circ$ " was given by students as "two right triangles form a rectangle with  $90 \times 4 = 360$ ,  $360 \div 2 = 180$ , so that the sum of three angles of a right triangle is  $180^\circ$ ." The irrelevant warrant for this conjecture given by the students in group 3 was "because there is a right angle in a right triangle."

Figure 2 shows students improving the accuracy of the warrants in the year, with the scores from 1.5, 1.7 to 2.8 for the three topics respectively. The quality of warrants given by the students at the end of grade 4 switched to using valid warrants for a conjecture from the level of irrelevant and plausible.

### Students' improvement on the types of rebuttals

The quality of rebuttals was improved by increasing the experience of conjecturing with the scores from 1.0, 1.2 to 1.8, depicted in the graph with slightly increasing up in Figure 2. The students in grade 4 were getting used to utilizing more than one



refutation against a disagreed warrant. They occasionally used the refutation ranked at level 3 against an invalid warrant. For instance, the student (S11) proposed a refutation ( $R_{S4}$ ) “*It is true for all triangles even they are not in the table.*” against an invalid refutation ( $R_{S3}$ ) “*all are in the table*”, which was given by a student (S9) against the warrant ( $W_{S13}$ ) “*a right angle is 90.  $180-90=180$ .*” proposed by a student (S5) for the conjecture ( $S_{15}$ ) “*adding two small angles of a right triangle is 90*”. The refutation ( $R_{S4}$ ) is against an invalid refutation ( $R_{S3}$ ). The ability to use rebuttals involves more complex skill than the use of warrants or claims (Kuhn 1991). The rebuttal for refuting an invalid rebuttal involves more complex skill than a rebuttal, since it is a prerequisite to have the ability to an invalid rebuttal.

### **Students’ improvement on the completeness of the elements in argumentation**

The various elements including warrants, rebuttals, backings used in argumentative discourse were improved to be more, with the scores from 0.8, 1.3 to 2.0 by increasing the experience of conjecturing, shown in the rising up graph in Figure 2. Overall, the conjectures students formulated at the beginning of grade 4 were frequently accompanied with a warrant, while the conjectures students formulated at the end of the year were intermediately followed by a warrant with backing. In the three topics, there were 9 unwarranted conjectures proposed by the students, 14 warranted conjectures supported by a backing or a rebuttal, and 4 warranted conjectures with more than one backing or rebuttal. For instance, a conjecture generated by #23 student “*each of the three angles in an equilateral triangle is 60*” is accompanied by a warrant “ *$180 \div 3 = 60$* ” and a backing “*the sum of three angles is 180*”.

It is noted that the more complete elements of argumentation, the higher the quality of the argumentation, but the completeness does not equal to the length of argumentation. A lengthy argumentation structure is not necessary to have high quality of argumentation. The more initial conjectures leading to more parallel argumentation streams in argumentation structure depended on the familiarity of the topics. Figure 2 indicates that not all initial conjectures effectively led to conclusions even though they have been engaged in conjecturing for a year. It seems that the validity of conjectures leading to conclusions depends on the mathematics topics. For instance, the triangles and quadrilaterals for the students who learned in grade 3 were more familiar with than rectangular solids, they formulated 8 initial conjectures for triangles and 9 for quadrilaterals which were more than the 3 initial conjectures for rectangular solids. It is possible that their weak knowledge resulted into their difficulties in making more initial conjectures that lead to conclusions.

### **CONCLUSIONS AND DISCUSSIONS**

The quality of argumentation for students in grade 4 initially ranked at level 1 is advanced into level 2 via conjecturing. The quality of argumentation was improved on three aspects: accuracy of warrants, types of rebuttals, and completeness of the elements of argumentation. The result indicates that the improvement of argumentation in grade 4 is possible if it is explicitly taught via conjecturing. The warrants for a



conjecture given by the students at the end of grade 4 were valid. By increasing the experience of conjecturing, the students were able to utilize at least one refutation against a disagreed warrant. The students improved the completeness elements of argumentation. The completeness referred to the study was warranted conjectures with backings for support or refuting a false conjecturing. It is not true that all initial conjectures were effectively turned into conclusions even though they have engaged in conjecturing in the year. It depends on the mathematics contents.

The study contributes to the improvement of students' argumentation as a result of conjecturing. In addition, the improvement was relevant to the mathematics contents. Thus, what mathematical contents might be more likely to promote students' argumentation? becomes a question to be answered for further study.

### Additional information

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# MODELING COGNITIVE DISPOSITIONS OF EDUCATORS FOR EARLY MATHEMATICS EDUCATION

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*Research shows that early mathematical abilities are important for future learning and that educators impact the quality of corresponding learning opportunities in kindergarten. However, it is an open question as to how subject-specific cognitive dispositions of educators can be described. Therefore, we adapt a model for mathematics teachers' subject-specific cognition for educators. Besides professional knowledge, the model comprises two components of reflective competence (RC) and action-related competence (AC) that are closely related to professional tasks concerning mathematical learning. Using video-based items, we developed a standardized test. Results on the quality of the measures with  $N = 112$  educators show the usability of the theoretical constructs for empirical investigations.*

## THEORETICAL BACKGROUND

Many studies indicate the importance of early mathematical abilities for future mathematical performance. Especially, the early number knowledge is considered as crucial for mathematical learning processes at school (e.g., Krajewski, & Schneider, 2009). Although early mathematical learning is often seen as a more implicit and self-regulated process than mathematical learning at school, impressive effects of the quality of the learning opportunities were described (e.g., Sylva et al., 2013). Hereby, the quality of available structures in respect to mathematics (materials, games, etc.) on the one hand and the quality of pedagogical processes (e.g., active mathematical learning support, van Oers, 2009; scaffolding, e.g., Wood et al., 1976) on the other hand are seen to play an important role. Accordingly, educators are an important factor for the quality of mathematical learning environments in kindergarten, as they are responsible for implementing those (for an overview cf. Gasteiger, 2012).

Despite of this consent, it is by far less clear what cognitive characteristics educators should hold in order to deal with the demands of offering these high-quality mathematical learning opportunities for children. Although early education has its own characteristics, we will in the following also use findings on teacher cognition, if analogies seem appropriate. Especially, we will adapt an extended model on teacher cognition comprising knowledge as well as components of competence for educators.

## Modeling cognitive dispositions of educators for early mathematics education

Recent research on educators' professional knowledge closely follows the rationale of research on professional knowledge of teachers and differentiates between content knowledge (CK) and pedagogical content knowledge (PCK, Shulman, 1986). CK – in our case mathematical knowledge – is knowledge about early mathematics, its scope,

forms, representations, and usage (e.g., Clements & Sarama, 2014; Tsamir et al., 2015). PCK is knowledge about the learning and teaching of mathematics in the early years, including, e.g., knowledge about typical student difficulties, indicators for at-risk students, and adequate learning tasks (e.g., Dunekacke, Jenßen, & Blömeke, 2015; for an international overview cf. Depaepe et al., 2013).

Following the paradigm of expertise research, this professional knowledge is seen as the relevant and specific knowledge base of teaching. Teachers' professional knowledge was accordingly found to predict student learning outcomes, aspects of high-quality instruction, and planning of instruction (Baumert et al., 2010; Hill et al., 2008; Dunekacke et al., 2015). However, research cannot explain how professional knowledge of teachers' accounts for these effects in detail. Especially, the relation between knowledge and different practical skills is not extensively investigated.

Teacher knowledge tests focus largely on the measurement of decontextualized, declarative knowledge, so that the demands during testing are very different from practical tasks. Hence, it is an open question if an extended view on teacher cognition that accounts for broader aspects of mathematics-specific teacher cognition could shed additional light on the complex relation between professional knowledge, instructional quality, and student achievement. A first step is to develop a suitable theoretical model including – besides knowledge – cognitive aspects with stronger connections to teaching demands, so that it covers the abilities to utilize this knowledge. Following this idea, we adapted a structure model of subject-specific teacher cognition (Lindmeier, 2011) for our research with educators.

Besides a component of subject-specific professional knowledge (basic knowledge, BK, as CK and PCK), this model uses two components of competence. Competences are here – in a European tradition – context specific and learnable cognitive performance dispositions that allow individuals to cope with certain situations in specific domains (cf., Koeppen et al., 2008), so they can be seen as the cognitive base for practical skills (related to early mathematical education in our case). Lindmeier (2011) differentiates for teachers between practical skills needed to plan, prepare, and post-process instruction and summarizes according cognitive dispositions as reflective competence (RC). In contrast, practical skills to implement high-quality instruction, e.g., to provide active mathematical learning support, are attributed to a different set of cognitive dispositions, called action-related competence (AC). Thus, the differentiation of reflective and action-related competences is theoretically legitimated by two very different groups of demands related to mathematical education: The provision and monitoring of mathematical learning (RC, pre- and post-active), as well as the active scaffolding of mathematical learning (AC, cf. also classroom interactions, Pianta et al., 2005). Educators are then expected to hold specific cognitive dispositions to master these tasks of early mathematical education.

The theoretical model was so far successfully used for research on the cognition of primary and secondary mathematics teachers: Reflective as well as action-related

competences proved to be related, but empirically separable components of mathematics teachers' cognition. They were, moreover, separable from professional knowledge (Kniesel et al., 2015; Lindmeier, 2011).

For the case of educators' cognition, research findings are less differentiated at the moment. Although there are good reasons to see analogies between the research on teachers' and educators' cognition, there are also some limitations to do so. Before expanding on the relation between professional knowledge and competence, we will introduce the organization of education for educators in our countries, as there are specific differences concerning the role of professional knowledge.

### **Education of early childhood educators in Germany and in Switzerland**

Unlike teachers for primary school (children aged 6+ years), early childhood educators in Germany and Switzerland are not necessarily educated specifically to teach mathematics. In fact, the acquisition of mathematics by children traditionally used to be of little interest, often being part of the educators' education only in relation to the general cognitive development of children. However, the situations in Germany and Switzerland started to diverge, as from 1998 to 2007, the education of pre-school educators (children aged 4-6) in Switzerland was transformed into an academic education (European Qualification Framework, EQF level 6) in parallel to primary teacher education. Hence, early mathematics and its acquisition are studied by the new generation of Swiss educators. In Germany, educators are still mostly educated in non-academic vocational schools (EQF level 4), where the focus lays on the acquisition of practical childcare skills. In sum, this accounts for big differences between the neighbouring countries, so that in 2012 only 3% of the kindergarten educators in Germany have an academic professional education, whereas since a decade all freshly educated Swiss educators have an academic background.

### **Professional knowledge and action-related competences/practical skills**

As explained above, the model we used assumes that the professional knowledge related to teaching mathematics (BK as CK and PCK) is a prerequisite for the two subject-specific competence components (RC, AC) with close relation to professional tasks. In fact, studies with mathematics teachers indicated medium to strong correlations between BK and the competence constructs, especially the relation between BK and RC was found to be substantive (Kniesel et al., 2015; Lindmeier, 2011). However, it is questionable if subject-specific professional knowledge of educators plays an equally important role as it does for teachers. At the one hand, subject-specific professional knowledge is not necessarily emphasized in the educators' education, as explained above. Especially, non-academic education should result in lower levels of knowledge about early mathematics education. On the other hand, knowledge concerning early mathematics (especially CK) has a substantial overlap with basic mathematical understanding, so that it is expected to be much less explicitly accessible for educators than professional knowledge concerning mathematics is for teachers. This effect may increase, if the education does not stress

this kind of knowledge (cf., Tsamir et al., 2015). Finally, whereas non-academic education is focused on the acquisition of practical skills, academic education is focused on the acquisition of advanced theoretical knowledge and critical thinking. Thus, it is an open question, if the findings of a strong common rooting of AC and RC in BK holds also true for educators.

## **RESEARCH QUESTIONS AND STUDY DESIGN**

The aim of our study was to adapt the three-component model of teacher cognition for kindergarten educators and develop standardized measures for the components in order to allow for a structural investigation of educators cognition. This first study focuses on the viability of the approach. Therefore, we worked on the following research question: (1) Is it feasible to develop valid and reliable instruments to assess the subject-specific components of educators as BK, RC, and AC? (2) Are the resulting measures sensitive to differences expected for groups with known characteristics (discriminant validity)?

### **Methods**

We decided to use a methodological approach that combines standardized assessment using traditional paper-pencil formats with innovative, video-based item formats (similar to Lindmeier, 2011). Video-vignettes of children involved in early mathematics should transport authentic professional demands, enable the educators to mentally engage with those, and thus elicit the target cognitive dispositions. In addition, the instrument was designed to proximally implement the characteristics of the targeted tasks, such as spontaneity (AC) through time constraints. We restricted the contents to the field of numbers and operations due to their importance for mathematical development. The assessment instrument was administered in small groups (up to 10 educators) together with a short questionnaire on personal background (gender, age, education, experience with early mathematics).

### **Sample**

The pilot study is based on the data of 112 participants from Switzerland ( $N = 82$ ) and Germany ( $N = 30$ ). All participants were active kindergarten educators and their age ranged between 20 and 59 years ( $M = 35.71$ ;  $SD = 10.58$ ). The majority of the participants was female (93.8%) and a subsample of 59 (52.7%) educators had a academic education. Our convenience sample did not represent the differences in respect to the academic background according to the framing conditions, as we aimed at a balancing of the contrasting groups and not nationally representative samples (academic track Germany: 56.7%; Switzerland: 51.2%). Of course, as a result of the recent changes, academic education is confounded with age in our sample.

### **Instruments**

In order to operationalize the adaption of the model of Lindmeier (2011), it is necessary to describe the mathematics-specific target tasks of the educators' work and implement them in the assessment instrument. For the assessment of action-related competence

(AC), the tasks arise when educators interact in pedagogical situations. Therefore, crucial abilities are to implement mathematical learning environments, provide active learning support, but also to identify and productively use mathematical learning opportunities for a child's development. A sample item targeting at the latter is:

*Item „set the table“:* In the video, two children set the table with miniature doll's dishes and silverware and notice that things are missing. The children say the number of plates and knives available. There are not enough knives and big plates. At the end, the children decide to use smaller plates. After the video, the educators are prompted: "The children already found out that there are things missing. Please ask them a question that converts the situation into a mathematical learning opportunity!"

The educators' open answers were coded according to partial credit scores with 0 credits for general, non-mathematical answers (e.g., "Can you tell me what you noticed?"). Answers that were mathematics-specific, but not transcending what children already noticed were credited with 1 (e.g., "How many plates do you have? How many knives?"). Full credits of 2 were given to mathematics-specific answers, that pick up children's actions and go beyond them (e.g., "You already noticed, that there are things missing. How many plates and knives do you need so that all children have a full setting?"). Overall, seven items for measuring AC were developed.

For the assessment of reflective competence (RC), the range of demands arises from the provision and monitoring of mathematical learning. There are, on the one hand, the preparing tasks to plan, organize, and prepare mathematical learning opportunities. On the other hand, educators are expected to assess and document children's mathematical development, including the task to diagnose specific learning difficulties in the domain. A sample item therefore is given in the following:

*Item „counting up to 40“:* In the video, a 6-year old counts up to 40 and shows characteristic difficulties when crossing the 10s: She falters and cannot name correctly the multiples of 10. Off screen, the educator scaffolds the student when she hesitates at a multiple of 10 by giving the correct name and encouraging: 'What comes next?' The child then autonomously counts on up to the next multiple of 10, but leaves out numbers with repeating digits (22, 33). After the video, the educators are prompted: "The child shows two systematic difficulties. What are they?"

The educators' open answers were coded according to a score with 0 credits for wrong answers or superficial answers not pointing to a mathematical error (e.g., „The child struggles with pronunciation.“). Each mathematics-specific difficulty that was identified was scored with 1, so that a maximum of 2 credits could be reached. Altogether, ten items for measuring RC were developed.

Finally, we conceptualized basic professional mathematical knowledge for educators (BK) as knowledge about early mathematics as well as knowledge about the teaching and learning of early mathematics. We did not differentiate between subject knowledge (CK) and pedagogical subject knowledge (PCK) in this study, as both kinds of knowledge were found to be heavily related in studies with teachers and time limitations forced us to lay the measurement focus on the new competence

components. Thus, we refrained from using a separate CK test, but integrated CK together with PCK as BK test. For example, we asked the educators to produce different representations for quantities like 8 (CK). PCK items required, e.g., knowledge about the average counting abilities of preschoolers (“Which of the following abilities are usually shown by preschoolers? Counting backwards from 10, 12, or 20”). Altogether, seven items for measuring BK were developed.

Two trained persons scored all answers according to a manual. The mean interrater-reliability was Cohen’s  $\kappa = .65$  ( $\kappa = .50-.80$ ) and considered sufficient in view of the mostly open item formats.

## RESULTS

To answer the first research question, we were examined the reliability of the scales. The 24 items showed an internal consistency of Cronbach’s  $\alpha = 0.88$  ( $r_{it} = .23-.61$ ) so that they could be seen as representing a single construct. However, the three subscales showed internal consistencies of comparable sizes ( $\alpha_{AC} = 0.77$ ,  $r_{it} = .42-.56$ ;  $\alpha_{RC} = 0.78$ ,  $r_{it} = .23-.59$ ;  $\alpha_{BK} = 0.71$ ,  $r_{it} = .30-.55$ ). By extrapolating the values according to the length of the overall test (24 items), the reliability of the subscales was estimated to be slightly better than the reliability of the overall scale (Spearman-Brown,  $\alpha_{AC24} = 0.92$ ,  $\alpha_{RC24} = 0.89$ ,  $\alpha_{BK24} = 0.89$ ). Thus, the use of subscales according to the theoretical model can also be justified. A preliminary confirmatory factor analysis indicates also a good fit of a three-dimensional model in line with the theoretical considerations and confirms the correlational findings.

The manifest correlations between the subscales were calculated in order to estimate the strength of relations between knowledge and competence components. A stronger relation between BK and RC ( $r = .70$ ,  $p < .01$ ) was found in comparison to the relations between BK and AC ( $r = .53$ ,  $p < .01$ ) and between AC and RC ( $r = .55$ ,  $p < .01$ ). Hence, the correlational patterns are in line with findings for teachers, indicating a strong common rooting of the competence components in knowledge with yet separable competence components. Reflective competence is stronger associated with knowledge than action-related competence.

Scale	M (SD)		t(110)	Effect size Cohen’s d
	Academic (n = 59)	Non-Academic (n = 53)		
AC	9.15 (2.85)	6.06 (2.95)	-5.649*	1.07
RC	12.27 (2.55)	8.17 (3.25)	-7.657*	1.45
BK	9.71 (2.63)	6.49 (2.86)	-6.199*	1.18
ALL	30.95 (6.38)	20.66 (7.10)	-8.078*	1.53

Table 1: Means, standard deviations and results of t-test for differences between the groups of educators with academic vs. non-academic education (\*  $p < .01$ )

The discriminant validity of the scales was examined by comparing two groups of educators (known-groups method, Table 1). The instrument was able to model the

expected group differences for the subscales (AC, RC, BK) and for the test in total (ALL). Differences were found to be significant between educators with academic and non-academic background. Thereby, educators with an academic degree achieved higher scores. Effect sizes showed that all differences could be interpreted as large effects.

## **DISCUSSION**

In this study, we adapted a model of teacher subject-specific cognition for the use with kindergarten educators. Besides professional knowledge for early mathematics education (BK), two components of competence closely related to the professional demands of preparing and post-processing mathematical learning (RC) and scaffolding mathematical learning (AC) were introduced. An according measurement instrument for the use in standardized assessment was developed. In order to elicit the targeted competences, our partly video-vignettes based method required the educators to act as-if in practical situations.

The answers of 112 active German and Swiss kindergarten educators were used to investigate the quality of the instrument based on 24 items. The analyses show satisfactory results for the internal consistency of the complete instrument, as well as for the intended subscales. In line with findings from research on teacher cognition, professional knowledge can be seen as an important base for the components of competence. Indeed, there are indications that knowledge is stronger associated to reflective competences than action-related competences. Additional analyses will be conducted with data from a more comprehensive study. We further investigated if our measures are sensitive in respect to expected differences of known groups. In line with the expectations, kindergarten educators with academic education outperform those without an academic background.

Although we can speak of a successful test development, there are also limitations to our study. First, our instrument is at the moment limited to a single, yet especially important mathematical content area (numbers and operations). Second, our instrument uses a combined component of professional knowledge (CK and PCK) due to pragmatic decisions (limited testing time, focus on competence components). Finally, due to a significant change of policies, we cannot disentangle effects of practical experience and (non-)academic education.

Nonetheless, this research paves way for the modeling and measuring of educators' cognitive dispositions for early mathematics education. In a following comprehensive study, we seek to investigate in more detail the relation between areas of educators' professional knowledge and competences as well as the predictive validity of our measures for practical skills. The research thus contributes to a better understanding of what kind of preparation educators need to master the challenging demands of high-quality early mathematics education in kindergarten.



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# EPISODIC FUTURE THINKING IN MATHEMATICAL SITUATIONS

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*Episodic future thinking is a process of mentally projecting one's self into a future event, allowing the event to be experienced before it actually occurs (Atance & O'Neill, 2001). The current study explores the possibility that students engage in episodic future thinking while solving mathematical tasks. Participating students were given mathematical situations and verbalized thoughts that emerged as they planned resolutions to the situations. All participants exhibited episodic future thinking and we present a categorization of these thoughts. Given extant results on the positive influence episodic future thinking has on general problem-solving ability, we propose that a similar influence might exist on mathematical problem solving.*

## INTRODUCTION

Many frameworks have been created to understand students' behaviours and thought processes that emerge when working with mathematics problems. Perhaps the most well-known are those in Schoenfeld's (1985) *Mathematical Problem Solving* and Pólya's (1945) *How to Solve It*; see (English & Sriraman, 2010) for a contemporary review. In both frameworks, "planning" is identified as an essential component of problem solving, or, more generally, while engaging with mathematical situations. Exactly how one should plan in a mathematical situation is not clear. Pólya's approach is in terms of *heuristics*, or general, situation-independent guidelines of how to act. Schoenfeld takes a different direction and introduces the notions of *resources* and *control*: problem solvers ought to draw on their acquired mathematical knowledge, knowing or determining what is relevant (resources) and what is extraneous (control). We take the view that neither perspective alone can account for the varied, idiosyncratic ways in which people actually navigate mathematical situations.

## Episodic and Semantic Memory

Schoenfeld's conceptualization of resources as mathematical knowledge has been used in a restricted sense as one's understanding of mathematical concepts and procedures. Phrased in terms of cognitive structures, "knowledge", mathematical or otherwise, is a part of *declarative memory*: memories, facts, and experiences that can be consciously recalled and verbalized or written. Declarative memory can be partitioned into *episodic* and *semantic* memory systems (Tulving, 1983). Semantic memory is the memory of facts—knowing that Szeged is in Hungary, for example—whereas episodic memory is the memory of specific personal experiences—remembering the moment you learned from your mother that Szeged is in Hungary. In this light, we propose a more general conceptualization of mathematical knowledge as comprising both episodic and semantic memories of mathematics.

When a student is engaged in learning any subject, including mathematics, they form both semantic and episodic memories; not only do they acquire knowledge of facts, definitions, procedures, and general processes, they also form memories of personal experiences of learning and using these facts and definitions, executing procedures, and engaging in mathematical thinking. Episodic memories of engaging with mathematics are not often reported in the literature, and typically viewed as irrelevant to the mathematics. We take the view that both semantic and episodic memories are vital to working in mathematical situations. The reliance on episodic memory may be particularly evident for mathematical situations that require some degree of planning, given recent evidence that episodic memory contributes to future thinking (Schacter, Addis & Buckner, 2007; Schacter et al., 2012).

### **Episodic Future Thinking**

Mathematics aside, what is known about how people plan for to-be-experienced events? Take, for example, planning a trip to Szeged. One may form a list of what to pack using only semantic knowledge—I will be away for ten days and therefore require ten undershirts, etc. One may also imagine themselves travelling to Szeged, even having never travelled there, and can form memories of these imagined future events. These simulations of the future, of experiencing the expected weather or being at the conference dinner or reconnecting with colleagues, can be used to plan for the trip. This pre-experiencing of the event may help one anticipate how the event will unfold, including what could go wrong and what may be required. The key point here is that the person does not necessarily plan for the to-be-experienced event semantically; they may also plan episodically. Indeed, this type of pre-experiencing of a future event, called *episodic future thinking* (Atance & O'Neill, 2001), is associated with more successful outcomes in various goal-directed activities including open-ended problem-solving (Taylor, Pham, Rivkin, & Armor, 1998; Schacter, 2012).

This paper explores the possibility that students engage in episodic future thinking when planning in mathematical situations. Participating student volunteers were given, sequentially, two mathematical situations and asked to think about what a solution could be. After mentally forming a solution, participants were interviewed about the thoughts and mental images they experienced when forming their solution. All students engaged, to varying degrees, in episodic future thinking.

The purpose of this paper is twofold: to 1) present evidence of people engaging in episodic future thinking while working in mathematical situations, and, 2) to highlight a number of methodological issues that must be addressed to further explore episodic future thinking in mathematics. This work is a part of a larger study intended to create a theory of future thinking in mathematics.

### **METHODS**

Student volunteers were recruited, via email and an in-class announcement, from a second year general mathematics course, covering multivariable calculus and elements of linear algebra, at the University of Auckland during the second semester. A total of nine students volunteered from this course. The recruitment email was sent

## Task 1

Graph a function  $h(x)$  that satisfies the following conditions.

- $h(0) = 2$
- $h'(x) > 0$  when  $x < -1$ ,  $h'(x) < 0$  when  $x > -1$
- $h''(x) > 0$  when  $x < -2$  and when  $x > 0$ ,  $h''(x) < 0$  when  $-2 < x < 0$
- $\lim_{x \rightarrow 0} h'(x) = \infty$

Imagine the graph of this function to the best of your ability and in as much detail as possible.

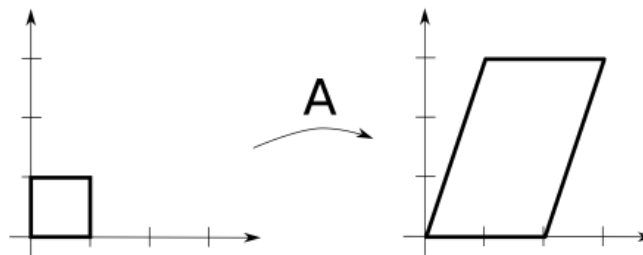
## Task 2

We are becoming familiar with the sight of algal blooms on our beaches in the Summer. We are told that these blooms are more frequent as a result of a number of factors, including warmer sea temperatures and greater nutrient concentrations. We observe that the algae congregates in certain places on our beaches and comes and goes over a period of days and weeks.

Your task is to construct a mathematical model of an algal bloom. That is, a mathematical description of the algal bloom that describes how it changes over time. Imagine this model to the best of your ability and in as much detail as possible.

## Task 3

Suppose the matrix  $A$  maps the unit square in  $\mathbb{R}^2$  to the parallelogram in the right half of the figure below.



What are the eigenvalues of  $A^{-1}$ ?

## Task 4

Suppose you work on a road construction site where fuel for all the machines is stored in a large cylindrical container. This cylinder is lying on its side. There is a small opening halfway along the tank facing upwards in which a measuring stick can be inserted to find the distance between the surface of fuel and the opening. What is an expression for the amount of fuel in the cylinder?

Figure 1: The mathematical tasks used in this study.

to an honours version of this course, yielding an additional two participants, for a total of eleven ( $N = 11$ ). Following informed consent, each student participant individually attended two sessions, each approximately one-half hour in duration, and was compensated with a university bookstore gift voucher for each session they attended. In the first session, they were asked to verbalize their thoughts about given mathematical topics. The results of this first set of interviews are reported in a

companion manuscript. The focus of the current is on the second session during which students were given, sequentially, two mathematical situations from the four presented in Figure 1, and asked to think about how they would solve each. They then verbalized their thinking with prompts from the interviewer.

Students' interviews were audio recorded and transcribed. The transcripts were analyzed using a Grounded Theory methodology (Strauss & Corbin, 2015). The initial analysis consisted of classifying the students' utterances as: 1) *episodic*, relating to events experienced, or to-be experienced, by the participant, 2) *semantic*, comprising factual statements, and 3) *other*, typically clarification or orientation statements. The episodic category was further refined into episodic memories—which naturally have occurred in the past—and episodic future thinking. Only those student utterances indicative of episodic future thinking are reported here. Following Tulving (1983) and Atance and O'Neill (2001), for an utterance to be classified as an episodic future thought, it must 1) refer to an event that is forecast to occur at some future time, and 2) involve the speaker.

## RESULTS

All participants engaged in some form of episodic future thinking while thinking about the provided mathematical situations. Episodic future thinking is as idiosyncratic as the participants' personal histories, making an authoritative categorization elusive. There were nevertheless similarities among some of the participants' utterances and these informed the creation of the descriptive categories reported below. Each category applies to the utterances of at least two participants. Representative quotes are provided, identified by participant numbers of the form Px.

### *Imagining Possible Actions (PA)*

This type of mental simulation involved anticipating what actions would need to be taken to complete the mathematical task. Participants described what they would need to write or operations they would perform to make progress on the task.

“So it's just sitting there trying to think about what you would do, what would be required, and how to piece all those little bits of information of what would be required together to get a correct solution. And then checking it back to make sure that that's actually what you've got.”(P2)

“So if I was to, like, flesh this out, I would write it in terms of 'n'.”(P1)

Responses in this category were not restricted to participants thinking about what to write or what methods to use, however. One participant, while recalling their thoughts about Task 1, indicated they thought about what movements their hand would need to take to form the graph.

“I was thinking of how to write it...what I imagine is a pen in hand...how to draw the graph.”(P8)

### *Imagining Future Social Situations (SS)*

A couple of participants imagined working the mathematical tasks in social settings.

“I imagined working with friends though...because I don't know how to do it. So I assume some of my friends would know how to do it...sitting around a circled desk...I would get this far and then they would help me along the way to find the formula.”(P0)

“I kind of just thought how I would, like, teach this to someone who didn't know what any of this meant...and I think before I started saying anything I just sort of planned out the structure of what I was going to say.”(P1)

### *Adapting a Past Experience (PE)*

A few of the participants recalled past events of working with similar mathematics. But these were not strict recollections of previous events; they were adapted to the context of the current mathematical situation.

“My first thought was to use triangles, because in the last year of school...we used to use triangles a lot in rates of change questions.”(P10)

“In my tutorial we had eigenvectors. They were given like this...but I couldn't remember what I did with the eigenvectors...I think you need to use one of them...sub it into this.”(P0)

One participant, while recalling their thoughts about Task 2, described their mental image of an expanding lily pad. Initially, the interviewer assumed this was spontaneously generated by the problem statement. The image was brought up again later in the interview and was revealed to be an actual problem the participant had previously encountered and was trying to modify to Task 2.

### *Experiencing Emotions (EE)*

Some students experienced emotions they associated with resolving the mathematical tasks. These were either inhibitory or conducive to engaging with the tasks.

“So I was just imagining myself in a room full of people with a test script in front of me and just freaking out a bit because it looks so hard. But also kind of confident because I know I've done problems like these before and I know I've always...got them right. So even though I'm looking at...no idea where to start, I know that I've felt like that before but I've always solved it.”(P1)

“I think about the endpoint. I think about...like the satisfaction feeling when you actually solve the problem.”(P11)

### *Anticipating Failure (AF)*

The final category of episodic future thoughts experienced by the participants involved them imagining working the mathematical task but being unable to make progress.

“I can recognize the components. And I can recognize how I might want them in there. I can't write them. So I was sitting here thinking, I really don't want to have to write this out properly because I don't know how to. ... you just think of all these things that you might have to do that you won't be able to do.”(P2)

“...I just pictured myself in a test room...or an exam room and just sort of like doing the first derivative test...I felt nervous, because I couldn't do it.” (P6)

### *Other Types of Episodic Future Thinking*

The participants verbalized an assortment of episodic future thoughts, such as: imagining what information and/or external resources they would need to draw on while forming the solution, mentally simulating the situation given in Tasks 2 and 4, modifying heuristics to the given task, and imagining a virtual path they would need to traverse to arrive at a solution. Each of these thoughts were verbalized by at most one participant, however, which lends support to the claim made above that episodic future thinking is highly idiosyncratic.

## **CONCLUSIONS AND FUTURE THOUGHTS**

This paper presents evidence that undergraduate students currently enrolled in a mathematics course do engage in episodic future thinking while working with mathematics. Episodic future thinking is just one possible process in which a student may engage and there are, of course, others: they may think of and apply the appropriate problem schema, or they may not be able to think mathematically at all. We present here a number of issues that ought to be addressed in order to further investigate episodic future thinking in mathematics.

The first is that episodic future thinking may not always arise in mathematical work—for example, when adding two single-digit numbers—and we are left wondering about the type of mathematical situation likely to prompt episodic future thinking. We argue, as in (Maciejewski & Barton, 2015), that the mental processes evoked depends on how the task relates to its solver. As Schoenfeld (1985) recognizes, a mathematical problem is only a *problem* if it relates to the solver in that way. If a task is too familiar to its solver, automaticity may be triggered and no thinking evoked. If the task is too unfamiliar to the solver, they may not be able to understand it, let alone make any progress in its resolution. Familiar, but not too familiar, tasks may invoke a problem schema. We suspect that unfamiliar-yet-understandable tasks, like the “problems” of Schoenfeld (1985), are the realm of those which elicit episodic future thinking. In such tasks, a solution, not readily apparent, must be consciously constructed and planned for; see, for example, (Lesh & Zawojewski, 2007). This notion of a *distance* between a problem and its solver, as proposed in (Maciejewski & Barton, 2015), appears new and may provide further insight into the genesis of the solver's actions. With this in mind, we are left desiring a more complete description of how students and problems might relate.

The second issue is the interaction between episodic future thinking and success in mathematical situations. Specifically, are students who are more able to engage in episodic future thinking, perhaps by forming more lucid mental simulations, better problem solvers? Previous work outside of mathematics does support the claim that more effective episodic future thinkers are more effective problem solvers (Taylor,

Pham, Rivkin & Armor, 1998; Schacter, 2012) but we are not aware of any work that considers discipline-specific problem-solving. We are also left wondering about the converse relationship: do effective mathematics problem solvers engage in episodic future thinking in their mathematical work? There is some affirmative evidence from mathematicians (Maciejewski & Barton, 2015); might the same be true for students?

The third issue concerns whether episodic future thinking in mathematics can be developed through training. As reviewed by Taylor, et al. (1998), psychology students who were trained to imagine the *process* of attaining an academic goal outperformed those with no training. In fact, those who were trained to imagine the *outcome* of attaining the goal (e.g., feeling relief or accomplishment) achieved at an even lower level. These results demonstrate that the development of episodic future thinking can, in some domains, be aided with an instructional intervention. We are optimistic that a similar result can be achieved in mathematics. Additionally, the results reported in (Taylor, et al., 1998) suggest that imagining a to-be-experienced event allows one to “pre-experience” the emotions associated with that event—as did the students in the *EE* category above—rendering one better able to manage those emotions if they do arise in the actual event. Such results are likely to have profound implications for students paralyzed by mathematical anxiety.

A number of methodological issues arose when planning, conducting, and interpreting the interviews in this study. The first, mentioned above, was the selection of the tasks. Of all the participants in this study, only one was able to completely resolve any of the tasks. All others made no or only slight progress. We are left wondering what type of episodic future thoughts would have emerged if more students were able to successfully engage with more of the tasks. Developing tasks that relate to students in this way is non-trivial: it requires an understanding of a student's personal mathematical history and their current mathematical understanding. It would seem, therefore, a futile endeavour. We suggest addressing this issue from another angle: have the students develop, or choose, their own tasks. This may mimic the process mathematicians undertake when choosing a research-level problem: they often choose problems they think they are able to contribute to, or are likely to lead to aesthetically pleasing resolutions, or be of interest to their colleagues—all of these choices involve some aspect of episodic future thinking.

The second methodological issue arose during the interviews. The participants appeared to be *primed* to respond “mathematically”. Simply stating that the task is mathematical appeared, to the interviewers, to prompt the participants into talking strictly about mathematics rather than describing any and all thoughts they were experiencing. As one participant said, “I pretty much just constrained myself to doing the question, nothing outside the question because those are really quite irrelevant to the question.” Therefore, we see a need for more naturalistic ways of observing students in problem-solving situations.



Finally, we address an anticipated criticism of the current work. The episodic future thoughts articulated by the participants contained, more often than not, scant mathematics. Therefore, it could be argued that episodic future thinking may not actually be relevant to working with mathematics. These thoughts were nevertheless experienced by our participants and we expect they are experienced more widely. Moreover, the thoughts, mathematical or otherwise, verbalized by the participants guided the mathematical choices they made while working the tasks. This leads us to consider possible implications for educational practice. Two are immediately apparent: instructors/teachers may 1) guide students to suppress or disregard seemingly irrelevant thoughts emerging while engaged with mathematics and revert to a “control/resource” approach similar to Schoenfeld's (1985), or 2) acknowledge that students might experience episodic future thoughts and assist them in making these vivid and productive. Since this paper is among the first that explore students' episodic future thinking in mathematical situations, it is too early to form concrete recommendations for educators. Nevertheless, we foresee training students to develop and hone their episodic future thinking in mathematical situations a productive way of significantly improving the efficacy of their mathematical problem solving.

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# TEACHER GESTURES AS PIVOT SIGNS IN SEMIOTIC CHAINS

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It is widely shown in literature that gestures take part in classroom interactions. Adopting the perspective of Theory of Semiotic Mediation, we illustrate how they can act as “pivot signs” in semiotic chains, namely they can be used by the teacher to link signs referring to a situated context (such as the activity with an artifact) to the mathematical domain. More specifically, by analysing a mathematical discussion in grade 5, we identify two different ways in which gestures intervene, together with other modalities, in this process: pointing as pivot sign and multimodal pivot signs.

## THEORETICAL FRAMEWORK

The construct of *semiotic chain* is used in many different ways in mathematics education literature. It usually refers to “a process in which a signifier in a previous sign combination becomes the signified in a new sign combination, and so on” (Presmeg, 2006, p.166) which can be used “as an instructional model that develops a mathematical concept starting with an everyday situation and linking the two”(ibid). Bartolini Bussi and Mariotti (2008) in their Theory of Semiotic Mediation (TSM) re-define semiotic chains in the particular case of classroom activities realized through the usage of artifacts. The starting point of TSM is that each artifact may be related to two different systems of signs: The culturally determined signs used to refer to mathematical contents connected with the artifact (*mathematical signs*), and the contingent signs produced by students when they use the artifact to face a particular task (*artifact signs*). The development of coherent relationships between the two systems of signs is thought as a fundamental educational goal, but empirical evidence shows that this process is not direct, rather passes through *pivot signs*, i.e. signs that, in a classroom community, may refer to the activity with the artifact *and also* to the mathematical domain. In other words, pivot signs are the fundamental “links” of *semiotic chains* evolving under the guidance of the teacher to foster the passage from situated signs to mathematical ones, especially during classroom discussions.

In our research study, we investigate the role of gestures within the semiotic chains as conceived in TSM. As widely recognized in literature, gestures play a role in classroom mathematical activities (e.g. see Arzarello et al., 2009; Radford, 2003; Roth, 2001). Referring to TSM, Bartolini Bussi and Baccaglini-Frank (2015) show that an iconic gesture recalling the activity with the artifact may work as a pivot sign when co-timed by the teacher with different words, initially referring to the situated context with the artifact and then referring to the mathematical domain. From our previous empirical studies, we have observed that gestures and gestural repetition or “catchment” appear during the construction of semiotic chains in mathematical discussions, and shape what we call *multimodal semiotic chains* (Maffia & Sabena, in press). The term

multimodality refers to the relevance and mutual coexistence of different cognitive, physical, and perceptual modalities playing a role in teaching-learning processes, including “oral and written symbolic communication as well as drawing, gesture, the manipulation of physical and electronic artifacts, and various kinds of bodily motion” (Radford et al., 2009, 91-92). The notion of catchment comes from gesture studies in psychology and is described by McNeill (2010) as the recurrence of some gesture feature during a conversation, with the function of giving cohesiveness to the discourse.

In order to carry out the analysis of the semiotic chains in a multimodal perspective, we rely on the notion of *semiotic bundle* (Arzarello et al., 2009), namely “a system of signs — with Peirce's comprehensive notion of sign — that is produced by one or more interacting subjects and that evolves in time” (ibid., p.100). Gestures and their mutual relationships constitute a *semiotic set* (Arzarello, 2006), interacting with the other ones that take part to the bundle (e.g. spoken words). Following McNeill (1992) we speak of *iconic* gestures when they depict the semantic content of the discourse, *metaphoric* gestures if the semantic content is abstract, and *deictic* gestures if they point to objects or positions in the space—these categories are not mutually exclusive.

In this paper, we focus on the role of gestures made by the teacher in the process of fostering the development of a multimodal semiotic chain. More specifically, we are driven by the following research question: *In which way(s) does the teacher use gestures as pivot signs in order to foster multimodal semiotic chains?*

## METHOD

For this study we did not produce specific data but we re-analyzed video-recordings taken from a previous long-term teaching experiment in which the first author acted as teacher (for more details see Maffia & Mariotti, in press). Since the camera is always recording the teacher, these data are suitable for our research question. As a counterpart, we have no information about students' gestures during the interaction.

Due to the limit of space, we present only two excerpts from a discussion that took place in a grade 5 class in Italy in February 2014. The students have just worked individually on a task with Geometrical Times-Table, that is a sort of  $10 \times 10$  visual times-table in which multiplications are represented by rectangles (according to the lengths of height and base). The task consists in (1) coloring two chosen cells in the same row of the table; then (2) cutting two slips of paper with the same dimensions of the colored cells, and gluing them together to form a new rectangle (step 3). Then pupils have to (4) color the cell in the table corresponding to the obtained rectangular piece of paper (Fig.1). Students have also to

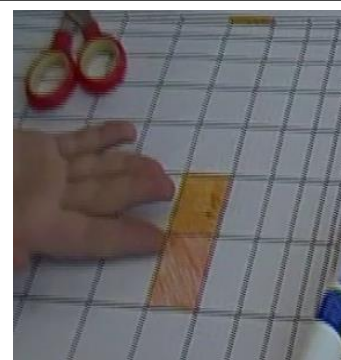


Fig. 1: A student colors the cells corresponding to  $6 \times 1$ ,  $6 \times 7$ , and  $6 \times 8$

write the corresponding multiplication on each colored cell (a request familiar for them, because they have already worked with this table).

Through the activity and the following class discussion, the students are guided to relate the gluing of rectangles to the idea of “sum of multiplications”, in order to recognize that the result of such an operation is still a multiplication: A common factor is given by the height of the rectangle, and the other factor is the sum of the factors given by bases of the rectangles in the initial multiplications. In other words, we wanted them to notice the structure of the already known distributive property.

According to the semiotic bundle approach (Arzarello, 2006), we carried out a *diachronic analysis* of the whole discussion to point out those episodes in which signs referring to different contexts (artifact and mathematical signs) are put in relation. The dialogues in these *critical events* (in the sense of Powell et al., 2003) have been transcribed. Then, we adopted a micro-analytical approach to point out the relationships between the different semiotic sets in the selected episodes (*synchronic analysis*). So, the transcript is enriched with descriptions and pictures of gestures and inscriptions. The word(s) uttered simultaneously to the *stroke phase* of gestures (according to McNeill, 1992, the central phase of the gesture, a peak between the preparation phase and the returning to the initial position) are underlined.

## DATA ANALYSIS AND RESULTS

In the beginning of the discussion, a student (we call him John) reports his particular example from the individual activity (he joined the rectangles corresponding to  $1 \times 1$  and  $1 \times 2$  obtaining  $1 \times 3$ ), but Alex shows some doubts about the fact that  $1 \times 3$  may be the result. The teacher intervenes to make Alex’s doubt explicit for all the students:

- 1 T: So, Alex tell me if I understood your doubt. You say: [...] I am not so sure about what comes out if I sum [*his hands move as if they were compressing something, Fig. 2a*] two multiplications. Isn’t it?
- 2 Alex: Yes, and an addition too.
- 3 T: Eh, then I do an addition, when I say “I sum” [*his hands seem to touch the surface of the compressed object, Fig. 2b*] I mean “to do an addition”.
- 4 Alex: Ah yes.
- 5 T: So, let’s try to consider John’s example to understand. Maybe he can help us to understand what he meant. He says: if I have  $1 \times 1$  [*he writes  $1 \times 1$  on the blackboard*] and  $1 \times 2$  [*he writes  $1 \times 2$  at a little distance*]
- 6 Fred:  $1 \times 1$  makes 1 and  $1 \times 2$  makes 2 and if you join them it makes 3.
- 7 T: Ok. This is it. [*He writes  $=1$  as vertical under  $1 \times 1$ , see Fig. 2d*] We know that  $1 \times 1$  makes 1 [*he moves his hand vertically, from the top to the bottom, pointing to the blackboard*] and [*he writes  $=$  in vertical under  $1 \times 2$* ]  $1 \times 2$  makes... [*he turns, facing the students*]
- 8 Pupils: Two!
- 9 T: Ok [*he writes 2, obtaining the inscription in Fig.2d*] then, what your peers were saying is that I can do the addition. [*moving his two hands vertically, with palms facing each other, Fig. 2c*] Is it right? [*murmurs*] How can I

signal that I have to do the addition [*repeating the same gesture close to the inscriptions on the blackboard, Fig. 2e*] between these two multiplications [*closing his hands, as grasping the inscriptions to make them closer, Fig. 2f; then he turns, Fig. 2g*]?

- 10 Mary: So...to write them?
- 11 T: Yes, how can I write it [*he points at the blackboard*]? Tell me Mary, I am listening to you.
- 12 Mary: You have to do...how is it called? You have to put a plus there.
- 13 T: Yes, I wanted to say that I would like to do this [*rotating his forefinger around the first multiplication*] plus [*he writes + just after the multiplication*] this [*rotating his forefinger around the second multiplication*] Is it ok if I write this way? [*he points to the inscription with the open hand, Fig. 2h*]
- 14 Pupils: Yes... No... Yes.
- 15 Erick: Actually it would be ok but we need brackets.
- 16 T: Ah!
- 17 Nick: Brackets!
- 18 T: Wait a minute. Erick was suggesting something. [...] Could you say me where do I have to put brackets?



Fig. 2: Teacher's gestures in the first excerpt

The teacher accompanies the reference to mathematical addition with the gesture of getting the two hands closer, as compressing something or putting together two things (lines 1-3-9, Fig. 2a-b-f). This gesture shows a catchment and appears synchronical with the words 'sum' and 'addition' (the second one is suggested by Alex, in line 2, and then used by the teacher too): It can be interpreted as a metaphoric gesture aimed to recall an intuitive model of sum as union of two sets.

In line 9, while keeping the catchment, the hands change their shape, opening vertically (Fig. 2c): They seem to indicate in iconic way the two sides of the rectangular figures. Through this iconic reference, the gesture recalls the activity with the artifact, and can therefore be interpreted as an artifact sign. This artifact sign is co-timed with the mathematical term 'addition'. Soon after, in the same utterance, the teacher faces the

blackboard, and repeats the reference to the addition (“I have to do the addition”), now accompanied with the grouping gesture pointed at the written multiplications  $1 \times 1$  and  $1 \times 2$  (Fig. 2e-f). We can notice that the word ‘addition’ (mathematical sign) is firstly used referring to the artifact context, and then with reference to the mathematical symbolism written at the blackboard, while similar gestures are performed. Gestures catchment helps to keep the reference to the activity with the artifact, allowing the teacher to link the gluing activity (also recalled in iconic way by the gesture) to the mathematical signs, given by the term ‘addition’ and by the written symbolic expressions. The matching of speech/inscriptions (mathematical signs) with the gesture (artifact sign) provides the opportunity to create a *pivot* within the multimodal semiotic chain.

In the teacher’s interventions, deictic gestures and words appear too: He points at the products, referring to them with the word ‘this’ (line 13) together with circular gestures indicating the whole multiplicative couples. The gestures suggest to consider them as whole objects. Students grasp the communicative aspect of the gesture-speech couple, and indeed they suggest to insert brackets (lines 15,17; Fig. 3e). As we know, from a mathematical perspective these brackets are not necessary. But they do have a cognitive role, serving to highlight the link with the activity with the artifact: The multiplications inside the brackets are the same that are written on the rectangular pieces of paper. In this sense, brackets act as pivot signs.

Some children would like to calculate the multiplications within the brackets and to write  $1+2=3$ , but the teacher focuses the attention on the relational level:

- 19 T: I would like to understand what happens with the multiplications [...] When we took the slip of paper  $1 \times 1$  [*placing his hand on the first bracket as grasping it, Fig.3a*] and we put it next to the slip of paper  $1 \times 2$  [*moving his hand on the second multiplication, Fig.3b*] what slip... well, these were cells [*he repeats the gesture pointing at the two multiplications in brackets*], then we pasted them [*getting his index fingers next one to another, Fig.3c*] and it comes out [*the right forefinger draws a circle on the imaginary plane created by the left hand, which is kept open and horizontal, Fig.3d*] a big single slip of paper [*repeating the same gesture*] That big slip of paper [*repeating again the same gesture*] corresponds to an operation: which one?
- 20 Pupils: One times three.
- 21 Dan: To three times one!
- 22 T: Daniel says  $3 \times 1$  [*the teacher writes  $3 \times 1$  on the blackboard, Fig.3e*]
- 23 Dan: Equals 3.
- 24 T: Do you agree?
- 25 Pupils: Yes.
- 26 John: No, it is  $1 \times 3$ .
- 27 T: John suggests  $1 \times 3$  instead.
- 28 Pupils: Yes, yes,  $1 \times 3$ .
- 29 T: Daniel, do you agree?



- 30 Dan: It is the same.  
 31 T: So, can I write  $1 \times 3$ , Daniel?  
 32 Dan: Yes, here we shouldn't put  $3 \times 1$  because we said that it is horizontal.



Fig. 3: Teacher's gestures in the second excerpt

The teacher makes different gestures. In line 19 he uses two gestures to point at the operations as written at the blackboard (Fig. 3a-b), followed by iconic gestures recalling the activity with paper and glue (Fig. 3c-d). The deictic gestures pointing to the mathematical expressions are performed while the words 'slip of paper' are uttered. Even if the focus of the discourse is on the mathematical relationships, as declared by the teacher ("I would like to understand what happens with the multiplications"), only his hand is pointing to an arithmetical operation: the whole sentence speaks about actions made with paper and glue ("we took the slip of paper...we put it next to...then we pasted them..."). The verbal artifact signs are linked to the mathematical signs on the blackboard through the pointing gesture. In other words, the deictic gestures of pointing act as *pivot signs* linking the artifact signs (spoken words) with the written operations (which are mathematical signs).

At the end of this intervention, the interpretation of the activity with the artifact in terms of mathematical meanings is made explicit through the question "That big slip of paper corresponds to an operation. Which one?". Again, words are accompanied by the repetition of a gesture (Fig. 3d) referring to the gluing of the pieces of paper.

The link between the mathematical signs on the blackboard and the activity with the artifact seems to be seized by the students. Indeed, the short exchange between John e Daniel (lines from 20 to 32) on the result of the "gluing" action ends when Daniel agrees on choosing  $1 \times 3$  because "it is horizontal". Line 32 has an hybrid nature and can work as a pivot sign: In fact, the term "horizontal" refers to the position of the rectangle within the table, 1 corresponding to the height and 3 to the base. The result of the mathematical operation is determined by Daniel with reference to the artifact (in which the rectangle is "horizontal" because the base is longer than the height): Even if the written signs introduced by the teacher are mathematical ones, the analysis with multimodal perspective shows that the reference to the artifact is constantly present and the shifting toward mathematical signs is still "under construction".

## DISCUSSION AND CONCLUSION

Considering the evolutions of signs in the classroom discussion, we could identify two different ways in which gestures function as pivot signs within the multimodal semiotic chains that link the artifact signs to the mathematical signs:

- *Pointing gestures indicating mathematical inscriptions* while speech refers to the artifact context: *gestures work as pivot signs*, because they link mathematical signs expressed in a semiotic modality to artifact signs expressed in a different modality. E.g., in line 19 gestures (Fig.3a-b) link the written arithmetical signs to the spoken artifact signs “slips of papers”.
- *Iconic gestures recalling the artifact and the related activity*, while co-timed speech contains mathematical terms, and/or are performed closely to mathematical symbolic inscriptions. E.g. in line 9, Fig. 2c: the hands refer to the vertical sides of the paper rectangles, while words refer to “do the addition”.

In this latter case, the link between artifact signs and mathematical signs is realized through their *simultaneous presence* by means of different semiotic sets in the semiotic bundle: It is the couple (gesture; word) or (gesture; symbol) which acts as a pivot, and we can speak of *multimodal pivot signs* to underline this aspect.

As described, a multimodal pivot sign can be observed when the different semiotic modalities refer to two different contexts, one referring to the artifact and the other to a mathematical domain (which is the goal of the learning activity). In gesture studies in psychology, some researchers speak about “gesture-speech mismatch” when gestures convey different information with respect to co-timed speech (e.g. Goldin-Meadow, 2003). Gesture-speech mismatches are claimed to be a useful tool in the mathematics teacher’s hands because they can convey two different pieces of information at the same time (Singer & Goldin-Meadow, 2005). In our study, we depart from cognitivist perspectives on gestures, to take a semiotic view integrating the TSM (Bartolini Bussi & Mariotti, 2008) with the multimodal analysis carried out with the lens of the Semiotic Bundle (Arzarello, 2006). From a semiotic perspective, our results seem to confirm the importance of mismatching or non-redundant gestures (Kita, 2000) in mathematical activities. On the other hand, including also other semiotic sets as the written mathematical inscriptions, we can speak more generally of mismatch *within* the semiotic bundle, but—more importantly in our view—we reinterpret mismatching cases in the light of the development of meanings from a situated context where activities with artifacts take place, to a mathematical domain, and to the teacher’s strategy to foster such development. We have empirical data supporting that gestures may be exploited by the teacher as pivot signs, in the two ways outlined in this paper. Our conviction is that this phenomenon is not limited to the teacher, but involves students as well: further research is needed to establish if this is true, and its influence on their learning.



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# HOW PRE-SERVICE TRAINING INFLUENCES FUTURE PRIMARY TEACHERS' ABILITY TO GRASP AND TEACH THE NOTIONS OF VARIATION AND COVARIATION THROUGH PATTERNS

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*In this paper we investigate the knowledge acquired by primary preservice teachers during their pre-service training with respect to teaching the notions of variation and covariation using growing patterns. The analysis is based on the anthropological approach of Chevallard, using the tools of personal and institutional relationship. Our results seem to indicate that mathematical and pedagogical activities involving growing patterns, variation and covariation are not sufficiently present in the training program of our participants and, consequently, most of them are unable to efficiently use these activities and notions.*

## INTRODUCTION AND BACKGROUND

Functions play a major role in school mathematics programs and are a part of secondary education curricula throughout the world. Students interested in pursuing scientific and technological careers must have a good grasp of functions when entering first-year university Calculus courses. However, research has identified a number of difficulties students encounter in learning functions in secondary school and at university. At the secondary level, students sometimes have difficulty identifying functional relationships when modelling situations involving the notion of rate of change (Oehrtman, Carlson & Thompson, 2008), when sketching the graph of a linear function, and when finding the equation of a given straight line (Weber & Thompson, 2014), to cite but a few examples.

Because students do not adequately learn the notion of function, research recommends, *inter alia*, that school curricula and instruction focus more on the ideas of variation and covariation to build a solid foundation for the notion of function (Hitt & González-Martín, 2015; Thompson & Carlson, *in press*). Although functions are not taught in primary school, this recommendation also applies to young children (7-8 years old); studies have shown that this age group can work with covariational relationships through activities involving growing patterns (Ellis, 2011; Walkowiak, 2014; Warren & Cooper, 2008). Such activities have long been considered effective for introducing algebraic notions, including the idea of function, to young children (Kieran, 2004; Radford, 2008).

Although numerous studies have examined children's use of growing patterns activities, little attention has been paid to preservice primary teachers (Makar & Canada, 2005). The few studies that do exist reveal many of the difficulties experienced

by primary teachers. According to Makar and Canada (2005), preservice primary teachers often lack the experience and knowledge to lead their pupils in tasks that promote the idea of function as a relation between varying quantities, using a patterning approach. These results echo more recent findings by Aljami (2015), who showed that elementary and middle school preservice teachers in Kuwait have difficulty generalising algebraic rules.

As growing patterns are now commonplace in curricula and textbooks worldwide, including in Québec (Canada), preservice teachers must be prepared to instruct their future pupils using growing patterns activities. Teachers must acquire a certain level of knowledge concerning the notions of variation and covariation in order to implement researchers' recommendations and help young children develop ideas related to the notion of function as a relationship between varying quantities. However, the scarce research performed on this topic has mostly adopted a cognitive perspective, and we have not found studies analysing these issues from an institutional point of view.

The purpose of this study is therefore to investigate the knowledge developed by preservice primary teachers during their training programs with respect to the notions of variation and covariation using growing patterns, and to see how this knowledge is influenced by institutional choices. We formulate our objective more precisely following the introduction of our theoretical framework.

## THEORETICAL FRAMEWORK

Because we are interested in drawing a line between primary preservice teachers' knowledge of activities involving variation and covariation (as well as of their didactic potential) and the institutional choices affecting their training, the anthropological theory of the didactic (ATD) (Chevallard, 1999) appears to be an adequate theoretical perspective. ATD attempts to understand the choices made by an institution in organising the teaching of mathematical notions and the consequences of these choices on learning. Any product of human activity can be modelled in terms of *praxeologies*. A *praxeology* is formed by a quadruplet  $[T/\tau/\theta/\Theta]$  consisting of a type  $T$  of task to be completed, a technique  $\tau$  which allows the task to be completed, a discourse  $\theta$ , and a theory  $\Theta$  that explains and justifies the discourse. Any institution  $I$  (such as a school or a training program) defines the positions  $p$  that individuals can occupy within it (based on the tasks they carry out), and imposes on its subjects ways of doing and thinking proper to the institution (a subject being any person  $x$  who occupies any of the possible positions  $p$  offered by  $I$ ).

At any institution concerned with mathematical notions, institutional choices direct the acquisition and development of individual knowledge (*connaissances* in French) and know-how. Thus, a dialectic exists between institutional and the personal perspectives regarding relationships with any mathematical notion or object of knowledge. To understand how an institution views a mathematical notion (or the role the notion plays within the institution), we need to define what an object  $o$  is, as well as the notions of personal and institutional relationship. An object is any entity, material or immaterial,

that exists for at least one individual; any intentional product of human activity is an object. Every subject  $x$  has a personal relationship with any object  $o$  –  $R(x, o)$  – as a product of all the interactions that  $x$  can have with the object  $o$  (using it, manipulating it, speaking of it, etc.). This personal relationship is created or modified through contact with  $o$  as it is presented in different institutions  $I$ , where  $x$  occupies a given position  $p$ . In the context of this personal relationship, a learner will develop what could be designated as knowledge, know-how, conceptions, competencies, mastery or mental images (Chevallard, 1989). As with individuals, an object  $o$  exists for an institution  $I$  if there is an institutional relationship with  $o$  in position  $p$  –  $R_I(p, o)$ . This relationship, which should ideally be that of the subjects in position  $p$  within  $I$ , roughly dictates what can be done with  $o$  within  $I$  and how  $o$  is implemented; in others words it determines the fate of  $o$  within  $I$ . By becoming a *subject* of  $I$  in position  $p$ , an individual  $x$  is subjected to the *institutional relationship*  $R_I(p, o)$ , which in turn re-models his/her own *personal relationship*. Taking these notions into account, the objective of the research presented herein is as follows: to identify those elements that characterise preservice primary teachers' *personal relationship* with the notions of variation and covariation and to relate it to their training program's *institutional relationship*.

## METHODOLOGY

Our volunteer participants were 34 preservice primary teachers (PT1 to PT34) in the last year of a four-year training program at the Université de Montréal. We first analysed the structure of their training program, which combines courses on general pedagogy and education, maths education, and school placements. The students take six courses in mathematics education (Figure 1) (18 credits out of the program's 123 credits – a credit being equivalent to 15 hours of courses and 30 hours of individual work). Note that only DID1000 is a mathematics course, whereas all the other courses are maths education courses.

Year	Course	Descriptor
1 <sup>st</sup>	DID1000	Review of basic mathematical notions in order to verify their understanding and clarify their functioning: numeracy, basic operations, fractions, proportionality and ratios, geometry in 2D, descriptive statistics.
1 <sup>st</sup>	DID4213	Problem solving. Mathematical reasoning. Modelling and simulating. Probabilities. Didactical situations.
2 <sup>nd</sup>	DID1204	The study of concepts, procedures, attitudes and mathematical reasoning applied to basic arithmetic (early learning in mathematics). Historical, epistemological and didactic background. Didactical situations, engineering and evaluation.
3 <sup>rd</sup>	DID2204	The study of concepts, procedures, attitudes and reasoning in arithmetic and statistics (subsequent learning). Historical, epistemological and didactic background. Didactical situations, engineering and evaluation.
3 <sup>rd</sup>	DID4203	The diversity of learners. The main difficulties in teaching and learning mathematics. The integration of at risk learners in regular classes. Evaluation and intervention adapted to suit the needs of diverse learners.
4 <sup>th</sup>	DID3204	The study of concepts, procedures, attitudes and reasoning in geometry. Historical, epistemological and didactic background. Euclidian and Cartesian perspective. Didactical situations, engineering and evaluation.

Figure 1. Distribution of maths education courses

In September 2015, we administered a questionnaire comprising six questions. The questions aimed to reveal elements of the participants' personal relationship with variation and covariation (knowledge, know-how, conceptions, etc.) through tasks

belonging to mathematical praxeologies as well as the praxeologies of a primary teacher's work. We also interviewed three participants to better pinpoint those elements that characterise their personal relationship with the notions of variation and covariation, and to identify which elements of this personal relationship are influenced by the institutional relationship present in their training program. We discuss herein the results of questions C, E and F (Figure 2), rounding out our analyses with excerpts from the interviews. The analysis of the questionnaire and interviews in their entirety will be the source of future papers.


Question C

Using sticks, we construct figures formed with triangles and obtain figural sequences as in both following examples.

a) How many sticks will be needed to construct the 6<sup>th</sup> figure?

b) What will be the position of a figure with 21 sticks?

c) For each growing pattern, could you give the algebraic expression that expresses the relationship between the number of sticks and the number of triangles?



Question E

Did you work this kind of activities during your training program of preservice primary teacher? If yes, within which course and for how long?

Question F

According to you, the kind of activities of question C could help prepare primary school pupils to be prepared to learn which notions of algebra at secondary? Explain.]

Figure 2. Questions C, E and F.

## DATA ANALYSIS

Figure 3 shows the distribution of answers to Question C (QC). We can see that only five participants were able to correctly respond to all the three items in QC, while another six participants gave correct answers but could only find one of the two algebraic expressions due to mistakes or omissions in their process. Moreover, eight participants did not complete the third item: two of them were only able to find the regularity (without establishing a clear covariation between the quantities), and the other six said they did not know how to answer. We can also observe that 14 participants (category “Other responses”) were unable to do more than respond to just one item. Furthermore, as only 11 participants addressed the three items of QC, we can see that 23 out of the 34 participants failed to generalise both patterns and provide an expression indicating how both quantities are related and how they vary in relation to one another. This seems to indicate that this knowledge and the ability to apply it are not part of their participants' personal relationship, in their last year of preservice training. This could be due to a lack of praxeologies that explicitly put these elements into play during their training.



Responses		Preservice teachers	Nb
Correct responses for the three items.	Correct responses for the three items a), b) and c). a) 18 and 13 sticks b) 10 <sup>th</sup> and 7 <sup>th</sup> position c) $y = 3x$ and $y = 2x + 1$	PT11, PT23, PT29, PT31, PT32	5
	Correct responses but with some errors or omissions	PT1, PT6, PT14 PT19, PT22, PT33	6
Correct responses just for item a) and b).	Presents just the regularity +3 and +2 to item c)	PT17, PT25	2
	Does not know how to answer to item c)	PT5, PT10, PT15, PT21, PT24, PT28	6
Other responses	At least one correct response	PT2, PT4, PT7, PT8, PT13, PT27, PT30, PT34	8
	One algebraic expression	PT3, PT16, PT20	3
	Wrong responses	PT9, PT12, PT26	3
"I don't know."		PT18	1

Figure 3: Responses of preservice teachers to question C.

With respect to the techniques used to tackle QC, three of the five participants (PT11, PT22, PT31) who were able to correctly respond to all three items did not write it explicitly. Most participants (18/34) used an empirical technique, writing the sequence of numbers or figures and deducing their response from it. Of these 18 participants, only four (PT1, PT14, PT23, PT29) were able to generate at least one algebraic expression. Only six participants (PT4, PT7, PT15, PT16, PT30, PT32) used an explicit technique through which they established a relationship between quantities, such as  $3 \times 6 = 18$  or  $2 \times 6 + 1 = 13$ . Of these six participants, only two (PT16, PT32) succeeded at writing the algebraic expressions. The remaining ten participants did not write any explicit technique in their attempts to solve QC. Although the explicit technique seems to be more appropriate for grasping the notions of variation and covariation, there are still some participants who use this technique and have difficulties generating algebraic expressions.

Regarding question E (QE), 19 participants responded "yes", stating that they had seen activities similar to those in QC in their training program. On the other hand, 12 participants responded "no", two gave unclear responses and one responded "I don't know". Figure 4 shows the distribution of answers to QE.

Responses	Courses		Preservice teachers	Nb
Yes	DID1000	Sure	PT4, PT11, PT18, PT23.	4
		uncertain	PT19, PT29.	2
	DID1204 & DID2204		PT6, PT16, PT25, PT27, PT32.	5
	DID4213		PT20, PT28, PT33.	3
	None precision		PT3, PT9, PT12, PT14, PT15.	5
No			PT1, PT2, PT5, PT8, PT10, PT13, PT17, PT21, PT22, PT24, PT26, PT34.	12
Other responses			PT7, PT30.	2
I don't know			PT31.	1

Figure 4: Responses preservice teachers to QE.

Of the 19 participants who responded “yes”, only four (PT11, PT23, PT29, PT32) were able to correctly respond to all the three items of QC, while four others (PT6, PT14, PT19, PT33) did so with errors or omissions; one of these 19 participants (PT18) expressed an inability to answer any item of QC. Given the inconsistencies in the participants’ responses as to whether tasks such as those presented in QC are present in their initial training, it seems unlikely that the institution includes them. And even if these tasks are taken into account by the institution, the impact on these future teachers seems very low, considering the number of participants who correctly responded to QC. Regarding the participants who responded “no”, only one (PT22) was able to generate both algebraic expressions.

When asked about the amount of time spent on activities dedicated to growing patterns in their initial training program, three participants stated having seen them in DID1000 “a little bit”, for “fifteen minutes” and “two hours”, respectively. Other responses were “probably”, “briefly”, and the comment “numerical sequences were shown in each course on didactics as an example of textbook activities, but not as a notion for teaching”. The number of contradictory responses is quite surprising and seems to indicate that the notions of variation and covariation, as well as activities concerning growing patterns, are not an important part of the institutional relationship of the training program; this could explain why students seem unable to clearly identify where it was presented. Moreover, if this content is presented, it seems that the time devoted to it does not exceed two hours (in hundreds of hours of training).

It therefore seems reasonable to conjecture that the initial training does not offer any praxeological organisation  $[T/\tau/\theta/\Theta]$  around activities involving growing patterns, including clear types of tasks, techniques to solve them and a discourse to explain the techniques. Without these tasks and the development of a discourse around them, the training could negatively affect the future teachers’ ability to implement these activities in their classrooms, as recommended by research.

Regarding question F (QF), we grouped responses according to their relevance in identifying elements related to functional relationships (Figure 5).

Very relevant	Notion of function	Variable, relationship, graph, and functions.	PT9, PT12, PT22, PT29, PT31.	5
Relevant	Notion of equation	Equivalence and the symbol « = ».	PT7, PT14, PT23, PT32.	4
	Unknown	Unknown, missing term.	PT10, PT27, PT30.	3
Other responses			PT1, PT2, PT6, PT8, PT11, PT13, PT16, PT25, PT26, PT34	10
I don't know			PT3, PT4, PT5, PT15, PT17, PT18, PT19, PT20, PT21, PT24, PT28, PT33.	12

Figure 5: Responses of preservice teachers to QF.

We see that only five participants named notions directly related to the notion of function. Only two of them (PT29, PT31) fully succeeded at solving QC, and their responses to QE were “yes” (in DID1000, PT29) and “I don’t know” (PT31).

Surprisingly, of the 11 participants who correctly responded to QC, five (PT1, PT6, PT11, PT19, PT33) were unable to link the notions involved to notions related to *early algebra* or functional thinking. It therefore appears that an ability to solve the mathematical task does not guarantee that the participant will know how to teach these notions or make clear connections with elements related to functions. Other responses related to notions of *early algebra* were given by seven participants. It is also surprising that 12 participants said they did not know how to connect the content of QC to any algebraic notion at the secondary level. Finally, it is worth noting that of the 22 participants who were unable to clearly relate the notions of QC to algebraic notions, 12 (PT3, PT4, PT6, PT11, PT15, PT16, PT18, PT19, PT20, PT25, PT28, PT33) mentioned that they had seen these notions in their training program.

The interviews helped explain the wide variety of responses and contradictions. One participant, PT29, had succeeded at QC, however she also indicated on the questionnaire that she was not sure if she had been exposed to the content in the training program (uncertain of having seen them in DID1000). When asked whether she had used content from her training program to solve QC, she responded: “No, I’d say some remnants from my secondary courses”. This could mean that these notions, while present in her personal relationship, were not derived from the training program; rather, she acquired them while at another institution (as a secondary student). When asked whether she would feel ready to use these notions in her practice as a primary teacher, she replied: “I certainly won’t feel strong enough [...] to create a learning situation based on that. I’d need to get more training, because my bachelor degree did not equip me to do that”. We are still analysing the data from the three interviews; they will be a source for future papers and help us better understand the data extracted from the questionnaires.

## FINAL REMARKS

Our data seems to indicate that the pre-service primary teachers in our sample are unable to clearly perceive two quantities that vary in relation to one another through growing patterns, nor can they identify the links between growing patterns and the notion of function. We also observed that the most popular technique was empirical, with participants needing to go term by term to respond to the task given in QC. Our analyses of the training program courses led us to conjecture that the institutional relationship with the notions of variation and covariation could be weak. The data from the questionnaires seem to confirm this, revealing a weak personal relationship with these notions among the participants.

In terms of praxeologies, the institution does not seem to give students tasks to solve activities involving growing patterns or activities to establish connections with algebraic notions and functions. This affects the participants’ personal relationship with these notions and impacts their ability to teach them. Those participants who were able to solve QC may have developed a personal relationship with these notions not as a result of their pre-service training, but because of their experience in secondary



school. Therefore, notwithstanding the official curricula's recommendation that teachers should be able to use growing patterns activities, there appears to be an institutional void that results in a lack of clear praxeologies to train future teachers to use these activities. We plan to deepen our conjecture by analysing the remaining questions of the questionnaire and cross-tabulating these data with the results of the interviews.

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# **A PRESERVICE SECONDARY SCHOOL TEACHER'S MATHEMATICS FOR TEACHING EXPONENTIAL EQUATIONS**

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*This qualitative study investigated a Malawian preservice secondary school teacher's mathematics for teaching exponential equations. The study was carried out with one preservice teacher. Data was generated using video lesson recordings and analysed using thematic analysis. The findings reveal that the preservice teacher was not able to unpack or decompress equation solving to make it accessible to students. Implications of these findings for mathematics teacher preparation are discussed.*

## **INTRODUCTION**

Teacher knowledge is important for students' learning. Empirical studies have shown that teacher knowledge influences and affects the quality of teaching and learning (Hill, Blunk, Charalambous, Lewis, Phelps, Sleep, & Ball, 2008). Studies of beginning and experienced teachers also reveal that teachers' understanding of and agility with the mathematical content affects the quality of their teaching. Thus, teachers must know the content thoroughly in order to be able to present it clearly, to make the ideas accessible to a wide variety of students, and to engage students in challenging work (Ball, Thames, & Phelps, 2008). Several researchers argue that knowing mathematics for teaching requires knowing in detail the topics and ideas that are fundamental to the school curriculum and beyond (Ball, et. al., 2008). In order to support students' learning of algebra and equation solving, teachers need to know and be able to do more than doing the mathematics for themselves. Despite this call by researchers, research reports on mathematical knowledge for teaching among Malawian preservice secondary school mathematics teachers are less available. Hence, this study examined the following question: What mathematics for teaching is displayed by a Malawian preservice secondary school mathematics teacher? The results from this study could inform preservice teacher educators about the content of mathematics teacher preparation.

## **THEORETICAL BACKGROUND**

Malawi is one of the countries which are experiencing challenges in the education system. Secondary school students' performance in national examinations continues to be poor with only around 50 percent of students passing end-of-cycle examinations (Ministry of Education Science and technology, 2008). Analyses of national examinations chief examiners' reports for years 2008 to 2013 also indicate students' poor performance in algebra in general and equations in particular. While explanations are available concerning system failure, (Ministry of Education Science and Technology, 2008), teacher knowledge is also vital in the development of students' conceptual understanding. Thus, in order to improve students' performance in

mathematics in general, teachers need to possess adequate and appropriate mathematics for teaching.

In teacher education, teaching practice is the most important aspect through which preservice teachers' performance is displayed. As such, Malawian preservice secondary school teachers practise teaching partly in their final semester and finally, for twelve weeks after their theory courses. In this paper, I describe a preservice secondary school teacher's mathematics for teaching that was revealed during teaching practice.

The theoretical framework for this study is that mathematical knowledge for teaching is situated in the practice of teaching (Ball et. al., 2008). In studying teaching, the study draws on Ball et. al. (2008) who suggest different tasks teachers need to perform and go about their work in the mathematics classroom. Kazima, Pillay and Adler (2008) condensed these into six categories. The six categories of tasks for teaching are as follows:

Defining: attempts to provide a definition

Explanations: teachers explain an idea or procedure

Questioning: teacher asks questions to move the lesson on

Representations: teachers represent ideas and in various ways

Working with learners' ideas: teachers engage with both expected and unexpected learners' mathematical ideas

Restructuring tasks: teachers change set tasks by scaling them either up or down

In this study, one preservice teacher's ability to define, explain, question, work with students' ideas and restructure tasks was investigated. I assumed a Vygotskian conception of mathematics teaching and learning as socially mediated (Vygotsky, 1978). Vygotsky believes that intellectual development is essentially a process of making meaning with others. He argues for the central role of discourse at all levels of education. Thus for teaching and learning to be meaningful for students, there should be collaboration between the teacher and the students and among students themselves.

## **METHOD**

This research falls into the qualitative design with case study approach. Merriam (2009) asserts that qualitative research focuses on process, meaning and understanding. In order to arrive at judgments regarding preservice teachers' mathematical knowledge for teaching "richly descriptive data" were needed (Merriam, 2009, p. 14). Words, therefore, were used instead of numbers to convey the outcomes of the research findings.

One male preservice secondary school mathematics teacher, Mwati (pseudonym), participated in the study. He was a Diploma in education student at a college of Education. The course program takes three years. Every year the college produces about 300 Diploma and Bachelors' Degree secondary school teachers through face to face programme and 200 diploma teachers through distance education programme.

These teachers are posted to various secondary schools in the country. Mwati was enrolled for the face to face programme. He was aged between 30 and 40. He was originally trained as a primary school teacher for two years under the Integrated Primary Teacher Education program (IPTE). The IPTE program in Malawi is run in such a way that students learn theory for one year and they go for teaching practice in the other year before they graduate. After Mwati graduated as a primary school teacher, he taught at one of the primary schools in the Malawi defence force for two years before joining the secondary school teacher education program. He was the best student in class. At the time the data was generated, there were only four mathematics education students at the college. They were all males. In the previous years, the college had stopped recruiting students for both Diploma in Education and Bachelor of Education programs because of some logistical issues.

I generated data using video lesson recordings. Using video allowed me to record the events thoroughly. It also allowed me to look at the episode multiple times for further analysis (Girden & Kabacoff, 2011). The lesson was for grade 11 (upper secondary school). It was about solving exponential equations. In the final semester of the final year of study, students at Mwati's institution are given an opportunity to practise teaching at a secondary school that is attached to the college. This teaching was, hence, part of Mwati's teacher training programme. Before beginning the data generation, I asked for permission from the principal of the concerned college, the head teacher and the mathematics teacher of the concerned class.

I analysed the data using thematic analysis. Firstly, I transcribed the video recordings in whole. Then, I analysed a transcript of a selected episode. I chose the particular episode because some elements that Mwati displayed in this episode were unusual for the Malawi context. I conducted both inductive and deductive thematic analysis (Powell, Francisco & Maher, 2003). I developed themes deductively from the theoretical framework. During the initial coding, some themes that were not in the theoretical framework were emerging from the data. Some of these themes were incorporated into the theoretical framework while others were regarded as separate categories of the characteristics of preservice teacher's mathematics for teaching exponential equations. Thus, I also developed some themes inductively. To achieve credibility of the results, another researcher analysed the data. We agreed in at least 90% of all the themes, with no discussion between the researchers. Furthermore, the findings were read and critiqued by other researchers.

## RESULTS AND DISCUSSION

### Lesson summary

The lesson consists of one event of which the object of learning is solving exponential equations by converting them to quadratic equations. The event is divided into four sub-events with a new sub-event marked by a key purpose. The first sub-event focused on a review of solving exponential equations by equating exponents. Mwati started by asking students to evaluate  $3^b \times 3^b$ ,  $5^{2a} \times 5^1$  and  $2^{x+1}$ . Students were required to use laws

of indices to work out these problems. In the second sub-event, Mwati solved the equation  $2^{2b} - 9 \times 2^b + 8 = 0$  which was discussed throughout the lesson. The equation was solved as a model example by the teacher. Mwati linked exponential equations to quadratic equations at a high level of abstraction. He also related the exponential equation to the quadratic equation  $x^2 - 9x + 8 = 0$  and continued with solving the two parallel equations. Mwati's presentation of the solution process to the equation differed from his plan which indicated that he would let  $x = 2^b$  so as to convert the equation to  $x^2 - 9x + 8 = 0$ . During his presentation of the example, the students seemed to follow how the teacher solved the two equations but they got lost in the middle and they explained to Mwati that they were confused. Despite the complaints raised by the students, Mwati gave them an exercise to solve in pairs. Sub-event 3 involved students practising the concept learnt by solving  $5^{2x} - 6(5^x) + 5 = 0$ . Few pairs solved the equation and one pair presented their work to the whole group. In sub-event 4, Mwati gave the students  $3^{2n} - 6(3^n) + 9 = 0$  and  $2^{2b+1} - 17(2^b) + 8 = 0$  to do as homework. I analysed the second sub-event and the results are discussed in the forthcoming paragraphs.

### Mathematics for teaching

The mathematics for teaching that Mwati demonstrated includes explanations, representations, questioning, working with students' ideas and restructuring tasks. There were no definitions of terms in the lesson hence definitions are not part of the analysis in this paper. The selected examples are not evaluative of the preservice teacher but help elucidate the knowledge for teaching demands and challenges faced by the preservice teacher.

### Explanations

In this study, any attempt to provide understanding of a problem to students was regarded as an explanation. During the teaching and learning process, Mwati gave several explanations to enhance students' understanding of the methods of solving exponential equations. Some explanations were comprehensible and useful for students unlike others. Firstly, Mwati explained that in an exponential equation, when the bases are the same, the powers are equal. Secondly, he displayed ability to connect a topic to prior years. He reduced an exponential equation to a quadratic one. When explaining about the use of quadratic equations to solve  $2^{2b} - 9 \times 2^b + 8 = 0$ , he stated that  $2^b$  should be regarded as the unknown. The following transcript of the second sub-event event provides explanations that Mwati gave about solving exponential equations:

- T: Let us consider the equation  $2^{2b} - 9 \times 2^b + 8 = 0$  and the term  $2^{2b}$ . Let us break it like the way we did with  $3^b \times 3^b = 3^{2b}$ . So what will it be equal to?
- Ralph: It will be  $2^{2b} = 2^b \times 2^b$ .
- T: It will be  $2^{2b} = 2^b \times 2^b$  or  $2^{2b} = (2^b)^2$ . Does that make sense?
- Students: Yes.

T: Ok. Let us look at the laws of indices.

[He brought the chart with the laws of indices that he used on first and second day.]

T: Law number 3. They are saying  $(x^a)^b = x^{ab}$ . Like in the problem we are solving  $2^{2b} = (2^b)^2$ . Therefore, the equation will be  $(2^b)^2 - 9(2^b) + 8 = 0$ . Now, from there, do you remember having done quadratic equations?

Students: Yes.

T: What we did on quadratic equations can be applied here. So we can make the equation be  $x^2 - 9x + 8 = 0$ . Have you seen that?

Students: Yes.

T: Therefore, this  $2^b$  has to be taken as unknown. Therefore, we can solve it as a quadratic equation because this  $2^b$  has been taken as unknown. Now, let us proceed working on  $(2^b)^2 - 9(2^b) + 8 = 0$ . What can we do from here? If you are given an equation like  $x^2 - 9x + 8 = 0$ . What is the next step here?

Rose: We need to factorise.

T: How can we factorise this one?

Charles: We can factorise by  $x^2 - 8x - x + 8 = 0$ .

T: Yes. Borrowing this information, can you apply it on  $(2^b)^2 - 9(2^b) + 8 = 0$ ?

George:  $(2^b)^2 - 8(2^b) - 2^b + 8 = 0$ .

T: Do we all agree?

Students: Yes (some), No (others).

T: Let us agree. I was saying we need to apply the same knowledge we know about  $x^2 - 8x - x + 8 = 0$  and apply that knowledge to  $(2^b)^2 - 9(2^b) + 8 = 0$ . Now this  $2^b$  has to be taken as unknown. Now from here, we can factorise the equation because we have  $2^{2b} = 2^b \times 2^b$ . So, we have  $2^b(2^b - 8) - 1(2^b - 8) = 0$ . Have you seen that?

Students: (Complaining, murmuring).

T: Look at  $x^2 - 8x - x + 8 = 0$ . Factor out  $x$  to have  $x(x - 8) - 1(x - 8) = 0$ .

[The teacher solved two parallel equations.]

Students: We are confused.

T: This  $(x(x - 8) - 1(x - 8) = 0)$  is more or less like  $(2^b(2^b - 8) - 1(2^b - 8) = 0)$ .

Yohave: I did not understand the second step. Where has the 9 gone?

T: At the second stage, where is the 9? [The teacher used  $x^2 - 8x - x + 8 = 0$ ].

Future:  $-8x - x$ .

T: Here in  $(2^b)^2 - 8(2^b) - 2^b + 8 = 0$ , the 9 has been substituted by  $-8(2^b) - 2^b$ . That means if we add or subtract  $-8(2^b) - 2^b$  we will get  $-9(2^b)$ . Have you seen that?

Students: Yes (some), (Others did not answer).

T: Ok. Let us proceed. Now from  $x(x-8) - 1(x-8) = 0$ , we need to factor out the bracket  $(2^b - 8)$ . This gives us  $(2^b - 8)(2^b - 1) = 0$ . This is the same as in  $x^2 - 8x - x + 8 = 0$ . In this one, we have  $(x-8)(x-1) = 0$ . So, we have either  $x-8=0$  or  $x-1=0$ . Isn't it? Now, taking the same knowledge to our exponential equation, we have either  $2^b - 8 = 0$  or  $2^b - 1 = 0$ .

Mwati continued solving the equations  $2^b - 8 = 0$  and  $2^b - 1 = 0$  until he found that either  $b = 3$  or  $b = 0$ . He seemed to have selected the methods of solving exponential equations with some insight and sophistication. However, he did not appear to be able to explain why the methods worked.

## Representations

There are five elements of knowledge of representations which have been identified by Ball et. al., (2008). These include representing ideas skilfully, mapping physical, graphical, symbolic notation, and operations, making connections among representations, recognising what is involved in using a particular representation, selecting representations for particular purposes and making and using mathematical representations effectively. As may be observed from the transcript, the representations used by Mwati were verbal and written symbols. He attempted to use questions to connect the verbal and written representations but he was not that successful because he asked low order questions. Hence, it was difficult for students to understand the method of solution of exponential equations under discussion.

## Questioning

Analysis of questions was mainly based on Bloom's taxonomy (Krathwohl, 2002). In this lesson, Mwati mostly asked factual and low order thinking questions. Other questions required 'yes' or 'no' answers. There was strong teacher guided instruction which made it difficult for students to construct their own comprehensive understanding of exponential equations. Mwati's questioning technique might reflect several issues. Firstly, it might be that emphasis was directed towards getting right answers. Secondly, it might reflect the teacher's procedural knowledge since he focused more on giving rules without reasons. It is also possible that he had not yet developed his questioning techniques. However, Mwati was a qualified primary school teacher and had taught for two years before he joined the secondary school teacher education programme. Moreover, he was a high academic achiever. As such, I would argue that he had limited ability to ask higher order questions and that he focused on procedural aspects of equation solving.

## Working with students' ideas

Working with students' ideas involves interpreting and making mathematical and pedagogical judgements about learners' questions, solutions, problems and insights, responding productively to learners' questions and assessing students' mathematical learning and deciding what to do next in class (Ball et. al., 2008). Of these, Mwati used questions to engage with students' ideas. As discussed in the preceding paragraph, the questions Mwati asked were of low order. He also mostly used procedural explanations which dominated the lessons such that students participated by responding to "yes" or "no" questions. The transcript above reveals that only one student asked a question which Mwati answered by explanation. Another important aspect that Mwati seems to lack is anticipating students' thinking. An important aspect of planning a lesson is engaging in solving the lesson problem in various ways. This enables teachers to anticipate students' thinking and the multiple ways they will devise to solve the problem. This also enables a teacher to plan the possible questions he/she may ask to stimulate thinking and deepen students' understanding. The findings, thus, reveal that Mwati displayed limited ability to work with students' ideas. For example, he ignored students' responses, questions and solutions and was not flexible enough to change the direction of the lesson to take into account the points that arose.

## Restructuring tasks

According to Ball et. al. (2008), restructuring tasks involves modifying tasks to make them easier or more difficult, making judgments about the mathematical quality of instructional materials and modifying as necessary and finding appropriate mathematical examples. When Mwati related  $(2^b)^2 - 9(2^b) + 8 = 0$  to  $x^2 - 9x + 8 = 0$ , he attempted to restructure the exponential equation to a quadratic form that students were familiar with. He gave explanations while he solved the two parallel equations but students could not understand. One of the statements by the teacher indicates that he realised that students did not understand but seemed not to have a remedy to the problem hence he forced them to write an exercise. Thus, he was not flexible enough to make the solution process for  $(2^b)^2 - 9(2^b) + 8 = 0$  to be transparent for the students to generalise from it. It could thus, be expected that he had limited knowledge necessary for teaching exponential equations.

The findings in this study illuminate the results of a study conducted by Adler and Davis (2006) who found out that compression or abbreviation in contrast to unpacking of mathematical ideas was dominant among teachers. Similar results were also found from a study conducted by Huang (2012) in the USA and China. Huang found out that participants had relatively limited knowledge of algebra for teaching.

## CONCLUSION

In this study, Mwati's mathematics for teaching was investigated. Findings reveal that Mwati selected the method of solving exponential equations with some insight and sophistication. However, he presented the method to students at a high level of



abstraction. This made him unable to unpack or decompress equation solving to make it accessible to students. Since this difficulty could be observed in a participant who was particularly good in class, it would indicate that his mathematics teacher education programme does not focus on mathematics for teaching. This might imply that teacher education needs to be adjusted to focus more on mathematics for teaching. With sufficient mathematics for teaching, teachers will be able to unpack or decompress mathematical ideas so that they are accessible to students. Teachers will also be able to interpret students' thinking, identify misconceptions, provide tasks and pose questions that will guide students' interpretations of mathematics. The teachers will also be able to find appropriate strategies for inducing cognitive conflict that will help students to deconstruct their naïve theories and reconstruct correct mathematical conceptions.

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# MATHEMATICAL COMPETENCES: STRUGGLING FOR A DEFINITION

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*The notion of competence is becoming more and more a central notion in education: it is used by policy makers with the aim of triggering and informing curriculum reforms. Therefore, the teachers are asked to design and carry on educational activities and to assess students focusing on the development of competences. We raise the problem of identification and definition of mathematical competence. In this developmental paper we show our reflections on the complexity of the idea of competence taking into account how it is presented in the European institutional documents and in some educational studies that deal with this topic.*

## INTRODUCTION

Over the last years, competence has become a central, debated, notion in the discourse around education and training promoted in Europe by the so-called Lisbon strategy. Political institutions (namely the European Parliament and Council) but also economical organizations (such as OECD) defend the adoption of the term, and the perspective it conveys, in order to highlight the importance of a holistic approach to teaching and learning not bound to specific discipline content knowledge. In consequence of such an approach, teachers have to cope with new professional challenges concerning the design of educational activities and assessment of students' learning by focusing on competence development.

Assuming and enacting competence-based perspective on teaching and learning appears as a challenging enterprise even because the notion of competence is not clear at all. In fact, the idea of competence is one of the most elusive in education (Kilpatrick, 2014). We contend that in order to cope effectively with those challenges the notion of competence needs to be clarified and elaborated more in depth in education and in mathematics education research, specifically.

This paper intends to trigger an explicit reflection on the idea of competence, aiming to unfold its complexity. To this end, we are going to examine, though in a partial way, the notion of competence from different perspectives: the notion of competence in the Institutional Context, in Pedagogy and in Mathematics Education. In fact this notion can be considered a “boundary object” (Cobb, McClain, Lamberg & Dean, 2003; Star and Griesemer, 1989) laying at the intersection of different fields. We are convinced that examining how the notion is conceptualised within these fields is a first step towards unfolding its meaning and complex structure.

The next sections present the conceptualisation of the notion of competence that emerges from the European institutional context, from the pedagogy research literature and from mathematics education research literature. As for mathematics education, we will dwell on the framework provided within KOM project (Niss, 2003; Niss & Højgaard, 2011), being it specifically developed for characterizing mathematical competence, and in consideration of its wide diffusion and influence.

In the final discussion we will try to draw some tentative and provisional conclusions on the complex structure of the notion of competence and to point out possible implications for mathematics education research on this theme; in particular, we will raise the question whether and how we can use the analytical theoretical tools developed in mathematics education research in order to frame and analyse the study of the development of mathematical competence in the classroom.

## THE NOTION OF COMPETENCE IN THE INSTITUTIONAL CONTEXT

The notion of competence has a decades-long history (see next section). However, in Europe it has come to the attention of those who operate in schools, of teachers in particular, and of the public opinion at large, in consequence of the resolutions concerning education and life-long learning adopted in the last years by the European Parliament within the so-called Lisbon strategy. In particular, the Recommendation 2006/962/EC of the European Parliament and of the Council of 18 December 2006 on key competences for lifelong learning [Official Journal L 394 of 30.12.2006] provides a framework of reference for the *key competences for lifelong learning*, in relation to which each State member is recommended to develop and articulate its own instruction system. This document defines a set of eight *key competences for lifelong learning*: a combination of knowledge, skills and attitudes appropriate to the context. They are particularly necessary for personal fulfilment and development, social inclusion, active citizenship and employment. *Mathematical competence and basic competences in science and technology* is one of them.

Mathematical competence is the ability to develop and apply mathematical thinking in order to solve a range of problems in everyday situations, with the emphasis being placed on process, activity and knowledge.

Some elements of this definition appear particularly interesting for the on-going discussion. The focus is on the use of mathematics and mathematical forms of thinking for solving problems arising from everyday situations – thus reflecting an increasing general trend – while the solution of intra-mathematical problems remains in the shadow. The emphasis is on thinking processes, the importance of which has been valued since many years by mathematics education researchers, but is sometimes overlooked in school practice. Mathematical competence is described in general unifying terms. By this, we mean that it is described independently from the specific fields in which mathematics as a discipline can be articulated (arithmetic, algebra, geometry...), from the possible fields of human experience in which mathematics can

be used, and from the fact that the used mathematics could be labelled as basic or advanced mathematics.

As mentioned above, this document is intended to provide a reference framework, which each State member is recommended to adopt to reform its own educational system. Actually, besides the declarations and intentions of the European Parliament, there can be a wide gap between the notion of competence as depicted in the framework and its enactment in national curricula. With this respect, the case of Italy is exemplar. Since 2007, the Italian Ministry of Education released four different documents setting the learning goals for the educational national system at different school levels. For instance the documents for the primary and middle school define the learning goals (at the end of grade 5 and of grade 8) in terms of competences, while for secondary high school (grade 13 of high schools) they describe specific learning goals linked to the different mathematical topics. Therefore, though assuming the European framework, these documents propose articulations of the notion of mathematical competence strongly inhomogeneous both between different school levels and at the same level between high schools and vocational and technical schools (Magenes & Maracci, in press).

Against such a composite background, teachers have to cope with new professional challenges concerning the design of educational activities and assessment of students' learning by focusing on competence development.

## **THE NOTION OF COMPETENCE IN PEDAGOGY**

The idea of competence begun to appear and be explored in pedagogy around the sixties and seventies, even if the origin of the idea can be traced back to Greek philosophy (Mulder, Weigel & Collins, 2006). Since its appearance, the idea of has undergone the influence of diverse learning theories. Weinert (2001) distinguishes in the existing literature nine diverse theoretical approaches towards competence (quoted in Mulder, Weigel & Collins, 2006). This suffices to have a flavour of the complexity of this notion that can only be far less than adequately presented in this section.

At the very beginning, the idea of competence was identified, under the influence of behaviourism, with the idea of performance that is a finished, observable and measurable behaviour (Pellerey, 2004). Over the years, the advancements in research on problem-solving and comparative studies on experts' and novices' performances led to distinguish between the two. Competence became thought of as the abstract capability held by an individual to accomplish a (possibly complex) task, while performance went to be considered as the possible manifestation of a competence (Bara, 1999). At the same, it was recognised that the capability of accomplishing a task does not depend only on the set of knowledge and skills that the individual might possess or not, but relies on a number of not directly observable factors. Drawing on the analysis of Mulder, Weigel and Collins (2006), Marzano and Iannotta (2015) suggest that the development of the notion of competence can be summarised in three main directions of evolution.

- *From the simple to the complex.* The notion of competence has been expanded to encompass the cognitive, the motivational and the emotional dimension.
- *From outside to inside.* Attention is drawn to subjective dimensions which are not directly observable from the outside, but that form the basis of individual behaviour.
- *From theoretical to pragmatic.* Competence is related to the individual's ability to use operational strategies for the solution of problems related to specific culture and contextual dimensions.

The definition of Pellerey (2004) seems to condense effectively this evolution:

“competence is the capability to cope with a task or set of tasks, being able to start and orchestrate one's own internal resources – cognitive, affective and volitional – and use external available ones in a consistent and fruitful way” (p. 12, our translation).

Hence, according to such an approach, the competence of an individual is not directly accessible.

Several authors use the metaphor of the iceberg in order to convey the idea that, besides what is observable, competence entails several not visible dimensions (Fig.1).

An important consequence of this conceptualization is that when we observe performances we only observe the (lack of) manifestation of a competence; the presence or the absence of a competence cannot be directly measured but only, possibly, inferred. The assessment of competences is based on inferential processes and cannot be based on automatic mechanical checks.



Figure 1: Adaptation of Castoldi's iceberg-model of competence (2006).

## THE NOTION OF MATHEMATICAL COMPETENCE IN MATHEMATICS EDUCATION

The elusiveness of the notion of competence is reflected also in mathematics education literature. Often the term competence is used without being defined: its meaning being assumed by authors as self-evident, intuitive or anyway shared by readers; or alternatively, it is defined making use of a host of near synonyms (e.g. mastery, proficiency, skill, ability, capability, etc.).

Kilpatrick (2014) provides a survey of the main frameworks for the notion of competence used in mathematics education, some of them are specifically developed for mathematics, and others are adapted from more general frameworks. These frameworks share the common feature of being purposefully designed to demonstrate

that “learning mathematics is more than acquiring an array of facts and that doing mathematics is more than carrying out well-rehearsed procedures” (p. 87).

One of the most recent frameworks specifically designed for mathematics was developed within the KOM project (Niss, 2003; Niss & Højgaard, 2011).

### The KOM project competence framework

The KOM project is promoted by the Danish Ministry of Education between 2000 and 2002 with the aim of identifying critical elements of the Danish educational system and developing suitable tools to address them. The heterogeneity with which mathematics is viewed at different school levels is one of main critical issues identified: *“mathematics is perceived and treated so differently at the different levels that one can hardly speak of the same subject”* (Niss 2003, p.3). Mathematical competence is assumed as an intended unifying notion, a tool to address the heterogeneity issue: that is to set the expected outcomes for school mathematics and frame the description of curricula and of students' learning progresses throughout the whole school system.

“Mathematical competence [then] means the ability to understand, judge, do and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role.” (Niss, 2003, p. 7).

The notion of competence is then articulated in “recognisable and distinct, major” components, named competencies: thinking mathematically, posing and solving mathematical problems, modelling mathematically, reasoning mathematically, representing mathematical entities, handling mathematical symbols and formalisms, communicating in, with, and about mathematics, and making use of aids and tools.

Each of these components has a double nature: it consists of analytical and productive aspects. The former involves understanding and evaluating the mathematical activities developed by others. The latter involves the direct accomplishment of mathematical activities. The eight components are closely related, but are independent from each other: they focus on different dimension of mathematics activity and none can be completely described in terms of the others. Niss and Højgaard (2011) propose to represent them as partly overlapping petals of a sole flower.

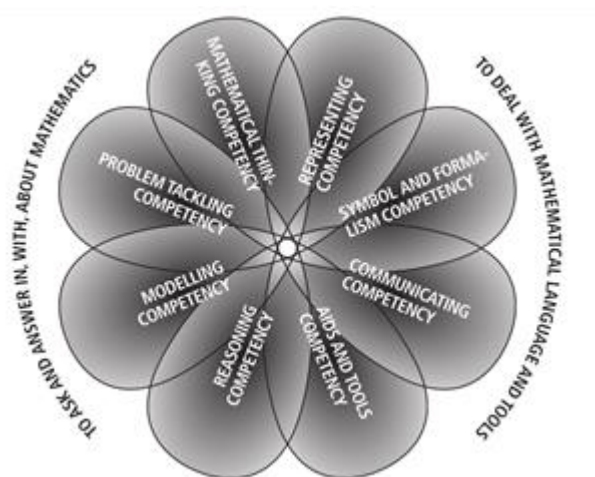


Figure 2: the flower of competence  
(Niss & Højgaard, 2011).

Summing up, the KOM project provides a description of mathematical competence which is in tune with those emerging from the institutional context and from the pedagogy literature. Moreover it aims at capturing, besides the general terms used, the

specificity of mathematics through highlighting the relevant aspects of mathematical activity. In addition, mathematical competence is described in terms of components which can characterise mathematical activity in any fields and at any school level. Mathematical activity is characterised by the synergetic enactment of all these components, and, at the same time, the development of mathematical competence requires the synergetic development of these components.

However, with respect to the approaches we examined in the previous section, we can notice that the KOM framework do not consider – at least not explicitly – metacognitive abilities and affective and volitional resources. The submerged part of the iceberg is apparently overlooked, even if these aspects are investigated in the educational studies in mathematics: the metacognitive skill, the system of beliefs and the attitude are acknowledged as fundamental in problem solving (Schoenfeld, 1992) and in all mathematical activities (Di Martino & Zan, 2014).

## **DISCUSSION**

The definitions of competence emerging from the institutional context, and from pedagogy and mathematics education literature are in tune to the extent to which they are all related to the individuals' capabilities of solving problems in a given context. Both in the approaches based on an iceberg-like view of competence and in the approach pursued within the KOM project (we could term it the flower approach), (mathematical) competence is conceptualized as a dynamic complex unit composed of major interacting components. But below this surface similarity, the two approaches show deep differences. The basic interacting components which are assumed to form a competence are essentially different in nature; the resulting definitions cannot be re-conducted one to the other. Hence, if one recognises, as we do, that both the approaches capture crucial aspects of the idea of competence that cannot be overlooked, then she/he is led to describe mathematical competence as complex integrated system. That raises, at least in principle, important interrelated methodological issues.

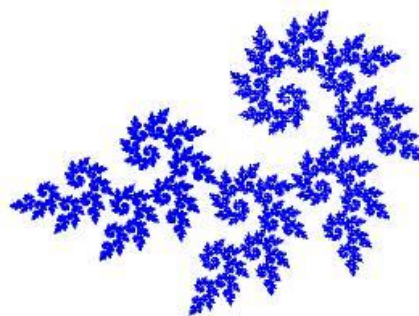
The first one concerns the choice of the unit of analysis in educational research on mathematical competence. The emerging notion of mathematical competence consists of a complex integrated system. When addressing research issues involving this notion, keeping such a complex system as a unit of analysis can be an overwhelming effort. At the same time, decomposing the idea of mathematical competence in smaller units can lead to fragmentary – if not misleading – results hard to recompose in a unitary whole. In such respect, Vygotsky's discussion on the importance of unit of analysis in psychological research is illuminating (1934/1986).

Intimately related with the previous one, another issue concerns the possibility of capitalizing on the existing research results in mathematics education. In fact, over the years, mathematics education research has focused on many of the components highlighted in the two approaches, e.g.: metacognitive abilities, attitudes, beliefs system, on the one hand; problem-solving processes, argumentation, representation and so on, on the other one. However, more often than not, these components have



been investigated as separate entities. Now, new pressing questions and needs are emerging about competence-centred approaches to mathematics teaching and learning: is it possible to establish a fruitful synergy among the advancements separately achieved in order to develop an effective synthesis and to provide theoretically grounded unified answers to these new questions? What networking strategies (Prediger et al. 2008) can be pursued to achieve such a synthesis?

A last issue, we wish to raise, concerns the “domain of a competence”. Since the notion of competence is explicitly related to the accomplishment of a set of tasks, one could refer competence to any domain of human culture or activity which is recognized and considered as unitary and homogeneous (such a domain could be perhaps referred to as a “field of experience” after Boero and colleagues, 1995). Hence one can speak of competences referring to any mathematical sector perceived as unitary and homogeneous, possibly not corresponding to the traditional organization of mathematics in subfields. How does the structure of competences referred to delimited fields of mathematical activity look like? Since mathematical problem solving always entails the activation of cognitive, affective and volitional resources, and is characterised by the synergetic enactment of the major components identified by Niss, we must conclude that the structure of competences referred to an even delimited mathematical field still consists in a complex integrated system of dynamic units composed by major interacting components, no matter how much delimited the field considered is; the structure of competence is identical at all scales. Pushing to extremes this idea we could say that mathematical competence has a *fractal* structure.



In other terms, the complexity of the notion competence in mathematical activities cannot be reduced through reducing the field of human activity at stake.

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# TEACHING FOR UNDERSTANDING: FOLDING BACK AND WORKING WITH PRIOR KNOWLEDGE

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*This paper considers ways in which teachers can encourage students to build on their prior knowledge when building an understanding for a new mathematical concept. Through drawing on the Pirie-Kieren theory and the associated notion of folding back we identify a number of characteristics of pedagogical actions that, we suggest, can help students to access, work with, and modify their existing understandings within a new context and support the emergence of more powerful and general understandings.*

## PRIOR KNOWLEDGE AND EXISTING UNDERSTANDINGS

The importance of prior knowledge in the process of learning mathematics is widely acknowledged (see Campbell & Campbell, 2009; Hatano, 1996; Mack, 2001). The NCTM ‘Learning Principle’ (NCTM, 2000), that focuses on the importance of conceptual understanding, states “students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge” (p. 20), and Krainer (2004) notes that “taking students’ prior knowledge into account is a prominent goal in mathematics education” (p. 87). More specifically An, Kulm, & Wu (2004) state that “using prior knowledge not only helps students to review and reinforce the knowledge being taught, but also helps them to picture mathematics as an integrated whole rather than as separate knowledge” (p. 165). However, there remains limited research on what “using prior knowledge” (or, as also described in the literature, accessing; building on; invoking; relating to; connecting; reviewing; reinforcing; solidifying; integrating; activating; tapping into, etc.) might involve and on the potential role of the teacher, and his or her associated pedagogical practices, in enabling this kind of process. As Mack (2001) asks, “how might students actually return to their initial understandings, and in what ways may this return influence the development of their understanding of the domain?” (p. 269). She also notes that, “students may not always return to their initial understandings on their own even when this return would be beneficial. Students may require guidance that leads them to reconsider these understandings at opportune times for growth to occur” (p. 269).

In this paper, we draw on the Pirie-Kieren Theory for the Dynamical Growth of Mathematical Understanding (Pirie & Kieren, 1991, 1992, 1994) and the associated theoretical metaphor of ‘folding back’ (Martin, 2008), and consider how folding back might offer a way for teachers both to think about the place of prior knowledge in the teaching of new concepts and to explicitly create opportunities for learners to engage with their existing understandings in new mathematical contexts. We illustrate these claims through considering data extracts taken from the classroom of Mort, a high

school teacher, as he works with a Grade 12 class on the concept of vectors in three-dimensional space.

### **FOLDING BACK AND GROWING UNDERSTANDING**

The Pirie-Kieren theory posits eight nested layers of understanding (illustrated as a diagrammatic representation or model in Figure 1) to describe ways in which a learner can be observed to act mathematically, and characterises the growth of mathematical understanding as emerging through the continual movement back and forth through the layers of knowing, as individuals reflect on and reconstruct their current understandings. The nesting of the layers illustrates the fact that growth in understanding need be neither linear nor mono-directional. In addition, each layer contains all previous layers and is included in all subsequent layers, to emphasize the embedded nature of mathematical understanding. Using the model, the growth of understanding of a learner, for a particular mathematical concept, can be mapped out. Figure 1 illustrates a diagrammatic representation of a hypothetical pathway of the plotted growth of understanding of a particular mathematical concept, for a learner or group of learners, over a particular period of time.

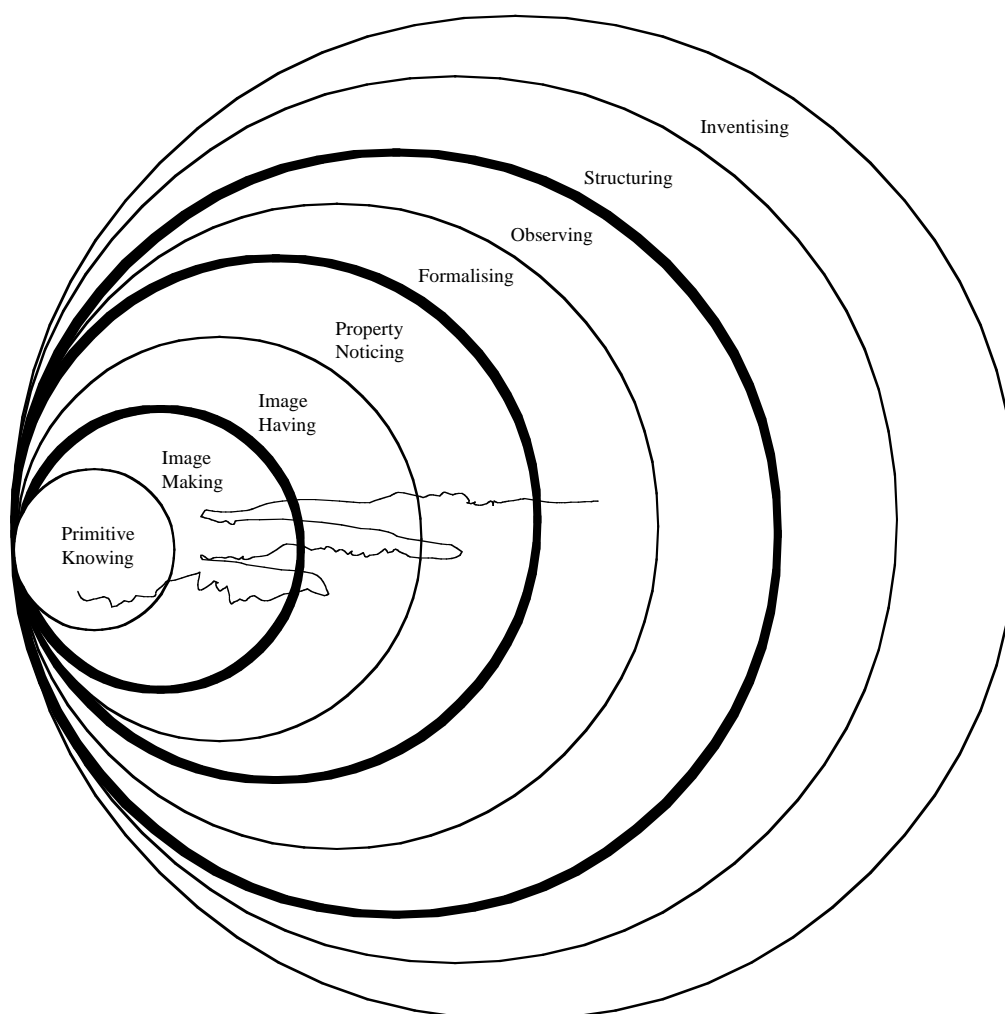


Figure 1. The Pirie-Kieren Model for the Dynamical Growth of Mathematical Understanding.

The definitions of the various layers of understanding actions have been fully set out in earlier work by Pirie and Kieren (Pirie & Kieren, 1992, 1994) and thus here only brief definitions of the layers relevant to the data analysed in this paper are offered. Primitive Knowing is seen as the starting place for the growth of understanding of any piece of mathematics. Primitive Knowing is observed to be everything that a learner knows (and can do) except the knowledge about the particular concept that is being considered by the observer. At Image Making the learner is engaging in activities aimed at helping him or her to developing particular representations for the topic and mathematical idea; to get an idea of what the concept is about. These images need not only be visual or pictorial in nature, and they can be ideas expressed in language or in action. By the Image Having stage the learner is no longer tied to an activity, he or she is now able to carry a mental plan for these activities with them and use it accordingly. This frees the mathematical activity of the learner from the need for particular actions or examples.

A key feature of the theory is the idea that a person functioning at an outer layer of understanding, when faced with a problem or a new context, needs to return to an inner layer of understanding to examine and modify their existing ideas and prior understandings and knowings related to the concept. This process, known as ‘folding back’ (Martin, 2008), implies that when a learner revisits existing understandings he or she carries with them the demands of the new situation and uses these to inform their new thinking at the inner layer, leading to what may be termed a ‘thicker’ understanding for the concept. It is the fact that the outer layer understandings are available to support and inform the inner layer actions that gives rise to the metaphor of *folding* and *thickening*. Folding back can be visualised as the folding of a sheet of paper in which a thicker piece is created through the action of folding one part of the sheet onto the other. Martin (2008) introduced a number of different forms of folding back which relate to the kind of cognitive actions the learner actually engages in at an inner layer. Of particular relevance to the analysis in this paper is the notion of collecting at an inner layer (Pirie & Martin, 2000). Collecting at an inner layer (folding back to collect is also an appropriate descriptive term) entails retrieving previous knowledge for a specific purpose and re-viewing or ‘reading it anew’ in light of the needs of current mathematical actions. Thus collecting is not simply an act of recall; it has the ‘thickening’ effect of folding back. In the data presented here we consider the specific situation of folding back to collect from the Primitive Knowing layer, in order to use this in Image Making actions for the new concept of vectors in three-dimensional space.

## METHODOLOGY AND METHODS

The larger study, from which this paper offers some initial findings, introduces teachers to the conceptual idea of folding back and, through video, traces its use as a pedagogical strategy in their lessons. The research design draws on elements of a learning study methodology (Lo, Marton, Pang, & Pong, 2004; Pang & Merton, 2003). The data collection had three main elements: a planning phase where the lessons were

designed using folding back as a theoretical tool; an implementation phase where the lesson sequence was taught; and an analysis phase focusing on the place of folding back in the lessons. The data set included video recordings and notes from the lesson planning and post-cycle meetings with the teachers, and video recordings (and associated copies of written work) of the whole-class and of smaller groups of students working in the research lessons.

The participating teacher on whom we focus in this paper, Mort, is an experienced teacher who, at the time of the study, was teaching in a private high school. A few years prior he had completed a higher degree, including taking a mathematics education class with the first author. He was also an active member of an informal research group involving the first author. Thus, prior to his participation in this project, he had a strong background in mathematics education research and had read and engaged with papers relating to folding back. As a part of the project he participated in a planning meeting with the research team, where potential lesson structures and plans were considered with a particular focus on where students might need to, or be encouraged to, fold back. Mort then taught two designed lessons, with two different classes, and these were observed and video-recorded by two members of the research team. The data were analysed through the repeated viewing of the videos and the creating of an increasingly detailed set of analytic notes. This process followed that outlined by Powell, Francisco and Maher (2003) and included identifying phases in the lessons where Mort was deliberately encouraging folding back and considering his actions through the lens of the Pirie-Kieren Theory.

## **DATA AND ANALYSIS**

We join Mort as he starts to introduce the content of the lesson, working from an interactive whiteboard at the front of the classroom. He begins by explicitly telling the students that he wants them to focus on making a connection between today's work and what they did in grade nine. However, he also sets this within the new context of three-dimensional space (R3), setting out clearly that the purpose of what they will be doing is to develop three forms of an equation of a line (referring here to vector, parametric and symmetric forms). He says "I want to take you all the way back to grade nine and I want to show you an equation of this form  $y = \frac{2}{3}x + 3$ " and asks "what form is that in?" Mort then works with the class in reviewing the  $y = mx + b$  equation form in two-dimensional space (R2). In terms of the Pirie-Kieren theory then, the students are positioned, by Mort, to work at the Image Making layer and to make an image for the equation(s) of a line in R3. He does not, however, simply start by talking about vectors or even R3. Instead, in order to help their Image Making, and to build a connected understanding to what they know about lines in R2, he explicitly prepares them to fold back to their existing understandings for this, their Primitive Knowing. We see the students working with Mort in revisiting and collecting existing understandings related to the equation of a line in R2. These include the  $y = mx + b$  equation form and, more significantly for what will follow, the meaning of the different

elements — in particular the notion that  $m$  represents slope. He is not yet working with these existing understandings in R3, but collecting them so as to be available to be read anew within this new context.

Mort asks “what does the slope tell me?”, follows up on a student’s offering of “direction”, and begins to explore the idea of steepness and then the need for a point to actually locate “where the line actually is”. He consolidates this to summarize that slope gives a sense of direction and the point defines the location of the line. He follows this with a very explicit statement that, having now reviewed all that they know about lines and their equations in R2, he would “love to” simply “throw a  $z$  into the equation” and hence have the equation in R3. However, Mort immediately points out that “it doesn’t quite work like that, and the reason is because one of the key concepts that we depend on falls apart in R3” asking “why does the idea of slope the way that we’ve defined it, the way that we understand it, not work for what we want to do in R3?” We see a student offering an answer of the same slope having different directions in R3, and providing a visual explanation of this using a pencil (holding it in space and rotating keeping it at the same angle to the ground) – something that Mort mimics with a metre stick so that the whole class can see. Mort continues to encourage the students to collect various prior understandings related to the equation of a line in R2. However, this is not simply a process of revisiting these existing understandings but of also considering which of these will be useful in R3 and in the developing of the vector equation at a later stage. The introduction of the idea of location of the line prepares Mort to return the students, albeit briefly, to Image Making in order to explicitly problematize this idea in R3. He points out that their existing understandings for the equation of a line in R2 cannot simply be transferred and used as an image in R3, particularly the idea of slope. However, he does not simply tell the students this and move on, but instead wants to explore why the idea of slope is no longer valid. Here, he is asking students to read anew, and to thicken the understandings they have collected, through thinking about them in their Image Making in R3.

Mort continues to probe other ideas the class may have, asking, “what other ways do you know that slope falls apart in R3?” and we see another student offer a complementary answer; one that draws on the way slope in R2 is commonly defined as ‘rise over run’. The student points out that this definition doesn’t work when you have an “out part” (referring to Mort’s visual modelling of the stick facing away from the board and into the room). Mort takes this up and explicitly revisits the equation for slope of a line in R2, to highlight that there is no third dimension in the equation (he asks “anywhere in there, did we say  $z$ ?”) and that slope remains a “planar concept”. Mort then poses the question of what other concept could be used instead of slope to describe and define the direction of a line, and he receives the answer of a vector. Following this Mort continues to work with the students in R2, developing the idea of needing an initial vector and a direction vector, before later extending this to R3 and to the general equation of a line in space. Here, Mort again makes clear the limitations of their existing understandings (in this case the equation for the slope of a line in R2)

and works with the class to see why this is problematic. Mort poses the questions of what to use instead of slope and he receives the answer that he knows will now allow a connected understanding to continue to grow for the class, a vector. With this idea having been offered by a student Mort is now ready to thicken the students' Primitive Knowing, which only defined direction as slope by instead defining it through the use of a vector. Once this idea is developed he then returns the class to R3 and is able to work with them again at the Image Making layer to develop the more general image of a line being defined through a vector equation.

## **FOLDING BACK AND TEACHING FOR UNDERSTANDING**

As noted earlier, while existing research notes the importance of teachers building on the prior knowledge of their students, there is little detail offered of what this might actually mean or involve. We suggest that the Pirie-Kieren theory and the notion of folding back, with its explicit consideration not only of the importance of earlier understandings but also of the need to actively collect, work on, read anew, and thicken these, in the kinds of ways illustrated by the teaching actions of Mort, offers a useful theoretical and practical way to more fully detail how prior knowledge can be accessed and built on in the mathematics classroom.

Mort, through his engagement with the Pirie-Kieren Theory, is sensitized to be aware of the existing understandings students may have and of how these need to be examined, modified and thickened within a new mathematical context. Through this theorising of his teaching actions he explicitly encourages and requires students to fold back and revisit their existing understandings that might help or hinder their learning and he is prepared to take time to allow folding back, collecting, and thickening to occur. He does not ignore those ideas that may be problematic in the new context nor simply point them out and then move on. Instead, he wants students to be aware of the potential limitations of their existing understandings, to spend time with them, and also to understand why such limitations exist. His approach to this is not a token gesture, for example, he does not simply tell the class that  $y = mx + b$  is not useful in R3, but takes time to explore the notion of slope and how and why it breaks down in the new context—deliberately allowing students to realize that their prior knowledge is problematic—thus providing a space for engaging with the more generalizable vector equation of a line.

This example also highlights that engaging meaningfully with and working on existing understandings and prior knowing is not a trivial process, but one that requires mathematical and pedagogical knowledge, as well a willingness to appreciate that such an approach will require classroom time (a concern noted by Mort) and thoughtful lesson planning. Mason (2015) talks of the need for teachers to undertake an a priori epistemological analysis of a mathematical topic prior to teaching it, suggesting that such an analysis can enable the overcoming of both epistemological and pedagogical obstacles (that is, those that arise from prior learning and teaching rather than the intrinsic complexity of the concept). Through the preparation for the teaching of this

lesson sequence, Mort engaged in such an analysis using the lens of the Pirie-Kieren Theory. This involved considering the prior understandings and primitive knowings the students would bring to the lessons, and how, in Image Making for the equation of a line in R3, they might need to engage with and modify these. In particular, Mort identified problematic prior knowings that would not be generalizable to the new context and planned points in his lessons where such potential obstacles to a connected understanding would be specifically exposed and addressed through deliberating encouraging acts of folding back and collecting and providing space for the necessary thickening (see Sabeti & Martin, 2014).

## CONCLUDING COMMENTS

Simon (2013) argues for the need to develop pedagogical theory that can inform instruction, calling for integrated theories of learning **and** teaching that are made up “of a theory of mathematics conceptual learning and a theory of instruction that builds on and is integrated with the theory of learning” (p. 95). In this paper we have advanced the use of a theoretical frame, initially conceived of as a means to focus on learning, positioning it to also be an integrated pedagogical tool. The Pirie-Kieren theory, and the notion of folding back, offers teachers a means to consider and identify specific existing understandings for a particular topic and to then design pedagogical actions that facilitate the engaging with and potential thickening of these within a new context, to enable a connected, deeper understanding. Although not the explicit focus of this paper, our research also suggests that for teachers to be able to engage with pedagogical theory in their own practice there is a need for substantial professional development. Mort had an in-depth knowledge of the Pirie-Kieren theory and the notion of folding back and it was this that enabled him to meaningfully engage in the kind of epistemological and pedagogical analysis suggested by Mason (2015). Such an analysis addresses the calls by Mack (2001) and Krainer (2004) for considering students’ growth of understanding as being central to pedagogical actions and, we suggest, has the potential to help teachers to enable students to build understandings that are connected, flexible, and integrated.

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# UNIVERSITY STUDENTS' DAILY EXPERIENCED EMOTIONS IN MATHEMATICS CLASSROOM

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*This research have two objectives: (1) identification of individual emotional experiences of students in their first year of university reported for several days in mathematics classes, and (2) identification of the antecedents of those individual emotional experiences. We apply a diary method to collect data. We managed to determine the appraisal structure of the students (i.e. the antecedents of their emotions) that support the emotional experiences reported by 15 students for seven days in their mathematics classes. The main achievement of this research is that all emotional experiences are consequence of the appraisal of situations that enable or avoid the accomplishment of four specific goals: 'to learn in each class', 'to solve exercises correctly in each class', 'to pass tests' and, implicitly, 'to pass the course'.*

## INTRODUCTION

Most of the research on students' emotions in the field of Mathematics Education focuses on their role in mathematical *problem-solving* (e.g. Goldin, 2000) and *mathematics anxiety*. Research also focuses, to a lesser extent, on students' emotions in *mathematical engagement* (e.g. Goldin, Epstein, Schorr, & Warner, 2011) and in the *emotions in mathematics classroom* (e.g. Lewis, 2013; Martínez-Sierra & García-González, 2015).

A group of investigation has paid attention to the emotional experiences of students in mathematics classes and the antecedents of those emotions (Martínez-Sierra & García González, 2014; Martínez-Sierra & García-González, 2015; Martínez-Sierra & García-González, 2016). This group has used the cognitive structure of emotions and the principles of appraisal theories to analyze the emotional experiences reported by focus groups. They found that the antecedents of emotional experiences are supported by a set of goals that is structured depending on the specific context of the students. Therefore, all the emotional experiences of the students reported are product of the appraisal of situations that enable or are proof of the achievement of goals (triggers positive emotions as satisfaction or pride) or prevent the scope of goals (triggers negative emotions as disappointment or self-reproach). All together these investigations show a common global appraisal structure in the different groups of participants. However, they were not able to identify the individual appraisal structure of every participant. Daily emotions of students in mathematics class and individual antecedents of those emotions are topics slightly studied. It is important to analyze the variability of emotional experiences among students because it can help to predict the

behaviour of the students and an increase of the variability of the emotional experiences can inform researches and teachers on the ability to adjustment of every student to the changing demands at class (Ahmed, Werf, Minnaert, & Kuyper, 2010). This research aims to start filling this gap by following these objectives: (1) Identification of individual emotional experiences of students in their first year of university –freshman students- reported for several days in mathematics classes, and (2) Identification of the antecedents of those individual emotional experiences.

## THEORY OF COGNITIVE STRUCTURE OF EMOTIONS

The *theory of the cognitive structure of emotions*—known as “OCC theory” for the initials of the surnames of the authors—it is an appraisal theory that is structured as a three-branch typology, corresponding to three kinds of stimuli: consequences of events, actions of agents, and aspects of objects. Each kind of stimulus is appraised with respect to one central criterion, called the central appraisal variable. An individual judges the following: (1) the *desirability of an event*, that is, the congruence of its consequences with the individual’s goals (an event is pleasant if it helps the individual to reach his goal, and unpleasant if it prevents him from reaching his goal), (2) the *approbation of an action*, that is, its conformity to norms and standards, and (3) the *attraction of an object*, that is, the correspondence of its aspects with the individual’s likes. In terms of the distinction between reactions to events, agents, and objects, we have three basic classes of emotions: “being *pleased* vs. *displeased* (reaction to events), *approving* vs. *disapproving* (reactions to agents) and *liking* vs. *disliking* (reactions to objects)” (Ortony et al., 1988, p. 33).

The OCC theory provides specifications for each class of emotions with three elements: (1) The type specification provides, in a concise sentence, the conditions that elicit an emotion of the type in question. (2) A list of tokens is provided, showing which emotion words can be classified as belonging to the emotion type in question. For example, ‘fright’, ‘scared’, and ‘terrified’ are all types of fear (of course, ‘fear’ is also a type of fear). (3) For each emotion type, a list of variables affecting intensity is provided. For example, the specification of the class of emotions labelled as ‘fear emotions’ is: (1) TYPE SPECIFICATION: (displeased about) the prospect of an undesirable event, (2) TOKENS: apprehensive, anxious, cowering, dread, fear, fright, nervous, petrified, scared, terrified, timid, worried, etc. y (3) VARIABLES AFFECTING INTENSITY: (a) the degree to which the event is undesirable (b) the likelihood of the event. In Table 1 we have summarized the type specifications of all 22 emotion types.

Joy:	(Pleased about) a desirable event
Distress:	(Displeased about) an undesirable event
Resentment:	(Displeased about) an event presumed to be desirable for someone else
Hope:	(Pleased about) the prospect of a desirable event
Fear:	(Displeased about) the prospect of an undesirable event
Satisfaction:	(Pleased about) the confirmation of the prospect of a desirable event
Disappointment:	(Displeased about) the disconfirmation of the prospect of a desirable event
Pride:	(Approving of) one's own praiseworthy action
Admiration:	(Approving of) someone else's praiseworthy action
Reproach:	(Disapproving of) someone else's blameworthy action
Remorse:	(Disapproving of) one's own blameworthy action and (being displeased about) the related undesirable event
Gratitude:	(Approving of) someone else's praiseworthy action and (being pleased about) the related desirable event
Liking (Love):	(Liking) an appealing object
Disliking	(Disliking) an unappealing object
(Hate):	

Table 1: The emotion type specifications of the OCC theory (A extract)

### Appraisals Structures

The OCC theory conceptualizes three support appraisals structures for changes in the world: (1) *structure of goals* to support appraisals of the desirability of events, (2) *structure of attitudes* to support appraisals of the appeal of objects, and (3) *structure of standards* to support appraisals of the praiseworthiness of actions.

The OCC theory defines *goals* as what one wants to achieve. There are three kinds of goals: active-pursuit goals (A-goals), interest goals (I-goals), and replenishment goals (R-goals). We reformulated the definition of these types of goals to adapt them to our data: A-goals represent the kind of things one wants to get done; a long period of time is needed to achieve these goals. Some examples are finishing high school or studying at university. I-goals are more routine goals and are necessary to achieve or support A-goals; they require a shorter time of period than A-goals. Some examples of I-goals are “understanding”, “solving a problem”, or “passing a course”. R-goals are the basic and necessary goals to accomplish all other types of goals. Some times they are so natural in the classroom that the subjects do not perceive them as goals. R-goals can be behaviour such as attending a class or bringing materials to work with (notebooks, books, notes).

For the OCC theory *standards* or norms represent the beliefs in terms of which decision assessments are made. We are concerned with *moral* or *quasi-moral* standards, standards of *behaviour*, and standards of *performance*. *Moral* or *quasi-moral* standards are the guidelines to approve or disapprove of the things someone is doing or did. *Behaviour* standards are conventions, norms, and other kinds of accepted regularities governing or characterizing social interactions. *Performance* standards are specific role-based norms; we understand them as the roles of being a teacher or student.

For the OCC theory a standard or norm can be *sufficient*, *necessary*, *facilitator* or *inhibitory*. The standard is said to be *sufficient* to achieve a goal of superior level if it is enough to achieve the goal; *necessary* if its observance is a requisite but not enough; *facilitator* when it increases the possibility to achieve the goal, but does not guarantee its observance; *inhibitory* in the case that it reduces the possibility to achieve the higher level goal.

The previous researches in which we based the present study (Martínez-Sierra & García González, 2014; Martínez-Sierra & García-González, 2015; Martínez-Sierra & García-González, 2016) used OCC theory to analyze the emotional experiences of the students. They show that the emotions are supported by a structure of goals, which is, in turn, supported by norms and attitudes.

### Research Questions

Considering the previous theoretical considerations, the research questions of the study reported in this paper is:

*RQ1. What are the daily individual emotional experiences of university students in mathematics classroom?*

The second question of this research is a consequence of the hypothesis of the cognitive structure of emotions theory that emotions are supported by appraisal structures (i.e. the antecedents of the emotions):

*RQ2. What are the individual appraisal structures that support the students' emotional experiences?*

## METHODOLOGY

### Participants and Context

By the time we carried out this investigation, the participants were attending their first mathematics course at the University (bachelor of international business at a Polytechnic University in a state at the north of Mexico City): Introduction to Mathematics. There are four “learning units” in this course to cover the typical mathematical topics of the Mexican high schools: (1) Algebraic expressions: addition, subtraction, multiplication and division of monomials and polynomials, (2) first and second-degree equations with one unknown variable, (3) Trigonometry and (4) Analytical geometry.

The teacher of the course is a woman with 11 years of experience at the Polytechnic University. She has a bachelor degree in administration and a master in regional economy. She referred that her courses are lectures but usually promoting mutual interaction among the students and the teacher. The usual structure of a class for a “new topic” is: (1) the teacher explains the topic, (2) the students ask questions about the topic and the teacher answers them, (3) the students do individual or team exercises on the topic, and (4) the group check the exercises and identify the mistakes of the students.

The system of evaluation in the course is summative as the institutional plan determines. The teacher grades the work of the students in class and their homework; these correspond to the 30% of the final qualification. Typically, the students develop a “practice”, or part of it, in each class. This practice consists of a series of tasks that the student must deliver in writing to the teacher so that she can evaluate it (also in a summative way). The other 70% of the evaluation in the course is determined by four tests, also evaluated in a summative way, corresponding to the four topics of the learning units.

### Data Gathering Procedure

Data was collected in order to obtain narratives of the emotional experiences (past experienced emotions as narrated by people) of the participants. We used two instruments to collect data: an open questionnaire and a *diary method*.

On the first day, we applied a questionnaire to collect the general data of the students. This data is synthesized in the description of the participants. This questionnaire also asked for their preferences (or not) to mathematics. The response threw us some emotional experiences of like and dislike.

We chose to use a particular type of diary method, the *event-based protocol* (Iida, Shrout, Laurenceau, & Bolger, 2012) where data collection is triggered by some focal experience of the participant, specifically the emotional experience of attending to a mathematics class. So, the teacher delivered a questionnaire that the students should answer ten or fifteen minutes before the class was over. The questions were: (1) What topics were discussed in class today?, (2) What did you learn in class today?, (3) What emotions or feelings did you experience today in class?, (4) Tell us about your positive experiences in class today, why were they positive?, (5) Tell us about your negative experiences in class today, why were they negative?, and (6) Did you feel motivated or did you lack of motivation today in class?, why did you feel that way?

Questions (1) and (2) were intended to become aware of the learning experience of the participants. Questions (3), (4), (5) and (6), following OCC theory, intend to provoke students to write about their emotional experiences in terms of the eliciting situations and their appraisals. This is the main reason to keep asking “Why do you feel this way?”

Finally, we collect 101 reports of the emotional experiences of 15 students attending the mathematical classes during 7 days: November 19, 21, 24, 25, 26, 28 and December 2, 2014.

### Data Analysis

We identify the emotional narratives of the participants with the name of the participants (pseudonyms): **Name-Rn** ( $n$ , for  $n=1, \dots, 7$ ) or **Name-C**.  $Rn$  denotes the number of the report for each day:  $n = 1$  corresponding to 19/11/2014,  $n = 2$  to 21/11/2014,  $n = 3$  v,  $n = 4$  to 25/11/2014,  $n = 5$  to 26/11/2014,  $n = 6$  to 28/11/2014 and  $n = 7$  to 02/12/2014.  $C$  denotes the response of the participant to the questionnaire.

Following OCC theory, we consider in our analysis two aspects to identify the type of emotion (we realize that the intensity of the emotions could not be identified since the beginning of our analysis, so, we did not consider this aspect): (1) ***Concise phrases that express the triggering situations*** of the emotional experiences. We emphasize these phrases in ***bold italics*** and (2) *Emotional phrases and words* that express the emotional experience from the emotional language of the participants or *phrases that show* the appraisal of the triggering situation. We emphasize these words or phrases in *italics*.

Given the above, we consider only the narratives that express an emotional experience in the data; this means that the phrase should consider at least one appraisal of the situation explicitly. So, we interpret in our analysis that the value of the appraisal of the triggering situation as positive or negative is established in the emotional words or phrases. Therefore, even if a participant used a specific word to label an emotional experience, we did not have to code it with the same name. Once we identified the triggering situations and the type of emotions, we proceeded to infer goals, norms and attitudes that support them. As a detailed example, we show the analysis of the reports of Candice.

## RESULTS: CANDICE'S CASE

We identified 24 emotional experiences in the reports of Candice. There were 23 daily reports (Table 2) and the questionnaire, where she reported she liked mathematics:

**Candice-C:** Yes [*I like mathematics*] ***because they are too easy and fun for me*** {Like}.

Report number	Extract of the report	Triggering situation
1	[My positive experience is] <b><i>I learned something new</i></b> {satisfaction} <b><i>and the teacher always solve any doubt</i></b> {Gratitude}	Learn something new The teacher solves doubts
	[My negative experience is] At the beginning I <b><i>did not understand how to do Gauss-Jordan method</i></b> {Deception}	Not understanding how to solve
	[I felt] motivated because, as I told earlier, <b><i>I got the chance to learn something new</i></b> {Satisfaction}.	Learn something new
2	[I felt] motivated, because <b><i>I understand better how to do it while solving more examples</i></b> {Satisfaction}.	Solve exercises Understand how to solve
	[My positive experience was that] I participated in class, <b><i>to understand the equations</i></b> better {Satisfaction}	Understand how to solve

Tabla 2: Dairy reports of Candice (Extract)

From Table 2, we organized the triggering situations according to the goal, norm or attitude associated. In this way, we bring together in the goal 'learn mathematics in each class' all the triggering situations that refer to the cognitive elements such as 'learn', 'understand', 'figure out', 'know', 'meet', 'find out', 'memorize', 'master a subject' or any other activity aimed to learn mathematics. In the same sense, we group

in the goal ‘solve exercise in class’ all the triggering situations that refer to ‘achieve’, ‘obtain’, ‘facilitate’, ‘practice’, ‘do’, ‘make’, ‘solving’ exercise in class.

We identified three norms in the structure of appraisal of Candice: two refer to the students and one to the teacher. We found that Candice refer to the following behaviour norms for students: ‘students should not miss their classes’ nor ‘be late’ and that ‘classmates should help each other’ –norm inferred by the situations she considered positive where the teacher or the students help solving exercises or give explanations of certain topic-. Candice referred that ‘the teacher must solve the doubts of the students’ (R1), so it is a global norm of the teachers in her structure of appraisals. Candice has an attitude of attraction towards mathematics because she expressed in the questionnaire that ‘mathematics is easy’ and ‘mathematics is fun’.

Finally we infer an appraisal structure that supports the emotional experiences of Candice (Figure 2). We represent the identified goals, norms and attitudes in the text boxes of Figure 2: Goals are represented in text boxes with a continuous border line, norms with a dotted line and attitudes with a point-line border. Attitudes are not related to goals and norms because they reflect a position of pleasure or displeasure towards an object such as mathematics, a test, some topics or the class. The arrows in the figures indicate what goals are *sufficient* (S), *necessary* (N), *facilitator* (F) or *inhibitory* (I) to achieve a higher-level goal. We put arrows signing up or down near the text box to highlight the positive or negative emotional experiences supported by the appraisal of fulfilling (or not) the goal or respect (or not) the norm. We indicate in parenthesis the labels of the reports corresponding of each type of emotion.

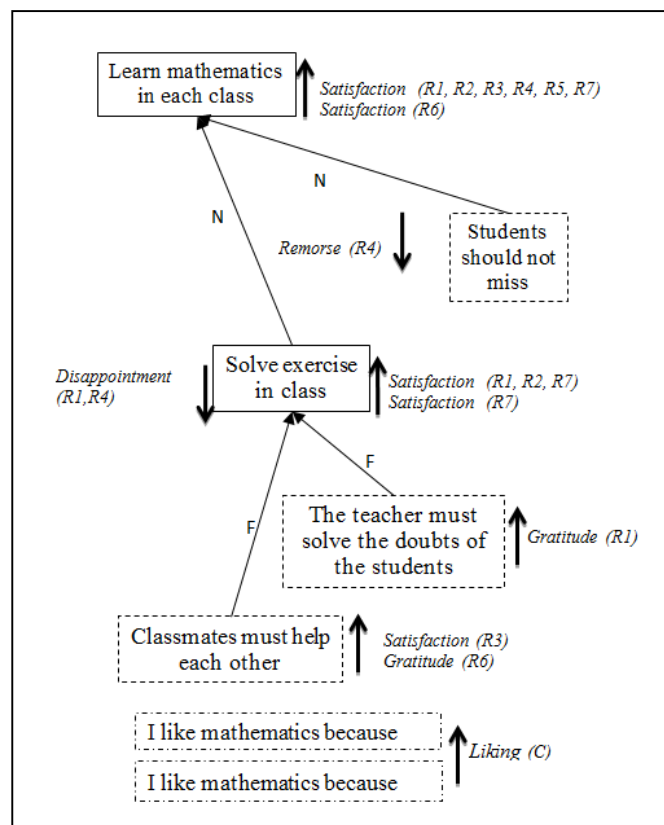


Figure 2. Candice' appraisal structure



## DISCUSSION

Results show that all emotions are related to the achievement of 4 goals of the students in mathematics class ('learn in each class', 'solve exercises correctly in each class', 'pass the tests' and, implicitly, 'pass the course'). All the participants experimented more or less the same type of emotions in the prosecution of the same goals. Satisfaction, disappointment and gratitude are the most frequent emotions. This suggests that the mathematics class is a highly organized context with clear goals to achieve; the causal succession is: First, I learn/understand the teacher's explanation, then I solve problems in class, then I pass the tests and, finally, I pass the course. The former considerations derive in one conclusion: the structures that support the emotional experiences of the students and their context are closely linked. Even if the emotional system of the people is part of our genetic inheritance as a species and the emotional experiences can be considered as individual phenomena, our results (together with the principles and results from appraisal theories) signed out that the appraisal structures of the emotional experiences of the participants are contextual.

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# CYCLES OF APOS-BASED RESEARCH: THE CASE OF FUNCTIONS OF TWO VARIABLES

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*This is a report of an Action-Process-Object-Schema Theory (APOS) based study consisting of three research cycles on student learning of the basic ideas of two-variable functions. Each of the research cycles used semi-structured interviews with students to test an initial conjecture of needed mental constructions, develop supporting classroom activities, and improve the conjecture. The article summarizes findings from each of the research cycles and shows the improvement in students' understanding of functions of two variables.*

## BACKGROUND AND RESEARCH QUESTIONS

Functions of several variables are a fundamental tool in the study of science and engineering. It has been shown in several studies that students show many difficulties when faced with them (Trigueros and Martínez-Planell, 2007, 2010; Kabael, 2011; Weber and Thompson, 2014). Some of these studies have shown that the way two-variable functions are introduced in textbooks and in class do not help students to develop a rich understanding of this important mathematical concept. In an effort to gain a better understanding of how students may learn this concept deeply, a project to develop and test a didactical approach based on APOS theory was conducted. The project involved three research cycles. The first two gave information about students' constructions which was used in the refinement of both the Genetic Decomposition and the sets of teaching activities (Martínez-Planell and Trigueros, 2012, 2013). The purpose of this report consists in presenting the characteristics of the whole design, results obtained in each cycle and, particularly, what was its impact in the design of the final didactical approach and on students' learning.

## THEORETICAL FRAMEWORK

As APOS is a well-known theory we will only discuss it briefly (for more detail see Arnon et al., 2013). In APOS theory an Action is a transformation of a mathematical object that the individual perceives as external. It may be a step by step transformation following an explicit algorithm or the use of a memorized fact. When the individual repeats an Action and reflects on it, the Action may be interiorized into a Process. The Process is now perceived as internal, under control of the individual, who may reflect on it without having to explicitly carry out all the steps of the transformation. A Process may be reversed and it may be coordinated with other Processes. When an individual needs to perform Actions on a Process, he/she may become aware of the Process as a totality, an entity in itself. In this case, it is said that the Process has been encapsulated into an Object and Actions can be applied to that Object. An Object may be de-

encapsulated into the Process it came from as needed in a problem situation. A mathematical Schema is a coherent collection of Actions, Processes, Objects, and other previously constructed Schemas, which are synthesized to form mathematical structures utilized in problem situations (Baker, Cooley, and Trigueros, 2000). The progression from Action, to Process, to Object, and to organize such constructions in Schemas is a dialectical progression where there may be passages and returns from one type of construction to the other (Czarnocha, Dubinsky, Prabhu, and Vidakovic, 1999). In APOS Theory it is stated that the overall tendency of the student when dealing with different problem situations involving the concept of interest will be different depending on the student's understanding of the concept as an Action, a Process, or an Object. Schema development is not considered in this article.

Research using APOS theory starts with a conjecture of the mental constructions that students need to do in order to understand a particular mathematical concept. The conjecture, called a genetic decomposition (GD), is based on the analysis of the mathematical concept itself, on classroom experience of the researchers, and results from any available data. The conjectured GD is then tested by analyzing students' responses to different research instruments. What typically happens is that students give evidence of doing some unexpected mental constructions and also show difficulties to use some of the conjectured constructions. This leads to refining the GD to better reflect the constructions students actually do and it also informs to the design and class testing of activities to give students' opportunities to make particular constructions to overcome the difficulties. At this point it is considered that a research cycle has been concluded and the next one may begin (see Figure 1). This cycle can be iterated until a GD that is stable, in the sense that no significant refinements can be expected from further studies, is obtained and it is used to predict those constructions that students can actually do to understand the mathematical concept of interest and it can be used to design instructional materials. The preliminary GD for function of two variables designed for this study and its refinements are omitted due to lack of space.

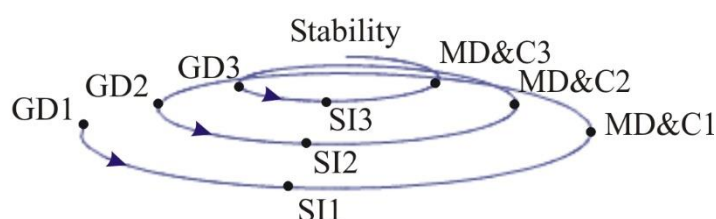


Figure 1: Research cycles in APOS: GD=Genetic decomposition, SI=Student interviews, MD&C= Materials' design and class testing.

## METHOD

In this project three research cycles were needed. In each cycle, students who had just finished taking a multivariable calculus course were chosen to be interviewed: 14 in the first cycle, 15 in the second cycle, and 19 in the third cycle. The third cycle included students from experimental and control sections. The experimental sections used specially designed activities to help students make the mental constructions

conjectured in the GD. Other than this, all students used the same textbook and syllabus. The participating professors were all experienced, taught the course regularly, were considered to be popular with students, and valued student learning. In each of the research cycles, interviewed students were chosen as to obtain a balance between above average, average and below average students in each of the sections. Students were given problems to solve and they were asked to produce a written answer while also describing their thinking out loud. The interviewer would intervene as necessary for clarifications regarding possible evidence of use of specific constructions conjectured in the GD. The interviews in each of the research cycles were audio recorded, transcribed, analyzed independently together with the written materials, and then discussed by the researchers until consensus was reached. Each of the interviews lasted from 45 minutes to one hour. Students' answers to interview questions were also graded taking into account the evidence about the possible constructions used by each student as shown in both the written and oral components.

The interview questionnaire for the first cycle included questions exploring students' construction of the Schema for  $R^3$ : locating and moving points in space, drawing graphs of simple relations, and intersecting fundamental planes (i.e. a plane given by  $x=c$ ,  $y=c$ ,  $z=c$ , where  $c$  is a constant) with surfaces given symbolically, or graphically. Other questions explored students' notion of the function definition by asking them to determine domain and range of functions of two variables given graphically, symbolically, or numerically, and by inquiring about uniqueness of functional value, the arbitrary nature of a functional relation, and the formal definition of function.

For the second research cycle, the interview questions focused on geometric issues mainly dealing with representing and intersecting fundamental planes with surfaces in different representations. It also included questions on cylinders and contour diagrams, topics omitted in the first cycle. Further, in response to informal classroom observations of students' work on activity sets, it explored student understanding of restricted domains and problems of conversion (Duval, 2006) between representations involving free variables.

The interview instrument for the third cycle included the following questions:

1. (1a) Draw in three-dimensional space, or represent on a physical manipulative, the collection of points in space that satisfy the equation  $y=2$ , and that are also on the graph of the function  $g(x,y)=x^2+x^3(y-2)+y^2$ . (1b) Do the same with the plane  $y=2$  and the graph of the function  $f(x,y)=x^2+y^2$ . (1c) If the domain of  $f$  is restricted to the pairs  $(x,y)$  that satisfy  $-1 \leq x \leq 1$  and  $-1 \leq y \leq 1$ , represent the domain of  $f$  as a subset of the Cartesian plane and (1d) find the range.
2. Given  $f(x,y)=x^2$ , (2a) Find the domain of  $f$ , (2b) the intersection of the plane  $y=1$  with the graph of  $f$ , (2c) draw the graph and (2d) draw the contour diagram of  $f$ .
3. Given  $S=\{(x,y,z): x^2+2x+y^2=3\}$ , draw in 3D space and describe the intersection of  $S$  with (3a) the  $x$  axis and (3b) the  $xz$  plane.

4. Find the intersection of the plane  $x=0$  with the graph of the function  $f(x,y)=x\sin(y)$  and represent the intersection in 3D space.
5. Choose the graph of (5a)  $g(x,y)=\sin(x)+y$  and of (5b)  $h(x,y)=\sin(xy)$  from six given surfaces. Justify your answer.
6. Draw the graph of  $f(x,y)=\sqrt{(x^2+y^2)}$ .

## RESULTS

Findings from the first research cycle put forward the realization that student difficulties in understanding functions of two variables are clearly due to their lack of construction of a strong Schema for  $\mathbb{R}^3$ . The need to construct fundamental planes as Objects and their role in the construction of a Process related to intersecting fundamental planes with surfaces and the possibility to place the resulting curve in its corresponding place in space was found to play a fundamental role in students' understanding of two-variable functions and its graphs. In particular, construction of this Process would enable students to distinguish between transversal sections and their projections onto the corresponding coordinate plane. It was also observed that the use of informal language in the class description of some mathematical processes can act as an obstacle to student learning. Hence, an important component in the construction of a Schema for  $\mathbb{R}^3$  is the introduction of a verbal language to mediate unambiguous Processes of conversions between symbolic, geometric, and physical representations. Another finding in the first cycle gave evidence that the construction of a Process for graphing functions of two variables requires students' recognition of transformations of a one-variable function or families of plane curves that may be obtained by substituting values for one of the variables in  $z=f(x,y)$ . This suggested the need to review through specific activities this topic in order to help students to construct a Process or Object conception of transformations. Another result of this cycle was the recognition that most students do not construct a Process conception of two-variable functions and that activities are needed to help students to do those constructions related to the Process of recognition of the domain of these functions as a collection of ordered pairs that can be geometrically represented in space as a subset of the  $xy$  plane. Students' difficulties associated to the construction of a Process of conversion from the symbolic to the geometric and/or physical representations, and the many difficulties students showed when facing functions with restricted domain were clearly related to this Process which is connected to the possibility of visualize the possible behavior of these functions. Students were not able to immediately generalize the definition of function from the context of one-variable functions to that of two-variable functions. For example, while in the case of functions of one variable students might be aware of the required uniqueness of functional value, and its interpretation as the rule of the vertical line, we found that this needs to be addressed explicitly in the new context of functions of two variables, in different representations. More generally, it was found that students need to do all constructions related to the function concept in different representations as they seem to be heavily dependent on each particular representation. All these results were used in the refinement of the preliminary GD and corresponding

pedagogical materials were designed and class-tested during two consecutive academic years.

The second research cycle uncovered several overlooked but important constructions that were deemed to be important in fully understanding functions of two variables. It was observed that after work with the designed activities, some students still showed they had not interiorized the Action of point by point evaluation of two-variable functions into a Process. This was found in analyzing students' difficulties with questions regarding the case where one of the independent variables is free. Evidence showed as well that the construction of a Process of intersecting fundamental planes with surfaces required considering explicitly situations that result in free variables. For example, situations in which a free variable is missing, which is the case of cylinders, and the case where both variables appear explicitly in the symbolic representation of a function but substituting a value for one of the variables allows the other to take on any value, as would be the case of  $z = x \sin(y)$  and  $x = 0$ , for example. Students' responses showed different behaviors for each case. Further, in the case of cylinders, this Process allows the recognition that their graphs can be obtained by intersecting with fundamental planes corresponding to the missing variable rather than by using a memorized technique. It was found that frequently students have difficulties interrelating differences in notation, as for example, when a student might have difficulty graphing  $f(x,y) = x^2$  but have no difficulty graphing  $z = x^2$ , or when a student might not immediately recognize that graphing the intersection of the plane  $x = 4$  with the surface  $z = xy^2$  is the same as representing graphically the set  $\{(x,y,z): z = xy^2, x = 4\}$ . Other interesting observations were that familiar algebraic expressions can act as obstacles in the interiorization of the Process of graphing functions of two variables, and that students showed a tendency to try to graph the entire surface in order to obtain a specifically requested transversal section. Results of this cycle led to further refinements in the GD and improvements in the design of classroom activities which were then class-tested for two consecutive semesters.

The third research cycle was considered to be the final research cycle since the analysis of students' responses to the interview questions did not show any result indicating the need to further refine the GD. Some results, however, led to improvements of the activity sets. Results obtained in the third research cycle gave evidence that students from the sections using the activity sets seemed to have constructed the Processes or Objects needed in a deep understanding of functions of two variables; they also confirmed that students who did not use the activity sets and collaborative work employed in the experimental sections showed the same difficulties that had been found in the first research cycle. The evidence on the differences in understanding two-variable functions was demonstrated not only by differences in scores for questions in the interview instrument but, and more importantly, the difference in students' responses to each question during the interview.

Questions 1a, 1b, 2b, and 4 dealt directly with intersecting a fundamental plane with a surface. Student performance can be summarized in the following table:

Section	Question	1a	1b	2b	4
Experimental	% tc	66.7%	91.7%	83.3%	66.7%
	% tpc	91.7%	91.7%	100%	91.7%
Control	% tc	0%	42.9%	28.6%	0%
	% tpc	42.9%	42.9%	28.6%	14.3%

Table1: Student performance (tc=totally correct; tpc=totally or partially correct)

The table shows clear differences between results obtained for these questions for students in control and experimental sections. However, the analysis of each student's response during the interviews revealed significant differences. For example consider Samuel's, a student from the experimental section, response:

Samuel: Draw in three-dimensional space or represent on the 3D Kit the collection of points in space that satisfy the equation  $y=2$  and that are also on the graph of the function  $f(x,y)=x^2+y^2$ . The first thing that comes to mind is that this is a circle, hmm, but if I substitute 2 in  $x^2+y^2$  I'm left with  $x^2+4$ , then what's left is a parabola here [He then drew the correct curve in space]. Ah! This is a bowl! Right? Yes, I think this is a bowl.

Observe that Samuel was initially drawn to think about a circle by the familiar algebraic expression  $x^2+y^2$ , but rather than conclude that the section is a circle or the surface a cylinder, as it frequently happened with students in the control section and in the second research cycle, he reflected on his initial suspicion by correctly using sections. Samuel showed not to be limited to an Action conception of the intersection based on memorized formulas, but rather, he showed in this and other responses a Process conception for intersecting fundamental planes with surfaces. This is further suggested when he was apparently able to imagine the surface to conclude "this is a bowl". Students from the control section were in some cases able to obtain the correct curve but then frequently were not able to place it on the corresponding fundamental plane in space. This is exemplified here by Damilette's response to the same question:

Interviewer: So you drew a parabola, looking upwards, that cuts the  $z$  axis at 4.

Damilette: Exactly, on the  $xz$  plane.

The following table shows students' performance on questions 2c, 5a, 5b, and 6 that dealt with the graphical representation of a function. Due to lack of space it is not possible to show other tables comparing student performance, but they are similar.

Section	Question	2c	2d	5a	5b	6
Experimental	% tc	91.7%	75%	75%	50%	66.7%
	% tpc	100%	100%	100%	83.3%	100%
Control	% tc	57.1%	14.3%	14.3%	28.6%	42.9%
	% tpc	71.4%	14.3%	57.1%	42.9%	57.1%

Table 2: Student performance (tc=totally correct; tpc=totally or partially correct)

Again it is more revealing to consider differences in the way students from different groups responded in the interviews. For example, when drawing the graph of  $f(x,y)=x^2$  Angel, from an experimental section, responded:

Angel: Now draw the graph of  $f$ . Ok, if we use several sections we can see that then  $y$  can be anything, and it would be the same graph extending to negative  $y$  infinity and also to positive  $y$  infinity. Ok, if we start at 0, with  $y$  equal to 0, ok, and we place  $y$  equal to 1, which we already have, and we can clearly see here that this keeps on extending. [See the figure below.]

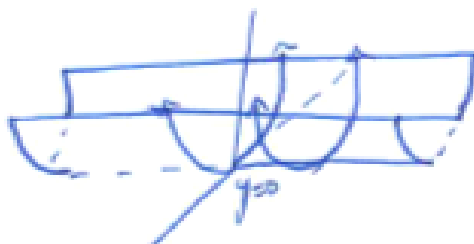


Figure 2: Angel's graph in question 2c

Responses like Angel's were typical in the case of students in the experimental section and evidence a Process conception for graphing cylinders. When Angel says "we can clearly see here that this keeps on extending" he shows he is able to imagine doing the intersection process many times without having to explicitly do it for many specific values. Students in experimental sections were able to justify their decisions and to relate fundamental planes with their effect when intersecting with the surface. Students in the control section showed typically to rely on memorization, as exemplified by Keyla's response to another question:

Keyla: Draw as carefully as you can the graph of  $f(x,y)=\sqrt{(x^2+y^2)}$ . This is the upper half of a cylinder... [after 1 minute thinking] No wait... what it forms is a circle... no that is not a cylinder, is it a hyperboloid? ...

## DISCUSSION AND CONCLUSIONS

This research project shows that students who worked with the final version of the activities and course design gave evidence of having a much better understanding of functions of two variables than students in other groups. They were able not only to answer interview questions successfully but showed their capability to reason to justify their responses, while students in other groups showed to rely more on memorized techniques or to be limited to respond some specific types of problems. This study also allows teachers and researchers to be aware of the difficulties involved in the design of an effective teaching sequence, in this case, through the evolution of an APOS based study in the course of three research cycles. Findings from the first two cycles resulted in successive refinements of the original GD and in the development and improvement of activity sets designed to help students in the development of the intended constructions. Work with the last sets of activities and class discussion enabled students in the third cycle to make the constructions involved in the understanding of the concept of function, their possibility to reason mathematically was also improved. This



approach involves collaborative work and group discussion that has proven to be effective and may invite teachers and researchers to try similar strategies to help students to understand this or other topics and to be able to deal with new and non-traditional problem situations.

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# HIGH SCHOOL MATHEMATICS TEACHERS' BELIEFS ABOUT ASSESSMENT IN MATHEMATICS

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*There are quite a few investigations on the beliefs of mathematics teachers in assessment despite the importance of assessment in the process of teaching and learning mathematics. This qualitative investigation aims to start filling this gap by identifying the beliefs in assessment in mathematics of a group of 18 high school mathematics teachers. The analysis of the data, collected through semi-structured interviews, was carried out with the theme analysis. The beliefs that we found are mostly on “what” to evaluate. So, the teachers belief they must evaluate: “the process of the solution of problems and not just the result”, “ the effort, attitudes and values of the students”, “if the students are able to ‘apply’ their knowledge” and “if it corresponds with the lessons of the teacher”.*

## INTRODUCTION

Much research in Mathematics Education has been conducted on the essential role of beliefs in learning and teaching mathematics (Leder, Pehkonen, & Törner, 2002; Philipp, 2007; Richardson, 1996; Thompson, 1992). These investigations show how important are the beliefs in mathematics of the teachers in their conceptions of teaching and learning mathematics and, therefor, in their pedagogical practice (Leder et al., 2002). So, for example, Liljedahl (2008) proposed that a teacher with a belief about mathematics as a *toolbox* —mathematics is seen as a set of rules, formulae, skills and procedures— will teach mathematics with emphasis on rules, formula, and procedures with an abundance of practice to enforce memorization and mastery.

Despite the importance of assessment in the process of teaching and learning in mathematics, as recognized by the extensive research in Mathematics Education (e.g. Houston, 2001; Iannone & Simpson, 2011; Niss, 1993), there are quite few investigation on beliefs of teachers (and students) in assessment in mathematics. We observed, for example, that there is no reference to this topic in the literature review on beliefs and affects of mathematics teacher made by Philipp (2007). Something similar occurs in the recent book of Bernack-Schüler, Erens, Leuders and Eichler (2015) dedicated exclusively to report investigations on beliefs in Mathematics Education; there is no article, not even in their references, on beliefs of teachers (or students) in assessment in mathematics.

The previous review indicates a gap on the research of the teachers' beliefs about assessment in mathematics. The goal of this research is to start filling this gap by attending the following research question:

*RQ: What are the beliefs of a group of high-school mathematics teacher about assessment in mathematics?*

## **THEORETICAL FRAMEWORK**

There is no agreement about the definition of a belief. However, according to Skott (2015) can be identified four key aspects that are at the core of the concept: (1) “beliefs are generally used to describe individual mental constructs, which are subjectively true for the person in question” (p.18), (2) “there are cognitive as well as affective aspects to beliefs, or at least beliefs and affective issues are viewed as inextricably linked, even if considered distinct” (p. 18), (3) “beliefs are generally considered temporally and contextually stable reifications that are likely to change only as a result of substantial engagement in relevant social practices”(p. 18) and (4) “beliefs are expected to significantly influence the ways in which teachers interpret and engage with the problems of practice” (p. 19).

Pajares (1992, p. 316) establishes that a belief is “an individual’s judgment of the truth or falsity of a proposition”. This is an operative definition for a narrative investigation because it is possible to identify quotes from interviews with this kind of judgements. Therefore, identification and coding of these quotes apply for a theme analysis of the narrative produced in the interviews, as we did in this research.

## **METHODOLOGY**

### **Participants and Context**

The eighteen in service mathematics teachers that participated in this research worked on the same high school located in a city near Mexico City. Some of them also worked in another school. The teachers aged between 26 and 67 years old. None of the participants were trained as high-school teachers, the rest are engineers who took the opportunity to work as teachers because “they fit the profile” (having a similar career to mathematics). It is worth to mention that in México there is no professional career to become high school mathematical teacher, so it is usual that engineers, mathematicians and other similar STEM professionals (Science, Technology, Engineers and Mathematics) perform this work. Only two of the participants have superior pedagogical education: one has a master degree in Education and the other one in Mathematical Education. Nine of the participants have less than 5 years in service, five between 10 and 20 years with groups, three between 21 and 30 years and 1 has more than 30 years of experience as a teacher. They attend the different mathematics courses in the school (Algebra, Geometry, Calculus) by working with the model of education based on ‘competencies’.

The competency-based learning model, according to the curricula of this school, promotes the development of apprenticeship considering the “know” (knowledge), the “know how to do” (application of knowledge) and the “know how to be” (conduct and attitudes). Therefore, the assessment of the development of the students must consider these three aspects. There are three types of assessment that are averaged: hetero

assessment (carried out by the teacher), the co evaluation (by the classmates) and the auto evaluation (by the student himself). The curricula propose several instruments of evaluation that intend to inform the student about his academic performance: cross-reference lists, rubrics, daily reports, portfolio of evidence and some others proposed by the teachers.

### **Data collection**

We carry out individual qualitative semi-structured interviews to collect data. Two postgraduate students in Mathematics Education with previous experience in qualitative interviews and the second, third and fourth authors of this paper did all the interviews on July 1 of 2015. The interviews lasted between 60 and 120 minutes, they were all videotaped. All the interviews were done in a classroom of the school in a private way. An intermediary teacher contacted his colleagues and stimulated them to be part of this research. The coordination of the mathematics academy (the group of teachers that teach mathematics) delivered the circular letters for the teachers with the specific schedules of the interviews. All of the participants voluntarily agreed to be interviewed.

At the beginning of each interview, the teachers detailed some personal and professional information that are summarised in the section of participants. The specific questions of assessment were guided to understand the whys, hows, whats, whos and whens of assessment in mathematics from the perspective of the participants: (1) How do you assess in your courses of mathematics? (2) What is an assessment in mathematics? (3) Why should an assessment in mathematics be made? (4) What is the purpose of assessment in mathematics? (5) When is the time to make an assessment in mathematics? (6) What things or aspects should be considered in an assessment in mathematics? (7) What tools or instruments should be considered to evaluate in mathematics? (8) How should the results of assessment in mathematics be expressed or communicated? (9) Who should make an assessment to the students? Who should make an assessment to the teachers? (10) What links are between learning, teaching and assessment in mathematics? (11) What could happen if there was none assessment in mathematics? (12) What could happen if there were no tests in mathematics?

### **Data analysis**

*Theoretical thematic analysis* (Braun & Clarke, 2006, 2012) was the strategy used to analyze the data in this research. The purpose of thematic analysis is to identify patterns of meaning (themes) throughout a data set provided by the answers to the research question addressed (Braun & Clarke, 2006, p. 82): “A theme captures something important about the data in relation to the research question and represents some level of *patterned* response or meaning within the data set” (emphasis in original). Patterns are identified through a rigorous process of data familiarization, data coding, and theme development and revision.

Themes or patterns within data can be identified in one of two primary ways in thematic analysis: in an inductive or ‘bottom up’ way, or in a theoretical or deductive or ‘top

down' way. We chose to make a theoretical thematic analysis: the codes and themes derive from concepts and ideas the researcher brings to the data; what the researcher maps during the analysis is not necessarily linked to the semantic data content. Thus, we focused on phrases where we identified "an individual's judgment of the truth or falsity of a proposition" (Pajares, 1992, p. 316) about "assessment in mathematics" (the use of this definition makes theoretical our theme analysis). Therefore, at the end each theme corresponds to a belief.

The stages in our analysis were (Braun & Clarke, 2006): (1) become familiar with the data, (2) generate initial codes, (3) search for themes, (4) review themes, (5) define and name themes, and (6) produce the report. The theme analysis was carried out in repeated workshops with all the authors of this paper.

## RESULTS

The theme analysis yields the results summarised on Table 1. We write the themes and sub themes so that they express a specific believe about assessment in mathematics expressed by a group of participants. We also group the themes in terms of the whys, hows, what, who and whens of assessment from the perspective of the participants.

Beliefs	F
<i>Beliefs about what to consider in an assessment</i>	
The attitudes, values and effort of the students should also be considered in an assessment and not only the resolution of problems	13
Assessment must show if the student is able to "apply" knowledge.	10
Assessment must also consider the process to solve problems and not only the results	9
Assessment must also consider the comprehension of the mathematics concepts	5
Assessment should be in correspondence to the lessons of the teacher	2
<i>Beliefs about the reasons to make an assessment</i>	
Assessment in mathematics is made to know what does the student learn	11
Assessment serves so that students pass the course	7
Assessment is useful for feedback because	
...it helps the students	6
...it helps the teachers in their practice	1
...it helps the student and the teacher	2
Assessment motivates students to...	
...study constantly	7
...participate in class, pay attention and take notes	6
<i>Beliefs about who should make an assessment</i>	
Students' assessment should be made by...	
...the teacher of the group because he knows the students better	4
...someone different than their teacher of the group but using the same instruments of assessment	3
... not only their teacher, but the student himself or someone else from the academy	4

Teachers' assessment should be made by	
...the students because they work together with the teachers	6
...people in charge of the guidelines of assessment, in addition to the students	10
...other teachers	1
<i>Beliefs about how to make an assessment</i>	
There should be different assessment instruments in the competency-based model	8
<i>Beliefs about when to make an assessment</i>	
Assessment must be constantly made...	
...in each class	12
...in terms of the topics in the class	4
An assessment could be made in class by just observing the daily participation of the students without the record in any instrument of evaluation	5

Note: F is the number of teachers in whom we identify this believe

Table 1. Teachers' beliefs about assessment in mathematics

We describe some of the beliefs of the teachers below. We just illustrate some of the themes and sub themes that we identified in most of the teachers.

### Beliefs about what to evaluate

*The attitudes, values and effort of the students should also be considered in an assessment and not only the resolution of problems*

Some teachers belief that it is necessary to evaluate the “know how to be” besides the “know how to do”. The behaviour in classroom, the attitude in class, the effort to make the daily activities, the ability to work as a team should also be considered even if the results of the tests are not completely satisfactory.

**Nadia:** Well, I don't know if we are right but I take into account both knowledge and attitudes; the attitude towards the class.

*Assessment must show if the student is able to “apply” knowledge*

Some teachers believe it is important that students build their own knowledge, but assessment should focus on its applications in different contexts: inside mathematics, in other disciplines and even in the “daily life” of the student.

**Ignacio:** Well, [evaluate a topic is] partially as the program says, my field of study, but assessment should also focus on the ability of the student to interact with his daily life. It is not just to acquire knowledge. There are some things in the tests that focus on the applications outside the school or maybe from inside the school.

*Assessment must also consider the process to solve problems and not only the results*

Some teachers believe that assessment must consider both the process to solve problems and the results. The aspects that should be taken into account in the process

are: the strategies that the students used to solve the problem, if the students used different strategies from the ones used by the teacher, the reasoning process to solve the problem and if the students are able to identify some application of the mathematics issues in his daily life. The teachers in this community consider this former aspect of high relevance because they are mostly engineers.

**Gonzalo:** [...] assessment in mathematics, why? Because many students are used to write the solutions but they don't know where does it came from. I tell them that I want the reasoning and the procedure used to obtain the solution because I want to know if they understand and the mathematical process they used to get to the solution.

### **Beliefs about the reasons to make an assessment**

*Assessment is useful for feedback because it helps students*

For some teachers, the result of the assessment is useful information that mainly helps the students and to a lesser extent to the teachers; some teachers even consider that the result of the assessment does not concern them.

**Ignacio:** [...] It is always necessary to have the antecedent of the assessment to identify the lacks of the students, what they have learned and the things that were more difficult for them, so they can have feedback.

*Assessment motivates students to participate in class, pay attention and take notes*

Some teachers believe assessment should be constant; one of the benefits identified in this type of assessment is that students study all time. The teachers argue that, without a continuous assessment, the students could have the opportunity to study just one day before the test so the only goal would be to pass the test and not to learn.

**Jaime:** [Without assessment in mathematics] There would be a lack of interest. Maybe, 20 percent of 40 students would take it seriously.

### **DISCUSSION**

The most frequently identified beliefs are those that consider what to evaluate. Some teachers consider assessment should focus on the comprehension of mathematical concepts, while others emphasize in the process of solution of problems; some others focus on the 'applications' of mathematical knowledge to real contexts and others consider attitudes, values and effort for the activities proposed. This variety can be explained in several ways:

There are two complementary explanations of the belief in the importance to evaluate if students are able to 'apply' their knowledge: (1) Almost all participants are engineers (in a wide sense, mathematics is conceptualized as an 'application' or 'tool' to solve engineering problems in Mexico), and (2) the influence on the competency-based model of this school because the curricula explicitly establishes the teaching and assessment of the "know how to do". This is why the teachers keep proposing the solutions of 'applied' problems to their students. Some even take their students outside

the school to perform specific activities where mathematical knowledge play special roll. In the same sense, the presence of the ‘know how to be’ in the curricula partially explains the belief that assessment should take into account the values, attitudes and effort made by the students.

There is heterogeneity on the beliefs about the reasons to make an assessment among the teachers. Some of them consider, in relation with the curricula, that assessment helps to feedback the students, the teachers or both. Other teachers believe assessment only serves to pass a course or from one educational level to another. Some others believe it motivates student to attend classes, take notes, participate and study; they even mention that without assessment the students would not do anything. Motivation seems to be an important topic for this former group of teachers.

On the other hand, some teachers consider that the teacher responsible of the group should be the responsible to make an assessment, this contrast with the curricula of the school. Others believe that someone else from the teacher responsible of the group should evaluate the students. Some possible evaluators could be the students themselves, the academy, the school management or even an external institution.

Some teachers argue that the large number of students and the limited time in mathematics class makes assessment a difficult activity. Some of them consider it is necessary to evaluate continuously but they do not have enough time to do it. They also face to the lack of honesty of the students in the co-evaluation. This causes some teachers to consider that the only person in charge of the assessment based on instruments, such as cross-reference lists, rubrics, daily reports, portfolio of evidence and some others proposed by the teachers, must be the teacher. The teachers use these tools to keep an individual register of the development of the students during the course. Other teachers believe it is possible to evaluate with just proper observations of the students because they are continuously in touch with them, so they can decide if the students deserve to pass the course or not.

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# FRAMEWORK FOR AN EARLY MATHEMATICAL PRESCHOOL CURRICULUM IN JAPAN

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*The purpose of this study is to propose a framework for an early mathematical preschool curriculum in Japan. Japanese preschool children do not learn elementary mathematics formally; thus, there is a problem connecting the mathematics learned in preschool and elementary school. Therefore, we have created an elementary mathematical programme to enable a senior preschool pupil (five to six years old) to learn mathematics smoothly in elementary school. This framework is constructed based on the five areas of kindergarten education in Japan, basic scholastic achievement in mathematical education in primary school, and mathematical literacy as described by Japanese scientists from the viewpoint of scope, as well as on Bruner's EIS principle from the viewpoint of sequence.*

## RESEARCH ON ELEMENTARY MATHEMATICS IN PRESCHOOL

Recently, many researchers have focused on elementary mathematics education for preschool children and younger elementary school students (e.g. Lin, 2013; Brandt, 2013). So far, there has been little research on elementary mathematical education because preschool teachers etc. do not pay attention to preschool mathematical education as related to mathematics learning in elementary school, and it is considered that preschool children have limited cognitive abilities (Inagaki, 1996), etc.

However, English and Mulligan (2013) note that their research over many years has revealed that young children have sophisticated mathematical minds and a natural eagerness to engage in a range of mathematical activities. Recent research shows that young children develop complex mathematical knowledge and abstract reasoning a good deal earlier than previously thought. Moreover, studies concerning the transition from preschool to elementary school in many countries or areas propose various kinds of curriculum or programme for elementary mathematics (Perry, MacDonald, & Gervasoni, 2015).

On the other hand, Japanese teachers have not paid attention to elementary mathematical education in preschool even in recent years, so there is little research on this topic (Matsuo, 2013). Nagaoka schools and kindergarten, which are attached to the Faculty of Education, Niigata University (2007), presented a mathematics and science curriculum connecting kindergarten education and education at the younger grades of elementary school. However, this curriculum has few concrete activities related to mathematics, particularly science, and it is not clear what kinds of mathematical content the activities described are related to. Funakoshi (2011) also proposed a preschool curriculum of elementary mathematics, but it is not a complex of preschool education and elementary school mathematical education. Moreover, he suggested a

programme based only on mathematics as a subject, not on activities that are fused with preschool daily life and did not make reference to grounds about the sequence of the programme. However, we can identify the foundations of elementary mathematics in various kinds of activity in which pupils play and learn by themselves in preschool; thus, we confirm that pupils' spontaneous activities related with mathematics can enrich the learning of elementary mathematics in preschool and complete the foundation of learning after finishing elementary school education by connecting these activities with the contents of mathematics teaching and learning in elementary school.

Based on the above, the purpose of this research is to propose a framework for constructing a feasible early mathematical curriculum in preschool in Japan to connect preschool education with elementary mathematics education in the early years of elementary school. In order to complete the framework, we first of all overview a framework for constructing an elementary mathematical preschool curriculum that was considered based on the perspective of scope. Next, from the sequential point of view, we clarify the need to add the developmental process of representation to the constructed framework as one of the factors we use in developing the curriculum. Finally, we complete the framework for constructing an early mathematical preschool curriculum based upon scope and sequence.

## **CURRENT SITUATION OF ELEMENTARY MATHEMATICAL EDUCATION IN JAPAN**

In Japan, there are mainly three kinds of preschool: kindergarten, nursery school, and mixed school (*kodomoen*). Usually, children attend preschool until the age of six. Let us focus on kindergarten, which many children attend in Japan, and examine the Course of Study for Kindergarten in Japan. In the Course of Study for Kindergarten, which the Ministry of Education, Culture, Sports, Science and Technology proposed in 2008, in particular in section 1–Basic Ideas of Kindergarten Education–of Chapter 1 General Provisions, it is said, “Education during childhood is extremely important in cultivating a foundation for lifelong character building....” Moreover, as regards curriculum formation, “Through such efforts, kindergartens should cultivate the foundation for compulsory and further education. ...” From these quotations, we can understand that the aim of kindergarten education in Japan is to cultivate the foundation for lifelong character building and elementary education.

The content of kindergarten education integrates *areas* of each child's development: Health; Human relationships; Environment; Language; and Expression. This means that the aims indicated in each area interrelate to allow children to acquire various experiences in kindergarten education, not from each area separately, and that the content of the curriculum is delivered in a comprehensive manner through the specific activities developed in relation to the children's learning environment. This differs from elementary education in which the curriculum comprises separate subjects.

Concretely, the content integrates the above five areas of each child's development. The Health area regards developing a healthy physical and mental condition, and

fostering children's abilities to independently maintain a healthy and safe life. The area of Human relationship regards developing self-reliance and fostering the ability to communicate with others in order to associate with and support each other. The area of Environment regards fostering children's abilities to relate to the environment with a sense of curiosity and inquisitiveness, and to incorporate this into their daily lives. The area of Language regards developing the will and attitude to verbally express experiences and thoughts in one's own words, as well as to listen to others' spoken words, and fostering an understanding of language and skills of expression. Finally, the area of Expression means developing rich feelings and the ability to express oneself, and enhancing creativity by expressing experiences and thoughts through one's words.

In particular, the aim of Environment as related to mathematics is as follows: (3) To enrich children's understanding of the nature of things, the concepts of numbers and quantities, written words, etc. through observing, thinking about, and dealing with surrounding things and experiences. The related content is written by (8) developing curiosity about the concepts of numbers and quantities, and figures in everyday life. Moreover, we will deal with the content as follows: (4) Children should be encouraged to place importance on their experiences based on the necessities of their own lives, so that interest, curiosity, and an understanding of the concepts of numbers and quantities and the written word can be fostered. Therefore, it is understood that kindergarten teachers will develop the children's foundation of learning of numbers, quantities, and geometric figures, which are the foundation of learning and life after elementary education with consideration to children's needs and interests.

From the above consideration, we understand that in particular we must treat strongly of the content related to elementary mathematics by focusing on the importance of children's needs and interests to learn about number, quantity, and geometric figures. However, based on the results of a paper-test survey and interview for preschool teachers, they do not always teach pupils about number, quantity, or geometric figures using conscious devices, with the aim of developing pupils' interest in the mathematical aspects of their activities (Matsuo, 2012).

Thus, we see that the content of the kindergarten curriculum in Japan is delivered in a comprehensive manner through the specific activities developed in the five areas. We must consider the elementary mathematical curriculum in preschool, particularly kindergarten, focusing on its relationship with the five areas. In elementary school in Japan, the contents of mathematics are treated separately as a school subject, and not integrated with all or some subjects. Therefore, when we consider teaching mathematics in preschool, we need to select mathematical contents together with other subjects or themes while considering cognitive development.

## **THE FRAMEWORK FOR CONSTRUCTING AN EARLY MATHEMATICAL PRESCHOOL CURRICULUM**

The framework for constructing an early mathematical preschool curriculum was established from the perspective of scope (Matsuo, 2014). In this paper, we complete the framework by considering the perspective of sequence.

### **From the perspective of scope**

This framework was established based on three points: the five areas of kindergarten education in Japan, basic scholastic achievement in mathematical education in elementary school, and mathematical literacy as proposed by Japanese scientists. First of all, we take up the five areas in the kindergarten curriculum. These areas are Health, Human relationships, Environment, Language, and Expression. The content integrates aspects of each child's development. In particular, content relating to elementary mathematics is involved in the area of Environment. The Health area is treated in order to consider the selection of an activity corresponding to the elementary mathematical content. Moreover, the areas of Language and Expression are considered with regard to the element of methods to learn elementary mathematics and the area Human relationships is considered with regard to all activities of children in preschool.

Secondly, we consider the points of basic scholastic achievement in mathematical education in elementary school, particularly types of student performance. These are established as a point for constructing the school curriculum, the so-called evaluation criteria of a lesson or unit of a lesson: knowledge and understanding of number, quantity, and geometric figures; thinking mathematically; mathematical skill; interest, willingness, and a positive attitude towards learning mathematics.

Third, we consider mathematical literacy. The contents of the elementary mathematical curriculum in preschool must be connected with mathematics in elementary school, while being general mathematics; thus, we must consider how it should be related with mathematical literacy learning in the future. In the project report, "wisdom of science and technology," produced by the Mathematical Science Expert Committee, the basic knowledge of science and technology, particularly mathematical knowledge, which every Japanese person needs to acquire, was indicated. Therefore, learning the basics of mathematical knowledge before entering elementary school is based on the idea that mathematical literacy is important with regard to looking forward to the future. We will examine numbers, quantities, and geometric figures, variation and relations, data and probability as the content of mathematical learning. We also consider two perspectives on mathematical learning methods: one is mathematical language, and the other is the process of mathematical problem solving.

From the above three points, we propose a framework for constructing an early mathematical preschool curriculum. With regard to the area of Health, we select the theme of preschool children's activities that are related to the elementary mathematics content, e.g. seasonal events and making rules in a game. On the other hand, the areas of Language and Representation are contained in the methods of mathematical problem solving. In the process of problem solving, children use various kinds of representation related to mathematics, including language. Moreover, the areas of Environment and Human relationships are taken up throughout the curriculum because the contents of the former in the Course of Study for Kindergarten are directly related with mathematics while children learn various kinds of activities through the latter in their kindergarten education. Therefore, a central part of the framework is embedded in the areas of Environment and Human relationships. Consequently, we will propose a framework for constructing an early mathematical preschool curriculum from the perspective of scope (figure 1). Furthermore, we select an activity theme that aims to enrich knowledge, understanding, thinking, skill, and attitude, so we can evaluate each of them based on the connection between elementary school and kindergarten. This curriculum is completed on the basis of these points in addition to the contents and methods of mathematics that we have already mentioned with regard mathematical literacy.

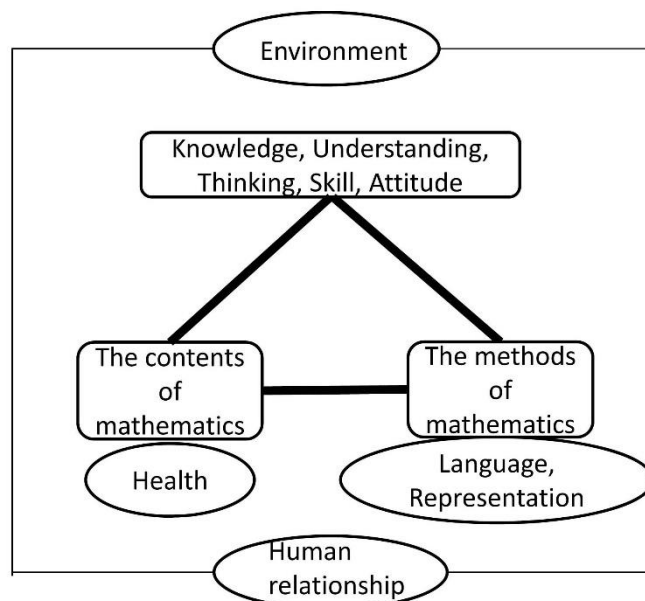


Figure 1: Model for considering the scope of the early mathematical preschool curriculum

### Bruner's principle of EIS and modes of representation

According to Bruner (1966), there are three methods of representation by which human beings translate experience into a model of the world. These methods involve storing and encoding information or knowledge in memory. The first is through action. We know many things for which we have no imagery and no words, and which are very difficult to teach using either words or diagrams and pictures. The second system of representation depends upon visual or other sensory organization and upon the use of summarizing images. We have come to talk about the first form of representation as enactive and the second as iconic. Iconic representation is principally governed by principles of perceptual organization and by the economical transformations in perceptual organization. On the other hand, enactive representation is based upon a learning response and forms of habituation. Finally, there is representation in words or language, the hallmark of which is that it is symbolic in nature, with certain features of

symbolic systems that are only now coming to be understood. Symbols (words) are arbitrary, they are remote in reference, and they are almost always highly productive or generative because symbolic representation expresses beyond what is possible through actions or images.

In mathematics education, children perform actions using manipulatives, for example, tiles, papers, coins, etc. in enactive representation. Iconic representation involves images or other visuals to represent the concrete situation enacted in the first mode. Children draw images of the objects on paper or visualise them in their heads. Otherwise, they could use shapes, diagrams, or graphs. The third mode, symbolic representation, takes the images from the second mode and represents them using words and symbols—for example characters, equations, etc.—so that children can explain acquired knowledge or how to learn it by using them.

### From the perspective of sequence

Basically, these three modes apply to the elementary mathematical curriculum in Japan; however, it seems that the term of use of the iconic mode is shorter than the other modes and children do not get enough images of mathematical knowledge or skill, thinking, etc. Most preschool children use representation in the enactive and iconic modes. Using the imagery of a piece of knowledge/skill, etc., people can understand it well and use it appropriately. Therefore, if children think or represent mathematical knowledge, skill, etc. sufficiently in the iconic mode, they can use it appropriately to solve various problems in the future. It is difficult for preschool children to use language proficiently because they do not have sufficient vocabulary. Thus, we propose that preschool children (five or six year-olds) become able to play or learn in the enactive and iconic modes with the aim of increasing their rate of use of the symbolic mode.

In order to consider a framework for constructing an early mathematical preschool curriculum from the perspective of sequence, we have to examine modes of representation. As mentioned in the above discussion, if we consider developmental stages, preschool children mainly think about and represent mathematical concepts, using both enactive and iconic modes.

In particular, we must focus on the iconic mode and recognize the importance of connecting it with the symbolic mode. Therefore, we present the activities conducted by four-year-old children using enactive representation, by five or six-year-old children using iconic representation, and by elementary school children using symbolic

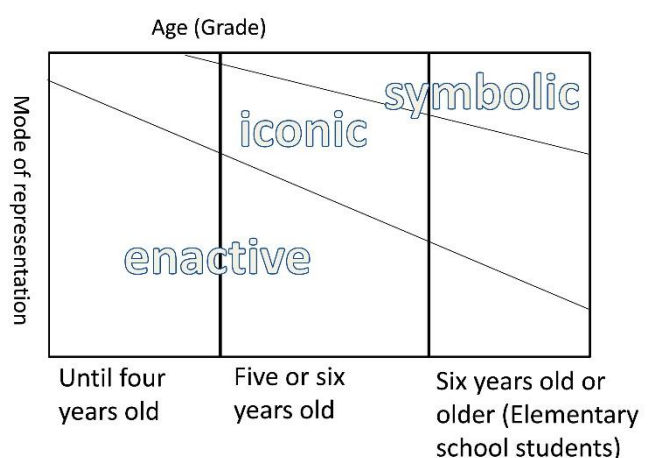


Figure 2: The relation between modes of representation and ages (grades)

representation at the degree to which the rate of something is high and three modes of representation properly (Figure 2). For example, when a pupil learns the measurement of length, until four years old, she/he measures the length through the direct-comparison method. At five or six years old, she/he begins to measure length through the indirect-comparison method, i.e. by using shorter units and counting the number of such units. Moreover, she/he understands the image of the method, but cannot explain why she/he uses the method or how to use the method sufficiently.

### **The framework for constructing an early mathematical preschool curriculum**

From the above discussion, we propose a framework for constructing an early mathematical preschool curriculum. From the perspective of scope, we set the content of activities in which children engage in preschool in figure 1. On the other hand, we constitute the learning process of those activities from the viewpoint of sequence based on the relation between age and modes of representation in figure 2 in this paper, and particularly focus on activities for five or six-year-old children and create a program tailored to them.

There are four characteristics of this framework. The first is the ability to implement the early mathematical curriculum by integrating it with daily life in preschool because the curriculum is considered from the five areas in the Course of Study for Kindergarten. The second is to be able to clarify the connection between preschool and elementary school education by showing the connection between the points of evaluation, which are the principles upon which the elementary school mathematics curriculum is constructed. The third is to be able to point out how to connect the curriculum with Japanese mathematical literacy which preschool children will learn in the future. The fourth is to be able to develop children's cognitive level corresponding to the modes of representation. It is not so difficult to learn mathematical knowledge through enactive and iconic representation, but using symbolic representation is difficult. This preschool curriculum is quite different from the elementary school curriculum because the aim or content of the curriculum is different even if the content seems to be the same.

Consequently, in this framework, we clarify the idea of linking mathematics education with lifelong learning because the curriculum is constructed based on the principles of each educational level to connect preschool education with compulsory education. Moreover, our framework is related to the aim of kindergarten education, i.e. integrated education, so we do not treat mathematical knowledge or thinking individually. This knowledge or thinking is naturally integrated into children's daily activities and they use the knowledge or thinking appropriately; thus, the possibility that this curriculum will be implemented in the present situation in Japan is high. In the future, we will complete the curriculum based on the framework and implement it.

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# SPATIAL METAPHORS: AN EMPIRICAL STUDY OF THE USE OF METAPHORS TO DESCRIBE SPATIAL-GEOMETRICAL CONFIGURATIONS

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*This study analyses how students use metaphors to describe spatial-geometrical configurations to get an insight of students' spatial thinking. Fifth-grade students were required to solve a spatial-geometrical task with language as a mode of representation, i.e. verbalizing the (re)construction of a spatial object and its spatial-geometrical characteristics. Findings suggest that whereas metaphors are extensively used by students to structure and communicate their spatial thinking, metaphors can also act as barriers in the spatial task solving process.*

## BACKGROUND OF STUDY

The use of metaphors in learning mathematics has been pointed out by different studies about language in mathematics classroom (Lakoff & Núñez 2000; Lakoff & Johnson 1980; Malle 2009; Pimm 1981). The aim of this study is to understand how students verbalize their spatial thinking, which should provide support for future learning in spatial geometry in mathematics. An analysis of metaphors in solving language-based spatial tasks should induce results regarding the following questions:

1. Which metaphors are used and for which purpose are these metaphors used to describe spatial-geometrical constellations?
2. What do metaphors reveal about student's spatial thinking?
3. To what extent is the use of metaphors helpful during the description of spatial-geometrical constellations?

The concept of spatial thinking has been chosen as a case to study the use of metaphors in mathematics learning, because it is an important factor for mathematics performance (Büchter 2011) and for learning and teaching of geometry (Maier 1999).

## THEORETICAL CONSIDERATIONS

### Metaphors

Mathematics and mathematical discourse is based on the use of metaphors, which can be defined as words or phrases with a “transferred” meaning, i.e. its meaning originates from another field, as in the following phrase: “an isosceles triangle is like a human standing on two feet, therefore both sides are equal” (Malle 2009, p. 10). Based on Lakoff and Johnson's work, Lakoff and Núñez (2000) define conceptual metaphors as mechanisms, which support the understanding of abstract ideas through concrete terms or experiences. Metaphors are described as projections from source domains to target domains, in which the source's properties and characteristics are assigned to the target

(Lakoff & Núñez 2000). Lakoff and Núñez (2000) differentiate between two types of conceptual metaphors: grounding and linking metaphors. Grounding metaphors are metaphors which project a source outside the fields of mathematics to a target within mathematics, e.g. “classes are containers” (see Table 1). In contrast, both source and target in linking metaphors originate from mathematics (Lakoff & Núñez 2000).

<i>Source Domain</i>	<i>Target Domain</i>
Container Schema	Classes
Interiors of container schema	Classes
Objects in an interior	Class members
Being an object in an interior	The membership relation
An interior of one container schema within a larger one	A subclass in a larger class

Table 2: An example of grounding metaphor “classes are containers”  
(Núñez 2000, p. 133)

Malle (2009) states that metaphors are important tools to facilitate mathematics learning, however, the heavy reliance on metaphors to learn mathematics can also lead to learning barriers. For instance, most pupils understand the mathematical concept of *diagonal* as *oblique* or *slanting*, as in the everyday-life meaning of the word *diagonal* and sometimes also in geometry classes (e.g. in rectangles and parallelograms) (ibid.). Whereas the use of such a metaphor is helpful in introducing diagonals, it might also be hindering, since not every slanting line is a diagonal (ibid.). Other examples of how metaphors can act as barriers in mathematical spatial discourse will be discussed later in this paper.

### Language in Mathematics Classroom

Language plays an important role in learning mathematics (Pimm, 1981). Based on Cummins’ (2000) research about language learning among non-native speaking students in classroom there are two major planes of languages in classrooms: Basic Interpersonal Communication Skills (BICS) and Cognitive Academic Language Proficiency (CALP). The former denotes the ability to communicate in everyday-life conversations and contexts and does not require complex syntactical structures or specific vocabulary, whilst the latter is used to understand academic concepts and abstract ideas and requires academic vocabulary, such as mathematical concepts (ibid.).

### Spatial Abilities and Spatial Language

The notion of spatial abilities has been a research object in psychology and in mathematics education. As one of the eight different intelligences, spatial abilities or knowledge is considered to be the ability to solve spatial tasks in navigation, visualize

objects in different angles and to recognize space and other spatial characteristics (Gardner 2006).

The solving of spatial tasks requires several cognitive process to be activated. In particular, students solving spatial tasks are required to visually percept spatial knowledge in a particular environment, which include figure-ground, spatial relations and position in space perception (Maier 1999). Figure-ground perception deals with the ability to identify a figure from a complex background. Spatial relations is the ability to visually identify and describe the relation between two or more spatial objects, and position in space is the ability to identify and describe the position of an object in space under consideration of one's own body (*ibid.*).

From a mathematics education perspective, Pinkernell (2003) describes spatial knowledge as the ability to mentally and really act on spatial objects in space, to recognize, understand, and describe spatial objects by referring to their geometrical properties, and by interpreting and constructing different forms of representation of spatial-visual objects (verbal, pictorial and action-based).

Spatial language, which is the language used to speak about spatial objects, their spatial position and spatial relations between two or more objects, is essential to describe spatial objects in space. Levinson (1996), Landau and Jackendoff (1993) emphasize the importance of an analysis of spatial language to understand the underlying spatial concepts. Coventry et al. (2009) highlight the important of spatial language to occur in a dialogue, rather than in a monologue, because the interaction in a dialogue gives the opportunity for the learners to participate more actively and creates a less artificial and less restricted setting. This is an important assumption which has been considered in the design of this study.

## METHODOLOGY

### Design of Spatial Task

Based on the theoretical background about spatial abilities and spatial language, a spatial task was designed to allow students to verbalize their spatial thinking and induce the use of spatial language in a communication-friendly setting. The setting for the implementation of the spatial tasks was the reconstruction method, which can be described as a learning environment in which two or more learners seated in a back-to-back arrangement communicate with each other to solve a task, based on constructivism and material-based learning. The task in the reconstruction method should promote communication among learners and can be dismantled in a series of steps, which one learner has to communicate to the other. The task involved two students – one, the giver or describer and the other, the receiver or the builder – working together to describe and reconstruct a spatial object. During the task, the giver was given a spatial object (see Figure 1) and given the following instruction:

“In this experiment you [the giver] will be given an object made up of these building blocks, which can be put together. You must give him/her [the receiver] instructions on how to build

this object, so that he/she [the receiver] can reconstruct the same object. The colour of the building blocks is not important and whilst you [the giver] are describing you can also touch and move the object as you like, but the object structure has to remain unchanged. At the end, the objects' structure must be identical."

Such a spatial task requires a suitable spatial object which the giver can describe to the receiver, so that the latter can reconstruct the spatial objects using the provided cubes according to the giver's instructions. The structure of the spatial object in the spatial task should activate student's spatial and geometrical knowledge and allow different use of metaphors during its description and its construction. In acknowledgement of the fact that there are indefinite ways of generating spatial objects from building cubes, I considered the following criteria to choose an adequate structure of the spatial object (see Figure 1): three-dimensionality (students are required to describe along the three dimensions), break-down (students can break down the object in different ways), and spatial relations (orthogonal spatial relation between different parts of the object).

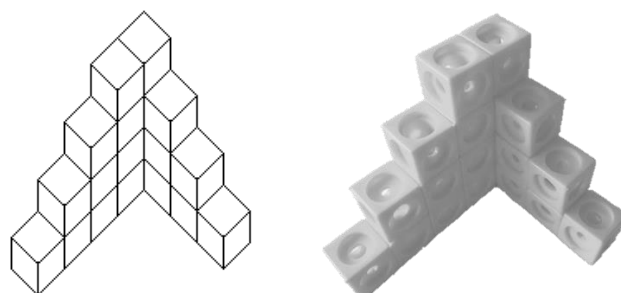


Figure 1: The structure of the spatial object of the spatial task in the reconstruction method

### Implementation and Data Analysis

Based on a theoretical sampling considering different influencing factors (language proficiency, spatial abilities and gender), 16 students attending the 5<sup>th</sup> Grade were chosen to describe the designed spatial object in the spatial task to another receiver in the reconstruction method. Both giver and receiver were seated in a back-to-back position and were given the previously mentioned instructions to solve the spatial task in a dialogue during the whole task solving process. The participants were filmed and their discourse was transcribed. Considering an interpretative qualitative approach (Jungwirth 2003), the collected data was analysed for use of metaphors regarding the above research questions and the underlying theoretical assumptions.

## RESULTS AND DISCUSSION

### On the Nature of Spatial Metaphors

Metaphors have been used by every student during the spatial task solving process in the reconstruction method. The first recognizable differentiation between the types of metaphors used was the reference to everyday-life situations and to mathematical terms and concepts. The former types of metaphors, everyday-life language (or BISC) spatial

metaphors are grounded metaphors, since they relate a target domain in mathematics (geometry and spatial abilities) to a source domain outside the world of mathematics (everyday-life context). In Table 2 a typical example of this spatial metaphor is “stairs” to describe the spatial object (in Figure 1).

<i>Source Domain</i>	<i>Target Domain</i>	<i>Transcript</i>
Balk-Stairs	Spatial Object	
Structure of balk-stairs	Properties of an internal part of spatial object	<i>“Okay, at the bottom it has... firstly it has only... here... only a <u>balk with four stones</u> and a <u>stairs downwards</u> with three, two and one.”</i>

Table 3: An example of grounding metaphor “stairs”

In the example in Table 2, the property of a stairs, which is the property of bridging a vertical distance by dividing it in smaller equidistant vertical distances, is projected on the internal part of the spatial object. Thus stairs is a grounding metaphor used by the student to describe the underlying structure of the spatial object.

Mathematical language (or CALP) spatial metaphors are linking metaphors, in which both source and target domain originate from mathematics, in particular, geometry. Consider the following mathematical spatial metaphor “pyramid” (a geometrical term rather than an everyday-life phenomenon), in which the giver uses it to describe the spatial object. In the case, the transferred property is that of convergence to a single point at the top, which is indicated by the describing student’s gestures.

<i>Source Domain</i>	<i>Target Domain</i>	<i>Transcript</i>
Pyramid	Spatial Object	
Structure of pyramid	Structure of the spatial object as a whole	<i>“Now do you almost have like a <u>pyramid</u>?” (Student points with his fingers on the top most cube of the object)</i>

Table 4: An example of a linking metaphor “pyramid”

In general, grounding and linking metaphors have been used to describe the following aspects: (i) the structure of the spatial object (see Table 2 and 3), (ii) the spatial position of the spatial object or its internal part and (iii) the spatial relation between two spatial objects or internal parts of the same spatial object.

The spatial position of the spatial object or its internal part was also described by students using everyday spatial metaphors, which mostly give the spatial object human characteristics, such as “looking” in Table 4.

<i>Source Domain</i>	<i>Target Domain</i>	<i>Transcript</i>
Human Body	Spatial object	
The ability of looking or seeing	Spatial position of an internal part of the object	“No, I mean that the foursome stairs should... <u>its top should look in your direction</u> , so straight, nor to the left and neither to the right”

Table 4: An example of spatial metaphor to describe the the spatial position of an internal part of the spatial object.

The next example in Table 5, shows how spatial metaphors, “wall” (everyday-life language) or “triangle” (mathematical language), can be used to describe the spatial relation between two internal parts of the spatial object.

<i>Source Domain</i>	<i>Target Domain</i>	<i>Transcript</i>
Wall; Triangle	Spatial object	
Property of two adjacent walls; Property of a triangle.	The spatial relation between two internal parts of the spatial object	“So a balk which is standing upwards with four stones, and then three stones at the side, and then two and one. And then on the other side like a wall... like a triangle and again four, then three, then two and then one.”

Table 5: An example of everyday-life metaphor “wall” and mathematical metaphor “triangle” to describe the spatial relation between two internal parts.

### Spatial Metaphors as Barriers

By referring to two episodes of students solving the spatial task in the reconstruction method, I will point out how the too frequent use of metaphors can also act as barriers during the spatial task solving process.

The designed spatial task requires the verbalization of spatial configurations and relations between one part of the object and another. This task demand is necessary due to the back-to-back position of the giver and the receiver in the reconstruction method. However, overall it has been noticed that students would rather rely on the use of metaphors, instead of focussing on verbalizing the spatial position and relations, such as right to, left to, on top on, behind of, etc... In the following conversation, the giver attempts to describe the spatial relation of the three cubes (which is the beginning of the reconstruction of second internal part) and another internal part of the object (“one foursome stairs”), however, upon the receiver’s question, the giver changes back to metaphors:

Giver: “*Then you have to do three cubes not next to it, but at the back where there are the last cubes, do three at the ground*”

Receiver: *"Where?"*

Giver: *"Like two stairs who are back-to-back."*

Receiver: *"This means that I need to do another stairs on top of it."*

Giver: *"No!"*.

The fact that the giver changes to the metaphor "stairs who are back-to-back" shows that the giver would rather rely on spatial metaphors to communicate spatial knowledge, rather than rephrasing and explaining again the spatial location where the three cubes must be constructed.

Another episode from the same student shows that the use of spatial metaphors is not always helpful. This can be led back to the receiver's misinterpretation of the giver's spatial metaphors, which is evident in the following transcript excerpt:

Giver: *"you have to... do it in a way that the foursome stairs looks like a wall, so that..."*

Receiver: *"this means that I have no cubes on which one can go up?"*

Giver: *"from your perspective it should be on the left side so that the foursome stairs..."*

Receiver: *"this means that I must have (counting) four, eight, twelve cubes together, like a bar of chocolate then"*

Giver: *"No, (...)"*.

During the second attempt to describe the orthogonal spatial relation between both internal parts of the object, the student refers to the metaphor "wall", whereas one internal part of the two should act as a wall in relation to the other. However, for the receiver a wall is rectangular, thus proposing the reconstruction of the "stairs" in a "bar of chocolate". This example shows how metaphors constructed in a dialogue induce different mental images in the student's minds and how they are not always successful alternatives to explicit descriptions of spatial relations based on spatial prepositions.

## CONCLUDING REMARKS

This empirical study suggests that the concept of spatial metaphors are important tools to understand student's spatial thinking and its verbalization. The study provides examples of spatial metaphors to describe the structure of a spatial object or a part, their spatial position, or their spatial relation, considering the differentiation between linking and grounding spatial metaphors. These types of metaphors enable an insight in student's spatial thinking about spatial objects and their construction, which can be used as a point of reference to improve and train student's spatial and geometrical thinking. Furthermore, this study also suggests that the use of metaphors on its own is not the best way to solve spatial tasks with language as a presentation mode. Thus this study highlights the importance of analysis of spatial metaphors in spatial language learning and its importance to solve verbal spatial-geometrical tasks. This study should



also highlight the importance of dialogue-based spatial tasks which are essential for the interpretation and the reflexion of use of spatial metaphors in the task solving process.

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# UNDERSTANDING MATHEMATICS FROM A HIGHER STANDPOINT AS A TEACHER: AN UNPACKED EXAMPLE

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*In this paper we characterize the kinds of connections that a teacher can establish in the classroom to address a mathematical situation. In particular, we show how, in the case of a classic problem, the different mathematical pathways that can be followed in a classroom require of the teacher the knowledge and ability to use different connections. Being able consciously to follow and exploit these connections in a classroom gives the teacher a privileged insight into the content, allowing him or her to understand it from a higher standpoint.*

*En este documento presentamos una caracterización del tipo de conexiones que un profesor puede establecer en el aula al abordar una situación matemática. En particular, mostramos como, en el caso de un problema clásico, las distintas vías matemáticas que pueden seguirse en un aula requieren en el profesor del conocimiento y la capacidad de uso de diferentes conexiones. La conciencia de poder seguir y explotar todas estas conexiones en un aula dan al profesor una visión privilegiada sobre el contenido, ya que le permiten comprenderlo desde una perspectiva superior.*

## INTRODUCTION

A deep knowledge of mathematics is an essential element in teaching the subject (Ma, 1999). In that sense, the topic of teachers' knowledge has been an important focus of research in teacher education. Several researchers have developed their work around the notion of a broader mathematical knowledge, understood as knowledge of the mathematical landscape (Ball & Bass, 2009), a peripheral vision of content (Foster, 2011), some familiarity with mathematics (Jakobsen, Thames, & Ribeiro, 2013), or a structural view of mathematics (Carrillo, Climent, Contreras, & Muñoz-Catalán, 2013). All these ways of perceiving teacher knowledge provide the researcher with an approach to the teacher's content knowledge that focuses on its relation to other mathematical content.

In the research literature on teachers' knowledge several conceptualizations have been developed (e.g., *the knowledge quartet*; Rowland, Turner, Thwaites, & Huckstep, 2009; *mathematical knowledge for teaching (MKT)*; Ball, Thames, & Phelps, 2008; and *mathematics teachers' specialised knowledge*; Carrillo et al., 2013). Such conceptualizations consider connections explicitly or implicitly—although with different interpretations and views. Rowland et al. (2009) consider connections in an explicit way as a knowledge domain; Carrillo et al. (2013) assume them to be part of some subdomains of teachers' knowledge, and Jakobsen et al. (2013) see them in a refinement of the existing model for MKT. Connections play an important role both

when deepening the coherence of planning one or more lessons and when sequencing the tasks in those lessons (Turner & Rowland, 2011). Connections are used when responding to students' comments or giving meaning to their productions (e.g., Ribeiro, Jakobsen, & Mellone, 2015).

Although connections are considered of crucial importance in and for teaching, they are generally perceived in a broad sense. We have adopted Kilpatrick's (2008) proposal to refine Klein's (1908) idea of understanding mathematics *from a higher standpoint*, using connections as a tool to achieve a nuanced understanding of this idea. To provide a deeper understanding of the role and importance of connections in the teaching and learning process, we present a categorization of the kinds of connections and then discuss them using a task (D'Angelo & West, 2000) that has been implemented with prospective primary teachers.

## A CATEGORIZATION OF KINDS OF CONNECTIONS

For a teacher, understanding mathematics from a higher standpoint involves being deeply familiar with the mathematical field being explored with students so that the teacher can guide their mathematical activity, preventing them (or the teacher) from becoming disoriented or taking a path that will be detrimental (Kilpatrick, 2008). To be such a guide, teachers need specialized knowledge and ability that will allow them to select, modify, and implement mathematical tasks (e.g., Charalambous, 2008; Ribeiro & Carrillo, 2011) in the classroom. This capacity is closely related to the connections that a teacher is able to establish, use, and manage with respect to the mathematical content being taught. Under the notion of connection as we see it, the elements being connected can be mathematical concepts, procedures, or properties, in the context of the problems, tasks, or situations available in a mathematics class.

In the following, in light of previous work (Montes, Carrillo, & Ribeiro, 2015), we present and discuss a characterization of connections having six dimensions: (i) *outside mathematics connections*; (ii) *intra-conceptual connections*; (iii) *transverse connections*; (iv) *auxiliary connections*; (v) *K-connections*; and (vi) *syntactic connections*. Discussing and interpreting the content of these different categories of connections allows us to delve into different ways of relating content that can contribute to improving instructional practice.

(i) *Outside mathematics connections*: These connections appear when a mathematical situation is related to a typical situation from contexts unrelated to mathematics (e.g., modelling real-world situations). Such connections are usually provided to evoke elements of the phenomenology of a school concept (Freudenthal, 1983). (ii) *Intra-conceptual connections*: In addressing a particular task, the teacher must often understand and use various aspects related to the same concept, which may be expressed in various representation systems, whether graphic, linguistic, or semiotic (e.g., Ribeiro et al., 2015), or diverse properties of the mathematical object being addressed, as different meanings of the concept can allow the teacher to evoke the concept image (Vinner, 1991). In addition, a teacher must know different features of a

mathematical object whose coordination in a given situation will allow students to extend their mathematical knowledge. (iii) *Transverse connections*: This type of connection is linked to the nature of some mathematical objects that are displayed in various forms and in different contexts throughout the various stages of education (e.g., infinity, proportionality, or the notion of equality). Thus, the teacher can make use of a mathematical concept like equality by evoking previously studied contexts and forms to foster a greater understanding of the task being addressed. (iv) *Auxiliary connections*: In some cases, the teacher can use mathematical concepts that are not directly linked to the situation being considered in the classroom. That allows the teacher to provide some additional mathematical context to enrich the activity or as a tool to solve part of the problem without making it the focus of attention. (v) *K-connections*: These connections are inspired by Klein's (1908) original work and the interpretation by the discipline that links it to contemplating the mathematics addressed in a specific context from the perspective of varied complexity. In the next section, we explore this idea further, allowing us to discuss two possible variations in complexity. These kinds of connections—and, in particular, the associated increase in complexity—were at the core of the conceptualization of *horizon content knowledge* (Ball & Bass, 2009). (vi) *Syntactic connections*: Schwab (1978) proposed a type of content knowledge linked to the rules governing how the discipline generates new knowledge. In the case of mathematics, for example, when a proof is developed, it has an underlying logic, such as proof by contradiction, proof by exhaustion, or generalization. This knowledge needs to be identified as usable in a specific situation, generating a syntactic connection as, for example, when one considers proving a property of the natural numbers and considers proof by induction as a kind of proof commonly used in this mathematical context.

This classification is not intended to establish a partition of the kinds of connections; instead, it should be considered a way of characterizing the different kinds perceived as intertwined. Figure 1 illustrates how the six different kinds of connections might be related to each other when school mathematics is viewed from a higher standpoint. Once students have undertaken a mathematical task, for example, the teacher might encourage them to make one or more intra-conceptual connections, representing the problem in different ways. Or an outside-mathematics connection might be made. The K-connections are quite important because they can work in either the direction of simplification or the direction of complexification, in each case leading to a modified task. The teacher might instead bring in an auxiliary connection to different but related mathematics or a transverse connection, which would be related to a previously encountered mathematical concept that could assist students in understanding the task. Finally, syntactic connections can permeate the task when students address the logic of what they are being asked to do. Consequently, various rules of reasoning can give rise to syntactic connections.

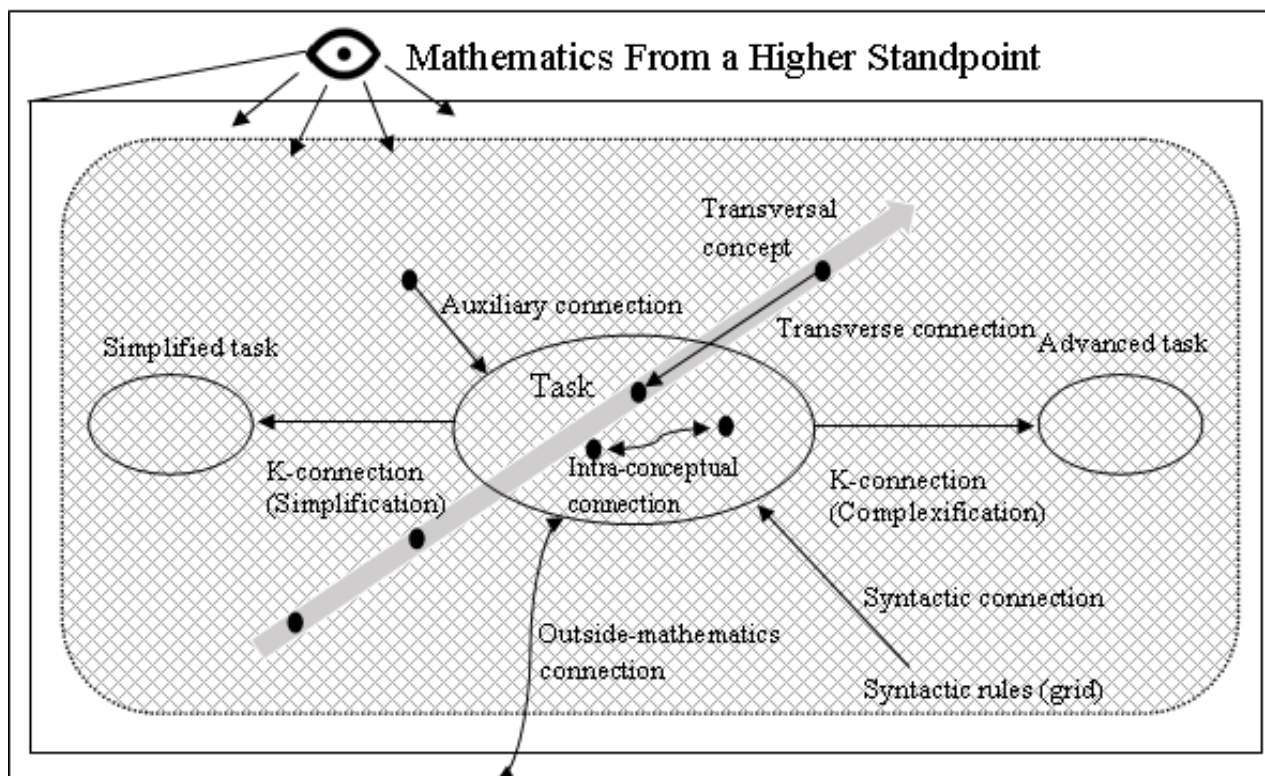


Figure 1: Intertwined connections and mathematics from a higher standpoint

### UNPACKING THE CONNECTIONS: AN EXAMPLE

In this section, we illustrate how a deep discussion of the different paths that can be taken to solve a specific problem illustrates the various connections that can be generated from it. The problem to be discussed is the following: “*At a meeting there are  $n$  people, and all greet each other with a handshake. How many handshakes are there?*” (D'Angelo & West, 2000; Kilpatrick, Swafford, & Findell, 2001, pp. 107–110).

We follow the process of *unpacking* (Ma, 1999) to show possible ways of exploring the mathematical content of the problem as linked to different kinds of connections. These possible connections arise from various real discussions in the initial training of teachers.

In particular, this problem allows for profound work in terms of *intra-conceptual connections*, in this case linked to the case of representation systems. The problem can be addressed, for example, from a numerical perspective by noting that each person shakes hands with  $n - 1$  people, which, after dividing by 2 to avoid repetition, yields  $n(n - 1)/2$  handshakes. One can also recognize that, if the first person we consider greets  $n - 1$  people, to avoid repeating handshakes the next person we consider greets  $n - 2$ , and so on, leading to the sum  $(n - 1) + (n - 2) + \dots + 2 + 1$ ; the result is equivalent to the previous expression. Another approach may be to link this situation to a geometric context in which one can think of the people as the  $n$  vertices of a polygon, which would require adding its sides and diagonals to get the total number of handshakes. Exploring how these solutions are interrelated leads to a deep

understanding of the problem without abandoning the mathematical concepts involved in it, which generates an intra-conceptual connection.

Also, this problem can be thought of from the perspective of its implementation in a classroom for special cases of  $n$  using real objects. For example, one might propose that the students form increasingly larger groups in which, instead of shaking hands, they recreate the interaction of two people by tossing a ball or holding the ends of a length of string. The relationship of these proposals to the problem itself is phenomenological, seeking real contexts in which the different aspects of the problem can be identified. This connection is *outside mathematics*, as the mathematical elements are linked to nonmathematical elements, although a deep understanding of those elements is required to establish their similarity.

Third, we find the *K-connections*. When the above problem is posed in the context of initial teacher training, the most common approach is to try to simplify it to particular cases. Beyond the validity of this approach, which we discuss under *syntactic connections*, the approach is one way to see a general problem from a simpler perspective. For example, we can consider the case of a group of 6 people greeting each other, and through a counting process, significantly less complex than the abstraction corresponding to the general case, arrive at 15 handshakes, establishing a *simplification connection*. These connections are useful to a teacher, for example, in modifying the difficulty of a classroom task, either reducing it through simplification or increasing it through a *complexification connection*.

To show the remaining kinds of connections, we resort to a variant of the same problem: *If at a meeting everyone shakes hands once, and there are 120 handshakes, how many people are there?* The problem is basically the same; it requires identifying the general expression for a number of people and then calculating the value of  $n$  for  $n(n+1)/2 = 120$ . That requires solving the quadratic equation using the usual formula for such situations. The use of the quadratic formula in this case has a utilitarian nature, generating an *auxiliary connection* between the problem and the solution of quadratic equations.

In attempting to help students move from the specific, 120, to the general,  $n$ , the teacher might invoke the concept of potential infinity, which the students had encountered previously, as a means of suggesting that the number of people at the meeting can keep increasing indefinitely, and the number of handshakes will keep increasing accordingly. By tying the generalization to infinity, the teacher is making a *transverse connection*.

Finally, discussions may emerge about elements linked to the validity of certain approaches. For example, one possible way to address the problem would be to give particular cases to a few people and try to generalize the counting procedure they use. In this case, one should consider whether this approach is successful in terms of the validity of the expression that one gets. That is, consider whether the expression of a generalized formula for a finite number of cases is mathematically correct. This is

related to knowledge detached from specific problem content that one can use with other mathematical content and that is closely linked to ways of demonstrating and establishing validity in mathematics. Therefore, a *syntactic connection* is generated between this kind of knowledge and the specific mathematical problem. This problem, as such, also raises the possibility of being linked to a heuristic problem-solving strategy: *look for a pattern*. That strategy is again detached from the specific content of the problem, and the naturally established connection between the strategy and the problem is syntactic knowledge.

## CONCLUSION

We have provided a classification of the different kinds of mathematical connections that a teacher can establish. Like Ma (1999), we believe that a teacher requires a thorough knowledge of mathematics, but even more, we believe that the practice of mathematics requires a panoramic insight into the content. That is, a teacher must not only be competent in the treatment of content but must also decide what kind of treatment to use at all times.

Klein (1924/1932) wrote: “The teacher’s knowledge should be far greater than that which he presents to his pupils. He must be familiar with the cliffs and the whirlpools in order to guide his pupils safely past them” (p. 162). A vision of the surrounding mathematical territory (from a higher standpoint), as well as a knowledge of the different paths one can take through it (connections), enables the teacher to perform the task of guiding pupils more safely and confidently given the familiarity and ease with which he or she can make decisions about which path to take and how to take it. Of special interest for this task are the K-connections, which allow the teacher to transform the content of the session by varying its difficulty. So in using a complexification connection, the teacher might link the content to other topics and courses, or at later stages, for example, might increase its difficulty. Also, this type of connection appears when the teacher knows a more advanced concept that could justify some mathematical fact that has come up during instruction. As for simplification connections, they have the same nature as the previous ones, based on looking for simpler versions of the concepts or mathematical tasks being addressed and attempting to justify an item, property, or mathematical result from a simpler conceptual perspective.

From our perspective, it is necessary for students to develop a connected knowledge of mathematics rather than understanding each concept as an isolated element. For that, the teacher should encourage students to solve complex problems, being aware of the content organization, heuristic strategies, and control of the solution process (Kilpatrick, 1985). We believe, therefore, that it is necessary to develop teachers’ ability to generate mathematically rich situations in their own teaching practice. For that, an in-depth discussion of problems such as the one shown here and all the possible solution paths one might pursue can not only nourish their mathematical knowledge but also develop their accompanying management skills.

For future research, we plan various courses of action such as designing, for initial and continuing teacher training, situations rich in possible connections in and from the situation itself, in order to work on developing teachers' reflective ability concerning mathematical content. Another way forward is to study and identify the kinds of connections that teachers activate and use in the classroom, observing whether there is any precedence of one over another, and the possible assessment of the actual use of connections by teachers at different educational levels.

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# GRAPHING AS FIGURATIVE AND OPERATIVE THOUGHT

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*Researchers have emphasized the importance of students conceiving graphs as representing quantities that vary in tandem. The same researchers have argued that much is left to understand relative to the extent that students conceive graphs as such. In this paper, I draw theoretical distinctions in students' graphing activities that provide insights into the extent that they conceive graphs as constituted by covarying quantities. Namely, I distinguish between students' graphing activities that foreground particular sensorimotor experience and students' graphing activities that foreground imagining a graph as an emergent trace of covarying quantities.*

## INTRODUCTION

Students' representational activities remain a pressing focus in mathematics education with researchers having problematized students' activities in the context of their problem-solving and learning. A subset of researchers' foci concerns students' graphing activities. For instance, researchers have investigated students' graphical "inventions" to organize and represent quantities' measures (diSessa, Hammer, Sherin, & Kolpakowski, 1991). Relevant to the present work, other researchers have characterized students' reasoning about quantities that change in tandem—termed *covariational reasoning*—with respect to their graphing activities (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Thompson, 2011). Collectively, these researchers have stressed the need to clarify nuances in students' graphing activities and how these nuances relate to the extent that covariational relationships constitute students' activities. The present work is both theoretically and empirically driven in this regard.

Compatible with Dewey's (1929) and Thompson's (1994) comments on the inseparable nature of theory and practice, what follows is a theoretical characterization of student' graphing activities inevitably shaped by constraints and affordances I have experienced working with students. The paper is theoretically driven in that I adopt a radical constructivist perspective on representations (von Glasersfeld, 1995) and use Moore and Thompson's (2015) distinctions in students' graphing activities—termed *static shape thinking* and *emergent shape thinking*—to characterize students' graphing activities. Furthermore, I consider Piagetian notions of figurative and operative thought (Piaget, 2001; Steffe, 1991; Thompson, 1985) to clarify Moore and Thompson's distinctions in students' graphing activities. The paper is empirically driven in that I draw from studies on student thinking to provide examples in which students' graphing activities are dominated by either figurative or operative elements of thought. I close with the implications of this theoretical report and identify potential future lines of empirical inquiry.

## THEORETICAL BACKGROUND

One theoretical approach to characterizing students' representational activities entails researchers making distinctions between *internal* and *external* representations. The former refer to various mental structures and the latter refer to artifacts existing in the material surrounding (see Izsák, 2004). An alternative theoretical approach entails researchers approaching students' representational activities as irreducible to internal and external representations (Piaget & Inhelder, 1967; Thompson, 2013; von Glasersfeld, 1987). This approach does not deny source material in the form of sensory input; but source material is perceived subjectively and constrained by one's operations of thought. Following this perspective, I approach artifacts (e.g., graphs, equations, and inscriptions) as inseparable from an individual's knowledge structures that constitute constructing or conceiving drawn or imagined artifacts.

An implication of approaching students representational activities as entailing active, subjective constructions is rejecting the notion that an observer (i.e., researcher) can describe another person's representational activities in terms of that person interpreting or filtering objective (or observer) defined artifacts in the material surrounding. Commenting on perspectives that do attempt to bridge conceptual structures and the ontological world, von Glasersfeld (1987) explained,

...it makes no sense to think of mental representations as any kind of...depiction of ontological reality...there is little benefit in speaking of "representations" or, indeed, "translation", where, as Kant's *Critique* has so irrefutably shown, there is no logically possible access to what they are supposed to represent.

That is, artifacts are defined by the beholder and thus it is an impossible task for an observer to identify or describe an artifact as it exists in the material surrounding—doing so would require that the observer be able to step outside her or his operations of thought. Rather, an observer can only hypothesize operations of thought that explain the observed actions of another while acknowledging that these hypotheses are constrained by the observer's subjective operations of thought (Steffe & Thompson, 2000; von Glasersfeld, 1995).

A theoretical distinction I use to develop hypothetical models of students' graphing activities is the distinction between *figurative* and *operative* thought (Piaget, 2001; Steffe, 1991; Thompson, 1985). Piaget primarily used the two forms of thought to distinguish thought based in and constrained to sensorimotor experience (including perception) from thought in which figurative elements are subordinate to mental operations. Steffe (1991) provided illustrative examples of these distinctions in explaining children's construction of number. He described that a figurative counting scheme is a scheme in which a child is able to count without perceptual material directly available, but he or she requires re-presenting particular sensorimotor actions (called figural unit items, which can be visual such as dots in an array or sensory such as finger taps) in thought when counting. Steffe defined children's operative counting schemes as entailing unitized records of counting that contain records of but are not

constrained by re-presenting particular perceptual material or sensorimotor experience. In this paper, I expand on these notions to describe the extent students' graphing activities are subordinate to figurative or operative elements of thought.

## COVARIATIONAL REASONING AND TWO PERSPECTIVES ON GRAPHS

Related to Steffe's (1991) expositions, Thompson (2011) described a model of a person reasoning mathematically via constructing measurable attributes (i.e., *quantities*) and relationships between quantities. One form of relationships between quantities is covariational reasoning, or "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354). Building on researchers' (Carlson et al., 2002; Thompson, 2011) descriptions of covariational reasoning, Moore and Thompson (2015) provided distinctions in students' graphing activities termed *emergent shape thinking* and *static shape thinking*.

Moore and Thompson (2015) described students' emergent shape thinking as conceiving a graph as a locus or trace via coordinating two quantities' magnitudes simultaneously. They explained, "emergent shape thinking entails assimilating a graph as a trace in progress (or envisioning an already produced graph in terms of replaying its emergence), with the trace being a record of the relationship between covarying quantities" (Moore & Thompson, 2015, p. 785). Figure 1 illustrates instantiations of emergent shape thinking. I emphasize emergent shape thinking entails imagining magnitudes (along the axes) in flux between instantiations, with a coordinate point representing a cognitive uniting of the magnitudes.

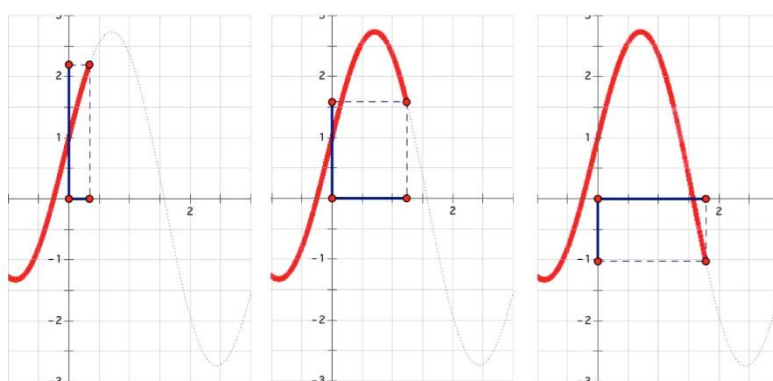


Figure 1: Instantiations of emergent shape thinking.

Moore and Thompson (2015) described students' static shape thinking as operating on a graph as an object in and of itself (i.e., graph-as-wire) and basing actions on perceptual cues and physical features of a graph. For example, the authors provided an example of students' static shape thinking in the form of students associating rate of change values with properties of direction (e.g., a student reasoning that a graph of  $y = 3x$  unquestionably implies a line sloping upward left-to-right regardless of coordinate system or orientation). As another example, Moore and Thompson described students associating shapes with various function names or classes regardless of coordinate

orientations or system (e.g., a graph that curves up unquestionably implies an exponential function).

I interpret Moore and Thompson's (2015) distinctions as compatible with distinctions between figurative and operative thought. In the case of students' static shape thinking, student actions are subordinate to perceptual (figurative) properties of shape. Students' emergent shape thinking, however, foregrounds the coordination of actions—specifically that of covarying quantities—so that figurative elements of their activity are subordinate to that coordination. Further illustrating relationships between figurative thought and students' static shape thinking, Moore and Thompson argued that if a students' graphing activities are dominated by static shape thinking, mathematical objects (e.g., rate of change) become properties of or subordinate to perceptual features of graphs (e.g., direction of a line). With respect to emergent shape thinking, mathematical objects become properties of the coordination of actions (e.g., rate of change as a measure of how quantities change), which enables a person to understand a variety of perceptually different situations (e.g., graphs in different orientations or coordinate systems) as representing an equivalent coordination of actions. In this paper, I expand on these notions to describe the extent that students' actions are subordinate to figurative or operative elements of thought. In doing so, I extend students' static shape thinking to include figurative aspects of thought (e.g., the physical drawing of a graph) not detailed by Moore and Thompson.

## DATA SOURCES

The data in this report are from clinical interviews (Ginsburg, 1997) or teaching experiments (Steffe & Thompson, 2000), each of which involved conceptual analysis techniques for the purpose of developing viable models of student thinking (von Glasersfeld, 1995). Although the examples in this paper stem from several studies and various methods, I conducted each of these studies with a common goal. Namely, I sought to develop explanatory models of U.S. collegiate students' thinking with attention to the extent that they conceptualize situations constituted by quantities (i.e., measurable attributes) and relationships between those quantities. I use the term situations to refer to students' uses of representations traditionally associated with mathematics (e.g., graphs, tables, and equations), as well as students' experiences with physical phenomena (e.g., an amusement park ride). For instance, colleagues and I characterized how students drew graphs among polar and Cartesian coordinate systems by coordinating covariational relationships between quantities over the course of a teaching experiment (Moore, Paoletti, & Musgrave, 2013). In another study, Carlson and I (2012) reported on clinical interviews to characterize how students' images of physical phenomena influence their constructions of graphs and formulas. I direct the reader to these studies for more detailed discussions of results and methods (including the validity of these methods).

In separating the following two sections according to figurative or operative thought, I do not intend to imply that a data example illustrating figurative thought is absent of

operative thought, and vice versa. I use the following two sections to illustrate the extent that particular figurative or operative elements of thought dominate students' graphing activities.

## GRAPHING AND OPERATIVE THOUGHT

Recall that a key aspect of operative thought is it is not constrained by sensorimotor experience (including perceptual material), although it contains records of such experience. An example of operative thought in the context of students' graphing activities involves the coordination of actions such that an individual comes to conceive equivalence among situations that are perceptually different or entail different sensorimotor experiences. For instance, in previous work, Paoletti, Musgrave and I (2013) illustrated students conceiving graphs in the polar and Cartesian coordinate systems as representing the same covariational relationships despite perceptual differences in graphs (e.g., a linear relationship being conveyed by a line in the Cartesian system and an Archimedean spiral in the polar system).

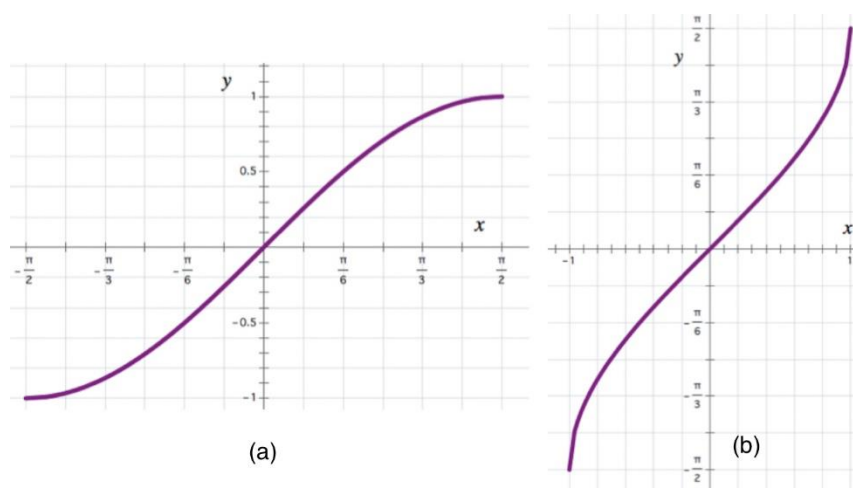


Figure 2: Two emergent traces of the sine relationship.

As another illustration, consider Noli's actions with respect to the graphs in Figure 2.

Noli: *[Noli has claimed that Figure 2a and 2b represent  $x = \sin^{-1}(y)$  and  $y = \sin^{-1}(x)$ , respectively] They're both representing the same thing just considering their outputs and inputs differently [referring to axes orientation]...So it's like here [referring to Figure 2b,  $y > 0$ ], with equal changes of angle measures [denoting equal changes along the vertical axis] my vertical distance is increasing at a decreasing rate [tracing graph]...here [referring to Figure 2a,  $x > 0$ ] it's doing the exact same thing. With equal changes of angle measures [denoting equal changes along the horizontal axis] my vertical distance is increasing at a decreasing rate [tracing graph]...this one looks like it's concave up [referring to Figure 2b from  $0 < x < 1$ ] and this one concave down [referring to Figure 2a from  $0 < x < \pi/2$ ], it's still showing the same thing.*

Noli's activity is an example of Thompson's (1985) description of operative thought as "actions of thought mov[ing] to the level of controlling schemata" (p. 195). In this

case, Noli held in mind that her activity, regardless of perceptual or sensorimotor differences among the graphs, represented an equivalent emergent trace of covarying quantities; it was this coordination of quantities that controlled her figurative activity.

## GRAPHING AND FIGURATIVE THOUGHT

As a contrast to Noli's activity, an example of a student's activity that is dominated by figurative thought would entail concluding the Figure 2 graphs cannot both represent the sine function due to the graphs "looking" different or having different concavities. In fact, some students' thinking is subordinate to figurative thought to the extent they maintain that Figures 2a and 2b unquestionably represent different functions despite their identifying that any coordinate pair on one graph can be related to a coordinate pair on the other graph (e.g.,  $(\pi/2, 1)$  in Figure 2a corresponds to  $(1, \pi/2)$  in Figure 2b). I interpret Moore and Thompson's (2015) description of students' static shape thinking to be compatible with this example of students' activities being subordinate to figurative thought; students' static shape thinking and this example foreground thought based in aspects of perception and shape.

Extending Moore and Thompson's (2015) description of students' static shape thinking, students' graphing activities can also be subordinate to other aspects of sensorimotor experience including the physical drawing of a graph. As one example, some students hold ways of thinking for graphs that entail the sequential actions of plotting a point along the vertical axis and drawing a graph left-to-right from that point. To illustrate, consider Belle's activity when attempting to graph a relationship between two distances (i.e., distance from city A, represented along the vertical axis; distance from city G, represented along the horizontal axis) such that the distance from G decreased at a constant rate with respect to an increasing distance from A.

Belle: Your distance from A starts at zero [*plots point at Cartesian origin*] because you're in A. Um, so as you get. Mmm, no, you're gonna start up here [*plots point on vertical axis to indicate non-zero distance from G*]. Ignore that [*covering origin*]. 'Cause, oh wait, no, stop [*crosses out second plotted point*]. No, you're here [*points at origin*]...[*Belle eventually settles on a point along the vertical axis indicating a non-zero value and suggests drawing a segment sloping downward left-to-right from her initial point*]. I wanted to show that the distance [from G] was decreasing [*motioning diagonally down and to the right from the point plotted on the vertical axis*], but that means that your distance from A is decreasing [*tracing along the vertical axis from the initial point to the origin*]...But your distance from A is growing. But your distance from G is decreasing. So, if that's growing [*draws arrow pointing upward beside the vertical axis label*] and that's decreasing, so [*draws arrow pointing downward beside horizontal axis label*]...[*the student works for six additional minutes without making progress*].

I highlight that although Belle had in mind the relationship she intended to represent (some quantity decreasing at a constant rate while the other quantity increased), her

actions were dominated by envisioning where to start a graph (i.e., along the vertical axis) and how to draw a graph (i.e., from left-to-right), which inhibited her ability to represent the intended relationship. Belle later identified a ‘starting’ point along the horizontal axis, however, she remained perturbed by the location of the point and having to draw a graph right-to-left from that point. She explained, “But I don’t want to start like, like I don’t like starting graphs...That’s weird...my graph is from right-to-left, which is probably not right.” Belle finished the task perturbed due to figurative elements of her graphing activity contradicting the graph representing the relationship she intended to represent, which suggests that her ways of thinking for graphs were subordinate to these figurative elements of thought.

## LOOKING FORWARD

Piaget (2001) and Steffe (1991) described cognitive development as a transition from thought dominated by figurative activity to thought dominated by operative activity. Due to the prevalence of graphical representations in students’ schooling, one would expect their graphing activities to become predominantly operative by college. In my work, however, I have repeatedly and frequently experienced constraints when studying undergraduate students’ thinking due to their graphing activities being dominated by figurative thought. Most significantly, this outcome can occur in cases in which students have conceptualized some relationship they intend to graph, but figurative elements of their thought inhibit their ability to construct a graph (see Belle). I interpret these findings to warrant deeper investigations into students’ development of graphing activities with attention to figurative and operative thought. One potential productive line of inquiry is investigating the cognitive reorganizations that occur as a student transitions from graphing activities dominated by figurative elements of thought to graphing activities dominated by operative elements of thought. I envision that researchers carrying out this line of inquiry will also provide insights into instructional supports that afford (or constrain) such transitions. Another productive line of inquiry might involve researchers investigating students’ graphing activities in order to characterize how figurative and operative elements of thought influence their learning of mathematical topics (see Thompson, 1994).

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# GENDER DIFFERENCES IN SOLVING OPEN-ENDED PROBLEMS IN MATHEMATICS

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*The impact of gender and classroom settings on students' solutions to open-ended mathematical problems is examined in this paper. The research was conducted with 103 fourth grade students from two schools in rural Jamaica. Students at one school were separated into gender classes while students at the other school remained in their coeducational class. The instruction was conducted using the open approach teaching method. Quantitative analysis suggested that overall, boys performed better than girls but gender differences varied with tasks. Qualitative analysis proved that boys and girls had many commonalities in their solution processes such as using similar strategies, and representations. A comparison of classroom settings shows that boys and girls in the coeducational classroom had greater improvement and more interactive discussions than those in single-sex classrooms.*

## INTRODUCTION

Globalization and increase in technology have caused a shift in emphasis from procedural understanding to conceptual understanding of mathematics (van Oers, 2002). The Ministry of Education (MOE) of Jamaica, concerned with Jamaican students' level of understanding of mathematical concepts, is creating different initiatives to support the improvement of students achievement outcomes. As a result, its new National Mathematics Policy calls for teachers to "use a flexible approach to the (teaching) of mathematics so that learners can be encouraged to develop their strategies for calculating and for problem-solving which they can explain to others" (MOE, 2013a, p.10). This, in effect, describes the open approach and reflects what countries such as Japan (Becker & Shimada, 1997); America (National Council of Teachers of Mathematics [NCTM], 2000); England (Cockcroft, 1982) as well as Singapore (Singapore Ministry of Education, 2010) have been advocating over the years. Researchers in these and other countries have accepted that open approach with open-ended problems can be used to develop conceptual understanding in learners of varying abilities in the same classroom. However the challenges that Jamaica faces is different from most of these countries. In Jamaica, girls are outperforming boys at all levels and in all subjects including Mathematics. Boys are also more likely to exhibit undesirable behaviors, repeat a grade, and be placed in special education programmes (Evan, 1999; MOE, 2013b). It is uncertain whether the open approach can develop understanding of mathematical concepts in both genders and subsequently reducing the gender disparity in Jamaican student.

## **LITERATURE REVIEW**

An open-ended mathematics problem, as discussed in this paper, is a question that is formulated to have multiple methods and one or more solutions (Nohda, 2000, Pehkonen, 2014). Becker and Shimada (1997) state that the goal of open-ended problems is not the final result but what can be learned during the solution process.

The open approach promotes understanding of mathematical concepts as students cannot rely on pre-determined rules and memorization to provide answers. Research has shown that students who learn mathematics with the open approach method can apply mathematical concepts better than those who learn with traditional method (Boaler, 1998); are able to solve problems in various situations long after learning (Cobb, 1988); analyze a situation and make the best decision based on given facts (Hitz & Scanlon, 2001). Most researchers have examined open approach and open-ended problems on their ability to develop conceptual understanding in both fast and slow learners simultaneously (Nohda, 1995, 2000; Chan, 2007; Pehkonen, 2014). However, additional information is needed on the gender responses in solving the open-ended problem.

The few studies about gender and academic performance in mathematics at the elementary level exist with different views and findings. Cai, (1995) compared 227 (96 girls and 131 boys) American grade 6 students' solutions to open-ended items from the QUASAR project and concluded that boys outperform girls on optional items. However, Wester and Henriksson (2000) converted selected mathematics items from TIMSS 1995 into open-ended problems and examined the response of 446 girls and 482 boys from grades 6 to 8 in Swedish schools. They concluded that females outperformed male on the open-ended items. Lindberg et al., (2010) stated, there existed no difference in mathematics performance from their meta-analysis of data from all the studies published from the year 1990 to 2007 in either gender. These studies have looked at students' response to open-ended items under test conditions only (Wester & Henriksson, 2000; Cia, 1995). Beller and Gafni, (2000) investigated the performance variance of boys and girls on multiple-choice as well as on open-ended items between 1988 and 1991 International Assessment of Educational Progress (IAEP) test on mathematics. Analysing the results of students from 20 participating countries, they argued that item format in totality cannot account for gender disparities in mathematics performance. Focusing on the test scores only might fail to reveal gender differences in students' understanding and patterns on problem solving. Using both quantitative and qualitative analysis to investigate students' response to open-ended problems within the classroom may help to reveal new insights into gender difference in solving open-ended items. Since gender performance is an issue, it seems appropriate to compare students in gender classes with those in coeducational class.

This paper examines the impact of gender and classroom settings on students' solutions to open-ended mathematical problems. Jamaica's National Grade Four Numeracy Test (MOE, 2013b) measures students' understanding of mathematical concepts. It shows

that students have the greatest difficulty with topics in the Number strand. Therefore, the research targeted topics in the Number strand in grade four.

## **METHOD**

One-hundred and three (103) fourth-grade students' from two elementary schools in rural Jamaica participated in the study. Students at one school were separated into gender classes. An all-girls class with 34 students and an all-boys class with 31 students. The 38 students at the other school remained in their mixed class. The schools were close (approximately 4 kilometers) to each other. Students attending these schools were from the same community, from households with the same Socio-Economic Status, and had the similar level of mathematical understanding according to their school records and teachers.

A pre-test was given to all participants, followed by a five-month intervention and post-test. During the intervention, teachers conducted three lessons a week with open-ended problems. Each lesson consisted of five different sections: (1) Introducing the problem (2) solving the problem by students; (3) presenting and discussing of students' solutions; (4) summarizing the lesson (5) reflecting on the lesson by students (Nohda, 2000; Lin et al., 2013). Female teachers taught the all-girls' and mixed classes while a male teacher taught the all-boys' class. Each teacher planned and implemented his or her lesson. Teachers taught the same topic during different sessions on the same day.

Collection of both qualitative and quantitative data was carried out. Pre-post tests were used for quantitative analysis. Each student's response to an open-ended problem was scored based on fluency, flexibility and originality (Becker and Shimada, 1997). Fluency is obtained through counting the correct responses. Flexibility refers to the number of different type of responses, and originality is the uniqueness or insightfulness of idea produced by the students. For qualitative analysis, each student response to test items was examined regarding solution strategies, representations, and justifications. Also, a qualitative study of peer interactions while solving open-ended problems was done.

## **RESULTS**

### **Quantitative Results**

The same items were used for both pre-test and post-test. The test consisted of both open-ended and closed questions. The main reason for the closed questions was to ensure students' computational ability while the open-ended questions tested their conceptual understanding as well as computational ability. Due to space limitation, solutions to open problems will be discussed. An analysis of boys' and girls' results on the pre and post-test show an increase in students' performance in all classes, see Table 1.

	Pre Test Mean	Standard Deviation	Post Test Mean	Standard Deviation
Single Boys	7	2.14	9.4	1.05
Coed Boys	7.5	2.31	11	1.92
Single Girls	8.5	1.88	10	2.13
Coed Girls	8	2.6	11	1.24

Table 1: Mean scores of boys and girls on the pre-posttests.

Students in the coed class had higher averages than students in the single-sex classes. The boys in the coed class show the largest increase in 3.5 increments from pre-test to post-test. Girls in the single-sex class had the lowest with 1.6 increments.

For ease of comparison, the total boys' score (single sex boys and coed boys) was compared with the total girls' score (single-sex girls and coed girls) for six open-ended items on the test-see table 2.

	Add or Subtract Problem	Place Value Problem*	Create and Solve Problem*	Number Pattern Problem*	Adding Fractions Problem*	Division Problem*
Total Boys	1.42	1.97	1.01	1.88	2.02	1.93
Total Girls	1.22	1.35	2.31	1.03	1.08	1.15

Table 2: Mean scores of boys and girls on six open-ended problems on the post-test.

For the problems with \*, the difference in mean scores between boys and girls is statistically significant ( $p < .05$ ). Table 2 shows that boys had a higher average on five of the six open problems while girls had a higher average on one open problem.

### Qualitative Results of Test

Two problems with gender differences, Create and Solve Problem and Division Problem, (see appendix) will be discussed in this paper. Both were modified from questions in student's mathematics textbook.

For the "Create and Solve Problem", students were shown a picture prompt of a boy selling different kinds of priced fruits and asked to use the information in the picture to create three questions and solve one. Students were expected to write or draw pictures representing statements using addition, subtraction, and multiplication about cost per item and number of items. Example: if one banana cost \$20, what is the cost of 4 bananas? More difficult questions would involve combining two or more operations in one question. Examples: what are the cost of 2 bananas and five oranges? And how much change should I receive from \$100, if I buy four bananas at \$20 each?

Over 89% of girls and 74% of boys were able to generate three problems ( $z = 2.31$ ,  $p < .05$ ). Also, girls were able to generate more complex problems than boys. Forty-six percent (46%) of problems generated by girls were complex problems (combining multiplication, addition, and subtraction) and 34% of boys problems were complex

problems (combining multiplication and addition). Girls in the single-sex class generated more questions than girls in the coed class, but the difference was not statistically significant.

Most boys 53% from the coed class and 61% of the single-sex class drew diagrams to represent their questions. This solution strategy may not have been a gender difference in reasoning, but because most students were lower average students and had difficulty with reading and writing. Few girls (15% from coed and 12 % from single-sex class) with low reading ability also used diagrams to represent their questions.

In solving the Division Problem, students were expected to show a correct computation procedure and interpret the computational results to give a realistic answer for the particular situation. Analysis of this problem was conducted from four aspects: (1) solution process, (2) numerical result of computation, and (3) final interpretation (reasoning).

A larger percentage of boys (69%) than girls (58%) produce the correct results of 6 busses ( $z = 2.04$ ,  $p < .05$ ). Boys (71%) and girls (67%) in the coed class had larger percentages than boys (61%) and girls (56%) in single-sex classes, but the difference was not statistically significant. Additionally, in both classroom settings, boys more than girls used diagrams to show their solution. That is, the boys drew circles to represent busses and dots or exes to represent passengers. Most girls used the traditional long division algorithm to show their solutions. The difference in strategies did not produce a difference in reasoning as there was no statistical significance between boys and girls who gave  $5\frac{1}{2}$  busses or 10 busses (the remainder) as their final answer. This was seen in both classroom settings. This result is similar to that of Cai (1995).

### **Qualitative Results: Peer Interactions**

A comparison of the single-sex classes revealed that there were richer class discussions in the all-boys and coed classes than in the all-girls class. Working in groups, the boys listened to each other's comments and tried to contribute to the discussions. The boys were helpful towards each other, preferred to use manipulative to assist them in solving the question, drew and created songs about what they were learning.

Students in the all-girls class preferred writing in their book and were cautious in their approach to solving open-ended questions. They liked explaining their work on the board, but they often copied work from their neighbor rather than discussed how to obtain the solution. This produces repetition of simple solutions as against in-depth and varied solutions. For example, when asked to give pairs of equivalent fractions by folding and shading a square paper, the all-girls class produced six different fractions while the all-boys class was able to show 13 and the coeducation group showed 18.

The girls in the all-girls were easily frustrated with each other. Frequently, the four-member group became pair work or individual work due to conflict within the group. The conflict occurred when two or more girls in the group nominated themselves as

leader of the group (groups should not have any leader). Disagreement among the “leaders” caused vexation and separation from the group.

This behavior is in contrast to Becker (2003) suggestions on how female learn mathematics; however, this contrast may have occurred as the females in this study were children and Becker were referring to the older female. Cultural and social factors may have also influenced this result or it could have been an oversight of the research planning. Conflict among the boys was planned for, but conflict among the girls was not expected. Similar to the all-boys’ class, the teacher of the all-girls class rearranged the students to minimize this problem.

Occasionally, the teacher of the coed class conducted gender grouping. The teacher created groups with girls only, groups with boys only and the remaining students in the coed group (each group had 3 or 4 students). Similarities with the created same-sex groups and the students in the single-sex classes were that the girls were organized and able to formulate a step by step plan to solve the problem whereas boys were more impetuous and often apply the trial and error strategy without thinking about efficiency or correctness.

It was also found that appealing to the sex-role stereotypes which define how male and female should behave impacted on how the students attempted the question and participated in the lesson. The male teacher encouraged boys of single-sex class by telling them they were competing against the girls in the single-sex group. This tactic was used to stem unwanted behavior and promote desired ones. In the coed class, if the teacher began the lesson by saying, “I think this will be exciting for the boys” normally, elicit more responses from the boys in the class. Boys participated more when the teacher established a practical reason for doing the activity and classes that had manipulatives-this was observed in both the single-sex and coed classes.

Another important observation was the used of the “I can’t” statement. “I can’t do it” was common among both sexes but for different reasons. Boys tended to say “I can’t” when they are asked to write down what they did; girls tended to say “I can’t” when they are requested to do or to use manipulatives to show what they were thinking. This observation is in line with Geist and King (2008) that boys prefer to show rather than write and girls prefer to write rather than to show.

## **CONCLUSIONS**

Instruction in the open approach with open-ended problems was able to increase students’ understanding of mathematical concepts and was most useful in co-ed classrooms than in single-sex classrooms. Boys and girls in the coed class than boys and girls in the single- sex class showed greater interactions in discussions and had fewer conflicts during lessons. With regards to gender, overall, boys perform better than girls, but the difference varies with tasks. Boys showed more tendency to use diagrams in their solution process whereas girls showed more tendency toward traditional methods. Boys often used the trial and error method while girls preferred to create a well-structured plan before attempting to solve the problem. Students,

especially male students, seem to show greater understanding of mathematical concepts when given the opportunity to solve mathematical problems using their method.

The results suggest that using the open approach with open-ended problems in the coeducation classroom may be a possible solution for reducing this gender disparity in mathematics.

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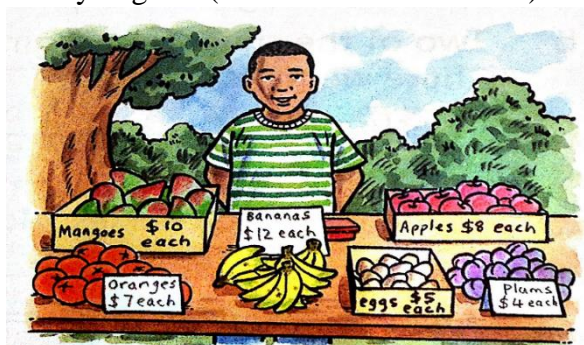
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## APPENDIX

**Creat and Solve Problem:** Give three math questions using the information in the picture below. Solve one of the three questions you gave. (Modified from textbook)



**Division Problem:** Students and teachers at Bath Elementary School will go to the beach by bus. There is a total of 110 students and teachers. Each bus holds 20 people. How many buses are needed? (Modified from textbook)

# OPPORTUNITIES FOR ALGEBRAIC REASONING THROUGH DIGITAL LEARNING RESOURCES

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*Formal algebra learning is well-known as being an area of difficulty for many students. This paper investigates 5<sup>th</sup> grade students' opportunities for algebraic reasoning through engaging in an algebra-related computer activity emphasizing non-formal algebraic notation and relationships. The activity has great potentials for algebraic reasoning important for in-depth learning of formal algebra. By drawing attention to the algebraic character in the variables, the relationship between them, and the logical consistency inherent in the equation system represented in the activity, the continuities between arithmetic and algebra is in focus and lays a foundation for later formal algebra learning. Teacher-led guidance is crucial for ensuring a realization of the learning potential.*

## RATIONALE

Algebra is an effective tool for exploring, analysing, and representing mathematical concepts and ideas, and for describing and modelling relationships in everyday phenomena (Kieran, 2007). Thus, learning algebra is important for succeeding in mathematics as well as other subjects that build upon mathematical knowledge. Unfortunately, algebra requires abstract thinking that makes it a well-known area of difficulty for some students (Booth, 1988; Kieran, 2007). Traditionally, arithmetic and algebra have been viewed as two distinct domains, with algebra instruction coming years after the first arithmetic concepts are taught (Carraher & Schliemann, 2007). Perhaps this segregation of algebra instruction from arithmetic instruction explains why formal algebra, typically introduced in lower secondary school, presents an immense obstacle for many students.

Over the past 10-15 years, there has been an increased acknowledgment of the importance of mastering a broader spectrum of mathematical competencies in order to succeed in mathematics (Kilpatrick, Swafford & Findell, 2001). In accordance with this trend, the definition of what constitutes effective mathematics teaching practices also has broadened (NCTM, 2014). Promoting, eliciting, and using students' reasoning, facilitating meaningful mathematical discourse, and using and connecting mathematical representations recur as very important methods. Further, we see a rapidly growing focus on how these ideas, intertwined with digital learning resources, can contribute to in-depth learning of mathematics, although further investigation by education experts is needed (Sinclair and Baccalini-Frank, 2016). Based on the observed difficulties many students have with the formal algebra, with particular

emphasis on the transition from arithmetic to the symbolic algebra, this paper investigates the following research question:

*What are the opportunities for algebraic reasoning through a digital learning resource emphasizing non-formal algebraic notation and relationships?*

## **PRIOR RESEARCH ON ALGEBRAIC REASONING AND LEARNING**

Following the traditional path of learning arithmetic first and algebra years later, students must overcome several discontinuities between the two domains when beginning algebra instruction (Booth, 1988; Kieran, 2007). Thus, many students experience formal algebra as meaningless manipulations of symbols, detached from reality. For example, the students must deal with new letters with new meanings (e.g.  $x$  representing an unknown number, a generalized number or a variable), which are used in unfamiliar operations. The students must transition from operating on given numbers to find a numerical answer in arithmetic, to operating with and on letters and accept that “the answer” is not necessarily numerical in algebra. Hence, the concept of the variable becomes highly important in algebra. A conceptual understanding of variables is crucial for succeeding with algebra, and it is an area which many students find particularly challenging (e.g. Knuth et al., 2005; Küchemann, 1981).

An increasing number of researchers argue for introducing and facilitating algebraic reasoning in arithmetic rather than treating arithmetic and algebra as two distinct domains (e.g. Carraher and Schliemann, 2007; 2016). These researchers stress that arithmetic has an inherently algebraic character, and that focusing on the functional character of arithmetic is a powerful way to facilitate in-depth learning of algebra. They argue that fostering students’ early understanding of functions and relations can increase their ability to make mathematical generalizations, deepen their understanding of variables as placeholders for elements of sets and, thus, better prepare them for a later, more formal introduction to algebra and functions. “Exposing the functional character of arithmetic explicitly helps elementary school children shift their focus from individual numbers to sets of numbers and quantities (i.e., variables) and their interrelations” (Carraher & Schliemann, 2016, p. 214). They stress that this shift leads students to consider and begin to understand relationships between variables. In order to develop an understanding of algebraic concepts and relationships, learning to represent, discuss and see the relationships between contextual, visual, verbal, physical and symbolic representations, is highlighted as meaningful (Arcavi, 2003; Carraher & Schliemann, 2016; NCTM, 2014).

## METHODS

The data for this study come from a larger research project, *ARK&APP*<sup>1</sup>, 2013-2015. *ARK&APP* investigates the use of educational resources in the planning, conducting and evaluation of teaching in four school subjects, and consists of two quantitative surveys and twelve qualitative case studies. This paper is based on one of three case studies in mathematics conducted in December 2014 in a Norwegian primary school. Two researchers observed one teacher and one group of 23 5<sup>th</sup> grade students (11 boys and 12 girls) in all of their mathematics lessons for three consecutive weeks. The topic was algebra. The data collected includes pre- and post-tests evaluating students' learning outcomes, observations and video recordings of various forms of classroom interactions as well as interviews with six students and the teacher. About 1/5 of the teaching time was used for two different algebra computer activities, and the students worked together in pairs. This paper reports on parts of the video-study findings concerning students' interactions and reasoning while engaging in one of these activities, called "The world of the symbols," part of a high-impact digital learning universe, "Salaby," from one of the main publishers for educational resources in Norway. In the activity, visual representations of physical objects such as strawberries or airplanes (see Figures 1 and 2) represent variables, which are elements in an algebraic structure representing equation systems with three equations and two variables. I will refer to each equation system as one "task", and each column of the task as "equation" 1, 2, and 3, respectively. The aim of each task is to find the value of the right-hand side expression of the third equation. The students cannot move on to the next task without correctly solving the current task.

Three cameras focused on three student pairs throughout all their work over the course of the three weeks. For this paper, the video recordings of all the three pairs' work with the computer activity were watched multiple times in order to get familiar with the students' reasoning and interactions in the activity. Further, extracts of students' reasoning and interaction that illuminate the research question were transcribed and analyzed in depth. The interaction between two students, John and Lisa, forms the basis for the main analysis in this paper. Episode 1 represents a common method of reasoning observed among the students in the class. Episode 2 represents a golden moment for algebra learning, and shows advanced algebraic reasoning in a non-symbolic environment. Both episodes, therefore, highlight potential for algebraic reasoning important for future formal algebra learning.

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<sup>1</sup> For more information on the research project, see  
<http://www.uv.uio.no/iped/english/research/projects/ark-app/>.

## FINDINGS AND DISCUSSION

In the first three tasks of the algebra computer activity, one of the two variables in the equation set has a particular value assigned to it (Figure 1). The first episode shows John and Lisa collaborating on this second task.

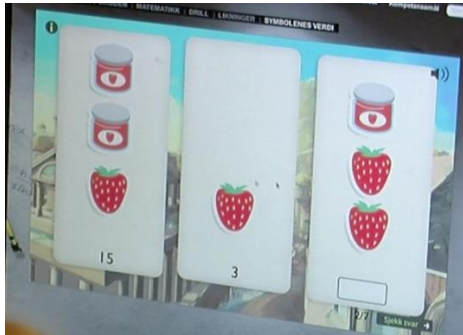


Figure 1: The second task of the algebra computer activity

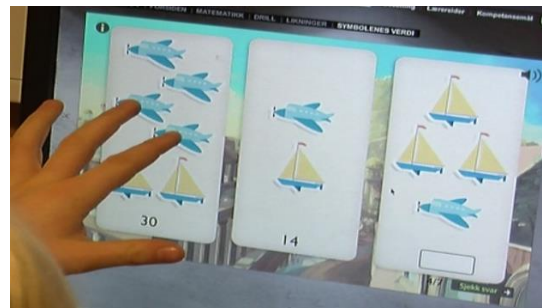


Figure 2: The fourth task of the algebra computer activity

### Episode 1

The excerpt illustrates a common way of reasoning among the students solving the first three tasks.

Lisa: Let us see... Uhm...

John: You can see that these are 15 (*points with one finger at the two jars and the one strawberry in the first equation of Figure 1*) and that one is 3 (*points with one finger at the strawberry in equation 2*). So, that one was 3 (*points with one finger at the strawberry in equation 1*).

Lisa: Then these are (*points the cursor at the jars in equation 1*), they have to be...

John: Then this has to be 6 (*places one finger on the upper jar in equation 1*). Because  $6 + 6$  (*points at the lower jar in equation 1*) is 12,  $+ 3$  (*points with one finger at the strawberry*) is 15.

Lisa: Mm [affirmative]. So, then this is (*points the cursor at the jar in the third equation*)  $6 + 3 + 3$  (*moves the cursor over the two strawberries*).

John: That is 12.

John and Lisa solved the task by using the information of the given variable in the second equation (one strawberry equals 3). By deductive reasoning, they found the value of the second variable. The values of the two variables were controlled for by calculating the value of the right hand-side expression of the first equation (two jars and one strawberry equals 15). Convinced that the two values of the variables were correct, they continued to equation three, and used both variables in order to find the value of the right hand-side expression, which was the solution of the task.

The excerpt shows the way the students' reasoning built on logical deduction based on prior arithmetical knowledge. The students search for unknowns, or "hidden," numbers behind visual representations of the variables in a non-formal system. They interacted with the objects based on the relationships between them and the numbers, taking into account the logical consistency in the relationship between the three columns. The students' work with relations has the possibility to better prepare them for more formal introduction of algebra and functions. Visual representations of physical objects, such as strawberries, function as placeholders for the concept of variables, and these variables are elements in a structure representing an equation system of three equations and two variables.

Learning to represent, discuss and see the relationships between different representations is beneficial for in-depth learning of mathematics in general (Arcavi, 2003; NCTM, 2014), and for algebra in particular (Carraher & Schliemann, 2016). Giving the students the opportunity to express relationships in formal symbolic representations built from work with other representations will foster algebraic proficiency. Further, viewing the objects as *variables* constrained to single values rather than *unknowns* would be particularly fruitful in order to provide a deeper understanding of relations and generalization ideas important for formal algebraic proficiency (ibid.).

## Episode 2

The second excerpt is from Lisa and John's discussion of task 4 (see Figure 2). This task requires a different solution strategy since there is no longer one particular value assigned to one of the variables. Both variables occur in all three equations.

John: If that is 14 (*points at the plane and the boat in equation 2 simultaneously with two fingers*), then perhaps they must be [short pause] wait a minute (*points one more time at the two objects with two fingers*)  $14 + 14$ , 28 (*points at the objects in equation 1 while saying 28*). [Short pause]. No, they cannot be 7... 4 on the plane perhaps?

Lisa: And 10 on the boats?

This excerpt shows John's initial response on this fourth task. He grouped the two objects – an airplane ("x") and a sailboat ("y") – and started operating on this new object ("if this is 14", *this* referring to  $x + y = z$ ). Further, saying " $14 + 14$ , 28" while pointing at the objects in equation 1, he operated on the new object in another context – the first equation. This solution strategy would be very efficient in order to solve this, and similar, algebra tasks. If John had continued operating on the new object, z, and seen that two airplanes and two sailboats is 28 ( $2x + 2y = 2(x + y) = 2 * 14 = 28$ ), it would be rather easy to deduct that the two airplanes left would have the value  $30 - 28 = 2$ , and therefore one airplane = 1. However, John and Lisa switched to a trial-and-error strategy at this point. In this particular task, it is challenging to find the correct

values for the objects by trial-and-error substitution. The students typically approached this fourth task by guessing values for the variables without the logical deduction seen in Episode 1.

Facilitating the calculation by making a new variable based on the two variables at play, and *using* this new variable in further calculation, can be characterized as a golden moment for fostering algebraic reasoning. If the algebraic character behind this intuitive strategy is made visible to the student, such an awareness may facilitate the learning of corresponding ideas represented in formal algebraic language and give a deeper meaning to such activities (Carraher & Schliemann, 2007; 2016). The algebra behind this way of reasoning is quite advanced, and has, represented with formal symbolic language, proven difficult for many students who are much older (Kieran, 2007). Equation systems with multiple variables represented with formal symbolism is typically taught in upper secondary school. Understanding of multiple strategies and the ability to decide which to use make it easier to see relationships between phenomena and operations, and contribute to a deeper understanding of the content (Kilpatrick et al., 2001; Rittle-Johnson & Star, 2009).

### **The importance of teacher-led guidance and digital resources**

Episodes 1 and 2 highlight several opportunities for exposure to algebraic reasoning important for formal algebra learning. However, in order for such learning to occur, the students need help to transfer their experiences and knowledge from working with specific activities to important elements for algebra learning. This is key to realizing the learning potential inherent in the activity. The teachers' role of eliciting and using evidence of students' reasoning in the teaching is paramount for facilitating a good learning progression (NCTM, 2014). For example, giving the students the opportunity to develop an understanding of formal symbolism based on their ideas and convictions in arithmetic can provide a meaningful foundation for further abstract learning (Carraher & Schliemann, 2007; 2016). The students' reasoning seen in Episode 1, and golden moments as the one seen in Episode 2, can have great possibilities for in-depth future learning of variables represented in formal symbolic language, of the relations and dependence between such variables, and what role they play in functional representation. Further, highlighting and contrasting the different solution strategies used by the students can contribute to greater flexibility in how the students approach problems and provide a deeper understanding of how concepts and procedures are related (Kilpatrick et al., 2001; Rittle-Johnson & Star, 2009).

Settings where the students are encouraged to make their reasoning visible through justification, argumentation and discussion are widely understood to foster crucial competencies important for developing a deeper understanding of mathematical concepts and ideas (Carraher & Schliemann, 2007; 2016; Kilpatrick et al., 2001; NCTM, 2014). Thus, there are many advantages for fostering algebraic reasoning

important for algebra learning in teacher-guided whole-class discussions highlighting explorations built on students' intuitive ideas (Carraher & Schliemann, 2007; 2016).

Digital learning resources have a focusing effect for students' group-based work (Çakir & Stahl, 2013) and can contribute to explorations of mathematical phenomena in a rich and visual way. Digital technologies can offer unique visual mediators that invite different ways of describing and comparing mathematical objects and relationships (Sinclair and Baccalini-Frank, 2016). The students worked much more eagerly and collaboratively when working in pairs on the computer, compared to how individually and non-communicatively they worked when sitting in pairs solving textbook tasks with pencil and paper. The engagement was also evident from the length of time the students spent reasoning and discussing possible solutions. John and Lisa spent more than 10 minutes working with the fourth task (Episode 2), and statements such as "Yess!" when they thought they had found the correct values, and "Ah! No..." when they discovered that the values were after all incorrect, showed that they invested feelings in their work with this task. When a frustrated fellow student grumbled about the impossibility of figuring out the problems, John replied: "Nothing is impossible, we just have to think... out loud." Supporting students to "engage in productive struggle as they grapple with mathematical ideas and relationships" (NCTM, 2014, p. 48) is seen as important for understanding and making sense of mathematical ideas and relationships.

## CONCLUDING REMARKS

This paper has shown multiple opportunities for algebraic reasoning among 5<sup>th</sup> grade students working with a computer activity emphasizing non-formal relationships between quantities. The discussion illuminates the importance of teacher-led guidance for ensuring that the algebraic character in the students' reasoning is pointed out for the students and built upon in further teaching. Golden moments for algebra learning are easily lost if the students thoughts and ideas are not uncovered and used in teaching, in interaction with important concepts, relationships and ideas in algebra.

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# CHALLENGING ABLEIST PERSPECTIVES ON THE TEACHING OF MATHEMATICS THROUGH SITUATION-SPECIFIC TASKS

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*CAPTeaM (Challenging Ableist Perspectives on the Teaching of Mathematics) investigates how ableist perspectives – according to which to be able-bodied is the norm and disability is a disadvantage that must be overcome – impact upon the teaching of mathematics. CAPTeaM is a partnership between researchers in the UK and Brazil which brings together approaches to investigating and transforming teacher beliefs and research into the mathematical learning of disabled students. We develop and trial situation-specific tasks that invite teacher reflection on incidents of disabled students' mathematical contributions. In this paper we present two types of tasks and illustrate analyses of data collected from 81 teachers that explore their views on including these contributions in their mathematics classrooms.*

## INTRODUCTION: ABLEISM AND THE TEACHING OF MATHEMATICS

As signatories of the United Nations Convention on the Rights of People with Disabilities (2006), both Brazil and the UK are committed to upholding the right of disabled people to education and to ensuring an inclusive education system at all levels. In order that this right be realised, signatory countries also undertake to implement appropriate measures to

“train professionals and staff who work at all levels of education. Such training shall incorporate disability awareness and the use of appropriate augmentative and alternative modes, means and formats of communication, educational techniques and materials to support persons with disabilities.” (Article 24, paragraph 4).

Within the mathematics education community, while social justice has been a concern for many researchers interested in building more equitable mathematics classrooms and in analysing and criticising the processes which sustain disadvantage, until recently, attention to disabled learners has been rare and, in particular, is almost non-existent in the literature related to mathematics teacher education. As Gervasoni and Lindenskov (2011) have argued, the discourses about disabled students that do exist have tended to underestimate their potential for learning mathematics. There are signs that this is beginning to change, with a small but growing body of research that points to how ableist<sup>1</sup> assumptions about what constitutes the normal body contribute to the

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<sup>1</sup> Campbell (2001) defines *ableism* as “a network of beliefs, processes and practices that produces a particular kind of self and body (the corporeal standard) that is projected as the perfect, species-typical and therefore essential and fully human. Disability then, is cast as a diminished state of being human.” (p. 44)

creation of learning environments which disable those whose cognitive, emotional, physical and or sensory configurations differ from what is currently defined as socially desirable (see, for example, Healy & Powell 2013). One message from this research is that, rather than being the consequence of internal, individual factors, students' underperformance in mathematics can result from "their explicit or implicit exclusion from the type of mathematics learning and teaching environment required to maximise their potential and enable them to thrive mathematically" (Gervasoni and Lindenstov 2011, p. 308).

To fulfil our commitments to disabled mathematics learners, then, we need to dismantle belief systems which "regard some students as being 'in need of fixing' or worse, as 'deficient and therefore beyond fixing'" (EADSNE 2010, p. 30). It was with this in mind that the project, CAPTeaM (Challenging Ableist Perspectives on the Teaching of Mathematics), was conceived.

The project aims to work with pre- and in-service teachers towards challenging the ableist assumptions that currently mediate our interpretations of mathematics learning and our practices as educators of mathematics. This involves the development and trialling of tasks that encourage teachers to reflect upon the challenges of teaching mathematics to disabled students. In this paper, we introduce the two types of Tasks (Type I and Type II) that our project deploys. By focusing on the responses of one participant and her colleagues to a task of each type, we illustrate how the design principles were played out in practice and the extent to which the tasks contributed to reflections about including students whose interactions with mathematics differ from those of students deemed more typical. We close by reflecting on findings that indicate the need to question more explicitly the notion of the typical mathematics classroom, a notion that might be seen as ableist itself.

## THE CAPTEAM PROJECT

CAPTeaM is a collaborative project involving researchers and pre- and in-service teachers in Brazil and the UK. A British Academy International Partnership and Mobility Scheme grant has enabled us to combine the different research foci of two research teams in a reciprocal manner. The UK component of CAPTeaM uses practice-based and research-informed Tasks which invite teachers to consider fictional mathematics teaching situations (scenarios) that are hypothetical, grounded on seminal learning and teaching issues and likely to occur in actual practice (Biza, Nardi & Zachariades 2007). This research programme aims at *transforming aspirations into strategies in context*. The Brazilian component of CAPTeaM, *Rumo à Educação Matemática Inclusiva (Towards an Inclusive Mathematics Education)* has been investigating forms of accessing and expressing mathematics that respect the diverse needs of all students with and without disabilities. Its aims include: understanding how mathematical learning processes are shaped by bodily, material and semiotic resources; contributing to the development of inclusive teaching strategies; and, exploring the relationships between sensory experience and mathematical knowledge.

## TYPE I AND TYPE II TASKS

In this paper we report from the first, one-year phase of CAPTeaM (2014-15) during which Type I and Type II tasks – aimed at providing opportunities for pre- and in-service teachers to reflect upon issues related to the inclusion of disabled mathematics learners in their classes – were designed and trialed. Data was collected in Brazil and the UK from a total of 81 teachers (60 from Brazil and 21 from the UK) who completed four tasks (three of Type I and one of Type II) in a three hour session. In Brazil, the participants included secondary mathematics teachers and primary teachers, while all the teachers who engaged with the tasks in the UK were secondary mathematics teachers. In both countries most of the participants were pre-service teachers although a small number of in-service teachers also completed the tasks. The tasks were designed to emphasise different issues related to inclusion and to challenge what we identify as ableist assumptions in different ways.

The design of the Type I tasks involved the selection by members of the Brazilian team of episodes of mathematical interactions between students and teachers from the database of video evidence collected in the different studies of their research programme. Following Healy and Powell (2013), the design principle behind the selection process was the idea of highlighting the mathematical agency of disabled students: instead of attempting to determine “normal” or “ideal” achievement and positioning those who deviate from supposed norms as problematic and in need of remediation, attention should be directed to how students’ mathematical ideas may develop differently and what pedagogical strategies are appropriate for supporting these developmental trajectories. The aim was hence to locate episodes representative of the successful mathematical practices associated with particular forms of interacting with the world – practices of learners who see with their hands and ears, who speak with their hands, whose visual memory is more efficient than their verbal memory, or, have other interesting ways of interacting with the world. We opted for episodes involving the use of interesting and valid mathematical strategies, but in which the properties and relations were expressed in unconventional or surprising forms.

Using the approach described by Biza, Nardi and Zachariades (2007), each episode was inserted as a video clip into a brief narrative about a fictional mathematics classroom. We then invited the participants to assume the role of the teacher of this class and evaluate the interactions of the disabled students that were presented in the video clips – first individually and in written responses to a set of questions and then in a group discussion (which we took observation notes from and also video-recorded).

We now present an example of a Type I task, *André and the pyramid*, and a sample response to the task. The video clip used in this task shows a short episode from an activity in which a blind student proposes a description of a square-based pyramid. More details on the research context in which this activity was used are in Healy and Fernandes (2011). An example of a Type II task, and also a sample response to it, follows.

### Example of Type I task: *André and the pyramid*

Imagine you are teaching a class about three-dimensional geometric figures. As the students work on exploring how they would describe what a square-based pyramid is to someone who doesn't know, you move around the class to observe their strategies. You notice many are counting faces, edges and vertices. André, who is blind, has been working with materials, such as 3D solids. He offers this description. [Video clip follows]

Questions:

- What is André proposing as a description of a square-based pyramid?
- What do you do next?
- What do you think are the issues in this situation?
- What prior experience do you have in dealing with these issues?
- What prior experience do you have in supporting the mathematical learning of blind students in your classroom?
- How confident do you feel about including blind students in your classroom?

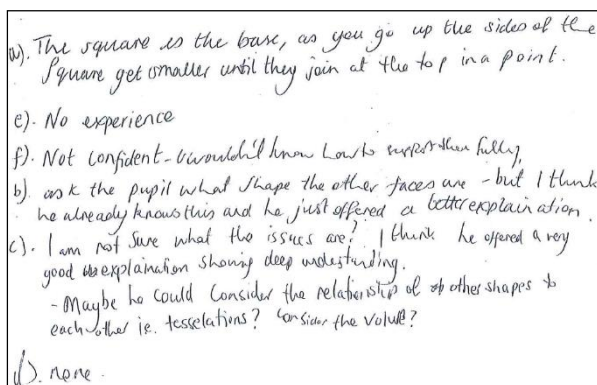
The 27sec video shows a blind student, André, describing his view of a square-based pyramid. As he spoke (the transcript is translated from Portuguese), André moved his fingers along the edges that join the vertices at the base of the pyramid to the vertex at its apex (stills from this video are presented in Nardi, Healy & Biza 2015, p. 55):

I would say that the part underneath is square... the base... is square...

And as you go up, they get, the sides of the square get smaller...

Until they form a point here on top (moves his fingers along the edges to the vertex at the apex of the pyramid).

### Example of response to the *André and the pyramid* Task: Beth



Beth recognises André's description: "The square is the base, as you go up the sides of the square get smaller until they join at the top in a point." (a). She intends to "ask the pupil what shape the other faces are" (b) but she thinks that "he already knows this and he just offered a better explanation." (b) She is "not sure what the issues are" and she thinks that André "offered a very good explanation showing deep understanding." (c) She also suggests that "maybe he could consider the relationship of the other shapes to each other

Figure 2: Sample response to the *André and the pyramid* Task (Beth).

i.e. tessellations? Consider the volume?" (c). Beth states that she has no experience in teaching students like André (d, e) and she is not confident about knowing "how to support them fully" (f). We return to a brief analysis of Beth's response in what follows.

### **Example of Task Type II: Artificially restricting mathematical interactions**

The Type II tasks were designed with the aim of provoking reflections about how access to mediational means differently shapes mathematical activity. Participants worked in groups of three. One member (A) acted as observer, the second (B) was asked to solve a mathematical problem whilst, temporarily and artificially, deprived of access to the visual field and the third member (C) acted as the teacher, with the role of communicating the problem intervening as judged necessary, but without speaking.

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For this activity we will split in groups of three.

One member of the group (A) is the observer.

A second group member (B) will temporarily lose access to the visual field (by shutting their eyes).

The third member (C) can see but cannot speak.

C will be given a piece of paper with the rest of the instructions.

*Instructions to C:* Your task is to ask (without speaking) B to multiply 347 by 35 and to indicate whether or not the answer suggested by B is correct.<sup>2</sup>

B should not have access to these instructions.

Once the task is complete, A, B and C have a short discussion about how the restrictions influenced their strategies.

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We stress that the aim of the task was not that the participants would attempt to role play the part of someone with a disability; rather, that the experience of doing mathematics without access to a resource that they are accustomed to use might heighten awareness of the importance (or not) of this resource as well as involve the participants in seeking new forms of expressing mathematics. Trios of participants engaged with the task for about 15 minutes. Then all convened for plenary discussion of the strategies that had emerged in the small groups. Small group activity as well as plenary discussions were video-recorded. We return to Beth –whose response to the *André and the pyramid* task we sampled earlier – this time as she collaborated with Mike and Ted.

### **Example response to Artificially restricting mathematical interactions: Beth's trio**

In Beth's trio, Mike assumed the role of observer (A), she had the responsibility of communicating the problem without speaking (C) and Ted for resolving it (B). Early difficulties in communicating the problem were eased after signs for “yes” (a gentle squeeze of the hand) and “no” (a gesture involving a rubbing out movement on the palm of Ted's hand) were established. Beth then traced out the digits 3, 4 and 7 on Ted's palm, used a cross to represent the operation and then traced out the 3 and the 5. Ted wrote out the multiplication correctly in columns even though he could not see

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<sup>2</sup> In each trio the problem involved multiplying a three-digit number by a two-digit number, although the numbers given varied across the groups.

what he had written. As he began to execute the long multiplication algorithm, he asked Beth to move his hand to record the results in the standard positions on the paper. They completed and recorded the two operations ( $347 \times 5$  and  $347 \times 30$ ) without difficulties, but by the time it came to adding the two results, Tom had forgotten what numbers he had written. Beth took his hand to remind him of the two results using the method of tracing numbers on his palm. He suggested it would be easier if she simply reminded him of the digits in each column, starting with the units. This would have worked very well, except that it turned out to be very difficult to communicate the digit “0”, as Ted first felt 6 and then 9 as it was traced out on his hand. Eventually, this difficulty was overcome, and with Beth once again guiding Ted’s hand to the correct location on the paper, he recorded the answer correctly. At this point, Ted removed his blindfold and the trio discussed the solution strategy. Beth was worried that Ted didn’t “know” what the answer was before removing the blindfold. Did his answer count? She and Mike questioned him about why he had insisted on writing when he couldn’t see what he recorded on paper. He answered “Yeah, yeah, because between us we had everything”. Indeed, Beth clearly felt some joint ownership of their production, saying “It’s quite neat, let’s show it to the camera”.

They agreed that Ted’s strategy had been strongly molded by his previous experiences, which led to reflections about other methods they might have used with someone without access to the visual field who did not already know the algorithm. Beth was concerned that it would be hard to solve mentally and felt she too would have relied on the same method in Ted’s place. Ted accepted that he could not have resolved the algorithm on his own and other methods of what they called “breaking the task down” were considered. As the discussion was finishing, Beth had another idea of how she might have presented the problem to Ted, using “mirror writing” then turning the paper over to set out the algorithm in a raised form that Ted could have read with his fingers, to which he replied “that would have been great, oh wow.”

## **SAMPLE ANALYSIS OF TASK TYPE I AND TYPE II DATA**

Thematic analysis of the data (written protocols of Type I tasks, video-recordings of small group engagement with Type II tasks and plenary discussions of both types of tasks) was carried out according to the five dimensions below and aimed to encourage participants’ to reflect on teaching mathematics to people with different disabilities.

1. *Value and Attuning*: to what extent the respondent attunes to and values the disabled learner’s contribution(s), and how, if at all, s/he attends to the particularities of their mathematical agency (Type I) or adapts to the restriction imposed on the communication (Type II);
2. *Classroom Integration and Benefit*: how the respondent manages the classroom after the contribution has been made (Type I) or comments on classroom management after engaging with Type II tasks;
3. *Experience and Confidence*: how experienced and confident the respondent claims to be in teaching students with the disability exemplified in the task (Type I and II);

4. *Institutional Possibilities and Constraints*: what institutional possibilities and constraints the respondent identifies as crucial to the teaching of students in the task (Type I) and reflection about the strategies developed in Type II tasks;
5. *Resignification*: evidence of respondent's reconsideration of their views and intended practices in the light of engaging with the tasks (Type I and II).

Returning to Beth's response to the Type I task, we see evidence of unconditional *valuing* of André's contribution ("he just offered a better explanation"; "very good explanation showing deep understanding") as well as *attuning* her further action to this contribution ("consider the volume?"). At the same time Beth mentions "the other shapes" (triangle faces of the pyramid) in order to *integrate* André's contribution, and bring it in line with, with what had been a prevalent preference in the classroom for seeing a pyramid in terms of its faces, edges and vertices. She does not seem intent though on switching his views to a more conventional description: her suggestion that he has "offered a better description" indicates that the video clip allowed her to focus on the mathematical agency of a student who saw with his fingers rather than with his eyes – André's difference hence was not viewed by Beth as a deficiency. Finally, in common with nearly all of the participants in the study, Beth states her lack of *experience and confidence* in teaching students like André.

In relation to the Type II task, Beth began by highlighting her preconception of how hard it would be to attune her interventions given the restrictions imposed on the interactions between her and Ted. As it turned out, once the initial system of signs had been established, with the exception of the problem with zero, she was able to create a variety of ways of communicating that made sense to Ted. The importance of establishing shared signs was mentioned frequently in post-Type II task discussions. To a certain extent, we might see this practice as the ultimate in *attuning*. Indeed we have argued elsewhere (Healy 2015) that both learners and teachers in inclusive settings attempt to make use of words and gestures in ways they believe will re-invoke the multimodal content they associate with mathematical concepts in ways that can be felt by others. The Type II task was not intended to involve Beth in simulating teaching mathematics to a blind student or working with someone who does not share the same spoken language; rather, it was to encourage reflections about different ways to permit others to feel mathematically. In this sense, the task achieved its aim. Issues related to *resignification* and to *institutional possibilities and constraints* also emerged. For example, Ted's comment that between he and Beth they had "everything", indicated that solving a mathematical problem under the conditions imposed became a joint activity, not an individual one, with Ted using Beth's sight to substitute the absence of his own. This is a powerful idea and one that, at least implicitly, challenges the notion that mathematical achievement necessarily be judged on what the student achieves individually. As the discussion progressed, however, this institutionally imposed norm was returned to place, with all three students remarking on how Ted could not have implemented his strategy alone.



## INCLUSIVE MATHEMATICS CLASSROOMS ARE NOT YET TYPICAL

Our evidence suggests that the participants in our study were encouraged to think about how the mathematical agency of disabled students might be supported or restricted by aspects of the learning environments in which they experience mathematics and to recognise that they are not *a priori* mathematically-deficient. Perhaps our tasks were even successful in motivating the pre- and in-service teachers to rethink the notion of the normal student. We believe that this is an important step towards preparing teachers to work with learners with disabilities and influencing how they choose to organise the learning activities they offer to all their students. On the other hand, our choice to embed the Type I tasks in classroom settings that the teachers are likely to experience (or have experienced), may have contributed to the edifying of a different norm, the normal classroom. Building an inclusive school mathematics requires the deconstruction of this notion too. This is no easy feat, but perhaps a small step in this direction would be to work towards a third type of task which involves us all in imagining what a truly inclusive mathematics classroom might look like.

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# THE IMPACT OF AN INTERVENTION ON THE DEVELOPMENT OF LOW-ACHIEVING SIXTH GRADE STUDENTS' ABILITIES RELATED TO FRACTION UNDERSTANDING

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*In the present study, we are interested in the impact of an intervention course on the development of low-achieving sixth grade students' abilities that are related to fraction understanding. An intervention course comprising of lessons for developing these abilities was implemented in a sample of low-achieving sixth grade students, while a control group was taught according to the Cyprus National Mathematics Curriculum. Three measurements took place, prior, immediately after and three months after the intervention course and growth analysis was used to examine growth rates and growth patterns for both the experimental and the control group. Growth rates and growth patterns are reported for both groups of students and the impact of the intervention course on growth rates and growth patterns is discussed.*

## INTRODUCTION

According to Ma (2005), one of the most important education issues is growth in academic achievement. A number of teaching experiments and research programs have been carried out in the past in order to examine fraction teaching and learning (e.g. Streefland, 1991). Moreover, a number of studies has examined growth patterns for mathematics achievement (e.g. Ding & Davison, 2005), but not for fraction understanding or fraction related abilities. In the present study, we are interested in the growth rates and growth patterns of abilities related to fraction understanding for low-achieving sixth grade students that participated in an intervention course comprising of lessons for developing these abilities. These abilities were explanations for fractions, representations of fractions and reflection during the solution of fraction problems. Additionally, comparison with a control group taught according to the Cyprus National Mathematics Curriculum (CNMC) took place.

## THEORETICAL BACKGROUND

*The rationale for focusing on the abilities and their description*

In a previous study (Nicolaou & Pitta-Pantazi, in press), we examined seven abilities that are related to fraction understanding of sixth grade students. The seven abilities were: fraction recognition, explanations for fractions, justifications about fractions, relative magnitude of fractions, representations of fractions, connections with decimals, percentages and division and reflection during the solution of fraction problems. In the present study, we focus on three of the abilities that correspond to higher order thinking skills (Stein & Lane, 1996) and that were found at the beginning

of the study to be among the most difficult for low-achievers: explanations for fractions, representations of fractions and reflection during the solution of fraction problems. In the literature, there is a debate whether higher order thinking is appropriate for low-achievers and whether they can benefit from its teaching (Zohar & Dori, 2003). However, Zohar and Dori (2003) have reported findings indicating that low-achievers have gained from projects whose goal was to teach higher order thinking skills in science classrooms. Additionally, Zohar and Dori (2003) claim for the importance of higher order thinking skills for all students including low-achievers. In the present study, we examine whether low-achieving sixth grade students have benefited from an intervention course that aimed at developing abilities that are related to fraction understanding and at the same time correspond to higher order thinking skills.

Explanations for fractions refer to students' ability to explain in their own words what a fraction is and also to explain in various ways (verbally, by using drawings, examples etc.) other issues concerning fractions and more specifically, fraction equivalence, comparison and density. Representations of fractions refer to students' ability to translate to visual, verbal and symbolic representations and their ability to construct drawings for fractions. Reflection during the solution of fraction problems refers to students' ability to reason their thinking and their answer while solving fraction problems, to support the reasonableness of their answer and verify a given answer.

### *Research questions*

The present study sought answers to the following research questions:

- (a) What are the growth rates and growth patterns of explanations for fractions, representations of fractions and reflection during the solution of fraction problems for low-achieving sixth grade students that took part in an intervention course for developing these abilities?
- (b) What are the growth rates and growth patterns of these abilities for the respective control group of low-achievers that were taught according to the CNMC?
- (c) Are the average growth rates of the experimental group higher than those of the control group?

## **METHODOLOGY**

### *Participants, instruments and procedure of the study*

In the present study, which was quantitative in nature, we analyze data for 148 low-achieving sixth grade students (87 students for the control group and 61 for the experimental group). A test comprising 37 tasks for measuring the seven abilities was developed and administered to a larger sample of sixth grade students three times (prior, immediately after and three months after the intervention course). After the first measurement, the sample of students was split into experimental and control group. On the basis of the computed score for fraction understanding in the first measurement, the students of the experimental and the control group were classified into three

categories by latent class analysis: high-achievers, medium-achievers and low-achievers (Nicolaou & Pitta-Pantazi, in press). In the present study, we focus on the low-achievers of the experimental and the control group respectively.

The intervention course comprised of nine lessons with duration 14x40 minutes. The implementation of the intervention course started at the end of October immediately after the pre-test and lasted for about nine weeks, until the end of January, just before the second administration of the test (post-test). The teachers that participated in the intervention course were asked to teach one lesson every week. The students of the control group during this time were taught according to the CNMC, which included topics such as recognizing representations of fractions, some activities of explaining what a fraction is, recognizing fractions as the division of the numerator by the denominator, equivalent fractions, fraction comparison and ordering, improper fractions and mixed numbers, decimals and percentages and their conversion to fractions. It must be noted that the total time devoted to fraction teaching was about the same for the two groups.

After the completion of the intervention course, all students (including those of the experimental group) were taught according to the CNMC until the end of the sixth grade. They practiced simplifying and comparing fractions, adding and subtracting fractions with the same or different denominator, adding and subtracting mixed numbers, solving fraction problems, fraction multiplication and division, solving problems of mixed numbers and multiplication and division of mixed numbers. About three months after the second measurement, a third measurement took place (retention-test) for examining the long term effects of the intervention course.

#### *The rationale of the intervention course – Its principles*

The intervention course comprised of activities which aimed at developing the seven abilities. The design of the intervention course was based on some principles for helping students of all categories to acquire fraction understanding, but especially the low-achievers that struggled the most. According to the first principle, the activities and the problems should be interesting, arise from everyday life and attract students' interest. The second principle referred to the sequence of the activities from the easiest to the more difficult ones. The third principle was about the way students worked; either individually or in small groups favoring discussion and exchange of ideas. After working individually or in small groups, students discussed their ideas in the whole classroom and this procedure helped them to share their views and interact. The fourth principle referred to the modification of the lesson plans according to the strengths and weaknesses of the students and their previous knowledge of fractions.

#### *The lessons*

In this section, we briefly describe some of the nine lessons that aimed at developing the three abilities. The first lesson aimed at developing students' ability in explaining what a fraction is, recognizing fractions in various representational systems, placing fractions on the number line and linking the concept of fractions to the division

numerator  $\div$  denominator. The third lesson cultivated students' ability to construct visual representations of fractions and acquire a feeling of the relative magnitude of fractions. The fourth lesson referred to the development of students' ability to convert verbal and symbolic representations to visual ones and vice versa. The fourth lesson was complementary to the first and third lessons regarding the development of students' abilities in representations of fractions. In some activities, problems of fractions were presented to students and they had to write the equation and construct drawing/drawings in order to solve them (translation from verbal to symbolic and visual representations). Other activities asked students to translate from symbolic to visual and verbal representations. In these activities, the equation was given and students had to write a problem that could be solved by this equation or construct a drawing. Finally, students were called to write problems on the basis of visual representations (from visual to verbal representation). Lessons 5 and 6, aimed at developing students' ability in reflection during the solution of fraction problems. Five carefully selected problems were presented to students and they had to solve them and were encouraged to reason their thinking, explain the strategy they used, express their confidence about their solution, examine whether the path they followed was correct or not and what they did correct and what wrong. Furthermore, they were called to think about the reasonableness of their answer and verify their answer. The seventh and eighth lessons aimed at developing students' abilities in fraction recognition, explanations for fractions and justifications about fractions.

### *Statistical analyses*

To answer the research questions, the mean values and standard deviations that emerged from the Confirmatory Factor Analysis utilized in a previous study carried out by the authors (Nicolaou & Pitta-Pantazi, in press) for the three abilities were used. To trace the development of the abilities, growth analysis was used. More specifically, we sought appropriate growth models that describe the development of the abilities for each group of students. In order to evaluate model fit, three widely accepted fit indices were computed: (a) the ratio of chi-square to its degrees of freedom, which should be less than 1.96 ( $\chi^2/\text{df} < 1.96$ ), (b) the Comparative Fit Index (CFI), the values of which should be equal to or larger than 0.90, and (c) the Root Mean Square Error of Approximation (RMSEA), with acceptable values less than or equal to 0.06 (Muthén & Muthén, 2007). We first examined linear growth models across the three measurements and in case such models did not fit to the data, we searched for curve growth models.

## **RESULTS**

Table 1 shows means and standard deviations for the three abilities for the low-achievers of each group for each of the three measurements.

Explanations for fractions						
Pre-test		Post-test		Retention test		
Group	$\bar{X}$	SD	$\bar{X}$	SD	$\bar{X}$	SD
Experimental ( $n=87$ )	0.21	0.28	0.52	0.42	0.44	0.39
Control ( $n=61$ )	0.22	0.36	0.27	0.39	0.35	0.38
Representations of fractions						
Pre-test		Post-test		Retention test		
Group	$\bar{X}$	SD	$\bar{X}$	SD	$\bar{X}$	SD
Experimental ( $n=87$ )	2.15	0.83	2.31	1.02	2.74	1.31
Control ( $n=61$ )	2.06	0.81	2.09	1.03	2.48	1.27
Reflection during the solution of fraction problems						
Pre-test		Post-test		Retention test		
Group	$\bar{X}$	SD	$\bar{X}$	SD	$\bar{X}$	SD
Experimental ( $n=87$ )	0.38	0.34	0.87	0.89	1.27	1.33
Control ( $n=61$ )	0.44	0.44	0.60	0.85	0.84	0.94

Table 1: Means and standard deviations of the three abilities for the low-achievers of each group

At the pre-test, the low-achievers of the experimental and the control group were approximately equivalent in the three abilities. At the post and the retention tests, however, the means of the experimental group were higher than the respective means of the control group. The results for the type of growth (linear or non-linear), the growth rates and the statistical indices, are shown in Table 2.

	Experimental	Control
Explanations for fractions	Curve (0, 0.25, 0.16) ( $\bar{S}=1.773$ , $Z=6.484$ , $CFI=1.000$ , $RMSEA=0.000$ , $\chi^2/df=0.447$ )	Linear (0, 1, 2) ( $\bar{S}=0.545$ , $Z=3.239$ , $CFI=1.000$ , $RMSEA=0.000$ , $\chi^2/df=0.204$ )
Representations of fractions	Curve (0, 1, 4) ( $\bar{S}=0.803$ , $Z=4.038$ , $CFI=1.000$ , $RMSEA=0.000$ , $\chi^2/df=0.016$ )	Curve (0, 1, 4) ( $\bar{S}=0.297$ , $Z=2.745$ , $CFI=1.000$ , $RMSEA=0.000$ , $\chi^2/df=0.416$ )
Reflection	Curve (0, 1, 2.56) ( $\bar{S}=1.828$ , $Z=6.721$ , $CFI=0.972$ , $RMSEA=0.094$ , $\chi^2/df=1.545$ )	Linear (0, 1, 2) ( $\bar{S}=0.572$ , $Z=3.821$ , $CFI=1.000$ , $RMSEA=0.000$ , $\chi^2/df=0.193$ )

Table 2: Growth patterns, growth rates and statistical indices for the development of the abilities for the low-achievers of the experimental and the control group

From Table 2, we observe that the mean values of slope were positive and statistically significant ( $Z>1.96$ ) for the three abilities and additionally, the mean values of slope for the experimental group were about three times more than the respective mean values of the control group. These findings indicate that both groups of students had progress in the three abilities, but the progress of the experimental group was greater.

Nevertheless, similarities and differences regarding the type of growth pattern and the duration of the effects of the intervention course were observed for the three abilities. More specifically, curve growth patterns were found for the low-achievers of the experimental group. The growth patterns were similar for explanations for fractions and reflection during the solution of fraction problems with initially higher growth rate that decreased with time, but differed for representations of fractions (initially lower growth rate, that increased with time). However, in the case of explanations for fractions, there was improvement during the period of the implementation of the intervention course and then a decline and three months after the end of the intervention course, the gap between the experimental and the control group diminished (see also the growth patterns in Figure 2). On the other side, the progress gained during the implementation of the intervention course was maintained for the other two abilities. Regarding the low-achievers of the control group, linear growth was found for explanations for fractions and reflection during the solution of fraction problems, while curve pattern was found for representations of fractions with initially lower growth rate that increased with time (growth patterns are presented in Figure 2).

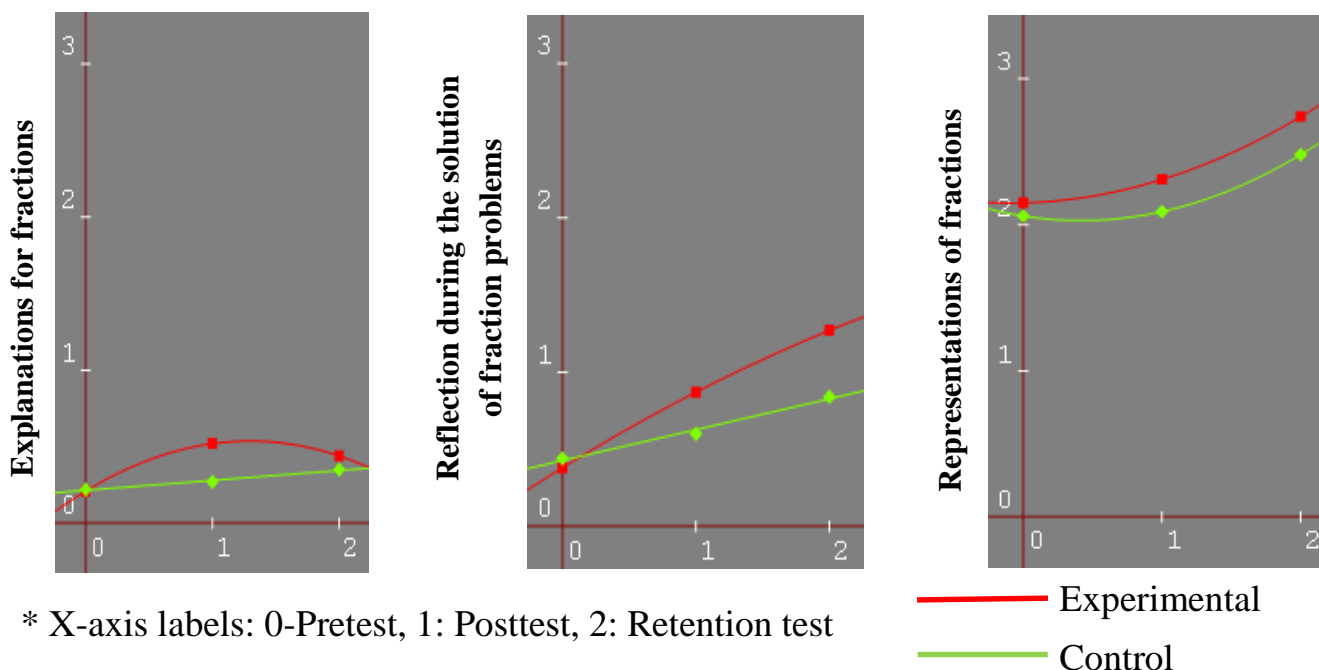


Figure 2: Growth patterns for the three abilities for the low-achievers of the experimental and the control group

## DISCUSSION

The present study provides interesting data about the development of abilities related to fraction understanding for two groups of low-achieving sixth grade students: those that took part in an intervention course for developing these abilities (experimental group) and those taught by the CNMC (control group). The results revealed that the mean growth rates for the experimental group were higher compared to the control group for the three abilities and thus provided evidence for the positive impact of the intervention course in developing low-achieving sixth grade students' abilities related

to fraction understanding and more specifically, explanations for fractions, representations of fractions and reflection during the solution of fraction problems. Nevertheless, some differences between the three abilities concerning the growth patterns and the duration of the effects of the intervention course were noticed.

Concerning representations of fractions, the experimental and the control group followed similar curve growth patterns with low initial positive growth rate that increased with time, that is acceleration was found. The growth pattern for representations of fractions was different compared to the other abilities and this is an interesting finding and could be an indication that the development of representations of fractions and more specifically translation among different representations of fractions might be more suitable during the second half of the sixth grade. Additionally, the superiority of the experimental group over the control group was maintained three months after the end of the intervention course. Regarding explanations for fractions and reflection during the solution of fraction problems, the students of the experimental group showed a similar curve growth pattern with initially higher growth rate that decreased with time. The students of the control group displayed linear growth with positive mean growth rate. The results might be explained by the fact that the intervention course boosted the students of the experimental group and they initially exhibited higher growth rate that decreased progressively. However, in the case of explanations for fractions, a decline in students' ability was noticed three months after the end of the intervention course and the superiority of the experimental over the control group diminished. On the other hand, in the case of reflection during the solution of fraction problems, the improvement continued after the end of the intervention course but in a slowing rate and the gap was maintained. The findings indicate that it might be more difficult to produce a permanent improvement for explanations for fractions compared to representations of fractions or reflection during the solution of fraction problems. The findings for the difficulty of explanations of fractions are in line with Newstead and Murray (1998) who referred to students' difficulties in explaining what a fraction is and other fraction related topics.

The importance of the present study lies in that the implementation of the intervention course improved low-achieving sixth grade students' abilities related to fraction understanding, and this group of students struggles the most and faces the majority of difficulties to acquire fraction understanding. Moreover, these abilities correspond to higher order thinking skills and were found to be among the most difficult for low-achievers. The results of the present study agree with Zohar and Dori (2003), who reported gains for low-achievers from projects whose goal was to teach higher order thinking skills in science classrooms. Additionally, the results imply that low-achievers can be benefited from intervention courses aiming to improve abilities corresponding to higher order thinking skills. The present study also provided evidence for the growth patterns of low-achieving sixth grade students that were taught by the intervention course and also those that followed the national curriculum (typical classes). Future studies could include larger samples and more measurements to reach more solid



conclusions and also could examine the growth patterns of medium and high achievers. Moreover, future studies could seek explanations for these growth patterns.

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# EFFECTS OF TEXTBOOKS ON MATHEMATICS TEACHING AND LEARNING IN GERMAN PRIMARY SCHOOLS

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*Empirical studies show that textbooks have an influence on teachers' instruction and that instruction affects students' achievement. However, little research has been conducted on the whole causal chain: the influence of textbooks on students' achievement mediated by teachers' instruction. We report findings from a re-analysis of a two-year longitudinal study with 75 primary school classes. Results of multilevel analyses indicate that the didactical structure and the content of a textbook influences students' achievement. Moreover, we provide evidence that the effect of the textbook content is mediated by instruction.*

## THEORETICAL BACKGROUND

In the last decades, a growing body of research has directed the attention to mathematics textbooks. Based on the assumption that textbooks have a substantial influence on the teaching and learning of mathematics, different kinds of research questions and methodologies were elaborated in mathematics education research (see the overview in Fan, Zhu, & Miao, 2013). Among others, textbooks were considered as artefacts with very different functions, for example, as learning material for students, as task collections or stimulus for teachers' lesson planning, or as tool for education administrations to communicate the intended curriculum (Fan et al., 2013). Accordingly, textbook research has addressed various questions concerning these functions, for example, the role of textbooks on teachers' lesson preparation and instruction (Krammer, 1985; Lepik, Grevholm & Viholainen, 2015) or the influence of teacher characteristics on the effects of textbooks on instruction (Remillard, 2005).

### The role of textbooks for teaching and learning: empirical results

In this section, we present exemplary empirical results which illustrate aspects of the interaction between the curriculum, textbooks, teachers and the learning activities of students. First we address results from studies investigating the effects of textbooks and, secondly, we elaborate on the question whether teacher characteristics (as moderator variables) influence textbook effects on the mathematics classroom.

#### *Effects of textbooks on mathematics teaching and learning*

Several studies, including international large scale assessments, showed that teachers frequently use textbooks for their instruction. For example, in TIMSS 2011 about 75% of the primary school teachers reported that they use the mathematics textbook as primary basis for their instruction (Mullis, Martin, Foy, & Arora, 2012). Similar results were presented by Lepik et al. (2015) for 402 secondary teachers from Estonia, Finland

and Norway. Comparisons of the influence of different factors on the teaching content indicated that the textbook has the strongest influence (e.g., Mullis et al., 2012).

These findings suggest a relation between textbooks and teachers' instruction. This assumption is supported by results from TIMSS 1995 which indicate a positive correlation between the space a topic covers in a textbook and the instructional time teachers using this textbook have dedicated to this topic in mathematics classroom (Schmidt et al., 2001). A classroom observation study by Krammer (1985) with 50 eighth-grade teachers provides evidence for a consistency between textbook features and teaching practices. The three teacher groups corresponding to three different textbooks significantly differ in their frequency of posing higher-order questions or the students' perception of remedial help. However, it is not clear if the textbook influenced the teachers or if teachers chose a textbook which suits their teaching style.

Van Steenbrugge, Valcke, and Desoete (2013) conducted one of the rare quantitative studies on the effects of mathematics textbooks on students' achievement. The cross-sectional study included 1579 students (grade 1 to 6) and their 90 teachers using five different textbooks. The authors did not find evidence for a substantial differential effect of textbooks on students' mathematics performance when controlling for teachers' experience. Törnroos (2005) also examined the influence of textbooks on students' achievement using data from TIMSS 1999 in Finland (nine different mathematics textbooks series used in 161 Finish secondary schools). It turned out that the amount of opportunities to learn a textbook provided specifically for the content of TIMSS items was significantly correlated with students' performance in the TIMSS test. In contrast to the results from van Steenbrugge et al. (2013), these findings suggest that textbooks have an effect on student achievement. Törnroos (2005) hypothesized that the textbook effect on student achievement is mediated by the teaching content.

#### *Teacher characteristics as moderators of textbook effects*

As previously mentioned, there is evidence for a connection between the textbook content and the content realized by teachers in classroom instruction (Schmidt et al., 2001). Case studies indicate that there are different patterns in textbook use so that use of the same textbook can result in a different quality of instruction (Fan et al., 2013). Hence, it seems to be promising to examine teacher characteristics as moderators of this relation. According to the review article of Remillard (2005) teachers' beliefs play a substantial role for teachers' decisions concerning the selection, design and enactment of mathematics tasks as well as the curriculum mapping. Moreover, there are plausible reasons for the influence of teacher knowledge on textbook use, though clear empirical evidence is still missing. In a study with 48 teachers on the quality of curriculum implementation, Stein and Kaufman (2010) did not find a correlation between teacher capacities (e.g., knowledge, professional development activities) and the quality of curriculum implementation (indicator: maintenance of cognitive demand from material to enactment phase of the lesson). In contrast, Hill and Charalambous (2012) presented a series of case studies indicating an influence of teacher knowledge

on the use of curriculum material. They hypothesize that instructional quality of teachers with low professional knowledge depends on the quality of the used curriculum material (which especially comprises the textbooks) whereas teachers with high professional knowledge are able to compensate low quality curriculum material.

## **RESEARCH QUESTIONS AND METHOD**

The previously presented research indicates that there are still open questions concerning the effect of textbooks on students learning in mathematics. Many research results are based on small scale studies using qualitative methods (cf. the overview in Fan et al., 2013). These results provide valuable insight into the interplay between textbooks, teacher and instruction and should be supplemented by corresponding results of quantitative studies. Only a few studies investigate the effects of textbooks on student achievement. Especially, we are not aware of longitudinal studies. Finally, most studies address textbooks from higher grade levels ( $>$  grade 5) and effects of textbooks in the first grades are not well investigated. To contribute to this field of research, we conducted a re-analysis of a data set from a longitudinal study (details below) which allows examining the following research questions:

1. Does the textbook used by primary mathematics teacher have an effect on the teaching content and on students' achievement at the beginning of primary school?
2. Are the effects of mathematics textbooks moderated by teacher characteristics?

### **Research context**

Basis for our analysis is an existing data set from a large two-year longitudinal study with primary school students from one federal state in Northern Germany. The sample consists of  $N = 2737$  students from 123 classes in 40 schools. It comprises student data from the beginning of grade 1 when students entered school (normally at the age of 6 years) to the end of grade 2. The original aim of the study is to address students' competence development in arithmetic which is the heart of primary mathematics.

There are specific characteristics of the national educational context framing the research and the interpretation of the results. In Germany, each federal state has a statewide curriculum which describes for each grade the content, skills and abilities teachers must address. The mathematics textbooks mirror these curricula and primary schools can select a mathematics textbook series for their grades 1-4. Our data set covers schools from one federal state following the same curriculum and using different mathematics textbooks. Hence, it is possible to investigate effects of different textbooks on teaching content and student achievement (cf. research question 1). Moreover, many primary school teachers in Germany teach mathematics without formal qualifications in mathematics and mathematics education. According to Richter, Kuhl, and Reimers (2012), students taught by these teachers show a lower mathematics achievement. Hence, we address teacher qualification as moderator for textbook effects on teaching content and student achievement (cf. research question 2).

## **Textbooks**

In a subsample of 75 classes four popular textbook series (denoted by A, B, C, D) were used. All textbooks series show a variation in the space used for the curricular topics and the emphasis of learning goals but they all mirror the prescribed curriculum. There are two striking features which allow a grouping of the textbooks for our study.

1. In contrast to textbook series A, B and C, series D prescribes a linear order which the students should follow when learning the arithmetical content. For grade 1, textbook series D consists of six and for grade 2 of five consecutively numbered booklets. For example, the first booklet in grade 1 covers the numbers 1-6, the second addition and subtraction with numbers 1-6, the third the numbers 7-13, the fourth the number domain 1-20 and related problems, the fifth addition and subtraction in the domain 1-20 etc. In each booklet the learning content is structured in small steps and students should work individually page by page on the presented mathematics problems. New concepts (numbers, operations) are introduced by a quick transformation from an iconic to the symbolic representation. Each problem type is dedicated 1-2 pages for practicing and connections between mathematically related topics are hardly addressed. The textbooks of series A, B and C do not prescribe in detail a specific learning trajectory for students. Arithmetic concepts are mostly introduced following an elaborated transformation from iconic and symbolic representations. Learning opportunities for procedural and conceptual knowledge are balanced and connections between related arithmetic topics are addressed. Based on these strong differences, we compared students taught by textbook series D with students taught by textbooks from series A, B or C to examine effects of the didactical structure of textbooks.

2. Another striking difference between the textbooks for grade 1 is a very specific content aspect: textbooks from series A and C do not introduce the number line as a representation in grade 1 (in these textbooks it appears at the beginning of grade 2 for the first time). In contrast, textbooks from series B and D suggest the introduction of the number line in the middle of grade 1. All textbooks address ordinal aspects of numbers (by counting or by ordering numbers). Based on these differences, we compared effects of textbook series A and C with effects of series B and D on the implemented teaching content and on student achievement related to the number line.

## **Instruments, data collection and statistical analysis**

Student and teacher data were collected by different tests and questionnaires. Data for controlling the learning prerequisites of the students (basic numerical skills, basic language skills, general cognitive abilities) were measured with approved standardized instruments at the beginning of grade 1. Data for the individual learning progress were collected at the end of grade 1 and 2 with grade-specific arithmetic tests on conceptual and procedural knowledge. The two tests on procedural knowledge comprised 57 and 45 items addressing e.g., doubling, halving, part-whole relations, addition, subtraction in the number domain 1-20 or 1-100 (depending on the grade). The two tests on

conceptual knowledge with 12 and 29 items covered grade-specific arithmetic content emphasizing e.g., models of subtraction, multiplication, division; comparing numbers, place value relations). Moreover, there was data from a number line test with eight items administered in the second half of grade 1 (allocating four numbers on a semi-structured number line and identifying four numbers on a structured number line). The data of each test were scaled based on Item-Response-Theory and all scales showed an acceptable reliability (EAP-PV between .79 and .96).

The teachers were asked for the textbook they used as well as for their teacher qualification (educated as mathematics teacher or not). Since the number line test was administered in the second half of grade 1 during the school year, the teachers were asked to what extent they already had addressed the number line at that time in their previous teaching. We took this as an indicator of teaching content implementation.

We conducted multilevel analyses which take into account the nested structure of the sample (students in classes). We included the variables for learning prerequisites at school entrance on the individual level and as aggregated value on the class level (as an indicator of group composition). Textbook type, implemented teaching content and teacher qualification were included on class level. For the number line context we analyzed the mediation *textbook*  $\rightarrow$  *implemented teaching content*  $\rightarrow$  *student achievement* by a multilevel 2-2-1-mediation structural equation model (following Preacher, Zyphur, & Zhang, 2010).

## RESULTS

To answer research question 1 we analyzed the effect of the didactical structure of the textbook on students' achievement. We considered the influence of textbook series A, B and C (textbook = 0) versus textbook series D (textbook = 1) on conceptual and procedural knowledge at the end of grade 1 and grade 2. Due to space limitation we only describe the relevant results for grade 1 and present a compact form of the multilevel analysis output for grade 2 (table 1).

At the end of grade 1 only 12% (respectively 13%) of the variance of students' achievement in procedural (conceptual) knowledge can be explained by the class level (ICC in the null model). It turns out that the teacher qualification and the textbook have no direct significant influence on the students' conceptual or procedural knowledge. However, there is a significant interaction effect teacher qualification  $\times$  textbook on students' procedural knowledge ( $\beta = -.55$ ,  $SE = .18$ ,  $p < .001$ ) explaining 29% of the variance on class level. There is no such interaction effect for conceptual knowledge.

At the end of grade 2, 15% of the variance of students' achievement in procedural and in conceptual knowledge can be explained by the class level (ICC in the null model). Table 1 shows the multilevel models for conceptual and procedural knowledge. In both cases textbook D has a direct negative effect. For the conceptual knowledge students of non-certified teachers show a low achievement independent of the textbook.

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Conceptual knowledge	Procedural knowledge
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		$\beta$ (SE)	$\beta$ (SE)
Individual level	Cognitive abilities	.27** (.04)	.35** (.04)
	Basic numerical skills	.19** (.04)	.20** (.04)
	Language skills	.14** (.03)	.09** (.03)
	R <sup>2</sup>	23%	29%
Class level	Cog. abilities (aggregated)	.35* (.17)	.39* (.18)
	Basic numeric. skills (agg.)	-.04 (.17)	-.08 (.20)
	Language (aggregated)	.12 (.16)	-.10 (.16)
	Teacher qualification	-.29* (.16)	-.05 (.14)
	Textbook	-.50** (.14)	-.28* (.14)
	Teacher qualification $\times$ Textbook	.14 (.18)	-.32 (.18)
	R <sup>2</sup>	46%	51%

\*\*  $p < .01$ , \*  $p < .05$ ; *textbook*: A, B & C = 0; D = 1; *teacher qualification*: certified mathematics teacher = 0, non-certified mathematics teacher = 1

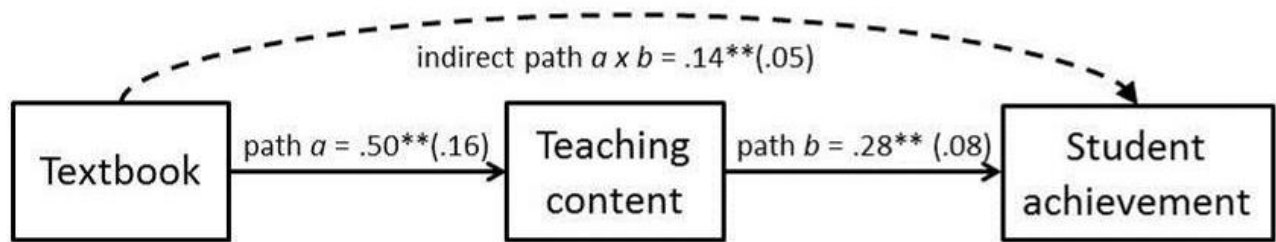
Table 1: Multilevel models for procedural and conceptual knowledge

For research question 2 we analyzed for the topic number line whether there is an effect of the textbook on students' achievement which is mediated by the teaching content. According to Preacher et al. (2010) the adequate model for nested data is a multilevel structural equation model. As presented in Figure 1, there are significant direct effects *textbook*  $\rightarrow$  *teaching content* and *teaching content*  $\rightarrow$  *student achievement* as well as a significant indirect effect *textbook*  $\rightarrow$  *student achievement* mediated by *teaching content*. Like in the models before the three variables for learning prerequisites are controlled on both levels. Here teacher qualification shows no influence.

## DISCUSSION

The results of our quantitative study with grade 1 and 2 students indicate that the textbook has a substantial influence on students' achievement in arithmetic in the first years of primary school. At least two different aspects of textbooks are relevant for this effect: the content covered by the textbook and its didactical structure.

From the mediation model presented in Figure 1 we conclude that the influence of the textbook content on students' achievement is mediated by the learning opportunities (i.e. learning content) that are offered to the students in the mathematics classroom. As described above, the textbook content influences the content teachers offer in mathematics lessons (e.g., Schmidt et al., 2001) and the vast majority of primary teachers report that they use textbooks as primary resource for teaching preparation (Mullis et al., 2012). Törnroos (2005) assumed learning opportunities as mediator for textbooks effects on secondary students' achievement but he had no data about the content taught to the students in his sample. Our results provide the missing evidence.



(RSMEA = .015, CFI = .99, TLI = .99, SRMR<sub>W</sub> < .001, SRMR<sub>B</sub> = .06);

*textbook*: A & C = 0; B & D = 1; *teaching content* as (i) allocating numbers on a semi-structured number line and (ii) identifying numbers on a structured number line: 0 = number line not addressed; 1 = one of two aspects addressed; 2 = both aspects addressed.

Figure 1: Mediation in a 2-2-1 design tested as multilevel structural equation model

In addition to the content aspect, our results in table 1 show that the didactical structure of a textbook, which suggests specific learning trajectories and learning activities, has a substantial impact on student achievement. Already Krammer (1985) presented findings for the secondary level that the didactical orientation of a textbook has a certain consistency with the observed teaching and learning activities. In our sample we found that a textbook which strongly prescribes learning activities in a specific linear order has negative effects on conceptual and procedural knowledge at the end of grade 2. Interestingly, in grade 1 we found only a strong negative effect for procedural knowledge, which is moderated by teacher qualification. Our interpretation is that the grade 2 content in mathematics (number domain 1-100, place value system, multiplication and division) is much more challenging than the grade 1 content so that the influence of instruction and the textbook increases. Our grade 2 findings differ from that of van Steenbrugge et al. (2013) who did not find differential effects of textbooks on achievement. We think that these contrary findings can be explained by the fact that we included longitudinal data and that we compared two contrasting groups of specific textbooks. Finally, concerning the didactical structure of a textbook the teacher qualification has only a moderating effect in grade 1 (in grade 2 there is a direct effect). For the influence of teaching content on student achievement, we did not find a moderating effect of teacher qualification. A possible reason is that the dichotomous variable teacher qualification was too coarse as an indicator for teacher professional knowledge.

There are several limitations of our study. Since we re-analyzed an existing data set we were not able to administer specific instruments for our research. In particular, the questionnaires do not provide fine-grained data on the implementation of the teaching content or the teacher knowledge. Moreover, the result for the mediation (figure 1) is restricted to the content “number line” because student and teacher data is available for this topic in the data set. Despite of these limitations the data set has the advantage that it covers a large sample taught by the same curriculum and allows multilevel analysis



with an adequate explanatory power. Accordingly, we were able to supplement and further develop existing research on the effects of textbooks on students' learning.

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# OPERATIONALISING CONSTRUCTIVIST THEORY USING POPPER'S THREE WORLDS

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*Adopting constructivist practices in mathematics without rejecting the integrity of fundamental structures of mathematical knowledge is acknowledged as a challenge of curriculum reform. This paper illustrates the use of an approach to operationalising constructivist theory that supports the description of both the structure of mathematical knowledge and the nature of individual's idiosyncratic and changing understanding.*

## INTRODUCTION

This paper applies the Alternative Theoretical Framework (ATF) proposed by Nutchey (2011) that was developed from Popper's three worlds (1978) and Piaget's reflective abstraction (1977/2001), to examine the interactions between a student and a researcher in a mathematics classroom. This paper serves to illustrate the usefulness of the theoretical framework in regard to operationalising constructivist theory. It shows how such analysis is beneficial in developing a deep understanding of the ways in which prior knowledge and the intervention of a more experienced individual (i.e., teacher, researcher) can influence a learner's development of mathematical understanding which is mutually compatible with other members of the community.

The paper presents a narrative of the interactions between the second author and a student, Zeke. The paper analyses this interaction in terms of the ATF. The narrative illustrates how the transformations of reflective abstraction can be supported by drawing upon students' apparent prior understanding of mathematical knowledge to support the development of new understanding. That is, how the target of learning (the primary structure) can be achieved by considering the corresponding scaffolding structures. Underpinning this paper is the conjecture that the desired acceleration is possible if due consideration is given to the structure of mathematics and the selection of an appropriate sequence of mathematical ideas to be taught. In this regard, the presented application of ATF is one way in which to support acceleration.

The paper firstly provides context for the narrative, then reviews informing theories (e.g., Popper, Piaget and Vygotsky), describes the ATF, applies the ATF to the narrative, and finally provides a brief discussion and conclusion.

## CONTEXT

The data reported upon in this paper is drawn from the *Accelerating the Mathematics Learning of Low Socio-Economic Status Junior Secondary Students (XLR8) project* (Cooper, Nutchey & Grant, 2013). The XLR8 project has developed an alternate curriculum that aims to accelerate the learning of under-performing junior secondary

students, such that they can successfully access further mathematics study and employment opportunities and thereby improve their life chances.

Zeke was a student in Year 8 and a participant in the XLR8 project. He believed that he could not do mathematics despite having demonstrated a capability for more complex thought and reasoning if prompted and scaffolded lightly. In previous lessons the students in Zeke's class had represented relationships, including linear relationships, on the Cartesian plane. They had also considered the representation of linear relationships (direct proportions) in tables and on dual-scale number lines, and considered the concept of rate of change and how this was represented on graphs and in tables of data and through the standard equation  $y = mx + c$ . In the cited lesson, students were practising rate-related calculations by attempting several worded questions.

One group of questions, which are presently reported, related to the purchase of apples. The students were informed that apples had the price of \$1.45 per kilogram. The first question asked how much would seven kilograms of apples cost. Using pen and paper, Zeke was observed to independently use a vertical addition algorithm to sum seven lots of \$1.45. He recorded the result in his workbook as "7kg is \$10.15". That is, his actions appeared to be based upon an understanding of multiplication as repeated-addition rather than the more sophisticated meaning of combining-equal-groups.

In the next question, students were asked "How many bags of apples can you buy with \$29?" Based upon work previously covered in class that involved similarly structured word problems, it was expected that students would solve this problem using a division calculation. As Zeke attempted this problem, the second author sat and discussed his work with him. After some thought and mumbled calculations, Zeke suggested the answer was 13, to which the researcher asked "How did you work that out?" It was unclear exactly how Zeke arrived at the answer of 13, although his explanation did involve doubling the cost and so it is assumed he also doubled the weight to get 13 (albeit incorrectly). His initial answer also indicated that rather than solve the problem using a division calculation, Zeke interpreted the question as one of multiplication with an unknown multiplier.

## **INFORMING THEORIES**

Constructivist theories, based upon the work of Piaget and Vygotsky, recognise the learner's role of actively constructing meaning from experience (Jardine, 2006), and that an individual's making of meaning will be guided by the activities of the individual in the social milieu (Davydov, 1995). However, numerous seminal thinkers in the field of education, including mathematics education (e.g., Baroody, 2003; Simon, Tzur, Heinz, & Kinzel, 2004; Steffe, 2004), have acknowledged that advances in teaching and learning practice have been impeded by difficulties in turning the constructivist theories into effective practice. A specific criticism is provided by English et al. (English, 2007; English & Sriraman, 2010) who have noted that some paradigms based

upon constructivist theory have been perceived to reject the integrity of fundamental structures of mathematical knowledge as a basis for learning.

From an educational practitioner's perspective, theories of learning must be consistent with the domain of learning and must lead to effective classroom practice. In regard to the learning of mathematics, these two objectives necessitate reconciliation between the existence of a discussable and improvable objective reality and the commonly accepted constructivist notion of learners constructing personal meaning through authentic domain experience. A common attitude regarding such challenges of contemporary mathematics education has been summarised by Baroody (2003, p. 29): "instruction cannot be significantly improved ... by turning away from more complex methods of instruction". And so, further effort to operationalise learning theory will potentially reduce a significant impediment to the improvement of mathematics education, and lead to practice which features teacher-guided, learner-centred cognitive activity (Mayer, 2004).

Popper (1978) conjectured the existence of three worlds of knowledge, in an attempt to overcome the limitations he perceived in the monist and dualist conceptualisations of knowledge implicit in most objectivist and constructivist theories. Popper accepted the existence of physical bodies, describing them as existing in World 1, and the existence of experiences, describing them as existing in World 2. Popper conjectured the existence of a third world – World 3 – that is comprised of the products of the human mind, including languages, scientific conjectures and mathematical theories, sculptures and feats of engineering. With specific regard to scientific conjecture and mathematical theory, Popper claimed that such World 3 objects can have causal effects upon the actions of World 1. In summary, Popper's World 3 is the world of ideas and the content of thought, World 1 is the world of physical actions and objects that embody World 3 ideas, and World 2 is the world of mental thoughts that operate over the ideas of World 3 and thus mediate between those ideas and the actions of World 1.

Popperian-based differentiation of objective World 3 knowledge from subjective World 2 knowledge provides an alternate lens through which to reconsider constructivist learning theories and practices. An example of such reconsideration has been proposed by Bereiter (2002), who has metaphorically described knowledge as a tool of which the learner builds understanding. This understanding can be characterised as the learner's manifold relationship to knowledge (Woodruff, 2005).

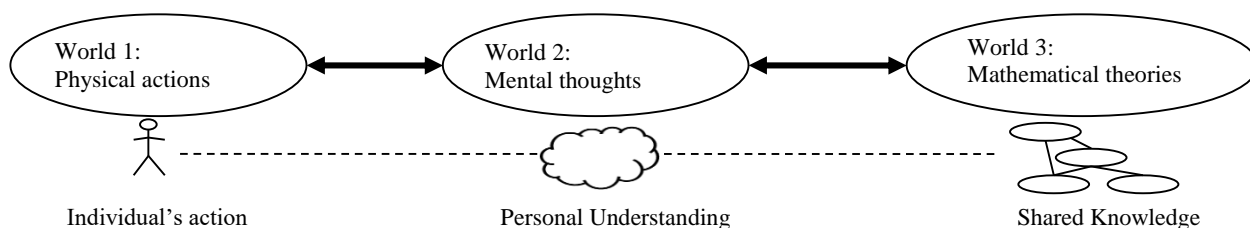
This paper is based upon the use of Popper's three-world model of knowledge to reconsider key aspects of Piagetian and Vygotskian learning theory. Piaget discussed the representation of knowledge in terms of the schema, which is the "structure, categorisation or organisation of thought or action" (Jardine, 2006, p. 5). Piaget presented schema as some simplified image of knowledge which guides the figurative aspects of thought (Piaget, 1970/2000). Piaget's schema was refined by Dubinsky to: "a coherent structure used by an individual to make sense of a perceived problem" (1991, p. 102). Piaget (1977/2001) extended his theory to consider the way in which

schema are transformed during conceptual development. Such transformation (the creation of relationships, based upon the observed properties, that modify the structure of knowledge) is referred to as abstraction (Dubinsky, 1991). In particular, Piaget considered the idea of reflective abstraction to be sufficiently powerful to describe the learner's entire conceptual development: "[it] ranges over ... all of the subject's cognitive activities [and] can be observed at every major stage of development" (Piaget, 1977/2001, p. 30). For Piaget, reflective abstraction explained the power of transforming the schema by combining or associating pre-existing conceptions in such a way that significantly extended the learner's ability (Dubinsky, 1991).

In learning environments focussed upon the creation of ideas, such as Scardamalia and Bereiter's (2006) computer-mediated knowledge building communities, the role of teacher is that of a more experienced individual. The teacher is in a position to scaffold enculturation by proposing abstract concepts, in particular those with which the learner has less experience and hence limited understanding, and by drawing upon the learner's extant understanding to make sense of the new concept. Such a scaffolding approach draws upon Vygotskian theory, in particular the zone of proximal development-based theoretical learning; the teacher creates opportunities for the learner to realise the organisation of concepts shared within the community.

## ALTERNATIVE THEORETICAL FRAMEWORK

Using Popper's three worlds as the basis for considering learning theories, Nutchey (2011) has proposed an alternative theoretical framework. This framework integrates Piaget's reflective abstraction as a fundamental theory with which to characterise a domain of World 3 knowledge. The reflective abstraction construct is also used to anticipate a learner's development of World 2 understanding with regard to World 3 knowledge, and thus informs how the transformation of understanding, or the enhancement of the learner-knowledge relationship, might be scaffolded. This synthesis of theory is summarised in Figure 1, and forms the basis for the elaboration of the operational model that will enable this alternative theoretical framework to be put into practice and which is reported upon, in part, in this paper. It illustrates how the learner's World 2 understanding, which may be inferred from past experience, mediates their World 1 actions with regards to the shared World 3 knowledge of the community. This in turn suggests that the careful design of future learning activities (experiences) may lead to further development of the learner's understanding of the shared knowledge.



*Figure 1.* Conceptualisation of knowledge and understanding adopted in ATF

Previously, Nutchey (2011) has described how Piaget's reflective abstraction has formed the basis for creating a graphical language to construct visual representations of the structure of mathematical knowledge, referred to as genetic decompositions. These examples of genetic decompositions identified mathematical ideas, or knowledge objects, and various associations that could be used to link the knowledge objects. These associations were based upon Piaget's five transformations of reflective abstraction. The use of genetic decompositions to characterise the structure of mathematical knowledge has also been demonstrated by Nutchey, Grant and Cooper (2014), in which they were used to aid the formulation of the XLR8 curriculum.

As was proposed in Nutchey (2011) and is illustrated in the narrative, the ATF provides a mechanism to guide (or examine) the transformation of an individual learner's World 2 understanding of World 3 knowledge. For each of the five transformations of reflective abstraction, patterns of knowledge objects and associations have been conjectured by Nutchey (2011) to correspond to the transformations. These patterns of knowledge objects and associations are referred to as the primary structures for each transformation. Patterns of knowledge objects that typically lay adjacent to the primary structures can also be identified. For each primary structure, the set of such generalised adjacent structures is referred to as the set of scaffolding structures for the transformation.

As well, the set of primary structures and associated scaffolding structures can then be used to analyse a learner's developmental trajectory (prior learning experiences) with regards to some specific knowledge object of interest for which a deepened understanding is desired. Firstly, the set of primary structures related to the knowledge object of interest can be identified, since these primary structures define the potential for the transformation of understanding. These primary structures can then be considered to determine which structure, and hence transformation, is of immediate importance in regard to the learner's conceptual development. Based upon this identification of the primary structures, a similar analysis of the developmental trajectory may also be performed to identify instances of the scaffolding structures, in particular those with which the learner has existing understanding (experience). Put simply, Nutchey's (2011) proposition allows for the systematic identification and description of the mathematical ideas and their structure that are to be understood, the ideas and structure that are already understood by the learner, and how what is already understood can be used to scaffold future learning.

## **APPLICATION**

To undertake such analysis, the XLR8 project has adopted design research as the methodology by which to iteratively propose, trial and refine theory and practice in regard to the XLR8 curriculum for accelerated learning. In particular, a variant of teacher experiment that is referred to as multi-faceted teaching experiment has been proposed and adopted, the details of which are discussed in Nutchey, Grant, Cooper and English (2015). In essence, this approach treats each participant (student, teacher

or researcher) as “a mathematician who is developing their understanding of the shared mathematical knowledge, albeit at different levels of sophistication” (Nutchey et al., 2015, p.5). To examine the implementation of the XLR8 curriculum, a variety of qualitative data was collected, including audio-recorded and transcribed classroom discourse, field notes and collected artefacts. To construct this paper’s narrative, these data were analysed by both authors and the agreed upon interpretation reported.

The narrative of Zeke’s work is now continued. The researcher queried Zeke’s initial response and probed the process he used. Referring to Zeke’s written answer to the previous question, the researcher stated, “so you doubled that and you doubled that” to which Zeke replied “oh yeah, 14”. The researcher added a line to Zeke’s workbook, showing the weight of 14 kilograms and the cost of \$20.30. She highlighted the doubling relationship within the quantities of the same type using a familiar tabular representation to scaffold manipulation of the proportion concept. Zeke then suggested a new answer of 19. Again, the researcher queried his answer, “how did you get 19”, to which Zeke replied “I don’t know”.

The researcher persisted, “So you are saying another five kilograms more? ... So how?” Zeke replied “\$1.45, so four more ... four more, is it seven?” Clearly Zeke estimated the answer of seven, and so the researcher asked Zeke to calculate the value, reminding him he could use a similar strategy to before (scaffolded coordination of place value and addition with symbolic representation). Zeke completed the sum of five lots of \$1.45 and arrived at the total of \$7.25. The researcher stepped out the sum thus far: \$20.30 for 14kg and the \$7.25 for the 5 kg. The researcher highlighted that the cost was still not up to \$29, and she then suggested “I am just thinking, you’re up to \$27.55, if you add one more kilo does that get you to \$29?”. The researcher then reiterated the sum in a way to query to the total “so you had 14 and 5 is 19 and 1 more?” to which Zeke replied “Oh, 20.”

## **DISCUSSION AND CONCLUSIONS**

To scaffold Zeke’s solution this problem, the researcher leveraged Zeke’s apparent understanding of repeated-addition meaning of multiplication. Rather than try to solve the problem using division, the researcher instead chose to coordinate this multiplication strategy with a strategy to decompose the problem into three simpler parts (the cost of 14kg, 5kg and 1kg).

To support this, the researcher augmented her verbal prompts and explanations with text written in Zeke’s workbook. After deducing that 14kg would cost \$20.30 Zeke wrote in his book “14kg is \$20.30”. The researcher drew upon this representation, recording directly under Zeke’s writing the weights (5kg, 19kg, 1kg, 20kg) and costs (\$7.25, \$27.55, \$1.45, \$29). This tabular-like representation of the quantities was referred to throughout the discussion. This expression of the proportional situation using the tabular representation became the scaffolding structure to help coordinate the direct proportion calculation. During this exchange, the researcher involved Zeke in the use of the table and performing calculations – a common action – that was based

upon the researcher's recognition of Zeke's current understanding and which incorporated the researcher's more complex understanding of proportional thinking (the primary structure to which the researcher intended Zeke to build a relationship).

This example of classroom interaction has been used to illustrate how a more experienced individual (teacher) can draw upon their own unique understanding of mathematics to shape the experience and hence understanding of a lesser-experienced individual (student). This relies firstly on the student engaging in some activity that exposes their understanding, an activity that can be observed and interpreted by the teacher. This prior understanding can be used as a scaffolding structure. The teacher can then compare their observation to their own understanding of the shared knowledge and craft the learning experience. This should take into consideration the cognitive mechanisms by which the students understanding may be transformed, that is, the five transformations of reflective abstraction and subsequent selection of a primary structure to which student understanding will be developed.

The crafting of such an intervention should aim to build connections between the learner's prior knowledge and new knowledge and so achieve the desired relational or structural understanding of mathematics. Through such carefully constructed co-activity, the learner and teacher come to a mutually compatible understanding of mathematics characterised by the understanding of a similarly connected set of mathematical concepts. By crafting such a learning activity that supports the student's action, the learner may develop an understanding that will underpin their future independent activity and the teacher may develop awareness and ability react to students' misconceptions. From the standpoint of the researcher, it is hoped that this type of detailed consideration may usefully inform the design and implementation of future mathematics curricula and classroom teaching.

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# SUSTAINING THE PROFESSIONAL GROWTH OF MATHEMATICS TEACHERS

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*Teacher professional learning is a complex activity based on the interaction of three sub-systems: the teacher, the school and the learning activity. Such learning evolves over time depending on salient outcomes and teacher value positions, and requires support that is responsive to the teacher's growth. This phenomenon has been observed within a group of teachers tasked with implementing an innovative curriculum. The curriculum involved an unfamiliar pedagogical practice that was designed to accelerate the mathematics learning of low socio-economic status junior secondary students. Based upon the observations, implications are drawn for the types of support needed to enhance teachers' capacity to engage with pedagogical and curricular changes, with the aim of sustained teacher professional growth.*

## INTRODUCTION

This paper presents an analysis of the professional learning (PL) activities that were provided to support a teacher's professional growth. These PL activities were provided for the teacher as she implemented a curriculum innovation as part of a project titled *Accelerating the Mathematics Learning of Low Socio-Economic Status Junior Secondary Students (XLR8)*. The activities were analysed in relation to Nutchey, Grant, Cooper and English's (2015) *interpretation continuum* that characterises teacher curriculum interpretation practices in terms of curriculum structure, sequence, pedagogy, resources, and assessment on a four point scale: Resister, Follower, Questioner, and Improver.

The paper first provides background to the project in relation to literature on teacher PL with respect to mathematics education. Drawing upon semi-structured interview data and classroom observations, the changing needs and teaching practices of a single teacher are described in terms of the support provided as the teacher implemented the XLR8 curriculum. This description shows how the teacher's practices evolved during the curriculum project and discusses the effectiveness of PL support. The effectiveness of the support provided is analysed in terms of the teacher's movement along the interpretation continuum. These results are used to propose conjectures as to the support needed to increase teacher professional growth when implementing a curriculum innovation.

## BACKGROUND

The XLR8 project has been designed to develop theory and practice regarding the acceleration of junior secondary students' (Years 8-9) mathematics proficiency in low-SES schools. The schools identified students to participate whose level of

mathematical achievement was assessed to be nominally at a mid-primary school level (Year 4) and who had not been ascertained with learning disabilities. The project has aimed to accelerate students' learning of mathematics such that they are able to enter Year 10 with the requisite knowledge to successfully study mathematics and ultimately enhance their capacity to engage with further study or employment.

To achieve this aim the project adopted design research (Cobb, Jackson & Munoz, 2016) as a basis for proposing, trialling and refining the XLR8 curriculum. The XLR8 curriculum has been designed to carefully explore the structure of mathematical knowledge in a nested, conceptually-focussed sequence (Cooper & Warren, 2011) that builds students' understanding from a low-achievement level to an age-appropriate level. The curriculum has employed a pedagogical framework (referred to as RAMR) which is comprised of teaching cycles of reality, abstraction, mathematics, and reflection learning activities (Cooper, Nutchey & Grant, 2013). The pedagogy is grounded in the students' reality, drawing upon suitable everyday life examples to situate learning. It provides a clear order of abstraction activities that progress through kinaesthetic-iconic-symbolic representations while also connecting to everyday and mathematical language. Mathematical activities build students' fluency with procedures and skills, promote conceptual understanding, and develop and reinforce connections to other mathematical ideas. During reflection, opportunities are made for students to reflect their learning back to reality and to transfer knowledge to new situations and form generalisations. Thus, the XLR8 curriculum has five features (Nutchey et al., 2015): (1) the structure of mathematical knowledge that underpins learning; (2) the conceptual sequence by which the mathematical structure is explored; (3) the RAMR pedagogy used to explore the structured sequence; (4) the resources provided to implement the RAMR pedagogy; and (5) the assessment materials that generate diagnostic, formative, and summative evidence of students' mathematical understanding.

As suggested by Postholm (2012), teachers participated in a range of professional learning activities to support their professional growth as they implemented the XLR8 curriculum: group-based professional learning sessions (e.g., workshops); demonstration and team teaching; personal reflections on practice, reflective conversations following classroom observations, and conversations with colleagues. Although professional learning providers intend to support change in teachers' practices, attitudes and/or beliefs and the learning outcomes of their students (Griffin, 1983), teachers are commonly described as expecting such activities to provide specific, practical ideas directly applicable to their classroom (Cooper, Baturu & Grant, 2006; Fullan & Miles, 1992). However, teachers have a tendency to appropriate their own meaning relating to the content of the professional learning activities (Warford, 2011). This occurs when teachers connect and integrate the activities with their previous experiences as learners and teachers of content, with their tacit and overt perceptions of pedagogy and with educational content and input from researchers,

colleagues, external teachers or other resource persons (the teacher-learning activity interaction in Opfer & Pedder, 2011).

Teachers can take time to develop mastery of the practical elements of curriculum innovations and may not engage fully with the underlying ethos of the curriculum innovation until a comfort level in pedagogical practice has been achieved (Clarke & Hollingsworth, 2002). Teacher professional growth is a gradual and difficult process that necessarily involves effort, anxiety and risk of failure (Guskey, 2002). Thus, teacher PL can fail to bring about change. To some educators, the answer to this is to develop collaborative relationships between researchers and teachers (Ward & Tikinoff, 1982) based on practical resources and activities that provide success (Baturu, Warren & Cooper, 2004). However, other educators argue that success needs PL to challenge teachers, set up cognitive conflict, and apply sustained pressure and support for experimentation (Huberman & Crandall, 1983; Guskey, 2002; Postholm, 2012) through, for example, observation (Zwart, Wubbels, Bergan & Bolhuis, 2009).

## **APPROACH**

Teachers were provided with modules, containing RAMR-based lessons and pre-post-tests, to trial in classrooms. These were accompanied with PL activities which were designed to: (1) meet teachers' needs in terms of mathematical content as well as pedagogy, and engage teachers as partners in decision-making; (2) acknowledge that change takes time and is an interaction of knowledge, beliefs and attitudes, classroom trials and reflection, sustained by observations of positive change; and (3) maintain continued dialogue with teachers and schools through action research approaches to trials (Baturu et al., 2004; Clarke & Hollingsworth, 2002; Cooper et al., 2006). Across 2013-2015, the PL experiences varied in response to school constraints and teacher needs. They included: (1) targeted PL at the start of each year to introduce the XLR8 pedagogy and modules and to experience, hands on, materials and resources for foundational concepts (e.g., place value) usually ignored in the secondary school and in secondary pre-service courses; (2) opportunities throughout the year for teachers to share experiences and explore the next modules; and (3) time before and after lesson observations for teachers and researchers to engage in reflective professional discussion about lesson implementation and student progress (referred to as coaching sessions).

Based upon feedback received from teachers in 2014, the research and PL focus changed at the start of 2015 to emphasise the coaching sessions. Researchers visited schools on a weekly basis at strategically chosen times so that teachers had a free period following their observed class for coaching sessions. These coaching sessions and the lesson observations were run by the same researcher, who was perceived by the teachers to have a practitioner rather than theoretical focus. The coaching sessions were conducted to meet the teacher's needs. Data were gathered by a variety of methods including: (1) field notes during lesson observations and coaching sessions; (2) video recordings of PL discussions; (3) individual semi-structured interviews; (4) teachers'

self-evaluations in terms of Nutchey et al.'s (2015) interpretation continuum; and (5) student responses to tests for each module. The researchers met regularly to share data and discuss classroom experiences. Data gathered from the interpretation continuum were particularly important because central to the underlying theory of XLR8 is that mathematics is a construction influenced by culture, and that the learning of mathematics should be matched to the reality (and culture) of the learner (Cooper et al., 2013). To this end, it was expected that each teacher in the project would critically adapt the presented curriculum so that it was suited to the needs of their students. As briefly described earlier, the interpretation continuum was composed of five subscales that reflect the five features of the curriculum (structure, sequence, pedagogy, resources and assessment). Along each subscale, the teachers could characterise themselves, in discussion with the researchers, as *Resister*, *Follower*, *Questioner* or *Improver* of the curriculum as a measure of the extent they engaged in the adoption and adaption of the XLR8 curriculum. Although it was anticipated that initially teachers might be followers of the XLR8 curriculum, it was hoped that over time that they would move along the continuum to become curriculum improvers.

In this paper the analysis of one teacher, Jackie, is reported upon. Jackie was selected as she is a typical case of the type of teacher for whom the XLR8 project was designed.

## RESULTS AND DISCUSSION

Jackie had three years of teaching experience within her school prior to beginning in the XLR8 project, she was a trained mathematics teacher, and she had taught almost exclusively in mathematics classrooms. In the XLR8 project, Jackie taught the same cohort of students for two years of the program (2014 and 2015) and, despite the population transience typical of low-SES schools, her class of students remained relatively constant across the two-year period. Jackie's class could be described as productive; class attendance was high and Jackie's behaviour management processes were well established.

Jackie initially found the whole group PL workshops useful in terms of initial understanding of RAMR pedagogy and explaining the use of resources and practical ways to assist her students to develop mathematical understanding.

I did do some of those PDs at the beginning of last year, I liked that because I felt like I went in prepared ... you can kind of bounce things off each other and have everyone come up with ideas.

Jackie described her teaching style before joining the XLR8 project as relatively didactic and text-book centred.

... because I've come from a background of very much text book, sit in rows, work out examples and multiple questions to get yourself through so it was really hard at the beginning for me to go from that.

The use of authentic, student-based reality along with inclusion of the Abstraction phase of RAMR prompted Jackie to alter her teaching style to incorporate kinaesthetic activities and hands-on materials.

Some of the mind, body, hand activities are really hard and it was a stretch to find something to do on certain topics ... I felt like is there any point me going into this stuff that doesn't really make sense in my head fully ... if I wasn't confident with something then I wasn't going to do it.

These aspects of the XLR8 curriculum were sufficiently foreign to Jackie's usual teaching style that they demanded more of her attention than mathematical structure and the XLR8 sequence during her initial implementation of the curriculum and caused her considerable anxiety.

In the beginning of the year I was absolutely petrified, mental breakdown, crying in the staffroom ... um I'm the type of person who wants everything to be perfect and I don't want to screw anything up, so if I feel like I might screw something up I get really upset.

In 2014, Jackie found class visits in this early part of implementation of the XLR8 curriculum to be difficult and stressful, particularly as they did not have time afterwards for coaching.

I think with the class visits, like I didn't mind having them done but I just felt like there was no time for me to talk to someone ... do you know what I mean, even if it was the day before just to be able to go well this is what I'm thinking of doing.

Jackie persevered with her implementation of the RAMR pedagogy into 2015 as she reported that she could see positive results in student learning and believed that her students enjoyed the activities. As the RAMR pedagogy became a more natural way of working for Jackie she indicated a more general change in her practice.

It's a big jump but now I have counters in my room and I use them for other classes and sometimes I just use them for an activity, yep this is what we're going to do and come up with my own thing, try to use the materials where I can so it helps me within my other classes, not just my year 8s.

The 2015 changes in support provided meant that an improved researcher-teacher relationship was built, which allowed Jackie to be more relaxed about class visits and provided the additional opportunity for collaborative planning and professional conversations. Also, she felt able to combine the best elements of her previous practice in a way that she felt improved the delivery of the curriculum for her students.

Just having another person in the room more than anything, I didn't feel judged like I did last year ... I just did what I did and asked for feedback. So now I don't feel like I'm going to be judged so yeah that's why I've moved up probably.

Once Jackie achieved a comfort level with the pedagogy, she was able to give more time and attention to developing a better understanding of the structured sequence. This allowed her to recognise and make connections to “what was coming next” and to

better select from her previous resources for use within the Mathematics phase of the RAMR to more effectively consolidate her students' learning.

I probably deviated ... a little bit more than I did the year before, found some of my own stuff as well as using the booklets. Instead of just doing it one way or just doing it the other way, I realised that you need both and if you put both together depending on the topic ... if you put both together the kids seems to grasp it more, you feel better about it.

At the end of 2015, Jackie was interviewed and asked to mark on the interpretation continuum her perception of her practice in regard to the five subscales. She was asked to mark what she believed to be her initial practice, *I* (at the beginning of 2014), and her final practice, *F* (at the end of 2015). Her self-evaluations are presented in Table 1. She felt her capacity had moved from resister to follower and finally to questioner and improver. Analysis of interview and classroom observation data gathered throughout the 2-year period led to the identification of shift in practice, labelled *M*. This seemed to occur when Jackie developed confidence in the RAMR pedagogy and so she became more able to consider the other features of the curriculum.

Continuum Sub-scale	Resister	Follower	Questioner	Improver
Structure	I	M	F	
Sequence		I	M	F
Pedagogy		I	M	F
Resources	I		M	F
Assessment	I	M	F	

Table 1: Interpretation profile of Jackie's professional growth.

## DISCUSSION AND CONCLUSIONS

Initially, the knowledge that classroom visits were for data gathering purposes caused Jackie stress in regard to whether the pedagogy was being implemented correctly and contributed to an overall perception of being judged rather than supported. The timing of visits did not coincide with gaps in her timetable leaving little time for relationship building or professional support for reflection or planning. Reflective conversations around mathematical structure and sequencing of concepts at this time did little to build Jackie's confidence as they were neither aligned with the focus of her energies nor reduced the cause of her anxiety. Theoretical discussions at this time compounded Jackie's anxiety as they highlighted additional factors to consider rather than addressing her immediate concerns.

In 2015, the changed focus on classroom observations together with professional and collaborative reflective and planning conversations (coaching sessions) allowed for an improved teacher-researcher relationship (cf. Baturo et al., 2004). It provided Jackie

with greater ownership of planning decisions (Cooper et al., 2006; Clarke & Hollingsworth, 2002). Alongside the changing support structure, Jackie's developing facility with changed pedagogical practices enabled her to feel more relaxed with day-to-day curriculum implementation allowing her time and energy to question and improve aspects of her pedagogy and the resources she implemented in her classroom.

Considering Jackie's experience with the implementation of an innovative curriculum program, introductory whole group intensive workshops are important to provide teachers with an initial understanding of the pedagogy and curriculum innovation and immediate practical teaching ideas that they can use with their students. Following initial practice with the pedagogy, classroom visits and observations are necessary to ensure teachers continue to trial the pedagogy but these visits need to be perceived as assistive and not judgemental. Collaborative planning before visits and reflective conversations after visits initially need to focus on the pedagogy and highlight positive aspects of student successes to build teachers' confidence and prompt extended practice with the pedagogy. Once the pedagogy has been mastered, attention to the sequence and structure of the mathematics may be included in professional conversations. Such conversations may help develop sufficient understanding of and fluency with the structure of mathematics, the sequence of the XLR8 curriculum and application of the RAMR pedagogy. With this increased understanding teachers are more likely to recognise the need for additional scaffolding or redirection as lessons progress, to both address students' misunderstandings and support students' acceleration to more complex levels of mathematical reasoning. This paper has illustrated the usefulness of the interpretation continuum as a tool to analyse and explain such changes to a teacher's practice in regard to the various features of a curriculum.

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# HOW DO PRIMARY SCHOOL CHILDREN SOLVE CONTINGENCY TABLE PROBLEMS THAT REQUIRE MULTIPLICATIVE REASONING?

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*Previous research has shown that primary school children are able to solve contingency table problems. However, they often use strategies that are generally invalid, such as simple frequency comparisons. In a recent study, we found that surprisingly many second- and fourth-graders made the correct decision when problems required multiplicative reasoning, while they did not refer to multiplicative strategies in their verbal responses. In this article, we analyse children's responses to these problems in more detail, to find out what reasoning processes made children succeed or fail, and how these processes depended on problem features. The results suggest that children struggle with verbalizing their reasoning when the ratio between the numbers cannot be described with number words (e.g., "half" or "double").*

## INTRODUCTION

### Understanding of contingency tables as a facet of mathematical competence

Understanding the concept of covariation is important in everyday life, and it is also a facet of mathematical competence in the domain of data and probability (e.g., CCSSI, 2010). This competence includes an ability to make decisions based on data that are represented in contingency tables. Figure 1 shows a 2x2-contingency table that represents the type of problems we used in the present study. There are two bags, *bag A* and *bag B*, that both contain blue and red chips. The total numbers of chips as well as the proportions of blue and red chips in each bag are unknown. In a (fictive) previous experiment, a number of chips have been drawn one by one and randomly from each bag. After each draw, the chip has been returned to the bag. The result of this previous experiment is represented in the figure. The question is: if you want to get a blue chip, is it better to draw from *bag A*, or from *bag B*, or does it make no difference?

A valid strategy to solve such problems is to compare the relative frequencies of blue chips between the two bags. In the above example, the relative frequency of blue chips in *bag A* is  $49/84 \approx 0.58$ , while the relative frequency of blue chips in *bag B* is  $19/23 \approx 0.83$ . As the latter is larger, it is better to choose *bag B*, because also the probability of getting a blue chip will be larger when drawing from *bag B* rather than *bag A*. While comparing the relative frequencies is a valid strategy irrespective of the specific frequencies given in the table, there are other, more straightforward strategies that are easier to apply in special cases. Research shows that children often make use of these other strategies.

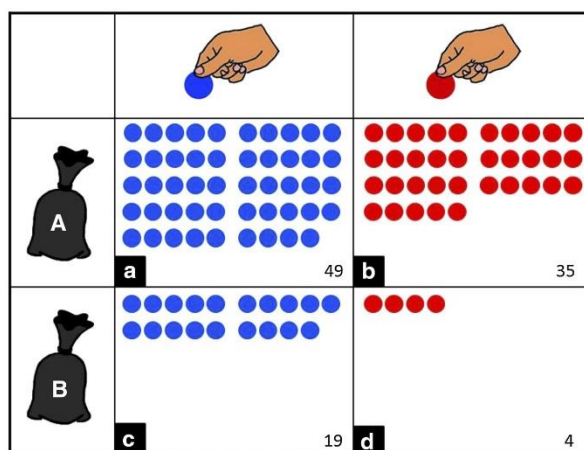


Figure 1: Example of a contingency table problem.

### Children's strategy use in solving contingency table problems

Previous research has shown that children have a basic understanding of covariation even before they are introduced to the topic at school (Reiss, Barchfeld, Lindmeier, Sodian, & Ufer, 2011; Shaklee & Paszek, 1985). Yet, they often use strategies that have been characterized as intuitive (Fischbein & Schnarch, 1997) and that are generally invalid (Batanero, Estepa, Godino, & Green, 1996). Among the predominantly used strategies are *single-cell strategies*, *two-cell strategies* and *additive strategies*. In single-cell strategies, children refer to only one cell to make their decision. In the example shown in Figure 1, they might choose *bag A* just because this bag contains so many blue chips. Two-cell strategies are strategies in which the decision is based on comparing the frequencies of only two of the four cells, while ignoring the other two cells. For example, children might argue that one should choose bag A because there are more blue chips in *bag A* than in *bag B*. Additive strategies are strategies that include addition or subtraction of cell frequencies. For example, an additive strategy would be to argue that the difference between blue and red chips is larger in *bag B* (15) rather than in *bag A* (14), so that one would—in this case correctly—choose *bag B*.

### Aim of the present study

In a recent study (Obersteiner, Bernhard, & Reiss, 2015), we used paper-pencil tests and individual interviews to investigate primary school children's strategy use in contingency table problems. Children worked on a variety of problems, some of which required multiplicative reasoning, while others could also be solved by other strategies. We found that children made use of a variety of strategies and rarely used multiplicative strategies. However, there were also children who made the correct decisions on multiplicative problems while in their verbal responses did not use multiplicative reasoning. This raises the question of how these children were able to solve these multiplicative problems, as only multiplicative strategies should lead to correct decisions. Accordingly, this study investigates children's reasoning more closely, to shed light on the cognitive mechanisms that allowed children to successfully

apply strategies that they were not able to explain verbally. First, we analyse the frequencies of strategy use in a quantitative manner. More precisely, we compare the frequencies of strategies that children use who made the correct decision with those of children who made the wrong decision. We expected that children who made the correct decision would use multiplicative strategies more often than children who made the wrong decision. Second, we use a qualitative approach to find out whether problem features (i.e., the specific numbers in the problems) affected children's reasoning. We focus our analysis on two multiplicative problems that differ greatly in the ratio between the relevant numbers (for details, see below). We expected that children would struggle less when the ratios can be described with number words (e.g., “a half”, “double”) than when this is not the case.

## METHOD

### Participants

The participants were 45 children from three German primary schools. Twenty-four of them were second-graders (11 male, 13 female; mean age 8.01 years,  $SD = 0.37$ ), and 21 were fourth-graders (11 male, 10 female; mean age 10.12 years;  $SD = 0.47$ ). None of these children had received any systematic instruction on contingency tables prior to assessment.

### Test problems

Altogether, the children in this study solved nine contingency table problems. The overall performance and self-reported strategies on all nine problems are described in Obersteiner et al. (2015). In the present article, we focus on only two of these problems, namely those that required multiplicative reasoning. As in the example described above (see Figure 1), the problems showed the result of a previous drawing from two bags A and B that contained blue and red chips. The children were asked to decide, whether it would be better to draw from *bag A*, from *bag B*, or whether it would not make no difference, when one wanted to get a blue chip. Table 1 displays the cell frequencies of the two problems.

In Problem 1, the correct response is “B”, because the ratio between blue and red chips is greater in *bag B* (20/21) than in *bag A* (30/11). Note that multiplicative reasoning is required to solve this problem successfully. Additive reasoning does not yield a correct response, because the difference between red and blue chips is equal (19) in both bags, leading to the response “no difference”. Choosing the cell with the highest frequency, or comparing the absolute number of blue chips between the bags is also invalid, because *bag A* contains the highest number of blue chips.

Problem 1	Blue	Red	Problem 2	Blue	Red
Bag A	30	11	Bag A	20	10
Bag B	20	1	Bag B	10	5

Table 1: The cell frequencies of Problem 1 (left) and Problem 2 (right).

In Problem 2, the correct response is “no difference”, because the ratios between blue and red chips in *bag A* (20/30) and *bag B* (10/15) are equal. Again, additive reasoning would lead to the incorrect response “A”, because the difference between blue and red chips is larger in *bag A* (10) than in *bag B* (5). Choosing the cell with the highest frequency or absolute comparison would also yield the incorrect response “A”, because bag A contains the highest number of blue chips.

### **Procedure**

Children were tested individually in a quiet room at their school. The interviewer introduced the problems using physical bags and chips. Then, the interviewer explained that the table shows the result of a previous drawing from the two bags, and that their classmate now wanted to draw a blue chip. The children were asked whether their classmate should draw from *bag A*, from *bag B*, or whether it would not make a difference. The interviewer also asked the children to explain how they came to their answer. If necessary, the interviewer asked clarifying questions. The complete interview was videotaped. For data analysis, we coded the correctness of answers and children’s explanations of their strategies.

### **RESULTS**

We first report on children’s accuracies on the two problems and then on the strategies they used according to their verbal responses. After that, we focus on individual children’s responses in more detail.

There were many more children who solved Problem 1 correctly compared to Problem 2. Seven (29%) out of the 24 second-graders and nine (43%) of the 21 fourth-graders solved Problem 1 correctly. Only two (8%) of the second-graders and seven (33%) of the fourth graders solved Problem 2 correctly. The solution rates for these multiplicative problems were the lowest compared to all other (non-multiplicative) problems that the children worked on (see Obersteiner et al., 2015). Low solution rates were not surprising because multiplicative reasoning is generally more demanding for children than additive reasoning or absolute number comparison. Moreover, the children were not experienced with solving contingency table problems, which probably made it particularly difficult for them to apply multiplication strategies in this unfamiliar problem situation.

We were interested in the strategies that children used to solve these multiplicative problems, and whether their strategies would yield correct or incorrect answers. Table 2 shows the overall frequencies of these strategies for the whole sample as well as the frequencies separately for the subsamples of children who gave the correct or the incorrect answer.

Overall, the majority of children’s responses exhibited two-cell and additive strategies. In Problem 1, none of the children referred to multiplicative reasoning. In Problem 2, only a small proportion of 14% of children mentioned multiplicative reasoning. This

means that multiplicative reasoning did not play an important role in children's argumentation.

Subsample	<i>n</i>	Frequency of Strategy (in %)				
		Single-Cell	Two-Cell	Additive	Multi-plicative	Other
<b>Problem 1</b>						
<i>Whole Sample</i>	<b>45</b>	<b>11</b>	<b>42</b>	<b>38</b>	<b>-</b>	<b>9</b>
Correct Responders	16	25	44	31	-	-
Incorrect Responders	29	3	41	41	-	15
<i>Second-Graders</i>	<b>24</b>	<b>17</b>	<b>50</b>	<b>29</b>	<b>-</b>	<b>4</b>
Correct Responders	7	57	14	29	-	-
Incorrect Responders	17	-	65	29	-	6
<i>Fourth-Graders</i>	<b>21</b>	<b>5</b>	<b>33</b>	<b>48</b>	<b>-</b>	<b>14</b>
Correct Responders	9	-	67	33	-	-
Incorrect Responders	12	8	8	58	-	26
<b>Problem 2</b>						
<i>Whole Sample</i>	<b>45</b>	<b>2</b>	<b>56</b>	<b>5</b>	<b>14</b>	<b>23</b>
Correct Responders	9	-	22	22	56	-
Incorrect Responders	36	3	65	-	3	29
<i>Second-Graders</i>	<b>24</b>	<b>-</b>	<b>68</b>	<b>-</b>	<b>14</b>	<b>18</b>
Correct Responders	2	-	-	-	100	-
Incorrect Responders	22	-	75	-	5	20
<i>Fourth-Graders</i>	<b>21</b>	<b>5</b>	<b>43</b>	<b>10</b>	<b>14</b>	<b>28</b>
Correct Responders	7	-	29	29	43	-
Incorrect Responders	14	7	50	-	-	43

Table 2: Percentages of the strategies the children referred to in their argumentations.

Surprisingly, even among those children who solved Problem 1 or Problem 2 correctly, only a small number gave verbal responses that included multiplicative reasoning. This finding is particularly striking in Problem 1, where seven children of grade two and nine children of grade four made a correct decision but none of their explanations was coded as multiplicative strategy. All of their answers were coded as single-cell, two-cell or additive strategies. This is surprising because these strategies are actually not valid to solve the problem correctly. Problem 2 shows a different pattern. Here, five of the nine children who solved Problem 2 correctly did reason in a multiplicative way in their verbal responses, while only one child who gave an incorrect response used multiplicative reasoning. This difference in the frequencies of multiplicative strategy use was as expected.

To shed more light on children's reasoning, Table 3 provides examples of those children's responses who made the correct decision. In Problem 1, all children referred to the fact that there is only one red chip *in bag B*. Presumably, this small number of

red chips was enough evidence for them to conclude that the risk to pick the red chip must be smaller in *bag B* than in *bag A*. This might have been the reason why the children did not explicitly mention the ratio between blue and red chip (and their strategy was therefore not coded as multiplicative), although they might have noticed the huge difference in ratios between *bags A* and *B*.

In Problem 2, almost all children who gave correct responses and used multiplicative strategies explicitly described the ratio between red and blue chips in each bag as “a half” or “double”. Presumably, the specific numbers in this problem (resulting in ratios of 0.5) allowed children to quickly recognize and verbalize that the ratios in the two bags were equal.

Strategy	Sample Responses
<b>Problem 1</b>	
<b>Second-Graders</b>	
Single-Cell	<i>Because here he has good chances. There is only one red chip.</i>
Two-Cell	Because in the table the red ones in A are eleven and in B only one.
Additive	Because there are twenty and there are only one and then one gets more blue chips there. And here are also many blue ones, but then one gets many red ones as well and there (points to bag B) it is easier to draw a blue one.
<b>Fourth-Graders</b>	
Two-Cell	<i>Because, uh, there is only one red one (points to bag B), and then it is not so dangerous to draw a red one, because it is only one. And up there (points to bag A) are more of the red ones.</i>
Additive	Because here are twenty blue chips (points to bag B), and these are ten less than here (points to bag A), but here is only one red one (points to bag B) than in A.
<b>Problem 2</b>	
<b>Second-Graders</b>	
Multiplicative	Because there are twenty and there are only ten. Or no, it does not make a difference. [I: How do you see that in the table that it does not make a difference] Because these are ten and this is a half, this is five (points to cells c and d) and this is twenty and a half is ten (points to cells a and b).
<b>Fourth-Graders</b>	
Two-Cell	Because twenty is more than ten and in B if one takes that, actually it doesn't make a difference. [I: No difference?] Yes.
Additive	Because in B there are fewer red ones but there are also fewer blue ones and in A there are more blue ones but also more red ones.
Multiplicative	Because in both it is a half. In A there are twice as many blue chips than in bag B, but in bag A there are also twice as many red chips than in bag B.

Table 3: Verbal responses of second- and fourth-graders that lead to correct decisions *Note:* I = Interviewer. The transcripts were translated from the German original by the authors.

## DISCUSSION

We analysed second- and fourth-graders' reasoning on contingency table problems that required multiplicative reasoning. We were particularly interested in the responses of children who made the correct decision. Many of them did not explicitly refer to multiplicative relations between numbers, which raised the question of how they were actually able to solve the problems correctly.

We found that children's responses depended strongly on problem type. When the ratio between the numbers was easy to describe with number words (e.g., "a half" or "double"), children were more often able to verbalize the relation between the numbers than when this was not the case. When the ratios between the numbers were more difficult to describe (here: 0.58 and 0.83), children were not able to refer to these ratios at all. Instead, most of them emphasized the small number of red chips (which was 1) to argue why it is better to choose the bag with the red chip. Presumably, the fact that the children did not reason multiplicatively in this problem does not mean that they did not notice the difference in ratios. If they had not noticed this difference, many more children would have made the wrong decision. Rather, it seems that children struggled with verbalizing the difference in ratios. In fact, many children in grades two and four are probably not familiar with terms such as "ratio", or lack the calculation skills to mentally calculate the ratios of the given numbers.

As our study suggests that children struggle particularly with verbalizing multiplicative relationships, classroom instruction might focus even more on argumentation skills, which are considered an important facet of mathematical competence (e.g., CCSSI, 2010). Another implication of our study concerns the introduction of contingency table problems in primary education. The challenge is to teach children solve covariation problems when they do not yet have the necessary skills to use multiplicative reasoning (see Van Dooren, De Bock, & Verschaffel, 2010). One option would be to use only problems that do not require multiplicative reasoning but that can be solved by other, more basic strategies. Following this approach, there is the risk that children might later on stick to these strategies even in problems in which these strategies are not applicable. In fact, focusing on additive strategies could lead to more severe problems later on (Batanero, Godino, & Estepa, 1998). The present study suggests another option, which is using problems with very simple ratios. With instruction, most children should be able to solve such problems. Using multiplicative problems has the advantage that these problems can make children aware that other, more basic strategies are not generally valid to solve contingency table problems and can therefore be beneficial to avoid misconceptions in children.

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# TYPES OF INTERACTION THAT PROMOTE OR HINDER THE NARRATIVE COHERENCE OF A MATHEMATICS LESSON

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*Classroom situations are characterized as processes of interactions in which the teacher and students collaboratively contribute to make a mathematical lesson a coherent story. The present paper clarifies the types of interactions that promote or hinder the narrative coherence of a mathematical lesson by analysing lessons conducted by different types of teachers. Our analysis identified interaction types that promote narrative coherence: multi-layered and consolidating interactions that soundly share each idea, and recursive and ZPD-making interactions that make the lesson plot coherent. In contrast, we found that one-response and answer-centred interactions in which some of the results are predetermined and the ideas are listed without connection result in students being less able to make sense of mathematics.*

## INTRODUCTION

In our attempts to clarify the qualities of the teaching of a mathematics lesson referred to as “structured problem solving” by Stigler et al. (1999), we have focused on a quality lesson having the structure of a narrative (Noe, 2005; Bruner, 1986). As Stigler et al. (1999) identified, a mathematics lesson of “structured problem solving” follows a script: reviewing the previous lesson, presenting the problem for the day, students working individually or in groups, discussing solution methods, and highlighting and summarizing the main point. However, this script should not be directly equated with an effective mathematics lesson, because there is a range of quality from effective to ineffective teaching, even if the pattern is indeed adopted by most teachers in Japan. We have shown that whether a lesson is developed like a narrative or just as ‘steps’ of a script greatly affects its quality by comparing the lessons conducted by experienced and novice teachers (Okazaki et al., 2014, 2015).

The present paper further examines how a coherent plot of a mathematics lesson can be produced. We focus on the patterns of teacher–student interactions, and clarify how they may promote or hinder the coherence of a lesson. We then characterize classroom situations as processes of interactions in which the teacher and students contribute to making a coherent plot of the lesson according to their sense and purpose of these events (Krummheuer, 2000).

## THEORETICAL BACKGROUND

### A quality mathematics lesson as a coherent narrative plot

A narrative is a speech act that fulfils the conditions of events, contexts, and temporal sequences. We may regard mathematics teaching as a teacher’s narrative act of communicating his/her rich mathematical experience to students. A narrative act here

means “a speech act that refers to the plural and temporally distant events and plots them along a temporal order of beginning–middle–end” (Noe, 2005, p. 326).

Bruner (1986) distinguished two modes of thought for the construction of our reality: paradigmatic and narrative modes. He wrote that the paradigmatic mode of thought “deals in general causes, and in their establishment, and makes use of procedures to assure verifiable reference and to test for empirical truth”, while the narrative mode “deals in human or human-like intention and action and the vicissitudes and consequences that mark their course. It strives to put its timeless miracles into the particulars of experience, and to locate the experience in time and place” (p. 13). We then consider that what drives the narrative is “Trouble: some misfit between Agents, Acts, Goals, Settings, and Means” (Bruner, 1996, p.94). In overcoming the trouble, we may examine how the zone of proximal development (ZPD), that is “the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more [sic] capable peer” (Vygotsky, 1978), can be constructed through teacher–student interactions.

We here mention the distinction between the materials and the plot in the narrative (Vygotsky, 1971). Materials refer to events, incidents and characters, which the writer adopts in making the narrative, and the plot refers to how these materials are composed as a narrative; i.e., the artistic links among materials. We therefore do not identify the factual and chronological sequences of the material with the plot because these often give rise to different emotions. Like the writer, experienced teachers may compose the lesson by relating the students’ ideas with each other to allow the students to see the meaning and effectiveness of mathematical ideas beyond just mastering the procedures. Likewise, Stigler and Perry (1988) referred to a well-formed story and an ill-formed story. The latter consists of a simple list of events strung together by phrases such as ‘and then...’, but with no explicit reference to the relations among events.

### **Interactions in the mathematics classroom**

Krummheuer (2000) conceived the classroom situation as processes of interaction in which students and teachers contribute according to their sense and purpose of these events. He characterized a plot as aspects of both something already fixed and something fragile, not yet entirely executed, still changeable, and stated “the tension between these two dimensions of this concept is crucial for its adaptation to classroom interaction and its function for leaning” (Krummheuer, 2000, p.25).

A narrative act is essentially a speech act for sharing with others. It is then important to analyse how the interactions for sharing knowledge come about. Voigt (1998, p.214) stated that “the negotiation of meaning is a necessary condition for learning when the students’ understandings differ from the understanding the teacher wants the students to gain. This difference is not necessarily a deficit characterized discourse in the classroom”. Additionally, he mentioned “the students and teacher achieve a thematic coherence in their discourse. Interactively, they constitute a mathematical theme that,

on the one hand depends on the participants' contributions, whereas on the other hand cannot be sufficiently explained by the thoughts and intentions of any one person alone" (Voigt, 1995, p.164). This suggests that the theme is not a fixed body of knowledge, but changes through the negotiation of meaning, and that an individual's learning is affected by the evolution of themes.

## METHODOLOGY

Our study involved three types of elementary school teachers: A) two experienced teachers who specialize in mathematics teaching (Mr. F and Mr. O), B) two experienced teachers who do not specialize in mathematics teaching (Ms. YM and Ms. YS), and C) two teachers who have a few years' experience (Mr. S and Mr. H). We asked each teacher to conduct a lesson.

We selected content from a fifth-grade mathematics textbook; i.e., 'the area of a parallelogram for which the height cannot be known from a straight line inside the parallelogram' (Fig. 1, right). We assumed that there are some differences in teachers' behaviour related to their students' difficulty in understanding the meanings of area and height. The lessons were recorded with video cameras and by taking field notes, and transcripts were made.

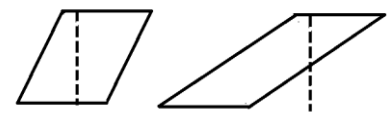


Figure 1

In our data analysis, we first interpreted all meaningful teacher–student interactions, discussed our interpretations, and integrated the interpretations in terms of triangulation. We next identified the lesson scenes that were discriminable as units of activity/discussion, and examined how they affect the subsequent lesson development, before trying to reconstruct the lesson 'plot' until all authors agreed. We in particular focused on how each interaction between teacher and students is related with other interactions and affects the whole lesson plot to clarify whether it promotes or hinders the coherence of a mathematics lesson of structured problem solving.

## RESULTS

### Multi-layered and recursive interactions

There were several interactions that enhanced the students' deeper understanding of the important idea in Mr. F's lesson. Mr. F employed one particular type of interaction in which he attempted to enrich the idea of using four  $3 \times 1$  parallelograms using plural voices (Fig. 2.1–2.4).

- Student 1: There are one, two, three, four unit parallelograms.
- Mr. F: What is this? (He circled the bottom unit parallelogram.) (Fig. 2.1)
- Student 2: The unit parallelogram is  $3 \times 1 = 3$  squares. As there are four unit parallelograms, the total number of squares is 12.
- Mr. F: Can anybody explain this in a similar way? (He wrote the formula  $3 \times 1 = 3$ .) (Fig. 2.2)

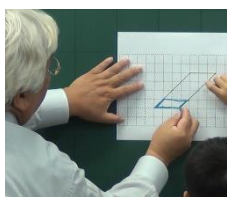


Figure 2.1

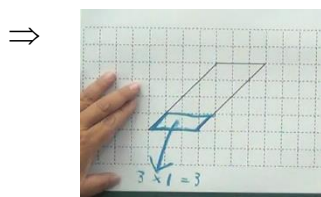


Figure 2.2

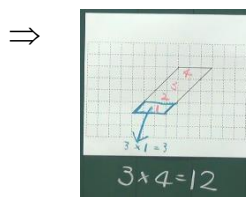


Figure 2.3

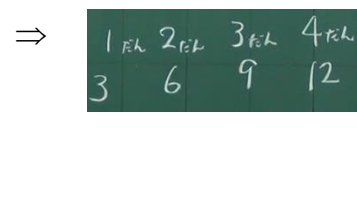


Figure 2.4

Student 3: The bottom unit parallelogram is 3 squares. There are four unit parallelograms, so  $3 \times 4 = 12$ . (Mr. F wrote the numbers 3, 6, 9, 12.) (Fig. 2.3)

Afterward, Mr. F confirmed with students that the area increases by 3 in each step of the unit parallelogram with a table (Fig. 2.4). Mr. F gradually wrote explanations on the blackboard to make the idea more detailed as he interacted with the different students, and consequently, the students seemed to share the meaning of this idea. We characterize this series of interactions as being multi-layered.

This idea was used to rethink today's point. When Mr. F asked the students again to find the height of the shape, they were not confident in their answer. Here, Mr. F reflected on the above idea with the students, and they said together, "The height of the smallest parallelogram is 1 cm, the height of the parallelogram one step higher is 2 cm..." Moreover, he modified the table by changing the word "step" to "cm" and newly adding cm<sup>2</sup> (Fig. 3). Consequently, the students reinterpreted one "step" as 1 cm of height and then understood that the area formula that they already knew was applicable to all parallelograms.

1 cm	2 cm	3 cm	4 cm
3	6	9	12
cm <sup>2</sup>	cm <sup>2</sup>	cm <sup>2</sup>	cm <sup>2</sup>

Figure 3

We observed an interaction in which the quality of an idea was improved by reinterpreting the knowns. (We call this a recursive interaction.) We considered that this interaction through which the essential themes (what the height is and how an area formula is conceptualized) emerged contributed to the coherence of the lesson plot.

### Consolidating and ZPD-making types of interaction

During the class discussion following individual activities in Ms. YM's class, Ms. YM asked student M to present his idea of dividing the parallelogram into top and bottom parts and arranging the parts alongside each other. Student M only stated the formula  $8 \times 3$  and the answer 24. Next, Ms. YM

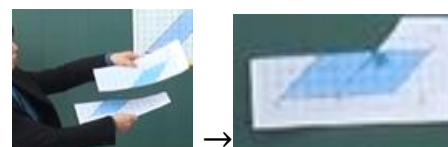


Figure 4

demonstrated the student's idea using paper she had already prepared (Fig. 4), and then invited the other students to share the idea by explaining it: "I think that he considered that the figure cannot be made into a rectangle even if it is cut vertically. He changed the figure into a parallelogram with a shape like a rectangle". Finally, Ms. YM confirmed the idea again using the formula  $8 \times 3 = 24$ . In summary, the interaction pattern for sharing an idea was 1) asking a student to present his/her idea, 2) clarifying the idea by demonstration, 3) inviting other students to explain the idea, and 4) confirming the idea using a formula. This interaction, which was common to all

experienced teachers, contributed to consolidating and sharing the idea among students. (We call this a consolidating interaction.)

We observed a teacher–student interaction in which the gap between two ideas proposed by students functions like a ZPD. Immediately after the above proposal, student S stated his procedural knowledge (without meaning) ‘multiplying the base BC by the height DE’, which is what appeared in his textbook (Fig. 5). Additionally, he emphasized that the height was the distance between parallel lines. Here, Ms. YM asked all students, “Why can we define the height as the line drawn from a vertex to the extended line of the base?” and did not authorize the procedure  $BC \times DE$ .

Ms. YM here tried to set a new goal of connecting the above two ideas. To do so, she reinforced the former idea presented by student M using student R’s idea of dividing the parallelogram into half but not moving the figure, and let students notice that the height was a sum of heights of two parallelograms. Ms YM wrote the formula  $4 \times 3 + 4 \times 3$ , transformed it into  $4 \times (3 + 3)$ , and explained that the height was 6.

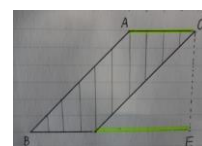


Figure 5

Given that both formulas became  $4 \times 6$ , the students could begin considering why the formula was true. Ms. YM then asked student KN to present his idea of isometric change by cutting the diagonal of the original parallelogram (Fig. 6). KN’s idea was well understood since the formula became  $4 \times 6$ , which was same as that above. Ms. YM here tried to integrate the previous ideas; i.e., KN’s idea is the same as M’s idea because their heights appeared inside the arranged figures and also the same as S’s idea because the height was the distance between the upper and lower bases.

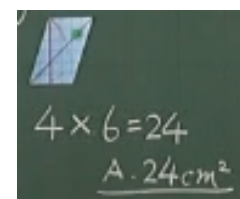


Figure 6

We see that when the students’ basic idea (with meaning) was compared with the scientific idea (without meaning) with similarity and a moderate distance between them, the students could set the new goal of connecting the ideas. We observed similar interactions in Ms. YS’s lesson, in which Ms. YS willingly used the exemplary students’ ideas written in a textbook as the latter, scientific idea. We refer to this type of interaction as a ZPD-making interaction, because of the learning goal of connecting two idea levels that emerge from different background knowledge.

### Predetermined outcome and answer-centred interaction

We found that the lessons of two teachers who have a few years’ experience (Mr. S and Mr. H) included similar interaction patterns.

Mr. H asked the students where the base and height are in the figure, and student IW responded, “If you extend the base BC and then extend it upward from this point, that is the height” (Fig. 7). Although the other students questioned the location of the height, Mr. H authorized IW’s opinion and asked, “How do you find the area when the height is not



Figure 7

inside the figure?” Here, we find that the height was conceived as being ‘already determined’.

Five ideas for obtaining the area were presented in Mr. H’s class. Since the placement of the height was predetermined, the interaction was centred on and restricted to whether the value of the area was correct. For example, the idea of dividing the original parallelogram into top and bottom parallelograms was confirmed as follows (Fig. 8).

Mr. H: I will introduce Student SA’s idea. He used this figure. Can anyone explain SA’s idea?

Student Y: In this situation we cannot multiply by the height. If we divide the parallelogram into halves and set the halves alongside each other, there is a base and also a height on the base.

Mr. H: How long is the length of the base?

Student Y: It’s 8 cm.

Mr. H: How long is the length of the height?

Student Y: It’s 3 cm.

Mr. H: What is the formula for the area?

Students: It’s 8 times 3.

Mr. H: Please raise your hand if you agree. Let’s move on to the next idea.



Figure 8

We note that Mr. H checked only the formula of the ‘transformed’ figure ( $8 \times 3$ ), and he had never referred to the area formula of the parallelogram in question ( $4 \times 6$ ).

This exchange was almost the same as that in Mr. S’s class. In Mr. S’s class, this exchange led to students’ doubts in the summary scene. When Mr. S stated, “the height is outside the figure”, several students questioned the statement, saying “what is the meaning of ‘outside’” and “I am not sure when the height is not on the base”. These students’ doubts suggest that the lesson was not a coherent story for the students.

### Listing ideas disconnectedly using a one-response interaction

We see that the above interaction consists of a simple one-response interaction between the teacher and particular students: 1) a student (or a substitute) presents the idea; 2) the teacher explains the idea or asks one or two simple questions to confirm the idea; and 3) the teacher authorizes the student’s opinion by emphasizing the formula of the post-transformed figure. Here, the main actor was the teacher, not the students.

Additionally, one interaction unit was disconnected with another. Indeed, both Mr. S and Mr. H used phrases such as “and then” when shifting between the interaction units, with no reference to the relations among ideas. Furthermore, these ideas were not used to formulate the area formula in the scene of the summary at the end of the lesson.

## DISCUSSION

In our attempt to clarify why differences in qualities of a mathematics lesson emerge even if the same script of “structured problem solving” is adopted, we identified several



interactions that functioned for and against the narrative coherence of the lesson by analysing lessons conducted by different types of teachers (Table 1).

	For	Against
Clarifying the meaning of an idea	Multi-layered interaction Consolidating interaction	One-response interaction Answer-centred interaction
Plotting the lesson	Recursive interaction ZPD-making interaction	Listing disconnected ideas Predetermining the outcome

Table 5: The interaction types for and against making the lesson narratively coherent.

To realize the coherent narrative of a lesson, the students' ideas for obtaining the area that constitute the narrative need to be understood by and shared with students as the actors. We observed two types of teacher–student interactions that contributed to clarifying the idea. One is what we call the multi-layered interaction, by which the meaning of one idea is gradually clarified using several students' multiple voices and with the process of constructing the idea. The other is what we call the consolidating interaction, which aims to clarify one student's idea through a network of demonstrating the idea, inviting others to explain the idea, and confirming the idea using a formula. Meanwhile, a one-response interaction was often observed in the novice teachers' classes, where the teacher's talk and questioning was followed by one student's short answer and then by the teacher's explanation. Given that the main actor was the teacher, we believe that the aim was not so much to share the idea among the students. Additionally, it is noted that the novice teachers mainly confirmed the value of the area (answer), in contrast to how the experienced teachers emphasized the 'method' of obtaining the answer with its meaning.

Even if each idea is shared by students, the whole lesson does not become a well-formed story for attaining the final learning goal. To realize a coherent plot of the lesson, we found that two types of interactions were used. One was what we call recursive interaction. The interaction is constructed as the ideas discussed in some lesson scene being used to solve some trouble in a later scene. In Mr. F's class, one fundamental idea for obtaining the area was finally used to construct the concept of the area formula, in particular that of height. Another is what we call a ZPD-making interaction, where the students' basic idea with meaning is compared with another formal idea without meaning at a moderate cognitive distance, and how to relate them with each other meaningfully is set as a new learning goal. We see the difference between the former and latter ideas as if they are the developmental potential in a ZPD. This interaction does not occur in a purely bottom-up manner, but aims to cross between two idea levels; this seems to contribute to making the lesson a coherent story.

However, determining the outcome from the start is not originally a characteristic of a narrative. Rather, it may constitute an ill-formed story. Only listing the ideas disconnectedly between events also seems incoherent from the perspective of the



narrative plot. As we saw in the lessons of Mr. S, such interaction may lead to student doubts owing to discrepancy between what the students explore during the lesson and the teacher's summary.

Finally, we mention our future tasks. In this paper, we mainly focused on a teacher's instructional acts. Conversely, the story-making process from the students' point of view has not yet been clarified sufficiently. In particular, we consider it important to explore how the students' learning goal may change and develop through teacher–student interactions during a lesson to further clarify the narrative coherence of a mathematics lesson.

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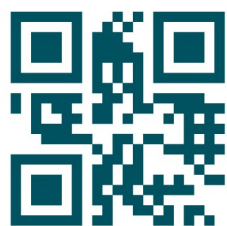
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